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TWO CONGRUENCE PROPERTIES OF LEGENDRE POLYNOMIALS

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1. Let $P_n(x)$ be the Legendre polynomial ($n \geq 0$, cf. Whittaker-Watson [2]) and p be an odd prime number. In this article we give two congruence properties of $P_n(x)$ modulo any power of p .

Our principal results are as follows:

Theorem 1. *For any $m \geq 1$, we have*

$$P_{mp^{v+1}-1}(x) \equiv P_{mp^v-1}(x^p) \pmod{p^{v+1}}.$$

Theorem 2.

$$P_{mp^{v+1}}(x) \equiv P_{mp^v}(x^p) \pmod{p^{v+1}}.$$

The $P_n(x)$ is defined as follows:

$$(1) \quad (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)h^n,$$

and can be written explicitly as

$$(2) \quad P_n(x) = \sum_{r=0}^m (-1)^r \frac{(2n-2r)!}{2^n \cdot r!(n-r)!(n-2r)!} x^{n-2r},$$

where $m=n/2$ or $(n-1)/2$ whichever is an integer.

We shall use the results of Honda [1] freely.

2. Proof of Theorem 1

Put
$$f(h) = \sum_{n=1}^{\infty} \frac{P_{n-1}(x)}{n} h^n.$$

Then we have

$$f'(h) = (1-2xh+h^2)^{-1/2}.$$

Now transform the variable h to t by

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$$h = \frac{2t(1-xt)}{1-t^2}.$$

Then

$$\frac{dt}{\sqrt{1-2hx+h^2}} = \frac{2dt}{1-t^2}$$

and $h=2t+\dots\in\mathbb{Z}[x][[t]]$. As is easily verified, the special element which kills $\int_0^t \frac{dt}{1-t^2} \pmod{p}$ is $p-T$ (cf. Honda [1]). Hence $f(h)$ is of type $p-T$, regarding $\mathbb{Z}_p[x]$ as a discrete valuation ring $\mathfrak{o}$ in the notation of [1, §2]. (In this case σ is given by $(k(x))^\sigma = k(x^p)$ for $k(x) \in \mathbb{Z}_p[x]$.) This proves

$$pf(h) \equiv f(h^p) \pmod{p},$$

i.e.,

$$P_{mp^{v+1}-1}(x) \equiv P_{mp^v-1}(x^p) \pmod{p^{v+1}}.$$

3. Proof of Theorem 2

Put

$$g(h) = \sum_{n=1}^{\infty} \frac{P_n(x)}{n} h^n.$$

Then

$$g'(h)dh = \{(1-2hx+h^2)^{-1/2}-1\} dh/h.$$

By the transformation

$$h = 2t(1-xt)/(1-t^2) = 2t + \dots,$$

we get

$$g'(h)dh = 2(x-t)dt/(1-t^2)(1-xt).$$

Furthermore we transform t to u by

$$1+u = \frac{\sqrt{1-t^2}}{1-xt}$$

and get

$$g'(h)dh = \frac{2du}{1+u}.$$

Since

$$h = 2xu + \dots,$$

the formal group whose transformer is $g(h)$ is of type $p-T$. This proves

$$pg(h) \equiv g(h^p) \pmod{p},$$

namely

$$P_{mp^{v+1}}(x) \equiv P_{mp^v}(x^p) \pmod{p^{v+1}}$$

which completes the proof of Theorem 2.

References

- [1] T. Honda: *On the theory of commutative formal groups*, J. Math. Soc. Japan **22** (1970), 213–246.
- [2] E.T. Whittaker and G.N. Watson: *A Course of Modern Analysis*, Cambridge, 1963.

