The structure of the bordism group $U_*(B_{Z_p})$

Author(s) Kamata, Masayoshi

Citation Osaka Journal of Mathematics. 7(2) P.409-P.416

Issue Date 1970

Text Version publisher

URL https://doi.org/10.18910/7257

DOI 10.18910/7257

Osaka University Knowledge Archive: OUKA

http://ir.library.osaka-u.ac.jp/dspace/

Osaka University
THE STRUCTURE OF THE BORDISM GROUP $U_*(BZ_p)$

Dedicated to Professor Keizo Asano on his 60th birthday

MASAYOSHI KAMATA

(Received March 10, 1970)
(Revised April 3, 1970)

In this paper, we determine the additive structure of the complex bordism group $U_*(BZ_p)$, where $BZ_p$ is a classifying space for $Z_p$, $p$ an odd prime. Conner-Floyd [1] computed the case $p=2$, and solved by a geometric method. Here we use the Mischenko series [4] instead of the geometric method of Conner-Floyd.

The author wishes to express his thanks to Professor S. Araki for his many valuable suggestions and discussions.

1. The order of the element $[L^{n-1}(p), i]$

We denote by $U_*(X, A)$ and $U^*(X, A)$ the complex bordism group and the complex cobordism group of a CW complex pair $(X, A)$ respectively. Let $L^n(p)$ be a $(2n+1)$-dimensional lens space defined by a rotation $\Gamma$ which acts on a $(2n+1)$-sphere $S^{2n+1}$ in complex coordinate by $\Gamma(z_0, \ldots, z_n) = (\rho z_0, \ldots, \rho z_n)$ with $\rho = \exp(2\pi i/p)$. $BZ_p$ is a CW complex of which the $(2n+1)$-skeleton is $L^n(p)$. The cell structure of $L^n(p)$ is given as follows:

$$L^n(p) = s^1 \cup \rho^2 \cup e^2 \cup \rho e^3 \cup \rho^2 e^{2n} \cup e^{2n+1}.$$  

Applying the exact sequence of the bordism group to a pair $(L^{n+1}(p), L^n(p))$, it follows immediately that

$$U_k(L^n(p)) \approx U_k(L^{n+1}(p)) \text{ for } k<2n+1.$$  

In this section we study the order of the element

$$[L^{n-1}(p), i] \in U_{2n-1}(L^n(p)) \approx \cdots \approx U_{2n-1}(BZ_p),$$

where $i:L^{n-1}(p) \to L^n(p)$ is the inclusion. In order to determine the order of $[L^{n-1}(p), i]$, we use the duality isomorphism between bordism groups and cobordism groups, and the relation between $K$-theories and cobordism theories.

**Theorem 1.1** (Atiyah-Kultze [3]). *If $X$ is an $n$-dimensional compact $U$-manifold, there is an isomorphism $D:U_k(X) \to U^{n-k}(X)$.*
$D[M^k, f]$ is given as follows. For a large integer $r$ such that $n+r-k$ is even, there is an embedding map $f^*: M^k \rightarrow S^r \times X^* - \{\ast\}$, which is homotopic to the map $f: M^k \rightarrow X \subset S^r \times X^* - \{\ast\}$, where $\ast$ denotes the base point. Denote by $N(M^k)$ the normal bundle of $M^k \subset S^r \times X^* - \{\ast\}$, and there is a bundle map $\varphi$ from $N(M^k)$ to the $(n+r-k)/2$-dimensional universal complex bundle $EU((n+r-k)/2)$. Then we can construct the map

$$d(f): S^r \times X^* \rightarrow T(N(M^k)) \rightarrow MU((n+r-k)/2),$$

where $T(N(M^k))$ and $MU((n+r-k)/2)$ are Thom complexes of $N(M^k)$ and $EU((n+r-k)/2)$ respectively. $D[M^k, f] = [d(f)]$.

The following theorem which connects $K$-theories with cobordism theories was given by Conner-Floyd [1].

**Theorem 1.2** (Conner-Floyd). If $X$ is a finite connected CW complex, the homomorphism $\rho: K(X) \rightarrow U^*(X)$ which maps $[\xi^n]_n$ into the 1-st cobordism Chern class $c_1(\xi^n)$ of $\xi^n$ is the monomorphism of $K(X)$ onto a direct summand of $U^*(X)$.

Let $\pi: L^n(p) \rightarrow CP^n$ be a canonical projection. If $\eta$ is a canonical complex line bundle over $CP^n$, $\tau_c(\mathbb{C}P^n) \oplus 1_{\mathbb{C}} = (n+1)\eta$ and $\tau(L^n(p)) \oplus 1 = \pi^{\ast}(\tau(\mathbb{C}P^n) \oplus 2)$, where $\tau(L^n(p))$ and $\tau(\mathbb{C}P^n)$ are tangent bundles over $L^n(p)$ and $\mathbb{C}P^n$ respectively, and lower index $c$ denotes a complex vector bundle. Therefore $L^n(p)$ is a $U$-manifold. Considering homomorphisms $D$ and $\rho$ of Theorems 1.1 and 1.2 for a space $L^n(p)$, we have the following

**Proposition 1.3.** $D[L^{n-1}(p), i] = \rho(\pi^*\eta - 1_c)$.

Proof. Let $\nu_c$ be the normal bundle of $CP^n$ in $CP^n$. Since $\tau_c(CP^n)| CP^n-1 = \tau_c(CP^n-1) \oplus \nu_c$,

$$r((\tau_c(CP^n)| CP^n-1) \oplus 1_c) = r(\tau_c(CP^n-1) \oplus 1_c \oplus \nu_c),$$

where $r$ denotes the real restriction. Moreover,

$$\pi^*r(\tau_c(CP^n) \oplus 1_c)| CP^n-1 = \pi^*r((\tau_c(CP^n-1) \oplus 1_c) \oplus \nu_c),$$

which implies that $\pi^*r\nu_c$ is the normal bundle of $L^{n-1}(p)$ in $L^n(p)$. The total space $E(\tau_c(CP^n))$ of $\tau_c(CP^n)$ can be represented as the set of all pairs $[\hat{u}, \hat{v}]$ with $||\hat{u}||=1$, $\hat{u} \in C^{n+1}$ and $\langle \hat{u}, \hat{v} \rangle = 0$ by the standard Hermitian metric of $C^{n+1}$, under the identification $(\hat{u}, \hat{v}) \equiv (\lambda \hat{u}, \lambda \hat{v})$ for all $\lambda \in \mathbb{C}$, $||\lambda||=1$. Now we define the Hermitian metric $F: E(\tau_c(CP^n)) \times E(\tau_c(CP^n)) \rightarrow C^1$ by

$$F([\hat{u}_1, \hat{v}_1], [\hat{u}_2, \hat{v}_2]) = \langle \hat{u}_1, \hat{u}_2 \rangle \langle \hat{v}_1, \hat{v}_2 \rangle.$$

Then the total space of $\nu_c$ is
STRUCTURE OF THE BORDISM GROUP $U^*(BZ_p)$

$E(\nu_c) = [[\bar{u}, \bar{v}] \in E(\tau_c(CP^n)) : \bar{u} \in C^n$, and $F([\bar{u}, \bar{v}], [\bar{u}, \bar{v}]) = 0$

for each $[\bar{u}, \bar{v}] \in E(\tau_c(CP^{n-1}))$,

that is, $E(\nu_c)$ consists of the elements $[\bar{u}, \bar{v}]$, where $\bar{v} = (0, \cdots, 0, z_n)$. Therefore $E(\pi*\nu_c)$ can be represented as the set of all pairs $[\bar{u}, \bar{v}]$ with $||\bar{v}|| = 1$, $\bar{u} \in C^n$ and and $\bar{v} = (0, \cdots, 0, z_n)$ under the identification $(\bar{u}, \bar{v}) \equiv (\rho \bar{u}, \rho \bar{v})$, $\rho = \exp(2\pi i/p)$.

Consider the open submanifold

$L^n(p) = \{[z_0, \cdots, z_n] \in L^n(p) ; |z_n| < 1\}$

of $L^n(p)$; there is a diffeomorphism

$g : L^n(p) \rightarrow E(\pi*\nu_c)$

given by

$g([z_0, \cdots, z_n]) = ([z_0/\lambda, \cdots, z_{n-1}/\lambda], (0, \cdots, 0, z_n/\lambda]), \lambda = \sqrt{\sum_{i=0}^{n-1} |z_i|^2}$,

that is, $L^n(p)$ is the tubular neighborhood of $L^{n-1}(p)$ in $L^n(p)$. We define the map

$f : E(\pi*\nu_c) \rightarrow \tilde{\eta}'$

by $f([z_0, \cdots, z_{n-1}], (0, \cdots, 0, z_n)) = ([z_0, \cdots, z_{n-1}], z_n)$, where $\eta'$ is a canonical complex line bundle over $CP^{n-1}$. Let $h$ be a standard homeomorphism between the Thom complex of $\eta'$ and $CP^n$. Then, for $[L^{n-1}(p), i] \in U_{2n-1}(L^n(p))$, we have

$d(i)([z_0, \cdots, z_n]) = [h \circ g([z_0, \cdots, z_n]), |z_n| \pm 1$.

$[0, \cdots, 0, 1], |z^n| = 1$.

It follows that $\pi = d(i)$. Since $\rho(\pi*\eta - 1_c) = [\pi]$, the proposition follows. q.e.d.

Kambe [2] showed that the order of $\pi*(\eta - 1_c) \in \tilde{K}(L^n(p))$ is $\rho^{(n-1)/(p-1)}$. Then we have the following

**Proposition 1.4.** $[L^{n-1}(p), i] \in U_{2n-1}(BZ_p)$ is of order $\rho^{(n-1)/(p-1)}$.

2. **The structure of $U_n(BZ_p)$**.

We consider the 2n-skeleton $L^o_n(p)$ of $L_n(p)$, that is,

$L^o_n(p) = s^i \cup p e^i \cup e^i \cup p \cdots \cup e^{2n-1} \cup p e^{2n}$.

Using the bordism exact sequence for a pair $(L^n(p), L^o_n(p))$, we have $U_k(L^o_n(p)) \approx U_k(L^n(p))$, for $k < 2n$. Therefore, for a large $n$

$U_{2k+1}(L^n(p)) \approx U_{2k+1}(L^o_n(p)) \approx U_{2k+1}(L^o_n(p)), U_{2k}(L^n(p)) \approx U_{2k}(L^o_n(p))$.

The bordism spectral sequence $\{E^*_{s,t}\}$ for $L^o_n(p)$ is trivial and if $s + t = 2k$, then
Lemma 2.1. If \( \alpha[L^r(p), i] = 0 \) in \( U_{2j+1}(L^n(p)) \) for a large \( n \) and \( \alpha \in U_{2j} \), then \( \alpha \in \mathfrak{p}U_{2j} \).

Proof. Since \( U_{2j+1}(L^n(p)) \approx U_{2j+1}(L_0^*(p)) \), we can assume that \( \alpha[L^r(p), i] \in U_{2j+1}(L_0^*(p)) \). Consider the reduced bordism spectral sequence \( \{ E^r_{s,t} \} \) for \( L_0^*(p) \), which is trivial. There is a filtration \( 0 = J_{0,0} \subset J_{1,0} \subset \cdots \subset J_{k,0} = U_k(L_0^*(p)) \) with \( J_{s,t} \neq 0 \) for \( s, t > 0 \). The multiplication

\[
m : U_s(L_0^*(p)) \otimes U_t \rightarrow U_{s+t}(L_0^*(p))
\]

induces the following commutative diagram

\[
\begin{array}{ccc}
U_1(L_0^*(p)) \otimes U_{2j} & \xrightarrow{\mu \otimes \text{id}} & U_1(L_0^*(p)) \otimes U_{2j} \\
\downarrow m_1 & & \downarrow m_2 \\
J_{1,2j} & \xrightarrow{\mu'} & H_1(L_0^*(p), U_{2j})
\end{array}
\]

where \( \mu \) is the edge homomorphism.

\[
\alpha \mu[L^r(p), i] = m_2(\mu \otimes \text{id})([L^r(p), i] \otimes \alpha) = \mu'(m_1([L^r(p), i] \otimes \alpha)) = \mu' \alpha[L^r(p), i] = 0.
\]

On the other hand, \( \alpha[L^r(p), i] \) is a generator of \( H_1(L_0^*(p)) \). Since \( H_1(L_0^*(p)) \) is \( \mathfrak{p} \)-torsion group, \( \alpha \in \mathfrak{p}U_* \). q.e.d.

Lemma 2.2. Suppose that \( X \) is an \( n \)-dimensional \( U \)-manifold. If \( [M_1, f_1], [M_2, f_2] \in U_\mathfrak{p}(X) \) are the elements represented by embedding maps \( f_k : M_k \rightarrow X \) \( (k = 1, 2) \). If the two embeddings are transversal to each other, then \( D[M_1, f_1]D[M_2, f_2] = D[M_1 \cdot M_2, f_1 \cdot f_2 | M_1 \cdot M_2] \), where \( M_1, M_2 \) is intersection manifold of \( M_1 \) and \( M_2 \) in \( X \).

Proof. We can suppose that \( M_1, M_2 \) is a submanifold satisfying \( N(M_1 \cdot M_2) = i_\ast N(M_1) \oplus i_\ast N(M_2) \), where \( i_k : M_1, M_2 \rightarrow M_k (k = 1, 2) \) is the inclusion map and \( N(M) \) is the normal bundle of \( M \) in \( X \). \( D[M_1 \cdot M_2, f_1 \cdot f_2 | M_1 \cdot M_2] \) is constructed by the bundle map

\[
\begin{array}{ccc}
N(M_1 \cdot M_2) & \xrightarrow{\Delta} & N(M_1) \times N(M_2) \\
\downarrow & & \downarrow \\
M_1 \cdot M_2 & \xrightarrow{\Delta} & M_1 \times M_2
\end{array}
\]

where \( \Delta \) is a diagonal map, and \( s \) and \( t \) are the dimensions of \( N(M_1) \) and \( N(M_2) \) respectively. In view of the definition of multiplication in the cobordism group, we complete the proof.

Suppose that \( \gamma \) is the canonical line bundle over \( CP^n \), it follows from Lemma 2.2 that \( \{ c_1(\pi^* \gamma) \}^* = D[L_0^*(p), i] \).

Mischenko obtained the following theorem [4], which plays an important
role to deduce some relations of the elements of $U_{n-1}(BZ_p)$.

**Theorem 2.3** (Mischenko). For a complex line bundle $\xi$ over a CW complex $X$, define a series $g(c_i(\xi))$ by

$$g(c_i(\xi)) = \sum_{k \geq 0} x_k c_i(\xi)^{k+1} \in U^*(X) \otimes \mathbb{Q}, \quad x_k = [CP^*].$$

This satisfies, for line bundles $\xi$ and $\eta$, the relation

$$g(c_i(\xi \otimes \eta)) = g(c_i(\xi))g(c_i(\eta)).$$

**Proposition 2.4.** There exists $\alpha_a \equiv 0 \mod p$ such that

$$p^a[L^a(p-1)(p), i] = \alpha_a[L^s(p), i].$$

Proof. The proof is by induction on $a$. Let $\eta$ be the canonical complex line bundle over $CP^p$. By Theorem 2.3

$$g(c_i(\eta^p)) = pg(c_i(\eta)) = p\left\{c_i(\eta) + \frac{x_1}{2} c_i(\eta)^2 + \cdots + \frac{x_{p-1}}{p} c_i(\eta)^p\right\}$$

and

$$(p-1)!g(c_i(\eta^p)) = p!c_i(\eta) + p(p-1)!x_1c_i(\eta)^2 + \cdots + (p-1)!x_{p-1}c_i(\eta)^p.$$ 

Since $U^*(CP^p)$ is torsion free, the above relation is an integral relation. Then, by the naturality of $g$ and $(\pi^*\eta)^p=1$,

$$p!c_i(\pi^*\eta) + p(p-1)!x_1c_i(\pi^*\eta)^2 + \cdots + (p-1)!x_{p-1}c_i(\pi^*\eta)^p = 0.$$ 

Using Lemma 2.2,

$$p![L^{p-1}(p), i] + p(p-1)!x_1[L^{p-2}(p), i] + \cdots + (p-1)!x_{p-1}[L^0(p), i] = 0.$$ 

Since the order of $[L^j(p), i]$ is $p$ for $j<p-1$ and the order of $[L^{p-1}(p), i]$ is $p^s$ by Proposition 1.4,

$$p![L^{p-1}(p), i] + (p-1)!x_{p-1}[L^0(p), i] = 0.$$ 

Since $p$ is prime, the case $a=1$ follows. Suppose our assertion is true for $b<a$. Let $\xi$ be the canonical line bundle over $CP^{a(p-1)+1}$. By Theorem 2.3,

$$g(c_i(\xi^p)) = p\left\{c_i(\xi) + \frac{x_1}{2} c_i(\xi)^2 + \cdots + \frac{x_{a(p-1)}}{a(p-1)+1} c_i(\xi)^{a(p-1)+1}\right\}.$$ 

Put $\{a(p-1)+1\}! = p^m m \equiv 0 \mod p$. If $n! = p^s n', \quad n' \equiv 0 \mod p$ then $u = \sum_{k \geq 1} [n/p^k]$. Hence

$$s = a \quad \text{if} \quad a = p^s + p^{s-1} + \cdots + 1, \quad s < a \quad \text{otherwise}.$$
Consider the following equation

\[ Ag(c_i(\xi^p)) = Ap\left\{ c_i(\xi) + \frac{x_1}{2}c_i(\xi)^2 + \cdots + \frac{x_a(p-1)}{a(p-1)+1}c_i(\xi)^{a(p-1)+1}\right\}, \]

where \( A = \{a(p-1)+1\}p^{-a-1} \).

This is an integral relation. Therefore, using \((\pi^*\xi)^p = 1\), the naturality of \( g \) and Lemma 2.2,

\[ p^m(L^{a(p-1)}(p), i) + \frac{p^m}{2}x_i[L^{a(p-1)-1}(p), i] + \cdots + \frac{p^m x_a(p-1)}{a(p-1)+1}[L^a(p), i] = 0. \]

Denote by \( o([L'(p), i]) \) the order of \([L'(p), i]\). Suppose that

\[ t = a(p-1) - (p^k n - 1), \quad n \equiv 0 \mod p, \]

By Proposition 1.4,

\[ o([L'(p), i]) = p^a \quad \text{if} \quad k = 1 \quad \text{and} \quad n = 1, \]

\[ o([L'(p), i]) = p^v, \quad v < a - k + 1 \quad \text{otherwise}. \]

Therefore,

\[ p^m[L^{a(p-1)}(p), i] + p^{a-1}mx_{p-1}[L^{a(p-1)-1}(p), i] = 0. \]

Since \( m \equiv 0 \mod p \), using the induction hypothesis, the proposition follows.

q. e. d.

Let \( \Gamma(p) \) be the polynomial subring of \( U_* \) generated by all \([Y_{ab}] \in U_{ab} \) with \( k \equiv p-1 \). We note that \( \Gamma(p)[CP^{p-1}] = U_* \).

**Proposition 2.5.** Suppose we are given a relation

\[ \sum_{k=0}^{a} [L^k(p), i][M^{2k-l-\k}] = 0, \]

with \([M^{2k-l-\k} \in \Gamma(p)\). Then \([M^{2k-l-\k} \in p^{[k(p-1)]+1}\Gamma(p).\)

Proof. The proof is by induction on \( n \). Lemma 2.1 implies that the case \( n = 0 \) is true. Suppose our assertion is true for \( m < n \). We consider

\[ \sum_{k=0}^{a} [L^k(p), i][M^{2k-l-\k}] = 0 \quad (1) \]

Applying Smith homomorphism to this equation, we have

\[ \sum_{k=0}^{a} [L^{k-1}(p), i][M^{2k-l-\k}] = 0. \]

By the induction hypothesis \([M^{2k-l-\k}] = p^{[k(p-1)]+1}[N^{2k-l-\k}]. \) Since \([\frac{k-1}{p-1}]\)
STRUCTURE OF THE BORDISM GROUP $U_n(BZ_p)$

For $k = a(p-1)$ and the order of $[L^h(p), i]$ is $p^{(k/p-1)+1}$ by Proposition 1.4, the equation (1) becomes

$$\sum_a p^a [L^h(p), i] [N^{2a-2a(p-1)}] = 0.$$  

From Proposition 2.4,

$$\sum_a \alpha_a [N^{2a-2a(p-1)}] [CP^{p-1}]^a [L^h(p), i] = 0.$$  

Since $\alpha_a \equiv 0 \mod p$, it follows from Lemma 2.1 that $[N^{2a-2a(p-1)}] \in pU_*$. This completes the proof.

Let $\Gamma_{2k}(p)$ consist of $2k$-dimensional homogenous polynomial. Finally we have the following

**Theorem 2.6.** The homomorphism

$$\Theta: \sum_{k=0}^{\infty} \Gamma_{2(n-k)}(p)/p^{(k/p-1)+1} \Gamma_{2(n-k)}(p) \to U_{2n+1}(BZ_p)$$

given by $\Theta(\sum_{k=0}^{\infty} [M^{k(n-k)}]) = \sum_{k=0}^{\infty} [M^{k(n-k)}][L^h(p), i]$ is isomorphism.

Proof. The Proposition 2.5 is precisely the statement that $\Theta$ is monomorphism. To check that $\Theta$ is epimorphism, we compute the order of the group

$$\sum_{k=0}^{\infty} \Gamma_{2(n-k)}(p)/p^{(k/p-1)+1} \Gamma_{2(n-k)}(p),$$

and compare it with that of $U_{2n+1}(BZ_p)$. The former is $p^\sigma$, $\tau = \sum_{k=0}^{\infty} t_k \left[ \frac{k}{p-1} \right] + 1$, where $t_k$ is the number of partitions of $k$, containing no $(p-1)$, the latter is $p^\sigma$, $\sigma = \sum_{k=0}^{\infty} s_k$, where $s_k$ is the number of partitions of $k$. Now

$$\sigma = \sum_{k=0}^{\infty} s_k = \sum_k \sum_a t_{k-a(p-1)} = \sum_j (\max \{a \mid j = k-a(p-1)+1\}) t_j = \sum_j \left( \left[ \frac{j}{p-1} \right] + 1 \right) t_j = \tau.$$  

Thus $\Theta$ is an isomorphism. q. e. d.

OSAKA CITY UNIVERSITY
References


