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Asymptotic Convergence of Solutions for Advection-Reaction-Diffusion Equations

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List of Publications

Journal Papers

1. S. Iwasaki, *Convergence of solutions to simplified self-organizing target-detection model*, Sci. Math. Jpn., Vol.e-2016, No.29, 2016.

2. S. Iwasaki, J. Yang, and T. Nakano, *A mathematical model of non-diffusion-based mobile molecular communication networks*, IEEE Commun. Lett., Vol.21, No.9, pp.1969–1972, 2017.

3. S. Iwasaki and T. Nakano, *Graph-based modeling of mobile molecular communication systems*, IEEE Commun. Lett., Vol.22, No.2, pp.376–379, 2018.

4. S. Iwasaki, *Exponential attractor for one-dimensional self-organizing target-detection model*, Funkcial. Ekvac. (accepted for publication).

5. S. Iwasaki, *Fractional powers of sectorial operators in network shaped domains*, Sci. Math. Jpn. (accepted for publication).

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7. S. Iwasaki, *Asymptotic convergence of solutions to Keller-Segel equation on network* (in preparation).

Proceedings

1. S. Iwasaki, J. Yang, A. O. Abraham, J. L. Hagad, T. Obuchi, and T.Nakano, *Modeling multi–target detection and gravitation by intelligent self–organizing bioparticles*, in Proc. of 2016 IEEE Global Communications Conference (GLOBECOM), 2016.

2. S. Iwasaki and T. Hatanaka, *Estimation distribution algorithm based on a molecular target detector model*, in Proc. of 48th ISCIE International Symposium on Stochastic Systems Theory and Its Applications, pp. 105–110, 2017.

3. S. Iwasaki, H. Xiao, T. Hatanaka, and T. Uchitane, *A general swarm intelligence model for continuous function optimization*, in Proc. of 11th International Conference, Simulated Evolution and Learning 2017, pp. 972–980, 2017.

International Conferences

1. S. Iwasaki, H.Xiao, T.Hatanaka, and T.Uchitane, *A basic study on function optimization based on general swarm model*, SICE Annual Conference 2016, Ibaraki, Sep., 2016.

2. S. Iwasaki and T.Hatanaka, *A study on estimation of distribution algorithm based on a partial differential equation model*, 2017 IEEE Symposium Series on Computational Intelligence (SSCI), Hawaii, Nov., 2017.

Domestic Conferences

1. 岩崎悟, Construction of exponential attractor for one-dimensional target-detection model, 2016 夏の偏微分方程式セミナー, 大阪, 8 月, 2016.

2. 岩崎悟, 畠中利治, 誘因・忌避モデルの解析とモデルに基づく分布推定アルゴリズムに関する考察, 第 11 回進化計算学会研究会, 兵庫, 9 月, 2016.

3. 岩崎悟, 畠中利治, 拡散・凝集の数理モデルに基づく複数解探索型の分布推定アルゴリズム, 進化計算シンポジウム 2016, 千葉, 12 月, 2016.

4. 岩崎悟, 畠中利治, 自己組織的ターゲット検出モデルによる分布推定アルゴリズムについて, 第 10 回コンピューテーショナル・インテリジェンス研究会, 富山, 12 月, 2016.

5. 岩崎悟, Mathematical analyses for simplified self-organizing target-detection model, 2017 秋の偏微分方程式セミナー, 大阪, 9 月, 2017.

6. 岩崎悟, Angelo Favini, 八木厚志, Asymptotic convergence of solutions to laplace reaction-diffusion equations, 第 43 回発展方程式研究会, 東京, 12 月, 2017.

Contents

Chapter 1

Introduction

In the natural world, one can see various static or dynamic patterns: animal skins, clouds, sand dunes, forest trees and so on. They actually take their forms automatically, but how are such patterns created? Furthermore, many organisms are known to exhibit complex aggregation behaviors and coordinations. These can be seen in schools of fish, honey bee colonies, and mounds built by termites and so on. Many examples are shown in [1] by Camazine, Franks, Sneyd, Bonabeau, Deneubourg, and Theraula. However, how do organisms create these seemingly intelligent behaviors? While, in applications, there are many cases that one has to predict or control those phenomena represented as weather forecasts. However, how can we predict or control the phenomena? In order to answer the above problems, it is important to understand essential mechanisms governing the phenomena. One of the methods to understand such mechanisms is to use favorable mathematical models describing the phenomena. Generally speaking, one considers models to be favorable if their solutions can show good agreements to the phenomena. Therefore, one has to reconstruct models again and again equations by comparing the properties of solutions derived from the models and the observations for natural phenomena.

As models describing various phenomena, many researchers have often utilized advection-reaction-diffusion equations. A. M. Turing suggested that a reaction-diffusion system with different diffusion coefficients may lead spatial inhomogeneity [2]. Many different kinds of advection-reaction-diffusion equations also show good agreements to various biological phenomena, and a lot of instances are shown in Murray's books [3, 4]. Furthermore, the concept of dissipative structures with mathematical models, which is introduced by Nicolis and Prigogine [5], contributes theoretical understandings for pattern formation or self-organization in the natural world such as Belousov-Zhabotinsky reactions and Rayleigh-Bénard convection.

Although there are many successful model equations, there are few equations whose solutions can be explicitly expressed by spatial and temporal variables, including advection-reaction-diffusion equations, for those equations are in general nonlinear. Under these circumstances, one has to investigate properties of solutions for advection-reaction-diffusion equations. In order to overcome these problems, two approaches are developed: numerical approaches and analytical approaches. Numerical approaches try to obtain approximate solutions as (visible) values by using computational methods, and these approaches are good at quantitative evaluations. In the meantime, analytical approaches are good at qualitative evaluations by using many kinds of mathematical theory. These two approaches are necessary to investigate properties of solutions for advection-reaction-diffusion equations.

One of the basic analytical studies for advection-reaction-diffusion equations is to prove existence and uniqueness of solutions. In order to show this importance, we give some examples here.

Although our main interest is to study advection-reaction-diffusion equations, we firstly consider examples of ordinary differential equations for simplicity. Consider the following ordinary differential equation

$$\frac{du}{dt} = u^2, \; t > 0, \qquad u(0) = u_0 > 0.$$

This equation is analytically solved, and the solution is given by $u(t) = 1/(1/u_0 - t)$. This solution satisfies that $u(t) \to \infty$ as $t \to 1/u_0 - 0$, therefore the solution uniquely exists on the interval $[0, 1/u_0)$ only and the length of existing interval depends on the magnitude of initial value u_0. We know from this example that, it is important to consider how long a solution exists even if a solution uniquely exists in a short time.

Secondly, consider the following ordinary differential equation

$$\frac{du}{dt} = 2u^{1/2}, \; t > 0, \qquad u(0) = 0.$$

This equation is also analytically solved, and the solutions are given by

$$\begin{cases} u_x(t) = (t - x)^2 & \text{for} \quad x < t, \\ u_x(t) = 0 & \text{for} \quad 0 \leq t \leq x \end{cases} \tag{1.1}$$

with arbitrary constant $x > 0$. Actually, each function $u_x(\cdot)$ satisfies the above differential equation. In this case, this equation does not possess uniqueness of solutions under the initial condition $u(0) = 0$. We know from this example that, it is important to consider when differential equations possess uniqueness of solutions.

Thirdly, as a well known example of non existence of solutions to a partial differential equation, Lewy [6] proved that there exists a function $F(x, y, z) \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that

$$-\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} + 2i(x + iy)\frac{\partial u}{\partial z} = F(x, y, z) \quad \text{in } \Omega$$

does not possess solutions for any domain $\Omega \subset \mathbb{R}^3$. This example shows us that, even if a differential equation exists, its meaningful solution may not exists.

Finally, consider the following partial differential equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{in } \mathbb{R}^2.$$

The solution of this equation is given by $u(x, y) = f(x + y)$ with arbitrary function $f \in \mathcal{C}^1(\mathbb{R})$. This means that, for each $C \in \mathbb{R}$, $u(x, y)$ have a uniform value $f(C)$ on each line $x + y = C$. Now, reconsider the above equation in a bounded domain, i.e.,

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{in } \Omega, \tag{1.2}$$

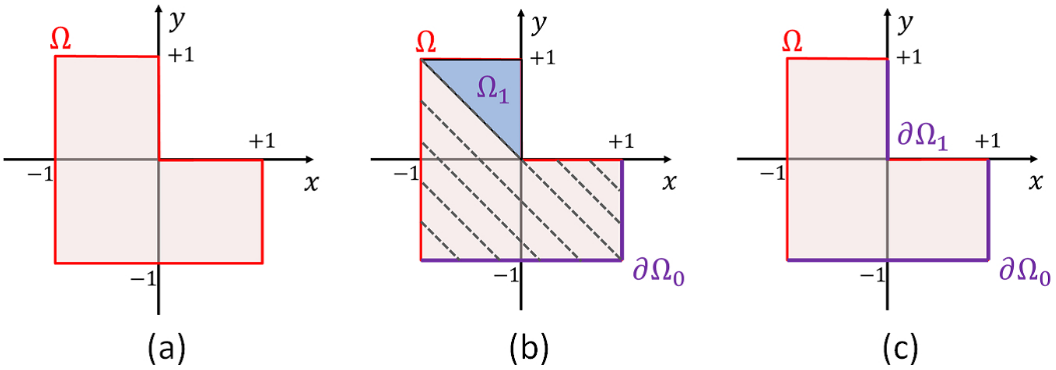

Fig. 1.1: Boundary conditions for the equation (1.2)

where Ω is given by the shape shown in Fig 1.1 (a). The solutions of this equation are also given by $u(x, y) = f(x + y)$ in Ω. To this problem, we want to impose boundary conditions which lead uniqueness of solutions. For example, consider imposing boundary values on the boundary $\partial\Omega_0$ shown in Fig 1.1 (b) only. Then, due to the restriction of values on each line $x+y = C$, the values in Ω_1 are not determined. In order to ensure the uniqueness in Ω, it is necessary to impose boundary values on the boundary $\partial\Omega_0 \cup \partial\Omega_1$ shown in Fig 1.1 (c). We know from this example that it is important to consider which boundary conditions lead unique existence of solutions of partial differential equations

As shown by the above examples, one can not always ensure existence and uniqueness of solutions to differential equations. Furthermore, even if a solution uniquely exists in a short time, it may blow-up in a finite time. So, investigating the existence and uniqueness of solutions to a model equation is first priority. In the words of J. S. Hadamard, it is important to show that the problem is "well-posed".

In order to construct a unique local solution for advection-reaction-diffusion equations, it is often convenient to formulate the equations as the Cauchy problem for an evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t \leq T, \\ U(0) = U_0, \end{cases} \tag{1.3}$$

in a Banach space X. Here, $U = U(t)$ is unknown function, A is a linear operator, and F is a nonlinear operator. If $X = \mathbb{R}^n$ and A is a matrix, the matrix exponential by the power series $e^{-tA} = \sum_{k=0}^{\infty}(-tA)^k/k!$ is always well-defined; therefore, we can discuss the unique existence of the local solution of (1.3) with this matrix exponential. However, in many cases, X is infinite dimensional Banach space and A is not bounded operator, so we have to be careful in the definition of e^{-tA}. To this end, we assume that the operator A is a sectorial operator with its angle less than $\pi/2$. For such operator A, we can consider the exponential functions e^{-tA}, $0 < t < \infty$, generated by $-A$. By using exponential functions and from Duhamel's principle, the solution of (1.3) is formally given by integral form $U(t) = e^{-tA}U_0 + \int_0^t e^{-(t-s)A}F(U(s))ds$. In order to ensure the unique existence of $U(t)$ satisfying the integral form, we have to use the Banach fixed-point theorem. To this end, as similar to the situation for ordinary differential equations, we have to assume a Lipschitz condition of $F(U)$, where the Lipschitz condition is formulated by using fractional powers of the sectorial operator A. From this point, it is important to consider sectorial operators A and the fractional powers of A.

Under the unique existence of solutions for advection-reaction-diffusion equations, our studies move on next stages, that is, we study properties of solutions. We now focus on asymptotic behaviors of solutions. As typical asymptotic behaviors of solutions, there are chaotic behavior, periodic behavior, and convergence to a stationary solution. In this doctoral thesis, our main theme is to study asymptotic convergence to a stationary solution (except for Chapter 5).

In the study of convergence to a stationary solution, [7, 8, 9] give interesting results: gradient systems whose solutions do not converge to a stationary solution. Here, gradient systems are differential equations that have the form

$$\frac{du}{dt}(t) = -\text{grad } \Phi(u(t))$$

with Φ a real valued function called a Lyapunov function. Since a solution $u(t)$ satisfies that

$$\frac{d}{dt}\Phi(u(t)) = \text{grad } \Phi(u(t)) \cdot \frac{du}{dt}(t) = -\|\text{grad } \Phi(u(t))\|^2,$$

$\Phi(u(\cdot))$ is strictly decreasing as time t increasing except at equilibria. One may expect that, for every gradient system, all solutions converge to a stationary solution $\overline{u}$ such that grad $\Phi(\overline{u}) = 0$. Actually, the value of a Lyapunov function $\Phi(u(t))$ converges to $\Phi(\overline{u})$ as $t \to \infty$ (if $\Phi(u(t))$ does not tend to $-\infty$). However, roughly speaking, the fact that $\Phi(u(t)) \to \Phi(\overline{u}) \in \mathbb{R}$ has only one dimensional information, so it is impossible to bind the behavior of $u(t)$ in a finite or infinite dimensional space. Indeed, [7, 8, 9] show that there are gradient systems such that their solutions do not converge to a stationary solution. The example [7, Section1.1, Example 3] gives such a gradient system of two ordinary differential equations, and [8, 9] give such gradient systems of partial differential equations. These facts show that additional assumptions for Lyapunov functions are required to prove convergence of solutions.

To this problem, one of the powerful techniques is to use the Łojasiewicz-Simon inequality with gradient systems. Łojasiewicz proved the following results: let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be a real analytic function, and $a \in U$; then, there exits constants $0 < \theta \leq 1/2$, $C > 0$, $r > 0$, such that

$$|f(x) - f(a)|^{1-\theta} \leq C\|\nabla f\| \quad \text{if} \quad \|x - a\| < r.$$

For the proof, we quote [10, 11]. The above inequality is called the Łojasiewicz inequality. Simon generalized the Łojasiewicz results to analytic functions on infinite-dimensional spaces [12], and the proof is simplified by Jendoubi [13]. The generalized Łojasiewicz inequality is called the Łojasiewicz-Simon inequality. The Łojasiewicz-Simon inequality has been successfully applied to proving convergence results for solutions of a variety of gradient systems; Cahn-Hilliard equations [14, 15], degenerate diffusion equations [16, 17], second order ordinary differential equations [18, 19], damped wave equations [20, 21], Ginzburg–Landau equations [22], and evolutionary integral equations [23, 24, 25]. From this applicability to convergence results, the Łojasiewicz-Simon inequality itself is studied; Chill [26, 27] introduced a set which is called a critical manifold to show the Łojasiewicz-Simon inequality. Furthermore, a non-smooth version of the Łojasiewicz-Simon inequality is considered in [28], and this result is applied to showing a convergence result for the Keller-Segel equations by Feireisl, Laurençot, and Petzeltová [29].

Contributions

This doctoral thesis presents analytical studies for four advection-reaction-diffusion equations; particularly, we focus on studying asymptotic behaviors of solutions. By applying the theory of abstract parabolic evolution equations, we construct a unique local solution to each equation. After that, by establishing a priori estimate, we extend the local solutions to global solutions. Further details of contributions in this thesis are as follows:

- In Chapter 3, we present results about strongly elliptic differential operators in a network shaped domain. Particularly, we intend to characterize the domains of fractional powers of sectorial operators determined from these differential operators, and the characterization results are applied to studying Keller-Segel equations in network shaped domains in Chapter 6. These results are obtained in [30].

- In Chapter 5, we study attraction-repulsion chemotaxis equations. It seems that there is no Lyapunov function for these equations, and we can not show the convergence of global solutions. However, we are able to prove that a dynamical system determined from these equations possesses exponential attractors. From this result, all solutions are attracted at exponential rates by a compact set with finite fractal dimension; therefore, we can indicate that their solutions show pattern formations. These results are obtained in [31].

- In Chapter 6, we study the Keller-Segel equations in network shaped domains. By applying the theory of abstract parabolic evolution equations, we construct strict solutions. Furthermore, by using a non-smooth version of the Łojasiewicz-Simon inequality, we conclude the asymptotic convergence of global solutions to a stationary solution. These results are obtained in [32].

- In Chapter 7, we study a quasilinear diffusion equation. To this problem, we can show that its stationary problem possesses a unique solution. This favorable property yields the convergence result without using the Łojasiewicz-Simon inequality. These results are obtained in [33].

- In Chapter 8, we study a Laplace reaction-diffusion equation. We also use the Łojasiewicz-Simon inequality for proving the convergence of global solutions to a stationary solution. To this problem, we use a concept of critical manifold (introduced by [26, 27]) with some modifications. These results are obtained in [34].

Outline

The outline of this doctoral thesis is as follows.

In Chapter 2, we prepares some preliminaries which are used in the rest parts of this thesis.

In Chapter 3, we present some results about strongly elliptic differential operators in a network shaped domain.

In Chapter 4, we present fundamentals of the theory of evolution equations of parabolic type and infinite dimensional dynamical systems. We apply existence and uniqueness results of local solutions to evolution equations to advection-reaction-diffusion equations in the subsequent chapters. A version of exponential attractor for non-autonomous dynamical systems is constructed for a dynamical system determined in Section 5.2.

In Chapter 5, after reviewing results about chemotaxis equations over the past years, we study attraction-repulsion chemotaxis equations. It seems that there is no Lyapunov function for these equations, but a dynamical system determined from these equations possesses exponential attractors.

In Chapters 6, 7, and 8, we show convergence results of global solutions to each advection-reaction-diffusion equations. Chapter 6 is devoted to studying the Keller-Segel equations in network shaped domains, Chapter 7 is devoted to studying a quasilinear diffusion equation, and Chapter 8 is devoted to studying a Laplace reaction-diffusion equation.

Finally, Chapter 9 ends this thesis with conclusions and future researches.

Chapter 2

Preliminaries

2.1 Function Spaces with Values in a Banach Space

Let X be a Banach space with norm $\|\cdot\|$, and let $[a,b]$ be a bounded closed interval. Let us introduce fundamental function spaces which will often appear in this thesis.

The space of uniformly bounded functions on $[a,b]$, which is denoted by $\mathcal{B}([a,b];X)$, is a Banach space with the norm

$$\|F\|_{\mathcal{B}} = \sup_{a\le t\le b} \|F\|.$$

For $m = 0, 1, 2, \ldots$, the space of m times continuously differentiable functions on $[a,b]$, which is denoted by $\mathcal{C}^m([a,b];X)$, is a Banach space with the norm

$$\|F\|_{\mathcal{C}^m} = \sum_{i=0}^{m} \max_{a\le t\le b} \|F^{(i)}(t)\|.$$

In particular, $\mathcal{C}^0([a,b];X)$ is simply denoted by $\mathcal{C}([a,b];X)$.

For $0 < \sigma < 1$, the space of σ-Hölder continuous functions on $[a,b]$, which is denoted by $\mathcal{C}^\sigma([a,b];X)$, is a Banach space with the norm

$$\|F\|_{\mathcal{C}^\sigma} = \|F\|_{\mathcal{C}} + \sup_{a\le s<t\le b} \frac{\|F(t)-F(s)\|}{(t-s)^\sigma}.$$

Finally, let us define the function space $\mathcal{F}^{1,\sigma}((a,b];X)$ for $0 < \sigma < 1$. Firstly, $\mathcal{F}^{1,\sigma}((a,b];X) \subset \mathcal{C}([a,b];X)$. In addition, $F \in \mathcal{F}^{1,\sigma}((a,b];X)$ if and only if the quantity $\omega_F(t)$ on $(a,b]$ satisfies that $\sup_{a\le t\le b} \omega_F(t) < \infty$ and $\lim_{t\to a+0} \omega_F(t) = 0$. The space $\mathcal{F}^{1,\sigma}((a,b];X)$ is a Banach space with the norm

$$\|F\|_{\mathcal{F}^{1,\sigma}} = \|F\|_{\mathcal{C}} + \sup_{a\le s<t\le b} \frac{(s-a)^\sigma \|F(t)-F(s)\|}{(t-s)^\sigma}.$$

We know that

$$F \in \mathcal{C}^\sigma([a,b];X) \text{ belongs to } \mathcal{F}^{1,\sigma}((a,b];X). \tag{2.1}$$

2.2 Functional Analysis

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The space of all bounded linear operators from X into Y, which is denoted by $\mathcal{L}(X,Y)$, is a Banach space with the norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{\|U\|_X \le 1} \|AU\|_Y.$$

In particular, when $Y = \mathbb{C}$, the space $\mathcal{L}(X,\mathbb{C})$ is called the dual space of X and is denoted by X'. Furthermore, when $X = Y$, $\mathcal{L}(X,X)$ is written as $\mathcal{L}(X)$ for brevity.

We say that $Y \subset X$ with dense embedding, if, for any $x \in X$, there exists a sequence $\{y_n\} \subset Y$ such that $\|x - y_n\|_X \to 0$ as $n \to \infty$. In addition, we say that $Y \subset X$ with continuous embedding, if there exists a constant $C \ge 0$ such that $\|y\|_X \le C\|y\|_Y$ for $y \in Y$.

2.2.1 Fréchet Derivative

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $\mathcal{O}$ be an open set of X, and let $F : \mathcal{O} \to Y$ be a continuous operator. For a point $U \in \mathcal{O}$, assume that there exists a bounded linear operator $A \in \mathcal{L}(X,Y)$ which satisfies

$$o(h) = \|F(U+h) - F(U) - Ah\|_Y, \quad \lim_{h\to 0} \frac{o(h)}{\|h\|_X} = 0,$$

for $h \in X$ such that $U + h \in \mathcal{O}$. Note that if such an A exists then the A is uniquely determined. Then, F is said to be Fréchet differentiable at a point U and A is called the Fréchet differential of F at U. The operator A is denoted by $F'(U)$ or $F'U$. When F is Fréchet differentialble at every point of $\mathcal{O}$, F is said to be Fréchet differentiable in $\mathcal{O}$. The mapping $F' : \mathcal{O} \to \mathcal{L}(X,Y)$, $U \mapsto F'U$ is called the Fréchet derivative of F. When the derivative F' is continuous from $\mathcal{O}$ into $\mathcal{L}(X,Y)$, we say that F is continuously Fréchet differentiable in $\mathcal{O}$.

2.2.2 Analytic functions in Banach Spaces

In this subsection, we demonstrate facts about analytic functions in Banach spaces, which are shown in [35] or [36, Chapter 8].

Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. For $n = 0, 1, 2, \ldots$, let a_n be a continuous, symmetric, and n-linear map of X^n into Y, which denotes $a_n \in \mathcal{L}_s^n(X,Y)$, with its norm

$$\|a_n\|_{\mathcal{L}_s^n(X,Y)} = \sup_{\substack{\|x_i\|_X = 1 \\ i=1,\ldots,n}} \|a_n(x_1, \ldots, x_n)\|_Y.$$

For simplicity, we denote $\|\cdot\|_{\mathcal{L}_s^n(X,Y)}$ as $\|\cdot\|_n$. For $a_n \in \mathcal{L}_s^n(X,Y)$ and $x \in X$, $a_n(x, x, \cdots, x)$ is written as $a_n x^n$ for brevity. A power series in $x \in X$ with values in Y is a series of the form $\sum_{n=0}^{\infty} a_n x^n$, where a_0 is a point of Y.

Consider a sequence $a_n \in \mathcal{L}_s^n(X,Y)$, $n = 0, 1, 2, \ldots$, and a scalar $r > 0$ such that $\sum_{n=0}^{\infty} \|a_n\|_n r^n < \infty$. Then, the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely and uniformly for $\|x\|_X < r$, and the power series is a continuous function with respect to x.

For an open set $\mathcal{O}$ of X, consider a function $F : \mathcal{O} \to Y$. We say that the function F is analytic at $x_0 \in \mathcal{O}$ if it coincides with a convergent power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ such that $\sum_{n=0}^{\infty} \|a_n\|_n \|x - x_0\|_X^n < \infty$ for x near x_0. Furthermore, F is analytic in $\mathcal{O}$ if it is analytic at each point of $\mathcal{O}$. Note that F is analytic at a point x_0 if and only if F is analytic on a neighborhood of x_0.

As for the Fréchet derivative of an analytic function, we know the following theorem. For the proof, see [35, Theorem].

Theorem 2.1. *Let $x_0 \in X$, and let a sequence $a_n \in \mathcal{L}_s^n(X, Y)$, $n = 0, 1, 2, \ldots$, and a scalar $r > 0$ satisfy $\sum_{n=0}^{\infty} \|a_n\|_n r^n < \infty$. Then, the analytic function $F(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ has a Fréchet derivative at any point x such that $\|x - x_0\|_X < r$, and the derivative (in $\mathcal{L}(X, Y)$) is $F'(x) = \sum_{n=1}^{\infty} n a_n(x - x_0)^n$. The new series converges for x such that $\|x - x_0\|_X < r$.*

We know the following identity theorem for analytic functions [35, Corollary].

Theorem 2.2. *If two analytic functions, $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$, coincide near x_0, then $a_n = b_n$ for all n.*

In the meantime, the following inverse mapping theorem holds true. For the proof, see [36, Corollary 4.37].

Theorem 2.3. *Let $F : X \to Y$ be an analytic function a point $x_0 \in X$. Let $F'(x_0) \in \mathcal{L}(X, Y)$, the first derivative of F at x_0, be a bijection from X onto Y. Then, F is a local analytic diffeomorphism at x_0. That is, there exist a neighborhood $\mathcal{O}_X(x_0)$ in X and a neighborhood $\mathcal{O}_Y(F(x_0))$ in Y such that F is a bijection from $\mathcal{O}_X(x_0)$ onto $\mathcal{O}_Y(F(x_0))$ and both F and F^{-1} are analytic functions.*

We give one example of analytic functions in a Banach space. Since this analytic function plays an important role in Sections 6 and 8, we give the proof of the analyticity.

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Then, the mapping $\mathcal{F}$ from $\mathcal{C}([0, 1]) = \mathcal{C}([0, 1]; \mathbb{R})$ to $\mathbb{R}$ given by*

$$\mathcal{F}(u) = \int_0^1 f(u(x))dx, \quad u \in \mathcal{C}([0, 1]),$$

is analytic in $\mathcal{C}([0, 1])$.

Proof. Let $u_0 \in \mathcal{C}([0, 1])$ be arbitrarily fixed. Put $m = \min_{0 \le x \le 1}\{u_0(x)\} > -\infty$ and $M = \max_{0 \le x \le 1}\{u_0(x)\} < \infty$. Then, for each $a \in [m, M]$, it follows from the analyticity of f that there exists a $r_a > 0$ such that

$$\sum_{n=0}^{\infty} \frac{|f^{(n)}(a)|}{n!} r_a^n < \infty$$

and

$$f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(y - a)^n \quad \text{for} \quad |y - a| < r_a.$$

Put $r = \min_{m \le a \le M}\{r_a\} > 0$. For such r, it is obvious that

$$\sum_{n=0}^{\infty} \frac{\max_{m \le a \le M} |f^{(n)}(a)|}{n!} r^n < \infty. \tag{2.2}$$

Note that, if $u \in \mathcal{C}([0,1])$ satisfies that $\|u - u_0\|_{\mathcal{C}} < r$, then

$$f(u(x)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(u_0(x))}{n!} (u(x) - u_0(x))^n$$

for every $x \in [0,1]$.

In the meantime, for $n = 1, 2, \ldots$, the n-th derivative of $\mathcal{F}(u)$ at u_0, $\mathcal{F}^{(n)}(u_0) \in \mathcal{L}_s^n(\mathcal{C}([0,1]); \mathbb{R})$, is given by

$$\begin{aligned} &\mathcal{F}^{(n)}(u_0)[h_1 h_2 \cdots h_n] \\ &= \int_0^1 f^{(n)}(u_0(x))[h_1(x) h_2(x) \cdots h_n(x)] dx, \quad (h_1, h_2, \ldots, h_n) \in [\mathcal{C}([0,1])]^n \end{aligned}$$

with its norm

$$\|\mathcal{F}^{(n)}(u_0)\|_n = \sup_{\substack{\|h_i\|_{\mathcal{C}} = 1 \\ i = 1, 2, \ldots, n}} \left| \int_0^1 f^{(n)}(u_0)[h_1 h_2 \cdots h_n] dx \right|.$$

Obviously, $\|\mathcal{F}^{(n)}(u_0)\|_n \le \max_{0 \le x \le 1} |f^{(n)}(u_0(x))| \le \max_{m \le a \le M} |f^{(n)}(a)|$. Therefore, it follows from (2.2) that

$$\sum_{n=0}^{\infty} \frac{\|\mathcal{F}^{(n)}(u_0)\|_n}{n!} r^n < \infty.$$

Furthermore, if $u \in \mathcal{C}([0,1])$ satisfies that $\|u - u_0\|_{\mathcal{C}} < r$, then we have

$$\mathcal{F}(u) = \int_0^1 \sum_{n=0}^{\infty} \frac{f^{(n)}(u_0(x))}{n!} (u(x) - u_0(x))^n dx = \sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}(u_0)}{n!} (u - u_0)^n,$$

which shows that $\mathcal{F}(u)$ is analytic at u_0. Since u_0 is arbitrary, we conclude the assertion. □

2.2.3 Interpolation Spaces

Let X_0 and X_1 be two Banach spaces with norms $\|\cdot\|_{X_0}$ and $\|\cdot\|_{X_1}$, respectively. We assume that $X_1 \subset X_0$ with dense and continuous embedding. For $0 \le \theta \le 1$, the (complex) interpolation space of X_0 and X_1 is denoted by $[X_0, X_1]_\theta$. For the definition of $[X_0, X_1]_\theta$, see [37, Subsection 1.5.1]. According to [38, Theorems 1.9.1 and 1.9.2], the space $[X_0, X_1]_\theta$ becomes a Banach space with a suitable norm.

We give the following interpolation theorem. For the proof, see [37, Theorem 1.15].

Theorem 2.4. *Let $X_1 \subset X_0$ (resp. $Y_1 \subset Y_0$) densely and continuously, and let $[X_0, X_1]_\theta$ (resp. $[Y_0, Y_1]_\theta$) be the interpolation space. If $T \in \mathcal{L}(X_0, Y_0)$ and $T \in \mathcal{L}(X_1, Y_1)$, then T belongs to $\mathcal{L}([X_0, X_1]_\theta, [Y_0, Y_1]_\theta)$ for any $0 < \theta < 1$ with the estimate*

$$\|T\|_{\mathcal{L}([X_0,X_1]_\theta, [Y_0,Y_1]_\theta)} \le \|T\|_{\mathcal{L}(X_0,Y_0)}^{1-\theta} \|T\|_{\mathcal{L}(X_1,Y_1)}^{\theta}. \tag{2.3}$$

2.2.4 Triplets of Spaces

Let Z and X be two Hilbert spaces with inner products $(\cdot,\cdot)_Z$ and $(\cdot,\cdot)_X$ and with norms $\|\cdot\|_Z$ and $\|\cdot\|_X$, respectively. We assume that $Z \subset X$ with dense and continuous embedding.

Then, there exists a Banach space Z^* (with its norm $\|\cdot\|_{Z^*}$) satisfying the following conditions. Firstly, $X \subset Z^*$ with dense and continuous embeddings. Secondly, $\{Z, Z^*\}$ forms an adjoint pair with duality product $\langle\cdot,\cdot\rangle_{Z\times Z^*}$, i.e., the mapping $\langle\cdot,\cdot\rangle_{Z\times Z^*} : Z \times Z^* \to \mathbb{C}$ satisfies that, for $\alpha, \beta \in \mathbb{C}$, $u, v \in Z$, and $\varphi, \psi \in Z^*$,

$$\langle \alpha u + \beta v, \varphi\rangle_{Z\times Z^*} = \alpha \langle u, \varphi\rangle_{Z\times Z^*} + \beta \langle v, \varphi\rangle_{Z\times Z^*},$$
$$\langle u, \alpha\varphi + \beta\psi\rangle_{Z\times Z^*} = \overline{\alpha} \langle u, \varphi\rangle_{Z\times Z^*} + \overline{\beta} \langle u, \psi\rangle_{Z\times Z^*},$$

and

$$|\langle u, \varphi\rangle_{Z\times Z^*}| \le \|u\|_Z \|\varphi\|_{Z^*},$$
$$\|u\|_Z = \sup_{\|\varphi_0\|_{Z^*}\le 1} |\langle u, \varphi_0\rangle_{Z\times Z^*}|, \qquad \|\varphi\|_{Z^*} = \sup_{\|u_0\|_Z \le 1} |\langle u_0, \varphi\rangle_{Z\times Z^*}|.$$

Finally, the duality product $\langle\cdot,\cdot\rangle_{Z\times Z^*}$ satisfies

$$\langle u, f\rangle_{Z\times Z^*} = (u, f)_X \quad \text{for all} \quad u \in Z,\ f \in X.$$

Such a Banach space can be always constructed in a unique way. For the detail, see [37, Chapter 1, Section 7]. Then the three spaces $Z \subset X \subset Z^*$ are called a triplet of spaces.

2.3 Function Spaces in Ω

Let Ω be a domain in $\mathbb{R}^n$ with the Lebesgue measure. For $1 \le p \le \infty$, $L_p(\Omega)$ denotes the complex valued L_p spaces. For $k = 0, 1, 2, \ldots$, we define

$$H^k(\Omega) = \{u \in L_2(\Omega);\ D^\alpha u \in L_2(\Omega) \text{ for } |\alpha| \le k\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denotes a multiindex, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ denotes the derivatives in the distribution sense. The space $H^k(\Omega)$ becomes a Hilbert space with the inner product

$$(u, v)_{H^k} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_{L_2}, \quad u, v \in H^k(\Omega).$$

The space $H^k(\Omega)$ is called the Sobolev space.

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. In order to extend the Sobolev space $H^k(\Omega)$ to that with fractional orders, we need the following extension operator in the next subsection. For the detail, see [39, Chapter VI, Theorems 5 and 5'].

Theorem 2.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, and let $k = 0, 1, 2, \ldots$. There exists a linear continuous operator $\mathcal{T} : H^k(\Omega) \to H^k(\mathbb{R})$ such that $(\mathcal{T}u)_{|\Omega} = u$ and $\|\mathcal{T}u\|_{H^k(\mathbb{R}^n)} \le C_k \|u\|_{H^k(\Omega)}$ for $u \in H^k(\Omega)$, where the constant $C_k \ge 0$ depends on k.*

2.3.1 Sobolev Spaces with Fractional Orders

Let $s \geq 0$ be a nonnegative number. The space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n)'; \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}u] \in L_2(\mathbb{R}^n)\},$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space, $\mathcal{S}(\mathbb{R}^n)'$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform of $\mathcal{S}(\mathbb{R}^n)'$, respectively. The space $H^s(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$(u, v)_{H^s} = ((1+|\xi|^2)^{s/2}\mathcal{F}u, (1+|\xi|^2)^{s/2}\mathcal{F}v)_{L_2}, \quad u, v \in H^s(\mathbb{R}^n).$$

By the theory of Fourier multipliers, we verify that the two definitions of $H^k(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ are equivalent for nonnegative integers $s = k$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Let us extend the definition of $H^k(\Omega)$ for fractional orders. For $s \geq 0$, we define

$$H^s(\Omega) = \{u \in L_2(\Omega); \exists U \in H^s(\mathbb{R}^n) \text{ such that } U|_\Omega = u \text{ almost everywhere in } \Omega\},$$

with its norm

$$\|u\|_{H^s(\Omega)} = \inf_{U \in H^s(\mathbb{R}^n), U|_\Omega = u} \|U\|_{H^s(\mathbb{R}^n)}.$$

With this norm, $H^s(\Omega)$ is a Banach space. Due to Theorem 2.5, we verify that the two definitions of $H^k(\Omega)$ and $H^s(\Omega)$ are equivalent for nonnegative integers $s = k$.

One see that, for $0 < s_0 < s_1 < \infty$,

$$H^{s_1}(\Omega) \subset H^{s_0}(\Omega) \subset L_2(\Omega) \text{ with continuous and compact embeddings.} \tag{2.4}$$

According to [38, Section 4.3.1, Theorem 1], we have the following theorem.

Theorem 2.6. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Let $0 \leq s_0 < s_1 < \infty$. Then*

$$[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta = H^s(\Omega) \quad \text{with norm equivalence,} \tag{2.5}$$

where $0 \leq \theta \leq 1$ and $s = (1-\theta)s_0 + \theta s_1$.

By using Theorem 2.4 and Theorem 2.6, we can verify the boundedness of the extension operator $\mathcal{T}$ from $H^s(\Omega)$ to $H^s(\mathbb{R}^n)$. This fact yields that $H^s(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^s(\Omega)} = (\mathcal{T}u, \mathcal{T}v)_{H^s(\mathbb{R}^n)}, \quad u, v \in H^s(\Omega).$$

The space $H^s(\Omega)$ is also called the Sobolev space.

2.3.2 Embedding Theorems

According to [38, Theorem 2.8.1/Remark 2 and Theorem 4.6.1], we get the following embedding theorem.

Theorem 2.7. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Let $n/2 < s < \infty$. Then*

$$H^s(\Omega) \subset \mathcal{C}(\overline{\Omega}) \quad \text{with continuous embedding.} \tag{2.6}$$

The following estimates are know as Gagliardo-Nirenberg's inequality. When $\Omega = \mathbb{R}$, the proof is given in [40, Theorem 3.3, 3.4, and 3.5]. Due to Theorem 2.5, the proofs for other cases are immediately reduced to this case.

Theorem 2.8. *Let Ω be a bounded interval in $\mathbb{R}$. Let $1 \le q \le 2$. Then, $H^1(\Omega) \cap L_q(\Omega) \subset L_r(\Omega)$ with the estimate*

$$\|u\|_{L_r} \le C_{q,r} \|u\|_{H^1}^a \|u\|_{L_q}^{1-a}, \qquad u \in H^1(\Omega) \cap L_q(\Omega),$$

where $q \le r \le \infty$, and a is given by $1/r = -a/2 + (1-a)/q$.

2.3.3 Spaces $\mathring{H}^s(\Omega)$ and $H^{-s}(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. By $\mathcal{D}(\Omega)$ we denote the space of all infinitely differentiable functions in Ω with compact support. For $s \ge 0$, the space $\mathring{H}^s(\Omega)$ is defined as the closure of the set $\mathcal{D}(\Omega)$ in the space $H^s(\Omega)$.

It is known from [38, Section 4.3.2, Theorem 1(a)] that

$$\mathring{H}^s(\Omega) = H^s(\Omega) \text{ for } 0 \le s \le 1/2. \tag{2.7}$$

However,

$$\mathring{H}^s(\Omega) \ne H^s(\Omega) \quad \text{for any } 1/2 < s < \infty.$$

For $s \ge 0$, the space $H^{-s}(\Omega)$ is defined as $\mathring{H}^s(\Omega)'$. Therefore, $\{\mathring{H}^s(\Omega), H^{-s}(\Omega)\}$ is an adjoint pair with duality product $\langle \cdot, \cdot \rangle_{\mathring{H}^s \times H^{-s}}$ on $\mathring{H}^s(\Omega) \times H^{-s}(\Omega)$. We know that $L_2(\Omega) \subset H^{-s}(\Omega)$ with the relation

$$(u, f)_{L_2} = \langle u, f \rangle_{\mathring{H}^s \times H^{-s}} \quad \text{for} \quad u \in \mathring{H}^s(\Omega),\ f \in L_2(\Omega). \tag{2.8}$$

Furthermore, for any $0 < s < \infty$, $\mathring{H}^s(\Omega) \subset L_2(\Omega) \subset H^{-s}(\Omega)$ becomes a triplet.

According to [41, Theorem 1.4.4.6], the following result is valid.

Theorem 2.9. *For any $-\infty < s < \infty$, $s \ne 1/2$, the partial derivation D_i $(i = 1, \dots, n)$ is a bounded operator from $H^s(\Omega)$ to $H^{s-1}(\Omega)$.*

By combining Theorems 2.7 and 2.9, we know that, for any $k = 0, 1, 2, \dots$,

$$H^s(\Omega) \text{ is continuously embedded in } \mathcal{C}^k(\overline{\Omega}) \text{ if } s > k + n/2. \tag{2.9}$$

2.4 Sectorial Operators

Let X be a Banach space with norm $\|\cdot\|$. Let A be a linear operator from a domain $\mathcal{D}(A) \subset X$ to X. The operator A is said to be a densely defined operator in X if the domain $\mathcal{D}(A)$ is dense in X. In the meantime, the operator A is said to be a closed linear operator in X if the graph $G(A) = \{(u, Au) \in X \times X;\ u \in \mathcal{D}(A)\}$ is closed in $X \times X$.

Let A be a densely defined, closed linear operator in X. We assume that the spectrum of A, which is denoted by $\sigma(A)$, is contained in an open sectorial domain such that

$$\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}, \quad 0 < \omega < \pi, \tag{2.10}$$

and its resolvent, which is denoted by $(\lambda - A)^{-1}$, satisfies the estimate

$$\left\|(\lambda - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}, \quad \lambda \notin \Sigma_\omega, \tag{2.11}$$

with some constant $M \geq 1$. We call such an operator A a sectorial operator of X.

For a sectorial operator A, we can define its angle ω_A by using analytic continuation of the resolvent $(\lambda - A)^{-1}$. For the definition of ω_A, see [37, Chapter 2]. We know that, for any $\omega_A < \omega' \leq \pi$, it holds that $\sigma(A) \subset \Sigma_{\omega'}$ and $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M_{\omega'}/|\lambda|$ for $\lambda \notin \Sigma_{\omega'}$ with some constant $M_{\omega'} \geq 1$.

2.4.1 Fractional Powers of Sectorial Operators

In this subsection, we present fundamentals of fractional powers of sectorial operators which is shown in [37, Chapter 2, Section 7].

Let X be a Banach space with norm $\|\cdot\|$. Let A be a sectorial operator of X with angle $0 \leq \omega_A < \pi$.

For any integer $n \in \mathbb{Z}$, the operator A^n is defined; indeed, when $n > 0$, A^n is a densely defined, closed operator of X; when $n < 0$, $A^n = (A^{-1})^{-n} = (A^{-n})^{-1}$ is a bounded operator of X; and, when $n = 0$, $A^0 = 1$ (the identity on X).

By ω we denote an angle such that $\omega_A < \omega < \pi$. By definition,

$$\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}, \quad \omega_A < \omega < \pi, \tag{2.12}$$

and

$$\left\|(\lambda - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M_\omega}{|\lambda|}, \quad \lambda \notin \Sigma_\omega, \quad \omega_A < \omega < \pi, \tag{2.13}$$

with some constant $M_\omega \geq 1$. Note that (2.12) implicitly means that $0 \in \rho(A)$, and that

$$\{\lambda \in \mathbb{C};\ |\lambda| \leq \delta\} \subset \rho(A), \tag{2.14}$$

provided that $0 < \delta < \|A^{-1}\|^{-1}$.

We define, for each complex number z such that $\mathrm{Re}\, z > 0$, the bounded linear operator

$$A^{-z} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-z} (\lambda - A)^{-1} d\lambda, \tag{2.15}$$

using the Dunford integral in $\mathcal{L}(X)$, where $\Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$ is an integral contour lying in the resolvent set $\rho(A)$ such that $\Gamma_+ : \lambda = re^{i\omega}$ for $\infty > r \geq \delta$, $\Gamma_0 : \lambda = \delta e^{i\theta}$ for $\omega \geq \theta \geq -\omega$, $\Gamma_- : \lambda = re^{-i\omega}$ for $\delta \leq r < \infty$. As a branch of the analytic function λ^{-z}, we take the principal branch $\mathbb{C} \setminus (-\infty, 0]$. In addition, Γ is oriented from $\omega e^{i\omega}$ to $\delta e^{i\omega}$, from $\delta e^{i\omega}$ to $\delta e^{-i\omega}$, and from $\delta e^{-i\omega}$ to $\infty e^{-i\omega}$.

On the other hand, since we can verify that A^{-z} is one-to-one for every $\mathrm{Re}\, z > 0$, its inverse

$$A^z = (A^{-z})^{-1} \quad \text{for} \quad \mathrm{Re}\, z > 0$$

is a single-valued linear operator of X.

According to the above definitions, for every real number $-\infty < x < \infty$, the fractional power A^x of A has been defined. As known properties, A^x are bounded operators on X

for $-\infty < x < 0$, $A^0 = 1$, and A^x are densely defined, closed linear operators of X for $0 < x < \infty$. For $0 < x < \infty$, the domain of A^x is denoted by $\mathcal{D}(A^x)$.

In the following, we present some properties of fractional powers. Firstly, for $0 \le \theta \le 1$, it holds that

$$A^\theta U = A^{\theta-1} A U = A A^{\theta-1} U, \quad U \in \mathcal{D}(A).$$

Secondly, for $0 \le \theta \le 1$, the following inequality called a moment inequality

$$\|A^\theta U\| \le C_\theta \|AU\|^\theta \|U\|^{1-\theta}, \quad U \in \mathcal{D}(A), \tag{2.16}$$

holds true. In addition, for $0 \le \theta_0 < \theta_1 < \infty$, a more generalized moment inequality

$$\|A^{\theta_0} U\| \le C_{\theta_0,\theta_1} \|A^{\theta_1} U\|^{\theta_0/\theta_1} \|U\|^{(\theta_1-\theta_0)/\theta_1}, \quad U \in \mathcal{D}(A^{\theta_1}), \tag{2.17}$$

also holds true.

2.4.2 Exponential Functions

Let X be a Banach space eith norm $\|\cdot\|$. Let A be a sectorial operator of X with angle $0 \le \omega_A < \pi/2$. We define the family of bounded operators e^{-tA} on X by the Dunford integral

$$e^{-tA} = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (\lambda - A)^{-1} d\lambda, \quad 0 < t < \infty,$$

in the space $\mathcal{L}(X)$. The integral contour in an infinite curve lying in $\rho(A)$ which surrounds $\sigma(A)$ counterclockwise. For example, we can take $\Gamma = \Gamma_- \cup \Gamma_+$, where $\gamma_\pm : \lambda = re^{\pm i\omega}$, $0 \le r < \infty$, which is oriented from $\infty e^{i\omega}$ to 0 and from 0 to $\infty e^{-i\omega}$ (note that $0 \in \rho(A)$). The integral along Γ is convergent in $\mathcal{L}(X)$. The family of operator e^{-tA} is called the exponential function generated by $-A$.

2.5 Sectorial Operators Associated with Sesquilinear Forms

Let $Z \subset X \subset Z^*$ be a triplet of spaces. We consider a sesquilinear form $a(U, V)$ defined on Z, that is, $a(U, V)$ is a complex-valued function defined for $(U, V) \in Z \times Z$ satisfying

$$\begin{cases} a(\alpha U + \beta \tilde{U}, V) = \alpha a(U, V) + \beta a(\tilde{U}, V), & \alpha, \beta \in \mathbb{C}, \quad U, \tilde{U}, V \in Z, \\ a(U, \alpha V + \beta \tilde{V}) = \overline{\alpha} a(U, V) + \overline{\beta} a(U, \tilde{V}), & \alpha, \beta \in \mathbb{C}, \quad U, V, \tilde{V} \in Z. \end{cases}$$

When $a(U, V)$ satisfies the condition

$$|a(U, V)| \le M \|U\|_Z \|V\|_Z, \quad U, V \in Z, \tag{2.18}$$

with some constant M, $a(U, V)$ is called a continuous form. In the meantime, when $a(U, V)$ satisfies the condition

$$\mathrm{Re}\ a(U, U) \ge \delta \|U\|_Z^2, \quad U \in Z, \tag{2.19}$$

with some positive constant $\delta > 0$, $a(U, V)$ is called a coercive form.

For continuous and coercive sesquilinear form $a(U,V)$ defined on Z, we can construct a linear isomorphism $\mathcal{A} : Z \to Z^*$ such that $a(U,V) = \langle \mathcal{A}U, V\rangle_{Z^*\times Z}$ for all $U, V \in Z$. Such $\mathcal{A}$ is called the linear operator associated with $a(U,V)$. Here, let us introduce two operators $\mathcal{A}_{|X}$ and $\mathcal{A}_{|Z}$. Firstly, we introduce the operator $\mathcal{A}_{|X}$ in X defined by

$$\begin{cases} \mathcal{D}(\mathcal{A}_{|X}) = \{U \in Z;\ \mathcal{A}U \in X\}, \\ \mathcal{A}_{|X}U = \mathcal{A}U, \quad \text{for } U \in \mathcal{D}(\mathcal{A}_{|X}). \end{cases}$$

The operator $\mathcal{A}_{|X}$ is called the part of $\mathcal{A}$ in X. In the meantime, we introduce the operator $\mathcal{A}_{|Z}$ in Z defined by

$$\begin{cases} \mathcal{D}(\mathcal{A}_{|Z}) = \{U \in Z;\ \mathcal{A}U \in Z\}, \\ \mathcal{A}_{|Z}U = \mathcal{A}U, \quad \text{for } U \in \mathcal{D}(\mathcal{A}_{|Z}). \end{cases}$$

The operator $\mathcal{A}_{|Z}$ is called the part of $\mathcal{A}$ in Z.

As for the $\mathcal{A}$ and its parts $\mathcal{A}_{|X}$ and $\mathcal{A}_{|Z}$, we know the following theorem. For details and its proof, see [37, Theorem 2.1].

Theorem 2.10. *Let $a(U,V)$ be a sesquilinear form on Z satisfying* (2.18) *and* (2.19)*, and let $\mathcal{A}$, $\mathcal{A}_{|X}$, and $\mathcal{A}_{|Z}$ be the linear operators of Z^*, X, and Z, respectively, which are determined from the form. Then, they satisfy* (2.10) *and* (2.11) *with angle $\omega = \pi/2$ and constant $\frac{M+\delta}{\delta}$. Hence, they are sectorial operators of Z^*, X, and Z, respectively, with angles $< \pi/2$.*

2.6 Sectorial Operators in $L_2(\Omega)$

2.6.1 Under the Periodic Conditions

Let us consider the case where $n = 1$. Let $I = (\alpha, \beta)$ be a bounded interval in $\mathbb{R}$. We introduce the following two spaces

$$H^1_P(I) = \{u \in H^1(I);\ u(\alpha) = u(\beta)\}$$

and

$$H^2_P(I) = \{u \in H^2(I);\ u(\alpha) = u(\beta) \text{ and } u'(\alpha) = u'(\beta)\}.$$

Note that $H^1_P(I)$ and $H^2_P(I)$ become Hilbert spaces with the inner products $(\cdot,\cdot)_{H^1}$ and $(\cdot,\cdot)_{H^2}$, respectively. Furthermore, we know that $H^1_P(I) \subset L_2(I)$ with dense and continuous embedding.

Consider the sesquilinear form

$$a(u,v) = \int_I a(x)u'\overline{v}'\,dx + \int_I u\overline{v}\,dx, \quad u, v \in H^1_P(I),$$

defined on $H^1_P(I)$. Here, $a(x)$ is a real-valued function in I satisfying the conditions

$$a \in H^1_P(I) \qquad \text{and} \qquad a(x) \geq \delta > 0 \text{ for all } x \in \overline{I} \tag{2.20}$$

with some constant $\delta > 0$. These conditions imply that $a(\cdot,\cdot)$ fulfills (2.18) and (2.19) on $H^1_P(I)$.

We consider the triplet $H^1_P(I) \subset L_2(I) \subset H^1_P(I)'$. The part of the operator associated with $a(\cdot,\cdot)$ in $L_2(I)$ is denoted by A. As for the domain $\mathcal{D}(A)$, we know the following shift property.

Theorem 2.11. *Let $I = (\alpha, \beta)$ be a bounded interval in $\mathbb{R}$. Let* (2.20) *be satisfied. Then, $\mathcal{D}(A) = H^2_P(I)$ with norm equivalence.*

The proof is quite similar to that of [37, Theorem 2.8], so we omit it.

2.6.2 Under the Dirichlet Conditions

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\partial\Omega$ be of $\mathcal{C}^2$ class. Consider the sesquilinear form

$$a(u,v) = a \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx, \quad u, v \in \mathring{H}^1(\Omega),$$

on the space $\mathring{H}^1(\Omega)$. Where, $\nabla u = {}^t(D_1 u, D_2 u, \dots, D_n u)$ and $\nabla u \cdot \overline{\nabla v} = D_1 u \overline{D_1 v} + D_2 u \overline{D_2 v} + \cdots + D_n u \overline{D_n v}$. We assume that $a > 0$. From the Poincaré inequality, we know that $\|u\|_{L_2} \le C \|\nabla u\|_{L_2}$ for $u \in \mathring{H}^1(\Omega)$ with some constant $C \ge 0$. Therefore, $a(\cdot,\cdot)$ fulfills (2.18) and (2.19) on $\mathring{H}^1(\Omega)$.

We consider the triplet $\mathring{H}^1(\Omega) \subset L_2(\Omega) \subset H^{-1}(\Omega)$. The part of the operator associated with $a(\cdot,\cdot)$ in $L_2(\Omega)$ is denoted by A. This operator A is regarded as realizations of the elliptic operator $-a\Delta$ in $L_2(\Omega)$, under the Dirichlet boundary conditions $\gamma u = 0$ on $\partial\Omega$, where γ is the trace operator defined by $\gamma u = u_{|\partial\Omega}$. From Theorem 2.10, we know that A is a sectorial operator of $L_2(\Omega)$ with angle $< \pi/2$.

We present characterization results of the domains of fractional powers A^θ for $0 \le \theta \le 1$. For $1/2 < s \le 2$, we define a closed subspace of $H^s(\Omega)$ by

$$H^s_D(\Omega) = \{u \in H^s(\Omega);\ \gamma u = 0\}.$$

By the shift property, we know that $\mathcal{D}(A) = H^2_D(\Omega)$ with norm equivalence. The following characterization is then shown. For the proof, see [42].

Theorem 2.12. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Then,*

$$\mathcal{D}(A^\theta) = [L_2(\Omega), H^2_D(\Omega)]_\theta = \begin{cases} H^{2\theta}(\Omega) & \text{if } 0 \le \theta < 1/4, \\ H^{2\theta}_D(\Omega) & \text{if } 1/4 < \theta \le 1, \end{cases} \tag{2.21}$$

with norm equivalence

$$C^{-1}\|u\|_{H^{2\theta}} \le \|A^\theta u\|_{L_2} \le C\|u\|_{H^{2\theta}}, \quad u \in \mathcal{D}(A^\theta),$$

$C > 0$ being determined by Ω and a.

2.6.3 Under the Neumann Conditions

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\partial\Omega$ be of $\mathcal{C}^2$ class. Consider the sesquilinear form

$$a(u, v) = a \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx + \int_\Omega u\overline{v}\, dx, \quad u, v \in H^1(\Omega),$$

on the space $H^1(\Omega)$. We assume that $a > 0$. Then, $a(\cdot, \cdot)$ fulfills (2.18) and (2.19) on $H^1(\Omega)$. Note that the Poincaré inequality dose not holds true when $u \in H^1(\Omega)$, so the sesquilinear form $a(\cdot, \cdot)$ must have the term $\int_\Omega u\overline{v}\, dx$ in order to prove the coerciveness.

We consider the triplet $H^1(\Omega) \subset L_2(\Omega) \subset H^1(\Omega)'$. The part of the operator associated with $a(\cdot, \cdot)$ in $L_2(\Omega)$ is denoted by A. This operator A is regarded as realizations of the elliptic operator $-a\Delta + 1$ in $L_2(\Omega)$, under the Neumann-type boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. From Theorem 2.10, we know that A is a sectorial operator of $L_2(\Omega)$ with angle $< \pi/2$.

For $3/2 < s \leq 2$, we define a closed subspace of $H^s(\Omega)$ by

$$H^s_N(\Omega) = \left\{ u \in H^s(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

By the shift property, we know that $\mathcal{D}(A) = H^2_N(\Omega)$ with norm equivalence. The following characterization is then shown. For the proof, see [37, Theorem 16.7].

Theorem 2.13. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Then,*

$$\mathcal{D}(A^\theta) = [L_2(\Omega), H^2_N(\Omega)]_\theta = \begin{cases} H^{2\theta}(\Omega) & \text{if } 0 \leq \theta < 3/4, \\ H^{2\theta}_N(\Omega) & \text{if } 3/4 < \theta \leq 1, \end{cases} \tag{2.22}$$

with norm equivalence

$$C^{-1}\|u\|_{H^{2\theta}} \leq \|A^\theta u\|_{L_2} \leq C\|u\|_{H^{2\theta}}, \quad u \in \mathcal{D}(A^\theta),$$

$C > 0$ being determined by Ω and a.

Chapter 3

Network Shaped Domains

In this chapter, we are concerned with strongly elliptic differential operators in a network shaped domain. In this doctoral thesis, we intend to characterize the domains of fractional powers of sectorial operators determined from these differential operators. Here, a network shaped domain means the pair of a set of nodes and a set of edges connecting the nodes. Network shaped domains come from a variety of areas: carbon nanostructure [43, 44], superconductivity [45], photonic crystals [46], network of beams [47], traffic flow on networks [48, 49], and blood vessel [50]. In the meantime, network shaped domains are justified as an approximation of narrow branching domains under suitable assumptions (see Fig. 3.1). For further topics, see a review paper [51] by Pokornyi and Borovskikh, and references therein.

A differential equation in a network shaped domain is a system of differential equations in edges which interact each other through connecting nodes. After the pioneering study on differential equations in network shaped domains by Lumer [52], they are studied by many researchers. Strum-Liouville eigenvalue problems are considered by Below [53]. Stability of steady states in reaction diffusion equations in network shaped domains is studied by Yanagida [54]. In addition, adjoint and self-adjoint differential operators in network shaped domains are studied by Carlson [55]. Recently, Camilli and Corrias [56] studied chemotaxis equations in network shaped domains and obtained global solutions in an integral sense. In the meantime, Kosugi [57] studied the reduction of thin domains in $\mathbb{R}^n$ ($n \geq 2$) to network shaped domains for a semilinear elliptic equation.

We want to study parabolic semilinear and quasilinear equations in network shaped domains by using the theory of abstract parabolic evolution equations. Particularly, in Chapter 6 of this thesis, we will consider the Keller-Segel equations in network shaped

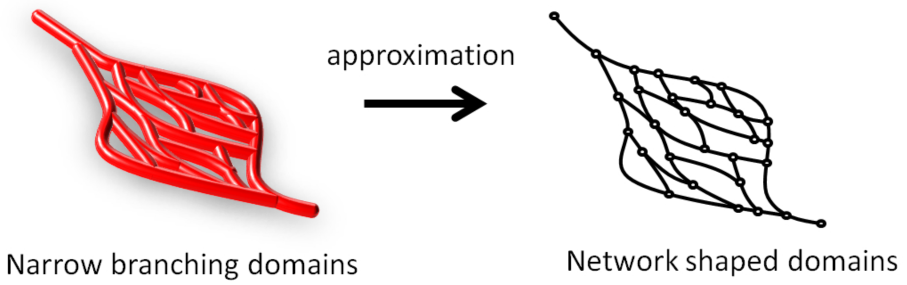

Fig. 3.1: Approximating narrow branching domains by network shaped domains

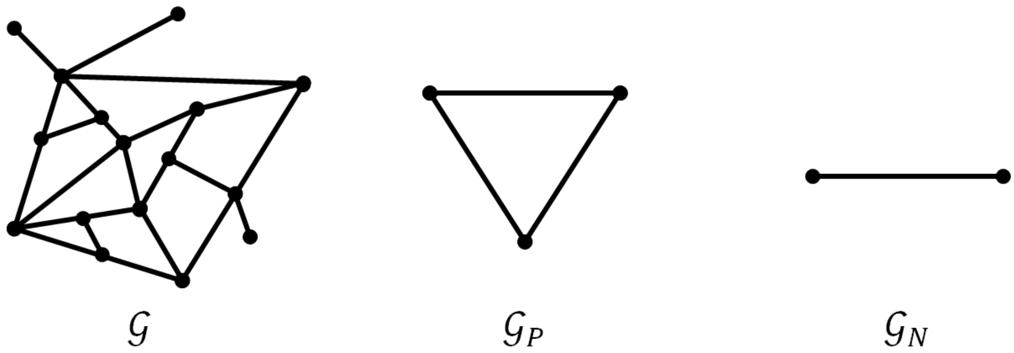

Fig. 3.2: Examples of network shaped domains

domains. To this end, it is important to investigate properties of fractional powers of sectorial operators determined from strongly elliptic differential operators. Yagi [42] has shown an application of the theory of H_∞ functional calculus to the characterization problem of the domains of the fractional powers of strongly elliptic differential operators in a bounded domain of $\mathbb{R}^n$ with the Dirichlet boundary conditions. By utilizing his techniques, we shall characterize the domains of the fractional powers of strongly elliptic differential operators in a network shaped domain. However, there are a few choices of the types of boundary conditions imposed at each node. One of the famous types is the Kirchhoff conditions, see [53, 54, 56], that is, the condition of continuity and the condition of the balance of flux at each node. Therefore, we will impose the Kirchhoff conditions for strongly elliptic differential operators.

The following results are obtained in [30].

3.1 Formulation of network shaped domains

We define network shaped domains as follows. Let $\mathcal{N} = \{N_j\}_j$ be a sequence of a finite number of points in $\mathbb{R}^3$. Let $\mathcal{E} = \{I_i\}_i$ be a sequence of segments in $\mathbb{R}^3$, where each segment connects a pair of two points of $\mathcal{N}$. We assume that every point has at least one connecting segment and there is no segment which intersects another segment. We call such a pair $\mathcal{G} = \{\mathcal{E}, \mathcal{N}\}$ a network shaped domain. In what follows, we call $I_i \in \mathcal{E}$ an edge of domain and $N_j \in \mathcal{N}$ an node of domain, respectively. As examples of network shaped domains, see Fig. 3.2.

For convenience, we consider a direction of I_i; $o(I_i) \in \mathcal{N}$ and $\omega(I_i) \in \mathcal{N}$ denote a start node and an end node of I_i, respectively. Since each $I_i \in \mathcal{E}$ has a Euclidian length $l_i > 0$, we identify I_i as the open interval $(0, l_i)$ with $o(I_i) = 0$ and $\omega(I_i) = l_i$. Furthermore, for each node $N_j \in \mathcal{N}$, let $o(N_j)$ and $\omega(N_j)$ be subsets of $\mathcal{E}$ given by

$$o(N_j) = \{I_i \in \mathcal{E}; o(I_i) = N_j\} \quad \text{and} \quad \omega(N_j) = \{I_i \in \mathcal{E}; \omega(I_i) = N_j\},$$

respectively. That is, $o(N_j)$ is the set of edges whose start nodes are N_j and $\omega(N_j)$ is the set of edges whose end nodes are N_j.

Let $f = \{f_i\}_{I_i \in \mathcal{E}}$ be a sequence of complex valued functions defined in each edge I_i. In what follows, we call that such f is a function in $\mathcal{G}$. As an example of functions in

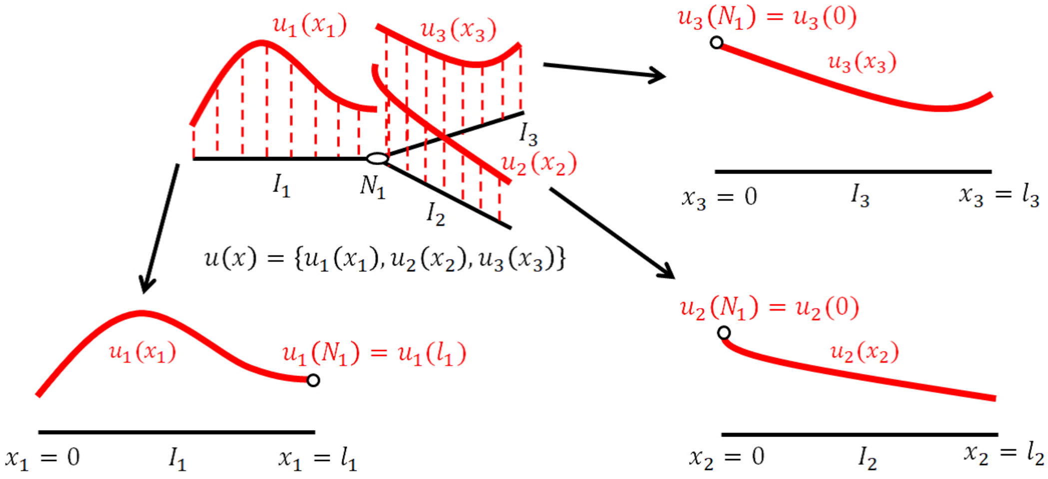

Fig. 3.3: An example of functions in a network shaped domain $\mathcal{G}$

$\mathcal{G} = \{\{I_1, I_2, I_3\}, \{N_1\}\}$, see Fig. 3.3. For each node N_j, we set $f_i(N_j) = f_i(0)$ if $I_i \in o(N_j)$ and $f_i(N_j) = f_i(l_i)$ if $I_i \in \omega(N_j)$. In addition, let us define the exterior normal derivative of f_i at N_j as

$$\frac{\partial f_i}{\partial n}(N_j) = -\lim_{\Delta x_i \to +0} \frac{f_i(\Delta x_i) - f_i(0)}{\Delta x_i} = -\frac{df_i}{dx_i}(0) \quad \text{if} \quad I_i \in o(N_j),$$
$$\frac{\partial f_i}{\partial n}(N_j) = \lim_{\Delta x_i \to +0} \frac{f_i(l_i) - f_i(l_i - \Delta x_i)}{\Delta x_i} = \frac{df_i}{dx_i}(l_i) \quad \text{if} \quad I_i \in \omega(N_j).$$

By using these definitions, for each N_j, we define

$$\frac{\partial f}{\partial n}(N_j) = \sum_{I_i \in o(N_j) \cup \omega(N_j)} \frac{\partial f_i}{\partial n}(N_j) \quad \text{for} \quad f = \{f_i\}.$$

3.2 Function Spaces in $\mathcal{G}$

Let us introduce function spaces in a network shaped domain $\mathcal{G} = \{\mathcal{E}, \mathcal{N}\}$. A notation $\{f_i\}_{I_i \in \mathcal{E}}$, which is a function in $\mathcal{G}$, is abbreviated by $\{f_i\}$, or it is simply denoted by f. Furthermore, the notation $\{1\}$, which is the set of functions identically one on each I_i, is abbreviated by 1.

Let f and g be functions in $\mathcal{G}$, and let $\alpha \in \mathbb{C}$. We define the following operations:

$$f + g = \{f_i + g_i\}, \quad \alpha f = \{\alpha f_i\}, \quad \overline{f} = \{\overline{f_i}\} \quad \text{and} \quad fg = \{f_i g_i\}.$$

Furthermore, for a function $\chi : \mathbb{C} \to \mathbb{C}$, we define $\chi(f) = \{\chi(f_i)\}$.

We often consider product spaces of Banach spaces $\prod_{I_i \in \mathcal{E}} X_i$, and they are abbreviated by $\prod X_i$.

3.2.1 L_p spaces and continuous function spaces

For $1 \leq p \leq \infty$, let $L_p(\mathcal{G})$ be the product space of spaces $\prod L_p(I_i)$ with the norm $\|f\|_{L_p(\mathcal{G})} = (\sum_{I_i \in \mathcal{E}} \|f_i\|^p_{L_p(I_i)})^{1/p}$. Particularly, $L_2(\mathcal{G})$ becomes the Hilbert space with the inner product $(f, g)_{L_2(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} (f_i, g_i)_{L_2(I_i)}$.

We introduce the space of continuous functions in $\mathcal{G}$ given by

$$\mathcal{C}(\mathcal{G}) = \left\{ f \in \prod \mathcal{C}(\overline{I_i});\ \begin{array}{l} \text{for each } N_j \in \mathcal{N}, \\ \quad \forall I_i \in o(N_j) \cup \omega(N_j),\ f_i(N_j) \text{ has a common value} \end{array} \right\}$$

with the norm $\|f\|_{\mathcal{C}(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} \|f_i\|_{\mathcal{C}(\overline{I_i})}$. In addition, the space of continuous functions satisfying the Kirchhoff conditions are given by

$$\mathcal{C}^1(\mathcal{G}) = \left\{ f \in \mathcal{C}(\mathcal{G}) \cap \prod \mathcal{C}^1(\overline{I_i});\quad \frac{\partial f}{\partial n}(N_j) = 0 \quad \text{for all } N_j \in \mathcal{N} \right\}$$

with the norm $\|f\|_{\mathcal{C}^1(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} \|f_i\|_{\mathcal{C}^1(\overline{I_i})}$. While, let us define the following space

$$\mathcal{C}^2(\mathcal{G}) = \left\{ f \in \mathcal{C}^1(\mathcal{G}) \cap \prod \mathcal{C}^2(\overline{I_i});\quad \begin{array}{l} \text{for each } N_j \in \mathcal{N},\ \forall I_i \in o(N_j) \cup \omega(N_j), \\ \dfrac{d^2 f_i}{dx_i^2}(N_j) \text{ has a common value} \end{array} \right\}.$$

Remark 3.1. *When N_j has only one connected edge, $u \in \mathcal{C}^1(\mathcal{G})$ satisfies the ordinary homogeneous Neumann boundary condition on N_j. Therefore, when $\mathcal{G}$ consists of only one edge (see $\mathcal{G}_N$ of Fig.3.2), $u \in \mathcal{C}^1(\mathcal{G})$ satisfies the ordinary Neumann boundary conditions. Furthermore, when $\mathcal{G}$ consists of only one loop (see $\mathcal{G}_P$ of Fig.3.2), $u \in \mathcal{C}^1(\mathcal{G})$ satisfies the periodic condition on a one-dimensional domain.*

For convenience, we use the following notations: for $f \in L_1(\mathcal{G})$,

$$\int_{\mathcal{G}} f dx = \sum_{I_i \in \mathcal{E}} \int_{I_i} f_i dx_i$$

and

"$f \geq 0$ for a.e. $\mathcal{G}$" if and only if "$f_i(x_i) \geq 0$ for a.e. $x_i \in I_i, \forall I_i \in \mathcal{E}$".

In particular, we write

$$(f, g)_{L_2(\mathcal{G})} = \int_{\mathcal{G}} f \overline{g}\, dx \quad \text{for} \quad f, g \in L_2(\mathcal{G}).$$

3.2.2 Sobolev spaces

Let $H^s(I_i)$ be the Sobolev space in I_i with fractional order $s > 0$. Let D_i be the differentiation in the sense of distribution on I_i, and let D be the following operation:

$$D : f = \{f_i\} \mapsto Df = \{D_i f_i\}.$$

In the following, we will introduce the Sobolev spaces $H^s(\mathcal{G})$ in $\mathcal{G}$ for the fractional order $0 < s \leq 3$. But in view of trace of the function $\prod H^s(I_i)$ for $1/2 < s \leq 3$, we have to consider the so-called compatibility conditions at nodes.

Firstly, we simply define $H^s(\mathcal{G}) = \prod H^s(I_i)$ for $0 < s \leq 1/2$. In the meantime, we define $H^{-s}(\mathcal{G}) = \prod H^{-s}(I_i)$ for $0 < s \leq 1/2$. As an immediate consequence of (2.7) and (2.8), for $0 < s \leq 1/2$, $H^s(\mathcal{G}) \subset L_2(\mathcal{G}) \subset H^{-s}(\mathcal{G})$ becomes a triplet with the relation

$$(u, f)_{L_2(\mathcal{G})} = \langle u, f \rangle_{H^s(\mathcal{G}) \times H^{-s}(\mathcal{G})} \quad \text{for} \quad u \in H^s(\mathcal{G}),\ f \in L_2(\mathcal{G}). \tag{3.1}$$

Secondly, let us define $H^s(\mathcal{G})$ for $1/2 < s \le 3/2$. (2.9) implies that

$$\prod H^s(I_i) \text{ is continuously embedded in } \prod \mathcal{C}(\overline{I_i}) \text{ if } s > 1/2 \tag{3.2}$$

with some embedding constant $c_s > 0$. In view of this fact, we define the space $H^s(\mathcal{G})$ for $1/2 < s \le 3/2$ as $H^s(\mathcal{G}) = \mathcal{C}(\mathcal{G}) \cap \prod H^s(I_i)$. Then, $H^s(\mathcal{G})$ is a closed linear subspace of $\prod H^s(I_i)$, so $H^s(\mathcal{G})$ becomes a Hilbert space with the inner product $(\cdot,\cdot)_{H^s(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} (\cdot,\cdot)_{H^s(I_i)}$.

Thirdly, let us define $H^s(\mathcal{G})$ for $3/2 < s \le 5/2$. (2.9) implies that

$$\prod H^s(I_i) \text{ is continuously embedded in } \prod \mathcal{C}^1(\overline{I_i}) \text{ if } s > 3/2 \tag{3.3}$$

with some embedding constant $c'_s > 0$. So, we define the space $H^s(\mathcal{G})$ for $3/2 < s \le 5/2$ as $H^s(\mathcal{G}) = \mathcal{C}^1(\mathcal{G}) \cap \prod H^s(I_i)$. As before, $H^s(\mathcal{G})$ is a Hilbert space with the inner product $(\cdot,\cdot)_{H^s(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} (\cdot,\cdot)_{H^s(I_i)}$.

Finally, we define the space $H^s(\mathcal{G})$ for $5/2 < s \le 3$ as $H^s(\mathcal{G}) = \mathcal{C}^2(\mathcal{G}) \cap \prod H^s(I_i)$. As before, $H^s(\mathcal{G})$ is also a Hilbert space with the inner product $(\cdot,\cdot)_{H^s(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} (\cdot,\cdot)_{H^s(I_i)}$.

We show the following two lemmas.

Lemma 3.1. *It holds that*

$$(\alpha Du, Dv)_{L_2(\mathcal{G})} = -(D[\alpha Du], v)_{L_2(\mathcal{G})} \quad \text{for } \alpha, v \in H^1(\mathcal{G}) \text{ and } u \in H^2(\mathcal{G}). \tag{3.4}$$

Proof. From integration by parts, it is observed that

$$\int_{\mathcal{G}} \alpha Du \overline{Dv}\, dx = \sum_{I_i \in \mathcal{E}} \Big[\alpha_i [D_i u_i] \overline{v_i}\Big]_{x_i=0}^{x_i=l_i} - \int_{\mathcal{G}} D[\alpha Du]\overline{v}\, dx.$$

Here, since $\alpha, v \in H^1(\mathcal{G})$ and $u \in H^2(\mathcal{G})$, we verify that

$$\sum_{I_i \in \mathcal{E}} \Big[\alpha_i [D_i u_i] \overline{v_i}\Big]_{x_i=0}^{x_i=l_i} = \sum_{N_j \in \mathcal{N}} \alpha(N_j)\overline{v(N_j)}\, \frac{\partial u}{\partial n}(N_j) = 0.$$

Therefore, (3.4) is obtained. □

Lemma 3.2. *Let $\alpha \in \mathcal{C}^1(\mathcal{G})$. Then, for any $0 \le s \le 1$, the multiplication $u \mapsto \alpha u$ is a bounded operator on $H^s(\mathcal{G})$ with the estimate*

$$\|\alpha u\|_{H^s(\mathcal{G})} \le C \|\alpha\|_{\mathcal{C}^1(\mathcal{G})} \|u\|_{H^s(\mathcal{G})}, \quad u \in H^s(\mathcal{G}). \tag{3.5}$$

Proof. It is clear that $\|\alpha u\|_{L_2(\mathcal{G})} \le \|\alpha\|_{\mathcal{C}(\mathcal{G})} \|u\|_{L_2(\mathcal{G})}$ for $u \in L_2(\mathcal{G})$ and that $\|\alpha u\|_{H^1(\mathcal{G})} \le C\|\alpha\|_{\mathcal{C}^1(\mathcal{G})} \|u\|_{H^1(\mathcal{G})}$ for $u \in \prod H^1(I_i)$. Then, the estimate is verified by applying Theorem 2.4 and Theorem 2.6, In addition, it is obvious that $\alpha u \in H^s(\mathcal{G})$ for $1/2 < s \le 1$, so the lemma is proved. □

3.2.3 Mean zero spaces

In this subsection, we introduce subspaces of $L_2(\mathcal{G})$ and $H^1(\mathcal{G})$ which play an important role in Sections 6.4 – 6.6 of Chapter 6. Let $L_{2,m}(\mathcal{G})$ be the space

$$L_{2,m}(\mathcal{G}) = \left\{ f \in L_2(\mathcal{G}); \int_{\mathcal{G}} f dx = 0 \right\}.$$

Clearly, $L_{2,m}(\mathcal{G})$ is a closed linear subspace of $L_2(\mathcal{G})$, so that $L_{2,m}(\mathcal{G})$ is a Hilbert space with the inner product $(\cdot,\cdot)_{L_{2,m}(\mathcal{G})} = (\cdot,\cdot)_{L_2(\mathcal{G})}$. We denote the orthogonal projection from $L_2(\mathcal{G})$ onto $L_{2,m}(\mathcal{G})$ by

$$Pf = f - \frac{1}{l_{total}} \int_{\mathcal{G}} f dx, \quad f \in L_2(\mathcal{G}). \tag{3.6}$$

Then, we know a version of Poincaré-Wirtinger inequality, i.e., there exists a constant C (depending on $\mathcal{G}$ only) such that

$$\|Pu\|_{L_{2,m}(\mathcal{G})} \le C \|Du\|_{L_{2,m}(\mathcal{G})}, \quad u \in H^1(\mathcal{G}). \tag{3.7}$$

The proof is omitted since it is quit analogous to that of the standard Poincaré-Wirtinger inequality (e.g. [58, Section 5.8, Theorem 1]).

Next, let us introduce $\mathcal{C}_m(\mathcal{G}) = \mathcal{C}(\mathcal{G}) \cap L_{2,m}(\mathcal{G})$. It is easy to see that $\mathcal{C}_m(\mathcal{G})$ is a Banach space with the norm $\|\cdot\|_{\mathcal{C}_m(\mathcal{G})} = \|\cdot\|_{\mathcal{C}(\mathcal{G})}$. Note that P given by (3.6) also becomes a bounded linear projection from $\mathcal{C}(\mathcal{G})$ onto $\mathcal{C}_m(\mathcal{G})$. So, we use the same notation P.

3.3 Sectorial Operators in $L_2(\mathcal{G})$

Note that $H^1(\mathcal{G}) \subset L_2(\mathcal{G})$ with dense and continuous embedding. Therefore, a triplet of spaces $H^1(\mathcal{G}) \subset L_2(\mathcal{G}) \subset H^1(\mathcal{G})'$ is constructed. Particularly, the duality product $\langle \cdot,\cdot \rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})}$ satisfies that

$$\langle f, u \rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})} = (f, u)_{L_2(\mathcal{G})} \quad \text{for all} \quad f \in L_2(\mathcal{G}), u \in H^1(\mathcal{G}). \tag{3.8}$$

Let us consider the following sesquilinear form

$$a(u, v) = \int_{\mathcal{G}} \alpha(x) Du \overline{Dv} dx + \int_{\mathcal{G}} \beta(x) u \overline{v} dx, \qquad u, v \in H^1(\mathcal{G}),$$

defined on $H^1(\mathcal{G})$. Here, $\alpha(x) = \{\alpha_i(x_i)\}$ is a set of real valued functions satisfying

$$\alpha \in \mathcal{C}^1(\mathcal{G}) \qquad \text{and} \qquad \alpha(x) \ge \alpha_0 \text{ in } \mathcal{G} \tag{3.9}$$

with some constant $\alpha_0 > 0$, and $\beta(x) = \{\beta_i(x_i)\}$ is a set of real valued functions satisfying

$$\beta \in L_\infty(\mathcal{G}) \qquad \text{and} \qquad \beta(x) \ge \beta_0 \text{ in a.e. } \mathcal{G}$$

with some constant $\beta_0 > 0$.

It is easy to see that $a(u,v)$ is continuous and coercive. More precisely, $a(u,v)$ satisfies that

$$|a(u,v)| \le \max\{\|\alpha\|_{L_\infty(\mathcal{G})}, \|\beta\|_{L_\infty(\mathcal{G})}\} \|u\|_{H^1(\mathcal{G})} \|v\|_{H^1(\mathcal{G})}, \qquad u,v \in H^1(\mathcal{G}), \tag{3.10}$$

and

$$\mathrm{Re}\, a(u,u) \ge \min\{\alpha_0, \beta_0\} \|u\|^2_{H^1(\mathcal{G})}, \qquad u \in H^1(\mathcal{G}). \tag{3.11}$$

Therefore, there exists a linear isomorphism $\mathcal{A} : H^1(\mathcal{G}) \to H^1(\mathcal{G})'$ such that

$$a(u,v) = \langle \mathcal{A}u, v\rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})}, \qquad u,v \in H^1(\mathcal{G}). \tag{3.12}$$

In what follows, let $A = \mathcal{A}_{|L_2(\mathcal{G})}$ be the part of $\mathcal{A}$ in $L_2(\mathcal{G})$. We verify by (3.8) that

$$u \in \mathcal{D}(A) \text{ if and only if } a(u,v) \text{ is continuous in } v \text{ with respect to the } L_2(\mathcal{G}) \text{ norm.} \tag{3.13}$$

Moreover, if $u \in \mathcal{D}(A)$ then $a(u,v) = (Au,v)_{L_2(\mathcal{G})}$ for all $v \in H^1(\mathcal{G})$.

3.3.1 Fundamental properties for A

We characterize A as a differential operator in $L_2(\mathcal{G})$.

Theorem 3.1. *It holds that $\mathcal{D}(A) = H^2(\mathcal{G})$ and, for $u \in \mathcal{D}(A)$,*

$$Au = -D[\alpha(x)Du] + \beta(x)u \quad \text{in} \quad L_2(\mathcal{G}) \tag{3.14}$$

with the norm equivalence

$$m^{-1} \|u\|_{H^2(\mathcal{G})} \le \|Au\|_{L_2(\mathcal{G})} \le m \|u\|_{H^2(\mathcal{G})}, \quad u \in \mathcal{D}(A), \tag{3.15}$$

where $m > 0$ depends on α_0, β_0, $\|\alpha\|_{\mathcal{C}^1(\mathcal{G})}$, and $\|\beta\|_{L_\infty(\mathcal{G})}$.

Proof. Let $u \in H^2(\mathcal{G})$. Then, it is observed from (3.4) that $a(u,v) = (-D[\alpha Du] + \beta u, v)_{L_2(\mathcal{G})}$ for $v \in H^1(\mathcal{G})$. Therefore, it follows from (3.13) that $u \in \mathcal{D}(A)$.

Let us show the opposite inclusion. For $u \in \mathcal{D}(A)$, there exists $f \in L_2(\mathcal{G})$ such that $Au = f$. Thus, $(f,v)_{L_2(\mathcal{G})} = (Au,v)_{L_2(\mathcal{G})} = a(u,v)$, that is,

$$(f - \beta u, v)_{L_2(\mathcal{G})} = (\alpha Du, Dv)_{L_2(\mathcal{G})} \quad \text{for all} \quad v \in H^1(\mathcal{G}). \tag{3.16}$$

Since $\prod \mathcal{C}_0^\infty(I_i) \subset H^1(\mathcal{G})$, we know that $\alpha_i D_i u_i \in H^1(I_i)$ for each I_i. Then, (3.9) implies that $D_i u_i \in H^1(I_i)$; therefore, $u \in \prod H^2(I_i)$. Using (3.16) again, we obtain that

$$(f + D[\alpha Du] - \beta u, v)_{L_2(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} \Big[\alpha_i [D_i u_i] \overline{v_i}\Big]_{x_i=0}^{x_i=l_i} \quad \text{for all} \quad v \in H^1(\mathcal{G}). \tag{3.17}$$

In (3.17), by choosing $v \in \prod \mathcal{C}_0^\infty(I_i)$, we obtain that $Au = f = -D[\alpha Du] + \beta u$ in $L_2(\mathcal{G})$. Furthermore, it follows from (3.17) that

$$\sum_{I_i \in \mathcal{E}} \Big[\alpha_i [D_i u_i] \overline{v_i}\Big]_{x_i=0}^{x_i=l_i} = \sum_{N_j \in \mathcal{N}} \alpha(N_j) \overline{v(N_j)} \frac{\partial u}{\partial n}(N_j) = 0 \quad \text{for all} \quad v \in H^1(\mathcal{G}).$$

Since $\overline{v(N_j)}$ are arbitrary and $\alpha(N_j)$ are positive, we deduce that $\frac{\partial u}{\partial n}(N_j) = 0$. We thus obtained that $\mathcal{D}(A) \subset H^2(\mathcal{G})$.

Finally, let us show the norm equivalence (3.15) for $u \in \mathcal{D}(A)$. It is obvious that $\|Au\|_{L_2(\mathcal{G})} \leq m \|u\|_{H^2(\mathcal{G})}$. On the other hand, for a fixed constant $k > 0$, it holds that

$$\begin{aligned}
&\|[A + (k - \beta)]u\|^2_{L_2(\mathcal{G})} = \int_{\mathcal{G}} [(D[\alpha Du])^2 - 2kuD[\alpha Du] + k^2u^2]dx \\
&= \int_{\mathcal{G}} [\alpha^2(D^2u)^2 + (D\alpha)^2(Du)^2 + 2(D\alpha)(Du)\alpha(D^2u) + 2k\alpha(Du)^2 + k^2u^2]dx \\
&\geq \int_{\mathcal{G}} [\frac{\alpha^2}{2}(D^2u)^2 + (2k\alpha - 7(D\alpha)^2)(Du)^2 + k^2u^2]dx.
\end{aligned}$$

Due to (3.9), we can choose sufficiently large $k > 0$ such that $2k\alpha - 7(D\alpha)^2 > 0$ in $\mathcal{G}$. Thus, we conclude that $m^{-1}\|u\|_{H^2(\mathcal{G})} \leq \|Au\|_{L_2(\mathcal{G})}$ due to (3.11). □

We know that A is a linear isomorphism from $H^2_{C,F}(\mathcal{G})$ onto $L_2(\mathcal{G})$. Consequently,

$$A^{-1} : L_2(\mathcal{G}) \to L_2(\mathcal{G}) \text{ is a compact operator,} \tag{3.18}$$

since $H^2_{C,F}(\mathcal{G})$ is compactly embedded in $L_2(\mathcal{G})$.

3.3.2 Fractional Powers A^θ

We want to characterize the domains of fractional powers A^θ. Due to Theorem 2.10, A is a sectorial operator of $L_2(\mathcal{G})$ with angle $\omega_A < \pi/2$, that is, the spectrum $\sigma(A)$ is contained in an open sectorial domain such that

$$\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\},$$

where $\omega_A < \omega \leq \pi/2$, and its resolvent $(\lambda - A)^{-1}$ satisfies the estimate

$$\left\|(\lambda - A)^{-1}\right\|_{\mathcal{L}(L_2(\mathcal{G}))} \leq \frac{M}{|\lambda|}, \quad \lambda \notin \Sigma_\omega, \tag{3.19}$$

with some constant $M \geq 1$. Then, fractional powers of A are defined as shown in Subsection 2.4.1. Note that, for each $0 \leq \theta \leq 1$, $\mathcal{D}(A^\theta)$ is a Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{D}(A^\theta)} = (A^\theta\cdot, A^\theta\cdot)_{L_2(\mathcal{G})}$.

It is easy to check that A is a positive definite self-adjoint operator in $L_2(\mathcal{G})$. From the results on square root problem for positive definite self-adjoint operators (for example, [37, Theorem 2.34]), we know that $\mathcal{D}(A^{1/2}) = H^1(\mathcal{G})$ with the norm equivalence

$$\min\{\alpha_0, \beta_0\}\|u\|^2_{H^1(\mathcal{G})} \leq \|A^{1/2}u\|^2_{L_2(\mathcal{G})} \leq \max\{\|\alpha\|_{L_\infty(\mathcal{G})}, \|\beta\|_{L_\infty(\mathcal{G})}\}\|u\|^2_{H^1(\mathcal{G})}$$

due to (3.10) and (3.11).

In addition, according to [37, Theorem 16.1], the domains of the fractional powers $\mathcal{D}(A^\theta)$ and the complex interpolation spaces $[L_2(\mathcal{G}), \mathcal{D}(A)]_\theta$ coincide; more precisely, for any $0 < \theta < 1$, $[L_2(\mathcal{G}), \mathcal{D}(A)]_\theta = \mathcal{D}(A^\theta)$ with isometry. In this case, it follows from Theorem 3.1 that

$$[L_2(\mathcal{G}), H^2(\mathcal{G})]_\theta = \mathcal{D}(A^\theta) \text{ with norm equivalence for any } 0 < \theta < 1. \tag{3.20}$$

According to [37, Theorem 16.3], (3.20) implies an integrable condition along the Γ_ω, i.e.,

$$\int_{\Gamma_\omega} |\lambda|^\theta| \left(A^{1-\theta}(\overline{\lambda}-A)^{-2}f,g\right)_{L_2(\mathcal{G})}| \, |d\lambda| \leq C_{\theta,\omega} \|f\|_{L_2(\mathcal{G})} \|g\|_{L_2(\mathcal{G})}, \quad f,g \in L_2(\mathcal{G}), \quad (3.21)$$

for $0<\theta<1$ with some constant $C_{\theta,\omega}>0$.

Our characterization result is given by the following theorem.

Theorem 3.2. *It holds that*

$$\mathcal{D}(A^\theta) = H^{2\theta}(\mathcal{G}) \quad \text{if} \quad \theta \neq 1/4, 3/4$$

with norm equivalence

$$m_\theta^{-1}\|u\|_{H^{2\theta}(\mathcal{G})} \leq \|A^\theta u\|_{L_2(\mathcal{G})} \leq m_\theta \|u\|_{H^{2\theta}(\mathcal{G})} \quad \text{for} \quad u \in \mathcal{D}(A^\theta),$$

$m_\theta > 0$ *being determined by* α_0, β_0, $\|\alpha\|_{\mathcal{C}^1(\mathcal{G})}$, *and* $\|\beta\|_{L_\infty(\mathcal{G})}$.

Proof. The proof will be divided into five Steps.

Step 1. Let us show that $\mathcal{D}(A^\theta) \subset \prod H^{2\theta}(I_i)$ for $0 \leq \theta \leq 1$. By applying the interpolation theorem for bounded operator, [37, Theorem 1.15], to the embedding $j : H^2(\mathcal{G}) \to \prod H^2(I_i)$, we see that $[L_2(\mathcal{G}), H^2(\mathcal{G})]_\theta \subset [L_2(\mathcal{G}), \prod H^2(I_i)]_\theta$ for any $0 \leq \theta \leq 1$ with

$$\|u\|_{[L_2(\mathcal{G}),\prod H^2(I_i)]_\theta} \leq \|u\|_{[L_2(\mathcal{G}),H^2(\mathcal{G})]_\theta}, \qquad u \in [L_2(\mathcal{G}), H^2(\mathcal{G})]_\theta.$$

Furthermore, by Theorem 2.6, it holds that $[L_2(\mathcal{G}), \prod H^2(I_i)]_\theta = \prod H^{2\theta}(I_i)$ with norm equivalence. Therefore, we conclude from (3.20) that $\mathcal{D}(A^\theta) \subset \prod H^{2\theta}(I_i)$ for every $0 \leq \theta \leq 1$ with the estimate

$$\|u\|_{H^{2\theta}(\mathcal{G})} \leq m_\theta \left\|A^\theta u\right\|_{L_2(\mathcal{G})}, \quad u \in \mathcal{D}(A^\theta). \quad (3.22)$$

Now, let $1/4 < \theta < 3/4$. It is known that $\mathcal{D}(A)$ is dense in $\mathcal{D}(A^\theta)$, namely, for every $u \in \mathcal{D}(A^\theta)$, there exists a sequence $u^{(n)} \in \mathcal{D}(A)$ such that $u^{(n)} \to u$ in $\mathcal{D}(A^\theta)$ as $n \to \infty$. Then, since $2\theta > 1/2$, we observe by (3.2) that, for each pair $I_i, I_k \in o(N_j) \cup \omega(N_j)$,

$$\begin{aligned} |u_i(N_j) - u_k(N_j)| &\leq |u_i(N_j) - u_i^{(n)}(N_j)| + |u_k^{(n)}(N_j) - u_k(N_j)| \\ &\leq \|u-u^{(n)}\|_{\mathcal{C}(\mathcal{G})} \leq c_{2\theta}\|u-u^{(n)}\|_{H^{2\theta}(\mathcal{G})} \\ &\leq c_{2\theta} m_\theta \|A^\theta(u-u^{(n)})\|_{L_2(\mathcal{G})} \to 0 \qquad \text{as} \qquad n \to \infty. \end{aligned}$$

Therefore, $u \in H^{2\theta}(\mathcal{G})$, so that $\mathcal{D}(A^\theta) \subset H^{2\theta}(\mathcal{G})$ for $1/4 < \theta < 3/4$.

Next, let $3/4 < \theta \leq 1$ and $u \in \mathcal{D}(A^\theta)$. By the density of $\mathcal{D}(A)$ in $\mathcal{D}(A^\theta)$, there exists a sequence $u^{(n)} \in \mathcal{D}(A)$ such that $u^{(n)} \to u$ in $\mathcal{D}(A^\theta)$ as $n \to \infty$. Then, since $2\theta > 3/2$, we observe by (3.3) that

$$\begin{aligned} \left|\frac{\partial u}{\partial n}(N_j)\right| &= \left|\frac{\partial u}{\partial n}(N_j) - \frac{\partial u^{(n)}}{\partial n}(N_j)\right| \\ &\leq \|u-u^{(n)}\|_{\mathcal{C}^1(\mathcal{G})} \leq c'_{2\theta}\|u-u^{(n)}\|_{H^{2\theta}(\mathcal{G})} \\ &\leq c'_{2\theta} m_\theta \|A^\theta(u-u^{(n)})\|_{L_2(\mathcal{G})} \to 0 \qquad \text{as} \qquad n \to \infty. \end{aligned}$$

Therefore, $u \in H^{2\theta}(\mathcal{G})$, so that $\mathcal{D}(A^\theta) \subset H^{2\theta}(\mathcal{G})$ for $3/4 < \theta \le 1$.

Step 2. Next, let us prove that $H^{2\theta}(\mathcal{G}) \subset \mathcal{D}(A^\theta)$ for $0 < \theta < 1/4$. From the definition (2.15), it follows that, for any $0 < \theta < 1$,

$$\begin{aligned} A^{\theta-1} &= \lim_{r\to\infty} \frac{1}{2\pi\theta i} \left[\lambda^\theta (\lambda - A)^{-1}\right]_{\lambda=re^{i\omega}}^{\lambda=re^{-i\omega}} + \frac{1}{2\pi\theta i} \int_{\Gamma_\omega} \lambda^\theta (\lambda - A)^{-2} d\lambda \\ &= \frac{1}{2\pi\theta i} \int_{\Gamma_\omega} \lambda^\theta (\lambda - A)^{-2} d\lambda \end{aligned}$$

due to integration by parts and (3.19).

Let $u \in H^{2\theta}(\mathcal{G})$ and $v \in \mathcal{D}(A)$. Then, we can write by (3.14)

$$\begin{aligned} \left(u, A^\theta v\right)_{L_2(\mathcal{G})} &= \left(u, A^{\theta-1} A v\right)_{L_2(\mathcal{G})} = \frac{1}{2\pi\theta i} \int_{\Gamma_\omega} \lambda^\theta \left(u, A(\lambda - A)^{-2} v\right)_{L_2(\mathcal{G})} d\lambda \\ &= \frac{1}{2\pi\theta i} \int_{\Gamma_\omega} \lambda^\theta \left(u, -D[\alpha D[(\lambda - A)^{-2} v]] + \beta(\lambda - A)^{-2} v\right)_{L_2(\mathcal{G})} d\lambda. \end{aligned} \tag{3.23}$$

Here, we use the following lemma.

Lemma 3.3. *It holds that*

$$|\left(u, D[\alpha D[A^{\theta-1} f]]\right)_{L_2(\mathcal{G})}| \le C_\theta \|u\|_{H^{2\theta}(\mathcal{G})} \|f\|_{L_2(\mathcal{G})} \quad \text{for all } f \in \mathcal{D}(A). \tag{3.24}$$

Proof of the lemma. It is observed from (3.1) that

$$\left(u, D[\alpha D[A^{\theta-1} f]]\right)_{L_2(\mathcal{G})} = \left\langle u, D[\alpha D[A^{\theta-1} f]]\right\rangle_{H^{2\theta}(\mathcal{G}) \times H^{-2\theta}(\mathcal{G})}.$$

So, it is enough to show that $f \mapsto D[\alpha D[A^{\theta-1} f]]$ is a bounded operator from $L_2(\mathcal{G})$ into $H^{-2\theta}(\mathcal{G})$. Indeed, $A^{\theta-1}$ is a bounded operator from $L_2(\mathcal{G})$ into $H^{2(1-\theta)}(\mathcal{G})$ due to (3.22). In addition, the differentiation D is a bounded operator from $H^{2(1-\theta)}(\mathcal{G})$ into $H^{1-2\theta}(\mathcal{G})$ due to (2.9), the multiplication of α is a bounded operator on $H^{1-2\theta}(\mathcal{G})$ due to (3.5), and D is a bounded operator from $H^{1-2\theta}(\mathcal{G})$ into $H^{-2\theta}(\mathcal{G})$ due to (2.9) again. □

Therefore, it follows from the Riesz representation theorem that there exists a unique $\tilde{u} \in L_2(\mathcal{G})$ such that

$$\left(u, D[\alpha D[A^{\theta-1} f]]\right)_{L_2(\mathcal{G})} = (\tilde{u}, f)_{L_2(\mathcal{G})} \quad \text{for all } f \in \mathcal{D}(A) \tag{3.25}$$

with the estimate

$$\|\tilde{u}\|_{L_2(\mathcal{G})} \le C_\theta \|u\|_{H^{2\theta}(\mathcal{G})}. \tag{3.26}$$

It follows from (3.24) and (3.25) that

$$\begin{aligned} \left(u, D[\alpha D[(\lambda - A)^{-2} v]]\right)_{L_2(\mathcal{G})} &= \left(u, D[\alpha D[A^{\theta-1} A^{1-\theta} (\lambda - A)^{-2} v]]\right)_{L_2(\mathcal{G})} \\ &= \left(A^{1-\theta} (\overline{\lambda} - A)^{-2} \tilde{u}, v\right)_{L_2(\mathcal{G})}, \end{aligned}$$

so,

$$\begin{aligned} &|\left(u, A^\theta v\right)_{L_2(\mathcal{G})}| \\ &\le C \int_{\Gamma_\omega} |\lambda|^\theta \left\{ |\left(A^{1-\theta} (\overline{\lambda} - A)^{-2} \tilde{u}, v\right)_{L_2(\mathcal{G})}| + |\left(u, \beta(\lambda - A)^{-2} v\right)_{L_2(\mathcal{G})}| \right\} |d\lambda|. \end{aligned}$$

Therefore, we obtain by (3.21) and (3.26) that

$$| \left(u, A^\theta v\right)_{L_2(\mathcal{G})} | \le C_{\theta,\omega}(\|\tilde{u}\|_{L_2(\mathcal{G})} + \|u\|_{L_2(\mathcal{G})}) \|v\|_{L_2(\mathcal{G})} \le C_{\theta,\omega} C_\theta \|u\|_{H^{2\theta}(\mathcal{G})} \|v\|_{L_2(\mathcal{G})} .$$

Consequently, since $v \in \mathcal{D}(A)$ is arbitrary and $\mathcal{D}(A)$ is dense in $L_2(\mathcal{G})$, we obtain that $u \in \mathcal{D}(A^\theta)$ with $\left\|A^\theta u\right\|_{L_2(\mathcal{G})} \le C_{\theta,\omega} C_\theta \|u\|_{H^{2\theta}(\mathcal{G})}$.

Step 3. Next, let us prove that $H^{2\theta}(\mathcal{G}) \subset \mathcal{D}(A^\theta)$ for $1/4 < \theta < 1/2$. Let $u \in H^{2\theta}(\mathcal{G})$ and $v \in \mathcal{D}(A)$. Then, as in Step 2, we can write (3.23), too. We use the following lemma.

Lemma 3.4. *It holds that*

$$| \left(u, D[\alpha D[A^{\theta-1} f]]\right)_{L_2(\mathcal{G})} | \le C_\theta \|u\|_{H^{2\theta}(\mathcal{G})} \|f\|_{L_2(\mathcal{G})} \quad \textit{for all } f \in \mathcal{D}(A). \tag{3.27}$$

Proof of the lemma. Since $2\theta \ne 1/2$, it follows from (2.9) that $Du \in H^{2\theta-1}(\mathcal{G})$. In the meantime, the map $f \mapsto \alpha D[A^{\theta-1} f]$ is a bounded operator from $L_2(\mathcal{G})$ into $H^{1-2\theta}(\mathcal{G})$. Indeed, $A^{\theta-1}$ is a bounded operator from $L_2(\mathcal{G})$ into $H^{2(1-\theta)}(\mathcal{G})$ due to (3.22). In addition, the differentiation D is a bounded operator from $H^{2(1-\theta)}(\mathcal{G})$ into $H^{1-2\theta}(\mathcal{G})$ due to (2.9), and the multiplication of α is a bounded operator on $H^{1-2\theta}(\mathcal{G})$ due to (3.5). Therefore, from (2.7), we know that

$$\left\langle Du, \alpha D[A^{\theta-1} f]\right\rangle_{H^{2\theta-1}(\mathcal{G}) \times H^{1-2\theta}(\mathcal{G})}$$

is well-defined. So,

$$\begin{aligned} &\left(u, D[\alpha D[A^{\theta-1} f]]\right)_{L_2(\mathcal{G})} \\ &= \sum_{I_i \in \mathcal{E}} \left[u_i \overline{\alpha_i D_i [A^{\theta-1} f]_i}\right]_{x_i=0}^{x_i=l_i} - \left\langle Du, \alpha D[A^{\theta-1} f]\right\rangle_{H^{2\theta-1}(\mathcal{G}) \times H^{1-2\theta}(\mathcal{G})} \\ &= - \left\langle Du, \alpha D[A^{\theta-1} f]\right\rangle_{H^{2\theta-1}(\mathcal{G}) \times H^{1-2\theta}(\mathcal{G})} \end{aligned}$$

due to $u \in H^{2\theta}(\mathcal{G})$ and $A^{\theta-1} f \in H^2(\mathcal{G})$, and as well the estimate is valid. □

So, by repeating the similar arguments to those in Step 2, we obtain that $u \in \mathcal{D}(A^\theta)$ with $\left\|A^\theta u\right\|_{L_2(\mathcal{G})} \le C_{\theta,\omega} C_\theta \|u\|_{H^{2\theta}(\mathcal{G})}$ for $1/4 < \theta < 1/2$.

Step 4. Let us prove that $H^{2\theta}(\mathcal{G}) \subset \mathcal{D}(A^\theta)$ for $1/2 < \theta < 3/4$. Let $u \in H^{2\theta}(\mathcal{G})$ and $v \in \mathcal{D}(A)$. In the present case, since

$$\left(u, -D[\alpha D[(\lambda - A)^{-2} v]]\right)_{L_2(\mathcal{G})} = \left(\alpha Du, D[(\lambda - A)^{-2} v]\right)_{L_2(\mathcal{G})} ,$$

we can write

$$\left(u, A^\theta v\right)_{L_2(\mathcal{G})} = \frac{1}{2\pi\theta i} \int_{\Gamma_\omega} \lambda^\theta \left\{ \left(\alpha Du, D[(\lambda - A)^{-2} v]\right)_{L_2(\mathcal{G})} + \left(u, \beta(\lambda - A)^{-2} v\right)_{L_2(\mathcal{G})} \right\} d\lambda. \tag{3.28}$$

In this case, we use the following lemma.

Lemma 3.5. *It holds that*

$$| \left(\alpha Du, D[A^{\theta-1} f]\right)_{L_2(\mathcal{G})} | \le C_\theta \|u\|_{H^{2\theta}(\mathcal{G})} \|f\|_{L_2(\mathcal{G})} \quad \textit{for all } f \in \mathcal{D}(A). \tag{3.29}$$

Proof of the lemma. Since $1 < 2\theta < 3/2$, it follows from (2.9) and (3.5) that the map $u \mapsto \alpha D u$ is a bounded operator from $H^{2\theta}(\mathcal{G})$ into $H^{2\theta-1}(\mathcal{G})$. In the meantime, the map $f \mapsto D[A^{\theta-1}f]$ is a bounded operator from $L_2(\mathcal{G})$ into $H^{1-2\theta}(\mathcal{G})$ due to (3.22) and (2.9). Therefore, it is observed from (3.1) that

$$\left(\alpha D u, D[A^{\theta-1}f]\right)_{L_2(\mathcal{G})} = \left\langle \alpha D u, D[A^{\theta-1}f]\right\rangle_{H^{2\theta-1}(\mathcal{G})\times H^{1-2\theta}(\mathcal{G})},$$

and as well the estimate is valid. □

So, by repeating the similar arguments to those in Step 2, we obtain that $u \in \mathcal{D}(A^\theta)$ with $\left\|A^\theta u\right\|_{L_2(\mathcal{G})} \le C_{\theta,\omega} C_\theta \left\|u\right\|_{H^{2\theta}(\mathcal{G})}$ for $1/2 < \theta < 3/4$.

Step 5. Finally, let us prove that $H^{2\theta}(\mathcal{G}) \subset \mathcal{D}(A^\theta)$ for $3/4 < \theta < 1$. Let $u \in H^{2\theta}(\mathcal{G})$ and $v \in \mathcal{D}(A)$. Then, by the same reason as in Step 4, we can write (3.28) again. In this case, we use the following lemma.

Lemma 3.6. *It holds that*

$$|\left(\alpha D u, D[A^{\theta-1}f]\right)_{L_2(\mathcal{G})}| \le C_\theta \|u\|_{H^{2\theta}(\mathcal{G})} \|f\|_{L_2(\mathcal{G})} \quad \textit{for all } f \in \mathcal{D}(A). \tag{3.30}$$

Proof of the lemma. Since $3/2 < 2\theta < 2$, we know that the map $u \mapsto D[\alpha D u]$ is a bounded operator from $H^{2\theta}(\mathcal{G})$ into $H^{2(\theta-1)}(\mathcal{G})$. Indeed, the differentiation D is a bounded operator from $H^{2\theta}(\mathcal{G})$ into $H^{2\theta-1}(\mathcal{G})$ due to (2.9), the multiplication of α is a bounded operator on $H^{2\theta-1}(\mathcal{G})$ due to (3.5), and D is a bounded operator from $H^{2\theta-1}(\mathcal{G})$ into $H^{2(\theta-1)}(\mathcal{G})$ due to (2.9) again. In the meantime, it is obvious that $A^{\theta-1}f \in H^{2(1-\theta)}(\mathcal{G})$. Therefore, from (2.7), we know that

$$\left\langle D[\alpha D u], A^{\theta-1}f\right\rangle_{H^{2(\theta-1)}(\mathcal{G})\times H^{2(1-\theta)}(\mathcal{G})}$$

is well-defined. So, we have

$$\begin{aligned}
&\left(\alpha D u, D[A^{\theta-1}f]\right)_{L_2(\mathcal{G})} \\
&= \sum_{I_i \in \mathcal{E}} \left[\alpha_i D_i u_i \overline{[A^{\theta-1}f]_i}\right]_{x_i=0}^{x_i=l_i} - \left\langle D[\alpha D u], A^{\theta-1}f\right\rangle_{H^{2(\theta-1)}(\mathcal{G})\times H^{2(1-\theta)}(\mathcal{G})} \\
&= -\left\langle D[\alpha D u], A^{\theta-1}f\right\rangle_{H^{2(\theta-1)}(\mathcal{G})\times H^{2(1-\theta)}(\mathcal{G})}.
\end{aligned}$$

Thus, the desired estimate is obtained. □

Repeating the similar arguments to those in Step 2, we obtain that $u \in \mathcal{D}(A^\theta)$ with $\left\|A^\theta u\right\|_{L_2(\mathcal{G})} \le C_{\theta,\omega} C_\theta \left\|u\right\|_{H^{2\theta}(\mathcal{G})}$ for $3/4 < \theta < 1$.

We have thus accomplished the proof of theorem. □

From this result, we show the following estimate

$$\|u\|_{H^1(\mathcal{G})} \le C \|u\|_{H^2(\mathcal{G})}^{2/3} \|u\|_{H^1(\mathcal{G})'}^{1/3} \quad \text{for} \quad u \in H^2(\mathcal{G}). \tag{3.31}$$

Indeed, this is observed from the property of the triplet: $\|u\|_{L_2(\mathcal{G})}^2 = |(u,u)_{L_2(\mathcal{G})}| = |\langle u, u\rangle_{H^1(\mathcal{G})'\times H^1(\mathcal{G})}| \le \|u\|_{H^1(\mathcal{G})'} \|u\|_{H^1(\mathcal{G})}$ for $u \in H^1(\mathcal{G})$, and the moment inequality for $A^{1/2}$ (see (2.16)): $\|A^{1/2}u\|_{L_2(\mathcal{G})} \le C\|Au\|_{L_2(\mathcal{G})}^{1/2} \|u\|_{L_2(\mathcal{G})}^{1/2}$ for $u \in H^2(\mathcal{G}) = \mathcal{D}(A)$.

3.4 Sectorial Operator in $H^1(\mathcal{G})$

In this section, let us assume that $\alpha > 0$ and $\beta > 0$ are positive constants on $\mathcal{G}$, that is, let us consider the following sesquilinear form:

$$a(u,v) = \alpha \int_{\mathcal{G}} Du\overline{Dv}dx + \beta \int_{\mathcal{G}} u\overline{v}dx, \qquad u,v \in H^1(\mathcal{G}),$$

defined on $H^1(\mathcal{G})$. Of course, as similar to the previous sections, we can construct a linear isomorphism $\mathcal{A} : H^1(\mathcal{G}) \to H^1(\mathcal{G})'$ such that

$$a(u,v) = \langle \mathcal{A}u, v\rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})}, \qquad u,v \in H^1(\mathcal{G}). \tag{3.32}$$

In what follows, let $\mathbb{A} = \mathcal{A}_{|H^1(\mathcal{G})}$ be the part of $\mathcal{A}$ in $H^1(\mathcal{G})$. Since

$$\langle u,v\rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})} = \langle u,v\rangle_{H^1(\mathcal{G}) \times H^1(\mathcal{G})'} \quad u,v \in H^1(\mathcal{G}),$$

we verify that

$$u \in \mathcal{D}(\mathbb{A}) \text{ if and only if } a(u,v) \text{ is continuous in } v \text{ with respect to the } H^1(\mathcal{G})' \text{ norm.} \tag{3.33}$$

Moreover, if $u \in \mathcal{D}(\mathbb{A})$ then $a(u,v) = \langle \mathbb{A}u, v\rangle_{H^1(\mathcal{G}) \times H^1(\mathcal{G})'}$ for all $v \in H^1(\mathcal{G})$.

In the following, we investigate properties of $\mathbb{A}$.

3.4.1 Fundamental properties for $\mathbb{A}$

We characterize $\mathbb{A}$ as a differential operator in $H^1(\mathcal{G})$.

Theorem 3.3. $\mathcal{D}(\mathbb{A}) = H^3(\mathcal{G})$ *with the norm equivalence*

$$\tilde{C}_{\alpha,\beta}^{-1} \|u\|_{H^3(\mathcal{G})} \le \|\mathbb{A}u\|_{H^1(\mathcal{G})} \le \tilde{C}_{\alpha,\beta} \|u\|_{H^3(\mathcal{G})}, \quad u \in H^3(\mathcal{G}). \tag{3.34}$$

Furthermore, for $u \in \mathcal{D}(\mathbb{A})$,

$$\mathbb{A}u = -\alpha D^2 u + \beta u \quad \textit{in} \quad H^1(\mathcal{G}).$$

Proof. Let $u \in H^3(\mathcal{G})$. Then, $a(u,v) = \langle -\alpha D^2 u + \beta u, v\rangle_{H^1(\mathcal{G}) \times H^1(\mathcal{G})'}$ for $v \in H^1(\mathcal{G})$. Therefore, it follows from (3.33) that $u \in \mathcal{D}(\mathbb{A})$.

Let us show the opposite inclusion. For $u \in \mathcal{D}(\mathbb{A})$, there exists $\tilde{u} \in H^1(\mathcal{G})$ such that $\mathbb{A}u = \tilde{u}$. As shown in Theorem 3.1, $u \in H^2(\mathcal{G})$ and

$$(\tilde{u} - \beta u, v)_{L_2(\mathcal{G})} = -\alpha \left(D^2 u, v\right)_{L_2(\mathcal{G})} \quad \text{for all } v \in H^1(\mathcal{G}).$$

Particularly, for $v \in \prod \mathcal{C}_0^\infty(I_i) (\subset H^1(\mathcal{G}))$, we see by $Dv \in H^1(\mathcal{G})$ that

$$(D\tilde{u} - \beta Du, v)_{L_2(\mathcal{G})} = \alpha \left(D^2 u, Dv\right)_{L_2(\mathcal{G})}.$$

Therefore, $u \in \prod H^3(I_i)$. Then, $-\alpha D^2 u = \tilde{u} - \beta u \in H^1(\mathcal{G})$, so that $u \in H^3(\mathcal{G})$.

Finally, for $u \in H^3(\mathcal{G})$,

$$\|\mathbb{A}u\|_{H^1(\mathcal{G})}^2 = \alpha^2 \|D^3 u\|_{L_2(\mathcal{G})}^2 + (\alpha^2 + 2\alpha\beta)\|D^2 u\|_{L_2(\mathcal{G})}^2 + (2\alpha\beta + \beta^2)\|Du\|_{L_2(\mathcal{G})}^2 + \beta^2 \|u\|_{L_2(\mathcal{G})}^2,$$

so, (3.34) is obtained. □

It is easy to check that $\mathbb{A}$ is a positive definite self-adjoint operator in $H^1(\mathcal{G})$. Furthermore, we know that $\mathbb{A}$ is a linear isomorphism from $H^3(\mathcal{G})$ onto $H^1(\mathcal{G})$.

3.4.2 Fractional powers $\mathbb{A}^\theta$

Next, we want to characterize the domains of fractional powers $\mathbb{A}^\theta$ for $3/4 < \theta < 1$. To this end, it is convenient to use the fact that $H^1(\mathcal{G}) = \mathcal{D}(A^{\frac{1}{2}})$ with norm equivalence due to Theorem 3.2. Due to [37, Theorem 2.1], $\mathbb{A}$ is a sectorial operator of $\mathcal{D}(A^{\frac{1}{2}})$ with angle $\omega_{\mathbb{A}} < \pi/2$, that is, the spectrum $\sigma(\mathbb{A})$ is contained in an open sectorial domain such that

$$\sigma(\mathbb{A}) \subset \Sigma_{\omega'},$$

where $\omega_{\mathbb{A}} < \omega' \leq \pi/2$, and its resolvent $(\lambda - \mathbb{A})^{-1}$ satisfies the estimate

$$\left\|(\lambda - \mathbb{A})^{-1}\right\|_{\mathcal{L}(\mathcal{D}(A^{\frac{1}{2}}))} \leq \frac{M'}{|\lambda|}, \quad \lambda \notin \Sigma_{\omega'}, \tag{3.35}$$

with some constant $M' \geq 1$. Then, fractional powers of $\mathbb{A}$ are defined as shown in Subsection 2.4.1. Note that, for each $0 \leq \theta \leq 1$, $\mathcal{D}(\mathbb{A}^\theta)$ is a Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{D}(\mathbb{A}^\theta)} = (A^{1/2}\mathbb{A}^\theta \cdot, A^{1/2}\mathbb{A}^\theta \cdot)_{L_2(\mathcal{G})}$.

By the same reasons as before, we know that

$$[H^1(\mathcal{G}), H^3(\mathcal{G})]_\theta = \mathcal{D}(\mathbb{A}^\theta) \text{ with norm equivalence for any } 0 < \theta < 1. \tag{3.36}$$

From this result, we can show the following theorem.

Theorem 3.4. *For $3/4 < \theta < 1$, it holds that $\mathcal{D}(\mathbb{A}^\theta) = H^{2\theta+1}(\mathcal{G})$ with norm equivalence.*

Proof. The proof will be divided into two Steps.

Step 1. Let us show that $\mathcal{D}(\mathbb{A}^\theta) \subset \prod H^{2\theta+1}(I_i)$ for $0 \leq \theta \leq 1$. By the same techniques in Step 1 of the proof of Theorem 3.2, we obtain that $\mathcal{D}(\mathbb{A}^\theta) = [H^1(\mathcal{G}), H^3(\mathcal{G})]_\theta \subset [\prod H^1(I_i), \prod H^3(I_i)]_\theta = \prod H^{2\theta+1}(I_i)$ for any $0 \leq \theta \leq 1$ with

$$\|u\|_{H^{2\theta+1}(\mathcal{G})} \leq C\|\mathbb{A}^\theta u\|_{\mathcal{D}(A^{\frac{1}{2}})}, \quad u \in \mathcal{D}(\mathbb{A}^\theta). \tag{3.37}$$

Now, let $3/4 < \theta \leq 1$ and $u \in \mathcal{D}(\mathbb{A}^\theta)$. Since $\mathcal{D}(\mathbb{A}^\theta) = \mathcal{D}(A^{\frac{1}{2}+\theta}) \subset \mathcal{D}(A)$, we already know that $u \in H^{2\theta+1}(\mathcal{G})$. In addition, $D^2u \in H^{2\theta-1}(\mathcal{G})$ is also proved due to the denseness of $\mathcal{D}(\mathbb{A}) \subset \mathcal{D}(\mathbb{A}^\theta)$.

Step 2. Conversely, let us prove that $H^{2\theta+1}(\mathcal{G}) \subset \mathcal{D}(\mathbb{A}^\theta)$ for $3/4 < \theta < 1$. Let $u \in H^{2\theta+1}(\mathcal{G})$ and $v \in \mathcal{D}(\mathbb{A})$. Then, we can write

$$\begin{aligned}
\left(u, \mathbb{A}^\theta v\right)_{\mathcal{D}(A^{\frac{1}{2}})} &= \left(u, \mathbb{A}^{\theta-1}\mathbb{A}v\right)_{\mathcal{D}(A^{\frac{1}{2}})} = \frac{1}{2\pi\theta i}\int_{\Gamma_{\omega'}} \lambda^\theta \left(u, \mathbb{A}(\lambda-\mathbb{A})^{-2}v\right)_{\mathcal{D}(A^{\frac{1}{2}})} d\lambda \\
&= \frac{1}{2\pi\theta i}\int_{\Gamma_{\omega'}} \lambda^\theta \left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}A(\lambda-A)^{-2}v\right)_{L_2(\mathcal{G})} d\lambda \\
&= \frac{1}{2\pi\theta i}\int_{\Gamma_{\omega'}} \lambda^\theta \left\{\alpha\left(Du, D[A(\lambda-A)^{-2}v]\right)_{L_2(\mathcal{G})} + \beta\left(u, A(\lambda-A)^{-2}v\right)_{L_2(\mathcal{G})}\right\} d\lambda.
\end{aligned}$$

Since $u \in H^{2\theta+1}(\mathcal{G})$ and $A(\lambda - A)^{-2} \in H^1(\mathcal{G})$,

$$\left(Du, D[A(\lambda-A)^{-2}v]\right)_{L_2(\mathcal{G})} = -\left(D^2u, A(\lambda-A)^{-2}v\right)_{L_2(\mathcal{G})}.$$

Note that $D^2u \in H^{2\theta-1}(\mathcal{G})$ and $A^{\theta-\frac{1}{2}}$ is a linear bounded operator from $H^{2\theta-1}(\mathcal{G})$ to $L_2(\mathcal{G})$ due to Theorem 3.2. So,

$$|(A^{\theta-\frac{1}{2}}D^2u, A^{\frac{1}{2}}w)_{L_2(\mathcal{G})}| \le C\|u\|_{H^{2\theta+1}(\mathcal{G})}\|w\|_{\mathcal{D}(A^{\frac{1}{2}})} \text{ for all } w \in \mathcal{D}(A^{\frac{1}{2}}).$$

It follows from the Riesz representation theorem that there exists a unique $\tilde{u} \in \mathcal{D}(A^{\frac{1}{2}})$ such that

$$(A^{\theta-\frac{1}{2}}D^2u, A^{\frac{1}{2}}w)_{L_2(\mathcal{G})} = (\tilde{u}, w)_{\mathcal{D}(A^{\frac{1}{2}})} \text{ for all } w \in \mathcal{D}(A^{\frac{1}{2}}) \tag{3.38}$$

with the estimate

$$\|\tilde{u}\|_{\mathcal{D}(A^{\frac{1}{2}})} \le C\,\|u\|_{H^{2\theta+1}(\mathcal{G})}\,. \tag{3.39}$$

It follows from (3.38) that

$$\begin{aligned}
&(Du, D[A(\lambda-A)^{-2}v])_{L_2(\mathcal{G})} = -(D^2u, A(\lambda-A)^{-2}v)_{L_2(\mathcal{G})}\\
&\quad = -(A^{\theta-\frac{1}{2}}D^2u, A^{\frac{1}{2}}A^{1-\theta}(\lambda-A)^{-2}v)_{L_2(\mathcal{G})} = -(\tilde{u}, \mathbb{A}^{1-\theta}(\lambda-\mathbb{A})^{-2}v)_{\mathcal{D}(A^{\frac{1}{2}})}\\
&\quad\quad = -(\mathbb{A}^{1-\theta}(\overline{\lambda}-\mathbb{A})^{-2}\tilde{u}, v)_{\mathcal{D}(A^{\frac{1}{2}})},
\end{aligned}$$

so,

$$\begin{aligned}
&|(u, \mathbb{A}^{\theta}v)_{\mathcal{D}(A^{\frac{1}{2}})}|\\
&\le C\int_{\Gamma_{\omega'}} |\lambda|^{\theta}\left\{|\left(\mathbb{A}^{1-\theta}(\overline{\lambda}-\mathbb{A})^{-2}\tilde{u}, v\right)_{\mathcal{D}(A^{\frac{1}{2}})}| + |\left(u, A(\lambda-A)^{-2}v\right)_{L_2(\mathcal{G})}|\right\}|d\lambda|.
\end{aligned}$$

According to [37, Theorem 16.3], (3.36) implies an integrable condition along the $\Gamma_{\omega'}$, i.e.,

$$\int_{\Gamma_{\omega'}} |\lambda|^{\theta}|\left(\mathbb{A}^{1-\theta}(\overline{\lambda}-\mathbb{A})^{-2}w, z\right)_{\mathcal{D}(A^{\frac{1}{2}})}||d\lambda| \le C\,\|w\|_{\mathcal{D}(A^{\frac{1}{2}})}\,\|z\|_{\mathcal{D}(A^{\frac{1}{2}})}, \quad w, z \in \mathcal{D}(A^{\frac{1}{2}}).$$

Therefore, we obtain by (3.39) that

$$|(u, \mathbb{A}^{\theta}v)_{\mathcal{D}(A^{\frac{1}{2}})}| \le C(\|\tilde{u}\|_{\mathcal{D}(A^{\frac{1}{2}})}\|v\|_{\mathcal{D}(A^{\frac{1}{2}})} + \|u\|_{H^{2\theta}(\mathcal{G})}\|v\|_{L_2(\mathcal{G})}) \le C\|u\|_{H^{2\theta+1}(\mathcal{G})}\|v\|_{\mathcal{D}(A^{\frac{1}{2}})}.$$

Consequently, $u \in \mathcal{D}(\mathbb{A}^{\theta})$ with $\|\mathbb{A}^{\theta}u\|_{\mathcal{D}(A^{\frac{1}{2}})} \le C\|u\|_{H^{2\theta+1}(\mathcal{G})}$.

We have thus accomplished the proof of theorem. □

We know that

$$\|u\|_{H^2(\mathcal{G})} \le C\|u\|_{H^3(\mathcal{G})}^{\frac{2}{3}}\|u\|_{L_2}^{\frac{1}{3}} \quad \text{for} \quad u \in H^3(\mathcal{G}). \tag{3.40}$$

Indeed, this is observed from the moment inequality for $\mathbb{A}^{\frac{1}{2}}$ (see (2.16)): $\|\mathbb{A}^{\frac{1}{2}}u\|_{H^1(\mathcal{G})} \le C\|\mathbb{A}u\|_{H^1(\mathcal{G})}^{\frac{1}{2}}\|u\|_{H^1(\mathcal{G})}^{\frac{1}{2}}$ for $u \in H^3(\mathcal{G}) = \mathcal{D}(\mathbb{A})$, and the norm equivalence $\|\mathbb{A}^{\frac{1}{2}}u\|_{H^1(\mathcal{G})} = \|Au\|_{L_2(\mathcal{G})}$.

Chapter 4

Evolution Equations of Parabolic Type

In this chapter, we present fundamentals of the theory of evolution equations of parabolic type. The first four sections are devoted to showing results about constructing local solutions for semilinear equations, quasilinear equations, and degenerate equations (the materials in first three sections are mainly shown in [37, Chapters 3, 4, and 5]). The results in Section 4.1 are utilized in Chapters 5 and 6; the results in Sections 4.2 and 4.3 are utilized in Chapter 7; and the results in 4.4 are utilized in Chapter 8. A version of exponential attractor for non-autonomous dynamical systems presented in section 4.5 is constructed for the dynamical system determined in Section 5.2.

4.1 Semilinear Evolution Equations

Let X be a Banach space with norm $\|\cdot\|_X$. We consider the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t \leq T, \\ U(0) = U_0, \end{cases} \tag{4.1}$$

in X. Here, A is a sectorial operator of X satisfying

$$\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}, \quad \omega_A < \omega < \frac{\pi}{2}, \tag{4.2}$$

and

$$\left\|(\lambda - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M_\omega}{|\lambda|}, \quad \lambda \notin \Sigma_\omega,\ \omega_A < \omega < \frac{\pi}{2}. \tag{4.3}$$

Meanwhile, F is a nonlinear operator from $\mathcal{D}(A^\eta)$ into X, where

$$0 \leq \eta < 1. \tag{4.4}$$

The operator F is assumed to satisfy a Lipschitz condition of the form

$$\begin{aligned} \|F(U) - F(V)\|_X \leq \phi(\|U\|_X + \|V\|_X) \\ \times \{\|A^\eta(U-V)\|_X + (\|A^\eta U\|_X + \|A^\eta V\|_X)\|U - V\|_X\}, \\ U, V \in \mathcal{D}(A^\eta), \end{aligned} \tag{4.5}$$

where $\phi(\cdot)$ is some increasing continuous function. The initial value U_0 is taken in X.

Then, there exists the general theorem about existence and uniqueness of local solutions to (4.1). The proof is given in [37, Theorem 4.4].

Theorem 4.1. *Let* (4.2), (4.3), (4.4), *and* (4.5) *be satisfied. Then, for any* $U_0 \in X$, (4.1) *possesses a unique local solution* U *in the function space:*

$$U \in \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}^1((0, T_{U_0}]; X),$$

where T_{U_0} *depends only on the norm* $\|U_0\|_X$. *In addition,* U *satisfies the estimates*

$$\|U(t)\|_X + t \left\| \frac{dU}{dt}(t) \right\|_X + t \|AU(t)\|_X \le C_{U_0}, \quad 0 < t \le T_{U_0},$$

with some constant $C_{U_0} > 0$ *depending on the norm* $\|U_0\|_X$.

4.2 Quasilinear Evolution Equations

Let X be a Banach space with norm $\|\cdot\|_X$. We consider the Cauchy problem for a quasilinear abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), & 0 < t \le T, \\ U(0) = U_0, \end{cases} \tag{4.6}$$

in X. Let Z be a second Banach space continuously embedded in X, and let K be the open ball of Z,

$$K = \{U \in Z; \|U\|_Z < R\}, \quad 0 < R < \infty.$$

For $U \in K$, $A(U)$ is a sectorial operator of X with angle $\omega_{A(U)} < \frac{\pi}{2}$ and with domain $\mathcal{D}(A(U))$.

We make the following assumptions. The spectrum $\sigma(A(U))$ is contained in a fixed open sectorial domain, i.e.,

$$\sigma(A(U)) \subset \Sigma_\omega = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}, \quad U \in K \tag{4.7}$$

with some angle $0 < \omega < \frac{\pi}{2}$, and the resolvent satisfies

$$\left\|(\lambda - A(U))^{-1}\right\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda|}, \quad \lambda \notin \Sigma_\omega,\ U \in K \tag{4.8}$$

with some constant $M \ge 1$. The domain $\mathcal{D}(A(U))$ satisfies

$$\mathcal{D}(A(U)) = \mathcal{D}(A(V)) \quad \text{for any pair of } U, V \in K. \tag{4.9}$$

In addition, $A(U)$ satisfies a Lipschitz condition of the form

$$\left\|[A(U) - A(V)]A(V)^{-1}\right\|_{\mathcal{L}(X)} \le N \|U - V\|_Y, \quad U, V \in K \tag{4.10}$$

with a constant $N > 0$. Here, Y is a third Banach space such that $Z \subset Y \subset X$ with continuous embeddings. The operator F is a nonlinear operator from another Banach space W into X, W being continuously embedded in Z. We assume that the Lipschitz condition

$$\begin{aligned}
&\|F(U) - F(V)\|_X \\
&\quad \le \phi(\|U\|_Z + \|V\|_Z) \times \{\|U - V\|_W + (\|U\|_W + \|V\|_W)\|U - V\|_Z\}, \\
&\qquad\qquad\qquad\qquad\qquad U, V \in W,
\end{aligned} \tag{4.11}$$

where $\phi(\cdot)$ is some continuous increasing function.

There are three exponents $0 \le \alpha < \beta < \eta < 1$ such that, for any $U \in K$, $\mathcal{D}(A(U)^\alpha) \subset Y$, $\mathcal{D}(A(U)^\beta) \subset Z$, and $\mathcal{D}(A(U)^\eta) \subset W$ with the estimates

$$\begin{cases}
\|\tilde{U}\|_Y \le D_1\|A(U)^\alpha \tilde{U}\|_X, & \tilde{U} \in \mathcal{D}(A(U)^\alpha), \quad U \in K, \\
\|\tilde{U}\|_Z \le D_2\|A(U)^\beta \tilde{U}\|_X, & \tilde{U} \in \mathcal{D}(A(U)^\beta), \quad U \in K, \\
\|\tilde{U}\|_W \le D_3\|A(U)^\eta \tilde{U}\|_X, & \tilde{U} \in \mathcal{D}(A(U)^\eta), \quad U \in K,
\end{cases} \tag{4.12}$$

$D_i > 0$ $(i = 1, 2, 3)$ being some constants. The initial function $U_0 \in K$ satisfies

$$U_0 \in \mathcal{D}(A(U_0)). \tag{4.13}$$

The exponents satisfy the relations

$$0 \le \alpha < \beta < \eta < 1. \tag{4.14}$$

Then, there exists the general theorem about existence and uniqueness of local solutions to (4.6). The proof is given in [37, Theorem 5.6].

Theorem 4.2. *Let* (4.7) – (4.14) *be satisfied. Then, there exists a unique local solution* U *to* (4.6) *on an interval* $[0, T_{U_0}]$ *in the function space:*

$$\begin{cases}
U \in \mathcal{C}([0, T_{U_0}]; \mathcal{D}(A(U))) \cap \mathcal{C}^{1-\alpha}([0, T_{U_0}]; Y) \cap \mathcal{C}^1((0, T_{U_0}]; X), \\
F(U) \in \mathcal{F}^{1,\sigma}((0, T_{U_0}]; X)
\end{cases} \tag{4.15}$$

with arbitrary $0 < \sigma < \min\{\beta - \alpha, 1 - \eta\}$, *where* T_{U_0} *is determined by the norm* $\|A(U_0)U_0\|_X$. *Furthermore,* U *satisfies the estimates*

$$\|F(U)\|_{\mathcal{F}^{1,\sigma}} + \max_{0 \le t \le T_{U_0}} \|A(U(t))U(t)\|_X \le C_{U_0} \tag{4.16}$$

with a constant C_{U_0} *determined by the norm* $\|A(U_0)U_0\|_X$.

4.3 Non-autonomous Evolution Equations

Let X be a Banach space with norm $\|\cdot\|_X$. We consider the Cauchy problem for a linear non-autonomous abstract evolution equation

$$\begin{cases}
\dfrac{dU}{dt} + A(t)U = F(t), \quad 0 < t \le T, \\
U(0) = U_0,
\end{cases} \tag{4.17}$$

in X, where $0 < T < \infty$ is a fixed time.

We make the following assumptions. For $0 \le t \le T$, the spectrum $\sigma(A(t))$ is contained in a fixed open sectorial domain, i.e.,

$$\sigma(A(t)) \subset \Sigma_\omega = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}, \quad 0 \le t \le T, \tag{4.18}$$

with some fixed angle $0 < \omega < \frac{\pi}{2}$, and the resolvent satisfies

$$\left\|(\lambda - A(t))^{-1}\right\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda|}, \quad \lambda \notin \Sigma_\omega,\ 0 \le t \le T, \tag{4.19}$$

with some fixed constant $M \ge 1$. The domain $\mathcal{D}(A(t))$ satisfies

$$\mathcal{D}(A(s)) \equiv \mathcal{D}(A(t)) \quad \text{for any pair of } 0 \le s, t \le T. \tag{4.20}$$

In addition, $A(t)$ satisfies a Hölder continuous condition of the form

$$\left\|[A(t) - A(s)]A(s)^{-1}\right\|_{\mathcal{L}(X)} \le N|t - s|^\mu, \quad 0 \le s, t \le T, \tag{4.21}$$

with some fixed exponent $0 < \mu \le 1$ and some constant $N > 0$.

We present two theorems which will be used in Chapter 7.

Theorem 4.3. *Let* (4.19)–(4.21) *be satisfied. Then, for any*

$$F \in \mathcal{F}^{1,\sigma}((0,T];X), \quad 0 < \sigma \le 1,$$

and any $U_0 \in X$, there exists a unique solution U to (4.17) *in the function space:*

$$U \in \mathcal{C}([0,T];X) \cap \mathcal{C}^1((0,T];X), \quad AU \in \mathcal{C}((0,T];X),$$

with the estimate

$$\|U(t)\|_X + t\left\|\frac{dU}{dt}(t)\right\|_X + t\|A(t)U(t)\|_X \le C(\|U_0\|_X + \|F\|_{\mathcal{F}^{1,\sigma}}), \quad 0 < t \le T.$$

Theorem 4.4. *Let* (4.19)–(4.21) *be satisfied. Let $U_0 \in \mathcal{D}(A(0))$, and let*

$$F \in \mathcal{F}^{1,\sigma}((0,T];X), \quad 0 < \sigma < \mu, \tag{4.22}$$

satisfy the spatial regularity condition

$$A(t)^\rho F \in \mathcal{B}([0,T];X), \quad 0 < \rho < \sigma. \tag{4.23}$$

Then, there exists a unique solution U to (4.17) *in the function space:*

$$AU \in \mathcal{C}([0,T];X), \quad \frac{dU}{dt}, AU \in \mathcal{F}^{1,\sigma}((0,T];X).$$

In addition, $A^{1+\rho}U$ belongs to $\mathcal{B}([0,T];X)$ with the estimate

$$\|A^{1+\rho}U\|_{\mathcal{B}} \le C[\|A(0)U_0\|_X + \|F\|_{\mathcal{F}^{1,\sigma}} + \|A^\rho F\|_{\mathcal{B}}]. \tag{4.24}$$

For the proof of Theorem 4.3, see that of [37, Theorem 3.9]. For the proof of Theorem 4.4, see that of [37, Theorem 3.10 and 3.11].

4.4 Degenerate Evolution Equations

This section is devoted to constructing the strict solution to the Cauchy problem for a degenerate abstract parabolic equation in Banach spaces. We essentially use the notion of multivalued linear operators and the theory of multivalued evolution equations of parabolic type that were monographed by Favini-Yagi [59].

4.4.1 Multivalued linear operators

Let X be a complex Banach space with norm $\|\cdot\|$ and let 2^X denote the family of all subsets of X. An operator $A\colon \mathcal{D}(A) \to 2^X - \{\emptyset\}$, where $\mathcal{D}(A)$ is a linear subspace of X, is called a multivalued linear operator of X if A satisfies

$$\begin{cases} Au + Av \subset A(u+v), & u, v \in \mathcal{D}(A), \\ \lambda Au \subset A(\lambda u), & \lambda \in \mathbb{C},\, u \in \mathcal{D}(A). \end{cases} \tag{4.25}$$

When A is a multivalued linear operator, $A0$ is always a linear subspace of X, and for $u \in \mathcal{D}(A)$ it holds that $Au = f + A0$ with arbitrary $f \in Au$. Analogously to the single valued linear operators, i.e., those of $A0 = \{0\}$, a number $\lambda \in \mathbb{C}$ is said to belong to the resolvent set $\rho(A)$ of A if the inverse of $\lambda - A$ is single valued and is a bounded linear operator of X. On the contrary, when $\lambda \notin \rho(A)$, the λ is said to belong to the spectrum $\sigma(A)$ of A. The $\mathcal{L}(X)$ valued analytic function $(\lambda - A)^{-1}$ defined in $\rho(A)$ is called the resolvent of A. Moreover, A is said to be a sectorial operator of X if there exists an open sectorial domain such that

$$\sigma(A) \subset \Sigma = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}, \tag{4.26}$$

where $0 < \omega < \pi$, and that $(\lambda - A)^{-1}$ satisfies

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \le \frac{D}{|\lambda|}, \qquad \lambda \notin \Sigma, \tag{4.27}$$

with some constant $D > 0$.

If $0 \in \rho(A)$, that is $A^{-1} \in \mathcal{L}(X)$, the graph

$$\mathcal{G}(A) = \{(f, u) \in X \times X;\ f \in Au\}$$

is a closed subspace of $X \times X$. Furthermore, $\{0\} \times A0$ is a closed subspace of $\mathcal{G}(A)$. Then, under $0 \in \rho(A)$, $\mathcal{D}(A)$ becomes a Banach space with the graph norm

$$\|u\|_{\mathcal{D}(A)} = \|Au\|_X \equiv \inf_{f \in Au} \|f\|, \qquad u \in \mathcal{D}(A) \tag{4.28}$$

(see [59, Proposition 1.1]).

If A is sectorial, $(\lambda - A)^{-1}$ satisfies the optimal decay estimate on the half line $(-\infty, 0]$; moreover, it is seen that

$$\lim_{\lambda \to -\infty} \lambda(\lambda - A)^{-1} f = f, \qquad f \in \overline{\mathcal{D}(A)}. \tag{4.29}$$

From this it is proved that $A0 \cap \overline{\mathcal{D}(A)} = \{0\}$. (In fact, if $f \in A0$, then $f \in (\lambda - A)0$ for every $\lambda \le 0$ and hence $(\lambda - A)^{-1} f = 0$; therefore, if $f \in \overline{D(A)}$ in addition, then

(4.29) implies $f = 0$.) This property furthermore provides that for any $u \in \mathcal{D}(A)$ and any $f \in X$, it holds that

$$[f - Au] \cap \overline{\mathcal{D}(A)} \quad \text{is a singleton if it is not empty.} \tag{4.30}$$

When X is a reflexive Banach space, it is known that

$$X = A0 + \overline{\mathcal{D}(A)} \tag{4.31}$$

(see Remark to [59, Proposition 2.1]). Then, if $g \in Au$, then $f - g = f' + f''$ with $f' \in A0$ and $f'' \in \overline{D(A)}$, i.e., $f - g - f' = f''$; on one hand, we have $f - g - f' \in f - Au$; on the other hand, $f'' \in \overline{D(A)}$. Thus, the condition that

$$[f - Au] \cap \overline{\mathcal{D}(A)} \quad \text{is a singleton} \tag{4.32}$$

holds automatically for any $u \in \mathcal{D}(A)$ and any $f \in X$.

When A is sectorial, its fractional powers for negative exponents are defined by the integrals

$$A^{-x} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-x} (\lambda - A)^{-1} d\lambda, \qquad x > 0, \tag{4.33}$$

in $\mathcal{L}(X)$, where $\Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$ is an integral contour lying in $\rho(A)$ such that $\Gamma_+ \colon \lambda = re^{i\omega}$ for $\infty > r \geq \delta, \Gamma_0 \colon \lambda = \delta e^{i\vartheta}$ for $\omega \geq \vartheta \geq -\omega, \Gamma_- \colon \lambda = re^{-i\omega}$ for $\delta \leq r < \infty$, $\delta > 0$ being a sufficiently small radius. It is easy to verify that A^{-x} satisfy the exponential law $A^{-(x+x')} = A^{-x}A^{-x'}$. The fractional powers for positive exponents are defined by $A^x = [A^{-x}]^{-1}$, $x > 0$; but of course A^x are multivalued linear operators of X. They also satisfy the law $A^{x+x'} = A^x A^{x'}$ in the sense of multivalued operators (see [59, Theorem 1.10]). As noticed by (4.28), each $\mathcal{D}(A^x)$ is a Banach space with the norm $\|A^x \cdot \|$. For $0 < x < y$, it is clear that $\mathcal{D}(A^y) \subset \mathcal{D}(A^x)$ with continuous embedding. Moreover, the moment inequality

$$\|A^x u\| \leq C_{x,y} \|A^y u\|^{x/y} \|u\|^{1-x/y}, \qquad u \in \mathcal{D}(A^y), \tag{4.34}$$

holds true. Indeed, if $f \in A^y u$, then $u = A^{-y} f = A^{-x} A^{x-y} f$; thereby, $A^{x-y} f \in A^x u$. Therefore, $\|A^x u\| \leq \|A^{x-y} f\| \leq C_{x,y} \|A^{-y} f\|^{1-x/y} \|f\|^{x/y} = C_{x,y} \|u\|^{1-x/y} \|f\|^{x/y}$. But, since $f \in A^y u$ is arbitrary, we observe (4.34) to be true.

Let now A be a sectorial operator with angle $\omega < \frac{\pi}{2}$. Then the analytic semigroup e^{-tA} generated by $-A$ is given by the integral

$$e^{-tA} = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (\lambda - A)^{-1} d\lambda, \qquad t > 0,$$

in $\mathcal{L}(X)$, the integral contour Γ being as above, with the norm estimate

$$\|e^{-tA}\|_{\mathcal{L}(X)} \leq Ce^{-\delta t}, \qquad 0 \leq t < \infty. \tag{4.35}$$

If $f \in A0$, then $e^{-tA} f = 0$ for all $t > 0$; therefore, as $t \to 0$, $e^{-tA} f$ does not converge to f in general. As a matter of fact, it is only verified like (4.29) that

$$\lim_{t \to 0} e^{-tA} f = f, \qquad f \in \overline{\mathcal{D}(A)} \tag{4.36}$$

(see the second Remark to [59, Theorem 3.5]). It is seen that $A^x e^{-tA}$ is single valued for every $t > 0$, although A^x is multivalued. Moreover, $A^x e^{-tA}$ satisfies the norm estimate h the estimate

$$\|A^x e^{-tA}\|_{\mathcal{L}(X)} \leq Ct^{-x}, \qquad 0 \leq x < \infty,\ 0 < t < \infty \tag{4.37}$$

(see [59, Proposition 3.2]).

4.4.2 Semilinear Degenerate Evolution Equations

Let Y be a complex Banach space with norm $\|\cdot\|_Y$. Consider the Cauchy problem for an abstract degenerate equation

$$\begin{cases} M\dfrac{du}{dt} + Lu = Mf(u), \qquad 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{4.38}$$

in Y. Here, L is a (single valued) sectorial operator of Y. That is, L is a densely defined, closed linear operator whose spectrum is contained in an open sectorial domain

$$\sigma(L) \subset \Sigma' = \{\lambda \in \mathbb{C};\ |\arg\lambda| < \omega'\} \tag{4.39}$$

with $0 < \omega' < \pi$ and whose resolvent satisfies the estimate

$$\|(\lambda - L)^{-1}\|_{\mathcal{L}(Y)} \le \frac{D'}{|\lambda|}, \qquad \lambda \notin \Sigma', \tag{4.40}$$

with some constant $D' > 0$.

Meanwhile, M is a bounded linear operator from X into Y, where X is another Banach space with norm $\|\cdot\|_X$ such that

$$\mathcal{D}(L) \subset X \subset \mathcal{D}(L^\alpha) \qquad \text{(continuously)} \tag{4.41}$$

with some $0 \le \alpha < 1$. It is assumed that M-spectrum of L is contained in an open sectorial domain

$$\sigma_M(L) \subset \Sigma = \{\lambda \in \mathbb{C};\ |\arg\lambda| < \omega\} \tag{4.42}$$

with some angle $0 < \omega < \frac{\pi}{2}$, and that the M-resolvent $(\lambda M - L)^{-1}$ of L satisfies

$$\|(\lambda M - L)^{-1}M\|_{\mathcal{L}(X)} \le \frac{D}{|\lambda|}, \qquad \lambda \notin \Sigma, \tag{4.43}$$

with some constant $D > 0$.

Finally, f is a nonlinear operator from $\mathcal{D}(f)$ ($\supset \mathcal{D}(L)$) into X. We assume that there is an exponent β such that $\alpha \le \beta < 1$ for which it holds that $\mathcal{D}(L^\beta) \subset \mathcal{D}(f)$ together with the Lipschitz condition

$$\|f(u) - f(v)\|_X \le \varphi(\|L^\beta u\|_Y + \|L^\beta v\|_Y)\|L^\beta(u - v)\|_Y, \qquad u, v \in \mathcal{D}(L^\beta), \tag{4.44}$$

where $\varphi(\cdot)$ is some continuous increasing function. It clearly follows that

$$\|f(u)\|_X \le \|f(0)\|_X + \varphi(\|L^\beta u\|_Y)\|L^\beta u\|_Y, \qquad u \in \mathcal{D}(L^\beta). \tag{4.45}$$

The initial value u_0 is taken in $\mathcal{D}(L^\beta)$. Under these structural assumptions (4.39)-(4.44), one can show local existence of strict solution for (4.38).

4.4.3 Semilinear Multivalued Evolution Equations

It is essentially convenient to treat the Cauchy problems like (4.38) as those of non-degenerate evolution equations by using the multivalued linear operators. Rewrite (4.38) into the form

$$\begin{cases} \dfrac{du}{dt} + Au \ni f(u), \qquad 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{4.46}$$

in the space X. Here, $A = M^{-1}L$ is a multivalued linear operator of X defined by

$$\begin{cases} \mathcal{D}(A) = \{u \in \mathcal{D}(L);\ \exists f \in X \text{ such that } Mf = Lu\}, \\ Au = \{f \in X;\ Mf = Lu\}. \end{cases} \tag{4.47}$$

It is easy to see that $(\lambda - A)^{-1} = (\lambda M - L)^{-1}M$. Consequently, (4.42) and (4.43) imply that (4.26) and (4.27) hold for A. Therefore, $-A$ generates an analytic semigroup e^{-tA} on X.

For $0 < \theta \le 1$, the fractional power $A^\theta = [M^{-1}L]^\theta$ is defined. It is very difficult to know A^θ in any precise way. However, since $\mathcal{D}(A^0) = X \subset \mathcal{D}(L^\alpha)$ due to (4.41) and $\mathcal{D}(A^1) = \mathcal{D}(A) \subset \mathcal{D}(L)$ due to (4.47), we can compare the domains $\mathcal{D}(A^{\tilde\theta})$ and $\mathcal{D}(L^\theta)$ as follows.

Proposition 4.1. *For $0 < \tilde\theta < 1$ and $\alpha < \theta < (1-\tilde\theta)\alpha + \tilde\theta$, it is true that $\mathcal{D}(A^{\tilde\theta}) \subset \mathcal{D}(L^\theta)$ with the estimate*

$$\|L^\theta u\|_Y \le C_{\tilde\theta,\theta}\|A^{\tilde\theta}u\|_X, \qquad u \in \mathcal{D}(A^{\tilde\theta}), \tag{4.48}$$

with some constant $C_{\tilde\theta,\theta} > 0$. Note that $\|A^{\tilde\theta}u\|_X$ is defined by (4.28),

Proof. Since

$$L(\lambda - A)^{-1} = L(\lambda M - L)^{-1}M = M[-1 + \lambda(\lambda - A)^{-1}],$$

it follows that

$$\|L(\lambda - A)^{-1}\|_{\mathcal{L}(X,Y)} \le (D+1)\|M\|_{\mathcal{L}(X,Y)}, \qquad \lambda \notin \Sigma.$$

Meanwhile, by (4.41) it follows that

$$\|L^\alpha(\lambda - A)^{-1}\|_{\mathcal{L}(X,Y)} \le \|L^\alpha\|_{\mathcal{L}(X,Y)}\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \le D\|L^\alpha\||\lambda|^{-1}, \qquad \lambda \notin \Sigma.$$

The moment inequality then yields for $\alpha < \theta < 1$ that

$$\begin{aligned} \|L^\theta(\lambda - A)^{-1}\|_{\mathcal{L}(X,Y)} &\le C\|L^\alpha(\lambda - A)^{-1}\|^{(1-\theta)/(1-\alpha)} \\ &\quad \times \|L(\lambda - A)^{-1}\|^{(\theta-\alpha)/(1-\alpha)} \le C|\lambda|^{-(1-\theta)/(1-\alpha)}, \qquad \lambda \notin \Sigma. \end{aligned}$$

By the definition (4.33), we see that

$$L^\theta A^{-\tilde\theta} = \frac{1}{2\pi i}\int_\Gamma \lambda^{-\tilde\theta}L^\theta(\lambda - A)^{-1}d\lambda.$$

Obviously, the integral is convergent in $\mathcal{L}(X,Y)$ if $(1-\tilde\theta)\alpha + \tilde\theta > \theta$. Meanwhile, $L^\theta A^{-\tilde\theta} \in \mathcal{L}(X,Y)$ immediately implies the desired inequality (4.48). □

Let us fix an exponent $0 < \tilde{\beta} < 1$ so that

$$\beta < (1 - \tilde{\beta})\alpha + \tilde{\beta}. \tag{4.49}$$

By the proposition, we have $\mathcal{D}(A^{\tilde{\beta}}) \subset \mathcal{D}(L^{\beta}) \subset \mathcal{D}(f)$.

It is now ready to construct local solution for (4.46). Since the equation in (4.46) is a semilinear parabolic equation (although it is multivalued), we can use analogous techniques as in the proof of [37, Theorem 4.1($\beta = \eta$)].

Theorem 4.5. *Under* (4.39)$\sim$(4.44)*, for any* $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$*,* (4.46) *possesses a unique local solution* u *in the function space:*

$$u \in \mathcal{C}([0, T_{u_0}]; \mathcal{D}(L^{\beta})) \cap \mathcal{C}((0, T_{u_0}]; \mathcal{D}(A)) \cap \mathcal{C}^1((0, T_{u_0}]; X), \tag{4.50}$$

$T_{u_0} > 0$ *being determined by the norm* $\|A^{\tilde{\beta}} u_0\|_X$ *alone.*

Moreover, the local solution u *satisfies the estimates*

$$\|A^{\tilde{\beta}} u(t)\| \leq C_{u_0}, \qquad 0 \leq t \leq T_{u_0}, \tag{4.51}$$

$$\|u'(t)\|_X + \|Au(t)\|_X \leq C_{u_0} t^{\tilde{\beta}-1}, \qquad 0 < t \leq T_{u_0}, \tag{4.52}$$

$C_{u_0} > 0$ *being determined by the norm* $\|A^{\tilde{\beta}} u_0\|_X$ *alone.*

Proof. For $0 < T < \infty$, we set a Banach space $\mathcal{X}(T)$ by

$$\mathcal{X}(T) = \mathcal{C}([0, T]; \mathcal{D}(L^{\beta}))$$

equipped with the norm $\|u\|_{\mathcal{X}} = \max_{0 \leq t \leq T} \|L^{\beta} u(t)\|_Y$. In addition, we set a closed ball $\mathcal{B}(T)$ of $\mathcal{X}(T)$ by

$$\mathcal{B}(T) = \{u \in \mathcal{X}(T); \|u\|_{\mathcal{X}} \leq R\},$$

where the radius $R > 0$ will be specified below.

For $u \in \mathcal{B}(T)$, we define a mapping

$$[\Phi u](t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(u(s)) ds, \qquad 0 \leq t \leq T.$$

Let us verify that, if R is suitably chosen and if T is sufficiently small, then Φ is a contraction of $\mathcal{X}(T)$ which maps $\mathcal{B}(T)$ into itself.

Step 1. The function $[\Phi u](t)$ is seen to be a Hölder continuous function with values in $\mathcal{D}(L^{\beta})$. To show this, we need to introduce an auxiliary exponent $\tilde{\beta}'$ such that $0 < \tilde{\beta}' < \tilde{\beta}$ but $\beta < (1 - \tilde{\beta}')\alpha + \tilde{\beta}'$ (see (4.49)). Then, it follows by Proposition 4.1 that

$$\mathcal{D}(A^{\tilde{\beta}}) \subset \mathcal{D}(A^{\tilde{\beta}'}) \subset \mathcal{D}(L^{\beta}).$$

Let g_0 be any element such that $g_0 \in A^{\tilde{\beta}} u_0$. Then, since $u_0 = A^{-\tilde{\beta}} g_0$, we have

$$L^{\beta}[e^{-tA} - e^{-sA}] u_0 = L^{\beta} A^{-\tilde{\beta}'} [e^{-(t-s)A} - 1] A^{\tilde{\beta}' - \tilde{\beta}} g_0.$$

Therefore, by [59, Theorem 3.5], we obtain that

$$\|L^{\beta}[e^{-tA} - e^{-sA}] u_0\|_Y \leq C \|g_0\|_X (t-s)^{\sigma}, \qquad 0 \leq s < t \leq T,$$

with the exponent $\sigma = \tilde{\beta} - \tilde{\beta}'$.

Meanwhile, we write

$$\int_0^t e^{-(t-\tau)A} f(u(\tau))d\tau - \int_0^s e^{-(s-\tau)A} f(u(\tau))d\tau$$
$$= \int_s^t e^{-(t-\tau)A} f(u(\tau))d\tau + [e^{-(t-s)A} - 1] \int_0^s e^{-(s-\tau)A} f(u(\tau))d\tau.$$

Then, in view of (4.37) and (4.45), the first term in the right hand side is estimated by

$$\begin{aligned}\left\| L^\beta \int_s^t e^{-(t-\tau)A} f(u(\tau))d\tau \right\|_Y &= \left\| L^\beta A^{-\tilde{\beta}'} \int_s^t A^{\tilde{\beta}'} e^{-(t-\tau)A} f(u(\tau))d\tau \right\|_Y \\ &\leq C \int_s^t (t-\tau)^{-\tilde{\beta}'} [\|f(0)\|_X + \varphi(R)R]d\tau \\ &\leq C[\|f(0)\|_X + \varphi(R)R](t-s)^{1-\tilde{\beta}'}, \qquad 0 \leq s < t \leq T.\end{aligned}$$

Similarly, the second term is estimated by

$$\begin{aligned}&\left\| L^\beta [e^{-(t-s)A} - 1] \int_0^s e^{-(s-\tau)A} f(u(\tau))d\tau \right\|_Y \\ &\quad = \left\| L^\beta A^{-\tilde{\beta}'} [e^{-(t-s)A} - 1] A^{\tilde{\beta}' - \tilde{\beta}} \int_0^s A^{\tilde{\beta}} e^{-(s-\tau)A} f(u(\tau))d\tau \right\|_Y \\ &\quad \leq CT^{1-\tilde{\beta}} [\|f(0)\|_X + \varphi(R)R](t-s)^\sigma, \qquad 0 \leq s < t \leq T.\end{aligned}$$

Hence, we have observed that

$$[\Phi u] \in \mathcal{C}^\sigma([0,T]; \mathcal{D}(L^\beta)) \qquad (\text{with } \sigma = \tilde{\beta} - \tilde{\beta}'). \tag{4.53}$$

In particular, Φ is a mapping from $\mathcal{B}(T)$ into $\mathcal{X}(T)$.

Step 2. Let us verify that Φ can map $\mathcal{B}(T)$ into itself. Using (4.45) and arguing in a similar way as above, we easily verify that

$$\|L^\beta [\Phi u](t)\|_Y \leq C' \|g_0\|_X + C'' T^{1-\tilde{\beta}'} [\|f(0)\|_X + \varphi(R)R], \qquad 0 \leq t \leq T, \tag{4.54}$$

with some positive constants C' and C''. Then, choose now R in such a way that

$$R = C' \|g_0\|_X + 1.$$

Furthermore, diminish $T > 0$ in such a way that

$$C'' T^{1-\tilde{\beta}'} [\|f(0)\|_X + \varphi(R)R] \leq 1.$$

Then, Φ maps $\mathcal{B}(T)$ into itself.

Step 3. In the meantime, Φ can be a contraction of $\mathcal{X}(T)$. In fact, for u, $v \in \mathcal{B}(T)$,

$$[\Phi u](t) - [\Phi v](t) = \int_0^t e^{-(t-\tau)A} [f(u(\tau)) - f(v(\tau))]d\tau.$$

Therefore, after some computations,

$$\|L^\beta\{[\Phi u](t)-[\Phi v](t)\}\|_Y \le C\varphi(2R)\int_0^t (t-\tau)^{-\tilde\beta'}\|L^\beta[u(\tau)-v(\tau)]\|_Y d\tau$$
$$\le C\varphi(2R)T^{1-\tilde\beta'}\max_{0\le\tau\le T}\|L^\beta[u(\tau)-v(\tau)]\|_Y, \qquad 0\le t\le T.$$

This shows that Φ is a contraction provided we further diminish $T>0$.

Step 4. By the fixed point theorem for contraction mappings, we conclude that Φ has a unique fixed point $u=[\Phi u]$ in $\mathcal{B}(T)$. (4.53) jointed with (4.44) then implies that $f(u)\in\mathcal{C}^\sigma([0,T];X)$. Thanks to [59, Theorem 3.7] on the linear multivalued equations, we obtain that u has the regularity $u\in\mathcal{C}^1((0,T];X)$ together with $u\in\mathcal{C}((0,T];\mathcal{D}(A))$ and satisfies the multivalued equation of (4.46). Moreover, according to the second Remark to [59, Theorem 3.7], $u'(t)$ is represented as

$$u'(t)=-Ae^{-tA}u_0+\int_0^t Ae^{-(t-s)A}[f(u(t))-f(u(s))]ds+e^{-tA}f(u(t)). \tag{4.55}$$

It is then verified that u is a strict solution to (4.46) belonging to (4.50).

Let us verify the estimates (4.51) and (4.52). Since

$$A^{\tilde\beta}u(t)\ni e^{-tA}g_0+\int_0^t A^{\tilde\beta}e^{-(t-s)A}f(u(s))ds,$$

we see that $\|A^{\tilde\beta}u(t)\|_X\le C(\|g_0\|_X+1)$ by the definition of the graph norm (4.28). Thereby, we obtain (4.51). Meanwhile, from (4.55) it follows that $\|u'(t)\|_X\le C(t^{\tilde\beta-1}\|g_0\|_X+1)$ (due to (4.37) with $x=\tilde\beta$). Hence the first estimate of (4.52) is observed. Noting that $Au(t)\ni -u'(t)+f(u(t))$, the second one of (4.52) is also observed.

We remember that all the constants appearing in the arguments were determined by the norm $\|g_0\|_X$ alone. But, since g_0 was any element of $A^{\tilde\beta}u_0$, it is possible to assert that T and C_{u_0} are determined by $\|A^{\tilde\beta}u_0\|_X$ alone.

Step 5. It remains to see uniqueness of the solution to (4.46) in the space (4.50). So, let v be any other solution lying in (4.50). Then, thanks to [59, Theorem 3.7] again, $v(t)$ must be equal to $[\Phi v](t)$ for any $0\le t\le T_{u_0}$. Thereby,

$$u(t)-v(t)=\int_0^t e^{-(t-s)A}[f(u(s))-f(v(s))]ds,$$
$$\|L^\beta[u(t)-v(t)]\|_Y\le C\int_0^t (t-s)^{-\tilde\beta'}\|L^\beta[u(s)-v(s)]\|_Y ds, \qquad 0\le t\le T_{u_0}.$$

For $0<S\le T_{u_0}$, we see that

$$\|L^\beta[u(t)-v(t)]\|_Y\le C\int_0^t (t-s)^{-\tilde\beta'}ds\max_{0\le s\le S}\|L^\beta[u(s)-v(s)]\|_Y$$
$$\le CS^{1-\tilde\beta'}\|u-v\|_{\mathcal{X}(S)}, \qquad 0\le t\le S.$$

This means that, if $S>0$ is sufficiently small, then $u(t)=v(t)$ for all $0\le t\le S$. Consequently,

$$u(t)-v(t)=\int_S^t e^{-(t-s)A}[f(u(s))-f(v(s))]ds, \qquad S\le t\le T_{u_0}.$$

We can repeat this argument to conclude that $u(t)=v(t)$ for all $0\le t\le T_{u_0}$. □

Remark 4.1. *We remark that the solution u may not continuous at $t = 0$ with respect to the graph norm of $\mathcal{D}(A^{\tilde{\beta}})$. Indeed, from*

$$A^{\tilde{\beta}}u(t) \ni e^{-tA}g_0 + \int_0^t A^{\tilde{\beta}}e^{-(t-s)A}f(u(s))ds,$$

it is observed that $u(t) \to u_0$ in $\mathcal{D}(A^{\tilde{\beta}})$ if $e^{-tA}g_0 \to g_0$ in X as $t \to 0$. But, in view of (4.36), *this is the case only when $g_0 \in \overline{\mathcal{D}(A)}$, i.e.,*

$$[A^{\tilde{\beta}}u_0] \cap \overline{\mathcal{D}(A)} \neq \emptyset. \tag{4.56}$$

It is however observed that, when X is a reflexive Banach space, this condition is automatically fulfilled for any $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$. Indeed, if $g_0 \in A^{\tilde{\beta}}u_0$, then $g_0 = f' + f''$ with $f' \in A0$ and $f'' \in \overline{D(A)}$ due to (4.31)*; since $A0 = A^{\tilde{\beta}}0$ in general, on one hand, we have $g_0 - f' \in A^{\tilde{\beta}}u_0$; on the other hand, $f'' \in \overline{D(A)}$. Hence,* (4.56) *is the case.* □

Remark 4.2. *Similarly, the solution u may not be differentiable at $t = 0$ even if u_0 is taken in $\mathcal{D}(A)$. However, if $u_0 \in \mathcal{D}(A)$ satisfies the compatibility condition*

$$[f(u_0) - Au_0] \cap \overline{\mathcal{D}(A)} \neq \emptyset, \tag{4.57}$$

then u is differentiable at $t = 0$, too, and satisfies the equation of (4.46) *at the initial time. In view of* (4.36) *this fact is verified by the second Remark to [59, Theorem 3.7]. As mentioned by* (4.32)*, when X is a reflexive Banach space, this condition is automatically fulfilled for any $u_0 \in \mathcal{D}(A)$.* □

Remark 4.3. *On the other hand, if* (4.57) *takes place, then $[f(u_0) - Au_0] \cap \overline{\mathcal{D}(A)}$ consists of a single element u_1 due to* (4.30)*. Since the solution satisfies the relation*

$$u'(0) + Au_0 \ni f(u_0) \qquad at \quad t = 0, \tag{4.58}$$

and since $u'(0) \in \overline{\mathcal{D}(A)}$, it must hold that $u'(0) = u_1$. In other words, $u'(0)$ satisfying (4.58) *is uniquely determined by u_0.* □

It is immediate to verify that the solution of (4.46) constructed above gives a unique solution to (4.38) lying in the function space:

$$u \in \mathcal{C}([0,T]; \mathcal{D}(L^{\beta})) \cap \mathcal{C}((0,T]; \mathcal{D}(L)) \cap \mathcal{C}^1((0,T]; X). \tag{4.59}$$

In fact, if u is a solution of (4.46), then it follows from $\frac{du}{dt}(t) - f(u(t)) \in M^{-1}Lu(t)$ that $M\left[\frac{du}{dt}(t) - f(u(t))\right] = Lu(t)$. In addition, u naturally belongs to (4.59). Conversely, if u is a solution to (4.38) lying in (4.59), then u actually belongs to (4.50) and satisfies the multivalued equation of (4.46).

Next, let us show continuous dependence of local solutions on the initial values. Set a subset of initial values

$$K_r = \{u_0 \in \mathcal{D}(A^{\tilde{\beta}}); \|A^{\tilde{\beta}}u_0\|_X \leq r\}$$

for $r > 0$. Theorem 4.5 provides for each $u_0 \in K(r)$, existence of a local solution in (4.50) on a unified interval $[0, T_r]$.

Theorem 4.6. *There exists a constant $C_r > 0$ such that, for u_0, $v_0 \in K_r$ and their local solutions u, v, respectively, it holds that*

$$\|A^{\tilde{\beta}}[u(t) - v(t)]\|_X \le C_r \|A^{\tilde{\beta}}(u_0 - v_0)\|_X, \qquad 0 \le t \le T, \tag{4.60}$$

provided that $0 < T(\le T_R)$ is suitably diminished.

Proof. Let $g_0 \in A^{\tilde{\beta}} u_0$ and $h_0 \in A^{\tilde{\beta}} v_0$. Then, $u_0 = A^{-\tilde{\beta}} g_0$ and $v_0 = A^{-\tilde{\beta}} h_0$. So,

$$u(t) - v(t) = A^{-\tilde{\beta}} \left[e^{-tA}(g_0 - h_0) + \int_0^t A^{\tilde{\beta}} e^{-(t-s)A}[f(u(s)) - f(v(s))]ds \right].$$

The norm is then estimated by

$$\begin{aligned} \|A^{\tilde{\beta}}[u(t) - v(t)]\|_X &\le C\|g_0 - h_0\|_X + C_r \int_0^t (t-s)^{-\tilde{\beta}} \|L^{\beta}[u(s) - v(s)]\|_Y ds \\ &\le C\|g_0 - h_0\|_X + C_r T^{1-\tilde{\beta}} \|u - v\|_{\mathcal{B}([0,T];\mathcal{D}(A^{\tilde{\beta}}))}, \qquad 0 \le t \le T. \end{aligned}$$

Since $g_0 - h_0 \in A^{\tilde{\beta}}(u_0 - v_0)$, the Lipschitz condition (4.60) holds true if $T > 0$ is sufficiently small. □

4.5 Non-autonomous Dynamical Systems and Exponential Attractors

Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{X}$ be a subset of X which is a metric space equipped with the distance $d(U, V) = \|U - V\|_X$. We consider a family of nonlinear operators $U(t, s)$ acting on $\mathcal{X}$ defined for $\Delta = \{(t, s); -\infty < s \le t < \infty\}$. The family $U(t, s)$ is called a continuous evolution operator on $\mathcal{X}$ if $U(t, s)$ satisfies that following three conditions; $U(s, s)$ is identity mapping on $\mathcal{X}$ for every $-\infty < s < \infty$; $U(t, r) \circ U(r, s) = U(t, s)$ for $-\infty < s \le r \le t < \infty$; and the mapping $((t, s), U_0) \in \Delta \times \mathcal{X} \mapsto U(t, s)U_0 \in \mathcal{X}$ is continuous. When $U(t, s)$ is a continuous evolution operator on $\mathcal{X}$, the triplet $(U(t, s), \mathcal{X}, X)$ is called a non-autonomous dynamical system.

Efendiev, Yamamoto, and Yagi [60] introduced a version of exponential attractor for non-autonomous dynamical systems. In their paper, a family $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ of subsets of $\mathcal{X}$ is called an exponential attractor for $(U(t, s), \mathcal{X}, X)$ if:

(i) Each $\mathcal{M}(t)$ is a compact set of X and its fractal dimension $d_F(\mathcal{M}(t))$ is finite and uniformly bounded, i.e., $\sup_{t\in\mathbb{R}} d_F(\mathcal{M}(t)) < \infty$.

(ii) It is positively invariant, i.e., $U(t, s)\mathcal{M}(s) \subset \mathcal{M}(t)$ for all $(t, s) \in \Delta$.

(iii) There exist an exponent $\alpha > 0$ and two monotone functions Q and τ such that

$$\forall B \subset \mathcal{X} \text{ bounded}, \ h(U(t, s)B, \mathcal{M}(t)) \le Q(\|B\|_X)e^{-\alpha(t-s)}, \\ s \in \mathbb{R}, \ s + \tau(\|B\|_X) \le t < \infty,$$

where $h(\cdot, \cdot)$ is the Hausdorff pseudo-distance defined by

$$h(A, B) = \sup_{U\in A} \inf_{V\in B} \|U - V\|_X, \quad A, B \subset \mathcal{X}.$$

Furthermore, they gave a sufficient condition to construct exponential attractors. To this end, they assume existence of a family $\{\mathcal{N}(t)\}_{t\in\mathbb{R}}$ of bounded closed subsets of $\mathcal{X}$ with the following properties:

(1) The diameter $\|\mathcal{N}(t)\|_X$ of $\mathcal{N}(t)$ is uniformly bounded, i.e., $\sup_{t\in\mathbb{R}} \|\mathcal{N}(t)\|_X < \infty$.

(2) It is invariant, i.e., $U(t,s)\mathcal{N}(s) \subset \mathcal{N}(t)$ for all $(t,s) \in \Delta$.

(3) It is absorbing in the sense that there is a monotone function σ such that

$$\forall B \subset \mathcal{X} \text{ bounded, } U(t,s)B \subset \mathcal{N}(t), \quad s \in \mathbb{R},\ s + \sigma(\|B\|_X) \le t < \infty.$$

(4) There is $\tau^* > 0$ such that, for every $s \in \mathbb{R}$, $U(\tau^* + s, s)$ is a compact perturbation of contraction operator on $\mathcal{N}(s)$ in the sense that

$$\begin{aligned} &\|U(\tau^* + s, s)U_0 - U(\tau^* + s, s)V_0\|_X \\ &\quad \le \delta \|U_0 - V_0\|_X + \|K(s)U_0 - K(s)V_0\|_X, \quad U_0, V_0 \in \mathcal{N}(s), \end{aligned}$$

where δ is a constant such that $0 \le \delta < 1/2$ and where $K(s)$ is an operator from $\mathcal{N}(s)$ into another Banach space Z which is embedded compactly in X and satisfies a Lipschitz condition

$$\|K(s)U_0 - K(s)V_0\|_Z \le L_1 \|U_0 - V_0\|_X, \quad U_0, V_0 \in \mathcal{N}(s),$$

with some constant $L_1 > 0$ independent of s.

(5) For any $s \in \mathbb{R}$ and any $\tau \in [0, \tau^*]$, the Lipschitz condition

$$\|U(\tau + s, s)U_0 - U(\tau + s, s)V_0\|_X \le L_2 \|U_0 - V_0\|_X, \quad U_0, V_0 \in \mathcal{N}(s),$$

holds with some constant $L_2 > 0$ independent of s and τ.

Under the above assumptions, a version of exponential attractor for non-autonomous dynamical systems is constructed. For the proof, see [60, Theorem 2.1].

Theorem 4.7. *Let $(U(t,s), \mathcal{X}, X)$ be a non-autonomous dynamical system in X. Assume that the above conditions (1)–(5) be satisfied. Then, one can construct an exponential attractor $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ for $(U(t,s), \mathcal{X}, X)$.*

Chapter 5

Chemotaxis Equations in One Dimensional Domains

In this chapter, we firstly review some results of chemotaxis model equations represented by Keller-Segel equations. Secondly, we present results of attraction-repulsion chemotaxis equations which is one of the extension of Keller-Segel equations.

5.1 Reviews of Chemotaxis Equations

After Keller-Segel [61] first introduced an advection-reaction-diffusion model for chemotactic phenomenon, this model was developed and studied further by many researchers [62, 63, 64, 65, 66, 67, 68]. The model

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1 \Delta u - \nabla \cdot [u \nabla \chi(v)] & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = a_2 \Delta v + g_1 u - dv & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & \text{in} \quad \Omega, \end{cases}$$

was first studied by Childress and Percus [69]. Here, Ω is a bounded two-dimensional domain with some regular boundary. The unknown functions $u = u(x, t)$ and $v = v(x, t)$ denote the density of bacteria and the concentration of chemical attractants, respectively, in the domain Ω at time t. The bacteria are motile in response to the gradients of $\chi(v)$, where $\chi(v)$ is a sensitivity function of bacteria to chemical attractants. The term $-\nabla \cdot [u \nabla \chi(v)]$ denotes the nonlinear advection which is affected by chemical attractants. Bacteria move preferentially towards higher concentration of chemical attractants. The term $g_1 u$ denotes that bacteria produce chemical attractants. The term $-dv$ denote decay rates of chemical attractants. The unknown functions u and v satisfy the Neumann boundary conditions on the boundary $\partial\Omega$.

When $\chi(v) = v$, the global existence for initial functions having small $L_1(\Omega)$ norm $\|u_0\|_{L_1}$ was obtained by Ryu and Yagi [70]. On the other hand, blowup of solutions was proved in [71, 72]. In addition, the stationary problem was studied by [73, 74, 75]. Feireisl, Laurençot, and Petzeltová [29] have shown convergence of global solutions to equilibria by using a non-smooth version of the Łojasiewicz-Simon inequality obtained in [28]. We quote [76, 77] for one-dimensional problem, and [56] for network shaped domain problem.

As another chemotaxis model:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1 \Delta u - \nabla \cdot [u \nabla \chi(v)] + cu - \gamma u^2 & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = a_2 \Delta v + g_1 u - dv & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & \text{in} \quad \Omega, \end{cases} \tag{5.1}$$

was presented by Mimura and Tsujikawa [78]. This model is obtained by simplifying a model introduced by Woodward, Tyson, Myerscough, Murray, Bedrene, and Berg [79]. As for analytical studies to (5.1), we quote [80, 81, 82, 83]. Numerical results to a slightly changed model (more precisely, the term $cu - \gamma u^2$ in the first equation of (5.1) is replaced by a cubic function $u^2(1-u)$) show very interesting pattern formations: stationary spot, stationary honeycomb, stationary stripe, stationary perforated stripe, moving spot, and moving perforated labyrinthine. These numerical results were obtained by Hai and Yagi [84]. These mathematical studies have contributed theoretical understandings to experimental observations observed by Budrene and Berg [85, 86].

5.2 Attraction-Repulsion Chemotaxis Equations

As shown in the previous section, there are many studies for chemotaxis equations. Almost all chemotaxis equations considered interactions between bacteria and chemical attractant. In order to describe a variety of chemotactic phenomena, another chemical substance, which is called a repellent, were considered. As the name suggests, repellents have the bacteria tend to move toward low concentrations of repellents. Some attraction-repulsion chemotaxis models have been proposed in [87, 88]. In these years, attraction-repulsion chemotaxis models were studied by many researchers [89, 90, 91].

Among them, Okaie et al. [92, 93] utilized an attraction-repulsion chemotaxis model for molecular communication. Roughly speaking, molecular communication is a means of communication among bioparticles. Bioparticles are very tiny biological machines whose size is nano to micro scale. Since bioparticles are made of biological materials, no traditional communication technology is applicable. Whereas, bioparticles can interact with each other by using signaling molecules. Molecular communication is expected to be applied to medical treatment. In particular, researchers try to develop a system delivering medicine to target (or affected area) directly, which is called Drug delivery system. Okaie et al. developed a mathematical model of mobile bionanosensor networks for target tracking. The mobile bionanosensor network which they proposed consists of bioparticles, targets, and two signaling molecules: attractants and repellents. Bioparticles are capable of moving around the environment, in addition to releasing two signaling molecules into the environment and reacting to the molecules in the environment. Targets are also mobile and their presence is assumed to be a potential thereat to the environment. When bioparticles gather around targets after a while, they considered the network is capable of detecting targets. At first, Okaie et al. developed an individual based model of mobile bionanosensor networks for target tracking [92]. After that they developed a macroscopic model (5.2) in [93]. Both of them mainly described numerical results for their mathematical model.

Based on (5.2), Iwasaki, Yang, Abraham, Hagad, Obuchi, and Nakano showed that, when model parameters are properly tuned, bioparticles express the ability to distribute evenly over multi-targets by numerical simulations [94]. Furthermore, Iwasaki, Yang, and Nakano presented a non-diffusion-based model of (5.2) in [95]. Iwasaki and Nakano extended this model to a problem in network shaped domains [50].

In the following subsections, we present analytical results to (5.2) obtained in [31].

5.2.1 Model Equations

In the consecutive subsections, we show analytical results to the following attraction-repulsion chemotaxis equations proposed by Okaie et al. [93]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1 \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial}{\partial x}\left[u\left(\dfrac{\partial}{\partial x}\chi_1(v) - \dfrac{\partial}{\partial x}\chi_2(w)\right)\right] & \text{in } I \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = a_2 \dfrac{\partial^2 v}{\partial x^2} + g_1 T(x,t)u - dv & \text{in } I \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = a_3 \dfrac{\partial^2 w}{\partial x^2} + g_2 u - hw & \text{in } I \times (0,\infty), \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = \dfrac{\partial w}{\partial x} = 0 & \text{on } \partial I \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in } I, \end{cases} \tag{5.2}$$

in the unit open interval $I = (0,1)$. The unknown functions $u = u(x,t)$, $v = v(x,t)$, and $w = w(x,t)$ denote the density of bioparticles, the concentration of chemical attractants, and the concentration of chemical repellents, respectively, in the interval I at time t. The bioparticles are motile in response to the gradients of $\chi_1(v)$ and $\chi_2(w)$, where $\chi_1(v)$ and $\chi_2(w)$ are sensitivity functions of bioparticles to chemical attractants and chemical repellents. The term $-\frac{\partial}{\partial x}\left[u\left(\frac{\partial}{\partial x}\chi_1(v) - \frac{\partial}{\partial x}\chi_2(w)\right)\right]$ denotes the nonlinear advection which is affected by chemical attractants and chemical repellents. Bioparticles move preferentially towards higher (resp. lower) concentration of chemical attractants (resp. repellents). The term $g_1 T(x,t)u$ denotes that bioparticles produce chemical attractant when they find the target $T(x,t)$. On the other hand, bioparticles always release chemical repellents by the production rate $g_2 u$. The terms $-dv$ and $-hw$ denote decay rates of chemical attractants and repellents. The unknown functions u, v, and w satisfy the Neumann boundary conditions at ∂I.

We assume that $\chi_1(v)$ and $\chi_2(w)$ are real smooth functions for $0 \le v < \infty$ and $0 \le w < \infty$, respectively, satisfying the condition

$$\sup_{0 \le v < \infty} \left| \frac{d^i \chi_1}{dv^i}(v) \right| < \infty \quad \text{and} \quad \sup_{0 \le w < \infty} \left| \frac{d^i \chi_2}{dw^i}(w) \right| < \infty \quad \text{for} \quad i = 1, 2. \tag{5.3}$$

We also assume that $T(\cdot, t)$ is in $H^1(I)$ for each $t \in (0,\infty)$ and satisfies the two conditions:

$$\sup_{0 < t < \infty} \|T(x,t)\|_{H^1} < \infty, \tag{5.4}$$

$$\|T(x,t_1) - T(x,t_2)\|_{H^1} \leq C|t_1 - t_2|, \quad \forall t_1, t_2 \in (0,\infty), \tag{5.5}$$

with some constant $C \geq 0$. The initial functions $u_0(x)$, $v_0(x)$, and $w_0(x)$ are nonnegative in I. Furthermore, a_i $(i = 1,2,3)$, g_i $(i = 1,2)$, d, and h are positive (> 0) constants.

We will construct exponential attractors for the non-autonomous dynamical system generated by (5.2). Due to this research, we indicate that bioparticles show aggregation behavior.

5.2.2 Abstract Formulation

Let us formulate (5.2) as the Cauchy problem for a non-autonomous semilinear abstract equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(t,U), \quad 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{5.6}$$

in X. Here, we set the underlying space X as

$$X = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \ u \in L_2(I), \ v \in H^1(I), \text{and } w \in H^1(I) \right\},$$

X being equipped with the norm

$$\|U\| = \|u\|_{L_2} + \|v\|_{H^1} + \|w\|_{H^1}.$$

Due to such a setting, the nonlinear advection term $-\frac{\partial}{\partial x}\left[u\left(\frac{\partial}{\partial x}\chi_1(v) - \frac{\partial}{\partial x}\chi_2(w)\right)\right]$ can be treated as a lower term. That is, we can formulate the quasilinear problem (5.2) as a semilinear problem of the form (5.6).

The linear operator A is given by $A = \mathrm{diag}\{A_1, A_2, A_3\}$ in X. The operator $A_1 = -a_1\frac{d^2}{dx^2} + 1$ is a positive definite self-adjoint operator of $L_2(I)$ with domain $\mathcal{D}(A_1) = H^2_N(I) = \{u \in H^2(I); u'(0) = u'(1) = 0\}$. In addition, the two operators $A_2 = -a_2\frac{d^2}{dx^2} + d$ and $A_3 = -a_3\frac{d^2}{dx^2} + h$ are positive definite self-adjoint operators of $H^1(I)$ with domain $\mathcal{D}(A_2) = \mathcal{D}(A_3) = H^3_N(I) = \{u \in H^3(I); u'(0) = u'(1) = 0\}$. Therefore, the domain of A is given by

$$\mathcal{D}(A) = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \ u \in H^2_N(I), v \in H^3_N(I), \text{and } w \in H^3_N(I) \right\},$$

and A is also a positive definite self-adjoint operator of X. Then, according to Theorem 3.2 and 3.4, we know the following characterization of the domains of the fractional powers

$$\mathcal{D}(A^\eta) = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \ u \in H^{2\eta}_N(I), v \in H^{1+2\eta}_N(I), \text{and } w \in H^{1+2\eta}_N(I) \right\}$$

with norm equivalence, where the exponent η is fixed in such a way that $3/4 < \eta < 1$.

In addition, the nonlinear operator $F : (0,\infty) \times \mathcal{D}(A^\eta) \to X$ is given by

$$F(t,U) = \begin{pmatrix} u - \dfrac{\partial}{\partial x}\left[u\left(\dfrac{\partial}{\partial x}\chi_1(\text{Re } v) - \dfrac{\partial}{\partial x}\chi_2(\text{Re } w)\right)\right] \\ g_1 T(x,t) u \\ g_2 u \end{pmatrix}, \quad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Finally, we set the space of initial values by

$$\mathcal{K} = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \quad 0 \le u \in L_2(I),\ 0 \le v \in H^1(I), \text{and } 0 \le w \in H^1(I) \right\}.$$

5.2.3 Local Solutions

Consider the Cauchy problem:

$$\begin{cases} \dfrac{dU}{dt} + AU = F(t,U), \quad s < t < \infty, \\ U(s) = U_s, \end{cases} \tag{5.7}$$

in X, with initial time $s \ge 0$. In order to construct local solutions to (5.7), we utilize the theory of non-autonomous semilinear abstract evolution equations. According to [37, pages 199 and 200], it is sufficient to prove that $F(t,U)$ satisfies the Lipschitz condition:

$$\begin{aligned} \|F(t_1,U_1) - F(t_2,U_2)\| &\le \phi(\|U_1\| + \|U_2\|) \\ &\quad \times \{\|A^\eta(U_1 - U_2)\| + (\|A^\eta U_1\| + \|A^\eta U_2\|)[|t_1 - t_2| + \|U_1 - U_2\|]\}, \\ &\qquad (t_1,U_1),(t_2,U_2) \in (s,\infty) \times \mathcal{D}(A^\eta), \end{aligned} \tag{5.8}$$

where $\phi(\cdot)$ is some increasing continuous function.

Proposition 5.1. *$F(t,U)$ satisfies (5.8) with some $\phi(\cdot)$ which does not depend on the initial time s.*

Proof. Since it holds for all $\theta \ge 0$ that

$$\max\{\|\text{Re } u\|_{H^\theta}, \|\text{Im } u\|_{H^\theta}\} \le \|u\|_{H^\theta} \le \|\text{Re } u\|_{H^\theta} + \|\text{Im } u\|_{H^\theta}, \quad u \in H^\theta(I),$$

it is sufficient to prove (5.8) in the case where U_1 and U_2 are real valued.

Since $\chi_i(\cdot)$, $i = 1, 2$, are smooth functions, we see from [37, (1.91) and (1.92)] that

$$\|\chi_i(v)\|_{H^1} \le \phi(\|v\|_{H^1}), \quad v \in H^1(I), \tag{5.9}$$

and

$$\|\chi_i(v_1) - \chi_i(v_2)\|_{H^1} \le \phi(\|v_1\|_{H^1} + \|v_2\|_{H^1})\|v_1 - v_2\|_{H^1}, \quad v_1, v_2 \in H^1(I). \tag{5.10}$$

In addition, we obtain that

$$\begin{aligned} \left\|\frac{d^2}{dx^2}\chi_i(v)\right\|_{L_\infty} &= \left\|\chi_i''(v)\left(\frac{dv}{dx}\right)^2 + \chi_i'(v)\frac{d^2 v}{dx^2}\right\|_{L_\infty} \\ &\le \phi(\|v\|_{H^1})(\|v\|_{H^{2\eta}}^2 + \|v\|_{H^{1+2\eta}}) \\ &\le \phi(\|v\|_{H^1})(\|v\|_{H^1}^{\frac{1}{\eta}}\|v\|_{H^{1+2\eta}}^{2-\frac{1}{\eta}} + \|v\|_{H^{1+2\eta}}) \\ &\le \phi(\|v\|_{H^1})\|v\|_{H^{1+2\eta}}, \qquad v \in H^{1+2\eta}(I). \end{aligned} \tag{5.11}$$

Let $U_1 = {}^t(u_1, v_1, w_1)$ and $U_2 = {}^t(u_2, v_2, w_2) \in \mathcal{D}(A^\eta)$, then

$$
\begin{aligned}
&\|F(t_1, U_1) - F(t_2, U_2)\| \le \|u_1 - u_2\|_{L_2} \\
&\quad + \left\| \frac{d}{dx} \left[u_1 \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2)) \right] \right\|_{L_2} + \left\| \frac{d}{dx} \left[(u_1 - u_2) \frac{d}{dx} \chi_1(v_2) \right] \right\|_{L_2} \\
&\quad + \left\| \frac{d}{dx} \left[u_1 \frac{d}{dx} (\chi_2(w_1) - \chi_2(w_2)) \right] \right\|_{L_2} + \left\| \frac{d}{dx} \left[(u_1 - u_2) \frac{d}{dx} \chi_2(w_2) \right] \right\|_{L_2} \\
&\quad + g_1 \|T(x, t_1) u_1 - T(x, t_2) u_2\|_{H^1} + g_2 \|u_1 - u_2\|_{H^1}. \qquad (5.12)
\end{aligned}
$$

Here, we easily obtain estimates for the first and last terms of the right hand side.

Let us estimate the second term. Obviously,

$$
\begin{aligned}
&\frac{d}{dx} \left[u_1 \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2)) \right] \\
&\quad = \frac{du_1}{dx} \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2)) + u_1 \frac{d^2}{dx^2} (\chi_1(v_1) - \chi_1(v_2)).
\end{aligned}
$$

Due to (5.10),

$$
\begin{aligned}
\left\| \frac{du_1}{dx} \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2)) \right\|_{L_2} &\le \left\| \frac{du_1}{dx} \right\|_{L_\infty} \|\chi_1(v_1) - \chi_1(v_2)\|_{H^1} \\
&\le \phi(\|v_1\|_{H^1} + \|v_2\|_{H^1}) \|u_1\|_{H^{2\eta}} \|v_1 - v_2\|_{H^1} \\
&\le \phi(\|U_1\| + \|U_2\|) \|A^\eta U_1\| \|U_1 - U_2\|.
\end{aligned}
$$

On the other hand, by using the same techniques as in (5.10), we have

$$
\begin{aligned}
&\left\| u_1 \frac{d^2}{dx^2} (\chi_1(v_1) - \chi_1(v_2)) \right\|_{L_2} \\
&\le \left\| u_1 \chi_1''(v_1) \left[\left(\frac{dv_1}{dx} \right)^2 - \left(\frac{dv_2}{dx} \right)^2 \right] \right\|_{L_2} + \left\| u_1 \left(\frac{dv_2}{dx} \right)^2 [\chi_1''(v_1) - \chi_1''(v_2)] \right\|_{L_2} \\
&\quad + \left\| u_1 \chi_1'(v_1) \left[\frac{d^2 v_1}{dx^2} - \frac{d^2 v_2}{dx^2} \right] \right\|_{L_2} + \left\| u_1 \frac{d^2 v_2}{dx^2} [\chi_1'(v_1) - \chi_1'(v_2)] \right\|_{L_2} \\
&\le \left\| u_1 \chi_1''(v_1) \left[\frac{dv_1}{dx} + \frac{dv_2}{dx} \right] \right\|_{L_\infty} \left\| \frac{dv_1}{dx} - \frac{dv_2}{dx} \right\|_{L_2} \\
&\quad + \left\| \left(\frac{dv_2}{dx} \right) \right\|_{L_2} \|u_1 [\chi_1''(v_1) - \chi_1''(v_2)]\|_{L_\infty} \\
&\quad + \|u_1\|_{L_2} \left\| \chi_1'(v_1) \left[\frac{d^2 v_1}{dx^2} - \frac{d^2 v_2}{dx^2} \right] \right\|_{L_\infty} + \|u_1\|_{L_2} \left\| \frac{d^2 v_2}{dx^2} [\chi_1'(v_1) - \chi_1'(v_2)] \right\|_{L_\infty} \\
&\le \phi(\|U_1\| + \|U_2\|) \{ \|A^\eta (U_1 - U_2)\| + (\|A^\eta U_1\| + \|A^\eta U_2\|) \|U_1 - U_2\| \}.
\end{aligned}
$$

By (5.9) and (5.11), the third term in the right hand side of (5.12) is estimated by

$$
\begin{aligned}
&\left\| \frac{d}{dx} \left[(u_1 - u_2) \frac{d}{dx} \chi_1(v_2) \right] \right\|_{L_2} \\
&\quad \le \left\| \frac{d^2 \chi_1}{dx^2}(v_2) \right\|_{L_\infty} \|u_1 - u_2\|_{L_2} + \left\| \frac{d\chi_1}{dx}(v_2) \right\|_{L_2} \left\| \frac{d}{dx}(u_1 - u_2) \right\|_{L_\infty} \\
&\quad \le \phi(\|U_2\|) (\|A^\eta U_2\| \|U_1 - U_2\| + \|A^\eta (U_1 - U_2)\|).
\end{aligned}
$$

The similar techniques are available to estimate the forth and fifth terms of the right hand side of (5.12).

Finally, from (5.4) and (5.5), we conclude that

$$
\begin{aligned}
&\|T(x,t_1)u_1 - T(x,t_2)u_2\|_{H^1} \\
&\quad \le \|T(x,t_1)\|_{H^1}\|u_1-u_2\|_{H^1} + \|u_2\|_{H^1}\|T(x,t_1)-T(x,t_2)\|_{H^1} \\
&\quad \le C(\|A^\eta(U_1-U_2)\| + \|A^\eta U_2\||t_1-t_2|).
\end{aligned}
$$

Note that this C does not depend on the initial time s. Therefore, we verify the desired estimate (5.8). □

Theorem 5.1. *Let $0 \le s < \infty$. For any $U_s \in \mathcal{K}$, there exists a unique local solutions to (5.7) in the function space:*

$$
0 \le U \in \mathcal{C}((s, s+T_{U_s}]; \mathcal{D}(A)) \cap \mathcal{C}([s, s+T_{U_s}]; X) \cap \mathcal{C}^1((s, s+T_{U_s}]; X), \tag{5.13}
$$

where T_{U_s} is determined by the norm $\|U_s\|$ alone. In addition,

$$
(t-s)\|AU(t)\| + \|U(t)\| \le C_{U_s}, \quad s < t \le s + T_{U_s}, \tag{5.14}
$$

where C_{U_s} is determined by the norm $\|U_s\|$ alone. In particular, T_{U_s} and C_{U_s} do not depend on the initial time s.

Proof. Thanks to Theorem 4.1, we conclude that for any initial value $U_s \in \mathcal{K}$, (5.7) possesses a unique local solution in the function space:

$$
U \in \mathcal{C}((s, s+T_{U_s}]; \mathcal{D}(A)) \cap \mathcal{C}([s, s+T_{U_s}]; X) \cap \mathcal{C}^1((s, s+T_{U_s}]; X)
$$

with the norm estimate (5.14).

We notice that $U(t)$ is real valued. Indeed, the complex conjugate $\overline{U(t)}$ of $U(t)$ is also a local solution of (5.7) with the same initial value U_s. So, the uniqueness of solution implies that $\overline{U(t)} = U(t)$; hence, $U(t)$ must be real valued.

The proof of nonnegativity is verified by easier argument than the proof of Theorem 6.3 (put $\delta = 0$ in the proof). So, the proof is omitted here. □

We verify Lipschitz continuity of solutions in the initial data. Let $0 < R < \infty$. Let $\mathcal{K}_R = \mathcal{K} \cap \overline{B}^X(0; R)$, where $\overline{B}^X(0; R)$ denotes a closed ball of X centered at 0 with radius R. Then, there is an interval $[s, s+T_R]$ on which (5.7) has a unique local solution for any $U_s \in \mathcal{K}_R$, where $T_R > 0$ is determined by R alone. Due to [37, Theorem 4.5], we have

$$
(t-s)^\eta\|A^\eta[U_1(t)-U_2(t)]\| + \|U_1(t)-U_2(t)\| \le C_R\|U_s^1 - U_s^2\|, \quad s < t \le s+T_R, \tag{5.15}
$$

where $U_1(t)$ (resp. $U_2(t)$) is a local solution to (5.7) for initial data $U_s^1 \in \mathcal{K}_R$ (resp. $U_s^2 \in \mathcal{K}_R$).

5.2.4 Global Solutions

This section is devoted to showing the global existence of solutions. For $U_0 = {}^t(u_0, v_0, w_0) \in \mathcal{K}$, let $U = {}^t(u, v, w)$ denote local solutions of (5.6) in the function space (5.13), i.e.,

$$0 \le U \in \mathcal{C}((0, T_U]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_U]; X) \cap \mathcal{C}^1((0, T_U]; X), \tag{5.16}$$

Here $[0, T_U]$ denotes the interval of existence of each $U = {}^t(u, v, w)$. We build up a priori estimates for the local solutions.

Proposition 5.2. *There exist a continuous increasing function $p(\cdot)$ and some positive exponent $\gamma > 0$ such that the estimate*

$$\|U(t)\| \le p(e^{-\gamma t} \|U_0\| + \|u_0\|_{L_1}), \quad 0 \le t \le T_U, \tag{5.17}$$

holds for any local solution U of (5.6) *in* (5.16)*, $p(\cdot)$ and γ being independent of T_U.*

Proof. In the proof, the notations $p(\cdot)$ and C stand for some continuous increasing functions and some constants, respectively, which are determined by the initial constants in (5.2) and by I in a specific way in each occurrence. We divide the proof into five steps.

Step 1. Integrate the first equation of (5.2) in I. Then, in view of $u \ge 0$,

$$\frac{d}{dt}\|u\|_{L_1} = 0,$$

hence,

$$\|u(t)\|_{L_1} = \|u_0\|_{L_1}, \quad 0 \le t \le T_U. \tag{5.18}$$

Step 2. Let us consider the linear problem:

$$\begin{cases} \dfrac{d}{dt}v + A_2 v = g_1 T(x,t)u, \quad 0 < t \le T_U, \\ v(0) = v_0 \end{cases} \tag{5.19}$$

in $H^1(I)'$, where $H^1(I)'$ is the dual space of $H^1(I)$. In (5.19), A_2 is regarded as a positive definite self-adjoint operator of $H^1(I)'$ with domain $H^1(I)$. The operator A_2 generates an analytic semigroup e^{-tA_2} on $H^1(I)'$ with the estimate

$$\|e^{-tA_2}\|_{\mathcal{L}((H^1)')} \le Ce^{-dt}, \qquad 0 \le t < \infty. \tag{5.20}$$

Then, the fractional power of A_2 satisfies

$$\mathcal{D}(A_2^\theta) = [H^1(I)', H^1(I)]_\theta = H^{1-2\theta}(I)', \quad 0 \le \theta < \frac{1}{2} \tag{5.21}$$

with norm equivalence.

Meanwhile, since

$$\begin{aligned} &\|g_1 T(x,t)u(t) - g_1 T(x,s)u(s)\|_{(H^1)'} \\ &\le C\left[\|T(x,t)u(t) - T(x,s)u(t)\|_{L_2} + \|T(x,s)u(t) - T(x,s)u(s)\|_{L_2}\right] \\ &\le C\left[\|u(t)\|_{L_2} \|T(x,t) - T(x,s)\|_{H^1} + \|T(x,s)\|_{H^1} \|u(t) - u(s)\|_{L_2}\right], \end{aligned}$$

we observe from (5.4), (5.5), and (5.16) that

$$g_1 T(x,t)u(t) \in \mathcal{C}([0,T_U];H^1(I)') \cap \mathcal{C}^{0,1}((0,T_U];H^1(I)').$$

Then, according to [37, Theorem 3.4], there exists a unique local solution v to (5.19) in the function space:

$$v \in \mathcal{C}((0,T_U];H^1(I)) \cap \mathcal{C}([0,T_U];H^1(I)') \cap \mathcal{C}^1((0,T_U];H^1(I)').$$

Moreover, v is necessarily given by the formula

$$v(t) = e^{-tA_2}v_0 + g_1\int_0^t e^{-(t-\tau)A_2}T(x,\tau)u(\tau)\,d\tau, \qquad 0 \le t \le T_U. \tag{5.22}$$

By the similar arguments, we obtain that

$$w(t) = e^{-tA_3}w_0 + g_2\int_0^t e^{-(t-\tau)A_3}u(\tau)\,d\tau, \qquad 0 \le t \le T_U. \tag{5.23}$$

Step 3. Let us estimate $v(t)$. It follows from (5.22) that

$$A_2 v(t) = e^{-tA_2}A_2 v_0 + g_1\int_0^t A_2^{\frac{7}{8}} e^{-\frac{(t-\tau)}{2}A_2} e^{-\frac{(t-\tau)}{2}A_2} A_2^{\frac{1}{8}}[T(x,\tau)u(\tau)]\,d\tau.$$

Note that $L^1(I) \subset H^{\frac{3}{4}}(I)'$ with continuous embedding, i.e., $\|\cdot\|_{(H^{\frac{3}{4}})'} \le C\|\cdot\|_{L_1}$. Then, by (5.20) and (5.21),

$$\|A_2 v(t)\|_{(H^1)'} \le C\left[e^{-dt}\|A_2 v_0\|_{(H^1)'} + \int_0^t (t-\tau)^{-\frac{7}{8}} e^{-\frac{d}{2}(t-\tau)}\|T(x,\tau)u(\tau)\|_{L_1}\,d\tau\right].$$

Therefore, we obtain by (5.4) and (5.18) that

$$\|v(t)\|_{H^1} \le C[e^{-dt}\|v_0\|_{H^1} + \|u_0\|_{L_1}], \quad 0 \le t \le T_U. \tag{5.24}$$

By the similar arguments, we obtain from (5.23) that

$$\|w(t)\|_{H^1} \le C[e^{-ht}\|w_0\|_{H^1} + \|u_0\|_{L_1}], \quad 0 \le t \le T_U. \tag{5.25}$$

Step 4. We shall use the notation

$$p_1(U) = p(\|v\|_{H^1} + \|w\|_{H^1} + \|u\|_{L_1}), \quad U = {}^t(u,v,w) \in X.$$

Multiply the second equation of (5.2) by $2\frac{\partial^2 v}{\partial x^2}$ and integrate the product in I. Then, by (5.4),

$$\begin{aligned}&\frac{d}{dt}\left\|\frac{\partial v}{\partial x}\right\|_{L_2}^2 + 2d\left\|\frac{\partial v}{\partial x}\right\|_{L_2}^2 + 2a_2\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L_2}^2\\ &\quad = -2g_1\int_I T(x,t)u\frac{\partial^2 v}{\partial x^2}\,dx \le a_2\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L_2}^2 + C\|u\|_{L_2}^2.\end{aligned}$$

Meanwhile, by Gagliardo-Nirenberg's inequality (Theorem 2.8),

$$\|u\|_{L_2} \leq C\|u\|_{H^1}^{\frac{1}{3}}\|u\|_{L_1}^{\frac{2}{3}} \leq \zeta_1\left(\|u\|_{L_2} + \left\|\frac{\partial u}{\partial x}\right\|_{L_2}\right) + C_{\zeta_1}\|u\|_{L_1}$$

with any $0 < \zeta_1 < 1$, hence,

$$\|u\|_{L_2} \leq \frac{\zeta_1}{1-\zeta_1}\left\|\frac{\partial u}{\partial x}\right\|_{L_2} + \frac{C_{\zeta_1}}{1-\zeta_1}\|u\|_{L_1}^2. \tag{5.26}$$

Therefore, we have

$$\frac{d}{dt}\left\|\frac{\partial v}{\partial x}\right\|_{L_2}^2 + 2d\left\|\frac{\partial v}{\partial x}\right\|_{L_2}^2 + a_2\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L_2}^2 - \zeta_2\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 \leq C_{\zeta_2}\|u_0\|_{L_1}^2 \tag{5.27}$$

with any $0 < \zeta_2 < 1$.

It is the same for $w(t)$. Hence,

$$\frac{d}{dt}\left\|\frac{\partial w}{\partial x}\right\|_{L_2}^2 + 2h\left\|\frac{\partial w}{\partial x}\right\|_{L_2}^2 + a_3\left\|\frac{\partial^2 w}{\partial x^2}\right\|_{L_2}^2 - \zeta_3\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 \leq C_{\zeta_3}\|u_0\|_{L_1}^2. \tag{5.28}$$

with any $0 < \zeta_3 < 1$.

In the meantime, multiply the first equation of (5.2) by $2u$ and integrate the product in I. Then,

$$\begin{aligned}\frac{d}{dt}\|u\|_{L_2}^2 + 2a_1\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 &= 2\int_I u\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial}{\partial x}(\chi_1(v) - \chi_2(w)\right)dx \\ &\leq a_1\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 + C\int_I u^2\left(\frac{\partial}{\partial x}(\chi_1(v) - \chi_2(w))\right)^2 dx.\end{aligned}$$

Here, on account of (5.24) and (5.25),

$$\begin{aligned}&\int_I u^2\left(\frac{\partial}{\partial x}(\chi_1(v) - \chi_2(w))\right)^2 dx \\ &\leq C\|u\|_{L_4}^2\left(\left\|\frac{\partial v}{\partial x}\right\|_{L_4}^2 + \left\|\frac{\partial w}{\partial x}\right\|_{L_4}^2\right) \\ &\leq C\|u\|_{H^1}\|u\|_{L_1}\left(\left\|\frac{\partial v}{\partial x}\right\|_{H^1}^{\frac{1}{2}}\left\|\frac{\partial v}{\partial x}\right\|_{L_2}^{\frac{3}{2}} + \left\|\frac{\partial w}{\partial x}\right\|_{H^1}^{\frac{1}{2}}\left\|\frac{\partial w}{\partial x}\right\|_{L_2}^{\frac{3}{2}}\right) \\ &\leq \zeta\left(\|u\|_{H^1}^2 + \left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L_2}^2 + \left\|\frac{\partial^2 w}{\partial x^2}\right\|_{L_2}^2\right) + C_\zeta p_1(U_0),\end{aligned}$$

with any $\zeta > 0$. Therefore, due to (5.26), we obtain that

$$\frac{d}{dt}\|u\|_{L_2}^2 + \|u\|_{L_2}^2 + (a_1 - \zeta)\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 - \zeta\left(\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L_2}^2 + \left\|\frac{\partial^2 w}{\partial x^2}\right\|_{L_2}^2\right) \leq C_\zeta p_1(U_0).$$

Now, we sum up this, (5.27), and (5.28). Then, by choosing ζ, ζ_2, and ζ_3 properly,

$$\frac{d}{dt}\left[\|u\|_{L_2}^2+\left\|\frac{\partial v}{\partial x}\right\|_{L_2}^2+\left\|\frac{\partial w}{\partial x}\right\|_{L_2}^2\right]+\delta\left[\|u\|_{L_2}^2+\left\|\frac{\partial v}{\partial x}\right\|_{L_2}^2+\left\|\frac{\partial w}{\partial x}\right\|_{L_2}^2\right]\le p_1(U_0),$$

with $\delta=\min\{1,2d,2h\}$. Thus, we arrive at the estimate

$$\|u(t)\|_{L_2}\le e^{-\delta t}[\|u_0\|_{L_2}+\|v_0\|_{H^1}+\|w_0\|_{H^1}]+p_1(U_0),\quad 0\le t\le T_U. \tag{5.29}$$

Step 5. Let $t\in[0,T_U]$ be fixed. Then, from (5.24) and (5.25),

$$\left\|v\left(\frac{t}{2}\right)\right\|_{H^1}+\left\|w\left(\frac{t}{2}\right)\right\|_{H^1}\le C\left[e^{-\frac{d}{2}t}\|v_0\|_{H^1}+e^{-\frac{h}{2}t}\|w_0\|_{H^1}+\|u_0\|_{L_1}\right]. \tag{5.30}$$

By regarding $U(\frac{t}{2})$ as an initial value, we obtain from (5.29) that

$$\begin{aligned}\|u(t)\|_{L_2}\le e^{-\frac{\delta}{2}t}&\left[\left\|u\left(\frac{t}{2}\right)\right\|_{L_2}+\left\|v\left(\frac{t}{2}\right)\right\|_{H^1}+\left\|w\left(\frac{t}{2}\right)\right\|_{H^1}\right]\\&+p\left(\left\|v\left(\frac{t}{2}\right)\right\|_{H^1}+\left\|w\left(\frac{t}{2}\right)\right\|_{H^1}+\|u_0\|_{L_1}\right).\end{aligned}$$

Then, due to (5.29) and (5.30),

$$\|u(t)\|_{L_2}\le p\left(e^{-\gamma t}[\|u_0\|_{L_2}+\|v_0\|_{H^1}+\|w_0\|_{H^1}]+\|u_0\|_{L_1}\right). \tag{5.31}$$

In this way, by (5.24), (5.25), and (5.31), we have established the desired a priori estimate (5.17). □

It is now possible to construct a global solution to (5.2) by the standard arguments.

Theorem 5.2. *For any initial value $U_0\in\mathcal{K}$, there exists a unique global solution $U={}^t(u,v,w)$ of* (5.2) *in the function space:*

$$0\le U\in\mathcal{C}((0,\infty);\mathcal{D}(A))\cap\mathcal{C}([0,\infty);X)\cap\mathcal{C}^1((0,\infty);X), \tag{5.32}$$

Proof. Let $U_0\in\mathcal{K}$ be initial value. Let us start with the local solution $U(t)$ to (5.6) on $[0,T_{U_0}]$ obtained in Theorem 5.1 with $s=0$. Set $U_s=U(s)$ for any $s\in(0,T_{U_0})$. On account of Proposition 5.2, we see that $\|U_s\|\le C_{U_0}$, where $C_{U_0}=p(\|U_0\|+\|u_0\|)$ and $p(\cdot)$ is the function in Proposition 5.2.

We here consider the Cauchy problem (5.7). Then, by using Theorem 5.1 again, there exists a unique local solution $V(t)$ on an interval $[s,s+t_{U_0}]$, where a length $t_{U_0}>0$ is determined by C_{U_0} alone. Note that t_{U_0} does not depend on the time s. Here, put

$$\tau=\begin{cases}t_{U_0}/2 & (\text{if}\quad t_{U_0}<T_{U_0}),\\ T_{U_0}/2 & (\text{if}\quad T_{U_0}\le t_{U_0}),\end{cases}$$

and $s=T_{U_0}-\tau$. Then, by the uniqueness of solution, we have $U(t)=V(t)$ for $T_{U_0}-\tau\le t\le T_{U_0}$; this means that we have constructed a local solution $V(t)$ to (5.6) on the interval $[0,T_{U_0}+\tau]$. Note that $U(t)$ and $V(t)$ have the same initial value U_0. Since we can choose the same constant C_{U_0} regardless of $U(t)$ or $V(t)$ due to Proposition 5.2, we can continue this extension procedure unlimitedly. Each time, the local solution is extended over the fixed length τ of interval. So, by finite times, the extended interval can cover any bounded interval $[0,T]$. □

For $U_0 \in \mathcal{K}$, let $U(t; U_0)$ be its global solution of (5.2) in (5.32). From Proposition 5.2, we obtain the estimate

$$\|U(t; U_0)\| \leq p(e^{-\gamma t}\|U_0\| + \|u_0\|_{L_1}), \quad 0 \leq t < \infty. \tag{5.33}$$

This jointed with (5.14) provides the following stronger dissipative estimate.

Theorem 5.3. *For $U(t; U_0)$, it holds that*

$$\|AU(t; U_0)\| \leq (1 + t^{-1})\tilde{p}(e^{-\gamma t}\|U_0\| + \|u_0\|_{L_1}), \quad 0 < t < \infty, \tag{5.34}$$

with some other continuous increasing function $\tilde{p}(\cdot)$.

Proof. In the proof, the continuous increasing function $\tilde{p}(\cdot)$ may vary from line to line. Let $s \in [0, \infty)$ and consider (5.7) with initial value $U_s = U(s; U_0)$. We then apply Theorem 5.1 to this problem to conclude that there exists $\tau > 0$ such that

$$\|AU(t; U_0)\| \leq (t - s)^{-1}\tilde{p}(\|U(s; U_0)\|), \quad s < t \leq s + \tau.$$

Note that τ depends only on $\|U(s; U_0)\|$ and hence due to (5.33) only on $p(\|U_0\|)$. First, applying this with $s = 0$, we see that

$$\|AU(t; U_0)\| \leq t^{-1}\tilde{p}(\|U_0\|), \quad 0 < t \leq \tau. \tag{5.35}$$

Second, taking $s = t - \tau$, we have

$$\begin{aligned} \|AU(t; U_0)\| &\leq \tau^{-1}\tilde{p}(\|U(t - \tau; U_0)\|) \\ &\leq \tau^{-1}\tilde{p}(p(e^{-\gamma t}e^{\gamma\tau}\|U_0\| + \|u_0\|_{L_1})), \quad \tau < t < \infty. \end{aligned}$$

Since τ depends only on $p(\|U_0\|)$, we obtain that

$$\|AU(t; U_0)\| \leq \tilde{p}(e^{-\gamma t}\|U_0\| + \|u_0\|_{L_1}), \quad \tau < t < \infty. \tag{5.36}$$

Combining (5.35) and (5.36), we conclude the desired estimate (5.34). $\square$

5.2.5 Exponential Attractors

Let us construct a non-autonomous dynamical system determined from (5.2). For this purpose, we consider, for any $-\infty < s < \infty$, the Cauchy problem:

$$\begin{cases} \dfrac{dU}{dt} + AU = \mathbb{F}(t, U), \quad s < t < \infty, \\ U(s) = U_s, \end{cases} \tag{5.37}$$

in X. Here, the nonlinear operator $\mathbb{F} : \mathbb{R} \times \mathcal{D}(A^\eta) \to X$ is defined by

$$\mathbb{F}(t, U) = \begin{cases} F(0, U), & -\infty < t < 0, \\ F(t, U), & 0 \leq t < \infty. \end{cases}$$

Since $\mathbb{F}(t, U)$ satisfies (5.8) in $\mathbb{R} \times \mathcal{D}(A^\eta)$, (5.37) possesses, for any $U_s \in \mathcal{K}$, a unique global solution U in the function space:

$$0 \leq U \in \mathcal{C}((s, \infty); \mathcal{D}(A)) \cap \mathcal{C}([s, \infty); X) \cap \mathcal{C}^1((s, \infty); X).$$

Let $U(\cdot, s; U_s)$ denote the global solution to (5.37) for initial function $U_s \in \mathcal{K}$ with initial time $s \in \mathbb{R}$. We set

$$U(t,s)U_s = U(t,s;U_s)$$

as a family of nonlinear operators $U(t,s)$ acting on $\mathcal{K}$ defined for $(t,s) \in \Delta = \{(t,s); -\infty < s \le t < \infty\}$. Then, the dissipative estimates (5.33) and (5.34) are rewritten as

$$\|U(t,s;U_s)\| \le p(e^{-\gamma(t-s)}\|U_s\| + \|u_s\|_{L_1}), \quad s \le t < \infty, \tag{5.38}$$

and

$$\|AU(t,s;U_s)\| \le (1 + (t-s)^{-1})p(e^{-\gamma(t-s)}\|U_s\| + \|u_s\|_{L_1}), \quad s < t < \infty. \tag{5.39}$$

For simplicity, $p(\cdot)$ is used instead of $\tilde{p}(\cdot)$.

Since it is clear from the uniqueness of solutions that $U(s,s) = I$ for $s \in \mathbb{R}$ and $U(t,s) = U(t,r) \circ U(r,s)$ for (t,r), $(r,s) \in \Delta$, $U(t,s)$ is an evolution operator acting on $\mathcal{K}$.

Proposition 5.3. *$U(t,s)$ is a continuous evolution operator on $\mathcal{K}$, i.e., the mapping $G : \Delta \times \mathcal{K} \to \mathcal{K}$, where $G(t,s;U_0) = U(t,s)U_0$, is continuous.*

Proof. Let $0 < R < \infty$ and $0 < T < \infty$ be arbitrarily fixed. We notice from (5.38) that $\|U(t,s)U_0\| \le p(2R)$ for any $0 \le t-s < \infty$ provided $U_0 \in \mathcal{K}_R$. For simplicity of notation, we rewrite $p(2R)$ to $p(R)$. By applying (5.15) with radius $p(R)$, we see that

$$\|U(t,s)U_0 - U(t,s)V_0\| \le C_{p(R)}\|U_0 - V_0\|, \quad U_0, V_0 \in \mathcal{K}_{p(R)},$$

provided that $0 \le t-s \le T_{p(R)}$.

Let next $T_{p(R)} \le t-s \le 2T_{p(R)}$. Then,

$$\begin{aligned}
&\|U(t,s)U_0 - U(t,s)V_0\| \\
&\quad = \|U(t,t-T_{p(R)})U(t-T_{p(R)},s)U_0 - U(t,t-T_{p(R)})U(t-T_{p(R)},s)V_0\| \\
&\quad \le C_{p(R)}\|U(t-T_{p(R)},s)U_0 - U(t-T_{p(R)},s)V_0\| \\
&\quad \le C_{p(R)}^2\|U_0 - V_0\|.
\end{aligned}$$

Repeating these arguments, we see that

$$\|U(t,s)U_0 - U(t,s)V_0\| \le C_{R,T}\|U_0 - V_0\|$$

for $0 \le t-s \le T$ with some constant $C_{R,T} > 0$.

On the other hand, we observe that $U(t,s)U_0$ satisfies the integral equation

$$U(t,s)U_0 = e^{-(t-s)A}U_0 + \int_s^t e^{-(t-\tau)A}F(\tau, U(\tau,s)U_0)d\tau, \quad s < t < \infty.$$

We can then verify that $U(t,s)U_0$ is continuous for (t,s) with values in X.

Therefore, as $((t_1,s_1),U_1) \to ((t_0,s_0),U_0)$,

$$\begin{aligned}
&\|G(t_1,s_1;U_1) - G(t_0,s_0;U_0)\| \\
&\quad \le \|G(t_1,s_1;U_1) - G(t_1,s_1;U_0)\| + \|G(t_1,s_1;U_0) - G(t_0,s_0;U_0)\| \to 0.
\end{aligned}$$

□

Hence, $(U(t,s), \mathcal{K}, X)$ becomes a non-autonomous dynamical system determined from (5.6).

We proceed to construct an exponential attractor. As shown in Section 4.5, Efendiev, Yamamoto, and Yagi [60] introduced a version of exponential attractor for non-autonomous equations. However, in this case, since the norm $\|u(t)\|_{L_1} = \|u_s\|_{L_1}$ is conserved for every $t \in [s, \infty)$, no compact set can attract every solution of (5.37) (cf. [77, Section 6]). Therefore, for each $\|u_s\|_{L_1} = l > 0$, we have to consider a space of initial values like $\mathcal{K}^l = \{\, {}^t(u,v,w) \in \mathcal{K}; \|u\|_{L_1} = l\}$ to reset. In the same way as above, for any $U_s \in \mathcal{K}^l$, (5.37) possesses a unique global solution in the function space:

$$U \in \mathcal{C}((s,\infty); \mathcal{D}(A)) \cap \mathcal{C}([s,\infty); \mathcal{K}^l) \cap \mathcal{C}^1((s,\infty); \mathcal{K}^l).$$

Moreover, we can construct the non-autonomous dynamical system $(U(t,s), \mathcal{K}^l, X)$ determined from (5.37).

Let us construct an exponential attractor for $(U(t,s), \mathcal{K}^l, X)$.

Theorem 5.4. *There exists an exponential attractor $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ for $(U(t,s), \mathcal{K}^l, X)$.*

Proof. We verify the sufficient conditions, namely, we show that there exists a family of closed bounded subset $\mathcal{X}(t)$, $t \in \mathbb{R}$, of $\mathcal{K}^l$ having the properties (1) $\sim$ (5) in Section 4.5.

In view of the dissipative estimate (5.39), we consider a subset

$$\mathcal{B} = \mathcal{K}^l \cap \overline{B}^{\mathcal{D}(A)}(0; 2p(2+l)).$$

This $\mathcal{B}$ is a compact set of X and is a bounded subset of $\mathcal{D}(A)$. Since $\mathcal{B}$ is also a bounded subset of $\mathcal{K}^l$, we observe from (5.39) that there exists a time $t_{\mathcal{B}}$ such that $U(t,s)\mathcal{B} \subset \mathcal{B}$ for every $t \geq t_{\mathcal{B}} + s$, where $t_{\mathcal{B}}$ is independent of s. We here set, for each $t \in \mathbb{R}$, that

$$\mathcal{X}(t) = \bigcup_{-\infty < s \leq t} U(t,s)\mathcal{B} = \bigcup_{t - t_{\mathcal{B}} \leq s \leq t} U(t,s)\mathcal{B}.$$

Let us see that the family $\mathcal{X}(t)$, $t \in \mathbb{R}$, fulfills all the desired conditions. It is clear that $\mathcal{X}(t) \subset \mathcal{K}^l$. In addition, since a mapping $g : [t - t_{\mathcal{B}}, t] \times \mathcal{B} \to \mathcal{K}^l$ such that $g(s, U_0) = U(t,s)U_0$ is continuous and the subset $[t - t_{\mathcal{B}}, t] \times \mathcal{B}$ is compact, its image $g([t - t_{\mathcal{B}}, t] \times \mathcal{B}) = \mathcal{X}(t)$ is also compact. Hence, the condition (1) is fulfilled.

By the definition of $\mathcal{X}(t)$,

$$\mathcal{X}(s) = \bigcup_{-\infty < r \leq s} U(s,r)\mathcal{B}.$$

For each r and t such that $-\infty < r \leq s \leq t < \infty$, it follows that $U(t,s) \circ U(s,r)\mathcal{B} = U(t,r)\mathcal{B} \subset \mathcal{X}(t)$. Hence, $U(t,s)\mathcal{X}(s) \subset \mathcal{X}(t)$, i.e., (2) is valid.

Consider any bounded subset B of $\mathcal{K}^l$. Thanks to (5.39), there exists a time t_B such that $U(t,s)B \subset \mathcal{B}$ for every $t \geq t_B + s$. Since $\mathcal{B} \subset \mathcal{X}(t)$, this means that the condition (3) is valid.

In the same way as [60, Proposition 5.1], we prove that the union $\cup_{t\in\mathbb{R}}\mathcal{X}(t)$ is a bounded subset of $\mathcal{D}(A)$. Hence, there is $R > 0$ such that $\cup_{t\in\mathbb{R}}\mathcal{X}(t) \subset \mathcal{K}^l_R = \mathcal{K}^l \cap \overline{B}^X(0; R)$. We here set $Z = \mathcal{D}(A^\eta)$. Then, (5.15) shows that the condition (4) is valid provided $\tau^* = T_R$,

$\delta = 0$, and $K(s) = U(s + T_R, s)$. The estimate provides also the Lipschitz condition of (5).

Therefore, we have verified that all the conditions (1)–(5) are fulfilled. Hence, [60, Theorem 2.1] yields existence of an exponential attractor $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ for $(U(t,s), \mathcal{K}^l, X)$. $\square$

Chapter 6

Keller-Segel Equations in Network Shaped Domains

In this chapter, we study the Keller–Segel equations in network shaped domains of the form (6.1). We use notations in Chapter 3. The following results are obtained in [32].

6.1 Model Equations

We are concerned with the classical Keller-Segel system on a network shaped domain $\mathcal{G} = \{\mathcal{E}, \mathcal{N}\}$.

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \dfrac{\partial^2 u_i}{\partial x_i^2} - \dfrac{\partial}{\partial x_i}\left[u_i \dfrac{\partial v_i}{\partial x_i}\right] & \text{in} \quad I_i \times (0,\infty), \quad I_i \in \mathcal{E}, \\ \dfrac{\partial v_i}{\partial t} = \alpha \dfrac{\partial^2 v_i}{\partial x_i^2} - \beta v_i + \gamma u_i & \text{in} \quad I_i \times (0,\infty), \quad I_i \in \mathcal{E}, \\ u_i(x_i, 0) = u_i^0(x_i) \geq 0, \quad v_i(x_i, 0) = v_i^0(x_i) \geq 0, & \text{in} \quad I_i, \quad I_i \in \mathcal{E}, \end{cases} \tag{6.1}$$

with the Kirchhoff conditions: for each $N_j \in \mathcal{N}$ and $t > 0$,

$$\forall I_i \in o(N_j) \cup \omega(N_j), u_i(N_j, t) \text{ has a same value (depending on } N_j \text{ and } t), \tag{6.2}$$

$$\forall I_i \in o(N_j) \cup \omega(N_j), v_i(N_j, t) \text{ has a same value (depending on } N_j \text{ and } t), \tag{6.3}$$

and

$$\frac{\partial u_i}{\partial n}(N_j, t) = 0, \tag{6.4}$$

$$\frac{\partial v_i}{\partial n}(N_j, t) = 0. \tag{6.5}$$

From the fact (6.16), we know that the total mass of $\{u_i\}_{I_i \in \mathcal{E}}$ is conserved for all time $t > 0$. Therefore, without loss of generality, it is possible to normalize the total mass of $\{u_i\}_{I_i \in \mathcal{E}}$; consequently, we can always assume that

$$\sum_{I_i \in \mathcal{E}} \int_{I_i} u_i^0 dx_i = 1. \tag{6.6}$$

Remark 6.1. *Instead of* (6.4)*, one may impose more natural condition for Keller-Segel system which describe the conservation of the total flux about u:*

$$\sum_{I_i \in o(N_j) \cup \omega(N_j)} \left[\frac{\partial u_i}{\partial n} - u_i \frac{\partial v_i}{\partial n} \right] (N_j, t) = 0, \quad \forall N_j \in \mathcal{N}, \quad \forall t > 0.$$

However, this condition and (6.5) *imply* (6.4)*. So, it is enough to consider* (6.4)*.*

6.2 Local solutions

Let us construct a local solution for (6.1) – (6.6). At first, note that the first two equations of (6.1) are simply denoted by

$$\begin{cases} \dfrac{\partial u}{\partial t} = D^2 u - D[uDv] & \text{in} \quad \mathcal{G} \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = \alpha D^2 v - \beta v + \gamma u & \text{in} \quad \mathcal{G} \times (0, \infty). \end{cases}$$

From this view point, we formulate (6.1) as the Cauchy problem for a semilinear abstract equation:

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{6.7}$$

in the underlying space $X = L_2(\mathcal{G}) \times H^1(\mathcal{G})$, X being equipped with the inner product

$$(U, \tilde{U})_X = (u, \tilde{u})_{L_2(\mathcal{G})} + (v, \tilde{v})_{H^1(\mathcal{G})}, \quad U = {}^t(u, v), \tilde{U} = {}^t(\tilde{u}, \tilde{v}) \in X.$$

By setting X as above, the nonlinear advection term $-D\,[uDv]$ can be treated as a lower term. That is, we can formulate the quasilinear problem as a semilinear problem of the form (6.7).

By using the results obtained in Chapter 3, the linear operator A is defined as follows: $A = \operatorname{diag}\{A_1, \mathbb{A}_2\}$ in X, where $A_1 = -D^2 + 1$ of $L_2(\mathcal{G})$ and $\mathbb{A}_2 = -\alpha D^2 + \beta$ of $H^1(\mathcal{G})$. From Theorems 3.1 and 3.3, the domain of A is given by

$$\mathcal{D}(A) = H^2(\mathcal{G}) \times H^3(\mathcal{G}) \tag{6.8}$$

with norm equivalence $\|U\|_{\mathcal{D}(A)} = \|u\|_{H^2(\mathcal{G})} + \|v\|_{H^3(\mathcal{G})}$ for $U = {}^t(u, v) \in \mathcal{D}(A)$. In addition, from Theorems 3.2 and 3.4, we know the following characterization of the domains of the fractional powers:

$$\mathcal{D}(A^\eta) = H^{2\eta}(\mathcal{G}) \times H^{2\eta+1}(\mathcal{G}) \quad \text{for} \quad 3/4 < \eta \le 1 \tag{6.9}$$

with norm equivalence $\|U\|_{\mathcal{D}(A^\eta)} = \|u\|_{H^{2\eta}(\mathcal{G})} + \|v\|_{H^{2\eta+1}(\mathcal{G})}$ for $U = {}^t(u, v) \in \mathcal{D}(A^\eta)$. In what follows, the exponent η is arbitrarily fixed such that $3/4 < \eta \le 1$.

The nonlinear operator $F : \mathcal{D}(A^\eta) \to X$ is given by

$$F(U) = {}^t(u - D\,[uDv]\,, \gamma u), \quad U = {}^t(u, v) \in \mathcal{D}(A^\eta).$$

Finally, we set the space of initial values by

$$\mathcal{K} = \left\{ U = \begin{pmatrix} u \\ v \end{pmatrix} \in X; \begin{array}{c} 0 \le u \text{ for a.e. } \mathcal{G} \quad \text{and} \quad 0 \le v \text{ for } \overline{\mathcal{G}}, \\ \int_{\mathcal{G}} u dx = 1 \end{array} \right\}.$$

In order to construct local solutions to (6.7), we apply Theorem 4.1. To this end, we prove that $F(U)$ satisfies the following Lipschitz condition.

Proposition 6.1. *$F(U)$ satisfies that, for $U, \tilde{U} \in \mathcal{D}(A^\eta)$,*

$$\begin{aligned} &\|F(U) - F(\tilde{U})\|_X \\ &\le C\left[(1 + \|U\|_X + \|\tilde{U}\|_X)\|U - \tilde{U}\|_{\mathcal{D}(A^\eta)} + (\|U\|_{\mathcal{D}(A^\eta)} + \|\tilde{U}\|_{\mathcal{D}(A^\eta)})\|U - \tilde{U}\|_X\right]. \end{aligned}$$

Proof. Let $U = {}^t(u, v)$ and $\tilde{U} = {}^t(\tilde{u}, \tilde{v})$. Since

$$\|F(U) - F(\tilde{U})\|_X \le \|u - \tilde{u}\|_{L_2(\mathcal{G})} + \|D[uDv] - D[\tilde{u}D\tilde{v}]\|_{L_2(\mathcal{G})} + \gamma\|u - \tilde{u}\|_{H^1(\mathcal{G})},$$

it is enough to show that

$$\begin{aligned} &\|D[uDv] - D[\tilde{u}D\tilde{v}]\|_{L_2(\mathcal{G})} \\ &\le C\left[(\|U\|_X + \|\tilde{U}\|_X)\|U - \tilde{U}\|_{\mathcal{D}(A^\eta)} + (\|U\|_{\mathcal{D}(A^\eta)} + \|\tilde{U}\|_{\mathcal{D}(A^\eta)})\|U - \tilde{U}\|_X\right]. \end{aligned} \tag{6.10}$$

We know that

$$\|D[uDv] - D[\tilde{u}D\tilde{v}]\|_{L_2(\mathcal{G})} \le \|D[(u - \tilde{u})Dv]\|_{L_2(\mathcal{G})} + \|D[\tilde{u}D(v - \tilde{v})]\|_{L_2(\mathcal{G})}.$$

Then,

$$\begin{aligned} \|D[(u - \tilde{u})Dv]\|_{L_2(\mathcal{G})} &= \left\|[D(u - \tilde{u})][Dv] + (u - \tilde{u})[D^2 v]\right\|_{L_2(\mathcal{G})} \\ &\le \|D(u - \tilde{u})\|_{\mathcal{C}(\overline{\mathcal{G}})} \|Dv\|_{L_2(\mathcal{G})} + \|u - \tilde{u}\|_{L_2(\mathcal{G})} \left\|D^2 v\right\|_{\mathcal{C}(\overline{\mathcal{G}})} \\ &\le C\left[\|v\|_{H^1(\mathcal{G})}\|u - \tilde{u}\|_{H^{2\eta}(\mathcal{G})} + \|v\|_{H^{2\eta+1}(\mathcal{G})}\|u - \tilde{u}\|_{L_2(\mathcal{G})}\right], \end{aligned}$$

here, the last inequality comes from (3.3). In the meantime,

$$\|D[\tilde{u}D(v - \tilde{v})]\|_{L_2(\mathcal{G})} \le C\left[\|\tilde{u}\|_{H^{2\eta}(\mathcal{G})}\|v - \tilde{v}\|_{H^1(\mathcal{G})} + \|\tilde{u}\|_{L_2(\mathcal{G})}\|v - \tilde{v}\|_{H^{2\eta+1}(\mathcal{G})}\right]$$

by the similar reasons as above. Therefore, due to (6.9), the desired estimate (6.10) is obtained. □

Then, let us show the local existence of solutions to (6.7).

Theorem 6.1. *For any $U_0 \in \mathcal{K}$, there exists a unique local solution $U = {}^t(u, v)$ to (6.7) in the function space:*

$$0 \le U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X), \tag{6.11}$$

where T_{U_0} is determined by the norm $\|U_0\|_X$ alone. In addition,

$$\|U(t)\|_{\mathcal{D}(A)} \le t^{-1} p_0(\|U_0\|_X), \quad 0 < t \le T_{U_0}, \tag{6.12}$$

where $p_0(\cdot)$ is an increasing continuous function.

Proof. Due to Theorem 4.1, we conclude that for any initial value $U_0 \in \mathcal{K}$, (6.7) possesses a unique local solution in the function space:

$$U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X)$$

with the estimate (6.12).

We notice that $U(t)$ is real valued. Indeed, the complex conjugate $\overline{U(t)}$ of $U(t)$ is also a local solution of (6.7) with the same initial value U_0. So, the uniqueness of solution implies that $\overline{U(t)} = U(t)$; hence, $U(t)$ must be real valued. Nonnegativity of local solutions is proved by similar techniques in the proof of Theorem 6.3. □

6.3 Global solutions

This section is devoted to showing the global existence of solutions. Let $U = {}^t(u, v)$ be any local solution to (6.1) in the function space (6.11), i.e.,

$$\begin{cases} 0 \le u \in \mathcal{C}((0, T_U]; H^2(\mathcal{G})) \cap \mathcal{C}([0, T_U]; L_2(\mathcal{G})) \cap \mathcal{C}^1((0, T_U]; L_2(\mathcal{G})), \\ 0 \le v \in \mathcal{C}((0, T_U]; H^3(\mathcal{G})) \cap \mathcal{C}([0, T_U]; H^1(\mathcal{G})) \cap \mathcal{C}^1((0, T_U]; H^1(\mathcal{G})). \end{cases} \tag{6.13}$$

Here $[0, T_U]$ denotes the interval of existence of each $U = {}^t(u, v)$. We then show the following a priori estimates.

Proposition 6.2. *Let $U_0 = {}^t(u_0, v_0) \in \mathcal{K}$. For any local solution U to* (6.1) *in* (6.13) *with initial value U_0, it holds that*

$$\|U(t)\|_X \le C(\|U_0\|_X + 1), \qquad 0 \le t \le T_U. \tag{6.14}$$

Remark 6.2. *More precisely, the constant C in* (6.14) *depends on the norm $\|u_0\|_{L_1(\mathcal{G})}$. However, from the view point of* (6.6), *we do not write the dependence.*

Proof. We divide the proof into four steps.

Step 1. Firstly, we observe that

$$\|u(t)\|_{L_1(\mathcal{G})} \equiv \|u_0\|_{L_1(\mathcal{G})} = 1, \qquad 0 \le t \le T_U. \tag{6.15}$$

Indeed, in view of $u \ge 0$, $u \in H^2(\mathcal{G})$, and $v \in H^3(\mathcal{G})$,

$$\frac{d}{dt}\|u\|_{L_1(\mathcal{G})} = \sum_{I_i \in \mathcal{E}} \frac{d}{dt}\int_{I_i} u_i dx_i = \sum_{I_i \in \mathcal{E}} [D_i u_i - u_i D_i v_i]_{x_i=0}^{x_i=l_i} = 0, \tag{6.16}$$

so (6.15) is valid.

Step 2. Considering that $(\frac{\partial u}{\partial t}, 2u)_{L_2(\mathcal{G})} = (D^2u - D[uDv], 2u)_{L_2(\mathcal{G})}$, we obtain from (3.4) that

$$\frac{d}{dt}\|u\|^2_{L_2(\mathcal{G})} + 2\|Du\|^2_{L_2(\mathcal{G})} = -\int_{\mathcal{G}} u^2[D^2v]dx.$$

Here, by Young's inequality,

$$-\int_{\mathcal{G}} u^2[D^2v]dx \le \int_{\mathcal{G}} \left(\varepsilon|u|^4 + C_\varepsilon[D^2v]^2\right)dx = \varepsilon\|u\|^4_{L_4(\mathcal{G})} + C_\varepsilon\left\|D^2v\right\|^2_{L_2(\mathcal{G})}$$

with any $\varepsilon > 0$. Then, by Gagliardo-Nirenberg's inequality and (6.15),

$$\|u\|^4_{L_4(\mathcal{G})} \le C\,\|u\|^2_{H^1(\mathcal{G})}\,\|u\|^2_{L_1(\mathcal{G})} \le C\,\|u\|^2_{H^1(\mathcal{G})}\,,$$

so,

$$\frac{d}{dt}\|u\|^2_{L_2(\mathcal{G})} - \varepsilon C\,\|u\|^2_{L_2(\mathcal{G})} + (2-\varepsilon C)\,\|Du\|^2_{L_2(\mathcal{G})} \le C_\varepsilon\,\left\|D^2 v\right\|^2_{L_2(\mathcal{G})}. \tag{6.17}$$

In addition, by Gagliardo-Nirenberg's inequality (Theorem 2.8) and (6.15) again,

$$\|u\|^2_{L_2(\mathcal{G})} \le C\,\|u\|^{\frac{2}{3}}_{H^1(\mathcal{G})}\,\|u\|^{\frac{4}{3}}_{L_1(\mathcal{G})} \le \varepsilon'\left(\|u\|^2_{L_2(\mathcal{G})} + \|Du\|^2_{L_2(\mathcal{G})}\right) + C_{\varepsilon'}$$

with any $0 < \varepsilon' < 1$, hence,

$$\|u\|^2_{L_2(\mathcal{G})} \le \frac{\varepsilon'}{1-\varepsilon'}\,\|Du\|^2_{L_2(\mathcal{G})} + \frac{C_{\varepsilon'}}{1-\varepsilon'}. \tag{6.18}$$

Therefore, by taking ε and ε' sufficiently small, it follows from (6.17) that

$$\frac{d}{dt}\|u\|^2_{L_2(\mathcal{G})} + \|u\|^2_{L_2(\mathcal{G})} + \|Du\|^2_{L_2(\mathcal{G})} \le C\left(\left\|D^2 v\right\|^2_{L_2(\mathcal{G})} + 1\right). \tag{6.19}$$

Step 3. Considering that $(\frac{\partial v}{\partial t}, 2v)_{L_2(\mathcal{G})} = (\alpha D^2 v - \beta v + \gamma u, 2v)_{L_2(\mathcal{G})}$, we obtain from (3.4) that

$$\frac{d}{dt}\,\|v\|^2_{L_2(\mathcal{G})} + 2\alpha\,\|Dv\|^2_{L_2(\mathcal{G})} + 2\beta\,\|v\|^2_{L_2} = 2\gamma\int_{\mathcal{G}} uv dx \le \beta\,\|v\|^2_{L_2(\mathcal{G})} + C\,\|u\|^2_{L_2(\mathcal{G})}\,.$$

It follows from (6.18) that

$$\frac{d}{dt}\,\|v\|^2_{L_2(\mathcal{G})} + 2\alpha\,\|Dv\|^2_{L_2(\mathcal{G})} + \beta\,\|v\|^2_{L_2} \le \xi_1\,\|Du\|^2_{L_2(\mathcal{G})} + C_{\xi_1} \tag{6.20}$$

with any $0 < \xi_1 < 1$.

Step 4. Considering that $(\frac{\partial v}{\partial t}, 2D^2 v)_{L_2(\mathcal{G})} = (\alpha D^2 v - \beta v + \gamma u, 2D^2 v)_{L_2(\mathcal{G})}$, we obtain from (3.4) that

$$\frac{d}{dt}\,\|Dv\|^2_{L_2(\mathcal{G})} + 2\alpha\,\left\|D^2 v\right\|^2_{L_2(\mathcal{G})} + 2\beta\,\|Dv\|^2_{L_2(\mathcal{G})} = -2\gamma\int_{\mathcal{G}} u[D^2 v]dx, \tag{6.21}$$

due to $\frac{dv}{dt} \in H^1(\mathcal{G})$. It follows by (6.18) that

$$-2\gamma\int_{\mathcal{G}} u[D^2 v]dx \le \alpha\,\left\|D^2 v\right\|^2_{L_2(\mathcal{G})} + C\,\|u\|^2_{L_2(\mathcal{G})} \le \alpha\,\left\|D^2 v\right\|^2_{L_2(\mathcal{G})} + \xi_2\,\|Du\|^2_{L_2(\mathcal{G})} + C_{\xi_2}$$

with any $0 < \xi_2 < 1$. Thus,

$$\frac{d}{dt}\,\|Dv\|^2_{L_2(\mathcal{G})} + \alpha\,\left\|D^2 v\right\|^2_{L_2(\mathcal{G})} + 2\beta\,\|Dv\|^2_{L_2(\mathcal{G})} \le \xi_2\,\|Du\|^2_{L_2(\mathcal{G})} + C_{\xi_2} \tag{6.22}$$

After multiplying a parameter $0 < \zeta < 1$ to (6.19), we add the product to (6.20) and (6.22) to obtain that

$$\frac{d}{dt}\left[\zeta \|u\|^2_{L_2(\mathcal{G})} + \|v\|^2_{L_2(\mathcal{G})} + \|Dv\|^2_{L_2(\mathcal{G})}\right] + \left[\zeta \|u\|^2_{L_2(\mathcal{G})} + \beta \|v\|^2_{L_2(\mathcal{G})} + (2\alpha + 2\beta) \|Dv\|^2_{L_2(\mathcal{G})}\right]$$
$$+ (\zeta - \xi_1 - \xi_2) \|Du\|^2_{L_2(\mathcal{G})} + (\alpha - \zeta C) \left\|D^2 v\right\|^2_{L_2(\mathcal{G})} \leq \zeta C + C_{\xi_1} + C_{\xi_2}.$$

Now, fix the parameters ζ, ξ_1, and ξ_2 so that $\alpha - \zeta C \geq 0$ and $\zeta - \xi_1 - \xi_2 \geq 0$. Then, we arrive at the differential inequality

$$\frac{d}{dt}\left[\zeta \|u\|^2_{L_2(\mathcal{G})} + \|v\|^2_{H^1(\mathcal{G})}\right] + \delta \left[\zeta \|u\|^2_{L_2(\mathcal{G})} + \|v\|^2_{H^1(\mathcal{G})}\right] \leq C$$

with $\delta = \min\{\zeta, \beta, 2\alpha + 2\beta\}$. Therefore, we conclude that

$$\zeta \|u(t)\|^2_{L_2(\mathcal{G})} + \|v(t)\|^2_{H^1(\mathcal{G})} \leq e^{-\delta t}\left[\zeta \|u_0\|_{L_2(\mathcal{G})} + \|v_0\|^2_{H^1(\mathcal{G})}\right] + C, \quad 0 \leq t \leq T_U,$$

so, the desired estimate (6.14) is established. □

It is now possible to construct a global solution to (6.7).

Theorem 6.2. *For any initial value $U_0 \in \mathcal{K}$, there exists a unique global solution U of* (6.7) *in the function space:*

$$0 \leq U \in \mathcal{C}((0,\infty); \mathcal{D}(A)) \cap \mathcal{C}([0,\infty); X) \cap \mathcal{C}^1((0,\infty); X). \tag{6.23}$$

Proof. Let $U(t; U_0)$ be the local solution with initial value U_0 to (6.7) on $[0, T_{U_0}]$ obtained in Theorem 6.1. Set $U_s = U(s; U_0)$ for any $s \in (0, T_{U_0})$. On account of the a priori estimate (6.14), we see that $\|U_s\|_X \leq C(\|U_0\|_X + 1)$. Then, by using Theorem 6.1 again with initial value U_s, there exists a unique local solution $U(t; U_s)$ on an interval $[s, s+T'_{U_0}]$, where a length $T'_{U_0} > 0$ is determined by $\|U_0\|_X$ alone (note that $\|U_s\|_X \leq C(\|U_0\|_X + 1)$). Here, put

$$\tau = \begin{cases} T'_{U_0}/2 & \text{if} \quad T'_{U_0} < T_{U_0}, \\ T_{U_0}/2 & \text{if} \quad T_{U_0} \leq T'_{U_0}, \end{cases}$$

and $s = T_{U_0} - \tau$. Then, by the uniqueness of solution, we have $U(t; U_0) = U(t; U_s)$ for $T_{U_0} - \tau \leq t \leq T_{U_0}$; this means that we have constructed a local solution $U(t; U_0)$ to (6.7) on the interval $[0, T_{U_0} + \tau]$. Due to the a priori estimate (6.14), we can continue this extension procedure unlimitedly. Each time, the local solution is extended over the fixed length τ determined by $\|U_0\|_X$ alone. So, by finite times, the extended interval can cover any bounded interval $[0, T]$. □

Obviously, it is true that

$$\|U(t)\|_X \leq C(\|U_0\|_X + 1), \qquad 0 \leq t < \infty. \tag{6.24}$$

Furthermore, we show the following strong estimate.

Proposition 6.3. *Let $U_0 \in \mathcal{K}$ be initial value. Then, the global solution $U(t) = U(t; U_0)$ of* (6.7) *in* (6.23) *holds that*

$$\|U(t)\|_{\mathcal{D}(A)} \leq (1 + t^{-1}) p_1(\|U_0\|_X), \quad 0 < t < \infty, \tag{6.25}$$

with some increasing continuous function $p_1(\cdot)$.

Proof. Let $s \in [0, \infty)$ and apply Theorem 6.1 with initial value $U_s = U(s; U_0)$ to conclude that there exists $\tau > 0$ (depending only on $\|U_s\|_X$ and hence only on $\|U_0\|_X$) such that

$$\|U(t)\|_{\mathcal{D}(A)} \leq (t - s)^{-1} p_0(\|U_s\|_X), \quad s < t \leq s + \tau.$$

First, applying this with $s = 0$, we see that

$$\|U(t)\|_{\mathcal{D}(A)} \leq t^{-1} p_0(\|U_0\|_X), \quad 0 < t \leq \tau.$$

Second, taking $s = t - \tau$, it follows by (6.24) that

$$\begin{aligned} \|U(t)\|_{\mathcal{D}(A)} &\leq \tau^{-1} p_0(\|U(t - \tau; U_0)\|_X) \\ &\leq \tau^{-1} p_0(C(\|U_0\|_X + 1)), \quad \tau < t < \infty. \end{aligned}$$

Combining these estimates, we conclude the desired estimate (6.25). □

In order to prove the convergence result, we need positivity of $u(t)$. For this purpose, we introduce the following space:

$$\mathcal{D}(A)_+ = \left\{ U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A); \begin{array}{c} 0 < u \text{ for } \overline{\mathcal{G}} \quad \text{and} \quad 0 \leq v \text{ for } \overline{\mathcal{G}}, \\ \int_{\mathcal{G}} u dx = 1 \end{array} \right\}.$$

Theorem 6.3. *Let $U(t) = {}^t(u(t), v(t))$ be the global solution in the function space* (6.23) *with initial value $U_0 \in \mathcal{K}$. Assume that there exists a $t_0 \in (0, \infty)$ such that $U(t_0) \in \mathcal{D}(A)_+$. Then, it holds that $u(t) > 0$ for $\overline{\mathcal{G}}$, for every $t \in [t_0, \infty)$.*

Proof. By slightly modifying the following techniques (particularly taking $t_0 = 0$ and $\delta = 0$), we can show that $u(t) \geq 0$ for $\overline{\mathcal{G}}$ for every $t \in [0, \infty)$. So, we prove the positivity of $u(t)$ under the condition $u(t) \geq 0$.

Put $\delta = \min_{I_i \in \mathcal{E}} [\min_{x_i \in \overline{I_i}} u_i(x_i, t_0)] > 0$. In addition, for arbitrarily fixed $\tau \in (t_0, \infty)$, put $C_\tau = \max_{I_i \in \mathcal{E}} [\max_{(x_i, t) \in \overline{I_i} \times [t_0, \tau]} D_i^2 v_i(x_i, t)]$. It follows from (6.25) that $|C_\tau| < \infty$.

Regard the solution $v \in \mathcal{C}((0, \infty); H^3(\mathcal{G}))$ as a known function and consider the linear diffusion equation on $\mathcal{G}$:

$$\frac{\partial u}{\partial t} = D^2 u + p D u + q u \quad \text{in} \quad \mathcal{G} \times (t_0, \infty),$$

where $p = \{-Dv\}$ and $q = \{-D^2 v\}$.

Now, introduce a cutoff function $H(\xi)$ such that

$$H(\xi) = \frac{1}{2} \xi^2 \text{ for } -\infty < \xi < 0 \quad \text{and} \quad H(\xi) = 0 \text{ for } 0 \leq \xi < \infty,$$

and the function

$$\varphi(t) = \int_{\mathcal{G}} H(u(t) - \delta e^{-C_\tau t}) dx, \quad t_0 \leq t \leq \tau.$$

Then, we know that

$$\begin{aligned}\varphi'(t) &= (H'(u-\delta e^{-C_\tau t}), \frac{\partial u}{\partial t} + \delta C_\tau e^{-C_\tau t})_{L_2(\mathcal{G})}\\ &= (H'(u-\delta e^{-C_\tau t}), D^2u + pDu + qu + \delta C_\tau e^{-C_\tau t})_{L_2(\mathcal{G})}.\end{aligned}$$

Firstly,

$$\begin{aligned}(H'(u-\delta e^{-C_\tau t}), D^2u)_{L_2(\mathcal{G})} &= -(DH'(u-\delta e^{-C_\tau t}), Du)_{L_2(\mathcal{G})}\\ &= -(H''(u-\delta e^{-C_\tau t})Du, Du)_{L_2(\mathcal{G})}\\ &= -\int_{\mathcal{G}} H''(u-\delta e^{-C_\tau t})[Du]^2dx \le 0.\end{aligned}$$

Secondly, since $p = \{-Dv\}$ and $v \in H^2(\mathcal{G})$,

$$\begin{aligned}&(H'(u-\delta e^{-C_\tau t}), pDu)_{L_2(\mathcal{G})}\\ &\quad = (pH'(u-\delta e^{-C_\tau t}), D[u-\delta e^{-C_\tau t}])_{L_2(\mathcal{G})}\\ &\quad = -(D[pH'(u-\delta e^{-C_\tau t})], u-\delta e^{-C_\tau t})_{L_2(\mathcal{G})}\\ &\quad = -([Dp]H'(u-\delta e^{-C_\tau t}) + pH''(u-\delta e^{-C_\tau t})Du, u-\delta e^{-C_\tau t})_{L_2(\mathcal{G})}.\end{aligned}$$

Since it follows from (6.25) that $\|p\|_{\mathcal{C}(\mathcal{G})} + \|Dp\|_{\mathcal{C}(\mathcal{G})} \le C\|p\|_{H^3(\mathcal{G})} \le C'_\tau < \infty$, we know that

$$\begin{aligned}&(H'(u-\delta e^{-C_\tau t}), pDu)_{L_2(\mathcal{G})} \le \frac{1}{2}\int_{\mathcal{G}} H''(u-\delta e^{-C_\tau t})[Du]^2dx\\ &\quad + C'_\tau \int_{\mathcal{G}} [H'(u-\delta e^{-C_\tau t})(u-\delta e^{-C_\tau t}) + H''(u-\delta e^{-C_\tau t})(u-\delta e^{-C_\tau t})^2]dx\\ &\quad = \frac{1}{2}\int_{\mathcal{G}} H''(u-\delta e^{-C_\tau t})[Du]^2dx + 4C'_\tau\varphi(t).\end{aligned}$$

Here, note that $H''(\xi)\xi^2 = H'(\xi)\xi = 2H(\xi)$ for any $\xi \in \mathbb{R}$.

Finally, it follows from $q + C_\tau \ge 0$ and $u \ge 0$

$$\begin{aligned}&(H'(u-\delta e^{-C_\tau t}), qu + \delta C_\tau e^{-C_\tau t})_{L_2(\mathcal{G})}\\ &= (H'(u-\delta e^{-C_\tau t}), [q+C_\tau]u)_{L_2(\mathcal{G})} - C_\tau(H'(u-\delta e^{-C_\tau t}), u-\delta e^{-C_\tau t})_{L_2(\mathcal{G})} \le 0.\end{aligned}$$

Combining above estimates, we obtain that $\varphi(t) \le 4C'_\tau\varphi'(t)$, so that $\varphi(t) \le \varphi(0)e^{4C'_\tau t}$ for $t_0 \le t \le \tau$. Since $\varphi(0) = 0$, it follows that $\varphi(t) \equiv 0$ for $t_0 \le t \le \tau$; consequently, we conclude that $u(t) \ge \delta e^{-C_\tau t}$ for $\overline{\mathcal{G}}$, for every $t \in [t_0, \tau]$. □

6.4 Lyapunov function

In what follows, let $U_0 \in \mathcal{K}$ be arbitrarily fixed and let $U(t) = {}^t(u(t), v(t))$ denote the global solution of (6.7) in the function space (6.23) starting from U_0. We can prove the convergence of global solution $U(t)$ under the assumption that

$$\exists t_0 \in [0,\infty) \text{ such that } u(t) > 0 \text{ for } \overline{\mathcal{G}}. \tag{6.26}$$

For such a $U(t)$, Theorem 6.3 and Proposition 6.3 ensure that

$$u(t) > 0 \text{ for } \overline{\mathcal{G}} \quad \text{for every } t \in [t_0, \infty), \tag{6.27}$$

and

$$\|U(t)\|_{\mathcal{D}(A)} \leq R_0 \quad \text{for every } t \in [t_0, \infty), \tag{6.28}$$

where $R_0 = (1 + t_0^{-1})p_1(\|U_0\|_X)$.

It is well known that the Keller-Segel system has Lyapunov functions, and that is same for our problem. By combining $(\frac{\partial u}{\partial t}, \log u - v)_{L_2(\mathcal{G})}$ and $\frac{1}{\gamma}\|\frac{\partial v}{\partial t}\|^2_{L_2(\mathcal{G})}$, we obtain that, for every $t \in [t_0, \infty)$,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{G}} \left[u\log u - u - uv + \frac{\alpha}{2\gamma}[Dv]^2 + \frac{\beta}{2\gamma}v^2 \right] dx \\ &= -\int_{\mathcal{G}} u[D[\log u - v]]^2 dx - \frac{1}{\gamma}\int_{\mathcal{G}} \left(\frac{\partial v}{\partial t} \right)^2 dx \end{aligned} \tag{6.29}$$

due to $u > 0$ for $\overline{\mathcal{G}}$. Therefore,

$$\begin{aligned} \Phi(U) &= \int_{\mathcal{G}} \left[u\log u - u - uv + \frac{\alpha}{2\gamma}[Dv]^2 + \frac{\beta}{2\gamma}v^2 \right] dx \\ &= \int_{\mathcal{G}} [u\log u - u - uv]dx + \frac{1}{2\gamma}\langle \mathcal{A}_2 v, v\rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})} \end{aligned}$$

is a global Lyapunov functional of the problem (6.7). It is easy to see that $\Phi(U(t)) \geq -C_{R_0}$ for all $t \in [t_0, \infty)$ with some constant $C_{R_0} \geq 0$ depending on R_0.

6.4.1 Stationary solutions

Let us show the following proposition.

Proposition 6.4. *The value $\Phi(U(t))$ is monotonously decreasing in $t \geq t_0$. Moreover, if $\frac{d}{dt}\Phi(U(t))\mid_{t=\bar{t}} = 0$ at some time $\bar{t} \geq t_0$, then $U(\bar{t})$ is a stationary solution of* (6.1).

Proof. The first assertion is obvious from (6.29). Assume that $\frac{d}{dt}\Phi(U(t))\mid_{t=\bar{t}} = 0$ at some time $\bar{t}$ and put $U(\bar{t}) = {}^t(\overline{u}, \overline{v})$. It follows from (6.29) that

$$D[\log \overline{u} - \overline{v}] = 0 \quad \text{and} \quad \frac{\partial \overline{v}}{\partial t} = -\mathbb{A}_2\overline{v} + \gamma\overline{u} = 0 \quad \text{for } \overline{\mathcal{G}}. \tag{6.30}$$

Since $\frac{\partial \overline{u}}{\partial t} = D[\overline{u}D(\log \overline{u} - \overline{v})] = 0$, $\overline{u}$ is a stationary solution of (6.1). □

As for stationary solutions of (6.7), we know the following proposition.

Proposition 6.5. *Assume that $\overline{U} = {}^t(\overline{u}, \overline{v}) \in \mathcal{D}(A)$ is a stationary solution of* (6.7), *i.e., $\overline{U}$ is a solution of the problem:*

$$\begin{cases} 0 = D^2\overline{u} - D\left[\overline{u}D\overline{v}\right] & \text{for a.e. } \mathcal{G}, \\ 0 = \alpha D^2\overline{v} - \beta\overline{v} + \gamma\overline{u} & \text{for } \overline{\mathcal{G}}, \\ \overline{u} \geq 0 \text{ and } \overline{v} \geq 0 & \text{for } \overline{\mathcal{G}}, \\ \int_{\mathcal{G}} \overline{u}dx = 1. \end{cases} \tag{6.31}$$

Then, it holds that $\overline{u} > 0$ for $\overline{\mathcal{G}}$.

Proof. Obviously, $\overline{v} \in \mathcal{C}^2(\mathcal{G})$ due to $H^3(\mathcal{G}) \subset \mathcal{C}^2(\mathcal{G})$. Furthermore, since $D^2\overline{u} = [D\overline{u}][D\overline{v}] + \overline{u}D^2\overline{v} \in \prod \mathcal{C}(\overline{I_i})$, it holds that $\overline{u} \in \mathcal{C}^1(\mathcal{G}) \cap \prod \mathcal{C}^2(\overline{I_i})$.

Let us show the positivity of $\overline{u}$ by contradiction. Assume that there exist $I_i \in \mathcal{E}$ and $a_i \in \overline{I_i}$ such that $\overline{u}_i(a_i) = 0$. Then, $D_i\overline{u}_i(a_i) = 0$ due to the nonnegativity of u_i (and (6.4) if $a_i \in \{0, l_i\}$). Regard $D_i\overline{v}_i \in \mathcal{C}(\overline{I_i})$ and $D_i^2\overline{v}_i \in \mathcal{C}(\overline{I_i})$ as known functions and consider the following Cauchy problem:

$$\begin{cases} D_i^2\overline{u}_i - [D_i\overline{v}_i][D_i\overline{u}_i] - [D_i^2\overline{v}_i]\overline{u}_i = 0 \quad \text{in} \quad \overline{I_i}, \\ \overline{u}_i(a_i) = 0, \quad D_i\overline{u}_i(a_i) = 0. \end{cases}$$

By the classical results for ordinary differential equations, the solution $\overline{u}_i$ is written in the form $\overline{u}_i = C_1\overline{u}_i^{(1)} + C_2\overline{u}_i^{(2)}$ with linear independent solutions $\overline{u}_i^{(1)}$, $\overline{u}_i^{(2)}$ and some constants $C_1, C_2 \in \mathbb{R}$. Then, considering the Wronskian of $\overline{u}_i^{(1)}$ and $\overline{u}_i^{(2)}$ at a_i, we know that $C_1 = C_2 = 0$. Therefore, $\overline{u}_i \equiv 0$ in $\overline{I_i}$. This implies that $\overline{u} \equiv 0$ for $\overline{\mathcal{G}}$, which contradicts $\int_{\mathcal{G}} \overline{u}dx = 1$. □

6.4.2 ω-limit set

We consider the ω-limit set defined by

$$\omega(U_0) = \{\overline{U} \in X; \exists t_n \nearrow \infty \text{ s.t. } U(t_n) \to \overline{U} \text{ in } X\}. \tag{6.32}$$

Since the closed ball $\overline{B}^{\mathcal{D}(A)}(0; R_0)$ of $\mathcal{D}(A)$ is a relatively compact set of X due to the compact embeddings $H^2(\mathcal{G}) \subset L_2(\mathcal{G})$ and $H^3(\mathcal{G}) \subset H^1(\mathcal{G})$, it is observed from (6.28) that $\omega(U_0) \neq \emptyset$. Furthermore, since $\|U(t)\|_{\mathcal{D}(A^{\frac{1}{2}})} \leq C\|U(t)\|_X^{\frac{1}{2}}\|U(t)\|_{\mathcal{D}(A)}^{\frac{1}{2}} \leq CR_0^{\frac{1}{2}}\|U(t)\|_X^{\frac{1}{2}}$ due to the moment inequality for $A^{\frac{1}{2}}$ (see (2.16)) and (6.28), we know that

$$\overline{U} \in \omega(U_0) \text{ if and only if } \exists t_n \nearrow \infty \text{ such that } U(t_n) \to \overline{U} \text{ in } \mathcal{D}(A^{\frac{1}{2}}). \tag{6.33}$$

It is clear that $\inf_{t_0 \leq t < \infty} \Phi(U(t)) > -\infty$. Meanwhile, $\Phi(U(t_n)) \to \Phi(\overline{U})$ as $n \to \infty$. Therefore, it follows that

$$\lim_{t \to \infty} \Phi(U(t)) = \Phi(\overline{U}) \quad \text{for any } \overline{U} \in \omega(U_0). \tag{6.34}$$

From these results, we obtain the following theorem.

Theorem 6.4. *The ω-limit set $\omega(U_0)$ contains at least one stationary solution of* (6.7).

Proof. Since $\Phi(U(\cdot)) \in \mathcal{C}^1((t_0, \infty); \mathbb{R})$ and $\lim_{t\to\infty} \Phi(U(t)) = \text{const.}$, there exists some increasing time sequence $t_n \nearrow \infty$ such that $\frac{d}{dt}\Phi(U(t)) \mid_{t=t_n} \to 0$ as $n \to \infty$. Therefore, it is observe from (6.29) that, as $t_n \to \infty$,

$$\begin{aligned} &\sqrt{u(t_n)}D[\log u(t_n) - v(t_n)] \to 0 \quad \text{in} \quad L_2(\mathcal{G}), \\ &\frac{\partial v}{\partial t}(t_n) \to 0 \quad \text{in} \quad L_2(\mathcal{G}). \end{aligned} \tag{6.35}$$

Note that (6.35) implies that

$$D^2u(t_n) - D[u(t_n)Dv(t_n)] \to 0 \quad \text{in} \quad H_C^1(\mathcal{G})' \tag{6.36}$$

since, for $w \in H^1_C(\mathcal{G})$,

$$\begin{aligned}&\left|\left\langle D^2 u(t_n) - D[u(t_n)Dv(t_n)], w\right\rangle_{H^1_C(\mathcal{G})'\times H^1_C(\mathcal{G})}\right| \\ &= \left|(\sqrt{u(t_n)}D[\log u(t_n) - v(t_n)], \sqrt{u(t_n)}Dw)_{L_2(\mathcal{G})}\right| \\ &\le C_{R_0}\|\sqrt{u(t_n)}D[\log u(t_n) - v(t_n)]\|_{L_2(\mathcal{G})}\|w\|_{H^1(\mathcal{G})}\end{aligned}$$

due to (6.28).

On the other hand, since $\overline{B}^{\mathcal{D}(A)}(0; R_0)$ is a relatively compact set of $\mathcal{D}(A^{\frac{1}{2}})$ and $\overline{B}^{\mathcal{D}(A)}(0; R_0)$ is sequentially weakly closed in $\mathcal{D}(A)$, there exists some subsequence $t_{n_k} \to \infty$ such that $U(t_{n_k})$ converges to a limit $\overline{U} = {}^t(\overline{u}, \overline{v}) \in \mathcal{D}(A)$ in $\mathcal{D}(A^{\frac{1}{2}})$ and weakly in $\mathcal{D}(A)$. Then, we know from (6.36) that $D^2\overline{u} - D[\overline{u}D\overline{v}] = 0$ in $H^1(\mathcal{G})'$. Consequently, $\overline{U}$ is a stationary solution of (6.7) and $\overline{U} \in \omega(U_0)$ by definition. □

In what follows, let $\overline{U} = {}^t(\overline{u}, \overline{v}) \in \omega(U_0)$ be fixed such that $\overline{U}$ is a stationary solution of (6.7). Then, Proposition 6.5 ensures that $\overline{U} \in \mathcal{D}(A)_+$. Now, our goal is to prove that $\omega(U_0) = \{\overline{U}\}$.

6.4.3 Some extension of $\Phi(U)$

Remember that $\int_{\mathcal{G}} u(t)dx \equiv 1$ for every $t \in [0, \infty)$ due to (6.15). In view of this fact, consider the decomposition

$$u^m(t) = u(t) - \overline{u} \quad \text{and} \quad v^m(t) = v(t) - \overline{v}.$$

Then, for every $t \in (0, \infty)$, $u^m(t)$ is in $\mathcal{C}_m(\mathcal{G})(\subset L_{2,m}(\mathcal{G}))$, where $L_{2,m}(\mathcal{G})$ and $\mathcal{C}_m(\mathcal{G})$ are defined in Subsection 3.2.3.

Since $\overline{U} \in \mathcal{D}(A)_+$, there exist constants $0 < \delta_0 < \delta_1$ and $r > 0$ such that

$$\delta_0 \le u^m + \overline{u} \le \delta_1 \text{ for } \overline{\mathcal{G}} \quad \text{if } u^m \in B^{\mathcal{C}_{C,m}(\overline{\mathcal{G}})}(0; r). \tag{6.37}$$

In view of this fact, we introduce a smooth extension of $(\xi \log \xi - \xi)$ on the whole line $\mathbb{R}$ such that

$$\chi(\xi) \equiv \xi \log \xi - \xi \quad \text{for} \quad \xi \in (\delta_0/2, 2\delta_1) \tag{6.38}$$

and the values $\chi(\xi)$ for $\xi \in (-\infty, \delta_0/2] \cup [2\delta_1, \infty)$ being defined suitably as follows; there exist constants $0 < \chi_0 < \chi_1$ such that

$$\chi \in \mathcal{C}^2(\mathbb{R}) \quad \text{and} \quad \chi_0 \le \chi''(\xi) \le \chi_1, \quad \xi \in \mathbb{R}. \tag{6.39}$$

Obviously,

$$\chi : (\delta_0/2, 2\delta_1) \to \mathbb{R} \text{ is analytic.} \tag{6.40}$$

Introduce the space

$$X_m = L_{2,m}(\mathcal{G}) \times H^1(\mathcal{G})$$

and the functional $\Psi : X_m \to \mathbb{R}$ given by, for $U^m = {}^t(u^m, v^m) \in X_m$,

$$\begin{aligned}&\Psi(U^m) \\ &= \int_{\mathcal{G}} [\chi(\overline{u} + u^m) - (\overline{u} + u^m)(\overline{v} + v^m)]dx + \frac{1}{2\gamma}\left\langle \mathcal{A}_2(\overline{v} + v^m), \overline{v} + v^m\right\rangle_{H^1(\mathcal{G})'\times H^1(\mathcal{G})}.\end{aligned}$$

It is observed from (6.39) that $\Psi(U^m)$ is defined for every $U^m \in X_m$. In addition, $\Psi(U^m) = \Phi(\overline{U} + U^m)$ if $u^m \in B^{\mathcal{C}_m(\mathcal{G})}(0; r)$ due to (6.37) and (6.38).

6.4.4 Differentiability of $\Psi(U^m)$

In order to prove the convergence result, the Fréchet derivative of $\Psi(U^m)$ plays an important role. Identifying $L_{2,m}(\mathcal{G})$ with its dual $L_{2,m}(\mathcal{G})'$, we regard $X'_m = L_{2,m}(\mathcal{G}) \times H^1(\mathcal{G})'$.

Proposition 6.6. *The function $\Psi : X_m \to \mathbb{R}$ is Fréchet differentiable with derivative*

$$\Psi'(U^m) = \begin{pmatrix} P[\chi'(\overline{u} + u^m) - (\overline{v} + v^m)] \\ \frac{1}{\gamma}\mathcal{A}_2(\overline{v} + v^m) - (\overline{u} + u^m) \end{pmatrix} \in X'_m \quad \text{for} \quad U^m = {}^t(u^m, v^m) \in X_m, \tag{6.41}$$

where P is given by (3.6). *In particular, $\Psi'(0) = 0$.*

Proof. From the simple calculation, we know that, for $G^m = {}^t(g^m, h^m) \in X_m$,

$$\begin{aligned} &\Psi(U^m + G^m) - \Psi(U^m) - \left\langle \begin{pmatrix} P[\chi'(\overline{u} + u^m) - (\overline{v} + v^m)] \\ \frac{1}{\gamma}\mathcal{A}_2(\overline{v} + v^m) - (\overline{u} + u^m) \end{pmatrix}, G^m \right\rangle_{X'_m \times X_m} \\ &= \int_{\mathcal{G}} [\chi(\overline{u} + u^m + g^m) - \chi(\overline{u} + u^m) - \chi'(\overline{u} + u^m) - g^m h^m]\, dx \\ &\quad + \frac{1}{2\gamma} \langle \mathcal{A}_2 h^m, h^m \rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})}. \end{aligned}$$

Due to (6.39),

$$\left| \int_{\mathcal{G}} [\chi(\overline{u} + u^m + g^m) - \chi(\overline{u} + u^m) - \chi'(\overline{u} + u^m)]\, dx \right| \le \chi_1 \|g^m\|^2_{L_2(\mathcal{G})},$$

so,

$$|\Psi(U^m + G^m) - \Psi(U^m) - \langle \Psi'(U^m), G^m \rangle_{X'_m \times X_m}| \le C \|G^m\|^2_{X_m}. \tag{6.42}$$

Hence, the Fréchet derivative (6.41) is obtained.

Remember (6.30). Particularly, since $\overline{u}, \overline{v} \in H^1(\mathcal{G})$, we know that $\log \overline{u} - v = \text{const}$ which does not depend on $I_i \in \mathcal{E}$. Hence, $P[\log \overline{u} - \overline{v}] = P[\chi'(\overline{u}) - \overline{v}] = 0$ due to (6.37); consequently, $\Psi'(0) = 0$. □

Due to (6.39), it is easy to see that Ψ' is Lipschitz continuous, i.e., there exists a constant $L_0 > 0$ such that

$$\|\Psi'(U^m) - \Psi'(\tilde{U}^m)\|_{X'_m} \le L_0 \|U^m - \tilde{U}^m\|_{X_m} \quad \text{for any } U^m, \tilde{U}^m \in X_m. \tag{6.43}$$

6.5 Asymptotic Convergence of Global Solutions

In this section, we conclude the main result of this chapter. As stating in Section 6.4, we assume that the global solution $U(t) = {}^t(u(t), v(t))$ starting from $U_0 \in \mathcal{K}$ satisfies (6.26). Let $\overline{U} = {}^t(\overline{u}, \overline{v}) \in \omega(U_0)$ be a stationary solution of (6.7) and $\overline{U} \in \mathcal{D}(A)_+$. Furthermore, $U^m(t) = {}^t(u^m(t), v^m(t)) = {}^t(u(t) - \overline{u}, v(t) - \overline{v})$.

We introduce the following two spaces

$$W = \mathcal{C}_m(\mathcal{G}) \times H^2(\mathcal{G}) \quad \text{and} \quad X^{-1} = H^1(\mathcal{G})' \times L_2(\mathcal{G}).$$

6.5.1 Key properties of Ψ

Let us show the following angle condition in W.

Proposition 6.7. *Let $t \geq t_0$. For sufficiently small $r_0 > 0$, there exists a constant $\varepsilon > 0$ such that*

$$-\frac{d}{dt}\Psi(U^m(t)) \geq \varepsilon \|\Psi'(U^m(t))\|_{L_{2,m}(\mathcal{G})\times L_2(\mathcal{G})} \left\| \frac{dU^m}{dt}(t) \right\|_{X^{-1}} \quad \textit{for} \quad U^m(t) \in B^W(0; r_0). \tag{6.44}$$

Proof. Firstly, we show that

$$-\frac{d}{dt}\Psi(U^m(t)) \geq \varepsilon' \|\Psi'(U^m(t))\|^2_{L_{2,m}(\mathcal{G})\times L_2(\mathcal{G})} \tag{6.45}$$

with some constant $\varepsilon' > 0$. Here, let r_0 be chosen small enough such that (6.37) is satisfied if $U^m(t) \in B^W(0; r_0)$. Then, by using a version of Poincaré-Wirtinger inequality (3.7), we obtain that

$$\begin{aligned} \|P[\chi'(u(t)) - v(t)]\|_{L_{2,m}(\mathcal{G})} &\leq C\|D[\chi'(u(t)) - v(t)]\|_{L_{2,m}(\mathcal{G})} \\ &\leq \frac{C}{\sqrt{\delta_0}} \|\sqrt{u(t)}D[\chi'(u(t)) - v(t)]\|_{L_{2,m}(\mathcal{G})}. \end{aligned}$$

Therefore, (6.45) is obtained from (6.29) and (6.41).

Secondly, we show that

$$-\frac{d}{dt}\Psi(U^m(t)) \geq \varepsilon'' \left\| \frac{dU^m}{dt}(t) \right\|^2_{X^{-1}} \tag{6.46}$$

with some constant $\varepsilon'' > 0$. Noting $\frac{dU}{dt} = \frac{dU^m}{dt}$, this is immediately observed from the estimate: for $w \in H^1(\mathcal{G})$,

$$\begin{aligned} &| \left\langle D^2u(t) - D[u(t)Dv(t)], w \right\rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})} | \\ &= |(Du(t) - u(t)Dv(t), Dw)_{L_2(\mathcal{G})}| \leq \sqrt{\delta_1} \|\sqrt{u(t)}D[\chi'(u(t)) - v(t)]\|_{L_2(\mathcal{G})} \|w\|_{H^1(\mathcal{G})}. \end{aligned}$$

Here, (6.37) is used again.

Combining (6.45) and (6.46), we obtain the desired angle condition (6.44). □

Next, we can also show the following Łojasiewicz-Simon inequality.

Proposition 6.8. *For sufficiently small $r_1 > 0$, there exist an exponent $0 < \theta \leq 1/2$ and a constant $\tilde{\varepsilon} > 0$ such that*

$$\|\Psi'(U^m)\|_{X'_m} \geq \tilde{\varepsilon} |\Psi(U^m) - \Psi(0)|^{1-\theta} \quad \textit{if} \quad U^m \in B^W(0; r_1). \tag{6.47}$$

The proof of the proposition will be given in Section 6.6.

6.5.2 Asymptotic convergence of $u(t)$ to $\overline{u}$

We give a main result of this chapter: the convergence of $U(t)$ to $\overline{U}$ in X^{-1} as $t \to \infty$.

If $\Psi(U(t)) = \Psi(U(s))$ for some $t_0 \le s < t$, then $\Psi(U(\tau))$ is constant with respect to $\tau \in [s,t]$ and Proposition 6.4 yields that $U(\tau)$ is a stationary solution, i.e., $\omega(U_0) = \{\overline{U}\}$. So, it is enough to consider the case where $\Psi(U(s)) > \Psi(U(t))$ for any pair of $t_0 \le s < t$.

Let us begin with proving the following crucial proposition. Let $r > 0$ be a radius so that (6.44) and (6.47) is satisfied in a ball $B^W(0; r)$.

Proposition 6.9. *Let $t_0 \le s < t < \infty$ be such that, for all $\tau \in [s,t]$, the values $U^m(\tau)$ stay in $B^W(0;r)$. Then, we have*

$$\|U^m(t) - U^m(s)\|_W \le C[\Psi(U^m(s)) - \Psi(0)]^{\frac{\theta}{3}}, \tag{6.48}$$

where $C > 0$ depends on θ, ε, $\tilde{\varepsilon}$, and R_0.

Proof. Since $\Psi(U^m(\tau)) > \Psi(0)$ for $s \le \tau \le t$, we observe from (6.44) that

$$\begin{aligned} -\frac{d}{d\tau}[\Psi(U^m(\tau)) - \Psi(0)]^\theta &= -\theta[\Psi(U^m(\tau)) - \Psi(0)]^{\theta-1}\frac{d}{d\tau}\Psi(U^m(\tau)) \\ &\ge \varepsilon\theta[\Psi(U^m(\tau)) - \Psi(0)]^{\theta-1}\|\Psi'(U^m(\tau))\|_{L_{2,m}(\mathcal{G})\times L_2(\mathcal{G})}\left\|\frac{dU^m}{dt}(\tau)\right\|_{X^{-1}}. \end{aligned}$$

Since $\|\Psi'(U^m(\tau))\|_{L_{2,m}(\mathcal{G})\times L_2(\mathcal{G})} \ge C\|\Psi'(U^m(\tau))\|_{X'_m}$, it follows from (6.47) that

$$-\frac{d}{d\tau}[\Psi(U^m(\tau)) - \Psi(0)]^\theta \ge C\tilde{\varepsilon}\varepsilon\theta\left\|\frac{dU^m}{dt}(\tau)\right\|_{X^{-1}}.$$

Integrate this inequality on $[s,t]$. Then,

$$\begin{aligned} [\Psi(U^m(s)) - \Psi(0)]^\theta - [\Psi(U^m(t)) - \Psi(0)]^\theta &\ge C\int_s^t \left\|\frac{dU^m}{d\tau}(\tau)\right\|_{X^{-1}} d\tau \\ &\ge C\|U^m(t) - U^m(s)]\|_{X^{-1}}. \end{aligned} \tag{6.49}$$

On the other hand, since $\mathcal{D}(A^{\frac{1}{2}})$ is continuously embedded in W, it is observed from (3.31) and (3.40) that

$$\|U^m(t) - U^m(s)\|_W \le C_{R_0}\|U^m(t) - U^m(s)\|_{X^{-1}}^{\frac{1}{3}}. \tag{6.50}$$

By combining (6.49) and (6.50), the desired estimate (6.48) is concluded. □

Due to (6.33), there exist some time sequence $t_n \nearrow \infty$ and an integer N such that, for all $n \ge N$, $\|U^m(t_n)\|_W \le \frac{r}{3}$ and $C[\Psi(U^m(t_n)) - \Psi(0)]^{\frac{\theta}{3}} \le \frac{r}{3}$, here C is the constant obtained in (6.48). Assume that, for all $\tau \in [t_N, t]$, the values $U^m(\tau)$ lie in $B^W(0;r)$. Applying (6.48) with $s = t_N$, we observe that

$$\begin{aligned} \|U^m(t)\|_W &\le \|U^m(t) - U^m(t_N)\|_W + \|U^m(t_N)\|_W \\ &\le C[\Psi(U^m(t_N)) - \Psi(0)]^{\frac{\theta}{3}} + \|U^m(t_N)\|_W \le \frac{2r}{3}. \end{aligned}$$

This means that, after the time t_N, the trajectory must remain in the ball $B^W(0; \frac{2r}{3})$ forever.

It is now ready to prove the asymptotic convergence of $U(t)$.

Theorem 6.5. *Let $U(t)$ be the global solution starting from $U_0 \in \mathcal{K}$ satisfying* (6.26) *and take a $\overline{U} \in \omega(U_0)$ which is a stationary solution of* (6.1). *Then, actually $\omega(U_0) = \{\overline{U}\}$ and it holds the following estimate*

$$\|U^m(t)\|_{X^{-1}} \leq C[\Psi(U^m(t)) - \Psi(0)]^\theta, \quad \forall t \geq t_N, \tag{6.51}$$

where $U^m(t) = U(t) - \overline{U}$ and θ is the exponent appearing in (6.47).

Proof. Let $t_N \leq s \leq t_n$; since $U^m(\tau) \in B^W(0; r)$ for all $\tau \in [s, t_n]$, the inequality (6.49) is available with $t = t_n$ to conclude that

$$[\Psi(U^m(s)) - \Psi(0)]^\theta - [\Psi(U^m(t_n)) - \Psi(0)]^\theta \geq C\|U^m(t_n) - U^m(s)\|_{X^{-1}}.$$

By taking limit $n \to \infty$, we obtain that

$$\|U^m(s)\|_{X^{-1}} \leq C^{-1}[\Psi(U^m(s)) - \Psi(0)]^\theta, \quad s \geq t_N,$$

since $U^m(t_n) \to 0$ in W. □

When $U_0 \in \mathcal{D}(A)_+$, the condition (6.26) is satisfied automatically with $t_0 = 0$. So, we obtain the following corollary.

Corollary 6.1. *For any $U_0 \in \mathcal{D}(A)_+$, the global solution $U(t)$ starting from U_0 converges to a stationary solution $\overline{U} \in \omega(U_0)$ and it holds the following estimate*

$$\|U^m(t)\|_{X^{-1}} \leq C[\Psi(U^m(t)) - \Psi(0)]^\theta, \quad \forall t \geq t_N. \tag{6.52}$$

6.6 Proof of Łojasiewicz-Simon inequality

In this section, we prove the Łojasiewicz-Simon inequality (6.47). The following proof is based on the techniques in [29, Section 4]. We divide the proof into five steps.

Step 1. For the eigenvalue problem

$$A_2^{-1} e_n = \mu_n e_n \quad \text{in} \quad L_2(\mathcal{G}),$$

due to a theory of compact operator and (3.18), we know that there exist a Hilbert basis $\{e_n\}_{n\in\mathbb{N}} (\subset H^2(\mathcal{G}))$ of $L_2(\mathcal{G})$ and positive eigenvalues $\{\mu_n\}_{n\in\mathbb{N}}$ such that $\mu_n \searrow 0$ as $n \to \infty$. For each $N \in \mathbb{N}$, considering orthogonal projection Q_N from $L_2(\mathcal{G})$ onto span $\{e_1, \ldots, e_N\}$, the estimate

$$\|v\|^2_{L_2(\mathcal{G})} \leq \|Q_N v\|^2_{L_2(\mathcal{G})} + \mu_{N+1} \langle \mathcal{A}_2 v, v \rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})}, \quad v \in H^1(\mathcal{G}) \tag{6.53}$$

is obtained.

From this result, it is observed that the mapping $\Theta : X_m \to X'_m$ given by, for $U = {}^t(u, v) \in X_m$,

$$\Theta(U) = \Psi'(U) + \begin{pmatrix} 0 \\ \Lambda Q_N v \end{pmatrix} = \begin{pmatrix} P[\chi'(\overline{u} + u) - (\overline{v} + v)] \\ \frac{1}{\gamma}\mathcal{A}_2(\overline{v} + v) - (\overline{u} + u) + \Lambda Q_N v \end{pmatrix},$$

is a coercive monotone operator if $N \in \mathbb{N}$ and $\Lambda > 0$ take sufficiently large.

Proposition 6.10. *For sufficiently large $N \in \mathbb{N}$ and $\Lambda > 0$, there exist $0 < L_1$ such that*

$$\frac{1}{L_1}\|U - \tilde{U}\|_X^2 \leq \left\langle \Theta(U) - \Theta(\tilde{U}), U - \tilde{U} \right\rangle_{X_m' \times X_m}, \quad U, \tilde{U} \in X_m. \tag{6.54}$$

Proof. Since $\chi'(\xi) - \chi'(\tilde{\xi}) = \int_0^1 \chi''(\theta\xi + (1-\theta)\tilde{\xi})d\theta \times (\xi - \tilde{\xi})$ for $\xi, \tilde{\xi} \in \mathbb{R}$, it is observed from (6.39) that $\chi_0(\xi - \tilde{\xi})^2 \leq (\chi'(\xi) - \chi'(\tilde{\xi}))(\xi - \tilde{\xi})$. From this, it is easy to obtain the estimate

$$\begin{aligned}&\left\langle \Psi'(U) - \Psi'(\tilde{U}), U - \tilde{U} \right\rangle_{X_m' \times X_m} \\ &\geq \frac{1}{\gamma}\left\langle \mathcal{A}_2(v - \tilde{v}), v - \tilde{v} \right\rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})} + \frac{\chi_0}{2}\|u - \tilde{u}\|_{L_2(\mathcal{G})}^2 - \frac{8}{\chi_0}\|v - \tilde{v}\|_{L_2(\mathcal{G})}^2.\end{aligned}$$

After multiplying $\Lambda > 0$ to (6.53), we add the product to this inequality. Then, by choosing $\Lambda > 8/\chi_0$ and taking $N \in \mathbb{N}$ sufficiently large (so that μ_{N+1} sufficiently small), we obtain the desired estimate (6.54). □

It follows by (6.54) that $\Theta : X_m \to X_m'$ is injective. In addition, by the Browder-Minty theorem (see [96, Theorem 9.45]), Θ is surjective. As for its inverse $\Theta^{-1} : X_m' \to X_m$, we obtain by (6.54) that

$$\|\Theta^{-1}(U^*) - \Theta^{-1}(\tilde{U}^*)\|_{X_m} \leq L_1\|U^* - \tilde{U}^*\|_{X_m'} \quad \text{for all} \quad U^*, \tilde{U}^* \in X_m'. \tag{6.55}$$

Step 2. We introduce $\partial\Theta^0 \in \mathcal{L}(X_m, X_m')$ defined by

$$\partial\Theta^0 = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \chi''(\overline{u}) & -1 \\ -1 & \frac{1}{\gamma}\mathcal{A}_2 + \Lambda Q_N \end{pmatrix} \in \mathcal{L}(X_m, X_m'). \tag{6.56}$$

By using the same techniques in the proof of Proposition 6.10, it is obtained that $\frac{1}{L_1}\|U\|_{X_m}^2 \leq \langle \partial\Theta^0 U, U \rangle_{X_m' \times X_m}$ for $U \in X_m$. Therefore,

$$\partial\Theta^0 \text{ is a linear isomorphism from } X_m \text{ onto } X_m'. \tag{6.57}$$

Here, we introduce the following two spaces:

$$Z_m = \mathcal{C}_m(\mathcal{G}) \times H^2(\mathcal{G}) \quad \text{and} \quad Y_m = \mathcal{C}_m(\mathcal{G}) \times L_2(\mathcal{G}).$$

As a restriction of Θ, consider the mapping $\tilde{\Theta} : Z_m \to Y_m$ such that

$$\tilde{\Theta}(U) = \begin{pmatrix} P[\chi'(\overline{u} + u) - (\overline{v} + v)] \\ \frac{1}{\gamma}A_2(\overline{v} + v) - (\overline{u} + u) + \Lambda Q_N v \end{pmatrix} \in Y_m, \quad U = \begin{pmatrix} u \\ v \end{pmatrix} \in Z_m. \tag{6.58}$$

Remember that P given by (3.6) becomes a bounded linear projection from $\mathcal{C}(\mathcal{G})$ onto $\mathcal{C}_m(\mathcal{G})$. Then, we can show the following proposition.

Proposition 6.11. $\tilde{\Theta} : \mathcal{U}(0) \subset Z_m \to Y_m$ *is an analytic function, where* $\mathcal{U}(0)$ *is a neighborhood of* 0 *in* Z_m.

Proof of the Proposition. It suffices to prove that the mapping $\mathcal{F} : B^{\mathcal{C}_m(\mathcal{G})}(0; r) \to \mathcal{C}_m(\mathcal{G})$ given by $\mathcal{F}(u) = P\chi'(\overline{u} + u)$ is an analytic mapping, where $r > 0$ is in (6.37). The proof is similar to that of Proposition 2.1, so we omit it. □

Particularly, its first derivative $\tilde{\Theta}' : \mathcal{U}(0) \to \mathcal{L}(Z_m, Y_m)$ is given by, for $U = {}^t(u, v) \in \mathcal{U}(0)$,

$$\tilde{\Theta}'(U) = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi''(\overline{u}+u) & -1 \\ -1 & \frac{1}{\gamma}A_2 + \Lambda Q_N \end{pmatrix} \in \mathcal{L}(Z_m, Y_m).$$

Here, we want to apply the inverse mapping theorem (Theorem 2.3). To this end, we show the following proposition.

Proposition 6.12. *$\tilde{\Theta}'(0) : Z_m \to Y_m$ is bijective.*

Proof of the Proposition. Let ${}^t(z, w) \in Y_m$. Then, since ${}^t(z, w) \in X'_m$, it follows from (6.57) that there exists a unique ${}^t(g, h) \in X_m$ such that

$$\partial\Theta^0 \begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} P[\chi''(\overline{u})g - h] \\ \frac{1}{\gamma}\mathcal{A}_2 h - g + \Lambda Q_N h \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}.$$

Firstly, since $\frac{1}{\gamma}\mathcal{A}_2 h = g - \Lambda Q_N h + w \in L_2(\mathcal{G})$, we know that $h \in H^2(\mathcal{G})$ due to Theorem 3.1. On the other hand, since $1/\chi''(\overline{u}) \in \mathcal{C}(\mathcal{G})$, we obtain that

$$g = \frac{1}{\chi''(\overline{u})}\left[h + z + \frac{\int_{\mathcal{G}}[\chi''(\overline{u})g - h]dx}{\sum_{I_i \in \mathcal{E}} l_i}\right] \in \mathcal{C}(\mathcal{G}).$$

Therefore, $\tilde{\Theta}'(0)$ is a bijection from Z_m onto Y_m. □

Due to Propositions 6.11 and 6.12, it follows from Theorem 2.3 that there exists a neighborhood $\mathcal{V}(0) \subset Y_m$ (note that $\tilde{\Theta}(0) = 0$ due to Proposition 6.6) such that

$$\tilde{\Theta} : \mathcal{U}(0) \to \mathcal{V}(0) \text{ is an analytic diffeomorphism,} \tag{6.59}$$

by retaking $\mathcal{U}(0)$ small enough.

Step 3. Consider the following finite dimensional linear space

$$E_N = \{0\} \times \text{span}\,\{e_1, \dots, e_N\} \subset Z_m$$

with its norm $\|\cdot\|_{E_N}$. From the fact that any two norms on a finite dimensional linear space are equivalent, we particularly know the norm equivalence

$$\|{}^t(0, v)\|_{E_N} = \|v\|_{H^1(\mathcal{G})} \quad \text{for } {}^t(0, v) \in E_N. \tag{6.60}$$

Then, we can show the following proposition.

Proposition 6.13. *There exist a constant $C > 0$ and a radius $r_1 > 0$ such that*

$$\begin{aligned} &\|(\Psi \circ \tilde{\Theta}^{-1})'({}^t(0, \Lambda Q_N v))\|_{E'_N} \\ &\geq C|\Psi \circ \tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)) - \Psi({}^t(0, 0))|^{1-\theta}, \quad \|v\|_{H^1(\mathcal{G})} < r_1. \end{aligned} \tag{6.61}$$

Proof of the proposition. It is observed from similar techniques to the proof of Proposition 6.11 that Ψ is analytic as a function from $\mathcal{U}(0) \subset Z_m$ to $\mathbb{R}$. By combining this fact and (6.59), we know that

$$\Psi \circ \tilde{\Theta}^{-1} : \mathcal{V}(0) \cap E_N \to \mathbb{R} \text{ is an analytic function from } E_N \text{ to } \mathbb{R}. \tag{6.62}$$

Therefore, we can apply the classical Łojasiewicz theorem [10] to obtain that there exist $\theta \in (0, 1/2]$ and a neighborhood $\mathcal{W}(0)$ in E_N such that

$$\begin{aligned}&\|(\Psi \circ \tilde{\Theta}^{-1})'(U^0) - (\Psi \circ \tilde{\Theta}^{-1})'(0)\|_{E_N'} \\ &\geq C|\Psi \circ \tilde{\Theta}^{-1}(U^0) - \Psi \circ \tilde{\Theta}^{-1}(0)|^{1-\theta}, \quad U^0 \in \mathcal{W}(0).\end{aligned}$$

Here, note that $(\Psi \circ \tilde{\Theta}^{-1})'(U^0) = \Psi'(\tilde{\Theta}^{-1}(U^0)) \circ (\tilde{\Theta}^{-1})'(U^0)$ by the chain rule, so that $(\Psi \circ \tilde{\Theta}^{-1})'(0) = 0$ due to $\tilde{\Theta}^{-1}(0) = 0$. Therefore, we obtain that

$$\|(\Psi \circ \tilde{\Theta}^{-1})'(U^0)\|_{E_N'} \geq C|\Psi \circ \tilde{\Theta}^{-1}(U^0) - \Psi(0)|^{1-\theta}, \quad U^0 \in \mathcal{W}(0).$$

Thus, (6.60) implies (6.61) □

Step 4. Let us give estimate of the left hand side of (6.61). It follows from (6.59) that $\|(\tilde{\Theta}^{-1})'({}^t(0, \Lambda Q_N v))\|_{\mathcal{L}(E_N, X_m)} \leq C_{r_1}$ if $\|v\|_{H^1(\mathcal{G})} < r_1$. Therefore,

$$\|(\Psi \circ \tilde{\Theta}^{-1})'({}^t(0, \Lambda Q_N v))\|_{E_N'} \leq C_{r_1} \|\Psi'(\tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)))\|_{X_m'}.$$

For arbitrarily fixed $u \in L_{2,m}(\mathcal{G})$, we know that

$$\|\Psi'(\tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)))\|_{X_m'} \leq \|\Psi'(\tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v))) - \Psi'({}^t(u, v))\|_{X_m'} + \|\Psi'({}^t(u, v))\|_{X_m'}. \tag{6.63}$$

Then, since it follows by (6.55) and the definition of Θ that

$$\|\tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)) - {}^t(u, v)\|_{X_m} \leq L_1 \|\Psi'({}^t(u, v))\|_{X_m'}, \tag{6.64}$$

we know from (6.43) that

$$\|\Psi'(\tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v))) - \Psi'({}^t(u, v))\|_{X_m'} \leq L_0 L_1 \|\Psi'({}^t(u, v))\|_{X_m'}.$$

Therefore, for ${}^t(u, v) \in X_m$ such that $\|v\|_{H^1(\mathcal{G})} < r_1$,

$$\|\Psi'({}^t(u, v))\|_{X_m'} \geq C|\Psi \circ \tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)) - \Psi({}^t(0, 0))|^{1-\theta}. \tag{6.65}$$

Step 5. From (6.42) with $U = {}^t(u, v)$ and $G = \tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)) - {}^t(u, v)$, it is observed that

$$\begin{aligned}|\Psi({}^t(u, v)) - \Psi \circ \tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v))| \leq \|\Psi'({}^t(u, v))\|_{X_m'} \|\tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)) - {}^t(u, v)\|_{X_m} \\ + C\|\tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)) - {}^t(u, v)\|_{X_m}^2.\end{aligned}$$

By using (6.64), we know that

$$|\Psi({}^t(u, v)) - \Psi \circ \tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v))| \leq C\|\Psi'({}^t(u, v))\|_{X_m'}^2. \tag{6.66}$$

Therefore, we obtain that, for ${}^t(u, v) \in X_m$ such that $\|v\|_{H^1(\mathcal{G})} < r_1$,

$$\begin{aligned}&|\Psi({}^t(u, v)) - \Psi({}^t(0, 0))| \\ &\leq |\Psi({}^t(u, v)) - \Psi \circ \tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v))| + |\Psi \circ \tilde{\Theta}^{-1}({}^t(0, \Lambda Q_N v)) - \Psi({}^t(0, 0))| \\ &\leq C[\|\Psi'({}^t(u, v))\|_{X_m'}^2 + \|\Psi'({}^t(u, v))\|_{X_m'}^{1/(1-\theta)}].\end{aligned}$$

Due to (6.43), it is possible to choose sufficiently small $r_1 > 0$ such that $\|\Psi'({}^t(u, v))\|_{X_m'}^2 \leq \|\Psi'({}^t(u, v))\|_{X_m'}^{1/(1-\theta)}$ for all ${}^t(u, v) \in B^{X_m}(0; r_1)$.

Since W is continuously embedded in X_m, we have established the desired estimate (6.47).

Chapter 7

Quasilinear Diffusion Equations

In this chapter, we consider the initial-boundary value problem for a quasilinear parabolic equation.

As a motivation to consider this problem (7.1), we want to study asymptotic behavior of solutions of (5.2). However, it is very difficult to investigate the time evolution of solutions in detail. One of the reasons for this difficulty is that (5.2) is an advection-reaction-diffusion equation with three components. Furthermore, the moving target $T(x,t)$ makes it hard to investigate the stationary state. Therefore, we intend to simplify the original model. First, we assume that the target is stationary, i.e., $T(x,t) \equiv T(x)$. Next, we assume that the sensitivity functions are of the forms $\chi_1(v) = V_1 v$ and $\chi_2(w) = V_2 w$, where V_1 and V_2 are some positive constants. Finally, we assume that g_1 and d are sufficiently large, so that v is in quasi-equilibrium state, i.e., $v = (g_1/d)T(x)u$. It is the same for g_2 and h; therefore, $w = (g_2/h)u$. By substituting these for $\chi_1(v)$ and $\chi_2(w)$ in the first equation of (5.2), we obtain that

$$\frac{\partial u}{\partial t} = a_1 \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left[u \frac{\partial}{\partial x}\left(\frac{g_1 V_1}{d} T(x) - \frac{g_2 V_2}{h}\right) u\right].$$

Putting $a_1 = a$, $G(x) = g_2 V_2/h - (g_1 V_1/d)T(x)$, we then arrive at (7.1).

On the other hand, Iwasaki and Hatanaka have applied these analytical results to theoretical understandings evolutionary computation, which is one of the famous computational algorithms to black box function optimization problems [97].

In [33], we consider the problem (7.1) with the Neumann boundary conditions, but we present some results with the periodic conditions in this chapter.

7.1 Model Equations

We are concerned with the initial-boundary value problem for a nonlinear diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}\left[u\dfrac{\partial}{\partial x}(G(x)u)\right] & \text{in } I \times (0,\infty), \\ u(0,t) = u(1,t) \quad \text{and} \quad \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) & \text{on } (0,\infty), \\ u(x,0) = u_0(x) & \text{in } I, \end{cases} \tag{7.1}$$

in the unit open interval $I = (0, 1)$. Note that the first equation of (7.1) is rewritten as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(a + G(x)u)\frac{\partial}{\partial x}u\right] + \frac{\partial}{\partial x}[G'(x)u^2]. \tag{7.2}$$

We assume that $a > 0$ is a positive constant. We also assume that

$$G \in \mathcal{C}_P^2(\overline{I}) \tag{7.3}$$

and there exists a positive constant $c > 0$ such that

$$c \le G(x) \quad \text{in } \overline{I}. \tag{7.4}$$

The space of initial functions is set by

$$\mathcal{K} = \{u_0 \in H_P^1(I); u_0(x) > 0 \text{ in } \overline{I}\}. \tag{7.5}$$

Here, $\mathcal{C}_P^2(\overline{I}) = \{u \in \mathcal{C}^2(\overline{I});\ u(0) = u(1),\ u'(0) = u'(1) \text{ and } u''(0) = u''(1)\}$ and $H_P^1(I) = \{u \in H^1(I);\ u(0) = u(1)\}$.

In Section 7.2, a local unique solution to (7.1) is constructed for each initial value $u_0 \in \mathcal{K}$. In Section 7.3, some regularity properties of local solutions are verified, which is necessary to obtain a priori estimates. In Section 7.4, we establish a priori estimates for local solutions to obtain the global existence of solutions. Section 7.5 is devoted to investigating the stationary problem of (7.1). Finally, in Section 7.6, we show that each global solution converges to a corresponding stationary solution.

7.2 Local Solutions

Note that $H_P^1(I) \subset L_2(I)$ with dense and continuous embedding. Therefore, a triplet of spaces $H_P^1(I) \subset L_2(I) \subset H_P^1(I)'$ is constructed.

Problem (7.1) is written as the Cauchy problem for an abstract equation

$$\begin{cases} \dfrac{du}{dt} + A(u)u = F(u), \quad 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{7.6}$$

in $X = H_P^1(I)'$. Here, $A(u)$ is a linear operator defined for $u \in Z = H_P^{\varepsilon_1}(I)$, where $1/2 < \varepsilon_1 < 1$. For $u \in Z$, let us consider the sesquilinear form

$$a(u; u_1, u_2) = \int_I (a + G(x)\chi(\text{Re } u))\frac{du_1}{dx}\overline{\frac{du_2}{dx}}dx + \int_I u_1\overline{u}_2 dx$$

on $H_P^1(I)$. Here $\chi(u)$ is a smooth cutoff function such that $\chi(u) = u$ for $u \ge 0$ and $\chi(u) \equiv -\delta$ for $u \le -\delta$, $\delta > 0$ being some small positive constant such that $2\|G\|_{\mathcal{C}}\delta \le a$. Then, due to (7.3) and (7.4), the sesquilinear form $a(u; \cdot, \cdot)$ is continuous and coercive. More precisely, $a(u; \cdot, \cdot)$ satisfies that

$$|a(u; u_1, u_2)| \le M_u \|u_1\|_{H^1} \|u_2\|_{H^1}, \qquad u_1, u_2 \in H_P^1(I), \tag{7.7}$$

and

$$\text{Re } a(u; u_1, u_1) \geq \min\{a/2, 1\} \|u_1\|_{H^1}^2, \qquad u_1 \in H^1_P(I), \tag{7.8}$$

where $M_u \geq 1$ depends on $\|u\|_Z$ (note that $\|u\|_{\mathcal{C}} \leq C\|u\|_Z$ due to (2.6)) and $\|G\|_{\mathcal{C}}$. Therefore, $a(u; \cdot, \cdot)$ defines sectorial operators $A(u) : H^1_P(I) \to H^1_P(I)'$ such that

$$a(u; u_1, u_2) = \langle \mathcal{A}(u)u_1, u_2\rangle_{H^{1\prime}_P \times H^1_P}, \qquad u_1, u_2 \in H^1_P(I). \tag{7.9}$$

with angles $\omega_{A(u)} < \pi/2$. We define $A(u)$ in (7.6) as this one.

The nonlinear operator $F : W \to X$ is given by

$$F(u) = u + \frac{\partial}{\partial x}[G'(x)u^2].$$

Here, $W = H^{\varepsilon_2}_P(I)$, where $\varepsilon_1 < \varepsilon_2 < 1$.

Let $0 < R < \infty$, and let $A(u)$ be defined for $u \in K_R = \{u \in Z; \|u\|_Z < R\}$. According to Theorem 2.10, we can see from (7.7) and (7.8) that the spectrum $\sigma(A(u))$ is contained in a fixed open sectorial domain, i.e.,

$$\sigma(A(u)) \subset \Sigma_{\omega_R} = \{\lambda \in \mathbb{C}; |\arg \lambda| < \omega_R\}, \quad u \in K_R \tag{7.10}$$

with some angle $\omega_{A(u)} < \omega_R < \pi/2$, and the resolvent satisfies

$$\left\|(\lambda - A(u))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M_R}{|\lambda|}, \quad \lambda \notin \Sigma_{\omega_R}, \quad u \in K_R, \tag{7.11}$$

with a constant $M_R \geq 1$. Furthermore, the domain of $A(u)$ satisfies

$$\mathcal{D}(A(u)) \equiv H^1_P(I), \quad u \in K_R. \tag{7.12}$$

Let us set $Y = H^{\varepsilon_0}_P(I)$ with a third exponent ε_0 chosen so that $1/2 < \varepsilon_0 < \varepsilon_1$. Thanks to the assumption of $\chi(u)$, it holds that

$$\begin{aligned}
&\|\chi(\text{Re } u) - \chi(\text{Re } v)\|_{\mathcal{C}} \\
&\quad = \left\| \int_0^1 \chi'(\theta \text{Re } u + (1-\theta)\text{Re } v)d\theta \cdot (\text{Re } u - \text{Re } v) \right\|_{\mathcal{C}} \\
&\quad \leq C(\|u\|_{\mathcal{C}} + \|v\|_{\mathcal{C}}) \|u - v\|_{\mathcal{C}} \\
&\quad \leq C_R \|u - v\|_Y, \qquad\qquad u, v \in K_R.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\langle [A(u) - A(v)]u_1, u_2\rangle_{H^{1\prime}_P \times H^1_P}| \\
&= \left| \int_I G(x)[\chi(\text{Re } u) - \chi(\text{Re } v)] \frac{du_1}{dx} \overline{\frac{du_2}{dx}} dx \right| \\
&\leq \|G\|_{\mathcal{C}} \|\chi(\text{Re } u) - \chi(\text{Re } v)\|_{\mathcal{C}} \left\| \frac{du_1}{dx} \right\|_{L_2} \left\| \frac{du_2}{dx} \right\|_{L_2} \\
&\leq C_R \|G\|_{\mathcal{C}} \|u - v\|_Y \|u_1\|_{H^1} \|u_2\|_{H^1}, \qquad u_1, u_2 \in H^1_P(I), \quad u, v \in K_R.
\end{aligned}$$

This inequality implies that

$$\left\| [A(u) - A(v)]\, A(v)^{-1} \right\|_{\mathcal{L}(X)} \\ \leq C_R \|G\|_{\mathcal{C}} \|u - v\|_Y, \quad u, v \in K_R. \tag{7.13}$$

The nonlinear operator F satisfies

$$\begin{aligned} &\|F(u) - F(v)\|_X \\ &\leq \|u - v\|_X + \left\| \frac{\partial}{\partial x}[G'(x)(u + v)(u - v)] \right\|_X \\ &\leq C\,[1 + \|G\|_{H^1}(\|u\|_Z + \|v\|_Z)]\, \|u - v\|_W, \quad u, v \in W. \end{aligned} \tag{7.14}$$

By similar techniques to the proof of Theorem 3.2, we verify that, for any $3/4 < \theta \leq 1$, $\mathcal{D}(A(u)^\theta) = [H^1_P(I)', H^1_P(I)]_\theta = H^{2\theta-1}_P(I)$ with a norm equivalence (although u is not in $H^1_P(I)$ but in $Z = H^{\varepsilon_1}_P(I)$). Thus, by setting $\alpha = (1 + \varepsilon_0)/2$, $\beta = (1 + \varepsilon_1)/2$, and $\eta = (1 + \varepsilon_2)/2$, we see that, for any $u \in K_R$, $\mathcal{D}(A(u)^\alpha) = Y$, $\mathcal{D}(A(u)^\beta) = Z$, and $\mathcal{D}(A(u)^\eta) = W$ with the estimates

$$\begin{cases} \|\tilde{u}\|_Y \leq D_1 \|A(u)^\alpha \tilde{u}\|_X, & \tilde{u} \in \mathcal{D}(A(u)^\alpha), \quad u \in K_R, \\ \|\tilde{u}\|_Z \leq D_2 \left\|A(u)^\beta \tilde{u}\right\|_X, & \tilde{u} \in \mathcal{D}(A(u)^\beta), \quad u \in K_R, \\ \|\tilde{u}\|_W \leq D_3 \|A(u)^\eta \tilde{u}\|_X, & \tilde{u} \in \mathcal{D}(A(u)^\eta), \quad u \in K_R, \end{cases} \tag{7.15}$$

$D_i > 0$ $(i = 1, 2, 3)$ being some constants depending on R. The initial value $u_0 \in K_R \cap \mathcal{K}$ satisfies

$$u_0 \in \mathcal{D}(A(u_0)) = H^1_P(I). \tag{7.16}$$

The exponents satisfy the relations

$$\frac{3}{4} < \alpha < \beta < \eta < 1. \tag{7.17}$$

Theorem 7.1. *For each $u_0 \in K_R \cap \mathcal{K}$, there exists a unique local solution to* (7.6) *in the function space:*

$$\begin{cases} 0 \leq u \in \mathcal{C}([0, T_{u_0}]; H^1_P(I)) \cap \mathcal{C}^{1-\alpha}([0, T_{u_0}]; Y) \cap \mathcal{C}^1((0, T_{u_0}]; H^1_P(I)'), \\ F(u) \in \mathcal{F}^{1,\sigma}((0, T_{u_0}]; H^1_P(I)'), \end{cases} \tag{7.18}$$

with $0 < \sigma < \min\{\beta - \alpha, 1 - \eta\}$, where T_{u_0} is determined by the norm $\|u_0\|_{H^1_P}$. Furthermore, u satisfies the estimates

$$\|F(u)\|_{\mathcal{F}^{1,\sigma}} + \max_{0 \leq t \leq T_{u_0}} \|A(u(t))u(t)\|_{H^{1\prime}_P} \leq R_0 \tag{7.19}$$

with a constant R_0 determined by the norm $\|u_0\|_{H^1_P}$.

Proof. Let us apply a Theorem 4.2 to construct local solutions to (7.6). Conditions (7.10) – (7.17) imply that the conditions of Theorem 4.2 are satisfied. Therefore, for any

$u_0 \in K_R \cap \mathcal{K}$, there exists an interval $[0, T_{u_0}]$ such that (7.6) possesses a unique local solution in the function space:

$$\begin{cases} u \in \mathcal{C}([0, T_{u_0}]; H^1_P(I)) \cap \mathcal{C}^{1-\alpha}([0, T_{u_0}]; Y) \cap \mathcal{C}^1((0, T_{u_0}]; H^1_P(I)'), \\ F(u) \in \mathcal{F}^{1,\sigma}((0, T_{u_0}]; H^1_P(I)'), \end{cases} \tag{7.20}$$

with $0 < \sigma < \min\{\beta - \alpha, 1 - \eta\}$, where T_{u_0} is determined by the norm $\|u_0\|_{H^1_P}$. Furthermore, u satisfies the estimates (7.19).

Let us show the nonnegativity of local solutions of (7.6). For $u_0 \in K_R \cap \mathcal{K}$, let $u(t)$ be the local solution of (7.6) constructed above in (7.20). Firstly, since u_0 is real valued, the uniqueness of the solution yields that $u(t)$ is real valued.

Let $H(u)$ be a $\mathcal{C}^{1,1}$ cutoff function given by $H(u) = u^2/2$ for $-\infty < u < 0$ and $H(u) \equiv 0$ for $0 \le u < \infty$. Since $u \in \mathcal{C}([0, T_{u_0}]; H^1_P(I)) \cap \mathcal{C}^1((0, T_{u_0}]; H^1_P(I)')$, we see that $\psi(t) = \int_I H(u(t))dx$ is continuously differentiable with the derivative

$$\psi'(t) = \langle H'(u(t)), u'(t)\rangle_{H^1_P \times H^{1\prime}_P}, \quad 0 < t \le T_{u_0},$$

where $\langle \cdot, \cdot \rangle_{H^1_P \times H^{1\prime}_P}$ is a duality product of $\{H^1_P(I), H^1_P(I)'\}$. Therefore, we have

$$\psi'(t) = \left\langle H'(u(t)), \frac{\partial}{\partial x}\left[(a + G(x)\chi(u(t)))\frac{\partial}{\partial x}u(t)\right]\right\rangle_{H^1_P \times H^{1\prime}_P} + \left\langle H'(u(t)), \frac{\partial}{\partial x}\left[G'(x)u(t)^2\right]\right\rangle_{H^1_P \times H^{1\prime}_P}.$$

Due to the assumption of $\chi(\cdot)$, we see that

$$\left\langle H'(u(t)), \frac{\partial}{\partial x}\left[(a + G(x)\chi(u(t)))\frac{\partial}{\partial x}u(t)\right]\right\rangle_{H^1_P \times H^{1\prime}_P} \le -\frac{a}{2}\int_I \left|\frac{\partial}{\partial x}H'(u(t))\right|^2 dx$$

and

$$\left\langle H'(u(t)), \frac{\partial}{\partial x}\left[G'(x)u(t)^2\right]\right\rangle_{H^1_P \times H^{1\prime}_P} \le \frac{a}{2}\int_I \left|\frac{\partial}{\partial x}H'(u(t))\right|^2 dx + C\|H'(u(t))\|^2_{L_2}\|G\|^2_{\mathcal{C}^2}.$$

Therefore, we obtain that

$$\psi'(t) \le C\|G\|^2_{\mathcal{C}^2}\psi(t), \quad 0 < t \le T_{u_0},$$

so,

$$\psi(t) \le \psi(0)\exp\left(C\|G\|^2_{\mathcal{C}^2}t\right), \quad 0 < t \le T_{u_0}.$$

Then, $\psi(0) = 0$ implies $\psi(t) \equiv 0$. Thus, $u(t) \ge 0$ for every $0 < t \le T_{u_0}$. □

Since $u(t) \ge 0$, it holds that $\chi(\mathrm{Re}\, u(t)) = u(t)$; this then means that the local solution of (7.6) is regarded as a local solution to the original problem (7.1).

7.3 Some Regularity Properties

As will be seen in the next section, we have to obtain the a priori estimate of the form (7.27) to show the global existence. To this end, it is necessary to verify some regularity property of local solutions in the variable t. Our goal is to prove that, for $u_0 \in \mathcal{K} \cap K_R$, the local solution constructed above enjoys the regularity

$$u \in \mathcal{C}((0, T_{u_0}]; H^2_P(I)) \cap \mathcal{C}^1((0, T_{u_0}]; L_2(I)),$$

so that $\frac{d}{dt}\left\|\frac{\partial u}{\partial x}\right\|^2_{L_2}$ is well-defined.

Firstly, we prove the following lemma.

Lemma 7.1. *Let* $0 < \tau < T_{u_0}$. *Then, it holds that*

$$\sup_{\tau \le t \le T_{u_0}} \|A(t)^{1+\rho}(u(t))\|_{H^{1\prime}_P} \le C_{R_0} \tag{7.21}$$

with some exponent $0 < \rho < \sigma$.

Proof. We regards $u(t)$ as a solution of the linear problem

$$\begin{cases} \dfrac{du}{dt} + A(t)u = F(t), \quad \tau < t \le T_{u_0} \\ u(\tau) = u_\tau, \end{cases} \tag{7.22}$$

in $H^1_P(I)'$. Here, the linear operator $A(t)$ is given by $A(t) = A(u(t))$, and the function $F(t)$ is given by $F(t) = F(u(t))$. In order to apply Theorem 4.4 (with $X = H_P(I)'$) to (7.22), we verify the conditions in the theorem.

The conditions (7.10), (7.11), and (7.12) implies that (4.18), (4.19), and (4.20), respectively. In addition, (7.13) and $u \in \mathcal{C}^{1-\alpha}([0, T_{u_0}]; Y)$ imply that

$$\left\|[A(t) - A(s)]A(s)^{-1}\right\|_{\mathcal{L}(H^{1\prime}_P)} \le N_{R_0}|t - s|^{1-\alpha}, \quad 0 \le s, t \le T,$$

with $N_{R_0} > 0$, that is, (4.21) holds true with $\mu = 1 - \alpha$.

The external force F satisfies (4.22) since $F(u) \in \mathcal{F}^{1,\sigma}((0, T_{u_0}]; H^1_P(I)')$ and $\sigma < \mu$. In the meantime, for all $\tau \le t \le T_{u_0}$, $u(t) \in H^1_P(I)$ implies that $F(t) \in L_2(I) = \mathcal{D}(A(t)^{1/2})$. Therefore, for arbitrary $0 < \rho < \sigma$,

$$\begin{aligned} \|A(t)^\rho F(t)\|_{H^{1\prime}_P} = \|A(t)^{\rho-1/2}A(t)^{1/2}F(t)\|_{H^{1\prime}_P} &\le \|A(t)^{\rho-1/2}\|_{\mathcal{L}(H^{1\prime}_P)}\|A(t)^{1/2}F\|_{H^{1\prime}_P} \\ &\le C_{R_0}\|F\|_{L_2}, \end{aligned}$$

thus, we verify (4.23). Finally, note that $u_\tau \in H^1_P(I) = \mathcal{D}(A(\tau))$.

Let us apply Theorem 4.4 to (7.22). Then, we obtain from (4.24) the desired estimate (7.21). □

Due to this Lemma, we have

$$\|u(t) - u(s)\|_{H^1_P} \le C_{R_0}|t - s|^\mu \tag{7.23}$$

with $\mu = \rho/(1+\rho)$. Indeed, by applying generalized moment inequality (2.17) with $\theta_0 = 1$ and $\theta_1 = 1 + \rho$,

$$\begin{aligned}\|u(t) - u(s)\|_{H^1_P} &\le C_{R_0}\|A(t)[u(t) - u(s)]\|_{H^{1\prime}_P} \\ &\le C_{R_0}C_\rho\|A(t)^{1+\rho}[u(t) - u(s)]\|^{1/(1+\rho)}_{H^{1\prime}_P}\|u(t) - u(s)\|^{\rho/(1+\rho)}_{H^{1\prime}_P} \\ &\le C'_{R_0}C_\rho\|u(t) - u(s)\|^{\rho/(1+\rho)}_{H^{1\prime}_P}.\end{aligned}$$

In addition, since $u \in \mathcal{C}^1([\tau, T_{u_0}]; H^1_P(I)')$, we obtain the desired estimate (7.23).

Now that we can show the following Theorem.

Theorem 7.2. *Let $0 < \tau < T_{u_0}$. Then, it holds that*

$$u \in \mathcal{C}((\tau, T_{u_0}]; H^2_P(I)) \cap \mathcal{C}^1((\tau, T_{u_0}]; L_2(I)) \tag{7.24}$$

with the estimate

$$\|u(t)\|_{L_2} + (t-\tau)\left\|\frac{du}{dt}(t)\right\|_{L_2} + (t-\tau)\|u(t)\|_{H^2_P} \le C_{R_0}, \quad \tau < t \le T_{u_0}. \tag{7.25}$$

Proof. We regards $u(t)$ as a solution of the linear problem

$$\begin{cases}\dfrac{du}{dt} + \tilde{A}(t)u = \tilde{F}(t), & \tau < t \le T_{u_0} \\ u(\tau) = u_\tau, \end{cases} \tag{7.26}$$

in $L_2(I)$. Here, the linear operator $\tilde{A}(t)$ is the part of $A(t)$ in $L_2(I)$, and the function $\tilde{F}(t)$ is given by $\tilde{F}(t) = \tilde{F}(u(t))$ in $L_2(I)$. In order to apply Theorem 4.3 (with $X = L_2(I)$) to (7.26), we verify the conditions in the theorem.

It is obvious from Theorem 2.10 that (4.18) and (4.19) are fulfilled. In addition, since $(a + G(x)u) \in H^1_P(I)$, it follows from Theorem 2.11 that $\mathcal{D}(\tilde{A}(t)) \equiv H^2_P(I)$ for all $\tau \le t \le T_{u_0}$; so, (4.20) holds true.

Meanwhile, for $v \in H^2_P(I)$,

$$[\tilde{A}(t) - \tilde{A}(s)]v = -\frac{d}{dx}\left[G(x)[u(t) - u(s)]\frac{dv}{dx}\right].$$

Therefore, due to (7.23),

$$\|[\tilde{A}(t) - \tilde{A}(s)]v\|_{L_2} \le C\|u(t) - u(s)\|_{H^1_P}\|v\|_{H^2_P} \le C|t-s|^\mu\|v\|_{H^2_P},$$

that is, (4.21) holds true.

Furthermore, due to (7.23) again,

$$\begin{aligned}&\|F(t) - F(s)\|_{L_2} \\ &\le \|u(t) - u(s)\|_{L_2} + \|\frac{d}{dx}[G'[u(t) + u(s)][u(t) - u(s)]]\|_{L_2} \\ &\le C_{R_0}\|u(t) - u(s)\|_{H^1_P} \le C_{u_0}|t-s|^\mu,\end{aligned}$$

thus, $F(t) \in \mathcal{C}^\mu([\tau, T_{u_0}]; L_2(I))$; this implies (4.22) (with $\sigma = \mu$) due to (2.1).

Let us apply Theorem 4.3 to (7.26). Then, we conclude the desired temporal regularity (7.24) with the estimate (7.25). □

7.4 Global Solution

For $u_0 \in \mathcal{K}$, let u denote any local solution of (7.6) on $[0, T_u]$ in the function space:

$$0 \le u \in \mathcal{C}((0,T_u]; H_P^2(I)) \cap \mathcal{C}([0,T_u]; H_P^1(I)) \cap \mathcal{C}^1((0,T_u]; L_2(I)). \tag{7.27}$$

We then show the following a priori estimates.

Proposition 7.1. *There exists a continuous increasing function $p(\cdot)$ such that, for any local solution u of* (7.6) *in* (7.27) *with initial value $u_0 \in \mathcal{K}$, it holds that*

$$\|u(t)\|_{H_P^1} \le p(\|u_0\|_{H_P^1}), \quad 0 \le t \le T_u. \tag{7.28}$$

Proof. In the proof, the notations C and $p(\cdot)$ stand for some constants and some continuous increasing functions, respectively, which are determined by the initial constants and $\|G\|_{\mathcal{C}^2}$ (see (7.3)) and by I in a specific way in each occurrence. In the following, we divide the proof into four steps.

Step 1. Let us integrate the first equation of (7.1) in I. Then, obviously, $\frac{d}{dt}\|u\|_{L_1} = 0$, i.e.,

$$\|u(t)\|_{L_1} = \|u_0\|_{L_1}, \quad 0 \le t \le T_u. \tag{7.29}$$

Step 2. Multiply the equation (7.2) by $2u$ and integrate the product in I. Then,

$$\frac{d}{dt}\|u\|_{L_2}^2 + 2\int_I (a + G(x)u)\left|\frac{\partial u}{\partial x}\right|^2 dx = 2\int_I u\frac{\partial}{\partial x}[G'(x)u^2]dx.$$

Here,

$$\begin{aligned}
2\int_I u\frac{\partial}{\partial x}[G'(x)u^2]dx &= -\frac{2}{3}\int_I \left[\frac{\partial}{\partial x}u^3\right] G'(x)dx \\
&= \frac{2}{3}\int_I u^3 G''(x)dx \\
&\le \zeta \|u\|_{L_4}^4 + C_\zeta \|G''\|_{L_4}^4 \\
&\le \zeta \|u\|_{H^1}^2 \|u\|_{L_1}^2 + C_\zeta \\
&\le \zeta \|u\|_{H^1}^2 + C_\zeta p(\|u_0\|_{L_1}),
\end{aligned}$$

with any $\zeta > 0$. Therefore, we get

$$\frac{d}{dt}\|u\|_{L_2}^2 + \|u\|_{L_2}^2 + a\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 \le p(\|u_0\|_{L_1}). \tag{7.30}$$

So, we obtain the following inequality

$$\|u(t)\|_{L_2}^2 \le e^{-t}\|u_0\|_{L_2}^2 + p(\|u_0\|_{L_1}), \quad 0 \le t \le T_u. \tag{7.31}$$

Step 3. In this step, we shall use the notation

$$P(u_0) = p(\|u_0\|_{L_2}).$$

Multiply the equation (7.2) by $4u^3$ and integrate the product in I. Then,

$$\frac{d}{dt}\|u\|_{L_4}^4 + 12\int_I (a+G(x)u)u^2\left|\frac{\partial u}{\partial x}\right|^2 dx = \frac{12}{5}\int_I u^5 G''(x)dx.$$

Here,

$$12\int_I (a+G(x)u)u^2\left|\frac{\partial u}{\partial x}\right|^2 dx = 3a\left\|\frac{\partial}{\partial x}u^2\right\|_{L_2}^2 + \frac{48}{25}\int_I G(x)\left|\frac{\partial}{\partial x}u^{\frac{5}{2}}\right|^2 dx,$$

and

$$\begin{aligned}\frac{12}{5}\int_I u^5 G''(x)dx &\le C\|u^5\|_{L_1}\|G''\|_{\mathcal{C}} \le C\left\|u^{\frac{5}{2}}\right\|_{L_2}^2\\ &\le \zeta_1\left\|\frac{\partial}{\partial x}u^{\frac{5}{2}}\right\|_{L_2}^2 + C_{\zeta_1}\left\|u^{\frac{5}{2}}\right\|_{L_1}^2\\ &\le \zeta_1\left\|\frac{\partial}{\partial x}u^{\frac{5}{2}}\right\|_{L_2}^2 + C_{\zeta_1}\|u\|_{L_2}^3\|u\|_{L_4}^2\\ &\le \zeta_1\left\|\frac{\partial}{\partial x}u^{\frac{5}{2}}\right\|_{L_2}^2 + C_{\zeta_1}P(u_0)\left\|u^2\right\|_{L_2}\\ &\le \zeta_1\left(\left\|\frac{\partial}{\partial x}u^{\frac{5}{2}}\right\|_{L_2}^2 + \left\|\frac{\partial}{\partial x}u^2\right\|_{L_2}^2\right) + C_{\zeta_1}P(u_0),\end{aligned}$$

with any $\zeta_1 > 0$. Therefore, we obtain the following differential inequality

$$\frac{d}{dt}\|u\|_{L_4}^4 + \|u\|_{L_4}^4 + a\left\|\frac{\partial}{\partial x}u^2\right\|_{L_2}^2 \le P(u_0). \tag{7.32}$$

Step 4. Multiply the equation (7.2) by $2\frac{\partial^2 u}{\partial x^2}$ and integrate the product in I. Then,

$$\begin{aligned}&\frac{d}{dt}\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 + 2\int_I (a+Gu)\left|\frac{\partial^2 u}{\partial x^2}\right|^2 dx\\ &= -2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}\frac{\partial}{\partial x}[Gu]dx - 2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial}{\partial x}[G'u^2]dx.\end{aligned} \tag{7.33}$$

First, by repeating integration by parts, we obtain

$$\begin{aligned}&-2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}\frac{\partial}{\partial x}[Gu]dx\\ &= -\frac{1}{3}\int_I G''\left[\frac{\partial}{\partial x}u^2\right]\frac{\partial u}{\partial x}dx - \frac{5}{3}\int_I G'\left[\frac{\partial}{\partial x}u^2\right]\frac{\partial^2 u}{\partial x^2}dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}&-2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial}{\partial x}[G'u^2]dx\\ &= -2\int_I G''u^2\frac{\partial^2 u}{\partial x^2}dx - 2\int_I G'\left[\frac{\partial}{\partial x}u^2\right]\frac{\partial^2 u}{\partial x^2}dx.\end{aligned}$$

Therefore,

$$\begin{aligned}
&-2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}\frac{\partial}{\partial x}[Gu]dx - 2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial}{\partial x}[G'u^2]dx \\
&= -\frac{1}{3}\int_I G''\left[\frac{\partial}{\partial x}u^2\right]\frac{\partial u}{\partial x}dx - \frac{11}{3}\int_I G'\left[\frac{\partial}{\partial x}u^2\right]\frac{\partial^2 u}{\partial x^2}dx - 2\int_I G''u^2\frac{\partial^2 u}{\partial x^2}dx.
\end{aligned}$$

So, we obtain by Young's inequality,

$$\begin{aligned}
&-2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}\frac{\partial}{\partial x}[Gu]dx - 2\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial}{\partial x}[G'u^2]dx \\
&\leq a\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2}^2 + C\left[\left\|\frac{\partial}{\partial x}u^2\right\|_{L_2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 + \|u\|_{L_4}^4\right].
\end{aligned}$$

From this inequality and (7.33), we have

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 + a\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2}^2 \leq C\left[\left\|\frac{\partial}{\partial x}u^2\right\|_{L_2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 + \|u\|_{L_4}^4\right].$$

Multiply a parameter $\xi > 0$ to the above differential inequality and add the product to (7.30) and (7.32). Then,

$$\begin{aligned}
&\frac{d}{dt}\left[\|u\|_{L_2}^2 + \|u\|_{L_4}^4 + \xi\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2\right] + \|u\|_{L_2}^2 + (1-\xi C)\|u\|_{L_4}^4 + (a-\xi C)\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2 \\
&+ (a-\xi C)\left\|\frac{\partial}{\partial x}u^2\right\|_{L_2}^2 \leq 2P(u_0).
\end{aligned}$$

Therefore, by choosing $\xi > 0$ such that $1 - \xi C > 0$ and $a - \xi C > 0$, we obtain

$$\frac{d}{dt}\left[\|u\|_{L_2}^2 + \|u\|_{L_4}^4 + \xi\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2\right] + \delta\left[\|u\|_{L_2}^2 + \|u\|_{L_4}^4 + \xi\left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2\right] \leq 2P(u_0)$$

with some constant $\delta > 0$. So, we conclude that

$$\begin{aligned}
&\left[\|u(t)\|_{L_2}^2 + \|u(t)\|_{L_4}^4 + \xi\left\|\frac{\partial u}{\partial x}(t)\right\|_{L_2}^2\right] \\
&\leq e^{-\delta t}\left[\|u_0\|_{L_2}^2 + \|u_0\|_{L_4}^4 + \xi\left\|\frac{\partial u_0}{\partial x}\right\|_{L_2}^2\right] + 2P(u_0), \quad 0 \leq t \leq T_u.
\end{aligned}$$

We have in this way established the desired a priori estimate (7.28). □

Thanks to Proposition 7.1, we conclude the global existence of solutions. The proof is quite similar to the argument in [37, Chapter 15, Section 4.1], we omit it here. Therefore, we know that, for any initial value $u_0 \in \mathcal{K}$, there exists a unique global solution u to (7.6) in the function space:

$$0 \leq u \in \mathcal{C}((0,\infty); H_P^2(I)) \cap \mathcal{C}([0,\infty); H_P^1(I)) \cap \mathcal{C}^1((0,\infty); L_2(I)). \tag{7.34}$$

It is obvious that

$$\|u(t)\|_{H_P^1} \leq p(\|u_0\|_{H_P^1}), \quad 0 \leq t < \infty. \tag{7.35}$$

Furthermore, we show the following strong estimate. By using Theorem

Proposition 7.2. *Let $u_0 \in \mathcal{K}$ be initial value. Then, the global solution $u(t) = u(t; u_0)$ of* (7.6) *in* (7.27) *holds that*

$$\|u(t)\|_{H^2_P} \le (1+t)p_1(\|u_0\|_{H^1_P}), \quad 0 < t < \infty, \tag{7.36}$$

with some increasing continuous function $p_1(\cdot)$.

Proof. Let $0 \le \tau < \infty$. Apply Theorem 7.1 and Theorem 7.2 with initial value $u_\tau = u(\tau; u_0)$ to conclude that there exists $T > 0$ (depending on $\|u(\tau)\|_{H^1_P}$ and hence on $\|u_0\|_{H^1_P}$ due to (7.35)) such that

$$\|u(t)\|_{H^2_P} \le (t-\tau)^{-1} p_0(\|u_0\|_{H^2_P}), \quad \tau < t \le \tau + T,$$

with some increasing continuous function $p_0(\cdot)$. First, applying this with $\tau = 0$, we see that

$$\|u(t)\|_{H^2_P} \le t^{-1} p_0(\|u_0\|_{H^2_P}), \quad 0 < t \le T.$$

Second, taking $\tau = t - T$, it follows by (7.35) that

$$\|u(t)\|_{H^2_P} \le T^{-1} p_0(\|u(t-T, u_0)\|_{H^1_P}) \le T^{-1} p_0(p(\|u_0\|_{H^1_P})), \quad T < t < \infty.$$

Combining these estimates, we conclude the desired estimate (7.36). □

In the meantime, we show the positivity of global solutions.

Theorem 7.3. *Let $u(t)$ be the global solution in the function space* (7.34) *with initial value $u_0 \in \mathcal{K}$. Then, it holds that $u(x,t) > 0$ for every $(x,t) \in \overline{I} \times [0, \infty)$.*

Proof. Put $\delta = \min_{x_i \in \overline{I}} u_0(x) > 0$. In addition, for arbitrarily fixed $\tau \in (0, \infty)$, put $C_\tau = \max_{(x,t) \in \overline{I_i} \times [0,\tau]} [-G''(x) u(x,t)] < \infty$.

Regard the solution $u(x,t)$ as a solution to the linear diffusion equation

$$\frac{\partial u}{\partial t} = p \frac{\partial^2 u}{\partial x^2} + q \frac{\partial u}{\partial x} + ru \quad \text{in} \quad I \times (0, \tau),$$

where $p = p(x,t) = a + G(x)u(x,t)$, $q = q(x,t) = 3G'(x)u(x,t) + G(x)\frac{\partial u}{\partial x}(x,t)$, and $r = r(x,t) = G''(x)u(x,t)$.

Now, introduce a cutoff function $H(\xi)$ such that

$$H(\xi) = \frac{1}{2}\xi^2 \text{ for } -\infty < \xi < 0 \quad \text{and} \quad H(\xi) = 0 \text{ for } 0 \le \xi < \infty,$$

and the function

$$\varphi(t) = \int_I H(u(x,t) - \delta e^{-C_\tau t}) dx, \quad 0 \le t \le \tau.$$

Then, we know that

$$\begin{aligned}
\varphi'(t) &= \int_I H'(u - \delta e^{-C_\tau t})(\frac{\partial u}{\partial t} + \delta C_\tau e^{-C_\tau t}) dx \\
&= \int_I H'(u - \delta e^{-C_\tau t})(p \frac{\partial^2 u}{\partial x^2} + q \frac{\partial u}{\partial x} + ru + \delta C_\tau e^{-C_\tau t}) dx.
\end{aligned}$$

Firstly,

$$\int_I H'(u-\delta e^{-C_\tau t})p\frac{\partial^2 u}{\partial x^2}dx$$
$$= -\int_I H''(u-\delta e^{-C_\tau t})\left|\frac{\partial u}{\partial x}\right|^2 p\ dx - \int_I H'(u-\delta e^{-C_\tau t})\frac{\partial p}{\partial x}\frac{\partial u}{\partial x}dx.$$

Here,

$$-\int_I H''(u-\delta e^{-C_\tau t})\left|\frac{\partial u}{\partial x}\right|^2 p\ dx \le -a\int_I H''(u-\delta e^{-C_\tau t})\left|\frac{\partial u}{\partial x}\right|^2 dx \le 0.$$

Secondly, let us estimate the following quantity

$$\int_I H'(u-\delta e^{-C_\tau t})(q-\frac{\partial p}{\partial x})\frac{\partial u}{\partial x}dx.$$

Note that $q-\frac{\partial p}{\partial x}=2G'(x)u$, and put $s=s(x,t)=2G'(x)u(x,t)$. Then, we have

$$\int_I H'(u-\delta e^{-C_\tau t})s\frac{\partial u}{\partial x}dx$$
$$= \int_I H'(u-\delta e^{-C_\tau t})s\frac{\partial}{\partial x}[u-\delta e^{-C_\tau t}]dx$$
$$= \int_I [H''(u-\delta e^{-C_\tau t})\frac{\partial u}{\partial x}sH'(u-\delta e^{-C_\tau t})\frac{\partial s}{\partial x}][u-\delta e^{-C_\tau t}]dx.$$

Therefore, since $\max_{t\in[0,\tau]}\|s(\cdot,t)\|_{\mathcal{C}^1(\overline{I})}<\infty$, we know that

$$\int_I H'(u-\delta e^{-C_\tau t})s\frac{\partial u}{\partial x}dx \le \frac{a}{2}\int_I H''(u-\delta e^{-C_\tau t})\left|\frac{\partial u}{\partial x}\right|^2 dx$$
$$+C'_\tau\int_I [H'(u-\delta e^{-C_\tau t})(u-\delta e^{-C_\tau t})+H''(u-\delta e^{-C_\tau t})(u-\delta e^{-C_\tau t})^2]dx$$
$$=\frac{a}{2}\int_I H''(u-\delta e^{-C_\tau t})\left|\frac{\partial u}{\partial x}\right|^2 dx + 4C'_\tau\varphi(t)$$

with some constant $C'_\tau>0$. Here, note that $H''(\xi)\xi^2=H'(\xi)\xi=2H(\xi)$ for any $\xi\in\mathbb{R}$.

Finally, it follows from $r+C_\tau\ge 0$ and $u\ge 0$ that

$$\int_I H'(u-\delta e^{-C_\tau t})(ru+\delta C_\tau e^{-C_\tau t})dx$$
$$=\int_I H'(u-\delta e^{-C_\tau t})(r+C_\tau)u\ dx - C_\tau\int_I H'(u-\delta e^{-C_\tau t})(u-\delta e^{-C_\tau t})dx \le 0.$$

Combining above estimates, we obtain that $\varphi(t)\le 4C'_\tau\varphi'(t)$, so that $\varphi(t)\le\varphi(0)e^{4C'_\tau t}$ for $0\le t\le\tau$. Since $\varphi(0)=0$, it follows that $\varphi(t)\equiv 0$ for $0\le t\le\tau$; consequently, we conclude that $u(x,t)\ge\delta e^{-C_\tau t}$ for $(x,t)\in\overline{I}\times[0,\tau]$. □

On the other hand, it follows from (7.29) that the quantity $\int_I u_0(x)dx$ is conserved for every time. Therefore, it is convenient to introduce the following initial value space

$$\mathcal{K}_l = \{u_0 \in \mathcal{K}; \int_I u_0(x)dx = l\}$$

for each $l > 0$. From the above results, we obtain the following theorem.

Theorem 7.4. *Let $0 < l < \infty$. Then, for any initial value $u_0 \in \mathcal{K}_l$, there exists a unique global solution u of* (7.1) *in the function space:*

$$0 < u \in \mathcal{C}((0,\infty); H_P^2(I)) \cap \mathcal{C}([0,\infty); H_P^1(I)) \cap \mathcal{C}^1((0,\infty); L_2(I)) \tag{7.37}$$

with

$$\int_I u(x,t)dx \equiv l \quad \textit{for all} \quad t \in [0,\infty).$$

7.5 Stationary Problem

In this section, we investigate the stationary problem of (7.1). As shown in Section 7.4, global solutions of (7.1) with initial value $u_0 \in \mathcal{K}_l$ conserve the quantity $\int_I u(t)dx \equiv \int_I u_0(x)dx$. Therefore, it is important to consider the following stationary problem for each fixed $0 < l < \infty$:

$$\begin{cases} a\dfrac{d^2\overline{u}_l}{dx^2} + \dfrac{d}{dx}\left[\overline{u}_l \dfrac{d}{dx}(G(x)\overline{u}_l)\right] = 0 & \text{in} \quad I, \\ \overline{u}_l(0) = \overline{u}_l(1) \quad \text{and} \quad \overline{u}_l'(0) = \overline{u}_l'(1), & \\ \displaystyle\int_I \overline{u}_l(x)dx = l, & \\ \overline{u}_l(x) \geq 0 & \text{in} \quad \overline{I}. \end{cases} \tag{7.38}$$

Firstly, we show the following proposition.

Proposition 7.3. *Assume that $\overline{u}_l \in H_P^2(I)$ is a solution of* (7.38)*. Then, $\overline{u}_l \in \mathcal{C}_P^2(\overline{I})$ and it holds that $\overline{u}_l(x) > 0$ for $x \in \overline{I}$.*

Proof. Put $p = p(x) = a + G(x)\overline{u}_l(x)$, $q = q(x) = 3G'(x)\overline{u}_l(x) + G(x)\overline{u}_l'(x)$, and $r = r(x) = G''(x)\overline{u}_l(x)$. Then, $\overline{u}_l'' = -(q/p)\overline{u}_l' - (r/p)\overline{u}_l \in \mathcal{C}_P(\overline{I})$, so $\overline{u}_l \in \mathcal{C}_P^2(\overline{I})$.

Let us show the positivity of $\overline{u}_l$ by contradiction. Assume that there exists $x_0 \in \overline{I}$ such that $\overline{u}_l(x_0) = 0$. Then, $\overline{u}_l'(x_0) = 0$ due to the nonnegativity of u_l. Consider the following Cauchy problem:

$$\begin{cases} p\overline{u}_l'' + q\overline{u}_l' + r\overline{u}_l = 0 & \text{in} \quad \overline{I}, \\ \overline{u}_l(x_0) = 0, \quad \overline{u}_l'(x_0) = 0. \end{cases}$$

By the classical results for ordinary differential equations, the solution $\overline{u}_l$ is written in the form $\overline{u}_l = C_1\overline{u}_l^{(1)} + C_2\overline{u}_l^{(2)}$ with linear independent solutions $\overline{u}_l^{(1)}$, $\overline{u}_l^{(2)}$ and some constants C_1, $C_2 \in \mathbb{R}$. Then, considering the Wronskian of $\overline{u}_l^{(1)}$ and $\overline{u}_l^{(2)}$ at x_0, we know that $C_1 = C_2 = 0$. Therefore, $\overline{u}_l \equiv 0$ in $\overline{I}$, which contradicts $\int_I \overline{u}_l \, dx = l > 0$. □

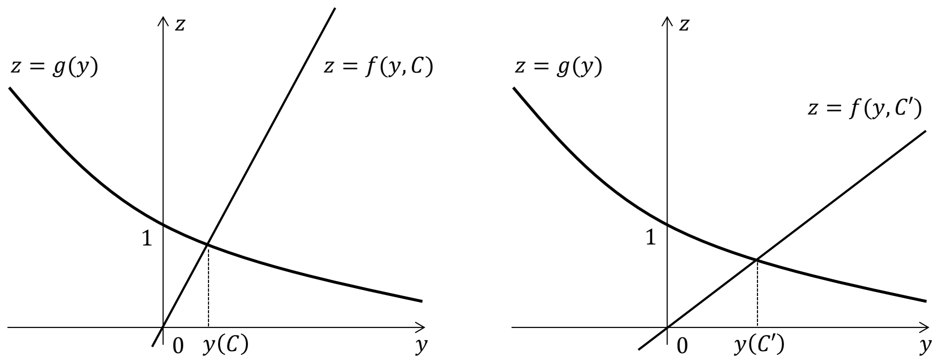

Fig. 7.1: Graphs of $f(y, C)$ and $g(y)$.

Furthermore, we can show the uniqueness of solutions to (7.38).

Theorem 7.5. *For each $l > 0$, the stationary problem* (7.38) *possesses a unique solution $\overline{u}_l(x)$. Moreover, $\overline{u}_l(x)$ is characterized by the functional equation*

$$\overline{u}_l(x) = \exp\left(\frac{C_l - G(x)\overline{u}_l(x)}{a}\right) \quad \text{in} \quad \overline{I},$$

where C_l is some constant determined by l.

The proof of Theorem 7.5 relies on the following two lemmas (see Fig. 7.1).

Lemma 7.2. *Let $x_0 \in \overline{I}$ be fixed. Then, for each $C \in \mathbb{R}$, a transcendental equation with respect to y :*

$$y = \exp\left(\frac{C - G(x_0)y}{a}\right) \tag{7.39}$$

possesses a unique positive solution.

Proof. Let us put $f(y, C) = y \exp(-C/a)$ and $g(y) = \exp(-G(x_0)y/a)$. Since $a > 0$ and $G(x_0) > 0$, we can observe that there exists a unique $y(C) > 0$ satisfying $f(y(C), C) = g(y(C))$. This $y(C)$ is the solution of (7.39). □

Lemma 7.3. *Let $x_0 \in \overline{I}$ be fixed. Then, the mapping*

$$y : C \in \mathbb{R} \mapsto y(C) \in (0, \infty),$$

where $y(C)$ is the solution of (7.39) *with C, is a strictly increasing continuous function satisfying*

$$y(-\infty) = 0 \quad \text{and} \quad y(\infty) = \infty.$$

Proof. When $-\infty < C < C' < \infty$, we obviously see that $f(y, C) > f(y, C')$ for all $y > 0$. Since $f(y, C)$ and $g(y)$ are continuous functions with respect to y, we observe that $y(C) < y(C')$ and $y(C) \to y(C')$ as $C \to C'$. Furthermore, $\exp -(C/a)$, i.e., the slope of $f(y, C)$, converges to ∞ (resp. 0) as $C \to -\infty$ (resp. $C \to \infty$). Thus, $y(-\infty) = 0$ and $y(\infty) = \infty$. □

Proof of Theorem 7.5. Due to the positivity of $\overline{u}_l$, the first equation of (7.38) is written as

$$\frac{d}{dx}\left[\overline{u}_l \frac{d}{dx}\left(a \log \overline{u}_l + G(x)\overline{u}_l\right)\right] = 0 \quad \text{in} \quad I.$$

Multiply this equation by $(a \log \overline{u}_l + G(x)\overline{u}_l)$ and integrate the product in I. Then, we obtain that

$$\int_I \overline{u}_l \left|\frac{d}{dx}\left(a \log \overline{u}_l + G(x)\overline{u}_l\right)\right|^2 dx = 0.$$

Therefore, $\overline{u}_l$ satisfies

$$a \log \overline{u}_l(x) + G(x)\overline{u}_l(x) = C \quad \text{in} \quad \overline{I}$$

with some constant $C \in \mathbb{R}$, i.e.,

$$\overline{u}_l(x) = \exp\left(\frac{C - G(x)\overline{u}_l(x)}{a}\right) \quad \text{in} \quad \overline{I}. \tag{7.40}$$

On account of Lemma 7.2, we verify the existence and uniqueness of $\overline{u}_l(x)$ satisfying (7.40). Furthermore, due to Lemma 7.3, the condition $\int_I \overline{u}_l(x)dx = l$ determines the constant C uniquely, and $\overline{u}_l$ with the C is the very solution of (7.38). □

7.6 Convergence to a Stationary Solution

In what follows, let $u_0 \in \mathcal{K}_l$ be arbitrarily fixed and let $u(t)$ denote the global solution of (7.1) in the function space (7.37) with initial value u_0. Now, we want to show the convergence of $u(t)$ to the stationary solution $\overline{u}_l$ obtained in Section 7.5. In is enough to consider the solution $u(t)$ after a fixed time, so we assume from (7.36) and Theorem 7.3 that

$$\min_{x \in \overline{I}} u(x,t) > 0 \text{ for every } t \in [0,\infty), \tag{7.41}$$

and

$$\|u(t)\|_{H^2_P} \leq R \text{ for every } t \in [0,\infty), \tag{7.42}$$

where $R = 2p_1(\|u_0\|_{H^1_P})$. Furthermore,

$$\int_I u(x,t)dx \equiv l, \quad \text{for all } t \geq 0. \tag{7.43}$$

7.6.1 Lyapunov function

In this subsection, let us construct a Lyapunov function for (7.1). As u is positive due to (7.41), the first equation of (7.1) is written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[u \frac{\partial}{\partial x}\left(a \log u + G(x)u\right)\right].$$

Multiply this equation by $(a \log u + G(x)u)$ and integrate the product in I. Then,

$$\frac{d}{dt}\int_I \left[a\left(u \log u - u\right) + \frac{G(x)}{2}u^2\right] dx = -\int_I u \left|\frac{\partial}{\partial x}\left(a \log u + G(x)u\right)\right|^2 dx. \tag{7.44}$$

Let $0 < s < t < \infty$. Integrating this inequality in $[s,t]$, we obtain that

$$\left[\int_I \left[a\left(u(\tau)\log u(\tau) - u(\tau)\right) + \frac{G(x)}{2}u(\tau)^2\right] dx\right]_{\tau=s}^{\tau=t} \leq 0.$$

If we set

$$\Phi(u) = \int_I \left[a\left(u\log u - u\right) + \frac{G(x)}{2}u^2\right] dx,$$

then $\Phi(u(t)) \leq \Phi(u(s))$. This means that Φ is a Lyapunov function for (7.1). It is easy to see that $\Phi(u(t)) \geq -C_R$ for all $0 \leq t < \infty$ with some constant $C_R \geq 0$ depending on R.

Let us show the following proposition.

Proposition 7.4. *If $\frac{d}{dt}\Phi(u(t))\,|_{t=\bar{t}} = 0$ at some time $\bar{t} \leq 0$, then $u(\bar{t})$ is a stationary solution of* (7.1)*. Consequently, it follows from Theorem 7.5 and* (7.43) *that $u(\bar{t}) = \overline{u}_l$, where $\overline{u}_l$ is the stationary solution of* (7.1) *obtained in Section 7.5.*

Proof. Put $\overline{u} = u(\bar{t}) > 0$. It follows from (7.44) that

$$a\log\overline{u} + G(x)\overline{u} = C$$

with some constant $C \in \mathbb{R}$. By similar proof of Theorem 7.5, we conclude that $\overline{u}$ is a stationary solution of (7.1). □

7.6.2 ω-limit set

We consider the ω-limit set defined by

$$\omega(u_0) = \{\overline{u} \in L_2(I); \exists t_n \nearrow \infty \text{ s.t. } u(t_n) \to \overline{u} \text{ in } L_2(I)\}. \tag{7.45}$$

Since the closed ball $\overline{B}^{H_P^2}(0;R_0)$ of $H_P^2(I)$ is a relatively compact set of $L_2(I)$ due to the compact embeddings (2.4), it is observed from (7.42) that $\omega(u_0) \neq \emptyset$. Furthermore, since $\|u(t)\|_{H_P^1} \leq C\|u(t)\|_{L_2}^{\frac{1}{2}}\|u(t)\|_{H_P^2}^{\frac{1}{2}} \leq CR^{\frac{1}{2}}\|u(t)\|_{L_2}^{\frac{1}{2}}$, we know that

$$\overline{u} \in \omega(u_0) \text{ if and only if } \exists t_n \nearrow \infty \text{ such that } u(t_n) \to \overline{u} \text{ in } H_P^1(I). \tag{7.46}$$

It is clear that $\inf_{0\leq t<\infty}\Phi(u(t)) > -\infty$. Meanwhile, $\Phi(u(t_n)) \to \Phi(\overline{u})$ as $n \to \infty$. Therefore, it follows that

$$\lim_{t\to\infty}\Phi(u(t)) = \Phi(\overline{u}) \quad \text{for any } \overline{u} \in \omega(u_0). \tag{7.47}$$

From these results, we obtain the following theorem.

Theorem 7.6. *The ω-limit set $\omega(u_0)$ contains $\overline{u}_l$, where $\overline{u}_l$ is the stationary solution of* (7.1) *obtained in Section 7.5.*

Proof. Since $\Phi(u(\cdot)) \in \mathcal{C}^1((0,\infty);\mathbb{R})$ and $\lim_{t\to\infty}\Phi(u(t)) = \text{const.}$, there exists some increasing time sequence $t_n \nearrow \infty$ such that $\frac{d}{dt}\Phi(u(t))\,|_{t=t_n} \to 0$ as $n \to \infty$. Therefore, it is observe from (7.44) that, as $t_n \to \infty$,

$$\sqrt{u(t_n)}\frac{\partial}{\partial x}[a\log u(t_n) + G(x)u(t_n)] \to 0 \quad \text{in} \quad L_2(I). \tag{7.48}$$

Note that (7.48) implies that

$$a\frac{\partial^2}{\partial x^2}u(t_n) + \frac{\partial}{\partial x}[u(t_n)\frac{\partial}{\partial x}[G(x)u]] \to 0 \quad \text{in} \quad H^1_P(I)' \tag{7.49}$$

since, for $v \in H^1_P(I)$,

$$\begin{aligned}
&\left|\left\langle a\frac{\partial^2}{\partial x^2}u(t_n) + \frac{\partial}{\partial x}[u(t_n)\frac{\partial}{\partial x}[G(x)u]], v\right\rangle_{H^{1\prime}_P\times H^1_P}\right| \\
&\quad = \left|(\sqrt{u(t_n)}\frac{\partial}{\partial x}[a\log u(t_n) + G(x)u], \sqrt{u(t_n)}v')_{L_2}\right| \\
&\quad \le C_R\|\sqrt{u(t_n)}\frac{\partial}{\partial x}[a\log u(t_n) + G(x)u]\|_{L_2}\|v\|_{H^1_P}
\end{aligned}$$

due to (7.42).

On the other hand, since $\overline{B}^{H^2_P}(0;R_0)$ is a relatively compact set of $H^1_P(I)$ and $\overline{B}^{H^2_P}(0;R_0)$ is sequentially weakly closed in $H^2_P(I)$, there exists some subsequence $t_{n_k} \to \infty$ such that $u(t_{n_k})$ converges to a limit $\overline{u} \in H^2_P(I)$ in $H^1_P(I)$ and weakly in $H^2_P(I)$. Then, we know from (7.49) that $\overline{u}'' - [\overline{u}[G(x)\overline{u}]']' = 0$ in $H^1_P(I)'$. Consequently, $\overline{u}$ is a stationary solution of (7.1) and $\overline{u} \in \omega(u_0)$ by definition. Therefore, Theorem 7.5 implies that $\overline{u}_l \in \omega(u_0)$. □

7.6.3 Asymptotic convergence of $u(t)$ to $\overline{u}_l$

Finally, we show the convergence of $u(t)$ to $\overline{u}_l$, that is, $\omega(u_0) = \{\overline{u}_l\}$.

Theorem 7.7. *Let $0 < l < \infty$. Let $u_0 \in \mathcal{K}_l$ and let $u(t)$ denote the global solution of* (7.1) *in the function space* (7.37) *with initial value u_0. Then, $u(t)$ converges to $\overline{u}_l$ in $H^1_P(I)$, where $\overline{u}_l$ is the stationary solution of* (7.1) *obtained in Section 7.5.*

To show the theorem, we use the following lemma.

Lemma 7.4. *$\omega(u_0)$ is a connected set with respect to $H^1_P(I)$ norm.*

The proof is quite similar to that of [37, Theorem 6.1], we omit it here.

Proof of Theorem 7.7. We already know from Theorem 7.6 that $\overline{u}_l \in \omega(u_0)$. Assume that $\omega(u_0) \neq \{\overline{u}_l\}$. Then, from Lemma 7.4 and the continuous embedding $H^1_P(I) \subset \mathcal{C}(\overline{I})$, we can take $\overline{u} \in \omega(u_0)$ such that $\overline{u} \neq \overline{u}_l$ and $\overline{u} > 0$. Consider the global solution $u(t;\overline{u})$ of (7.1) with initial value $\overline{u}$. Then, since $\Phi(u(t;\overline{u})) \equiv \text{const.}$ for all $0 \le t < \infty$, it holds that $\frac{d}{dt}\Phi(u(t;\overline{u}))\,|_{t=0} = 0$. So, Proposition 7.4 implies that $u(0;\overline{u}) = \overline{u} = \overline{u}_l$, which is a contraction. □

Chapter 8

Laplace Reaction-Diffusion Equations

In this chapter, we study the initial-boundary value problem for a Laplace reaction-diffusion equation of the form (8.1).

First, we construct a unique local solution for (8.1). We will regard the equation as a degenerate evolution equation of parabolic type whose linear problems have been systematically studied by the monograph [59]. Use of the multivalued linear operators enable us to rewrite the degenerate equation into a multivalued evolution equation but of nondegenerate form. The reduced multivalued evolution equation can then be solved locally by analogous techniques to the usual (single valued) evolution equations. Those have been described in Section 4.4.

Second, we show that any bounded global solution to (8.1) if it exists necessarily converges to a stationary solution of (8.1) as t tends to infinity under the assumption that $f(u)$ is analytic for u which varies in a neighborhood of the ω-limit set of the global solution. The reduction of (8.1) into a degenerate evolution equation enables us also to use the theory of infinite-dimensional Łojasiewicz-Simon gradient inequality which was developed by Chill [26], Chill-Haraux-Jendoubi [98], Haraux-Jendoubi [99], Jendoubi [13] and others.

When $m(x) \equiv 1$, Matano [100] first established in one-dimensional case (i.e., $\Omega \subset \mathbb{R}$) such asymptotic convergence without assuming the analyticity of $f(u)$. For higher dimensional cases, the papers [98, 13] suggested that the infinite-dimensional Łojasiewicz-Simon gradient inequality can derive the asymptotic convergence of the global solutions. However, in higher dimensional cases, the analyticity of $f(u)$ does not directly imply that of the nonlinear operator $u \mapsto f(u)$ acting from $H^1(\Omega)$ into $H^{-1}(\Omega)$ because of $H^1(\Omega) \not\subset \mathcal{C}(\overline{\Omega})$. So, its application is not straightforward and some devices like in Section 8.6 seem to be necessary.

The following results are obtained in [34].

8.1 Model Equations

We study the initial-boundary value problem for a Laplace reaction-diffusion equation

$$\begin{cases} m(x)\dfrac{\partial u}{\partial t} = a\Delta u + m(x)f(u) & \text{in} \quad \Omega\times(0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{8.1}$$

in a three-dimensional bounded domain Ω of $\mathcal{C}^2$ class. Here, $m(x)$ is a given function in $L_\infty(\Omega)$ such that

$$0 \le m(x) \le 1 \quad \text{and} \quad m(x) \not\equiv 0. \tag{8.2}$$

For the reaction function $f(u)$, it is assumed that

$$f(u) \text{ is a real valued function for } -\infty < u < \infty \text{ of class } \mathcal{C}^2 \text{ satisfying the condition } f(0) = 0. \tag{8.3}$$

The unknown function $u = u(x,t)$ is imposed the homogeneous Dirichlet conditions on the boundary $\partial\Omega$. In addition, $u_0(x)$ is a real initial function in Ω. The diffusion coefficient $a > 0$ is a fixed constant.

Such an elliptic-parabolic equation arises in the study of heat conduction in the composite media consisting of several materials that have their own heat conductivity (cf., [101, 102, 103, 104, 105]). Let $\Omega \subset \mathbb{R}^3$ denote such a composite medium and let Ω be divided into the direct sum of sub-domains Ω_i, $1 \le i \le n$, Ω_i denoting a material with a constant heat conductivity $a_i > 0$. Then the equation describing heat conduction in Ω is given by

$$\frac{\partial u}{\partial t} = \nabla\cdot[a(x)\nabla u] + f(u) \qquad \text{in} \quad \Omega\times(0,\infty),$$

where $a(x)$ is a step function such that $a(x) \equiv a_i$ for $x \in \Omega_i$, $1 \le i \le n$, and where $f(u)$ denotes a nonlinear heat controller. For a test function $\varphi(x) \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\begin{aligned} \langle \nabla\cdot[a(x)\nabla u], \varphi\rangle &= -\langle a(x)\nabla u, \nabla\varphi\rangle = -\sum_{i=1}^n \int_{\Omega_i} a_i \nabla u\cdot\nabla\varphi\, dx \\ &= \sum_{i=1}^n \int_{\Omega_i} a_i[\Delta u]\varphi\, dx - \sum_{i=1}^n \int_{\partial\Omega_i} a_i \frac{\partial u}{\partial n_i}\varphi\, dx, \end{aligned}$$

where n_i denotes the outer normal vector of $\partial\Omega_i$. We here assume on each interface $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \neq \emptyset$ that

$$a_i\frac{\partial u}{\partial n_i} + a_j\frac{\partial u}{\partial n_j} = 0 \qquad \text{on } \Gamma_{ij},$$

which means the continuity interface condition imposed in many problems (f.e., [106, 107, 108]). Under the assumption, the heat equation takes the form

$$\frac{\partial u}{\partial t} = a(x)\Delta u + f(u). \tag{8.4}$$

We also want to consider the case where some material may possess extremely larger conductivity than others, say, (for simplicity) $a_i = \infty$ for some i. In such a sub-domain, the equation is no longer a heat equation but is a Laplace equation. Then, instead of (8.4), it is convenient to rewrite the equation into the form

$$m(x)\frac{\partial u}{\partial t} = a\varDelta u + m(x)f(u) \qquad \text{in} \quad \Omega \times (0, \infty),$$

where $a = \min_{1 \le i \le n} a_i$ is a positive number and $m(x)$ is the function $a/a(x)$ for $x \in \Omega$. Clearly, $m(x)$ satisfies (8.2). We are going to seek a continuous solution $u(x,t)$ with respect to the variable x in the whole Ω; therefore, our solution satisfies the free boundary value conditions and the no crossing flux conditions on the interfaces of sub-domains Ω_i.

8.2 Local Solutions

We begin with constructing a local solution for (8.1) by employing the general theory of semilinear abstract degenerate evolution equations explained in Section 4.4.

Let us formulate (8.1) as the Cauchy problem for an abstract evolution equation of the form (4.38), i.e.,

$$\begin{cases} M\dfrac{du}{dt} + Lu = Mf(u), \qquad 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{8.5}$$

in the underlying space

$$Y \equiv L_2(\Omega). \tag{8.6}$$

Here, L is a realization of $-a\varDelta$ in $L_2(\Omega)$ under the homogeneous Dirichlet conditions on $\partial\Omega$ with $\mathcal{D}(L) \equiv H^2(\Omega) \cap H^1_0(\Omega)$, $H^1_0(\Omega)$ being a closure of $\mathcal{C}^\infty_0(\Omega)$ in the Sobolev space $H^1(\Omega)$. Of course, L is a self-adjoint operator of Y. By Poincaré's inequality, there exists a positive constant c such that

$$(-a\varDelta u, u) = a\|\nabla u\|^2_{L_2} \ge ac\|u\|^2_{L_2}, \qquad u \in H^2(\Omega) \cap H^1_0(\Omega). \tag{8.7}$$

Consequently,

$$\|u\|_{H^2} \le C\|Lu\|_{L_2}, \qquad u \in \mathcal{D}(L). \tag{8.8}$$

Hence, L is positive definite in Y; and L satisfies the conditions (4.39)-(4.40) in Subsection 4.4.2.

According to Theorem 2.12, the domains of its fractional powers L^θ are known as

$$\mathcal{D}(L^\theta) = \begin{cases} H^{2\theta}(\Omega) \qquad \text{if } 0 \le \theta < \frac{1}{4}, \\ H^{2\theta}_D(\Omega) \equiv \{u \in H^{2\theta}(\Omega); u_{|\partial\Omega} = 0\} \qquad \text{if } \frac{1}{4} < \theta \le 1. \end{cases} \tag{8.9}$$

In particular, we have $\mathcal{D}(L^{\frac{1}{2}}) = H^1_0(\Omega) = H^1_D(\Omega)$.

Meanwhile, the second Banach space X is set by

$$X \equiv H^1_0(\Omega), \tag{8.10}$$

noting that (4.41) is verified with $\alpha = \frac{1}{2}$ (due to (8.9)). The operator M is then a multiplicative operator by the function $m(x)$ from $H_0^1(\Omega)$ into $L_2(\Omega)$. As verified in [59, Example 3.4], M and L satisfy (4.42)-(4.43) with some angle $\omega < \frac{\pi}{2}$. Notice that these conditions may fail in $L_2(\Omega)$; so, the settings (8.6) and (8.10) are essential.

Finally, $f(u) \equiv f(\mathrm{Re}\, u(x))$ denotes a nonlinear operator with $\mathcal{D}(f) \equiv \mathcal{D}(L^\beta) = H_D^{2\beta}(\Omega)$ (due to (8.9)), where β is some fixed exponent such that $\frac{3}{4} < \beta < 1$. It is known that $H^{2\beta}(\Omega) \subset \mathcal{C}(\overline{\Omega})$. Then f is a mapping from $\mathcal{D}(f)$ into X. Moreover, since

$$\nabla[f(\mathrm{Re}\, u) - f(\mathrm{Re}\, v)] = [f'(\mathrm{Re}\, u) - f'(\mathrm{Re}\, v)]\nabla \mathrm{Re}\, u + f'(\mathrm{Re}\, v)\nabla \mathrm{Re}\,(u - v),$$

we observe that

$$\begin{aligned}\|\nabla[f(\mathrm{Re}\, u) - f(\mathrm{Re}\, v)]\|_{L_2} \le {} & [\max_{|r| \le \|u\|_{\mathcal{C}} + \|v\|_{\mathcal{C}}} |f''(r)|]\, \|u - v\|_{\mathcal{C}}\, \|\nabla u\|_{L_2} \\ & + [\max_{|r| \le \|v\|_{\mathcal{C}}} |f'(r)|]\, \|\nabla(u - v)\|_{L_2}, \qquad u, v \in \mathcal{D}(f).\end{aligned}$$

From this, it is readily verified that the Lipschitz condition (4.44) takes place. In this way, all the structural assumptions (4.39)–(4.44) in Subsection 4.4.2 are fulfilled by the operators L, M and f.

The problem (8.5) is equivalently rewritten in the form

$$\begin{cases} \dfrac{du}{dt} + Au \ni f(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{8.11}$$

in the space X. Here, A is a multivalued linear operator of X which is given by

$$\mathcal{D}(A) = \{u \in H_D^2(\Omega);\ \exists f \in H_0^1(\Omega) \text{ such that } m(x)f = -a\Delta u\},$$

i.e., $A \equiv M^{-1}L$. We fix the third exponent $\tilde{\beta}$ in such a way that $\tilde{\beta}$ satisfies

$$2\beta - 1 < \tilde{\beta} < 1.$$

Then, α, β and $\tilde{\beta}$ satisfy the relation (4.49). By virtue of (4.48) in Proposition 4.1 ($\theta = \beta$), we have $\mathcal{D}(A^{\tilde{\beta}}) \subset \mathcal{D}(L^\beta)$ and

$$\|u\|_{\mathcal{C}} \le C\|L^\beta u\|_{L_2} \le C\|A^{\tilde{\beta}} u\|_{H_0^1}, \qquad u \in \mathcal{D}(A^{\tilde{\beta}}). \tag{8.12}$$

Theorem 4.5 in Subsection 4.4.2 then provides, for any $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$ ($\subset H_D^{2\beta}(\Omega)$), that there exists a unique local solution to (8.11) in the function space:

$$u \in \mathcal{C}([0, T_{u_0}]; \mathcal{D}(A^{\tilde{\beta}})) \cap \mathcal{C}((0, T_{u_0}]; \mathcal{D}(A)) \cap \mathcal{C}^1((0, T_{u_0}]; X),$$

$T_{u_0} > 0$ being determined by the norm $\|A^{\tilde{\beta}} u_0\|_{H_0^1}$ alone. The continuity of $u(t)$ at $t = 0$ with respect to the graph norm of $\mathcal{D}(A^{\tilde{\beta}})$ is verified by Remark 4.1 because of (8.10).

As noticed in the remark of Theorem 4.5, this $u(t)$ is a unique solution to (8.5) lying in the function space:

$$u \in \mathcal{C}([0, T_{u_0}]; H_D^{2\beta}(\Omega)) \cap \mathcal{C}((0, T_{u_0}]; H_D^2(\Omega)) \cap \mathcal{C}^1((0, T_{u_0}]; H_0^1(\Omega)). \tag{8.13}$$

If $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$ is real, then the complex conjugate $\overline{u(t)}$ of the solution $u(t)$ is also a solution of (8.5) lying in (8.13). Then, by the uniqueness, we must have $\overline{u(t)} \equiv u(t)$, that is, the following result is proved.

Theorem 8.1. *Let* $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$ *be real. Then,* (8.5) *possesses a unique real local solution in the function space* (8.13). *Here, the existence interval* $[0, T_{u_0}]$ *is determined by the norm* $\|A^{\tilde{\beta}} u_0\|_{H_0^1}$ *alone.*

Let $H^s(\Omega; \mathbb{R})$ be the real Sobolev space in Ω with exponent $0 \le s < \infty$. It is naturally true that $H^s(\Omega) = H^s(\Omega; \mathbb{R}) + iH^s(\Omega; \mathbb{R})$ (see [37, Theorem 1.34]). The operator L is then a real sectorial operator from $H_D^2(\Omega; \mathbb{R})$ into $L_2(\Omega; \mathbb{R})$. Furthermore, according to [109, Theorem 3.3], the fractional power L^β is also a real operator and $L^\beta u = L^\beta(\mathrm{Re}\, u) + iL^\beta(\mathrm{Im}\, u)$ for $u \in \mathcal{D}(L^\beta)$. This together with (8.9) yields that $\mathcal{D}(L^\beta) \cap L_2(\Omega; \mathbb{R}) = H_D^{2\beta}(\Omega; \mathbb{R})$.

In what follows, we handle only real Sobolev spaces $H^s(\Omega; \mathbb{R})$. In addition, these spaces are denoted by $H^s(\Omega)$ for simplicity.

8.3 Lyapunov Function

Let $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$ be real. Let $u(t)$ be any local solution of (8.5) on an interval $[0, T]$ which lies in the space:

$$u \in \mathcal{C}([0, T]; H_D^{2\beta}(\Omega)) \cap \mathcal{C}((0, T]; H_D^2(\Omega)) \cap \mathcal{C}^1((0, T]; H_0^1(\Omega)). \tag{8.14}$$

As noticed above, $u(t)$ is a real valued function.

In view of (8.14), multiply the equation of (8.5) by $\frac{\partial u}{\partial t}$ and integrate the product in Ω. Then,

$$\int_\Omega m \left| \frac{\partial u}{\partial t} \right|^2 dx + \frac{a}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 dx = \frac{d}{dt} \int_\Omega mF(u(t)) dx,$$

or

$$\frac{d}{dt} \int_\Omega \left[\frac{a}{2} |\nabla u|^2 - mF(u(t)) \right] dx = -\int_\Omega m \left| \frac{\partial u}{\partial t} \right|^2 dx, \tag{8.15}$$

where $F(u) = \int_0^u f(v) dv$ is a primitive function of $f(u)$.

This then shows that the function

$$\Psi(u) = \int_\Omega \left[\frac{a}{2} |\nabla u|^2 - mF(u) \right] dx, \qquad u \in H_D^{2\beta}(\Omega),$$

is a Lyapunov function for local solutions of (8.5) for real initial value $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$. The following property of $\Psi(\cdot)$ is easily verified.

Proposition 8.1. *For the local solution* $u(t)$ *above, the value* $\Psi(u(t))$ *is monotonously decreasing for* $0 \le t \le T$. *If* $\frac{d}{dt}[\Psi(u(t))] = 0$ *at some time* $t = \bar{t}$, *then* $\overline{u} = u(\bar{t})$ *is a stationary solution of* (8.5).

Proof. The first assertion is obvious from (8.15). So, let us prove the second assertion. Assume that $\frac{d}{dt}[\Psi(u(t))] = 0$ at some time $t = \bar{t}$. From (8.15) it follows that $m(x)|\frac{\partial u}{\partial t}(\bar{t})|^2 = 0$ in Ω; naturally, it is the same for $m(x)\frac{\partial u}{\partial t}(\bar{t})$. Hence,

$$a\Delta u(\bar{t}) + m(x) f(u(\bar{t})) = 0 \qquad \text{in } \Omega,$$

which means that $u(\bar{t})$ is a stationary solution of (8.5). □

8.4 Asymptotic convergence of global solutions

Let $u_0 \in \mathcal{D}(A)$ and u_0 be real. Assume for this u_0 that (8.5) possesses a global solution lying in the function space

$$u \in \mathcal{C}([0,\infty); H_D^2(\Omega)) \cap \mathcal{C}^1([0,\infty); H_0^1(\Omega)) \tag{8.16}$$

and satisfying the global estimate

$$\|u'(t)\|_{H_0^1} + \|Au(t)\|_{H_0^1} \le R_0, \qquad 0 \le t < \infty, \tag{8.17}$$

with some constant $R_0 > 0$. Under above assumptions, it is easy to see that $\Psi(u(t)) \ge -C_{R_0}$ for all $0 \le t < \infty$ with some constant $C_{R_0} \ge 0$ depending on R_0.

Here, note that, under suitable growth conditions on $f(u)$ for $0 < u < \infty$, one can show existence of global solutions lying in (8.16) and satisfying (8.17). Assume that $f(u)$ is a $\mathcal{C}^3$ function for $-\infty < u < \infty$ and that $f(u)$ satisfies

$$\begin{aligned} -D_1'u(u^{p-1}+1) \le f(u) \le D_2'u(1+u)^{-1}, \qquad & 0 \le u < \infty, \\ -D_3'(u^{p-1}+1) \le f'(u) \le D_4', \qquad & 0 \le u < \infty, \end{aligned}$$

with some exponent $p \ge 2$ and some constants $D_i' > 0\,(i = 1,2,3,4)$. Then, if an initial function $u_0 \in \mathcal{D}(A^{\tilde{\beta}})$ is real and satisfies $m(x)u_0 \ge 0$ in Ω, then (8.5) possesses a unique real global solution in the function space (8.16) and the solution satisfies the global estimate (8.17). Of course, the Theorem 8.2 is applicable to these global solutions. The results about the existence of global solution is shown in [110].

8.4.1 ω-limit set

For the trajectory $u(t)$, we consider its ω-limit set defined by

$$\omega(u_0) = \{\overline{u} \in \mathcal{D}(A^{\tilde{\beta}});\ \exists t_n \nearrow \infty,\ \ u(t_n) \to \overline{u} \ \text{ in } \ \mathcal{D}(A^{\tilde{\beta}})\}.$$

It is seen that $\omega(u_0) \ne \emptyset$. Indeed, from (8.17), $\sup_{0\le t<\infty} \|Au(t)\|_{H_0^1} < \infty$; in the meantime, $\mathcal{D}(A)$ is compactly embedded in $\mathcal{D}(A^{\tilde{\beta}})$ by the following lemma.

Lemma 8.1. *For any $0 \le \theta < 1$, the embedding from $\mathcal{D}(A)$ into $\mathcal{D}(A^\theta)$ is compact.*

Proof. We notice that $\mathcal{D}(A) \subset \mathcal{D}(L) = H_D^2(\Omega)$ is compactly embedded in $H_0^1(\Omega)$. Then, the desired result is verified by the moment inequality (4.34). □

It is clear that $\inf_{0\le t<\infty} \Psi(u(t)) > -\infty$. Meanwhile, $\Psi(u(t_n)) \to \Psi(\overline{u})$. Therefore, it follows that

$$\lim_{t\to\infty} \Psi(u(t)) = \Psi(\overline{u}) \qquad \text{for any} \quad \overline{u} \in \omega(u_0). \tag{8.18}$$

Moreover, the standard arguments show the following result.

Proposition 8.2. *The set $\omega(u_0)$ consists of stationary solutions of* (8.5).

Proof. Let $\overline{u} \in \omega(u_0)$ and let $u(t_n) \to \overline{u}$ with some increasing time sequence $t_n \nearrow \infty$. In view of Theorem 8.1, there exists time $T > 0$ such that the equation of (8.5) for each initial value $u_n = u(t_n)$ and (8.5) possesses a local solution $u(t; u_n)$ and $u(t; \overline{u})$, respectively, on the interval $[0, T]$ uniformly.

Since $u_n \to \overline{u}$ in $\mathcal{D}(A^{\tilde{\beta}})$ it is possible to apply Theorem 4.6 to obtain that $u(t; u_n) \to u(t; \overline{u})$ in $\mathcal{D}(A^{\tilde{\beta}})$ for any $t \in [0, T]$. By the Markov property, we have $u(t + t_n) = u(t; u_n)$; therefore, $u(t + t_n) \to u(t; \overline{u})$, i.e., $u(t; \overline{u}) \in \omega(u_0)$ for any $t \in [0, T]$.

It then follows from the fact (8.18) that $\Psi(u(t; \overline{u}))$ is constant with respect to $t \in [0, T]$. Consequently, $\frac{d}{dt}\Psi(u(t; \overline{u})) \equiv 0$. Proposition 8.1 then yields that $\overline{u} = u(0; \overline{u})$ must be a stationary solution to (8.5). □

8.4.2 Fundamental properties for $\overline{u}$

In view of (8.12) and (8.17), we know that there exists a number $\rho > 0$ such that

$$\sup_{0 \le t < \infty} \|u(t)\|_{\mathcal{C}} \le \rho \qquad \text{and} \qquad \sup_{\overline{u} \in \omega(u_0)} \|\overline{u}\|_{\mathcal{C}} \le \rho.$$

Knowing this, we introduce a cutoff $\widetilde{f}(u)$ of $f(u)$ such that $\widetilde{f}(u) = f(u)$ in a neighborhood of the interval $-\rho \le u \le \rho$ and $\widetilde{f}(u)$ is a $\mathcal{C}^2$ function for $-\infty < u < \infty$ satisfying

$$|\widetilde{f}^{(i)}(u)| \le D_i, \qquad -\infty < u < \infty, \tag{8.19}$$

for $i = 0, 1, 2$. Let $\widetilde{F}(u) = \int_0^u \widetilde{f}(v)dv$ be its primitive; then, $|\widetilde{F}(u)| \le D_0|u|$ for $-\infty < u < \infty$. In addition, let us introduce a new function

$$\widetilde{\Psi}(u) = \int_\Omega \left[\frac{a}{2}|\nabla u|^2 - m\widetilde{F}(u)\right] dx, \qquad u \in H_0^1(\Omega).$$

Since $\widetilde{F}(u) = F(u)$ for $-\rho \le u \le \rho$, $\widetilde{\Psi}(u)$ still plays the role of a Lyapunov function for the trajectory $u(t)$.

As we are going to argue only for this trajectory, $\widetilde{f}(u)$, $\widetilde{F}(u)$ and $\widetilde{\Psi}(u)$ will be simply denoted by $f(u)$, $F(u)$ and $\Psi(u)$, respectively, as before.

We can then prove that Ψ is differentiable in $H_0^1(\Omega)$.

Proposition 8.3. $\Psi : H_0^1(\Omega) \to \mathbb{R}$ *is Fréchet differentiable with the derivative*

$$\Psi'(u) = -[a\Delta u + m(x)f(u)] \in H^{-1}(\Omega), \qquad u \in H_0^1(\Omega). \tag{8.20}$$

In particular, $\Psi'(\overline{u}) = 0$ *for any* $\overline{u} \in \omega(u_0)$.

The proof of this proposition will be given in Section 8.5.

In addition to (8.19), let us assume that

$$f(u) \text{ is real analytic for } u \text{ in a neighborhood of } [-\rho, \rho]. \tag{8.21}$$

Under this assumption, we can prove the Łojasiewicz-Simon gradient inequality.

Proposition 8.4. *Let* $\overline{u} \in \omega(u_0)$*. There exists an exponent* $0 < \theta \le \frac{1}{2}$ *and a neighborhood* $U(\overline{u})$ *of* $\overline{u}$ *in* $H_0^1(\Omega)$ *in which it holds true that*

$$\|\Psi'(u)\|_{H^{-1}} \ge C\,|\Psi(u) - \Psi(\overline{u})|^{1-\theta}, \qquad u \in U(\overline{u}). \tag{8.22}$$

The proof of this proposition will be described in Section 8.6.

8.4.3 Asymptotic convergence of $u(t)$ to $\overline{u}$

Using Propositions 8.3 and 8.4, we can establish convergence of $u(t)$ to $\overline{u}$ in $H^1_0(\Omega)$ and estimate the rate of convergence.

If $\Psi(u(t)) = \Psi(u(s))$ for some $s < t$, then $\Psi(u(\tau))$ is constant with respect to $\tau \in [s,t]$ and Proposition 8.1 yields that $u(\tau)$ is a stationary solution, i.e., $\omega(u_0) = \{\overline{u}\}$. So, it suffices to consider the case that $\Psi(u(s)) > \Psi(u(t))$ for any pair of $s < t$.

Let us begin with proving the following crucial proposition.

Proposition 8.5. *Let $r > 0$ be a radius so that* (8.22) *is satisfied in a ball $B^{H^1_0}(\overline{u}; r)$, i.e., $B^{H^1_0}(\overline{u}; r) \subset U(\overline{u})$. Let $0 \le s < t < \infty$ be such that, for all $\tau \in [s,t]$, the values $u(\tau)$ stay in $B^{H^1_0}(\overline{u}; r)$. Then, we have*

$$\|u(t) - u(s)\|_{H^1_0} \le C[\Psi(u(s)) - \Psi(\overline{u})]^{\frac{\theta}{2}}. \tag{8.23}$$

Proof. Since $\Psi(u(\tau)) > \Psi(\overline{u})$ for $s \le \tau \le t$, we observe from (8.15) that

$$\begin{aligned}
-\frac{d}{d\tau}[\Psi(u(\tau)) - \Psi(\overline{u})]^\theta &= -\theta[\Psi(u(\tau)) - \Psi(\overline{u})]^{\theta-1}\frac{d}{d\tau}\Psi(u(\tau)) \\
&\ge \theta\|m\|^{-1}_{L_\infty}[\Psi(u(\tau)) - \Psi(\overline{u})]^{\theta-1}\left\|M\frac{du}{d\tau}(\tau)\right\|^2_{L_2}.
\end{aligned}$$

Since $M\frac{du}{d\tau}(\tau) = -Lu(\tau) + Mf(u(\tau)) = -\Psi'(u(\tau))$ due to (8.20), it follows that

$$-\frac{d}{d\tau}[\Psi(u(\tau)) - \Psi(\overline{u})]^\theta \ge \theta\|m\|^{-1}_{L_\infty}[\Psi(u(\tau)) - \Psi(\overline{u})]^{\theta-1}\|\Psi'(u(\tau))\|_{L_2}\left\|M\frac{du}{d\tau}(\tau)\right\|_{L_2}.$$

Here (8.22) is available to obtain that

$$-\frac{d}{d\tau}[\Psi(u(\tau)) - \Psi(\overline{u})]^\theta \ge C\left\|M\frac{du}{d\tau}(\tau)\right\|_{L_2}$$

with some constant $C > 0$. Integrate this inequality on $[s,t]$. Then,

$$\begin{aligned}
[\Psi(u(s)) - \Psi(\overline{u})]^\theta - [\Psi(u(t)) - \Psi(\overline{u})]^\theta &\ge C\int_s^t \left\|M\frac{du}{d\tau}(\tau)\right\|_{L_2} d\tau \\
&\ge C\|M[u(t) - u(s)]\|_{L_2}.
\end{aligned} \tag{8.24}$$

Therefore,

$$\|M[u(t) - u(s)]\|_{L_2} \le C^{-1}[\Psi(u(s)) - \Psi(\overline{u})]^\theta.$$

Meanwhile, in view of (8.7),

$$\begin{aligned}
a\|\nabla[u(t) - u(s)]\|^2_{L_2} &= (-a\Delta[u(t) - u(s)], u(t) - u(s)) \\
&= (M[-u'(t) + f(u(t)) + u'(s) - f(u(s))], u(t) - u(s)) \\
&\le 2[\sup_{0\le\tau<\infty}\|u'(\tau)\|_{L_2} + \|f(u(\tau))\|_{L_2}]\|M[u(t) - u(s)]\|_{L_2}.
\end{aligned}$$

Hence, the desired estimate (8.23) is concluded by (8.17). □

By definition, there exists a time sequence $t_n \nearrow \infty$ such that $u(t_n) \to \overline{u}$ in $\mathcal{D}(A^{\tilde{\beta}})$, *a fortiori* in $H_0^1(\Omega)$. As above, let $r > 0$ be such that $B^{H_0^1}(\overline{u}; r) \subset U(\overline{u})$. Then, there exists an N such that for all $n \geq N$, $\|u(t_n) - \overline{u}\|_{H_0^1} \leq \frac{r}{3}$ and that $C[\Psi(u(t_n)) - \Psi(\overline{u})]^{\frac{\theta}{2}} \leq \frac{r}{3}$, here C is the constant obtained in (8.23). Assume that, for all $\tau \in [t_N, t]$, the values $u(\tau)$ lie in $B^{H_0^1}(\overline{u}; r)$. Applying (8.23) with $s = t_N$, we observe that

$$\begin{aligned}\|u(t) - \overline{u}\|_{H_0^1} &\leq \|u(t) - u(t_N)\|_{H_0^1} + \|u(t_N) - \overline{u}\|_{H_0^1} \\ &\leq C[\Psi(u(t_N)) - \Psi(\overline{u})]^{\frac{\theta}{2}} + \|u(t_N) - \overline{u}\|_{H_0^1} \leq \frac{2r}{3}.\end{aligned}$$

This means that, after the time t_N, the trajectory must remain in the ball $B^{H_0^1}(\overline{u}; \frac{2r}{3})$ for ever.

It is now ready to prove the asymptotic convergence of $u(t)$.

Theorem 8.2. *Under* (8.2) *and* (8.3), *let* $u(t)$ *be a global solution of* (8.5) *lying in* (8.16) *and satisfying* (8.17). *Let* (8.21) *be satisfied. Then, it holds true that*

$$\|u(t) - \overline{u}\|_{H_0^1} \leq C[\Psi(u(t)) - \Psi(\overline{u})]^{\frac{\theta}{2}}, \qquad \forall t \geq t_N, \tag{8.25}$$

where θ *is the exponent appearing in the gradient inequality* (8.22).

Proof. Let $t_N \leq s < \infty$. Let t_n be the time sequence mentioned above. Let $s \leq t_n$; since $u(\tau) \in B^{H_0^1}(\overline{u}; r)$ for all $\tau \in [s, t_n]$, the inequality (8.24) is available with $t = t_n$ to conclude that

$$[\Psi(u(s)) - \Psi(\overline{u})]^\theta - [\Psi(u(t_n)) - \Psi(\overline{u})]^\theta \geq C\|M[u(t_n) - u(s)]\|_{L_2}.$$

Let $n \to \infty$. Then, as $u(t_n) \to \overline{u}$ in $\mathcal{D}(A^{\tilde{\beta}})$, we obtain that

$$\|M[u(s) - \overline{u}]\|_{L_2} \leq C^{-1}[\Psi(u(s)) - \Psi(\overline{u})]^\theta, \qquad s \geq t_N.$$

Arguing in the same way as in the proof of Proposition 8.5, we can verify that

$$\|u(s) - \overline{u}\|_{H_0^1}^2 \leq C\|M[u(s) - \overline{u}]\|_{L_2}, \qquad s \geq 0.$$

These estimates then yield the desired estimate (8.25). □

As a corollary, we obtain the asymptotic convergence of $u(t)$ in $\mathcal{D}(A^{\tilde{\beta}})$.

Corollary 8.1. *Under the same situations as in Theorem 8.2, we have*

$$\|u(t) - \overline{u}\|_{\mathcal{D}(A^{\tilde{\beta}})} \leq C[\Psi(u(t)) - \Psi(\overline{u})]^{\frac{\theta(1-\tilde{\beta})}{2}}, \qquad \forall t \geq t_N.$$

Proof. The result is immediately verified by (8.17) and (8.25) if we use the moment inequality (4.34). □

8.5 Differentiability of Ψ

Let us verify Fréchet differentiability of the Lyapunov function.

Proof of Proposition 8.3. For u, $h \in H_0^1(\Omega)$, we have

$$\frac{a}{2}\left[\int_\Omega |\nabla(u+h)|^2 dx - \int_\Omega |\nabla u|^2 dx\right] = \langle -a\Delta u, h\rangle_{H^{-1}\times H_0^1} + \frac{a}{2}\int_\Omega |\nabla h|^2 dx.$$

Meanwhile, it is seen from (8.19) that

$$|F(u+h) - F(u) - f(u)h| \le D_1|h|^2, \qquad -\infty < h,\, u < \infty.$$

Therefore,

$$\left|\int_\Omega m(x)\left[F(u+h) - F(u) - f(u)h\right] dx\right| \le C\|m\|_{L_\infty}\int_\Omega |h(x)|^2 dx.$$

Hence, we obtain that

$$\left|\Psi(u+h) - \Psi(u) - \langle -[a\Delta u + mf(u)], h\rangle_{H^{-1}\times H_0^1}\right| \le C\|h\|_{H_0^1}^2,$$

which is the assertion to be proved. □

Furthermore, Ψ' is also verified to be differentiable.

Proposition 8.6. $\Psi' : H_0^1(\Omega) \to H^{-1}(\Omega)$ *is Fréchet differentiable with the derivative*

$$\Psi''(u) = -[a\Delta + mf'(u)] \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega)), \qquad u \in H_0^1(\Omega), \tag{8.26}$$

here $mf'(u)$ is a multiplicative operator by the function $mf'(u)$ from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

Proof. Since

$$|f(u+h) - f(u) - f'(u)h| \le D_2|h|^2, \qquad -\infty < h,\, u < \infty,$$

due to (8.19), we have

$$\begin{aligned}\|m[f(u+h) - f(u) - f'(u)h]\|_{H^{-1}} &\le C\|m[f(u+h) - f(u) - f'(u)h]\|_{L_2}\\ &\le C\|m\|_{L_\infty}\|h\|_{L_4}^2, \qquad u,\, h \in H_0^1(\Omega).\end{aligned}$$

Therefore,

$$\|\Psi'(u+h) - \Psi'(u) + [a\Delta h + mf'(u)h]\|_{H^{-1}} \le C\|m\|_{L_\infty}\|h\|_{L_4}^2, \qquad u,\, h \in H_0^1(\Omega).$$

Because of the embedding $H_0^1(\Omega) \subset L_6(\Omega)$, we verify the assertion of proposition. □

8.6 Łojasiewicz-Simon Gradient Inequality

Proof of Proposition 8.4. We will follow the similar procedure of arguments presented in Chill [26].

Put $T = \Psi''(\overline{u})$. As known by (8.26), $T = L - mf'(\overline{u})$ is a bounded linear operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Here, L is a realization of $-a\Delta$ under the homogeneous Dirichlet conditions in the space $H^{-1}(\Omega)$, L being an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. It is easily verified that T is a symmetric operator in the sense that

$$\langle Tu, v\rangle_{H^{-1}\times H_0^1} = \langle u, Tv\rangle_{H_0^1\times H^{-1}}, \qquad u, v \in H_0^1(\Omega), \tag{8.27}$$

for $mf'(\overline{u})$ is a multiplicative operator by the function $mf'(\overline{u})$. Since T is written in the form $T = [I - K]L$, where $K = mf'(\overline{u})L^{-1}$, and K is a compact operator of $H^{-1}(\Omega)$, the Riesz-Schauder theory provides that T is a Fredholm operator with index 0, i.e., $\dim \mathcal{K}(T) = \operatorname{codim} \mathcal{R}(T) = N$.

Since $\mathcal{K}(T)$ is a finite-dimensional space, we can regard it as a closed subspace of each Banach space of the triplet $H_0^1(\Omega) \subset L_2(\Omega) \subset H^{-1}(\Omega)$. Let $P: L_2(\Omega) \to \mathcal{K}(T)$ be the orthogonal projection in $L_2(\Omega)$. Then, P becomes a bounded operator from $H_0^1(\Omega)$ into itself which is a projection from $H_0^1(\Omega)$ onto $\mathcal{K}(T)$ in the space $H_0^1(\Omega)$. On the other hand, since P is symmetric in $L_2(\Omega)$, P can be extended as a projection from $H^{-1}(\Omega)$ onto $\mathcal{K}(T)$ in the space $H^{-1}(\Omega)$, too. These projections naturally induce the following topological direct sums for $H_0^1(\Omega)$ and $H^{-1}(\Omega)$:

$$H_0^1(\Omega) = H_1 + \mathcal{K}(T), \qquad \text{where } H_1 = (I-P)H_0^1(\Omega),$$
$$H^{-1}(\Omega) = H_{-1} + \mathcal{K}(T), \qquad \text{where } H_{-1} = (I-P)H^{-1}(\Omega),$$

respectively. Of course, P is symmetric in the sense that

$$\langle P\varphi, u\rangle_{H^{-1}\times H_0^1} = \langle \varphi, Pu\rangle_{H^{-1}\times H_0^1}, \qquad u \in H_0^1(\Omega), \quad \varphi \in H^{-1}(\Omega). \tag{8.28}$$

Lemma 8.2. *H_{-1} actually coincides with $\mathcal{R}(T)$ and T is an isomorphism from H_1 onto H_{-1}.*

Proof of lemma. By (8.27) and (8.28), we observe that $TP = PT = 0$ on $H_0^1(\Omega)$. Therefore, $T = (I-P)T$ on $H_0^1(\Omega)$; this means that $\mathcal{R}(T) \subset H_{-1}$; in particular, T transforms H_1 into H_{-1}. Meanwhile, $\operatorname{codim} \mathcal{R}(T) = N = \operatorname{codim} H_{-1}$. This implies that $\mathcal{R}(T)$ and H_{-1} coincide. Clearly, T is injective on H_1 and $T(H_1) = T(H_0^1(\Omega)) = \mathcal{R}(T) = H_{-1}$. □

We now introduce the critical manifold of $\overline{u}$ defined by

$$S = \{u \in H_0^1(\Omega); (I-P)\Psi'(u) = 0\}. \tag{8.29}$$

Lemma 8.3. *In a neighborhood of $\overline{u}$, S is a $\mathcal{C}^1$-manifold of dimension N.*

Proof of lemma. Indeed, consider the operator $G: H_0^1(\Omega) \to H_{-1}$ given by $G(u_1, u_2) = (I-P)\Psi'(u_1 + u_2)$ for $u_1 \in H_1$ and $u_2 \in \mathcal{K}(T)$. Then, since $D_1G(u_1, u_2) = (I - P)\Psi''(u_1 + u_2)_{|H_1}$, yields that $D_1G(\overline{u}_1, \overline{u}_2) = T_{|H_1}$ is an isomorphism from H_1 onto H^{-1}, where $\overline{u} = \overline{u}_1 + \overline{u}_2$, $\overline{u}_1 \in H_1$, $\overline{u}_2 \in \mathcal{K}(T)$. So, the implicit function theorem yields that in a neighborhood of $\overline{u}$, S can be represented as

$$S = \{(g(u_2), u_2); u_2 \in \mathcal{K}(T) \text{ and } g: \mathcal{K}(T) \to H_1\},$$

where g is a $\mathcal{C}^1$ mapping defined in a neighborhood of $\overline{u}_2 \in \mathcal{K}(T)$. □

According to [26, Theorem 2], we can state the following proposition.

Proposition 8.7. *Assume that the restriction of Ψ on S satisfies* (8.22) *in a subset $U(\overline{u}) \cap S$, where $U(\overline{u})$ is some neighborhood of $\overline{u}$ in $H_0^1(\Omega)$, with exponent $\theta \in (0, \frac{1}{2}]$. Then, Ψ itself satisfies* (8.22) *in a neighborhood of $\overline{u}$ in $H_0^1(\Omega)$ with the same exponent θ.*

The desired inequality (8.22) on S can generally be verified, as mentioned in [26, Corollary 3], from analyticity of the Lyapunov function $\Psi(u)$. This is, however, not true in the present case, for the transform $u \mapsto \int_\Omega mF(u)dx$ may not be analytic in $H_0^1(\Omega)$ due to the fact that $H^1(\Omega) \not\subset \mathcal{C}(\overline{\Omega})$. So, we have to utilize upper shifting of the spaces.

We introduce two auxiliary spaces

$$H_D^{2\beta}(\Omega) \quad \text{and} \quad H^{2(\beta-1)}(\Omega),$$

where β is the same exponent as in Section 8.2, i.e., $\frac{3}{4} < \beta < 1$. We then observe the analyticity of $\Psi(u)$ in a neighborhood of $H_D^{2\beta}(\Omega)$.

Lemma 8.4. $\Psi : H_D^{2\beta}(\Omega) \to \mathbb{R}$ *is an analytic function for $u \in B^{H_D^{2\beta}}(\overline{u}; r)$ if $r > 0$ is sufficiently small.*

Proof of lemma. It suffices to prove that the function $\mathcal{F}(u) = \int_\Omega mF(u)dx$ is analytic in $B^{H_D^{2\beta}}(\overline{u}; r)$ if $r > 0$ is sufficiently small. The proof is similar to that of Proposition 2.1, so we omit it. □

Lemma 8.5. *The manifold S defined by* (8.29) *is analytic in a neighborhood $U(\overline{u})$ of $\overline{u}$ in $H_0^1(\Omega)$.*

Proof of lemma. We use the shift property of L, that is, L is an isomorphism from $H_D^{2\beta}(\Omega)$ $(\subset H_0^1(\Omega))$ onto $H^{2(\beta-1)}(\Omega)$ $(\subset H^{-1}(\Omega))$ on account of (8.9). Naturally, $T = L - mf'(\overline{u})$ is a bounded operator from $H_D^{2\beta}(\Omega)$ into $H^{2(\beta-1)}(\Omega)$.

Since $\mathcal{K}(T) \subset H_D^2(\Omega) \subset H_D^{2\beta}(\Omega)$, P is still a projection operator both in $H_D^{2\beta}(\Omega)$ and $H^{2(\beta-1)}(\Omega)$. The property $PT = TP = 0$ holds true on $H_D^{2\beta}(\Omega)$, too. Then, as before, P induces the following two topological direct sums:

$$H_D^{2\beta}(\Omega) = H_{2\beta} + \mathcal{K}(T), \quad \text{where } H_{2\beta} = (I - P)H_D^{2\beta}(\Omega)\ (\subset H_1),$$
$$H^{2(\beta-1)}(\Omega) = H_{2(\beta-1)} + \mathcal{K}(T), \quad \text{where } H_{2(\beta-1)} = (I - P)H^{2(\beta-1)}(\Omega)\ (\subset H_{-1}).$$

In addition, it is verified that T is still an isomorphism from $H_{2\beta}$ onto $H_{2(\beta-1)}$. Indeed, since $T \in \mathcal{L}(H_D^{2\beta}(\Omega), H^{2(\beta-1)}(\Omega))$, T is a bounded operator from $H_{2\beta}$ into $H^{2(\beta-1)}(\Omega)$. Moreover, since $PT = 0$, T transforms $H_{2\beta}$ into $H_{2(\beta-1)}$. It is already known that T is injective on H_1, and hence on $H_{2\beta}$. Finally, to see that T is surjective, let $\varphi \in H_{2(\beta-1)}$. Then, there exists an element $v \in H_1$ such that $Tv = \varphi$; since $Lv = \varphi + mf'(\overline{u})\varphi \in H^{2(\beta-1)}(\Omega)$, it follows that $v \in H_D^{2\beta}(\Omega)$; therefore, if $u = (I - P)v \in H_{2\beta}$, then $Tu = T(I - P)v = Tv = \varphi$ due to $TP = 0$.

It is now ready to prove the lemma. We first observe that $S \subset H_D^2(\Omega)$. Indeed, $u \in S$ means that $\Psi'(u) = P\Psi'(u)$; hence, $Lu = mf(u) + P\Psi'(u) \in L_2(\Omega)$, i.e., $u \in H_D^2(\Omega)$. Moreover, when u varies in a neighborhood of $\overline{u}$ in $H_0^1(\Omega)$, S is contained in some bounded set of $H_D^2(\Omega)$. Therefore, S is equally given by

$$S = \{u \in H_D^{2\beta}(\Omega); (I - P)\Psi'(u) = 0\}.$$

Moreover, if $\tilde{r} > 0$ is sufficiently small, then

$$S \cap B^{H_0^1}(\overline{u};\tilde{r}) \subset S \cap B^{H_D^{2\beta}}(\overline{u};r),$$

where r is the radius obtained in Lemma 8.4, since $\|u\|_{H_D^{2\beta}} \leq C\|u\|_{H_D^2}^{2\beta-1}\|u\|_{H_0^1}^{2(1-\beta)}$, $u \in H_D^2(\Omega)$. We next remark that S is equally given by $(g(u_2), u_2) \in H_{2\beta} + \mathcal{K}(T)$, where $u_1 = g(u_2)$ is an implicit mapping of $G(u_1, u_2) = 0$ for the mapping $G\colon H_D^{2\beta}(\Omega) \to H_{2(\beta-1)}$ such that $G(u_1, u_2) = (I - P)\Psi'(u_1, u_2)$ for $(u_1, u_2) \in H_{2\beta} + \mathcal{K}(T)$. As verified, $D_1G(\overline{u}_1, \overline{u}_2) = T_{|H_{2\beta}}$, where $\overline{u} = \overline{u}_1 + \overline{u}_2$, is an isomorphism from $H_{2\beta}$ onto $H_{2(\beta-1)}$. Lemma 8.4 provides that $\Psi'(u)$ and hence $G(u)$ is analytic for $u \in B^{H_D^{2\beta}}(\overline{u};r)$. Consequently, the implicit mapping $u_1 = g(u_2)$ is also analytic in a neighborhood of $\overline{u}_2 \in \mathcal{K}(T)$.

In this way, we have completed the proof of the lemma. □

By two Lemmas 8.4 and 8.5, $\Psi(u)$ is analytic with respect to u when u varies in a neighborhood of $\overline{u}$ on S which is a finite-dimensional analytic manifold. Then, Łojasiewicz' classical result [10] is available to conclude that there exists some exponent $\theta \in (0, \frac{1}{2}]$ for which it holds that

$$\|\Psi'(u)\|_{H^{-1}} \geq C|\Psi(u) - \Psi(\overline{u})|^{1-\theta}, \qquad u \in B^{H_0^1}(\overline{u};r) \cap S.$$

As stated above, Proposition 8.7 thus provides the desired inequality (8.22). □

Chapter 9

Conclusions and Future Researches

In this doctoral thesis, we mainly handled four problems in Chapters 5,6,7, and 8. In order to show the existence and uniqueness of solutions to each equation, we utilized the theory of evolution equations of parabolic type summarized in Chapter 4. After that, we focused on asymptotic behaviors of solutions to model equations.

In Chapter 5, we studied attraction-repulsion chemotaxis equations of the form (5.2). As for these equations, we could construct exponential attractors for a non-autonomous dynamical system determined from (5.2). Roughly speaking, this result means that there exists a compact set with finite fractal dimension and the set attracts all global solutions to (5.2) at exponential rates. From this result, we expect that the global solutions show spatial pattern structures as time increasing.

In Chapters 6, 7, and 8, we showed, for each equation, that the global solutions converge to a stationary solution. Obviously, the Lyapunov functions play an important role for showing convergence results. Let us consider what properties of Lyapunov functions bring us success.

In Chapter 6, we studied the Keller-Segel equations in network shaped domains of the form (6.1). As shown in Section 6.4, (6.1) has a Lyapunov function of the form

$$\Phi((u,v)) = \int_{\mathcal{G}} [u \log u - u] dx - \int_{\mathcal{G}} uv \ dx + \frac{1}{2\gamma} \langle \mathcal{A}_2 v, v \rangle_{H^1(\mathcal{G})' \times H^1(\mathcal{G})} .$$

In order to obtain the convergence result, we showed that this Lyapunov function satisfies the Łojasiewicz-Simon inequality (6.47). For the proof of (6.47), we utilizing the techniques in [29]. We summarize essential points of the proof. Firstly, the inverse operator $A_2^{-1} : L_2(\mathcal{G}) \to L_2(\mathcal{G})$ of the differential operator A_2 in $L_2(\mathcal{G})$ becomes a positive definite self-adjoint compact operator; so, we could obtain the inequality (6.53). Secondly, as shown in Proposition 6.10, the convexity of $\int_{\mathcal{G}} [u \log u - u]dx$ and (6.53) yield that the monotonicity of $\Theta : X_m \to X_m'$ (which is corresponding to the first derivative of Φ). This fact enables us to use the Browder-Minty theorem for Θ, so that Θ is bijection. From this result and inverse mapping theorem (Theorem 2.3), we could apply the classical Łojasiewicz theorem [10] to obtain Proposition 6.13. After some calculations, we conclude the Łojasiewicz-Simon inequality (6.47).

In Chapter 7, we studied a quasilinear diffusion equation of the form (7.1). As shown in Section 7.5, the stationary problem of (7.1) possesses a unique solution (with characterization by a functional equation). This favorable property yields the convergence result.

In Chapter 8, we studied a Laplace reaction-diffusion equation of the form (8.1). This equation has a Lyapunov function of the form

$$\Psi(u) = \int_\Omega \left[\frac{a}{2}|\nabla u|^2 - mF(u)\right] dx.$$

In this problem, the analyticity of $\int_\Omega mF(u)dx$ with respect to u plays an important role. The analyticity of $F : \mathbb{R} \to \mathbb{R}$ does not always ensure that $\int_\Omega mF(\cdot)dx : X \to \mathbb{R}$ is analytic. Whether $\int_\Omega mF(\cdot)dx$ is analytic or not depends on the norm of the Banach space X. In the present case, $\int_\Omega mF(\cdot)dx$ may not be analytic in $H_0^1(\Omega)$ due to the fact that $H^1(\Omega) \not\subset \mathcal{C}(\overline{\Omega})$. With this fact in mind, we considered the shifted triplet $H_D^{2\beta}(\Omega) \subset H_D^{2\beta-1}(\Omega) \subset H^{2(\beta-1)}(\Omega)$ and a critical manifold (introduced by [26, 27]), so that the Łojasiewicz-Simon inequality (8.22) is verified; consequently, we proved the convergence result. By the way, $H_0^1(\Omega) \subset \mathcal{C}(\overline{\Omega})$ when $\Omega \subset \mathbb{R}$; then, without considering the shifted triplet, we can directly show the convergence result by using the theorems in [26]. This is an example of the difficulties by increasing dimension of the domain Ω.

In Chapters 6 and 8, note that we obtain the convergence results with decay estimates (6.51) and (8.25). This is one of the benefit of the Łojasiewicz-Simon inequality. On the contrary, in Chapter 7, we did not know decay estimates, since we obtained the convergence result without proving the Łojasiewicz-Simon inequality.

Finally, let us suggest further study on the problems in this thesis. Although we showed that every global solution converges to a stationary solution in Chapters 6,7, and 8, we still not know the detail of the stationary solution except for Chapter 7. Generally speaking, a stationary problem of a parabolic partial differential equation becomes a elliptic partial differential equation, and the investigating solutions of the a elliptic partial differential equation is a hard work. So, we may investigate solutions of difficult elliptic partial differential equations through limits of global solutions for a corresponding parabolic partial differential equations. In the meantime, it is one of the interesting problems to identify initial values which converge to a common stationary solution. We address these problems with numerical analysis. Since we have obtained our convergence results in the Sobolev norm topology (that is, H^1 topology), we will be able to apply the theory of the finite element method (FEM).

In Chapter 8, we could extend the results to critical manifolds introduced in [26, 27]; so, the techniques must have a lot of potential to become successful. In this thesis, sectorial operators, analytic property of functions, and convex property of functions leaded the Łojasiewicz–Simon inequality, but they have complicated relationships. If we can untie them, then we will be able to construct frameworks which are applicable to a variety of parabolic partial differential equations.

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