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Title	On groups with a standard component of known type. II
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Citation	Osaka Journal of Mathematics. 1981, 18(3), p. 703-723
Version Type	VoR
URL	https://doi.org/10.18910/7260
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Osaka J. Math.
18 (1981), 703-723

ON GROUPS WITH A STANDARD COMPONENT OF KNOWN TYPE, II

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(Received November 21, 1979)
(Revised December 23, 1980)

In [3] we considered those finite groups G having a standard subgroup A, such that $m_2(C_G(A))>1$ and $A/Z(A)$ is of known type. The goal of this paper is to settle certain ambiguities that were not dealt with in [3]. In the case $A\cong G_2(4)$ we showed that G was "of Conway type", although we did not actually prove that $G\cong Co_1$. For the case $A/Z(A)\cong L_3(4)$ we appealed to the results of Nah [7] to conclude that $\langle A^G\rangle\cong Suz$ or He. However, there were errors in [7] which put the results in question. Our main result is the following:

Theorem. *Let A be a standard subgroup of the finite group G. Suppose that $m_2(C_G(A))>1$ and $A/Z(A)\cong L_3(4)$ or $G_2(4)$. Then one of the following holds:*

i) $A\trianglelefteq G$;
ii) $A\cong G_2(4)$ *and* $\langle A^G\rangle\cong Co_1$;
iii) $A\cong L_3(4)$ *or* $SL_3(4)$ *and* $\langle A^G\rangle\cong Suz$ *or* Suz/Z_3; *or*
iv) $A/Z(A)\cong L_3(4)$, $Z(A)\cong Z_2\times Z_2$, *aand* $\langle A^G\rangle\cong He$.

The method of proof is to choose certain 2-groups in $AC_G(A)$ and push-up their normalizers. Eventually, we determine the structure of the centralizer of a central involution at which point we can quote an appropriate recognition theorem.

Throughout the paper we use the following notation. A is a standard subgroup of G, $R\in Syl_2(C_G(A))$ and $m(R)>1$. We assume $A\not\trianglelefteq G$ and that G is a minimal counterexample to this theorem.

Much of the mathematics in this paper was completed at the 1979 Summer Institute on Finite Simple Groups in Santa Cruz, sponsored by the American Mathematical Society and funded by the National Science Foundation. We began the project as a result of conversations at the conference with various colleagues. Moreover our four weeks in Santa Cruz provided a good opportunity

* Research supported in part by N.S.F. grant 7721554
** Research supported in part by N.S.F. grant 78-01944

for collaboration.

1. Pushing-up and cores

We have $A/Z(A) \cong L_3(4)$ or $G_2(4)$. In the first case let E, F be 2-subgroups of AR such that $R \leq E \cap F$ and such that $Z(A)E/RZ(A)$ and $Z(A)F/RZ(A)$ are the two E_{16} subgroups in a Sylow 2-subgroup of $RA/RZ(A)$. If $A/Z(A) \cong G_2(4)$, let $A_1/Z(A)$ be the subgroup generated by all long root subgroups in a fixed system of root subgroups of $G_2(4)$. Then $A_1/Z(A) \cong SL_3(4)$ and we may choose corresponding subgroups E and F of A_1R.

The first stage of the development of the 2-local structure of G is concerned with the groups $N_G(E)$ and $N_G(F)$. In this section we study these groups and make certain other observations that apply to each of the possible configurations. In later sections we look at individual cases.

(1.1) (i) R is elementary abelian.
(ii) There exists $g \in G - N(A)$ with $R^g \leq C(R)$. For any such g, $R^g \leq AR$.

Proof. The second assertion in (ii) follows from (20.1) of [2]. The rest of (ii) then follows from (3.3) of [3]. Also, (3.2) of [3] gives (i).

(1.2) Let X be a quasisimple group with $Z(X)$ an elementary abelian 2-group and $X/Z(X) \cong L_3(4)$. Let $H/Z(X)$ and $K/Z(X)$ be the E_{16} subgroups in a Sylow 2-subgroup of $X/Z(X)$. Then
(i) H and K are elementary abelian;
(ii) $H \cap K = Z(HK)$; and
(iii) $N_X(H)$ (resp. $N_X(K)$) is the split extension of H (resp. K) by $L_2(4)$.

Proof. $N_X(H)/Z(X)$ is the split extension of $H/Z(X)$ by $L_2(4)$, and $H/Z(X)$ is the natural module for $L_2(4)$. In particular, $N_X(H)$ is transitive on $(H/Z(X))^\#$. Thus, each coset of $Z(X)$ in H consists of involutions. This proves (i). (ii) follows from (i) and the fact that $(H \cap K)/Z(X) = Z(HK/Z(X))$. (iii) holds since a Sylow 2-subgroup of a complement to $H/Z(X)$ in $N_X(H)/Z(X)$ is conjugate to $(H \cap K)/Z(X)$.

(1.3) (i) $R \cong E_4$.
(ii) $R^gRZ(A)/RZ(A)$ is a root subgroup of $RA/RZ(A)$, and for suitable choice of g, it is a long root subgroup.
(iii) If $A/Z(A) \cong G_2(4)$, then $Z(A) = 1$.
(iv) If $|Z(A)|$ is odd, then $R^g \cap A = 1$ provided R^g projects to a long root subgroup of $A/Z(A)$.

Proof. (i) follows from (ii). Suppose $A/Z(A) \cong L_3(4)$. Choose $g \in G - N(A)$ with $R^g \leq AR$, and let $1 \neq x \in R^g$. By (1.2) we have x central in a Sylow 2-

subgroup, say D, of AR. Then $D \leq N(C(A^g))$. As D is generated by elementary subgroups of order $2^4|R|$, we conclude that $D \leq A^g R^g \leq C(R^g)$. (ii) follows. Suppose that $|Z(A)|$ is odd. Then $D \in \mathrm{Syl}_2(AR) \cap \mathrm{Syl}_2(A^g R^g)$ and $D' = Z(D) \cap A = Z(D) \cap A^g$. Consequently, (iv) holds.

Suppose $A/Z(A) \cong G_2(4)$. Then (iii), (ii), and (iv) follow from (8.3), (8.9), and (8.6) of [3], respectively.

(1.4) NOTATION. If $A/Z(A) \cong L_3(4)$, let $A_1 = A$. If $A/Z(A) \cong G_2(4)$, then $Z(A) = 1$ and we let A_1 be the group generated by all long root subgroups in a fixed system of root subgroups of A. In either case A_1 is quasisimple and $A_1/Z(A_1) \cong L_3(4)$. In the second case $A_1 \cong SL_3(4)$. Choose a fixed Sylow 2-subgroup of $A_1 R$ and let E/R and F/R be the corresponding E_{16} subgroups. By (1.2) and (1.3) $E \cong F \cong E_{64}$ and $E \cap F = Z(EF)$. Moreover, we may take $g \in G$ such that $E \cap F = R \times R^g$.

Let $\Omega = E^G \cup F^G$. We will refer to elements of Ω as *planes*, elements of R^G as *points*, and elements of $(E \cap F)^G$ as *lines*.

(1.5) Suppose that $|Z(A)|$ is odd. Then

(i) $E - A$ is partitioned by its 16 points.

(ii) $N(E) = P_0(N(E) \cap N(R))$, with $P_0 \trianglelefteq N(E)$ and $P_0/C_{P_0}(E) \cong E_{16}$, regular on the 16 points of E. $P_0 = O(C_G(E)) \times O_2(P_0)$.

Proof. By (1.3) (iv) and (3.6) of [3], $E \cap F$ contains 4 points and the non-identity elements of these points partition $(E \cap F) - A$. Now E contains 5 lines that contain R, these being conjugate under $N_{A_1}(E)$. This proves (i).

Since $R^g \cap A_1 = 1$, $N_{A_1}(E)$ is transitive on the 15 points of E, other than R. Since $E \cong E_{64}$, $E \leq A^g R^g$ and $N_{A^g}(E)$ is transitive on the 15 points of E other than R^g. Thus, $N(E)$ is 2-transitive on the 16 points in E. The 16 points and 20 lines in E form an affine plane, so all but the last sentence of (ii) follows from Theorem 1 of [8]. $P_0 = [N_A(E), P_0] C_{P_0}(E)$ and $C_{P_0}(E) = EO(C_G(E))$ with $[O(C_G(E)), N_A(E)] \leq [O(N_G(R)), N_A(E)] = 1$, so $P_0 = O(C(E)) \times O_2(P_0)$.

(1.6) Suppose $|Z(A)|$ is even. Then

(i) $R \leq A$.

(ii) E contains 6 points.

(iii) $N(E)/C(E)$ contains $\hat{A}_6$, the 3-fold cover of A_6, as a normal subgroup.

(iv) There is a 3-element acting as an outer diagonal automorphism of A and transitive on $R^\sharp$.

Proof. By (3.6) of [3] $E \cap F$ contains either 4 points or 2 points. In the first case we argue as in (1.5) to conclude that $N(E)$ is 2-transitive on the 16 points of E and there exists $D \trianglelefteq N(E)$ with D inducing a regular normal subgroup on $R^G \cap E$. Then $[D, E] \trianglelefteq N_G(E)$ and one checks that $E = [D, E] \times R$. But then EF

splits over R, contradicting $|Z(A)|$ even. Therefore, $E \cap F$ contains exactly 2 points, E contains exactly 6 points, and (ii) holds.

Let $L \in \mathrm{Syl}_3(N_A(EF))$. Then $L \leq N_G(E \cap F)$, so L stabilizes each of the two points in $E \cap F$. Therefore, $R^g = [L, E \cap F]$. By symmetry (iv) holds, and since $|Z(A)|$ is even, $R \leq A$, proving (i). Now $N(E) \cap N(R)$ contains a subgroup inducing $A_5 \times Z_3$ on E, where the Z_3 factor stabilizes each point in E. Since $N_{A^g}(E)$ moves R, we conclude that $N(E)$ induces S_6 or A_6 on the points of E. Since $O^2(N(E))$ acts irreducibly on E as an $\boldsymbol{F}_2$-space, and since $N(E)/C(E)$ contains a normal subgroup of order 3, we see that E may be regarded as 3-dimensional $\boldsymbol{F}_4$-space for either $3 \cdot A_6$ or $A_6 \times Z_3$. But $SL_3(4) \not\geq A_6 \times Z_3$, so the latter case is not possible. This proves (iii).

(1.7) Let $X \in N_G^*(E, 2')$ and $Y = \langle A^{N(X)} \rangle$. Then either

(i) $X = 1$; or

(ii) $Y/Z(Y) \cong Suz$, He, or Co_1, and $X = O(C_G(A))$.

Proof. Suppose $X \neq 1$. Then $X = \Gamma_{1,R}(X) \leq N(A)$, and since $\mathcal{H}^*_{N(A)}(E, 2') = \{O(C(A))\}$, $X = O(C(A))$. Similary, $X = O(C(A^g))$ for each $g \in N(E)$. As $N_G(E) \not\leq N(A)$, (ii) holds by minimality of $|G|$.

(1.8) Suppose G contains a 2-central involution, z, such that $(C_G(z)/OC_G(z)))^{(\infty)}$ is isomorphic to the centralizer of a 2-central involution in one of the groups Suz, He, or Co_1. Then $O(C_G(z)) = 1$.

Proof. We may assume that $z \in E$ is a 2-central involution in $N(A)$, and as $C_G(z)^{(\infty)}$ is 2-constrained, z is not conjugate to an involution in R. As $E \leq N(O_G(C(z)))$, (1.7) imples that $O(C_G(z)) \leq O(C_G(\boldsymbol{L}))$. Suppose $O(C_G(z)) \neq 1$, let $X = O(C_G(A))$ and $Y = \langle A^{N(X)} \rangle$. Then $[X, Y] = 1$.

Suppose $R \leq N(Y^g)$. As $|\mathrm{Aut}(Y^g) : Y^g| \leq 2$, $R \cap Y^g$ contains an involution, r. Then $E(C_Y(r)) \cong A$, so that $X^g \leq C(AR)$. Thus $X = X^g$ and $Y = Y^g$. That is, R fixes precisely one point in Y^G. Now suppose $z \in N(Y^g)$. Then z centralizes a Y^g-conjugate of R^g, and it follows from Gleason's lemma that $\langle R^{C_G(z)} \rangle$ is transitive on the elements of Y^G fixed by z. But $\langle R^{C_G(z)} \rangle \leq Y$. So z fixes a unique element of Y^G and the result follows from Holt's Theorem [6].

For the remainder of this section we operate under the following hypotheses:

(1.9) (i) z is a 2-central involution in G;

(ii) There is an extraspecial subgroup $X \leq C_G(z)$ such that $|X| = 2^7$ or 2^9 and $\langle z \rangle \in \mathrm{Syl}_2(C(X))$;

(iii) X is weakly closed in a Sylow 2-subgroup of $C_G(z)$, with respect $C_G(z)$; and

(iv) If $g \in C_G(z)$ and $m(X \cap X^g) > 1$, then $X = X^g$.

(1.10) Assume Hypothesis (1.9). Then X is strongly closed with respect to $C_G(z)$ in a Sylow 2-subgroup of $C_G(z)$.

The proof of (1.10) will be carried out in a series of steps. Assume the result to be false.

(1.11) There exists $g \in C_G(z)$ such that setting $Y=\langle X, X^g\rangle$, $B=N_X(X^g)$, $D=N_{X^g}(X)$, and $I=X\cap X^g$, the following hold:

(i) $Y/BD \cong L_2(2^n)$, $Sz(2^n)$, or D_{2n} for n odd;

(ii) BD/I is the sum of natural modules for Y/BD; and

(iii) $I<D$.

Proof. Use (2.4) of [12].

(1.12) $I \cong Z_2$, Z_4, or Q_8.

Proof. This is (iv) of Hypotheses (1.9).

(1.13) $I \ncong Z_2$.

Proof. Suppose otherwise and let bars denote images in $C(z)/\langle z\rangle$. We have $m(\bar{X})=m(\bar{B})+m(\bar{X}/\bar{B})=m(\bar{D})+m(\bar{X}/C_{\bar{X}}(\bar{D}))$. Also, $m(\bar{D})\geq m(\bar{X}/\bar{B})=m(\bar{X}/C_{\bar{X}}(\bar{D}))$. For $\bar{d}\in\bar{D}^\sharp$, $[\bar{X},\bar{D}]=\bar{B}=C_{\bar{X}}(\bar{d})$, so by (7.6) of [2], B is abelian. We conclude from these facts that either $|X|=2^7$ with $m(\bar{D})=3$, or $|X|=2^9$ with $m(\bar{D})=4$. The first case is out since this would force each $1\neq\bar{d}\in\bar{D}$ to act on $\bar{X}$ as a b_3 involution of $O_6^\pm(2)$, whereas $\Omega_6^\pm(2)$ contains no such involutions. Hence $|X|=2^9$.

Now $Y/BD\cong L_2(2^4)$ and BD/I is the natural module, so there exists a subgroup $J\leq Y$ such that J induces Z_{15} on each of $\bar{B}$, $\bar{D}$, and $\bar{X}/\bar{B}$. Viewing $J\leq \mathrm{Aut}(X)$, we see that $\mathrm{Aut}(X)/\mathrm{Inn}(X)\cong O_8^+(2)$, $\bar{B}$ is a singular 4-space of $\bar{X}$, and $\bar{D}$ is contained in the unipotent radical of the stabilizer in $O_8^+(2)$ of $\bar{B}$. Let T be this unipotent radical. Then $T^\sharp$ consists of 28 a_4 involutions and 35 remaining involutions of type a_2. Also, $T=D\times D_1$, where $D_1\cong E_4$ and J induce Z_3 on D_1. Therefore, $D_1^\sharp$ consists of the 3 a_4 involutions fixed by $O_5(J)$ and J acts semiregularly on the a_2 involutions in T. This is numerically impossible.

(1.14) (i) $Y=C_Y(I)\circ I$ if and only if $I\cong Q_8$.

(ii) $O^2(Y)\leq C(I)$.

(iii) If $Y=O^2(Y)I$, then $I\cong Q_8$.

Proof. If $Y=C_Y(I)I$ and $I\cong Z_4$, then $X\leq Y\leq C(I)$, a contradiction. On the otherhand, if $I\cong Q_8$, then $Q=C_Q(I)I$, so $Y=C_Y(I)I$. Thus (i) holds. (iii) follows from (i) and (ii), and (ii) follows from the fact that Y centralizes

both $\bar{I}$ and $\langle z \rangle$.

(1.15) $|X: B| = 2$.

Proof. Suppose false. Then Y/BD is a Bender group and $Y = O^2(Y)I$. By (1.14) (iii) $I \cong Q_8$, and by (1.14) (i) $Y = C_Y(I)I$. Set $W = C_Y(I)$ and $V = W \cap X$. Then $m(\bar{V}) = 4$ or 6, and one of the following holds:

(a) $|\bar{X}| = 2^6$, $W/O_2(W) \cong L_2(4)$, and $O_2(\bar{W})$ the natural module; or
(b) $|\bar{X}| = 2^8$, $W/O_2(W) \cong L_2(8)$, and $O_2(\bar{W})$ is the natural module; or
(c) $|\bar{X}| = 2^8$, $W/O_2(W) \cong L_2(4)$, and $O_2(\bar{W})$ is the sum of two copies of the natural module.

Set $E = D \cap W$ and consider the action of $\bar{E}$ on $\bar{X}$. Since $E \leq C(I)$, either $\bar{E} \leq O_4^{\pm}(2)$ or $\bar{E} \leq O_6^{\pm}(2)$, according to $|\bar{X}| = 2^6$ or 2^8. If (b) holds, then $\bar{E}$ consists of b_3 involutions in $O_4^{\pm}(2)$, whereas $\Omega_4^{\pm}(2)$ contains no b_3 involutions. If (c) holds then $\bar{E} \cong E_{16}$ and $\bar{E} \leq C(\bar{B})$. Since $\bar{B}$ is a 4-space in the 6-space $\bar{V}$, $\bar{E}$ centralize a proper non-degenerate subspace of $\bar{V}$. However, $m(O^{\pm}(l, 2)) < 4$ for $l < 6$. Therefore, (c) does not hold. Suppose (a) holds. Then $O_2(W) \cong E_{32}$, $B \cap W \cong E_8$, and we may regard $\bar{E} \leq O_4^+(2)$. Then each $\bar{e} \in \bar{E}^\sharp$ is an a_2 involution in $O_4^+(2)$, and so $\bar{E} \leq \Omega_4^+(2) \cong S_3 \times S_3$. But then $\bar{E}$ is a Sylow 2-subgroup of $\Omega_4^+(2)$, whereas $\Omega_4^+(2)$ contains c_2 involutions. This is a contradiotion.

(1.16) $I \cong Z_4$.

Proof. Otherwise $I \cong Q_8$ and by (1.15) $m(D\bar{X}/\bar{X}) = 3$ or 5, according to whether $|\bar{X}| = 2^6$ or 2^8. By (1.14) (i), $\bar{D}$ centralize $\bar{I}$, so $\bar{D} \leq O_4^{\pm}(2)$, or $O_6^{\pm}(2)$, respectively. But $m(O_4^{\pm}(2)) = 2$ and $m(O_6^{\pm}(2)) = 4$. This is impossible.

(1.17) $I \ncong Z_4$.

Proof. Suppose $I \cong Z_4$. Then by (1.15), $m(D/I) = m - 2$, $m = m(\bar{X})$ while by (1.11), $B/I = C_{X/I}(D)$. This is impossible as $(\mathrm{Aut}(X) \cap N(I))/C(X/I) \cong Sp_{m-2}(2)$ is of 2-rank $m-3$.

In view of (1.16) and (1.17), the proof of (1.10) is now complete.

2. Suz

In this section we assume that $|Z(A)|$ is odd and $A/Z(A) \cong L_3(4)$. That is $A \cong L_3(4)$ or $SL_3(4)$. We maintain the notation of §1. In addition, we set $P = O_2(P_0)$, where P_0 is as in (1.5). Set $Z = A \cap Z(EF)$ and $S = FC_P(RZ/Z)$.

(2.1) (i) $E = C_{PF}(E)$;
(ii) $P/E = O_2(N_G(E)/E) \cong E_{16}$ and $P_0 = P \times O(C_G(E))$, so $P = O_2(N_G(E))$.
(iii) $(S \cap P)/E \cong E_4$; and

(iv) $S/E \cong E_{16}$

Proof. These are all clear, given 1.5.

(2.2) (i) $S = N_{PS}(F)$;
(ii) $|F^P| = 4$.
(iii) S is a Sylow 2-subgroup of $C(Z) \cap C(RZ/Z) \cap N(E) \cap N(F)$.
(iv) $|\langle (F \cap A)^P \rangle| \geq 4^4$.

Proof. Since $S/E \cong E_{16}$, $EF \trianglelefteq S$. The groups E and F are the unique subgroups of EF isomorphic to E_{64}, and $S \leq N(E)$. Therefore, $S \leq N(F)$. (i) follows from this and the fact that $S/E = N_{PS/E}(FE/E)$. (ii) follows from (i). Let $S \leq T$, with T Sylow in $C(Z) \cap C(RZ/Z) \cap N(E) \cap N(F)$. As S is transitive on the points in RZ, $T \leq SN_T(R)$. But $N_T(R) = EF$, so (iii) holds.

To obtain (iv) let $T = \langle (F \cap A)^P \rangle$. Since $TE/E = S/E \cong E_{16}$, it will suffice to show that $E \cap A \leq T$. Suppose otherwise and let $W = [P, I]$, where $I \in \mathrm{Syl}_3(N_A(EF))$. $P/(E \cap A)$ is abelian since $N_A(E)$ is transitive on $(P/E)^\#$. Thus $|W| = 4^4$ and $W \cap R = 1$. As $Z \leq T$ and T is I-invariant, $T \cap (E \cap A) = Z$ and $T = (F \cap A)W_1$, where $W_1 = T \cap W$. As I acts irreducibly on W_1/Z and on Z, W_1 is abelian. Also $W_1 = T \cap W \trianglelefteq W$. Choosing an appropriate conjugate of F we obtain $W_2 \in W_1^{N(E)}$ with $W_2 \trianglelefteq W$ and $W_1 \cap W_2 = 1$. Therefore, W is abelian.

We show W is elementary abelian as follows. Let $f \in (F \cap A) - Z$. Let $g \in P$ such that $f^g = fw_1$, with $w_1 \in W_1 - Z$. As f^g is an involution, f inverts w_1. If W is not elementary, then $|w_1| = 4$ an**Z** letting g vary, f inverts W_1. Now let f vary and obtain a contradiction.

Consider $N = N(W)$ and let bars denote images in N/W. The involutions in WR are in $W \cup E$, so $R^G \cap WR = R^W$. We conclude that $\bar{N}$ has a standard subgroup $\bar{L} \cong L_2(4)$ with $\bar{R} \in \mathrm{Syl}_2(C_{\bar{N}}(\bar{L}))$. By [1], $E(\bar{N}) \cong L_2(4)$, A_9, HJ, or M_{12}. As $|W| = 2^8$ and 11 does not divide $|GL(8, 2)|$, $E(\bar{N}) \ncong M_{12}$. Suppose $E(\bar{N}) \cong A_9$. Then $\bar{R} \sim \bar{F}$ in $\overline{N(W)}$ and it follows that $R^G \cap A \neq \emptyset$, which is not the case. Next, suppose $E(\bar{N}) \cong HJ$. For $f \in (F \cap A) - Z$, we have $[f, W] = W_1 = C_W(f)$, and $\bar{f}$ is a 2-central involution of $E(\bar{N})$. Viewing $\bar{N} \leq \mathrm{Aut}(W)$ we then have $E(\bar{N}) = \langle C_{\bar{N}}(\bar{f}) \mid f \in (F \cap L) - Z \rangle \leq N(W_1)$. This is impossible.

We are left with the case $E(\bar{N}) \cong L_2(4)$. Clearly, W is weakly closed in a Sylow 2-subgroup of $N(W)$, and applying Theorem 4 of [5] we conclude that W is strongly closed in a Sylow 2-subgroup of C. The main theorem of [5] gives a contradiction.

Define $P(F) = O_2(N_G(F))$, so that (P, E) is symmetric to $(P(F), F)$. By 2.2 (i) and (iii), $S = FC_P(RZ/Z) = EC_{P(F)}(RZ/Z)$.

(2.3) Let $x \in P(F) - S$, $F_0 = (E \cap A)(E^x \cap P)$, and $H = \langle P, P(F) \rangle$. Then
(i) $E^x \cap E = Z$ and $S = EE^x$

(ii) $P \cap S = EF_0$ and E and F_0 are the maximal elementary abelian 2-subgroups of $P \cap S$. Also $E \cong F_0$.

(iii) $F^H = \{F_0, F^P\}$ and $E^H = \{E, (E^x)^P\}$.

(iv) $\Omega \cap S = F^H \cup E^H$ and $N_G(S)$ act on $\{F^H, E^H\}$.

(v) H induces A_5 on E^H.

Proof. Let $h \in P - S$. $F \cap F^h \cap E = Z$ and $F \cap F^h \leq E$, so $F \cap F^h = Z$. Then $|S| = |FF^h|$, so $S = FF^h$. So (i) follows from (2.2) (iii) which guarantees symmetry between E and F. (i) implies (ii).

If $U \cap P \neq 1$ for some point U in E^x, then $U \cap F_0 \neq 1$, so as $m(F_0) = 6$, $U \leq F_0$ and F_0 is a plane. On the otherhand if $U \cap P = 1$ for each point U in E^x and each $x \in P(F) - S$, then $\langle (E \cap A)^{P(F)} \rangle = F_0$ is of order 64, contradicting 2.2 (iii) and (iv).

So F_0 is a plane. By (1.3) $E \cap A$ intersects each point of G trivially, and so $F_0 - (E \cap A)$ is partitioned by its points and $E \cap A = F_0 \cap A^y$ for each point $R^y \leq F_0$. $F_0 E \trianglelefteq P$ so by (ii), $F_0 \trianglelefteq P$. Then $P \leq O^{2'}(C(F_0 \cap A^y) \cap N(F_0)) = P(F_0)$, so $P = P(F_0)$.

Let V be a plane in S. If $V \leq P$, then $V = E$ or F_0 by (ii). Suppose $V \not\leq P$. $V = O^{2'}(C_G(V))$, so $Z \leq V$. As $V \not\leq P$ and $P = C_{SP}(e)$ for $e \in (E \cap A) - Z$, $V \cap (E \cap A) = Z$. If $V \cap E \neq Z$, then V contains some point R^j of E, for $j \in P$. Then $RZ \leq V^{j^{-1}}$, so $V \in F^P$. This leaves the case $V \cap E = Z$. The involutions in $S \cap P$ are $F_0^\# \cup E^\#$. Hence $|F_0 : V \cap F_0| = 4$, and as $F_0 - E$ is partitioned by its points, $V \cap F_0$ is a line. However, P is transitive on the lines in F_0, through Z, so $V \cap F_0 \in (E^x \cup F_0)^P$. It follows that $V \in (E^x)^P$. It has now been shown that

$$\Omega \cap S = \{E, F_0\} \cup F^P \cup (E^x)^P .$$

Notice that $(E^x)^P$ is precisely the set of $V \in S \cap \Omega$ such that $V \cap E = Z$, while $F_0 \cap F = Z$. By symmetry between E and F, $\{F_0\} \cup F^P = (F) \cup (F^h)^{P(F)}$, for $h \in P - S$. Therefore, $\{F_0\} \cup F^P = F^H$. By symmetry, $E^H = \{E\} \cup (E^x)^P$, and so (iii) and (iv) hold. (v) follows from (iii).

(2.4) S is special with $Z(S) = Z$.

Proof. $E/Z \leq Z(S/Z)$, so by (2.3) (i), $[S, S] \leq Z$. $[R, S] = Z$ so $[S, S] = \Phi(S) = Z$. $Z(S) \leq C_S(R) = EF$ with $C_E(S) = Z$, so the lemma holds.

(2.5) $Z(SP/Z) = (E \cap A)/Z$.

Proof. Set $SP/Z = \overline{SP}$. Then $Z(SP/E) = (S \cap P)/E$ so $Z(\overline{SP}) \leq (S \cap P)/Z$. $C_{\overline{E}}(P) = C_{\overline{F}_0}(P) = (E \cap A)/Z$, since P is transitive on the lines through Z on E and F_0. On the otherhand if $x \in N_A(E)$ is of order 3 then $C_{S \cap P}(x) = R$ and $[S \cap P, x] = F_0$, so as $C_{\overline{R}}(\overline{SP}) = 1$, $Z(\overline{SP}) = [Z(\overline{SP}), x] \leq \overline{F}_0$. Therefore $Z(\overline{SP}) =$

$C_{\bar{F}_0}(\bar{P})=(E\cap A)/Z$.

(2.6) Choose notation as in (2.3) and set $\bar{S}=S/Z$ and $A(S)=\mathrm{Aut}(S)/C_{\mathrm{Aut}(S)}(\bar{S})$. Then

(i) S is the central product of two copies of the Sylow 2-group of $L_3(4)$.

(ii) $\bar{S}$ is an orthogonal space over $GF(4)$ with $(\bar{s},\bar{t})=0$ if and only if $[s,t]=1$ and $\bar{s}$ singular if and only if $s^2=1$. $\mathrm{Aut}(S)\cap C(Z)$ preserves this structure and $C_{A(S)}(Z)\cong O_4^+(4)$. $A(S)$ is $Z_3\times C_{A(S)}(Z)$ extended by a field automorphism of order 2, with $O_3(A(S))$ inducing scalar action on $\bar{S}$ corresponding to a generator of $GF(4)^\sharp$. $C_{\mathrm{Aut}(S)}(\bar{S})=V=\bar{S}\times U$, where $\bar{S}\cong U=C_V(O_3(A(S))\trianglelefteq\mathrm{Aut}(S)$ and for $z\in Z^\sharp$, the map $\bar{s}\to C_U(s\langle z\rangle)$ is a $C_{A(S)}(Z)$-isomorphism of $\bar{S}$ with the dual of U.

(iii) $H/S\cong A_5$ and $C_H(S)=Z\in\mathrm{Syl}_2(C_G(S))$ and $S\in\mathrm{Syl}_2(C_G(\bar{S}))$.

(iv) H is irreducible on $\bar{S}$ as a $GF(4)$-module.

(v) $\bar{S}$ is the sum of two natural modules for $S/H\cong A_5$, as a $GF(2)$-module.

(vi) $H\trianglelefteq N_G(S)$.

Proof. Let $S_0=\langle E\cap A, F\cap A\rangle$ and $S_1=\langle I,R\rangle$, where I is $F_0\cap C(F\cap A)$. Clearly S_0 is isomorphic to a Sylow 2-subgroup of $L_3(4)$ and this also holds for S_1 as $S_1=IR$ and $[i,R]=Z=Z(S_1)$ for $i\in I-Z$. Moreover, S is the central product of S_0 and S_1, proving (i). (i) implies (ii); the first two sentences of (ii) are reasonably clear; we supply a proof of the rest. Let $S=T_1*T_2$ with, $T_i\cong S_0$. Let $E_{16}\cong X_{ij}\leq T_i$, $i,j\in\{1,2\}$. Each $v\in V^\sharp$ acts faithfully on some X_{ij}, say X. As $[v,S]\leq Z$, $v\in C(Z)$. This determines $V/C_V(X)$ in $GL(X)\cong L_4(2)$, and we find $V/C_V(X)\leq E_{16}$, and hence $|V|\leq 2^{16}$. On the otherhand in the split extension of X_{ij} by $L_4(2)$ there is U_{ij} with $[U_{ij},X_{i3-j}]=1=U_{ij}\cap T_i=[U_{ij},y_{ij}]$, $[U_{ij},T_i]\leq Z$, and $U_{ij}\cong E_4$, where y_{ij} is of order 3 with $C_{T_i}(y_{ij})=1$. Embed U_{ij} in $\mathrm{Aut}(S)$ by taking $[U_{ij},T_{3-i}]=1$; set $U=\langle U_{ij}: i,j\rangle$. $[U_{ij},U_{rs}]\leq C(T_1)\cap C(T_2)=1$ for $(i,j)\neq(r,s)$, so U is elementary abelian. Similarly $U\cong E_{2^8}$ and $U\cap\bar{S}=1$. So $U\bar{S}\cong E_{2^{16}}$ and as $|V|\leq 2^{16}$, $V=U\bar{S}$. Let y of order 3 with $\langle y\rangle V/V=O_3(A(S))$. Then $\langle y\rangle V/C_V(X_{ij})=\langle y_{ij}\rangle V/C_V(X_{ij})$, so $[y,U]=1$ and hence $U=C_V(y)\trianglelefteq\mathrm{Aut}(S)$. Finally let $z\in Z^\sharp$. If $s\in S$ with $[U,s]\leq\langle z\rangle$, then as $C_{\mathrm{Aut}(S)}(z)$ is irreducible on $\bar{S}$, $[U,S]\leq\langle z\rangle$, a contradiction. Thus $|U: C_U(s\langle z\rangle)|=2$, completing the proof of (ii).

Since $E\in\mathrm{Syl}_2(C_G(E))$, $Z\in\mathrm{Syl}_2(C_G(S))$. $C_G(\bar{S})\leq N_G(R)S$ and $N_S(R)=EF\in\mathrm{Syl}_2(C_G(EF/Z)\cap N(R))$ so $S\in\mathrm{Syl}_2(C_G(\bar{S}))$. Thus $C_H(S)=XZ$, where $X=O(C_H(S))$. By (1.7) $X\leq Z(H)$. We have $|PS/S|=4$ and $PS/S=[PS/S,u]$, when u is a 3-element in $N_A(S)$. So by (ii) together with (2.3) (v) and $H=O^{2'}(H)$, we have $H/S\cong A_5$. Therefore, (iii) holds.

By (ii) one of the following holds: H/S stabilizes a nonsingular 1-space of $\bar{S}$, H/S stabilizes a pair of complementary totally singular 2-spaces of $\bar{S}$, or H/S is irreducible on $\bar{S}$. The first two cases do not occur because of (2.5). There-

fore, (iv) holds, and (iv) implies (v). Finally, (vi) follows from (2.3) (iv) and (1.7).

(2.7) Choose $u \in N_A(S)$ with $|u|=3$ and $[E,u] \neq 1$, and let $y \in N_H(R) \cap C(u)$ with $|y|=3$. Then $u=xy^{\pm 1}$, where $|x|=3$, x induces scalar action on S/Z as an $\boldsymbol{F}_4$-module, and $Z=[Z,x]$.

Proof. $Z=[Z,u]$ and $y \in C(Z)$, so $u \neq y$. Also, u acts on H and acts nontrivially on PS/S. Hence $u=xy^i$ w th x of order 3 in $C(H/S)$ and $i=\pm 1$. By (2.6) (v) $H\langle x\rangle$ acts irreducibly on S/Z as an $\boldsymbol{F}_2$-module, so Schur's lemma shows that x induces an $\boldsymbol{F}_4$ scalar on S/Z.

(2.8) Let $T_0 \in \mathrm{Syl}_2(N_G(S))$ and $\bar{T}_0 = T_0/Z$. Then

(i) $\bar{S}=J(\bar{T}_0)$;

(ii) $T_0 \in \mathrm{Syl}_2(G)$; and

(iii) $Z \trianglelefteq N_G(T_0)$.

Proof. By 2.6. iii, $S=C_{T_0}(\bar{S})$. Thus if (i) fails there is a nontrivial elementary abelian 2-subgroup U of $\mathrm{Aut}_G(\bar{S})$ with $|U| \geq |\bar{S}: C_{\bar{S}}(U)|$, which is impossible from the structure of $\mathrm{Aut}(S)$ described in 2.6. ii.

Let $g \in N_G(T_0)$. We claim $Z^g=Z$. Either $Z=Z(T_0)$, in which case the claim is clear, or $|Z: Z(T_0)|=2$.

In the latter case, $Z(T_0) \leq Z^g$ and $Z^g/Z(T_0) \leq Z(T_0/Z(T_0))$. But using (i) and 2.6 (i), we see that $Z/Z(T_0)=Z(T_0/Z(T_0))$. This proves the claim, and so (ii) follows from (i).

(2.9) (i) $P \cap \Omega = \{E, F_0^{N(E)\cap N(P)}\}$ has order 6.

(ii) $P \in \mathrm{Syl}_2(C_G(E \cap A))$.

(iii) $N_G(P)$ is transitive on $P \cap \Omega$.

Proof. Let $V \in P \cap \Omega$ and B a point of V. Conjugating by $N(E) \cap N(P)$ we may take $B \cap S \neq 1$. Then $B \cap S \leq E$ or $B \cap S \leq F_0$ by (2.3) (ii). As each elementary subgroup of $N(R)$ of rank 6 is a plane through R, $B \leq E$ or $B \leq F_0$, so $V=(E \cap A)B=E$ or F_0. Hence (i) holds.

Clearly $P \in \mathrm{Syl}_2(C_G(E \cap A) \cap N(E))$. So if (ii) is false there is a 2-element $g \in N(P) \cap C(E \cap A)$ such that $E^g \neq E$. Therefore, $N(P)^{(P\cap\Omega)}=A_6$ or S_6. Let $I=N(P) \cap C(E \cap A)$. Then $I^{(P\cap\Omega)} \neq 1$ and is normal in $N(P)^{(P\cap\Omega)}$. So, $I^{(P\cap\Omega)} \geq A_6$ and this forces $S \leq I$, a contradiction. This proves (ii). (iii) now follows from (i), (ii), and the symmetry between E and F_0.

(2.10) $\Omega = E^G$.

Proof. See (2.9) and (2.3) (iii).

(2.11) Set $K=O^2(N_G(P))$. Then

(i) $K/PO(K) \cong 3A_6$.
(ii) $[y, K] \leq PO(K)$.
(iii) $P/(E \cap A)$ is the natural module for $K/PO(K)$.
(iv) $E \cap A$ is the natural module for $K/PO(K)\langle y\rangle \cong A_6$.

Proof. $N_K(E)^{(P\cap\Omega)} \geq A_5$, so by (2.9) $K^{(P\cap\Omega)} = A_6$. $N_K(E) \neq (N_K(E) \cap C(E \cap A))K_{P\cap\Omega}$ so $K \neq C_K(E \cap A)K_{P\cap\Omega}$. Hence $K/C_K(E \cap A) \cong A_6$ acts naturally on $E \cap A$.

$(KP)_{P\cap\Omega} = P(N_{KP}(R)_{P\cap\Omega})$ while $(N_{KP}(R)_{P\cap\Omega})/O(K)R$ acts faithfully on RZ, and hence is a subgroup of E_9. Thus $KP/PO(K)$ is a subgroup of $A_6 \times E_9$ or of $3A_6 \times Z_3$. Choose y as in 2.7. $y \in N_H(R) \leq N(E) \leq N(P)$, while by 2.6 parts (ii) and (v), $(E \cap A)/Z = [P, E/Z] \leq C_{P/Z}(y)$ and $C_{P/Z}(y)$ is a complement to R in $C_S(Z)$. Thus $[y, K] \leq PO(K)$, so $P/E \cap A$ is a faithful $GF(4)$-module for $K/PO(K)$, so $K/PO(K) \leq GL_3(4)$. Then as $K/PO(K) \leq A_6 \times E_9$ or $3A_6 \times Z_3$, the lemma holds.

(2.12) Let $PS \geq T_0 \in \mathrm{Syl}_2(N_G(S))$. Then
(i) $T_0 \in \mathrm{Syl}_2(G)$;
(ii) $SP \leq T = T_0 \cap O^2(N_G(P))$, $|T_0 : T| \leq 2$, and $H\langle x\rangle T/S \cong S_3 \times A_5$;
(iii) $Z_2(T) = Z \neq Z(T)$;
(iv) $E^T = \{E, F_0\}$; and
(v) $P \leq T_0$.

Proof. (i) is just (2.8) (ii). $(E \cap A)/Z = Z(PS/Z)$, so $E \cap A \leq T_0$. Thus (v) follows from (2.9) (ii). By (2.11) and (1.7), $O^2(N_G(P)) = I \times O(C(R))$ where $y \in I$ is the split extension of P by A_6/Z_3. Let J be the setwise stabilizer in $O^2(N_G(P))$ of $\{E, F_0\}$. $PS \leq T = T_0 \cap J \in \mathrm{Syl}_2(J)$, while with (2.11) (ii), $\langle y\rangle(N_A(S) \cap N(P))O(C(A))$ contains a Hall $2'$-group of J, so $J \leq N(S)$. $J/O_2(J)O(C(A)) \cong Z_3 \times S_3$ with $[y, J] \leq O_2(J)O(C(A))$, so $JH/SO(C(R)) = S_3 \times A_5$. Of course $JH = \langle X\rangle TH$. $T_0JH/S \leq S_3 \times S_5$, so $|T_0 : T| \leq 2$. Hence (ii) and (iv) hold. Finally J induces S_3 on Z, so $Z \neq Z(T)$. On the otherhand $Z_2(T) \leq C_{HT}(S/Z) = S$ while by (2.6) (i), $Z(S/Z(T)) = Z/Z(T)$. Hence (iii) holds.

(2.13) Let $K = HT\langle x\rangle$. Then K is the semidirect product of $N_K(\langle x\rangle)$ with S and $N_K(\langle x\rangle)$ is determined up to conjugation in $\mathrm{Aut}(S)$, so that the isomorphism class of K is determined.

Proof. $C_S(x) = 1$ and $C_K(S) = Z$, so K is the semidirect product of $N_K(\langle x\rangle)$ with S by a Frattini argument and we may regard K as a subgroup of $W = N_{\mathrm{Aut}(S)}(\langle x\rangle)$. Choose notation as in 2.6. ii and set $W^* = W/U$. By 2.6. ii and (v), and as the 1-cohomology of the natural module for A_5 is trivial, U is transitive on the complements to U in UH. Thus it remains to show K^* is determined up to conjugacy in W^*, since $C_U(H) = 1$.

Let $t \in T$ invert x with $t^2 \in S$ and $[H, t] \leq S$. As $C_{W^*}(t^*)^\infty \neq 1$, t interchanges the components of W^*, and then as t inverts x, t^* is determined up to conjugacy in W^*. Then $K^* = E(C_{W^*}(t^*))\langle t^* \rangle \langle x^* \rangle$ is determined up to conjugacy in W^*.

(2.14) (i) There exists a unique subgroup Q of T isomorphic to the central product of three quaternion groups and invariant under $\langle y \rangle$.

(ii) $Q \trianglelefteq HT$.

(iii) $|E \cap A : E \cap A \cap Q| = 2$.

Proof. Let $D = Suz$. By (2.13) we may take $HT\langle x \rangle \leq D$. Set $\langle z \rangle = Z(T)$, $C = C_D(z)$ and $Q = O_2(C)$. Then $Q \cong (Q_8)^3$. Set $\tilde{C} = C/\langle z \rangle$ and $C^* = C/Q$. Suppose $B \leq T$ with $B \cong Q \neq B$. Then $\tilde{B} \cong E_{64}$, so $|B^*| \geq |\tilde{Q} : C_{\tilde{Q}}(B)|$. So as $C^* \cong \Omega_6^-(2)$ acts naturally on $\tilde{Q}$, $E_8 \cong B^* \leq O_2(C_{C^*}(Z(T^*))$ with $B^* = C_{C^*}(B^*)$. Suppose $\langle y \rangle \leq N(B)$. Set $C_{C^*}(Z(T^*)) = K^*$ and $\bar{K} = K^*/Z(T^*)$. Then $\bar{B}$ is a 4-subgroup of $\bar{K}$ invariant under $\langle \bar{y} \rangle$, so $\bar{B} = Z(\bar{T}^k)$ for some $k \in C_K(y)$, or $B^* \cong Q_8$. As $B^* \cong E_8$, the first case holds. But then $B^* \neq C_G(B^*) \cong E_{16}$. Thus Q is uniquely determined.

As $Q \trianglelefteq C \geq HT$, $Q \trianglelefteq HT$. $(Q \cap S)/Z$ is an irreducible $GF(2)$-module of S/Z of rank 4 for H/S, so $(E \cap A \cap Q)/Z = C_{Q \cap S/Z}(P)$ is of order 2, and (iii) holds.

(2.15) Set $K = O^2(N_G(P))$ and $\langle z \rangle = Z(T)$. Then

(i) $T \in \mathrm{Syl}_2(K)$.

(ii) $E \cap A \cap Q = Z_3(T) \cap E \cap A$.

(iii) $Q \trianglelefteq C_K(z)$.

Proof. $T_0 \leq N(K)$ by (2.13) and $T \in \mathrm{Syl}_2(K)$ from the definition of T. By (2.11) (iv), $Z_3(T) \cap E \cap A$ is a hyperplane of $E \cap A$. By (2.14) (iii), $E \cap A \cap Q$ is a hyperplane of $E \cap A$ in $Z_3(T)$, so (ii) holds. Then $[Q, Z_3(T) \cap E \cap A] \leq \langle z \rangle$, so $Q \leq O_2(C_K(z))$ by (2.11) (iv). But $C_K(z) = O_2(C_K(z))C_K(\langle z, y \rangle)$, and for $g \in C_K(\langle z, y \rangle)$, $Q^g \leq T$ and $y \in N(Q^g)$, so $Q = Q^g$ by (2.14) (i). Thus $Q \trianglelefteq C_K(z)$.

(2.16) Set $M = \langle T^{N(Q)} \rangle$. Then $M/QO(Z(M)) \cong \Omega_6^-(2)$ acts naturally on $Q/\langle z \rangle$.

Proof. Out $(Q) \cong O_6^-(2)$ with HT/Q a maximal parabolic of $E(\mathrm{Out}(Q))$. So by (2.15) (iii), $\mathrm{Out}_M(Q) \cong \Omega_6^-(2)$. $C_M(Q) = O(M)\langle z \rangle$ and by (1.7), $O(M) \leq Z(M)$.

(2.17) (i) M is transitive on $Z^{C(z)} \cap Q$

(ii) $N(Z) \cap C(z)$ is transitive on the $C(z)$-conjugates of Q containing Z.

Proof. (2.16) implies (i) and (i) implies (ii),

(2.18) (i) $N_G(Z) = HT_0\langle x \rangle O(N_G(Z))$ with $HT \trianglelefteq N_G(Z)$.

(ii) If $g \in C(z)$ and $m(Q \cap Q^g) > 1$, then $Q = Q^g$.

Proof. Set $X=N(Z)$, $\bar{X}=X/Z$. Then by (2.8), $\bar{S}=J(\bar{T}_0)$, so $\bar{S}$ is weakly closed in $N_{\bar{X}}(\bar{S})$. We next show $\bar{S}$ to be strongly closed. If not by Corollary 4 of [5], there is $\bar{B}\leq\bar{S}$ and $g\in X$ such that $\bar{D}=\bar{B}^g\not\leq\bar{S}$ and for $d\in\bar{D}-\bar{S}$, $m([\bar{S},d])\leq m(\bar{D}/\bar{D}\cap\bar{S})$. But $m([\bar{S},t])\geq 2$ for each involution $t\in T_0-S$ by (2.6), so $m(\bar{D}/\bar{D}\cap\bar{S})>1$. Hence by (2.6) there is $d\in\bar{D}-\bar{S}$ with $m([\bar{S},d])=4$, so $m(\bar{D}/\bar{D}\cap\bar{S})\geq 4>m(T_0/S)$, a contradiction.

So $\bar{S}$ is strongly closed. Now by Goldschmidt's fusion Theorem [5], and the action of H on $\bar{S}$, $\bar{S}O(\bar{X})\trianglelefteq\bar{X}$. By (1.7), $S\trianglelefteq X$, so (i) follows from (1.7) and (2.6).

Choose g as in (ii). Then as $m(Q\cap Q^g)>1$, we may take $Z\leq Q^g$. So by (2.17) we may take $g\in X$. Now as $C_X(z)=C_H(z)T_0O(N(Z))$ with $[HT, O(C(Z))]=1$ and $Q\trianglelefteq C_H(z)T_0$, $Q=Q^g$.

Set $X=C(z)$, $\tilde{X}=X/\langle z\rangle$, $N_X(Q)^*=N_X(Q)/Q$.

(2.19) Q is weakly closed in X.

Proof. If $g\in X$ with $Q\neq Q^g\leq N(Q)$, then $\tilde{Q}^g\cong E_{64}$, so as $N_X(Q)^*\cong O_6^-(2)$ or $\Omega_6^-(2)$ acts naturally on $\tilde{Q}$, $m(Q\cap Q^g)>1$. This contradicts (2.18) (ii).

We can now obtain a contradiction. By (2.18) (ii), (2.19), and (1.10), Q is strongly closed in $C_G(z)$. So by Goldschmidt's fusion theorem [5], $QO(X)\trianglelefteq X$. Then (2.16) and (1.8) imply $O(X)=1$, and $M\trianglelefteq X$. By Theorem 2 in [11], and Theorem B of [10], we have $\langle A^G\rangle\cong Suz$, which we are assuming false.

3. Co_1

In this section we assume $A/Z(A)\cong G_2(4)$ and obtain a contradiction; we continue the notation in §1. In particular, let $A_1\cong SL_3(4)$ be as in (1.4) and $\langle x\rangle=Z(A_1)$. Inaddition we set $B=E(C_G(x))$ By (1.3) (iii) $Z(A)=1$.

(3.1) $B=3Suz$, the covering group of the Suzuki group.

Proof. This follows from (8.14) of [3] and the result established in §2.

Since A_1 is standard in B and $R\in\mathrm{Syl}_2(C_B(A_1))$ the entire analysis of §2 applies to the triple (R, A_1, B), replacing (R, A, G). We will make use of the subgroups Z, E, F, P, Q, and T as defined in §1 or constructed in §2. Then $E\cap A$ is the direct product of two long root subgroups of A(or A_1). Let $B_0=C_B(z)'$, for $z\in Z^\sharp$. Then $Q\trianglelefteq B_0$ and $B_0/Q\cong\Omega_6^-(2)$.

(3.2) Let $I=N_A(E\cap A)$.

(i) $I=D(J\times\langle x\rangle)$, where $D=O_2(I)$ and $J\cong SL_2(4)$.

(ii) $(E\cap A)=Z(D)$, $D/(E\cap A)$ is elementary of order 4^3, and $D/(E\cap A)$

is generated by the images of 3 short root subgroups.

(iii) $Z(DJ/(E\cap A))=U_\alpha(E\cap A))=U_\alpha(E\cap A)/(E\cap A)$ for U_α a short root subgroup.

(iv) $[D, U_\alpha]=E\cap A$.

(v) I/D acts indecomposably on $D/(E\cap A)$.

(vi) I/D acts on $D/U_\alpha(E\cap A)$ as on the natural module for $GL_2(4)$.

Proof. These facts are elementary consequences of the Chevalley commutator relations for $G_2(4)$.

(3.3) (i) $E-(E\cap A)$ is partitioned by the sixteen members of $R^G\cap E=\Delta$.

(ii) $N(E)=D(N(E)\cap N(\langle x\rangle))$. In particular $N(E)^\Delta=(N(E)\cap N(\langle x\rangle))^\Delta$, $N_B(E)^\Delta\trianglelefteq N(E)^\Delta$, and $N_B(E)^\Delta$ is $GL_2(4)$ acting on its natural module.

(iii) $\hat{P}=PD=O_2(N(E)\cap C(E\cap A))\in \mathrm{Syl}_2(N(E)\cap C(E\cap A))$ and $\hat{P}^\Delta=P^\Delta$ is regular.

(iv) $C_{\hat{P}}(E)=DC_P(E)=D\times R$.

Proof. (i) is just (1.5) (i). $X=N(E)\cap N(\langle x\rangle)$ is transitive on Δ, so $N(E)=X(N(E)\cap N(R))$. By a Frattini argument and (3.2) (i), $N(E)\cap N(R)=DN_X(R)$, so $N(E)=DX$. Now (ii) follows, and implies (iii) and (iv).

(3.4) (i) $\hat{P}=DP$ with $D\cap P=E\cap A$.

(ii) $D=[\hat{P}, x]$.

(iii) $Z(\hat{P}/(E\cap A))\geq U_\alpha R(E\cap A)/(E\cap A)$.

(iv) $[\hat{P}, \hat{P}]\leq U_\alpha(E\cap A)$.

Proof. By 3.3) (iii), $\hat{P}=DP$, while $D\cap P=C_{P\cap A}(E)=E\cap A$. By (i), $[\hat{P}, x]\leq D$, while by (3.2), $D=[D, x]$, so (ii) holds. J acts on $C_{D/(E\cap A)}(P)$, so by (3.2), $[U_\alpha, P]\leq E\cap A$. Of course $[P, R]\leq E\cap A$, so (iii) holds. Then (3.2) (vi) implies $[P, D]\leq U_\alpha(E\cap A)$, while by (3.2) (ii), $[D, D]\leq E\cap A$, and by (2.11) (iii), $[P, P]\leq E\cap A$. Hence (iv) holds.

(3.5) $D=O_2(C_G(P))\in \mathrm{Syl}_2(C_G(P))$.

Proof. We first show that $[D, P]=1$. Choose $Y\leq C_G(x)$ such that $|Y|=3$, Y is transitive on $R^\sharp$ and $[R, A]=1$ (for example $Y=\langle y\rangle$, with y as in (2.11)). Then $Y\times\langle x\rangle$ contains a subgroup Y_1 of order 3 such that $Y_1\leq C_G(A)$. Then Y_1 acts on $\hat{P}$, $[Y_1, D]=1$ and $[Y_1, P]=P$. Therefore, $[P, Y_1, D]=[P, D]$, $[Y_1, D, P]=[1, P]=1$, and $[D, P, Y_1]\leq[D, Y_1]=1$. By the 3-subgroups lemma, $[P, D]=1$.

Finally, $C_G(P)\leq C_G(R)$ so that $C_G(P)=C(P)\cap C(R)=C_D(P)O(C(A))=DO(C(A))$ by (1.7), so the lemma holds.

(3.6) Let $T_1=T\cap J\in \mathrm{Syl}_2(J)$, $V_0/(E\cap A)=C(T_1)\cap D/E\cap A)$, and $V=[V_0, \langle x\rangle]$.

Then V contains a unique $\langle x\rangle$-invariant subgroup Q_0 such that $Q_0 \cong Q_8$ and $Z(Q_0)=Z(Q)$.

Proof. The action of $J\times\langle x\rangle$ on $D/(E\cap A)$ is easily determined from the Chevalley commutator relations. The group V_0 is the product of $E\cap A$ together with the product of two short root subgroups, where the short roots add to a long root. Then V is the group generated by these two short root subgroups.

The group $V/Z(V)\cong E_{16}$ and $Z(V)=C_V(x)$ is a long root subgroup. Since $\langle x\rangle$ acts without fixed points on $V/Z(V)$, $\langle x\rangle$ stabilizes precisely five 4-subgroups of Y/Z. Aside from the images of the two short root subgroups, there are three subgroups each having preimage containing a unique $\langle x\rangle$-invariant Q_8 and having center of order 2 in $Z(V)=Z$. Since $Z(Q)\leq Z$, the result follows.

(3.7) (i) $T\leq C_G(Q_0)$.
(ii) Q_0Q is extraspecial of order 2^9.
(iii) $Q_0\in \mathrm{Syl}_2(C_G(Q))$.
(iv) $B_0\leq C_G(Q_0)$.

Proof. By (3.5) and the fact that $PT_1\trianglelefteq T$, we have $T\leq N(V_0)$. Since also $T\leq C(x)$, by (3.6), $T\leq N(Q_0)$. As $\langle x\rangle\times T$ acts on Q_0, we necessarily, have (i). In particular, $Q\leq C(Q_0)$, proving (ii).

Let $C=C_G(Q)$ and suppose $Q_0\notin \mathrm{Syl}_2(C)$. Consider $N_{C\langle r\rangle}(Q_0\langle r\rangle)=N$, where $r\in R^\sharp$. First we claim that $Q_0\langle r\rangle$ has index at most 2 in a Sylow 2-subgroup of N. So suppose otherwise and let $Y=C\cap N\cap C(r)$. Then $|C\cap N: Y|\leq 2$ so $Q_0O(Y)<Y$ and $Y\leq C(Q)<C(E\cap A\cap Q)$. By (2.14)(iii), $Y\leq DRO(C(A))\langle x\rangle$. By (1.7) $Y=(\langle x\rangle O_2(Y))\times O(Y)$. Now $AR\cap C(Q)\trianglelefteq AR\cap N(Q)$, so it follows from (3.2) and $Y^X=Y$, that $U_\omega\leq Y$. However, the commutator relations show $U_\omega\not\leq N(Q_0)$, a contradiction. Therefore, the claim holds. We conclude that $N/Q_0\langle r\rangle$ has a 2-complement of index 2.

Both N and $Q_0\langle r\rangle$ are invariant under $\langle x\rangle\times E$. By (1.7) and the above claim we conclude that $|N_C(\langle x\rangle)|$ is divisible by 4. As $N(\langle x\rangle)/O(N(\langle x\rangle))\leq \mathrm{Aut}(Suz)$, this is impossible. This establishes (iii).

To obtain (iv) consider the group C. If $O(C)\neq 1$, the assertion follows from (1.7) and the structure of Co_1. Suppose $O(C)=1$. If $E(C)=1$, then $C=Q_0\langle x\rangle$ and (iv) holds. If $E(C)\neq 1$, then $O^{2'}(C)\cong SL_2(q)$ for some $q\equiv 3,5$ (mod 8) and $[Q_0, B_0]\leq [O^{2'}(C), B_0]=1$.

(3.8) Let $F=N_G(Q_0Q)^{(\infty)}$. Then $Q_0Q\trianglelefteq F$ and $F/Q_0Q\cong \Omega_8^+(2)$.

Proof. By (3.7) $\langle x\rangle\times B_0\leq N_G(Q_0Q)$. Let $M=O^2(N_G(Q_0Q)/C_G(Q_0Q/\langle z\rangle))$. Then $M\leq \Omega_8^+(2)$ and $\langle x\rangle\times B_0$ induces a subgroup M isomorphic to $Z_3\times\Omega_6^-(2)$. Easy arguments show that $(Z_3\times\Omega_6^-(2))\langle t\rangle=M_1$ is maximal in $\Omega_8^+(2)$, where

τ inverts the Z_3 factor and induces a transvection on the $\Omega_6^-(2)$ factor. It will suffice to show that M contains such an element τ and $M > M_1$.

To get τ, use the fact that $N_A(\langle x\rangle)$ contains an involution inverting x. Thus $M_1 \leq M$. The argument in the first paragraph of the proof of (3.7) shows that $[V, T] \leq V$. Since $\langle x\rangle$ acts irreducibly on V/Q_0Z, $[V, T] \leq Q_0Z \leq Q_0Q$. Hence $V \leq N_G(Q_0Q)$ and V induces on $Q_0Q/\langle z\rangle$ a subgroup of M not contained in M_1. This proves (3.8).

(3.9) $N_G(Q_0Q)/Q_0QO(N_G(Q_0Q) \cong \Omega_8^+(2)$.

Proof. Otherwise $\langle x\rangle \times B_0\langle g\rangle \leq C_G(x)$, where g induces a transvection on $Q/\langle z\rangle$. On the otherhand $N_G(\langle x\rangle)/O(N_G(\langle x\rangle) = \mathrm{Aut}(Suz)$, so no such g exists.

(3.10) $C_F(Z)$ contains a normal subgroup $\hat{S}$ such that

(i) $\hat{S}$ is special with $Z(\hat{S}) = Z$, and $\hat{S}$ is the central product of three copies of a Sylow 2-group of $L_3(4)$.

(ii) $C_F(Z)/\hat{S} \cong \Omega_6^+(2)$ has two noncentral chief factors on $\hat{S}/Z$, both of which are natural.

(iii) $\hat{S}$ is weakly closed in $N_G(\hat{S})$ with respect to $N_G(Z)$.

(iv) $N_G(\hat{S})/\hat{S}O(C_G(\hat{S})) \cong S_3 \times \Omega_6^+(2)$ and $C_F(Z)O(C(Z)) = N_G(\hat{S}) \cap C(Z)$.

Proof. F acts on $Q_0Q/\langle z\rangle$ as the natural module for $\Omega_8^+(2)$ and the image of Z is a singular point. So $N_F(Z)/Q_0Q$ is a parabolic subgroup of $\Omega_8^+(2)$ isomorphic to $Q_6^+(2)$ on its natural module. Set $U = C_{Q_0Q}(Z)$ and $\hat{S} = O_2(C_F(Z))$. Then $1 \trianglelefteq Z \trianglelefteq U \trianglelefteq \hat{S} \trianglelefteq C_F(Z)$ is a normal series with U/Z and $\hat{S}/U$ the natural module for $C_F(Z)/\hat{S} \cong \Omega_6^+(2)$. That is (ii) holds.

Next $S = C_{\hat{S}}(x)$ and $\hat{S} = S[\hat{S}, x]$. Moreover by 2.6, $B_1 = C_B(Z)^\infty$ is a subgroup of F acting as $\Omega_4^-(2)$ on S/Z as the sum of two natural modules, and S is the central product of two copies of the Sylow 2-group of $L_3(4)$. Also there is $g \in C_F(Z)$ with $[\hat{S}, x] \leq C_{\hat{S}}(x^g) = S^g$, so $[\hat{S}, x]$ is isomorphic to a Sylow 2-group of $L_3(4)$. As $[\hat{S}, x, B_1] = 1$, $S = [S, B] \leq C([\hat{S}, x])$. Therefore (i) holds.

$V = \hat{S}/Z$ is elementary abelian and if $g \in N(Z)$ with $\hat{S}^g \leq F$ and $\hat{S} \neq \hat{S}^g$, then $V \neq V^g$ and $m(V^g/V \cap V^g) = m(V/V \cap V^g) \geq m(V/C_V(V^g))$, which is impossible by (ii). Thus $\hat{S}$ is weakly closed in F with respect to $N(Z)$.

Let $j \in Q_0Q - C(Z)$ be an involution. Then $[Z, j] = z$ and $[j, C_F(Z)] \leq C_{Q_0Q}(Z) \leq \hat{S}$, so $j \in N(\hat{S})$ and $\langle C_F(Z), j\rangle/\hat{S} \cong Z_2 \times \Omega_6^+(2)$. However from (i), $\mathrm{Out}(\hat{S})$ is the extension of $Z_3 \times O_6^+(4)$ by a field automorphism, so as $[Z, j] \neq 1$, j induces a field or graph-field automorphism, and as j centralizes $C_F(Z)/\hat{S}$, it is the former. In particular $C_F(Z)/\hat{S} = E(\mathrm{Out}(\hat{S})) \cap C(j)$ is maximal in $E(\mathrm{Out}(\hat{S}))$, so if $N_G(\hat{S})^\infty \neq C_F(Z)$, then $N_G(\hat{S})^\infty/\hat{S} \cong \Omega_6^+(4)$. But then as RZ/Z and $(E \cap A)/Z$ are singular points in $\hat{S}/Z$, $RZ \in (E \cap A)^{N(\hat{S})}$, contradiction.

So $C_F(Z)=N_G(\hat{S})^\infty$, and hence by (1.7), $N_G(\hat{S})\cap C(Z)=C_F(Z)O(N_G(\hat{S}))\langle t\rangle$, where either $t=1$ or t induces a $GF(4)$-transvection on $\hat{S}/Z$. In the latter case t acts on $\langle j, U\rangle=Q_0Q$, and (3.9) supplies a contradiction. In particular the second part of (iv) holds. In addition as $\hat{S}$ is weakly closed in F with respect to $N(Z)$, (iii) holds. There is an element of order 3 in A acting nontrivially on Z, so by (iii) and a Frattini argument some 3-element in $N(\hat{S})$ is nontrivial on Z, so that the proof of (iv) is complete.

(3.11) (i) $\hat{S}=O_2(C_G(Z))$.

(ii) $N_G(\hat{S})$ contains a Sylow 2-group of G.

Proof. Claim $\hat{S}$ is strongly closed in $N(\hat{S})$ with respect to $C(Z)$. Assume not. By 3.10, $\hat{S}$ is weakly closed, while $V=\hat{S}/Z$ is an elementary subgroup of $C(Z)^*=C(Z)/Z$. So by Theorem 4 in [5] there is $U\leq V$ and $W=U^g\leq N(V)$ such that $m([V,w])\leq m(W/W\cap V)$ for each $w\in W$. But by 3.10, $m([V,w])\geq 4$ for each involution $w\in N(V)/V$, so $m(W/W\cap V)\geq 4$. As $(N(V)\cap C(Z)^*)/V\cong\Omega_6^+(2)$ has 2-rank 4, $m(W/W\cap V)=4$ and $WV/V=O_2(X/V)$ where X is the stabilizer of a singular point of V. Now if $w\in W-V$ then $m(C_V(w))=8$, so by symmetry between V and V^g, $m(W)\geq 8$. Thus $m(C_V(W))\geq m(V\cap W)\geq 4$, impossible as $m(C_V(W))=2$.

So the claim is established. Now by Goldschmidt's fusion theorem [5] and (1.7) and (3.10) (iv), (i) holds. Moreover if $I\in \mathrm{Syl}_2(N_G(Z))$, then $Z=Z_2(I)$, so (3.10) (iii) and (i) imply (ii).

(3.12) Let $Q_1=Q_0Q$.

(i) If $g\in C(z)$ and $m(Q_1\cap Q_1^g)>1$, then $Q_1=Q_1^g$.

(ii) Q_1 is weakly closed in a Sylow 2-subgroup of $C_G(z)$.

Proof. Suppose $g\in C(z)$ and $m(Q_1\cap Q_1^g)>1$. By (3.8) we may assume $Z\leq Q_1\cap Q_1^g$, and applying (3.8) to $N(Q_1^g)$ we may take $g\in N(Z)$. By (3.11) and (3.10) $C_G(z)\cap N(Z)=C_F(Z)\langle j\rangle O(N(Z))$ and by (1.7) $[O(N/Z)), C_F(Z)\langle j\rangle]=1$. Since $C_F(Z)\langle j\rangle\leq N(Q_1)$ we conclude that $g\in N(Q_1)$, proving (i).

To prove (ii), suppose $g\in C(z)$ and $Q_1^g\leq N(Q_1)$. By (3.9) $Q_1^gQ_1/Q_1\leq\Omega_8^+(2)$. If $Q_1^g\neq Q_1$, then by (i) $m(Q_1\cap Q_1^g)=1$, so $(Q_1^g\cap Q_1)/\langle z\rangle$ is an anisotropic 1-space or 2-space. In the first case $m(Q_1^gQ_1/Q_1)=7$ and $Q_1^gQ_1/Q_1$ is a subgroup of $Sp_6(2)$, while in the second case $m(Q_1^gQ_1/Q_1)=6$ and $Q_1^gQ_1/Q_1$ is a subgroup of $O_6^-(2)$. In either case we have a contradiction.

As in §2 we can now reach a contradiction. By (3.12) and (1.10), Q_1 is strongly closed in $C_G(z)$, so by Goldschmidt's fusion theorem [5] $Q_1O(C_G(z))\trianglelefteq C_G(z)$. By (1.7) and (1.8) $O(C_G/(z)=1$. Finally, (3.9) and Patterson's theorem [9] yield $G\cong Co_1$, which we have assumed to be false.

4. He

In this section we assume $|Z(A)|$ is even. By (1.6) $Z_2\times Z_2\cong R\in \mathrm{Syl}_2(Z(A))$.

(4.1) (i) $N(E)/C(E)$ contains $3A_6$ and induces S_6 on $R^G\cap E$. Similarly for F.
(ii) There is an element g of order 3 and an involution y such that $\langle g, y\rangle\cong S_3$ and $\langle g, y\rangle$ induces S_3 on R.

Proof. By (1.6) $N(E)/C(E)$ and $N(F)/C(F)$ contain $3A_6$. By (1.7) $N(E)^{(\infty)}=EL$, where $L\cong 3A_6$ and $\langle g\rangle=Z(L)$ acts as an outer diagonal automorphism of A. Now $C_A(g)\cong A_5$ and we may assume that $F_1=F\cap C_A(g)\in \mathrm{Syl}_2(C_A(g))$. Set $J=N_L(F_1)\cong S_4\times Z_3$. Then $EJ\leq N(C_E(F_1)F_1)=N(F)$.

Let bars denote images in $N(F)/C(F)$ and suppose $\overline{N(F)}=3A_6$. Then $\overline{EJ}\cong S_4\times Z_3$ and $Z(\overline{EJ})=Z(\overline{N(F)})$. This forces $\langle \bar{g}\rangle=Z(\overline{N(F)})$, whereas $[\bar{E}, \bar{g}]=\bar{E}$. Consequently $N(F)$ induces S_6 on $R^G\cap F$. By symmetry, (i) holds. Consequently, $N(E)\cap N(R)$ induces S_5 on $R^G\cap E-\{R\}$. and (ii) follows.

(4.2) Let $S=EF$ and $y\in S_1\in \mathrm{Syl}_2(N(S))$. Then either
(i) $S_1\in \mathrm{Syl}_2(G)$ and $S_1/S\cong E_4$, or
(ii) $S_1/S\cong D_8$ and $E\sim F$ in $N(A)$.

Proof. Let $S_2=N_{S_1}(E)$. By 4.1, $S_2/S\cong E_4$. As E and F are the unique elementary abelian subgroups of S of order 2^6 we conclude either $S_1/S\cong D_8$ or $S_1=S_2$. In the first case $E\in F^{N(S)}$ and as $N(E)$ is transitive on $R^G\cap E$, $N(R)$ is transitive on $E^G\cap N(R)$, so $E\in F^{N(A)}$ and (ii) holds. In the second case we show $S=J(S_1)$, to conclude $S_1\in \mathrm{Syl}_2(G)$, so that (i) holds. If not there exists $E_{2^6}\cong U\leq S_1$ with $U\neq E$ or F. Then

$$(*)\qquad |\mathrm{Aut}_U(E)|\geq |E: C_E(U)|.$$

But by 4.1.i, the representation of $\mathrm{Aut}_G(E)$ on E is determined and (*) forces $\mathrm{Aut}_U(E)=\mathrm{Aut}_F(E)$, so that $U\leq UE=FE=S$.

(4.3) $S_1\in \mathrm{Syl}_2(G)$.

Proof. Suppose otherwise and let $g\in N(S_1)-S_1$ with $g^2\in S_1$. Then $S^g\neq S$. Let $Z=Z(S)=E\cap F$. If $Z^g=Z$, then g stabilizes the two element set $R^G\cap Z$. So, for some $s\in S_1$, $gs\in N(R)$ and it follows that $g\in S_1$. Suppose, then, that $Z^g\neq Z$.

We have $Z=S'$, so $Z^g=(S')^g$. By (4.2) $|E^g\cap S|\geq 2^4$ and so either $(E^g\cap S)Z$ or $(F^g\cap S)Z$ is elementary of order at least 2^5, say the former. Therefore, $(E^g\cap S)Z\leq E$ or F and $S^g\leq N(E)$ or $N(F)$. Apply (4.2) to conclude that $Z^g=(S^g)'\leq S$. Now $S\cap S^g\leq C(ZZ^g)$ and $ZZ^g\leq E$ or F. Since $|S^gS: S|\leq 4$ we necessarily have $|S\cap S^g|=2^6$ and $|S^gS: S|=4$. Then

$S \cap S^g = E$ or F, so $g \in N(E)$ or $N(F)$. But this is not the case.

(4.4) (i) $N_G(S)/SO(C(S)) \cong S_3 \times S_3$ or $S_3 \wr Z_2$

(ii) The structure of S_1 is uniquely determined by $|S_1| = 2^{10}$ or 2^{11}.

Proof. Let $A(S) = \mathrm{Aut}(S)/C_{\mathrm{Aut}(S)}(S/Z(S))$. As $S \in \mathrm{Syl}_2(A)$ and $E_4 \cong R \in \mathrm{Syl}_2(Z(A))$ with $A/Z(A) \cong L_3(4)$, we may calculate in A to determine $Z(S) = E \cap F$ is partitioned by

$$\{R, R_0\} \cup \{[E, s]: s \in S\}$$

where $R_0 = [Z(S), x]$ and x is of order 3 in $N_A(S) - Z(A)$. $N_G(S) \leq N_G(Z(S))$, so $N_G(S)$ acts transitively on the two member set $R^G \cap Z(S) = \{R, R_0\}$ and $|N_G(S): N(R) \cap N_G(S)| = 2 = |\mathrm{Aut}(S): N_{\mathrm{Aut}(S)}(R)|$. $\mathrm{Out}_{\mathrm{Aut}(A)}(S) \cong S_3 \times S_3 \cong A(S/R)$ and $N_{A(S)}(R)$ is isomorphic to a subgroup of $A(S/R)$, so $A(S) \cong S_3 \wr Z_2$ and $N_{A(S)}(R) \cong S_3 \times S_3$. $\mathrm{Out}_{N(E)}(S) \cong S_3 \times S_3$, so (i) holds.

Let $T \in \mathrm{Syl}_3(A\langle g \rangle \cap N(S))$, and choose T so that $S_1 = SN_{S_1}(T)$. $C_S(T) = 1 = C(T) \cap C_{\mathrm{Aut}(S)}(S/Z(S))$ as T is irreducible on $S/Z(S)$. Thus the product is semidirect and $N_{S_1}(T) \leq A(S) \cong S_3 \wr Z_2$. Next by 4.2, $N_{S_1}(T) \cong E_4$ or D_8, and in the former case $N_{S_1}(T) \leq N(E)$. Thus $|S_1| = 2^{10}$ or 2^{11}, $TN_{S_1}(T) = N_{A(S)}(E)$ or $A(S)$, and $S_1 T$, and hence also S_1, is uniquely determined by $|S_1|$.

(4.5) (i) S_1 is isomorphic to a Sylow 2-group of He or $\mathrm{Aut}(He)$.

(ii) S_1 contains a unique extraspecial 2-subgroup Q of order 2^7 with $Z(Q) = Z(S_1)$.

(iii) $Q \leq N(E) \cap N(F)$.

(iv) $S_1/Q \cong D_8$ or D_{16}.

(v) $Q \cong (D_8)^3$.

Proof. (i) follows from (4.4) and the fact that the results obtained so far apply to He and $\mathrm{Aut}(He)$. In particular we can embed S_1 as a Sylow 2-group of $G_1 = He$ or $\mathrm{Aut}(He)$. Let $\langle z \rangle = Z(S_1)$, $C = C_{G_1}(z)$, and $Q = O_2(C)$. Then (iii), (iv), and (v) follow from the structure of G_1. Moreover $C/Q \cong L_3(2)$ or $PGL_2(7)$, with $E(C/Q)$ acting on $V = Q/\langle z \rangle$ as the sum of the natural module and its dual. In particular this forces $V = J(S/\langle z \rangle)$, so Q is unique, and (ii) holds.

(4.6) Let $\langle z \rangle = Z(Q)$, $X = E$ or F, and $I_X = O^{2'}(C(z) \cap N(X))$. Then

(i) $I_X \cong E_{64}(S_4 \times Z_2)$.

(ii) $I_X \not\leq N(R)$.

(iii) $|Q \cap X| = 16$.

(iv) $Y = \langle I_E, I_F \rangle \leq N(Q)$.

Proof. By (4.1) and (1.7) $O^{2'}(N_G(X)) = L \cong S_6/Z_3/E_{64}$, and $E(L/X)$ acts naturally on X. In particular $I_X = C_L(z) \cong E_{64}(S_4 \times Z_2)$. As $S_1 \cap N(X) \not\leq N(R)$,

(ii) holds. By (4.5) (v), $m(X \cap Q) \leq 4$ and by (4.5) (iv), $m(X/X \cap Q) \leq 2$, so (iii) holds. By (iii), $QX/X \cong E_8$, so as $L/X \cong S_6/Z_3$, $N_L(QX)/X \cong Z_2 \times S_4$. As $\langle z \rangle = Z(QX)$, $\langle z \rangle \trianglelefteq N_L(QX)$, so $I_X = N_L(QX)$. Hence (iv) holds.

(4.7) (i) $Y/Q \cong L_3(2)$.
(ii) $Q/\langle z \rangle$ is the sum of the natural module for Y/Q and its dual.
(iii) $N(Q)/QO(N(Q)) \cong L_3(2)$ or $PGL_2(7)$.

Proof. By (1.7) we may take $O(N(Q))=1$. Embed S_1 in G_1 as in 4.5, and adopt the notation of that lemma. Let V_1 and V_2 be the two $E(C/Q)$-chief factors in $V = Q/\langle z \rangle$. Then EQ/Q centralizes a hyperplane E_1 of V_1 and a point E_2 of V_2, with $E_1E_2 = [V, E]$. As $[E, Q] \leq E \cap Q \cong E_{16}$, $E_1E_2 = (E \cap Q)/\langle z \rangle$. In particular each member of $E - Q$ induces an involution of type a_2 on V, and EF induces automorphisms in $\Omega_6^-(2)$ on V. Therefore $Y = \langle E^Y, F^Y \rangle$ induces automorphisms in $\Omega_6^-(2) \cong A_8$ on V. $EFQ = S_1 \cap YQ$ with $EF/Q \cong D_8$ and $Y = O^{2'}(Y) = O^2(Y)$, so $YQ/Q \cong A_6$, A_7, or $L_3(2)$. However there is one class each of A_6's and A_7's and two classes of $L_3(2)$'s in A_8. As the involutions in EFQ/Q are of type a_2, we conclude (i) and (ii) holds. Similarly as $S_1/Q \cong D_8$ or D_{16} and $Y/Q \cong L_3(2)$ is a transitive subgroup of $N_G(Q)^\infty/Q \leq A_8$, (iii) holds.

(4.8) Q is strongly closed in S_1 with respect to $C(z)$.

Proof. By (4.5) (ii), Q is weakly in S_1 with respect to $C(z)$. Set $\bar{N}(Q) = N(Q)/QO(N(Q))$ and $C(z)^* = C(z)/\langle z \rangle$, so that $V = Q^* \cong E_{64}$. Assume Q is not strongly closed. By (2.4) of [12], there exists $g \in C(z)$ such that, setting $L = \langle Q, Q^g \rangle$, $B = N_Q(Q^g)$, $D = Q^g \cap N(Q)$, and $I = Q \cap Q^g$, the following hold:
(1) $L/BD \cong L_2(2^n)$, $Sz(2^n)$, or D_{2m}, m odd;
(2) BD/I is the sum of natural modules for L/BD; and
(3) $I \neq D$.

$m(\bar{D}) \leq m(\bar{S}) = 2$. But by Corollary 4 in [5], $m([V, d]) \leq m(\bar{D})$ for each $d \in D - I$, while by (4.7), $m([V, s]) \geq 2$ for each $s \in S_1 - Q$. Hence $m(\bar{D}) = 2$ and $m([V, d]) = 2$ for each $d \in D - I$. By (4.7) it follows that $\bar{D} \leq E(\bar{N}(Q))$ and that $[D, V] = C_V(D)$ is of rank 3. But $B = [Q, V]I$, so $B^* = C_V(D)$ is of codimension at most 2 in V, a contradiction.

(4.9) (i) $Q = F^*(C_G(z))$.
(ii) $C_G(z)/Q \cong PGL_2(7)$.

Proof. By 4.8 and Goldschmidt's fusion theorem [5], $QO(C(z)) \trianglelefteq C(z)$. By (4.7) and (1.8) $O(C(z)) = 1$. If $C(z) = Y$, then by [4], $G \cong He$, contrary to our assumption that G is a counter example to the Main Theorem. So (4.7) completes the proof.

(4.10) $G \neq O^2(G)$.

Proof. All involutions in EF are fused to z or $r \in R^\sharp$ in $N_G(E)$ and $N_G(F)$. All involutions in Y are fused into EF under Y. But by (4.9) (ii) $|S_1|=2^{11}$, so as $R^G \cap Z(S)$ is of order 2, $|S_1 \cap N(R)|=2^{10}$. In particular some involution $t \in S_1 \cap N(R)-Y$ induces a graph-field automorphism on A. Then $[R, t]=1$ and $C_A(t)/R \cong E_9 Q_8$. Then $m_3(C_G(t))>1$, so by (4.9) $t \notin z^G$. Hence if (4.10) is false, $t \in r^G$ by Thompson transfer. As $[R, t]=1$, this contradicts (1.1).

As G is simple, (4.10) yields a contradiction. This completes the proof of the Main Theorem.

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