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$q$ -crystal structure on type  $B$  signed unimodal  
factorizations of reduced words and type  $D$  signed  
unimodal factorizations of flattened words

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This thesis is based on the following paper.

## Paper related to the thesis

T. Hiroshima,  *$\mathfrak{q}$ -crystal structure on primed tableaux and on signed unimodal factorizations of reduced words of type  $B$* , to appear in Publ. RIMS Kyoto Univ.

## Presentations

- T. Hiroshima, *The branching rule of  $C_n$ -type and Kashiwara crystals*, Algebraic Lie Theory and Representation Theory 2016, Nagano, Japan, June 10, 2016.
- T. Hiroshima, *Crystal interpretation of a formula on the branching rule of types  $B_n$ ,  $C_n$  and  $D_n$* , Algebraic Lie Theory and Representation Theory 2017, Shizuoka, Japan, June 12, 2017.
- T. Hiroshima,  *$\mathfrak{q}$ -crystal structure on primed tableaux and on signed unimodal factorizations of reduced words of type  $B$* , Algebraic Lie Theory and Representation Theory 2018, Nagano, Japan, May 28, 2018.

# Abstract

Crystal basis theory for the queer Lie superalgebra was developed in [10, 11], where it was shown that semistandard decomposition tableaux admit the structure of crystals for the queer Lie superalgebra or simply  $\mathfrak{q}$ -crystal structure. In this thesis, we explore the  $\mathfrak{q}$ -crystal structure of signed unimodal factorizations of reduced words of type  $B$  as well as that of signed unimodal factorizations of flattened words of type  $D$ . To do so, we construct the explicit odd Kashiwara operators on primed tableaux [14] (semistandard marked shifted tableaux [5],  $P$ -tableaux [23], or semistandard shifted Young tableaux [29]). The forms of the highest and lowest weight vectors of primed tableaux are also given. We clarify the relation between signed unimodal factorizations of reduced (flattened) words of type  $B$  (resp.  $D$ ) and the type  $B$  (resp.  $D$ ) Coxeter-Knuth relation. We also give the explicit algorithms for odd Kashiwara operators on signed unimodal factorizations of reduced words of type  $B$ , which are also applicable on signed unimodal factorizations of flattened words of type  $D$  without any alterations.

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# Chapter 1

## Introduction

Kashiwara introduced the theory of crystal bases of an integrable highest weight module over the quantum group  $U_q(\mathfrak{g})$  for a symmetrizable Kac-Moody algebra  $\mathfrak{g}$  [18, 19]. In particular, crystal graphs are extremely well-behaved with respect to tensor products, which makes Kashiwara crystals or simply crystals one of the most important and successful tools to study representations and combinatorics. So far, many combinatorial models have been introduced for crystals, which include Young tableaux model [20, 26], Littelmann path model [27], the alcove model [25], etc. In [28], Morse and Schilling gave a crystal structure in the decreasing factorizations of reduced words of the symmetric group or reduced words of type  $A$ , where Edelman-Greene (EG) insertion [8] plays a crucial role. In this thesis, we will extend their results beyond type  $A$ . In the type  $B$  (resp.  $D$ ) setting, a signed unimodal factorization of reduced (resp. flattened) words of type  $B$  (resp.  $D$ ) is the counterpart of the decreasing factorization of reduced words of type  $A$  and the primed Kraśkiewicz insertion of type  $B$  (resp.  $D$ ) [14] plays the role of EG insertion.

We will show that signed unimodal factorizations of reduced (resp. flattened) words of type  $B$  (resp.  $D$ ) admit the structure of crystals for the queer Lie superalgebra or simply  $\mathfrak{q}$ -crystals. The theory of  $\mathfrak{q}$ -crystals was developed by Grantcharov et al. and semistandard decomposition tableaux were shown to admit the  $\mathfrak{q}$ -crystal structure [10, 11]. Recently, Hawkes et al. [14] gave a bijection between the set of semistandard unimodal tableaux and that of primed tableaux of the same shape. The semistandard unimodal tableaux are the same as the semistandard decomposition tableaux originally introduced by Serrano [29] and a bijection between the set of

semistandard decomposition tableaux defined in [10, 11] and that defined in [29] was established by Choi et al. [7]. Combining these results, it follows that primed tableaux form a  $\mathfrak{q}$ -crystal. Recently, Assaf and Oğuz [1, 2] constructed the even (or ordinary) Kashiwara operators on semistandard marked shifted tableaux by using Stembridge's local axioms [30]. They further augmented the even Kashiwara operators they gave to the odd Kashiwara operators and constructed a  $\mathfrak{q}$ -crystal structure on semistandard marked shifted tableaux [1, 2]. We construct the odd Kashiwara operators on primed tableaux along the lines of [14]. That is, we give the odd Kashiwara operators on primed tableaux by translating the  $\mathfrak{q}$ -crystal structure of words through Serrano's semistandard mixed shifted insertion [29]. The method of [1, 2] and that in this thesis are completely different but the results are the same. By exploiting the  $\mathfrak{q}$ -crystal structure of primed tableaux thus obtained, we impose a  $\mathfrak{q}$ -crystal structure on signed unimodal factorizations [14] (with the fixed number of factors) of reduced words of type  $B$  and on those of flattened words of type  $D$ .

In more detail, the main results of the present thesis are summarized as follows: Firstly, as in [14], we employ the semistandard mixed shifted insertion to show that the set of primed tableaux forms a  $\mathfrak{q}$ -crystal and that the odd Kashiwara operators on a primed tableau are independent of a choice of a recording tableau in the above insertion scheme (Theorem 3.3). Secondly, we calculate the highest and lowest weight vectors of primed tableaux by using the rules of ordinary Kashiwara operators on a primed tableau developed in [14] (Theorem 3.4). Thirdly, by the  $\mathfrak{q}$ -crystal structure of primed tableaux thus obtained and by the type  $B$  (resp.  $D$ ) primed Kraśkiewicz insertion, we show that the set of signed unimodal factorizations (with the fixed number of factors) of type  $B$  (resp.  $D$ ) Coxeter-Knuth related reduced (resp. flattened) words admits a  $\mathfrak{q}$ -crystal structure (Theorems 4.4 and 5.4). We also give the explicit rules of odd Kashiwara operators on a signed unimodal factorization of reduced words of type  $B$ , which are also applicable to a signed unimodal factorization of flattened words of type  $D$ .

The thesis is organized as follows: In Chapter 2, we review  $\mathfrak{q}$ -crystals and the model of semistandard decomposition tableaux as a realization of the  $\mathfrak{q}$ -crystal. The main combinatorial objects in Chapter 3 are primed tableaux or semistandard marked shifted tableaux. We give the odd Kashiwara operators on a primed tableau by using the semistandard mixed shifted insertion and also give the forms of the highest and lowest weight vectors of primed tableaux. Chapters 4 and 5 concern signed unimodal factorizations of re-

duced words of type  $B$  and those of flattened words of type  $D$ , respectively. We point out that such factorizations with the fixed number of factors form a  $\mathfrak{q}$ -crystal. The explicit algorithms for odd Kashiwara operators are also given in Chapter 4.

## Chapter 2

# Crystals for the queer Lie superalgebra

### 2.1 Crystals for the queer Lie superalgebra

Let us start by briefly sketching crystals for the general linear Lie algebra  $\mathfrak{gl}(n)$  or simply  $\mathfrak{gl}(n)$ -crystals [4, 16, 19]. Let  $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$  be the weight lattice and  $P^\vee = \bigoplus_{i=1}^n \mathbb{Z}k_i$  be the dual weight lattice with  $\langle \epsilon_i, k_j \rangle = \delta_{ij}$  for  $(i, j = 1, \dots, n)$ . Let  $\{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n\}$  be the set of simple roots and  $\{h_i = k_i - k_{i+1} \mid 1 \leq i < n\}$  be the set of simple coroots. Let

$$P^+ = \{\lambda \mid \lambda \in P, \langle \lambda, h_i \rangle \geq 0 \ (i = 1, \dots, n-1)\}$$

be the set of dominant integral weights.

**Definition 2.1** *A set  $B$  together with maps  $\text{wt} : B \rightarrow P$  and  $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \sqcup \{\mathbf{0}\}$  is called a  $\mathfrak{gl}(n)$ -crystal if the following properties are satisfied for  $i = 1, \dots, n-1$ : when we define*

$$\varepsilon_i(b) = \max \{k \geq 0 \mid \tilde{e}_i^k b \in B\},$$

*and*

$$\varphi_i(b) = \max \left\{ k \geq 0 \mid \tilde{f}_i^k b \in B \right\},$$

*for  $b \in B$ , then*

$$(1) \ \varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z}_{\geq 0} \text{ and } \varphi_i(b) = \varepsilon_i(b) + \langle \text{wt}(b), h_i \rangle,$$

- (2) if  $\tilde{e}_i b \neq \mathbf{0}$ , then  $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ ,  $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ , and  $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ ,
- (3) if  $\tilde{f}_i b \neq \mathbf{0}$ , then  $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ ,  $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ , and  $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ ,
- (4) for  $b, b' \in B$ ,  $\tilde{f}_i b = b' \iff \tilde{e}_i b' = b$ .

The maps  $\tilde{e}_i$  and  $\tilde{f}_i$  are called Kashiwara operators and  $\text{wt}(b)$  is called the weight of  $b$ . In Definition 2.1,  $\mathbf{0}$  is a formal symbol;  $b = \mathbf{0}$  implies  $b \notin B$ .

**Remark 2.1** *More precisely, crystals in Definition 2.1 are called semiregular [16] or seminormal [4]. Throughout this thesis, we consider only semiregular or seminormal crystals.*

**Definition 2.2 (tensor product rule)** *Let  $B_1$  and  $B_2$  be  $\mathfrak{gl}(n)$ -crystals. The tensor product  $B_1 \otimes B_2$  is defined to be the set*

$$B_1 \times B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$$

*whose crystal structure is defined by*

- (1)  $\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2)$ ,
- (2)  $\varepsilon_i(b_1 \otimes b_2) = \max \{\varepsilon_i(b_2), \varepsilon_i(b_1) - \langle \text{wt}(b_2), h_i \rangle\}$ ,
- (3)  $\varphi_i(b_1 \otimes b_2) = \max \{\varphi_i(b_1), \varphi_i(b_2) + \langle \text{wt}(b_1), h_i \rangle\}$ ,
- (4)  $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \end{cases}$
- (5)  $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \end{cases}$

*for  $i = 1, \dots, n-1$ .*

**Remark 2.2** *Note that this definition is different from that by the convention of Kashiwara. Since we use Kashiwara operators on primed tableaux which are constructed by the anti-Kashiwara convention for the tensor product rule [14], we adopt Definition 2.2.*

Next, let us describe crystals for the queer Lie superalgebra  $\mathfrak{q}(n)$  or simply  $\mathfrak{q}(n)$ -crystals introduced in [10, 11].

**Definition 2.3** *A  $\mathfrak{q}(n)$ -crystal is a set  $B$  together with the maps  $\text{wt} : B \rightarrow P$ ,  $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z}_{\geq 0}$  and  $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \sqcup \{\mathbf{0}\}$  for  $i \in I := \{1, \dots, n-1, \bar{1}\}$  satisfying the following conditions:*

- (1)  $B$  is a  $\mathfrak{gl}(n)$ -crystal with respect to  $\text{wt}$ ,  $\varepsilon_i$ ,  $\varphi_i$ ,  $\tilde{e}_i$  and  $\tilde{f}_i$  for  $i = 1, \dots, n-1$ ,
- (2)  $\text{wt}(b) \in \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \epsilon_i$  for  $b \in B$ ,
- (3)  $\text{wt}(\tilde{e}_{\bar{1}}b) = \text{wt}(b) + \alpha_1$ ,  $\text{wt}(\tilde{f}_{\bar{1}}b) = \text{wt}(b) - \alpha_1$  for  $b \in B$ ,
- (4)  $\tilde{f}_{\bar{1}}b = b'$  if and only if  $b = \tilde{e}_{\bar{1}}b'$  for all  $b, b' \in B$ ,
- (5) for  $3 \leq i \leq n-1$ , we have
  - (i) the operators  $\tilde{e}_{\bar{1}}$  and  $\tilde{f}_{\bar{1}}$  commute with  $\tilde{e}_i$  and  $\tilde{f}_i$ ,
  - (ii) if  $\tilde{e}_{\bar{1}}b \in B$ , then  $\varepsilon_i(\tilde{e}_{\bar{1}}b) = \varepsilon_i(b)$  and  $\varphi_i(\tilde{e}_{\bar{1}}b) = \varphi_i(b)$ .

The crystal associated with an irreducible highest weight module  $V(\lambda)$  in the category  $\mathcal{O}_{int}^{\geq 0}$  [9, 12] of tensor representations over the  $\mathfrak{q}(n)$ -quantum group  $U_q(\mathfrak{q}(n))$  is denoted by  $B_n(\lambda)$ . We identify the partition  $\lambda$  with  $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in P^+$  in a usual way [16]. In  $B_n(\lambda)$ , the relevant partition  $\lambda$  is a strict partition  $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_l > \lambda_{l+1} = 0)$ , where  $l \leq n$ . A crystal  $B$  can be viewed as an oriented colored graph with colors  $i \in I$  when we define  $b \xrightarrow{i} b'$  if  $\tilde{f}_i b = b'$  ( $b, b' \in B$ ). This graph is called a crystal graph. The crystal graph of  $B_n(\square)$ , i.e.,  $B_n(\lambda)$  for  $\lambda = \epsilon_1 \in P^+$ , is given by

$$\begin{array}{c} \boxed{1} \xrightleftharpoons[\bar{1}]{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \\ \cdot \end{array}$$

For  $\mathfrak{q}(n)$ -crystals  $B_1$  and  $B_2$ , the tensor product  $B_1 \otimes B_2$  is a  $\mathfrak{gl}(n)$ -crystal, where actions of  $\tilde{e}_{\bar{1}}$  and  $\tilde{f}_{\bar{1}}$  on  $b_1 \otimes b_2$  ( $b_1 \in B_1$ ,  $b_2 \in B_2$ ) are prescribed as

$$\tilde{e}_{\bar{1}}(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_{\bar{1}}b_2, & \text{if } \langle \text{wt}(b_1), k_1 \rangle = \langle \text{wt}(b_1), k_2 \rangle = 0, \\ \tilde{e}_{\bar{1}}b_1 \otimes b_2, & \text{otherwise,} \end{cases}$$

$$\tilde{f}_1(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{f}_1 b_2, & \text{if } \langle \text{wt}(b_1), k_1 \rangle = \langle \text{wt}(b_1), k_2 \rangle = 0, \\ \tilde{f}_1 b_1 \otimes b_2, & \text{otherwise.} \end{cases}$$

Under these rules,  $B_1 \otimes B_2$  is also a  $\mathfrak{q}(n)$ -crystal.

**Remark 2.3** *These rules are different from the ones given in [6, 10, 11] because we adopt the anti-Kashiwara convention for the tensor product rule.*

Let  $B$  be a  $\mathfrak{q}(n)$ -crystal and suppose that  $B$  is in a class of normal  $\mathfrak{gl}(n)$ -crystal [4]. Here, the connected components in  $B$  are referred to as the maximal subcrystals of  $B$ , where all the elements are connected by even Kashiwara operators. We define the automorphism  $S_i$  on  $B$  by

$$S_i(b) = \begin{cases} \tilde{f}_i^{\langle \text{wt}(b), h_i \rangle} b, & \text{if } \langle \text{wt}(b), h_i \rangle \geq 0, \\ \tilde{e}_i^{-\langle \text{wt}(b), h_i \rangle} b, & \text{if } \langle \text{wt}(b), h_i \rangle < 0, \end{cases}$$

for  $b \in B$  and  $i = 1, 2, \dots, n-1$ . Let  $w$  be an element of Weyl group  $W_n$  of  $\mathfrak{gl}(n)$  which is generated by simple reflections  $s_i$  for  $i = 1, \dots, n-1$ . For  $w \in W_n$ , the length of  $w$  is defined to be  $l(w) = \min\{l \mid \exists i_1, \dots, i_l, w = s_{i_1} \cdots s_{i_l}\}$ . There exists a unique action  $S_w : B \rightarrow B$  of  $W$  on  $B$  such that  $S_{s_i} = S_i$  for  $i = 1, \dots, n-1$  [19]. Let us set  $w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}$ . Then  $w_i$  is the element of shortest length in  $W_n$  such that  $w_i(\alpha_i) = \alpha_1$  for  $i = 2, \dots, n-1$ . We define new operators by

$$\tilde{e}_i = S_{w_i^{-1}} \tilde{e}_1 S_{w_i}, \quad \tilde{f}_i = S_{w_i^{-1}} \tilde{f}_1 S_{w_i},$$

where  $S_{w_i} = S_2 \cdots S_i S_1 \cdots S_{i-1}$  and similar for  $S_{w_i^{-1}}$ . These operators together with  $\tilde{e}_1$  and  $\tilde{f}_1$  are called *odd* Kashiwara operators, while  $\tilde{e}_i$  and  $\tilde{f}_i$  ( $i = 1, \dots, n-1$ ) are called *even* Kashiwara operators.

**Theorem 2.1** ([10, 11]) *Let  $B_n(\lambda)$  be a  $\mathfrak{q}(n)$ -crystal. There is a unique element  $b \in B_n(\lambda)$  such that  $\tilde{e}_i b = \tilde{e}_i b = \mathbf{0}$  for all  $i = 1, \dots, n-1$ , which is called a  $\mathfrak{q}(n)$ -highest weight vector and there is a unique element  $b \in B_n(\lambda)$  called a  $\mathfrak{q}(n)$ -lowest weight vector such that  $S_{w_0} b$  is a  $\mathfrak{q}(n)$ -highest weight vector, where  $w_0$  is the element of longest length in  $W_n$ .*

## 2.2 Semistandard decomposition tableaux

One of the combinatorial models of  $\mathfrak{q}(n)$ -crystals is the model of semistandard decomposition tableaux [10, 11], which we will describe in this section. To begin with, we introduce necessary notation and definitions, some of which will be used in subsequent chapters.

Let  $\mathcal{P}^+$  denote the set of strict partitions,  $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l > \lambda_{l+1} = 0)$ . For  $\lambda \in \mathcal{P}^+$ , the length  $l(\lambda)$  of  $\lambda$  is defined as the number of positive parts of  $\lambda$  and the size of  $\lambda$  is defined to be  $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$ . For  $m = |\lambda|$ , we write  $\lambda \vdash m$ . The *shifted diagram* of shape  $\lambda \in \mathcal{P}^+$  is defined to be the set

$$S(\lambda) = \{(i, j) \in \mathbb{N}^2 \mid i \leq j \leq \lambda_i + i - 1, 1 \leq i \leq l(\lambda)\}.$$

A filling  $T$  of  $S(\lambda)$  with letters is called a *shifted tableau*. We denote by  $T_{i,j}$  the entry at  $(i, j)$ -position of  $T$  and by  $T_i$  the sequence of entries in the  $i$ -th row of  $T$ , reading from left to right, i.e.,  $T_{i,i}T_{i,i+1} \cdots T_{i,i+\lambda_i-1}$ . A shifted tableau without any boxes is denoted by  $\emptyset$ .

**Example 2.1** For a strict partition  $\lambda = (5, 3, 1)$ ,

$$S(\lambda) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad T = \begin{array}{|c|c|c|c|c|} \hline T_{1,1} & T_{1,2} & T_{1,3} & T_{1,4} & T_{1,5} \\ \hline & T_{2,2} & T_{2,3} & T_{2,4} & \\ \hline & & T_{3,3} & & \\ \hline \end{array}$$

**Definition 2.4** A skew Young diagram  $\lambda/\mu$  is obtained by removing a shape  $\mu$  from a larger shape  $\lambda$  containing  $\mu$ .

**Definition 2.5** (1) A horizontal strip is a skew shifted Young diagram where each column can have at most one box.

(2) A vertical strip is a skew shifted Young diagram where each row can have at most one box.

**Definition 2.6** (1) A word  $\mathbf{a} = a_1 \cdots a_s$  is called a hook word if it satisfies  $a_1 \geq a_2 \geq \cdots \geq a_k < a_{k+1} < \cdots < a_s$  for some  $1 \leq k \leq s$ .

(2) For  $\lambda \in \mathcal{P}^+$ , let  $T$  be a filling of  $S(\lambda)$  with letters from the alphabet  $X_n := \{1, 2, \dots, n\}$ . Then,  $T$  is called a semistandard decomposition tableau of shape  $\lambda$  if

- (i)  $T_i$  is a hook word of length  $\lambda_i$  for  $i = 1, \dots, l(\lambda)$ .
- (ii)  $T_i$  is a hook subword of maximal length in  $T_{i+1}T_i$ , the concatenation of  $T_{i+1}$  and  $T_i$ , for  $i = 1, \dots, l(\lambda) - 1$ .

We denote by  $\text{SSDT}_n(\lambda)$  the set of semistandard decomposition tableaux of shape  $\lambda$ .

**Remark 2.4** Definition 2.6 is due to Grantcharov, Jung, Kang, Kashiwara, and Kim [10] and is different from the one originally introduced in [29] and the one used in [5, 14]. However, the bijection among these two models is established in [7], where the word and the tableau defined in Definition 2.6 are called the reverse hook word and the reverse semistandard decomposition tableau, respectively. The hook word and the semistandard decomposition tableau defined in [5, 29] are also called the weakly unimodal word and the semistandard unimodal tableau, respectively [14]. The definition here is more adequate to describe the  $\mathfrak{q}(n)$ -crystal structure because the  $\mathfrak{q}(n)$ -highest and lowest weight vectors take simpler forms.

We define the *reading word* of  $T \in \text{SSDT}_n(\lambda)$  as follows: We read entries through each row from right to left starting from the topmost row to the bottommost one. We denote by  $\text{rw}(T)$  the reading word of  $T$ . This definition is opposite to that given in [6, 10] because we follow the anti-Kashiwara convention for the tensor product rule. The weight of  $T$  is defined to be  $\text{wt}(T) = \sum_{i=1}^n a_i \epsilon_i$ , where  $a_i$  is the number of letters  $i$  in  $\text{rw}(T)$ .

Let  $\text{rw}(T) = a_1 a_2 \cdots a_m$  be the reading word of  $T \in \text{SSDT}_n(\lambda)$ . We identify  $T$  with the tensor product,

$$\boxed{a_1} \otimes \cdots \otimes \boxed{a_m}$$

by embedding

$$\text{SSDT}_n(\lambda) \rightarrow B_n(\square)^{\otimes |\lambda|}.$$

The  $\mathfrak{q}(n)$ -crystal structure of  $B_n(\square)^{\otimes |\lambda|}$  is given by the crystal graph of  $B_n(\square)$  equipped with the tensor product rule. Thus,  $\text{SSDT}_n(\lambda)$  is endowed with the  $\mathfrak{q}(n)$ -crystal structure (Fig. 2.1). The actual computation of the crystal graph can be done by the bracketing rule on  $\text{rw}(T)$  (see Section 3.2).

The  $\mathfrak{q}(n)$ -highest weight vector in  $\text{SSDT}_n(\lambda)$  is the semistandard decomposition tableau whose subtableau with entry  $l(\lambda) - i + 1$  is a connected border strip of size  $\lambda_{l(\lambda)-i+1}$  starting at  $T_{i,i}$  for  $i = l(\lambda), \dots, 1$  and the  $\mathfrak{q}(n)$ -lowest weight vector is the one where the  $i$ -th row is filled with entry  $n - i + 1$  for  $i = 1, \dots, l(\lambda)$  [6, 10].

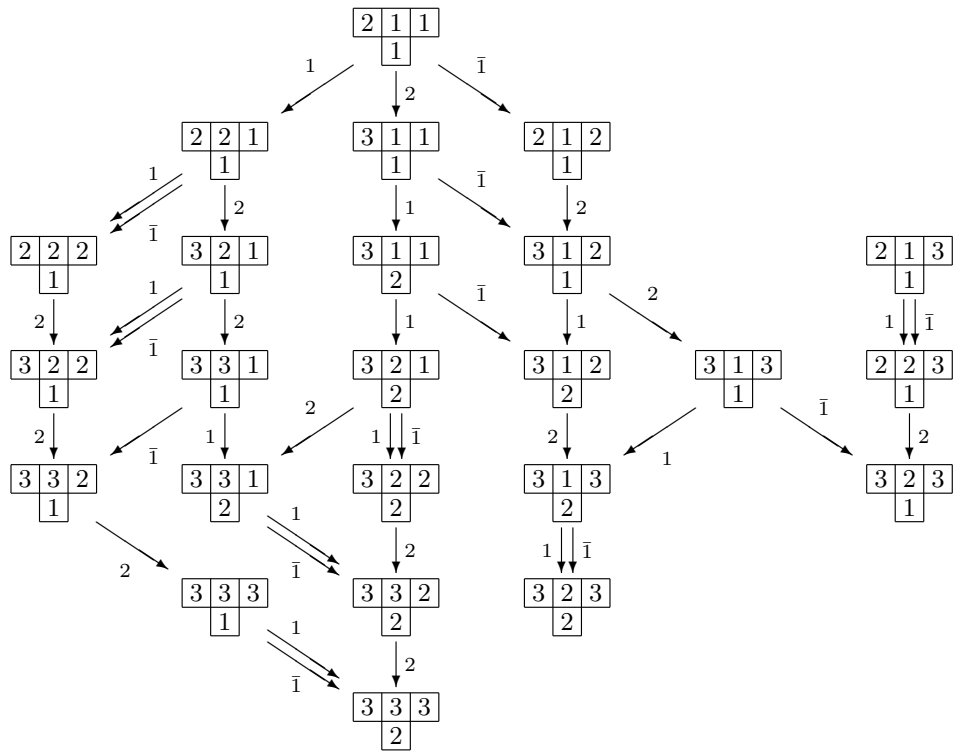


Figure 2.1:  $q(3)$ -crystal structure of  $\text{SSDT}_3(\lambda = (3, 1))$ .

**Example 2.2** Let  $n = 4$  and  $\lambda = (5, 3, 1)$ . Then the  $\mathfrak{q}(n)$ -highest weight vector (left) and the  $\mathfrak{q}(n)$ -lowest weight vector (right) are

$$\begin{array}{ccccc} \boxed{3} & \boxed{2} & \boxed{2} & \boxed{1} & \boxed{1} \\ & \boxed{2} & \boxed{1} & \boxed{1} & \\ & & \boxed{1} & & \end{array} \quad \begin{array}{ccccc} \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} \\ & \boxed{3} & \boxed{3} & \boxed{3} & \\ & & \boxed{2} & & \end{array} .$$

# Chapter 3

## $\mathfrak{q}$ -crystal structure on primed tableaux

In this chapter, we give a  $\mathfrak{q}$ -crystal structure on primed tableau, which will be the foundation to construct a  $\mathfrak{q}$ -crystal structure on signed unimodal factorizations of reduced (resp. flattened) words of type  $B$  (resp.  $D$ ) in Chapter 4 (resp. 5).

### 3.1 Primed tableaux and the semistandard shifted mixed insertion

**Definition 3.1** *A shifted tableau  $T$  with  $n$  boxes is called a standard shifted Young tableau if*

- (1) *the entries are all distinct ranging from 1 to  $n$ ,*
- (2) *the numbers in each rows are strictly increasing, and*
- (3) *the numbers in each column are strictly increasing.*

*We denote by  $\text{ST}_n(\lambda)$  the set of standard shifted Young tableaux with  $n$  boxes and shape  $\lambda$ .*

**Definition 3.2** *A primed tableau  $T$  of shape  $\lambda$  is a filling of  $S(\lambda)$  with letters from the alphabet  $X'_n := \{1' < 1 < 2' < 2 < \cdots < n\}$  such that:*

- (1) *the entries are weakly increasing along each column and each row of  $T$ ,*

- (2) each row contains at most one  $i'$  for every  $i = 1, \dots, n$ ,
- (3) each column contains at most one  $i$  for every  $i = 1, \dots, n$ , and
- (4) there are no primed letters on the main diagonal.

We denote by  $\text{PT}_n(\lambda)$  the set of primed tableaux of shape  $\lambda$ . For  $T \in \text{PT}_n(\lambda)$ , we define the reading word  $\text{rw}(T)$  as follows: We first read primed letters through each column from top to bottom starting from the rightmost column to the leftmost one, and remove all the primes. Next, we read unprimed letters through each row from left to right starting from the bottommost row to the topmost one.

**Remark 3.1** A primed tableau is also called a semistandard marked shifted tableau [5], a  $P$ -tableau [23], or a semistandard shifted Young tableau [29].

We denote by  $\mathcal{B}_n^m$  the set of words consisting of  $m$  letters from the alphabet  $X_n$ . For a word  $\mathbf{b} = b_1 \cdots b_m \in \mathcal{B}_n^m$ , we recursively construct a sequence of pairs of tableaux,

$$(\emptyset, \emptyset) = (T^{(0)}, Q^{(0)}), (T^{(1)}, Q^{(1)}), \dots, (T^{(m)}, Q^{(m)}) = (T, Q), \quad (3.1)$$

where  $T^{(i)} \in \text{PT}_n(\lambda^{(i)})$  and  $Q^{(i)} \in \text{ST}_n(\lambda^{(i)})$  by the insertion algorithm due to Serrano [29]. The algorithm to obtain  $(T^{(i)}, Q^{(i)})$  from  $(T^{(i-1)}, Q^{(i-1)})$  is as follows:

- (1) Insert  $b_i$  into the first row of  $T^{(i-1)}$ , bumping out the leftmost entry  $a$  that is strictly greater than  $b_i$  (in the order given by the alphabet  $X'_n$ ). If such an  $a$  does not exist, append  $\boxed{b_i}$  to the end of the first row of  $T^{(i-1)}$  and let  $T^{(i)}$  be this new tableau. To get  $Q^{(i)}$ , we add  $\boxed{i}$  to  $Q^{(i-1)}$  such that  $T^{(i)}$  and  $Q^{(i)}$  have the same shifted shape. Stop. If such an  $a$  exists, do as follows:
  - (2) (i) if  $a$  is not on the main diagonal, do as follows:
    - (i-1) if  $a$  is not primed, insert it into the next row, bumping out the leftmost entry that is strictly greater than  $a$  from that row  $R$ .
    - (i-2) if  $a$  is primed, insert it into the next column to the right, bumping out the topmost entry that is strictly greater than  $a$  from that column  $C$ .

(ii) if  $a$  is on the main diagonal, then it must be unprimed.

Prime it and insert it into the column to the right, bumping out the topmost entry that is strictly greater than  $a$  from that column  $C$ .

(3) If the bumped element does not exist, append  $\boxed{a}$  to the end of the row  $R$  (in case (i-1)) or the column  $C$  (in case (i-2) or (ii)), and let  $T^{(i)}$  be this new tableau. To get  $Q^{(i)}$ , we add  $\boxed{i}$  to  $Q^{(i-1)}$  such that  $T^{(i)}$  and  $Q^{(i)}$  have the same shifted shape. Stop. If the bumped element exists, repeat (2) and (3) with  $a$  being this bumped element.

We write the above process as  $T^{(i-1)} \leftarrow b_i = T^{(i)}$ . We call  $T$  (resp.  $Q$ ) in (3.1) the *mixed insertion* (resp. *recording*) tableau. This algorithm, the *semistandard shifted mixed insertion* introduced by Serrano [29], is a semistandard generalization of the mixed insertion introduced by Haiman [13].

**Example 3.1** The word  $\mathbf{b} = 333323212$  has the following mixed insertion (left) and recording (right) tableaux.

$$\begin{array}{cccc} \boxed{1} & \boxed{2'} & \boxed{2} & \boxed{3'} & \boxed{3} \\ & \boxed{2} & \boxed{3'} & \boxed{3} & \\ & & \boxed{3} & & \end{array} \quad \begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{6} \\ & \boxed{5} & \boxed{7} & \boxed{9} & \\ & & \boxed{8} & & \end{array} .$$

## 3.2 $\mathfrak{q}$ -crystal structure on primed tableaux

The set  $\mathcal{B}_n^m$  is a  $\mathfrak{gl}(n)$ -crystal and is also a  $\mathfrak{q}(n)$ -crystal. The  $\mathfrak{gl}(n)$  (or  $\mathfrak{q}(n)$ )-crystal structure of  $\mathcal{B}_n^m$  is given by identifying  $\mathbf{b} = b_1 b_2 \cdots b_m \in \mathcal{B}_n^m$  with the tensor product of  $\mathfrak{gl}(n)$  (or  $\mathfrak{q}(n)$ )-crystal  $B_n(\square)$ ,

$$\boxed{b_1} \otimes \cdots \otimes \boxed{b_m}.$$

In practice, the tensor product rule for actions of  $\tilde{e}_i$  and  $\tilde{f}_i$  ( $i = 1, \dots, n-1$ ) is rephrased by the following *bracketing rule* on the word  $\mathbf{b}$ : The Kashiwara operators  $\tilde{e}_i$  and  $\tilde{f}_i$  depend only on the letters  $i$  and  $i+1$  in  $\mathbf{b}$ . We successively bracket any adjacent pairs  $(i+1, i)$  and remove these pairs from the word. We call this procedure the  $(i+1, i)$  bracketing. If all letters  $i+1$  are bracketed, then  $\tilde{e}_i \mathbf{b} = \mathbf{0}$ . Otherwise,  $\tilde{e}_i$  changes the leftmost unbracketed letter  $i+1$  to  $i$ . If all letters  $i$  are bracketed, then  $\tilde{f}_i \mathbf{b} = \mathbf{0}$ . Otherwise,  $\tilde{f}_i$  changes the

rightmost unbracketed letter  $i$  to  $i + 1$ . The rules of action of odd Kashiwara operators  $\tilde{e}_{\bar{1}}$  and  $\tilde{f}_{\bar{1}}$  on the word  $\mathbf{b}$  are quite simple. If the letter 2 does not exist in  $\mathbf{b}$ , then  $\tilde{e}_{\bar{1}}\mathbf{b} = \mathbf{0}$ . Let  $b_k$  be the leftmost 2 in  $\mathbf{b}$ . If the subword  $b_1 \cdots b_{k-1}$  contains the letter 1, then  $\tilde{e}_{\bar{1}}\mathbf{b} = \mathbf{0}$ . Otherwise,  $\tilde{e}_{\bar{1}}$  changes  $b_k = 2$  to 1. If the letter 1 does not exist in  $\mathbf{b}$ , then  $\tilde{f}_{\bar{1}}\mathbf{b} = \mathbf{0}$ . Let  $b_k$  be the leftmost 1 in  $\mathbf{b}$ . If the subword  $b_1 \cdots b_{k-1}$  contains the letter 2, then  $\tilde{f}_{\bar{1}}\mathbf{b} = \mathbf{0}$ . Otherwise,  $\tilde{f}_{\bar{1}}$  changes  $b_k = 1$  to 2. One can easily check that the conditions (3), (4), and (5) in Definition 2.3 are satisfied, where the weight of  $\mathbf{b} \in \mathcal{B}_n^m$  is defined to be  $\text{wt}(\mathbf{b}) = \sum_{i=1}^n a_i \epsilon_i$  with  $a_i$  being the number of letters  $i$  in  $\mathbf{b}$ .

As an element of the  $\mathfrak{gl}(n)$ -crystal, a word  $\mathbf{b}$  is highest weight, i.e.,  $\tilde{e}_i\mathbf{b} = \mathbf{0}$  for all  $i = 1, 2, \dots, n-1$ , if and only if it is a Yamanouchi word [20]. That is, the weight of the subword  $b_k b_{k+1} \cdots b_m$  is a partition for all  $k = 1, 2, \dots, m$ . However, Yamanouchi words are not always highest weight in a sense of  $\mathfrak{q}(n)$ -crystals. For example, the word  $\mathbf{b} = 321121$  is Yamanouchi, but  $\tilde{e}_{\bar{1}}\mathbf{b} = 311121 \neq \mathbf{0}$ .

**Theorem 3.1** ([14, 29]) *The semistandard shifted mixed insertion gives a bijection*

$$\text{HM} : \mathcal{B}_n^m \xrightarrow{\sim} \bigcup_{\lambda \vdash m} [\text{PT}_n(\lambda) \times \text{ST}_n(\lambda)]. \quad (3.2)$$

Suppose that  $\text{HM}(\mathbf{b}) = (T, Q)$  for  $\mathbf{b} \in \mathcal{B}_n^m$ . We define  $P_{\text{HM}}(\mathbf{b}) = T$  and  $R_{\text{HM}}(\mathbf{b}) = Q$ .

**Theorem 3.2** *The recording tableau  $R_{\text{HM}}(\mathbf{b}) = Q$  is constant for all the elements  $\mathbf{b}$  in each connected component of the crystal  $\mathcal{B}_n^m$ . Here and hereafter, the connected component of a  $\mathfrak{q}$ -crystal  $B$  is referred to as the maximal subcrystal of  $B$  where all the elements are connected by even and odd Kashiwara operators.*

*Proof.* It was proven that any elements  $\mathbf{b}$  of  $\mathcal{B}_n^m$  that are connected by even Kashiwara operators have the same mixed recording tableau  $R_{\text{HM}}(\mathbf{b}) = Q$  [14]. Thus, it is sufficient to show that for  $\mathbf{b} \in \mathcal{B}_n^m$  and  $\mathbf{b}' = \tilde{f}_{\bar{1}}\mathbf{b} (\neq \mathbf{0}) \in \mathcal{B}_n^m$ , the mixed recording tableau of  $\text{HM}(\mathbf{b})$  and that of  $\text{HM}(\mathbf{b}')$  are identical.

We write the sequence of tableaux (Eq. (3.1)) for  $\mathbf{b}$  as

$$(\emptyset, \emptyset) = (T^{(0)}, Q^{(0)}), (T^{(1)}, Q^{(1)}), \dots, (T^{(m)}, Q^{(m)})$$

and that for  $\mathbf{b}'$  as

$$(\emptyset, \emptyset) = (T^{(0)'}, Q^{(0)'}), (T^{(1)'}, Q^{(1)'}), \dots, (T^{(m)'}, Q^{(m)'})$$

The claim is that  $Q^{(m)} = Q^{(m)'}.$  Let us write  $\mathbf{b} = b_1 b_2 \cdots b_m$  and suppose that  $b_{k_1}, b_{k_2}, \dots, b_{k_l}$  are 1 ( $k_1 < k_2 < \cdots < k_l$ ). If the letter 2 exists in the subword  $b_1 \cdots b_{k_1-1}$ , then  $\tilde{f}_1 \mathbf{b} = \mathbf{0}$  so that  $2 \notin \{b_1, \dots, b_{k_1-1}\}$ . According to the rule of  $\tilde{f}_1$ ,  $\mathbf{b}' = b'_1 \cdots b'_{k_1-1} b'_{k_1} b'_{k_1+1} \cdots b'_m = b_1 \cdots b_{k_1-1} 2 b_{k_1+1} \cdots b_m$ .

If  $k_1 = 1$ , it is obvious that  $Q^{(k_1-1)} = Q^{(k_1-1)'}$ . If  $k_1 = 2$ , then  $T^{(k_1-1)} = T^{(k_1-1)'} = \boxed{x}$ , where  $x \geq 3$  and is unprimed. Since

$$\boxed{x} \leftarrow i = \boxed{i \mid x}$$

( $i = 1, 2$ ), we have  $Q^{(k_1-1)} = Q^{(k_1-1)'}$ . Suppose that  $k_1 \geq 3$ . Clearly  $T^{(k_1-1)} = T^{(k_1-1)'}$  and  $Q^{(k_1-1)} = Q^{(k_1-1)'}$ . The configuration of the first row of  $T^{(k_1-1)}$  ( $= T^{(k_1-1)'}$ ) is

$$\boxed{x \mid y \mid \cdots},$$

where  $3 \leq x \leq y$  and  $x$  is unprimed. When  $b_{k_1} = 1$  (resp.  $b'_{k_1} = 2$ ) is inserted into the first row of  $T^{(k_1-1)}$  (resp.  $T^{(k_1-1)'}$ ), the entry  $x (\geq 3)$  on the main diagonal is bumped out and  $x'$  is inserted into the second column, where the entry  $y$  is bumped out since  $x' < y$ . The subsequent process of  $T^{(k_1-1)} \leftarrow 1$  is the same as that of  $T^{(k_1-1)'} \leftarrow 2$  so that  $Q^{(k_1)} = Q^{(k_1)'}$ . If  $l = 1$ , then  $Q^{(m)} = Q^{(m)'}$  because  $b_{k_1+1}, \dots, b_m \geq 2$ .

Next, we assume that  $l \geq 2$ . Since  $b_{k_1+1}, \dots, b_{k_2-1} \geq 2$ , we have  $Q^{(k_2-1)} = Q^{(k_2-1)'}$ .

If  $k_2 = 2$ , then  $T^{(k_2-1)} = \boxed{1}$ ,  $T^{(k_2-1)'} = \boxed{2}$ , and  $b_{k_2} = b'_{k_2} = 1$ , which implies  $Q^{(k_2)} = Q^{(k_2)'}$ . If  $k_2 > 2$ , then the configuration of the first row of  $T^{(k_2-1)}$  is

$$\boxed{1 \mid x \mid \cdots},$$

and that of  $T^{(k_2-1)'}$  is

$$\boxed{2 \mid x \mid \cdots},$$

where  $x \geq 3'$ . When  $b_{k_2} = 1$  is inserted into the first row of  $T^{(k_2-1)}$ , the entry  $x$  is bumped out. When  $b'_{k_2} = 1$  is inserted into the first row of  $T^{(k_2-1)'}$ , the entry 2 on the main diagonal is bumped out and  $2'$  is inserted into the second column. Since  $2' < x$ , the entry  $x$  is bumped out. The subsequent process of  $T^{(k_2-1)} \leftarrow 1$  is the same as that of  $T^{(k_2-1)'} \leftarrow 1$  so that  $Q^{(k_2)} = Q^{(k_2)'}$ . Thus, we have  $Q^{(k_2)} = Q^{(k_2)'}$  for  $k_2 \geq 2$ . If  $l = 2$ , then  $Q^{(m)} = Q^{(m)'}$  because  $b_{k_2+1}, \dots, b_m \geq 2$ .

Finally, we assume that  $l \geq 3$ . Since  $b_{k_2+1}, \dots, b_{k_3-1} \geq 2$ , we have  $Q^{(k_3-1)} = Q^{(k_3-1)'}$ .

If  $k_3 = 3$ , then

$$T^{(k_3-1)} = \boxed{1 \mid 1} \quad \text{and} \quad T^{(k_3-1)'} = \boxed{1 \mid 2'}.$$

If  $k_3 > 3$ , then the configuration of the first row of  $T^{(k_3-1)}$  is

$$\boxed{1 \mid 1 \mid x \mid \cdots}$$

and that of  $T^{(k_3-1)'}$  is

$$\boxed{1 \mid 2' \mid x \mid \cdots},$$

where  $x \geq 3'$ .

Now we assume the following: for  $i \geq 3$ ,  $Q^{(k_i-1)} = Q^{(k_i-1)'}$  and when  $k_i = i$ ,

$$T^{(k_i-1)} = \boxed{1 \mid \cdots \mid 1}^{i-1} \quad \text{and} \quad T^{(k_i-1)'} = \boxed{1 \mid \cdots \mid 1 \mid 2'}^{i-1},$$

and when  $k_i > i$ , the configuration of the first row of  $T^{(k_i-1)}$  is

$$\boxed{1 \mid \cdots \mid 1 \mid x \mid \cdots}^i$$

and that of  $T^{(k_i-1)'}$  is

$$\boxed{1 \mid \cdots \mid 1 \mid 2' \mid x \mid \cdots}^i,$$

where  $x \geq 3'$ . This assumption is satisfied for  $i = 3$ . We proceed by induction.

When  $k_i = i$ , we have

$$T^{(k_i)} = \boxed{1 \mid \cdots \mid 1}^i \quad \text{and} \quad T^{(k_i)'} = \boxed{1 \mid \cdots \mid 1 \mid 2'}^i$$

so that  $Q^{(k_i)} = Q^{(k_i)'} (b_{k_i} = b'_{k_i} = 1)$ . The equality  $Q^{(k_i)} = Q^{(k_i)'}$  also holds for  $k_i > i$ . This is shown as follows: When  $b_{k_i} (= 1)$  is inserted into the first row of  $T^{(k_i-1)}$ , the entry  $x$  is bumped out. When  $b'_{k_i} (= 1)$  is inserted into the first row of  $T^{(k_i-1)'}$ , the entry  $2'$  in the  $(i-1)$ -st box is bumped out and is inserted into the  $i$ -th column. Since  $2' < x$ , the entry  $x$  is bumped out. The subsequent process of  $T^{(k_i-1)} \leftarrow 1$  is the same as that of  $T^{(k_i-1)'}$  so that  $Q^{(k_i)} = Q^{(k_i)'}$ . Thus, we have  $Q^{(k_i)} = Q^{(k_i)'}$  for  $k_i \geq i$ . Since  $b_{k_i+1}, \dots, b_{k_{i+1}-1} \geq 2$ , we have  $Q^{(k_{i+1}-1)} = Q^{(k_{i+1}-1)'}$ . Furthermore, when  $k_{i+1} = i+1$ ,

$$T^{(k_{i+1}-1)} = \boxed{1 \cdots \boxed{1}^i} \quad \text{and} \quad T^{(k_{i+1}-1)'} = \boxed{1 \cdots \boxed{1 \ 2'}^i},$$

and when  $k_{i+1} > i+1$ , the configuration of the first row of  $T^{(k_{i+1}-1)}$  is

$$\boxed{1 \cdots \boxed{1 \ x}^{i+1} \cdots}$$

and that of  $T^{(k_{i+1}-1)'}$  is

$$\boxed{1 \cdots \boxed{1 \ 2' \ x}^{i+1} \cdots},$$

where  $x \geq 3'$ . Hence, by induction, we have  $Q^{(k_i)} = Q^{(k_i)'}$ . Since  $b_{k_i+1}, \dots, b_m \geq 2$ , we have  $Q^{(m)} = Q^{(m)'}$ . This completes the proof.  $\square$

Let us fix a mixed recording tableau  $Q_\lambda \in \text{ST}_n(\lambda)$  and define a map  $\Psi_\lambda : \text{PT}_n(\lambda) \rightarrow \mathcal{B}_n^m$  as  $\Psi_\lambda(T) = \text{HM}^{-1}(T, Q_\lambda)$ . By this map, we may define

$$\begin{aligned} \tilde{e}_i^P(T) &= (\Psi_\lambda^{-1} \circ e_i \circ \Psi_\lambda)(T), \\ \tilde{f}_i^P(T) &= (\Psi_\lambda^{-1} \circ f_i \circ \Psi_\lambda)(T), \end{aligned}$$

( $i = 1, \dots, n-1, \bar{1}$ ). By construction, these operators may depend on the choice of  $Q_\lambda$ . Nonetheless, they can be constructed independent of the choice of  $Q_\lambda$  so that they can be regarded as Kashiwara operators on  $T \in \text{PT}_n(\lambda)$ . The weight of  $T \in \text{PT}_n(\lambda)$  is defined to be  $\text{wt}(T) = \sum_{i=1}^n a_i \epsilon_i$ , where  $a_i$  is the number of letters  $i$  in  $\text{rw}(T)$ . We also write  $\text{wt}(T) = (a_1, a_2, \dots, a_n)$ .

**Theorem 3.3 ([15])** *The set of primed tableaux  $\text{PT}_n(\lambda)$  admits a  $\mathfrak{q}(n)$ -crystal structure by the  $\mathfrak{q}(n)$ -crystal isomorphism  $\text{HM}$  (Eq. (3.2)).*

*Proof.* It was shown that operators  $\tilde{e}_i^P$  and  $\tilde{f}_i^P$  ( $i = 1, \dots, n-1$ ) are independent of the choice of  $Q_\lambda \in \text{ST}_n(\lambda)$  in Eq. (3.2) and that the bijection HM is a  $\mathfrak{gl}(n)$ -crystal isomorphism. We claim that  $\tilde{e}_1^P$  and  $\tilde{f}_1^P$  are also independent of the choice of  $Q_\lambda$  and that HM is  $\mathfrak{q}(n)$ -crystal isomorphism. The claim is the consequence of the following two lemmas (Lemmas 3.1 and 3.2).  $\square$

The explicit description of the operators  $\tilde{e}_1^P$  and  $\tilde{f}_1^P$  resulting from Lemma 3.1 and 3.2 is the same as in Definition 4.5 in [2]. Nevertheless, we present the following two lemmas with proofs because the way of construction is completely different from that in [1, 2].

**Lemma 3.1 ([15])** *For  $T \in \text{PT}_n(\lambda)$ , the operator  $\tilde{e}_1^P$  on  $T$  is given by the following rule: If  $T_{1,1} = 2$ , change  $T_{1,1}$  to 1. If  $T_{1,i} = 2'$  ( $i \geq 2$ ), change  $T_{1,i}$  to 1. Otherwise,  $\tilde{e}_1^P T = \mathbf{0}$ .*

*Also,  $P_{\text{HM}}(\tilde{e}_1^P \mathbf{b}) = \tilde{e}_1^P T$  and  $\tilde{e}_1^P \mathbf{b} = \mathbf{0}$  if and only if  $\tilde{e}_1^P T = \mathbf{0}$  for  $P_{\text{HM}}(\mathbf{b}) = T$ .*

Note that the action of  $\tilde{e}_1^P$  on  $T \in \text{PT}_n(\lambda)$  depends only on  $T$ .

*Proof of Lemma 3.1.* Let  $\mathbf{b} = b_1 b_2 \dots b_m \in \mathcal{B}_n^m$  be the word such that  $\text{HM}(\mathbf{b}) = (T, Q_\lambda)$  ( $P_{\text{HM}}(\mathbf{b}) = T$  for a fixed  $Q_\lambda$ ). Let  $b_k$  be the leftmost 2 in  $\mathbf{b}$  ( $1 \leq k \leq m$ ). In the following, we assume that the subword  $b_1 b_2 \dots b_{k-1}$  does not contain 1; otherwise  $\tilde{e}_1^P \mathbf{b} = \mathbf{0}$ . Suppose that the number of letters 1 in the subword  $b_{k+1} b_{k+2} \dots b_m$  is  $i-1$ .

(1):  $i = 1$ . The first row of  $T$  has the following configuration.

$$\boxed{2 \mid x \mid \dots},$$

where  $x \geq 2$ .

(2):  $i \geq 2$ . The first row of  $T$  has the following configuration.

$$\boxed{1 \mid \dots \mid 1 \mid \overset{i}{2'} \mid \dots}.$$

Since  $\tilde{e}_1^P \mathbf{b} = b_1 \dots b_{k-1} 1 b_{k+1} \dots b_m = \mathbf{b}'$ , the first row of  $T' = P_{\text{HM}}(\mathbf{b}')$  has the configuration,

$$\boxed{1 \mid x \mid \dots}$$

for Case (1) and

$$\begin{array}{|c|c|c|c|c|} \hline & & & i & \\ \hline 1 & \cdots & 1 & 1 & \cdots \\ \hline \end{array}$$

for Case (2). In both cases,  $T'_{1,j} = T_{1,j}$  for  $j \in \{1, 2, \dots, \lambda_1\} \setminus \{i\}$  and  $T'_i = T_i$  for  $i = 2, \dots, l(\lambda)$ .

It is clear from the above arguments that  $\tilde{e}_1^P$  does not depend on the choice of the recording tableau  $Q_\lambda$  in Eq. (3.2) and that  $P_{\text{HM}}(\tilde{e}_1 \mathbf{b}) = \tilde{e}_1^P T$ .

If  $\tilde{e}_1^P T = \mathbf{0}$ , then the first row of  $T$  does not contain the letter 2 or the configuration of the first row of  $T$  is

$$\begin{array}{|c|c|c|c|c|} \hline & & & i & \\ \hline 1 & \cdots & 1 & 2 & \cdots \\ \hline \end{array}$$

( $i \geq 2$ ). This implies that  $2 \notin \{b_1, \dots, b_m\}$  or  $\mathbf{b} = \underbrace{1 \cdots 1}_{i-1} 2 \cdots$  so that  $\tilde{e}_1 \mathbf{b} = \mathbf{0}$ .

The converse is obvious. □

**Lemma 3.2 ([15])** *For  $T \in \text{PT}_n(\lambda)$ , the operator  $\tilde{f}_1^P$  on  $T$  is given by the following rule: If the first row of  $T$  does not contain letters 1, then  $\tilde{f}_1^P T = \mathbf{0}$ . Let  $(1, i)$  be the position of the rightmost 1 in the first row.*

(1):  $i = 1$ . If  $T_{1,2} = 2'$ , then  $\tilde{f}_1^P T = \mathbf{0}$ . Otherwise, change  $T_{1,1} = 1$  to 2.

(2):  $i \geq 2$ . If  $T_{1,i+1} = 2'$ , then  $\tilde{f}_1^P T = \mathbf{0}$ . Otherwise, change  $T_{1,i} = 1$  to  $2'$ .

Also,  $P_{\text{HM}}(\tilde{f}_1 \mathbf{b}) = \tilde{f}_1^P T$  and  $\tilde{f}_1 \mathbf{b} = \mathbf{0}$  if and only if  $\tilde{f}_1^P T = \mathbf{0}$  for  $P_{\text{HM}}(\mathbf{b}) = T$ .

Again, note that the action of  $\tilde{f}_1^P$  on  $T \in \text{PT}_n(\lambda)$  depends only on  $T$ .

*Proof of Lemma 3.2.* Let  $\mathbf{b} = b_1 b_2 \cdots b_m \in \mathcal{B}_n^m$  such that  $\text{HM}(\mathbf{b}) = (T, Q_\lambda)$  ( $P_{\text{HM}}(\mathbf{b}) = T$  for a fixed  $Q_\lambda$ ). Let  $b_k$  be the leftmost 1 in  $\mathbf{b}$  ( $1 \leq k \leq m$ ). In the following, we assume that the subword  $b_1 b_2 \cdots b_{k-1}$  does not contain 2; otherwise  $\tilde{f}_1 \mathbf{b} = \mathbf{0}$ . Suppose that the number of letters 1 in the subword  $b_{k+1} b_{k+2} \cdots b_m$  is  $i - 1$ .

(1):  $i = 1$ . The first row of  $T$  has the following configuration.

$$\begin{array}{|c|c|c|} \hline 1 & x & \cdots \\ \hline \end{array}$$

Here,  $x \geq 2$ . This is shown as follows: For  $k = 1, 2$ , this is obvious so that we assume  $k \geq 3$ . Before inserting  $b_k = 1$ , entries  $T_{1,1}^{(k-1)}$  (unprimed) and  $T_{1,2}^{(k-1)}$  are greater than or equal to 3 because  $2 \notin \{b_1, \dots, b_{k-1}\}$ . Then, after inserting  $b_k = 1$ ,  $T_{1,1}^{(k)} = 1$  and  $T_{1,2}^{(k)} \geq 3'$ . Subsequent insertion of a letter 2 (if it exists) changes  $T_{1,2}^{(k)}$  to 2 but not to  $2'$  due to the step (1) in the semistandard shifted mixed insertion.

(2):  $i \geq 2$ . The first row of  $T$  has the following configuration.

$$\boxed{1 \mid \cdots \mid \overset{i}{1} \mid x \mid \cdots}.$$

Here,  $x \geq 2$  as in (1).

In both cases,  $x \neq 2'$ . Since  $\tilde{f}_1 \mathbf{b} = b_1 \cdots b_{k-1} 2 b_{k+1} \cdots b_m = \mathbf{b}'$ , the first row of  $T' = P_{\text{HM}}(\mathbf{b}')$  has the configuration,

$$\boxed{2 \mid x \mid \cdots}$$

for Case(1) and

$$\boxed{1 \mid \cdots \mid \overset{i}{2'} \mid x \mid \cdots}$$

for Case (2). In both cases,  $x \neq 2'$ ; otherwise  $\tilde{f}_1^P T \notin \text{PT}_n(\lambda)$ , i.e.,  $\tilde{f}_1^P T = \mathbf{0}$ . Furthermore, in both cases,  $T'_{1,j} = T_{1,j}$  for  $j \in \{1, 2, \dots, \lambda_1\} \setminus \{i\}$  and  $T'_i = T_i$  for  $i = 2, \dots, l(\lambda)$ .

It is clear from the above arguments that  $\tilde{f}_1^P$  does not depend on the choice of the recording tableau  $Q_\lambda$  in Eq. (3.2) and that  $P_{\text{HM}}(\tilde{f}_1 \mathbf{b}) = \tilde{f}_1^P T$ .

If  $\tilde{f}_1^P T = \mathbf{0}$ , then the first row of  $T$  does not contain the letter 1 or the configuration of the first row of  $T$  is

$$\boxed{1 \mid \cdots \mid 1 \mid \overset{i}{2'} \mid \cdots}$$

( $i \geq 1$ ). This implies that  $1 \notin \{b_1, \dots, b_m\}$  or  $2 \in \{b_1, \dots, b_{k-1}\}$  where  $b_k = 1$  is the leftmost 1 in  $\mathbf{b}$  so that  $\tilde{f}_1 \mathbf{b} = \mathbf{0}$ . The converse is obvious.  $\square$

The explicit rules of  $\tilde{f}_i^P$  on  $T \in \text{PT}_n(\lambda)$  are described below [14], which will be used in the proof of Theorem 3.4. Another but equivalent rule is given in [1, 2].

We first apply the  $(i+1, i)$ -bracketing on the reading word  $\text{rw}(T)$ . If all letters  $i$  are bracketed in  $\text{rw}(T)$ , then  $\tilde{f}_i^P T = \mathbf{0}$ . Otherwise, the rightmost unbracketed letter  $i$  in  $\text{rw}(T)$  corresponding to an  $i$  or an  $i'$  in  $T$ , which we call bold unprimed  $i$  or bold primed  $i$  respectively.

- (1) If the bold letter  $i$  is unprimed, denote the cell it is located in as  $x$ .
- (2) If the bold letter  $i$  is primed, conjugate the tableau  $T$  first. The conjugation of a primed tableau  $T$  is obtained by reflecting the tableau over the main diagonal, changing all primed entries  $k'$  to  $k$  and changing all unprimed entries  $k$  to  $(k+1)'$ . Denote the resulting tableau as  $T^*$ . The bold primed  $i$  in  $T$  is now the bold unprimed  $i$  in  $T^*$ , Denote the cell it is located in as  $x$ .

Given any cell  $z$  in a signed primed tableau  $T$  (or its conjugation  $T^*$ ), denote by  $c(z)$  the content of  $z$ ,  $z_E$  the cell to the right of  $z$ , and  $z_W$  the cell to its left.

The algorithm of action of  $\tilde{f}_i^P$  on  $T$  is as follows:

- (1) If  $c(x_E) = (i+1)'$ , change  $c(x)$  to  $(i+1)'$  and change  $c(x_E)$  to  $i+1$ .
- (2) If  $c(x_E) \neq (i+1)'$  or  $x_E$  is empty, then there is a maximal connected ribbon (expanding in South and West directions) with the following properties:
  - (i) The North-Eastern most box of the ribbon (the tail of the ribbon) is  $x$ .
  - (ii) The contents of all boxes within a ribbon besides the tail are either  $(i+1)'$  or  $i+1$ .

Denote the South-Western most box of the ribbon (the head) as  $x_H$ .

- (2-a) If  $x_H = x$ , change  $c(x)$  to  $i+1$ .
- (2-b) If  $x_H \neq x$  and  $x_H$  is on the main diagonal (in case of a tableau  $T$ ), change  $c(x)$  to  $(i+1)'$ .
- (2-c) Otherwise, change  $c(x)$  to  $(i+1)'$  and change  $c(x_H)$  to  $i+1$ .

In the case when the bold  $i$  is unprimed, we apply the above rules on  $T$  to find  $\tilde{f}_i^P T$ . In the case when the bold  $i$  is primed, we first conjugate  $T$  and then apply the above rules on  $T^*$ , before reversing the conjugation.

**Lemma 3.3** ([14]) *As an element of a  $\mathfrak{gl}(n)$ -crystal  $T \in \text{PT}_n(\lambda)$  is a highest weight element, i.e.,  $\tilde{e}_i^P T = \mathbf{0}$  for all  $i = 1, \dots, n-1$ , if and only if its reading word  $\text{rw}(T)$  is a Yamanouchi word.*

**Theorem 3.4** ([15]) *Let us consider  $\text{PT}_n(\lambda)$  as a  $\mathfrak{q}(n)$ -crystal. The  $\mathfrak{q}(n)$ -highest weight vector in  $\text{PT}_n(\lambda)$  is a tableau whose  $i$ -th row is filled with letters  $i$  and the  $\mathfrak{q}(n)$ -lowest weight vector in  $\text{PT}_n(\lambda)$  is a tableau where the subtableau with entries  $n - l(\lambda) + i$  or  $(n - l(\lambda) + i)'$  is a connected border strip of size  $\lambda_{l(\lambda)-i+1}$  starting at  $T_{i,i} = n - l(\lambda) + i$  for  $i = l(\lambda), \dots, 1$ .*

Note that the entries in the subtableau in the  $\mathfrak{q}(n)$ -lowest weight vector are uniquely determined.

*Proof of Theorem 3.4.* Since the  $\mathfrak{q}(n)$ -highest (lowest) weight vector is unique in a  $\mathfrak{q}(n)$ -crystal [10, 11], it is sufficient to show that the tableaux described in Theorem 3.4 are the  $\mathfrak{q}(n)$ -highest weight vector and  $\mathfrak{q}(n)$ -lowest weight one. Let  $T \in \text{PT}_n(\lambda)$  be a primed tableau whose  $i$ -th row is filled with letters  $i$  ( $i = 1, \dots, l(\lambda)$ ). It is clear by Lemma 3.3 that  $\tilde{e}_i^P T = \mathbf{0}$  ( $i = 1, \dots, n-1$ ) because the reading word  $\text{rw}(T)$  is a Yamanouchi word. It is also obvious that  $\tilde{e}_1^P T = \mathbf{0}$  by Lemma 3.1.

Let  $i \in \{2, \dots, n-1\}$  and let us write  $\text{wt}(T) = (\mu_1, \mu_2, \dots, \mu_n)$ . Then,  $\mu_1 = \lambda_1, \dots, \mu_l = \lambda_l, \mu_{l+1} = \dots = \mu_n = 0$ , where  $l = l(\lambda)$ .

Firstly, we determine the configuration of the tableau  $S_1 \cdots S_{i-1} T$ . Let us write  $\text{wt}(S_{i-1} T) = (\mu_1^{(i-1)}, \mu_2^{(i-1)}, \dots, \mu_n^{(i-1)})$ . Since

$$S_{i-1} T = \left( \tilde{f}_{i-1}^P \right)^{(\mu_{i-1} - \mu_i)} T = \left( \tilde{f}_{i-1}^P \right)^{(\lambda_{i-1} - \lambda_i)} T,$$

we have  $\mu_{i-1}^{(i-1)} = \lambda_{i-1} - (\lambda_{i-1} - \lambda_i) = \lambda_i$  and  $\mu_i^{(i-1)} = \lambda_i + (\lambda_{i-1} - \lambda_i) = \lambda_{i-1}$ . The other components of  $\text{wt}(S_{i-1} T)$  are the same as those of  $\text{wt}(T)$ . Set  $T^{(k)} = S_k \cdots S_{i-1} T$  and  $\text{wt}(T^{(k)}) = (\mu_1^{(k)}, \mu_2^{(k)}, \dots, \mu_n^{(k)})$  for  $k = i-1, \dots, 2$ . Assume that  $\mu_k^{(k)} = \lambda_i, \mu_{k+1}^{(k)} = \lambda_k, \dots, \mu_i^{(k)} = \lambda_{i-1}$ , and that the other components of  $\text{wt}(T^{(k)})$  are the same as those of  $\text{wt}(T)$ . This assumption is satisfied for  $k = i-1$ . Let  $\text{wt}(T^{(k-1)}) = \text{wt}(S_{k-1} T^{(k)})$  be  $(\mu_1^{(k-1)}, \mu_2^{(k-1)}, \dots, \mu_n^{(k-1)})$ . Since

$$S_{k-1} T^{(k)} = \left( \tilde{f}_{k-1}^P \right)^{(\mu_{k-1}^{(k)} - \mu_k^{(k)})} T^{(k)} = \left( \tilde{f}_{k-1}^P \right)^{(\lambda_{k-1} - \lambda_i)} T^{(k)},$$

we have that  $\mu_{k-1}^{(k-1)} = \lambda_{k-1} - (\lambda_{k-1} - \lambda_i) = \lambda_i$ ,  $\mu_k^{(k-1)} = \lambda_i + (\lambda_{k-1} - \lambda_i) = \lambda_{k-1}$ , and the other components of  $\text{wt}(S_{k-1} T^{(k)})$  are the same as those of

$\text{wt}(T^{(k)})$ . That is,  $\mu_{k-1}^{(k-1)} = \lambda_i, \mu_k^{(k-1)} = \lambda_{k-1}, \dots, \mu_i^{(k-1)} = \lambda_{i-1}$ , and the other components of  $\text{wt}(T^{(k-1)})$  are the same as those of  $\text{wt}(T)$ . Hence, by induction, we have  $\text{wt}(T^{(1)}) = \text{wt}(S_1 \cdots S_{i-1}T) = (\mu_1^{(1)}, \mu_2^{(1)} \dots, \mu_n^{(1)}) = (\lambda_i, \lambda_1, \dots, \lambda_{i-2}, \lambda_{i-1}, \lambda_{i+1}, \dots)$ . The weight of this form determines the configuration of  $T^{(1)}$  uniquely. The configuration of the  $(k-1)$ -st and  $k$ -th rows is

$$\begin{array}{ccccccc}
& & & & \vdots & & \\
\hline
k-1 & \cdots & k-1 & k' & k & \cdots & k \\
\hline
& k & \cdots & k & (k+1)' & k+1 & \cdots \\
& & & \uparrow & & & \\
& & & \lambda_i & & & \\
& & & \vdots & & & 
\end{array}$$

for  $k = 2, \dots, i - 2$ , where the rightmost  $k$  in the  $k$ -th row is located in the  $\lambda_i$ -th box from the left, i.e.,  $T_{k, k + \lambda_i - 1}^{(1)} = k$  and the all boxes of the right part from the  $(\lambda_i + 2)$ -nd position in the  $k$ -th row are filled with letters  $k + 1$ . Since the number of  $(k + 1)'$  or  $k + 1$  in  $T^{(1)}$  is  $\lambda_k$ ,  $T_{k+1, k + \lambda_i}^{(1)}$  must be  $k + 1$  so that the entry just above is  $T_{k, k + \lambda_i}^{(1)} = (k + 1)'$ . The letters  $k'$  and  $k$  appear only in these two rows. The configuration of the  $(i - 1)$ -st and  $i$ -th rows is

$$\begin{array}{ccccccc} & & \vdots & & & & \\ i-1 & \cdots & i-1 & i' & i & \cdots & i \\ & i & \cdots & i & & & \\ & & \vdots & & & & \end{array}$$

where letters  $i'$  and  $i$  appear only in these two rows. The configuration below the  $i$ -th row of  $T^{(1)}$  is the same as that of  $T$ . Note that the number of  $k$  in the  $k$ -th row is  $\lambda_i$  irrespective of  $k$  for  $k = 1, 2, \dots, i$ .

Secondly, we compute the configuration of the tableau

$$S_2 \cdots S_i S_1 \cdots S_{i-1} T.$$

Let us write  $\text{wt}(S_i T^{(1)}) = (\nu_1^{(i)}, \nu_2^{(i)}, \dots, \nu_n^{(i)})$ . Since

$$S_i T^{(1)} = \left( \tilde{f}_i^P \right)^{(\mu_i^{(1)} - \mu_{i+1}^{(1)})} T^{(1)} = \left( \tilde{f}_i^P \right)^{(\lambda_{i-1} - \lambda_{i+1})} T^{(1)},$$

we have  $\nu_i^{(i)} = \lambda_{i-1} - (\lambda_{i-1} - \lambda_{i+1}) = \lambda_{i+1}$  and  $\nu_{i+1}^{(i)} = \lambda_{i+1} + (\lambda_{i-1} - \lambda_{i+1}) = \lambda_{i-1}$ . The other components of  $\text{wt}(S_i T^{(1)})$  are the same as those of  $\text{wt}(T^{(1)})$ .

In particular,  $\nu_{i-1}^{(i)} = \lambda_{i-2}$ . In this step, the left  $\lambda_{i+1}$  letters  $i$  in the  $i$ -th row in the configuration above is bracketed in  $\text{rw}(T^{(1)})$  by the  $(i+1, i)$ -bracketing so that the number of unbracketed letters  $i'$  and  $i$  is  $\lambda_{i-1} - \lambda_{i+1}$ . Since  $\tilde{f}_i^P$  is applied  $\lambda_{i-1} - \lambda_{i+1}$  times in this step, all of these unbracketed letters  $i'$  and  $i$  are updated and the configuration of the  $(i-1)$ -st and  $i$ -th rows of  $S_i T^{(1)}$  turns out to be

$$\begin{array}{c}
 \vdots \\
 \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
 \hline
 i-1 & \cdots & & i-1 & (i+1)' & i+1 & \cdots & i+1 \\
 \hline
 i & \cdots & i & (i+1)' & i+1 & \cdots & i+1 & \\
 \hline
 \end{array} \\
 \uparrow \\
 \lambda_{i+1} \qquad \qquad \qquad \vdots
 \end{array}$$

The computation can be done step by step following the rules for  $\tilde{f}_i^P$ . The letter  $i'$  disappears in the new configuration and the number of letters  $i$ , all of which are located in the  $i$ -th row, is reduced to  $\lambda_{i+1}$ .

Let us write  $S_{k+1} \cdots S_i T^{(1)} = T^{(k+1)'}$  and

$$\text{wt}(T^{(k+1)'}) = (\nu_1^{(k+1)}, \nu_2^{(k+1)}, \dots, \nu_n^{(k+1)})$$

for  $k = i-1, \dots, 2$ . For  $k = i-1, \dots, 2$ , assume that the configuration of the  $(k-1)$ -st and  $k$ -th rows of  $T^{(k+1)'}$  is

$$\begin{array}{c}
 \vdots \\
 \begin{array}{|c|c|c|c|c|c|c|}
 \hline
 k-1 & \cdots & k-1 & k' & k & \cdots & k \\
 \hline
 & k & \cdots & k & (k+2)' & k+2 & \cdots \\
 \hline
 \end{array} \\
 \uparrow \\
 \lambda_{i+1} \qquad \vdots
 \end{array}
 ,$$

the number of letters  $(k+1)$ , all of which are located in the left side of the  $(k+1)$ -st row, is  $\lambda_{i+1}$  and the letter  $(k+1)'$  does not appear in  $T^{(k+1)'}$  so that  $\nu_{k+1}^{(k+1)} = \lambda_{i+1}$ , and letters  $k'$  and  $k$  appear only in these two rows so that  $\nu_k^{(k+1)} = \lambda_{k-1}$ . For  $k = i-1$ , this configuration is consistent with the previous one and other assumptions are satisfied. Let us write  $\text{wt}(S_k T^{(k+1)'}) = (\nu_1^{(k)}, \nu_2^{(k)}, \dots, \nu_n^{(k)})$ . Since  $T^{(k)'} = S_k T^{(k+1)'} = \left(\tilde{f}_k^P\right)^{(\nu_k^{(k+1)} - \nu_{k+1}^{(k+1)})} T^{(k+1)'} = \left(\tilde{f}_k^P\right)^{(\lambda_{k-1} - \lambda_{i+1})} T^{(k+1)'}$ , we have  $\nu_k^{(k)} = \lambda_{k-1} - (\lambda_{k-1} - \lambda_{i+1}) = \lambda_{i+1}$  and  $\nu_{k+1}^{(k)} = \lambda_{i+1} + (\lambda_{k-1} - \lambda_{i+1}) = \lambda_{k-1}$ . The other components of  $\text{wt}(S_k T^{(k+1)'})$

$$\begin{array}{ccccccc}
& & & & & & \vdots \\
\boxed{k-2} & \cdots & \boxed{k-2} & \boxed{(k-1)'} & \boxed{k-1} & \cdots & \boxed{k-1} \\
& \boxed{k-1} & \cdots & \boxed{k-1} & \boxed{(k+1)'} & \boxed{k+1} & \cdots \\
& & & \uparrow & & & \\
& & & \lambda_{i+1} & & & \vdots
\end{array}$$
$$\lambda_i$$

|   |     |   |    |   |     |   |
|---|-----|---|----|---|-----|---|
| 1 | ... | 1 | 3' | 3 | ... | 3 |
|---|-----|---|----|---|-----|---|

Let  $T \in \text{PT}_n(\lambda)$  be a primed tableau where the subtableau with entries  $n - l(\lambda) + i$  or  $(n - l(\lambda) + i)'$  is a connected border strip of size  $\lambda_{l(\lambda)-i+1}$  starting at  $T_{i,i} = n - l(\lambda) + i$  ( $i = l(\lambda), \dots, 1$ ). Then,  $\text{wt}(T) = (\underbrace{0, \dots, 0}_{n-l}, \lambda_l, \dots, \lambda_1, 0, \dots)$ , where  $l = l(\lambda)$ . One of the reduced expressions of  $w_0$  is

so that  $S_{w_0}T = (S_1S_2 \cdots S_{n-1}) \cdots (S_1S_2 \cdots S_{n-l})T$ . Since  $S_{n-l}T = (\tilde{e}_{n-l}^P)^{\lambda_l}T$ , we have  $\text{wt}(S_{n-l}T) = (\underbrace{0, \dots, 0}_{n-l-1}, \lambda_l, 0, \lambda_{l-1}, \dots, \lambda_1, 0, \dots)$ . Repeating the simi-

$$\text{wt}(S_1 S_2 \cdots S_{n-l} T) = (\lambda_l, \underbrace{0, \dots, 0}_{n-l}, \lambda_{l-1}, \dots, \lambda_1, 0, \dots).$$

Set  $T^{(k)} = (S_1 S_2 \cdots S_{n-l+k}) \cdots (S_1 S_2 \cdots S_{n-l}) T$  and assume that

$$\text{wt}(T^{(k)}) = (\underbrace{\lambda_{l-k}, \lambda_{l-k+1}, \dots, \lambda_l}_{k+1}, \underbrace{0, \dots, 0}_{n-l}, \lambda_{l-k-1}, \dots, \lambda_1, 0, \dots)$$

for  $k = 0, \dots, l-2$ . This assumption is satisfied for  $k = 0$ . Since  $S_{n-l+k+1} T^{(k)} = (\tilde{e}_{n-l+k+1}^P)^{\lambda_{l-k-1}} T^{(k)}$ , we have

$$\begin{aligned} \text{wt}(S_{n-l+k+1} T^{(k)}) \\ = (\underbrace{\lambda_{l-k}, \dots, \lambda_l}_{k+1}, \underbrace{0, \dots, 0}_{n-l-1}, \lambda_{l-k-1}, 0, \lambda_{l-k-2}, \dots, \lambda_1, 0, \dots). \end{aligned}$$

Repeating the similar calculations, we have

$$\begin{aligned} \text{wt}(S_{k+2} \cdots S_{n-l+k+1} T^{(k)}) \\ = (\underbrace{\lambda_{l-k}, \dots, \lambda_l}_{k+1}, \lambda_{l-k-1}, \underbrace{0, \dots, 0}_{n-l}, \lambda_{l-k-2}, \dots, \lambda_1, 0, \dots). \end{aligned}$$

Now since

$$S_{k+1}(S_{k+2} \cdots S_{n-l+k+1} T^{(k)}) = (\tilde{e}_{k+1}^P)^{\lambda_{l-k-1} - \lambda_l} S_{k+2} \cdots S_{n-l+k+1} T^{(k)},$$

we have

$$\begin{aligned} \text{wt}(S_{k+1} \cdots S_{n-l+k+1} T^{(k)}) \\ = (\underbrace{\lambda_{l-k}, \dots, \lambda_{l-1}}_k, \lambda_{l-k-1}, \lambda_l, \underbrace{0, \dots, 0}_{n-l}, \lambda_{l-k-2}, \dots, \lambda_1, 0, \dots). \end{aligned}$$

Repeating the similar calculations, we obtain

$$\begin{aligned} \text{wt}(T^{(k+1)}) &= \text{wt}(S_1 \cdots S_{n-l+k+1} T^{(k)}) \\ &= (\underbrace{\lambda_{l-k-1}, \dots, \lambda_l}_{k+2}, \underbrace{0, \dots, 0}_{n-l}, \lambda_{l-k-2}, \dots, \lambda_1, 0, \dots). \end{aligned}$$

Hence, by induction, we have

$$\text{wt}(S_{w_0} T) = \text{wt}(T^{(l-1)}) = (\lambda_1, \lambda_2, \dots, \lambda_l, 0, \dots),$$

which clearly indicates that all the entries in the  $i$ -th row of  $S_{w_0} T$  are letters  $i$  for  $i = 1, 2, \dots, l$  so that  $S_{w_0} T$  is the  $\mathfrak{q}(n)$ -highest weight vector in  $\text{PT}_n(\lambda)$ . This finishes the proof of the second part.  $\square$

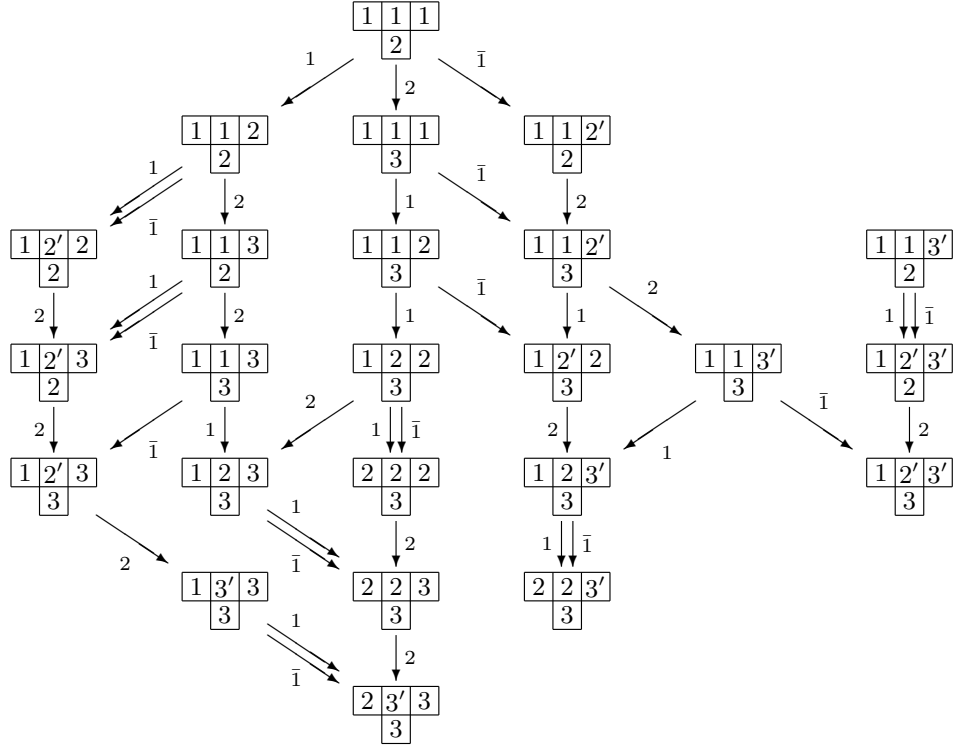


Figure 3.1:  $\mathfrak{q}(3)$ -crystal structure of  $\text{PT}_3(\lambda = (3, 1))$ .

**Example 3.2** Let  $n = 5$  and  $\lambda = (5, 3, 1)$ . Then the  $\mathfrak{q}(n)$ -highest weight vector (left) and the  $\mathfrak{q}(n)$ -lowest weight vector (right) are

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & & \\ \hline 3 & & & & \\ \hline \end{array}
 \qquad
 \begin{array}{|c|c|c|c|c|} \hline 3 & 4' & 4 & 5' & 5 \\ \hline 4 & 5' & 5 & & \\ \hline 5 & & & & \\ \hline \end{array}
 .$$

Figure 3.1 is an example of  $\mathfrak{q}(n)$ -structure of  $\text{PT}_n(\lambda)$ , where  $n = 3$  and  $\lambda = (3, 1)$ .

# Chapter 4

## **q-crystal structure on signed unimodal factorizations of reduced words of type $B$**

In this chapter, we give a  $\mathfrak{q}(m)$ -crystal structure on signed unimodal factorizations with  $m$  factors of reduced words of type  $B$  by translating the  $\mathfrak{q}(m)$ -crystal structure of primed tableaux obtained in Section 3.2 through the type  $B$  primed Kraśkiewicz insertion. The key ingredients are the type  $B$  Kraśkiewicz insertion and its properties as well as the concept of unimodal words.

### **4.1 The type $B$ Kraśkiewicz insertion**

Let us start by defining reduced words of type  $B$ . Let  $W_n^B$  be the Coxeter group of type  $B_n$  (or type  $C_n$ ). In the representation of signed permutations,  $W_n^B$  is generated by simple reflections,

$$\begin{aligned} s_0 &= \bar{1}2 \cdots n, \\ s_i &= 12 \cdots i - 1 \ i + 1 \ i \ i + 2 \cdots n \quad (i = 1, 2, \dots, n-1), \end{aligned}$$

where we use the one line notation for signed permutations [17], e.g.,  $3\bar{2}4\bar{1}$  stands for  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & \bar{2} & 4 & \bar{1} \end{pmatrix}$ . We identify the product of simple reflections with the sequence of their indices or the word. For example,  $s_0 s_1 s_2 s_0 s_1 s_3$  may be written as the word 012013. For  $w \in W_n^B$ , the length of  $w$  is defined to

be  $l(w) = \min\{l \mid \exists i_1, \dots, i_l, w = s_{i_1} \cdots s_{i_l}\}$  and a word  $w$  of length  $l(w)$  is referred to as a *reduced word*. We denote by  $|\mathbf{a}|$  the number of letters in a word  $\mathbf{a}$ . The set of all reduced words of the element  $w \in W_n^B$  is denoted by  $R_B(w)$ . The elements of  $R_B(w)$  are classified by the type  $B$  Coxeter-Knuth relations [21].

**Definition 4.1** Let  $\mathbf{a}, \mathbf{b} \in R_B(w)$  for  $w \in W_n^B$ . Suppose that  $\mathbf{a}$  and  $\mathbf{b}$  are written, respectively, in the forms  $\mathbf{a} = \mathbf{cxd}$  and  $\mathbf{b} = \mathbf{cyd}$ , where  $\mathbf{x}$  and  $\mathbf{y}$  are the subwords of four adjacent letters of  $\mathbf{a}$  and  $\mathbf{b}$ , respectively. If  $\mathbf{x} \sim \mathbf{y}$  or  $\mathbf{y} \sim \mathbf{x}$  is one of the relations listed below, we say that  $\mathbf{a}$  is elementary type  $B$  Coxeter-Knuth related to  $\mathbf{b}$ :

- (1)  $0101 \sim 1010$
- (2)  $ab(b+1)b \sim a(b+1)b(b+1)$
- (3)  $ba(b+1)b \sim b(b+1)ab$
- (4)  $a(a+1)ba \sim a(a+1)ab$  with  $a+1 < b$
- (5)  $(a+1)ab(a+1) \sim (a+1)ba(a+1)$  with  $a+1 < b$
- (6)  $abdc \sim adbc$
- (7)  $acdb \sim acbd$
- (8)  $adcb \sim dacb$
- (9)  $badc \sim bdac$

and their reverses, where  $a < b < c < d$  unless otherwise stated.

Let  $\mathbf{a}, \mathbf{b} \in R_B(w)$ . If there exists a sequence of reduced words,

$$\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k = \mathbf{b}$$

such that  $\mathbf{a}_i$  is elementary type  $B$  Coxeter-Knuth related to  $\mathbf{a}_{i+1}$ , we say that  $\mathbf{a}$  and  $\mathbf{b}$  are type  $B$  Coxeter-Knuth related and write  $\mathbf{a} \stackrel{B}{\sim} \mathbf{b}$ .

**Example 4.1** We have  $R(\bar{3}2\bar{1}) = \{01210, 02120, 02102, 20120, 20102\}$ . Since  $0121 \sim 0210$  by (2) of Definition 4.1 with  $a = 0$  and  $b = 1$ ,  $01210 \stackrel{B}{\sim} 02120$ . Since  $0210 \sim 2010$  and  $0102 \sim 0120$  by (4) of Definition 4.1 with  $a = 0$  and  $b = 2$ ,  $02102 \stackrel{B}{\sim} 20102 \stackrel{B}{\sim} 20120$ . However,  $02120 \not\stackrel{B}{\sim} 02102$ .

**Definition 4.2** A word  $\mathbf{a} = a_1 a_2 \cdots a_l$  is called unimodal if there exists  $k$  such that  $a_1 > a_2 > \cdots > a_k < \cdots < a_l$ . We call  $\mathbf{a}_\downarrow = a_1 a_2 \cdots a_k$  the decreasing part of  $\mathbf{a}$  and  $\mathbf{a}_\uparrow = a_{k+1} a_{k+2} \cdots a_l$  the increasing part of  $\mathbf{a}$ .

**Definition 4.3** Let  $P$  be a shifted tableau of shape  $\lambda$ . If

- (1)  $P_{l(\lambda)} P_{l(\lambda)-1} \cdots P_1$  is a reduced word of  $w \in W_n^B$  and
- (2)  $P_i$  is a unimodal word of maximum length in  $P_{i+1} P_i$  for  $i = 1, \dots, l(\lambda) - 1$ ,

then  $P$  is called a type  $B$  standard decomposition tableau of  $w$ . We denote the set of such tableaux by  $\text{SDT}_w^B(\lambda)$ .

**Remark 4.1** A type  $B$  standard decomposition tableau is also called a reduced unimodal tableau [14].

Let  $\mathbf{a} = a_1 a_2 \cdots a_m \in R_B(w)$  for  $w \in W_n^B$ . We recursively construct a sequence of pairs of tableaux,

$$(\emptyset, \emptyset) = (P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \dots, (P^{(m)}, Q^{(m)}) = (P, Q),$$

by the insertion algorithm due to Kraśkiewicz [21, 22]. The algorithm to obtain  $(P^{(i)}, Q^{(i)})$  from  $(P^{(i-1)}, Q^{(i-1)})$  is as follows:

- (1) Let  $a = a_i$  and  $R$  be the first row of  $P^{(i-1)}$ .
- (2) Insert  $a$  into  $R$  as follows:
  - (i) If  $Ra$  is unimodal ( $R$  may be empty), append  $\boxed{a}$  to  $R$  and let  $P^{(i)}$  be this new tableau. To get  $Q^{(i)}$ , we add  $\boxed{i}$  to  $Q^{(i-1)}$  such that  $P^{(i)}$  and  $Q^{(i)}$  have the same shifted shape. Stop.
  - (ii) If  $Ra$  is not unimodal, let  $b$  be the smallest element in  $R_\uparrow$ , the increasing part of  $R$ , larger than or equal to  $a$ . Such a  $b$  always exists.
    - (ii-1):  $a = 0$  and  $R$  contains 101 as a subsequence. Leave  $R$  unchanged and go to (2) with  $a = 0$  and  $R$  being the next row.
    - (ii-2):  $b \neq a$ . Put  $a$  in  $b$ 's position and let  $c = b$ .
    - (ii-3):  $b = a$ . Leave  $R_\uparrow$  unchanged and let  $c = a + 1$ .

We insert  $c$  into  $R_{\downarrow}$ , the decreasing part of  $R$ . Let  $d$  be the largest element in  $R_{\downarrow}$  which is smaller than or equal to  $c$ . Such an element always exists.

(ii-4):  $d \neq c$ . Put  $c$  in  $d$ 's place and let  $a' = d$ .

(ii-5):  $d = c$ . Leave  $R_{\downarrow}$  unchanged and let  $a' = c - 1$ .

(3) Repeat (2) with  $a = a'$  and  $R$  being the next row.

The tableau  $P$  (resp.  $Q$ ) is called the insertion (resp. recording) tableau.

**Remark 4.2** *The above described Kraśkiewicz insertion of type  $B$  is a  $B_n$  (or  $C_n$ )-analogue of the Edelman-Greene insertion [8] for the words of the symmetric group [23].*

**Example 4.2** *For a reduced word  $012013 \in R_B(3\bar{2}4\bar{1})$ , we have*

$$P = \begin{array}{|c|c|c|c|} \hline 2 & 0 & 1 & 3 \\ \hline 0 & 1 & & \\ \hline \end{array} \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & & \\ \hline \end{array}.$$

**Theorem 4.1** ([21, 22]) *The Kraśkiewicz insertion of type  $B$  gives a bijection between  $R_B(w)$  and pairs of tableaux  $(P, Q)$  where  $P \in \text{SDT}_w^B(\lambda)$  and  $Q \in \text{ST}_n(\lambda)$  for  $w \in W_n^B$ .*

$$K_B : R_B(w) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathcal{P}^+} [\text{SDT}_w^B(\lambda) \times \text{ST}_n(\lambda)].$$

**Theorem 4.2** ([23]) *Two reduced words in  $R_B(w)$  are type  $B$  Coxeter-Knuth related if and only if they have the same insertion tableau.*

**Remark 4.3** *In this thesis, we use only “if” part of Theorem 4.2.*

**Example 4.3** *We continue Example 4.1. Reduced words 01210 and 02120 in  $R(\bar{3}2\bar{1})$  have the insertion tableau*

$$\begin{array}{|c|c|c|} \hline 2 & 1 & 0 \\ \hline 0 & 1 & \\ \hline \end{array},$$

*whereas reduced words 02102, 20120, and 20102 have the insertion tableau*

$$\begin{array}{|c|c|c|c|} \hline 2 & 1 & 0 & 2 \\ \hline 0 & & & \\ \hline \end{array}.$$

**Definition 4.4** Suppose that the entries  $i, i+1, \dots, j$  appear in  $Q$  in the following manner: There exists some  $i \leq k \leq j$  such that:

- (1) the entries  $i, i+1, \dots, k$  form a vertical strip,
- (2) these entries are increasing down the vertical strip,
- (3) the entries  $k, k+1, \dots, j$  form a horizontal strip,
- (4) these entries are increasing from left to right, and
- (5) each box in the vertical strip is left of these boxes in the horizontal strip that are on the same row.

We say that these entries form a vee in  $Q$ .

**Lemma 4.1** ([24]) Let  $\mathbf{a} = a_1 a_2 \dots a_m \in R_B(w)$  and  $K_B(\mathbf{a}) = (P, Q)$ . The subword  $a_i a_{i+1} \dots a_j$  is unimodal if and only if the entries  $i, i+1, \dots, j$  form a vee in  $Q$ .

Let  $(x_i, y_i), \dots, (x_j, y_j)$  be the sequence of positions of a vee  $i, i+1, \dots, j$  in  $Q$ . Then, by Definition 4.4, there exists an index  $k$  such that  $x_i < \dots < x_k \geq \dots \geq x_j$  and  $y_i \geq \dots \geq y_k < \dots < y_j$ . We call such a  $k$  the *bottom index*.

**Example 4.4** In Example 4.2, the entries 3, 4, 5, 6 in  $Q$  form a vee and the corresponding subword 2013 of  $w = 012013$  is unimodal. In this example, the bottom index is 2.

## 4.2 q-crystal structure on signed unimodal factorizations of reduced words of type $B$

For a reduced word  $\mathbf{a} \in R_B(w)$ , a *signed unimodal factorization* of  $\mathbf{a}$  is a factorization  $\mathbf{A} = (s_1 \mathbf{a}_1)(s_2 \mathbf{a}_2) \dots$ , where  $\mathbf{a}_i$  are unimodal words and  $s_i$  are signs. Factors  $(s_i \mathbf{a}_i)$  can be empty. In this case, the sign is taken to be 0. Otherwise, signs are either  $+$  or  $-$ . The weight of a unimodal factorization, denoted by  $\text{wt}(\mathbf{A})$ , is defined to be  $\text{wt}(\mathbf{A}) = \sum_{i=1}^n |\mathbf{a}_i| \epsilon_i$ . We denote by  $U_m^{B\pm}(w)$  the set of all signed unimodal factorizations with  $m$  factors of reduced words of  $w \in W_n^B$ .

**Definition 4.5** A signed primed tableau  $T$  of shape  $\lambda$  is a filling of  $S(\lambda)$  with letters from the alphabet  $X'_n$  such that:

- (1) the entries are weakly increasing along each column and each row of  $T$ ,
- (2) each row contains at most one  $i'$  for every  $i = 1, \dots, n$ , and
- (3) each column contains at most one  $i$  for every  $i = 1, \dots, n$ .

We denote by  $\text{PT}_n^\pm(\lambda)$  the set of signed primed tableaux of shape  $\lambda$ .

**Remark 4.4** A signed primed tableau is also called a  $Q$ -tableau [23].

Now, let us describe the *primed Kraśkiewicz insertion* introduced in [14], which we call the primed Kraśkiewicz insertion of type  $B$ . Let  $\mathbf{A} \in U_m^{B^\pm}(w)$  be a signed unimodal factorization with  $m$  unimodal factors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ . We recursively construct a sequence of pairs of tableaux,

$$(\emptyset, \emptyset) = (P^{(0)}, T^{(0)}), (P^{(1)}, T^{(1)}), \dots, (P^{(m)}, T^{(m)}) = (P, T),$$

where  $P^{(i)}$  is computed by the type  $B$  Kraśkiewicz insertion of letters of  $\mathbf{a}_i$ , one by one from left to right into  $P^{(i-1)}$ . The tableau  $T^{(i)}$  is a signed primed tableau of shape  $\lambda^{(i)}$  and computed by the following algorithm:

Let  $(x_1, y_1), \dots, (x_r, y_r)$  be the ordered sequence of positions of boxes newly appeared on  $P^{(i)}$ . By Lemma 4.1, there exists an index  $k$  such that  $x_1 < \dots < x_k \geq \dots \geq x_r$  and  $y_1 \geq \dots \geq y_k < \dots < y_r$ . The tableau  $T^{(i)}$  is obtained from  $T^{(i-1)}$  by adding the boxes of  $(x_1, y_1), \dots, (x_{k-1}, y_{k-1})$  each filled with  $i'$  and the boxes of  $(x_{k+1}, y_{k+1}), \dots, (x_r, y_r)$  each filled with  $i$ . If the sign of the factor  $\mathbf{a}_i$  is  $-$  (resp.  $+$ ), the box  $(x_k, y_k)$  is filled with the letter  $i'$  (resp.  $i$ ).

As in the Kraśkiewicz insertion, we call such a  $k$  the bottom index of the sequence  $(x_1, y_1), \dots, (x_r, y_r)$  and say that the sequence of entries

$$\underbrace{i', \dots, i'}_{k-1}, i^{(\iota)}, \underbrace{i, \dots, i}_{r-k}$$

forms a vee in  $T$ , where  $i^{(\iota)}$  is either  $i'$  or  $i$ . We call the leftmost entry in the above sequence a *head* of the vee.

**Example 4.5** For a signed unimodal factorization  $\mathbf{A} = (+01)(-2013) \in U_2^{B^\pm}(3\bar{2}4\bar{1})$  of the reduced word in Example 4.2, we have

$$P = \begin{array}{|c|c|c|c|} \hline 2 & 0 & 1 & 3 \\ \hline & 0 & 1 & \\ \hline \end{array} \quad T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2' & 2 \\ \hline & 2' & 2 & \\ \hline \end{array}.$$

The entries  $2', 2', 2, 2$  whose positions are  $(1, 3), (2, 2), (2, 3), (1, 4)$  in this order form a vee in  $T$ . The head of the vee is the entry  $2'$  at the position  $(1, 3)$ .

**Theorem 4.3 ([14])** *The primed Kraskiewicz insertion of type  $B$  gives a bijection between  $U_m^{B\pm}(w)$  and pairs of tableaux  $(P, T)$  where  $P \in \text{SDT}_w^B(\lambda)$  and  $T \in \text{PT}_m^\pm(\lambda)$  for  $w \in W_n^B$ .*

$$K'_B : U_m^{B\pm}(w) \xrightarrow{\sim} \bigsqcup_{\lambda \in P^+} [\text{SDT}_w^B(\lambda) \times \text{PT}_m^\pm(\lambda)]. \quad (4.1)$$

Since  $\text{PT}_m(\lambda)$  is a  $\mathfrak{q}(m)$ -crystal, so is  $\text{PT}_m^\pm(\lambda)$ . The Kashiwara operators  $\tilde{e}_i^{\pm P}$  and  $\tilde{f}_i^{\pm P}$  on  $T \in \text{PT}_m^\pm(\lambda)$  ( $i = 1, \dots, m-1, \bar{1}$ ) are given by

$$\tilde{e}_i^{\pm P} T = \text{PR} \circ \tilde{e}_i^P \circ \text{DPR}(T), \quad (4.2)$$

$$\tilde{f}_i^{\pm P} T = \text{PR} \circ \tilde{f}_i^P \circ \text{DPR}(T), \quad (4.3)$$

where DPR removes primes on the main diagonal of  $T$  and PR recovers primes at the original positions on the main diagonal. Let us define a *prime type* of  $T \in \text{PT}_m^\pm(\lambda)$  as the set of indexes  $i$  such that the entry of  $(i, i)$ -position is primed. The tableaux in  $\text{PT}_m^\pm(\lambda)$  of the same prime type form a connected  $\mathfrak{q}(m)$ -crystal. The  $\mathfrak{q}(m)$ -highest (resp. lowest) weight vector in  $\text{PT}_m^\pm(\lambda)$  is easily obtained by the  $\mathfrak{q}(m)$ -highest (resp. lowest) weight vector in  $\text{PT}_m(\lambda)$  and the given prime type.

**Example 4.6** *Let  $m = 5$ ,  $\lambda = (5, 3, 1)$ , and the prime type be  $\{1, 3\}$ . Then the  $\mathfrak{q}(m)$ -highest weight vector (left) and the  $\mathfrak{q}(m)$ -lowest weight vector (right) in  $\text{PT}_m^\pm(\lambda)$  are*

$$\begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline & 2 & 2 & 2 & \\ \hline & & 3' & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 3' & 4' & 4 & 5' & 5 \\ \hline & 4 & 5' & 5 & \\ \hline & & 5' & & \\ \hline \end{array}.$$

**Theorem 4.4 ([15])** *The set of all signed unimodal factorizations  $U_m^{B\pm}(w)$  with  $m$  factors of reduced words of  $w \in W_n^B$  admits a  $\mathfrak{q}(m)$ -crystal structure. Each connected component of  $U_m^{B\pm}(w)$  consists of the signed unimodal factorizations of type  $B$  Coxeter-Knuth related reduced words.*

The even Kashiwara operators on  $U_m^{B\pm}(w)$  are given by

$$\begin{aligned}\tilde{e}_i^F &= K_B'^{-1} \circ (\text{Id}, \tilde{e}_i^{\pm P}) \circ K_B', \\ \tilde{f}_i^F &= K_B'^{-1} \circ (\text{Id}, \tilde{f}_i^{\pm P}) \circ K_B',\end{aligned}$$

( $i = 1, \dots, m-1$ ) and the action of odd Kashiwara operators are given by the following rules: Let  $\mathbf{A} = (s_1 \mathbf{a}_1)(s_2 \mathbf{a}_2) \cdots \in U_m^{B\pm}(w)$  be a signed unimodal factorization with the unimodal word  $\mathbf{a}_i$  and its sign ( $i = 1, 2, \dots, m$ ) for  $w \in W_n^B$ . The action of the odd Kashiwara operator  $\tilde{e}_1^F$  on  $\mathbf{A}$  is given by the following rule:

(e-0):  $\tilde{e}_1^F$  always changes the first two factors if  $\tilde{e}_1^F \mathbf{A} \neq \mathbf{0}$ .

(e-1):  $\mathbf{a}_1 \mathbf{a}_2$  is not unimodal. In this case,  $\tilde{e}_1^F \mathbf{A} = (s_1 \mathbf{a}'_1)(s_2 \mathbf{a}'_2) \cdots$ , where  $\mathbf{a}'_2$  is obtained by dropping the first letter of  $\mathbf{a}_2$  and  $\mathbf{a}'_1$  is the word obtained by appending that letter to  $\mathbf{a}_1$ . Here,  $\mathbf{a}'_1$  is assumed to be unimodal. Otherwise,  $\tilde{e}_1^F \mathbf{A} = \mathbf{0}$ .

(e-2):  $\mathbf{a}_1 \mathbf{a}_2$  is unimodal. In this case, If  $\mathbf{a}_2 = \emptyset$  or  $s_1 \neq 0$  and  $s_2 = +$ , then  $\tilde{e}_1^F \mathbf{A} = \mathbf{0}$ . Otherwise,  $\tilde{e}_1^F \mathbf{A} = (s'_1 \mathbf{a}'_1)(s'_2 \mathbf{a}'_2) \cdots$ , where  $\mathbf{a}'_1$  and  $\mathbf{a}'_2$  are the same as in (e-1) and signs  $s'_1$  and  $s'_2$  are given by the following rule:

When  $\mathbf{a}_1 = \emptyset$ ,

$$(s'_1, s'_2) = \begin{cases} (s_2, 0), & |\mathbf{a}_2| = 1, \\ (s_2, +), & |\mathbf{a}_2| \geq 2. \end{cases}$$

When  $\mathbf{a}_1 \neq \emptyset$ ,

$$(s'_1, s'_2) = \begin{cases} (s_1, 0), & |\mathbf{a}_2| = 1, \\ (s_1, +), & |\mathbf{a}_2| \geq 2. \end{cases}$$

The action of the odd Kashiwara operator  $\tilde{f}_1^F$  on  $\mathbf{A}$  is given by the following rule:

(f-0):  $\tilde{f}_1^F$  always change the first two factors if  $\tilde{f}_1^F \mathbf{A} \neq \mathbf{0}$ .

(f-1):  $\mathbf{a}_1 \mathbf{a}_2$  is not unimodal. In this case,  $\tilde{f}_1^F \mathbf{A} = (s_1 \mathbf{a}'_1)(s_2 \mathbf{a}'_2) \cdots$ , where  $\mathbf{a}'_1$  is obtained by dropping the last letter of  $\mathbf{a}_1$  and  $\mathbf{a}'_2$  is the word obtained by prepending that letter to  $\mathbf{a}_2$ . Here,  $\mathbf{a}'_2$  is assumed to be unimodal. Otherwise,  $\tilde{f}_1^F \mathbf{A} = \mathbf{0}$ .

(f-2):  $\mathbf{a}_1 \mathbf{a}_2$  is unimodal. If  $\mathbf{a}_2 \neq \emptyset$  and  $s_2 = -$ , then  $\tilde{f}_1^F \mathbf{A} = \mathbf{0}$ . Otherwise,  $\tilde{f}_1^F \mathbf{A} = (s'_1 \mathbf{a}'_1)(s'_2 \mathbf{a}'_2) \cdots$ , where  $\mathbf{a}'_1$  and  $\mathbf{a}'_2$  are the same as in (f-1) and signs  $s'_1$  and  $s'_2$  are given by the following rule:

$$(s'_1, s'_2) = \begin{cases} (0, s_1), & |\mathbf{a}_1| = 1, \\ (s_1, -), & |\mathbf{a}_1| \geq 2. \end{cases}$$

Under these Kashiwara operators  $\tilde{e}_i^F$  and  $\tilde{f}_i^F$  ( $i = 1, \dots, m-1, \bar{1}$ ) together with  $\tilde{e}_i^{\pm P}$  and  $\tilde{f}_i^{\pm P}$  ( $i = 1, \dots, m-1, \bar{1}$ ) on  $\text{PT}_m^\pm(\lambda)$ , the bijection  $K'_B$  (Eq. (4.1)) is a  $\mathfrak{q}(m)$ -crystal isomorphism.

*Proof.* Let us fix  $P \in \text{SDT}_w^B(\lambda)$  in  $K'_B$  (Eq. (4.1)). We can define even Kashiwara operators on  $U_m^{B\pm}(w)$  as

$$\begin{aligned} \tilde{e}_i^F &= K_B'^{-1} \circ (\text{Id}, \tilde{e}_i^{\pm P}) \circ K'_B, \\ \tilde{f}_i^F &= K_B'^{-1} \circ (\text{Id}, \tilde{f}_i^{\pm P}) \circ K'_B, \end{aligned}$$

( $i = 1, \dots, m-1$ ). Then, the bijection  $K'_B$  turns out to be a  $\mathfrak{gl}(m)$ -crystal isomorphism. Now, since we have established the  $\mathfrak{q}(m)$ -crystal structure on  $\text{PT}_m^\pm(\lambda)$ , the bijection  $K'_B$  is shown to be actually a  $\mathfrak{q}(m)$ -crystal isomorphism. Let us determine the odd Kashiwara operators  $\tilde{e}_1^F$  and  $\tilde{f}_1^F$  on  $\mathbf{A} \in U_m^{B\pm}(w)$  satisfying

$$\begin{aligned} \tilde{e}_1^F &= K_B'^{-1} \circ (\text{Id}, \tilde{e}_1^{\pm P}) \circ K'_B, \\ \tilde{f}_1^F &= K_B'^{-1} \circ (\text{Id}, \tilde{f}_1^{\pm P}) \circ K'_B. \end{aligned}$$

Let  $T \in \text{PT}_m^\pm(\lambda)$  be a signed primed tableau such that  $K'_B(\mathbf{A}) = (P, T)$ , where  $P \in \text{SDT}_w^B(\lambda)$ . Let us write  $\mathbf{a}_1 = u_1^{(1)} \cdots u_r^{(1)}$  and  $\mathbf{a}_2 = u_1^{(2)} \cdots u_s^{(2)}$ .

Firstly, let us determine the action of  $\tilde{e}_1^F$ . Suppose that  $T' = \tilde{e}_1^{\pm P} T \neq \mathbf{0}$ . Namely, we have the following three cases:

- (i):  $T_{1,1} = 2'$ ,  $T'_{1,1} = 1'$ , and  $T_{i,j} = T'_{i,j}$  for  $(i, j) \neq (1, 1)$ .
- (ii):  $T_{1,1} = 2$ ,  $T'_{1,1} = 1$ , and  $T_{i,j} = T'_{i,j}$  for  $(i, j) \neq (1, 1)$ .
- (iii):  $T_{1,k} = 2'$ ,  $T'_{1,k} = 1$  ( $\exists! k \geq 2$ ), and  $T_{i,j} = T'_{i,j}$  for  $(i, j) \neq (1, k)$ . In this case, the entry  $2'$  does not appear in the first row of  $T$  except at  $(1, k)$ .

Let  $\mathbf{A}' = (s'_1 \mathbf{a}'_1)(s'_2 \mathbf{a}'_2) \cdots \in U_m^{B\pm}(w)$  be a signed unimodal factorization such that  $K'_B(\mathbf{A}') = (P, T')$ .

In Case (i), since neither  $1'$  nor  $1$  appears in  $T$ , the first factor of  $\mathbf{A}$  is empty;  $(s_1 \mathbf{a}_1) = ()$ . The entry  $T_{1,1} = 2'$  appears first when the letter  $u_1^{(2)}$  is inserted and neither  $1'$  nor  $1$  appears at the position except at  $(1, 1)$  in  $T'$  so that  $\mathbf{a}'_1 = u_1^{(2)}$  and  $\mathbf{a}'_2 = u_2^{(2)} \cdots u_s^{(2)}$ . Only the first two factors of  $\mathbf{A}$  change under the action of  $\tilde{e}_1^F$  if  $\tilde{e}_1^F \mathbf{A} \neq \mathbf{0}$ .

In Case (ii), we have that  $\mathbf{a}'_1 = u_1^{(2)}$  and  $\mathbf{a}'_2 = u_2^{(2)} \cdots u_s^{(2)}$  and that only the first two factors of  $\mathbf{A}$  change under the action of  $\tilde{e}_1^F$  if  $\tilde{e}_1^F \mathbf{A} \neq \mathbf{0}$  as in Case (i).

In Case (iii), the primed entry  $T_{1,k} = 2'$  is the head of the vee starting from  $2'$  in  $T$ . Otherwise, there exists an entry  $2'$  in  $T$  at the position  $(x_1, y_1) \neq (1, k)$  with  $x_1 < 1$  and  $y_1 \geq k$ , which is impossible. Since  $T_{1,k} = 2'$  is changed to  $T'_{1,k} = 1$ , we have that  $\mathbf{a}'_1 = \mathbf{a}_1 u_1^{(2)}$  and  $\mathbf{a}'_2 = u_2^{(2)} \cdots u_s^{(2)}$  and that only the first two factors of  $\mathbf{A}$  change under the action of  $\tilde{e}_1^F$  if  $\tilde{e}_1^F \mathbf{A} \neq \mathbf{0}$ .

We first claim that  $\tilde{e}_1^F \mathbf{A} = \mathbf{0}$  if  $\mathbf{a}'_1$  is not unimodal. Note that  $\mathbf{a}'_2$  is always unimodal. This is verified as follows: Suppose that  $\mathbf{a}'_1$  is not unimodal. Then  $\mathbf{a}_1$  consists of at least two letters and  $T$  has the following configuration. (Note that  $\mathbf{a}_1$  is unimodal).

$$\begin{array}{ccccccc} & & & r & & & \\ & & & & & & \\ x & 1 & \cdots & 1 & \cdots & & \\ & y & & & & & \\ & & \ddots & & & & \end{array},$$

where  $r \geq 2$ , the entry  $x$  is either  $1'$  or  $1$ , and the entry  $y$ , which appears when the letter  $u_1^{(2)}$  is inserted, is either  $2'$  or  $2$ . The entry  $y$  does not appear in the first row; otherwise  $\mathbf{a}'_1$  would be unimodal. When  $|\mathbf{a}_2| = 1$ , the entry  $2'$  does not appear in the first row of  $T$  so that  $\tilde{e}_1^{\pm P} T = \mathbf{0}$  by Lemma 3.1 with Eq. (4.2) and therefore  $\tilde{e}_1^F \mathbf{A} = \mathbf{0}$ . When  $|\mathbf{a}_2| \geq 2$ , suppose that the entry  $z$  ( $z$  is either  $2'$  or  $2$ ) appears in the first row of  $T$ :

$$\begin{array}{ccccccc} & & & r & & & \\ & & & & & & \\ x & 1 & \cdots & 1 & z & \cdots & \\ & y & & & & & \\ & & \ddots & & & & \end{array}.$$

The entry  $z$  appears after  $y$  appears in the course of the type  $B$  primed Kraśkiewicz insertion of  $\mathbf{a}_2$ . The position of  $y$  is  $(2, 2)$  and that of  $z$  is

$(1, r+1)$ . Let  $k^{(2)}$  be the bottom index of the sequence of boxes which appear in the insertion of  $\mathbf{a}_2$ . Since  $2 \geq 1$  and  $2 < r+1$ ,  $k^{(2)} = 1$  so that  $z = 2$  (not  $2'$ ). Therefore,  $\tilde{e}_1^{\pm P} T = \mathbf{0}$  so that  $\tilde{e}_1^F \mathbf{A} = \mathbf{0}$ .

Next, we verify the rule of signs. We assume that  $\mathbf{a}'_1$  is unimodal. Let

$$(x_1^{(1)}, y_1^{(1)}), \dots, (x_r^{(1)}, y_r^{(1)})$$

be the ordered sequence of the positions of boxes which appear in the insertion of the unimodal word  $\mathbf{a}_1$  with the bottom index  $k^{(1)}$ . Similarly, let

$$(x_1^{(2)}, y_1^{(2)}), \dots, (x_s^{(2)}, y_s^{(2)})$$

be the ordered sequence of the positions of boxes which appear in the insertion of the unimodal word  $\mathbf{a}_2$  with the bottom index  $k^{(2)}$ . Let  $k^{(1)'} (resp. k^{(2)'})$  be the bottom index of the sequence of positions of boxes which appear in the insertion of the unimodal word  $\mathbf{a}'_1 (resp. \mathbf{a}'_2)$ .

(e-1):  $\mathbf{a}_1 \mathbf{a}_2$  is not unimodal. In this case,  $\mathbf{a}_1 \neq \emptyset$  and  $\mathbf{a}_2 \neq \emptyset$ . Since  $\mathbf{a}'_1$  is unimodal, we have  $1 = x_1^{(1)} = \dots = x_r^{(1)} = x_1^{(2)}$  and  $y_1^{(1)} < \dots < y_r^{(1)} < y_1^{(2)}$  so that  $k^{(1)} = 1$  and  $k^{(1)'} = 1$ . If the bottom index  $k^{(2)} = 1$ , then  $1 = x_1^{(2)} = \dots = x_s^{(2)}$  and  $y_1^{(2)} < \dots < y_s^{(2)}$  so that

$$1 = x_1^{(1)} = \dots = x_r^{(1)} = x_1^{(2)} = \dots = x_s^{(2)}$$

and

$$y_1^{(1)} < \dots < y_r^{(1)} < y_1^{(2)} < \dots < y_s^{(2)}.$$

This implies that the sequence of entries  $\underbrace{1^{(r)}, \dots, 1}_r, \underbrace{2^{(s)}, \dots, 2}_s$  forms a vee in

$T$ , where  $i^{(r)}$  is either  $i'$  or  $i$  ( $i = 1, 2$ ). Consequently,  $\mathbf{a}_1 \mathbf{a}_2$  is unimodal, which contradicts the assumption of (e-1). Hence,  $k^{(2)} \geq 2$ . Therefore, the entry at the position  $(x_1^{(2)}, y_1^{(2)}) = (1, y_1^{(2)})$  in  $T$  is  $2'$ , which is the head of the vee starting from  $2'$  in  $T$  and is the only  $2'$  in the first row of  $T$ . The operator  $\tilde{e}_1^{\pm P}$  changes this  $2'$  to 1 and leaves other entries unchanged. Here,  $s'_1 = s_1$  and  $s'_2 = s_2$ . This can be shown as follows: Since  $\mathbf{a}'_1$  is unimodal, the configuration of the first row of  $T$  is

$$\boxed{x \mid 1 \mid \cdots \mid 1 \mid \overbrace{2'}^{r+1} \mid \cdots},$$

where  $r \geq 1$  and  $x$  is either  $1'$  or 1 so that the first row of  $T'$  has the following configuration.

$$\overline{x \mid 1 \mid \cdots \mid 1 \mid 1 \mid \cdots}^{r+1}.$$

Then, we have that  $s'_1 = s_1$ ; otherwise  $T'_{1,1} \neq x$  because  $k^{(1)} = 1$  and  $k^{(1)'} = 1$ . By noting  $k^{(2)} \geq 2$ , we also have that  $s'_2 = s_2$  because the entries  $2'$  and  $2$  at the same positions except at  $(1, r+1)$  in  $T$  and  $T'$  coincide.

(e-2):  $\mathbf{a}_1 \mathbf{a}_2$  is unimodal. In this case,

$$1 = x_1^{(1)} = \cdots = x_r^{(1)} = x_1^{(2)} = \cdots = x_s^{(2)}$$

and

$$y_1^{(1)} < \cdots < y_r^{(1)} < y_1^{(2)} \cdots < y_s^{(2)}$$

so that  $k^{(2)} = 1$ . If  $\mathbf{a}_1 \neq \emptyset$  and  $s_2 = +$ , then the configuration of the first row of  $T$  is

$$\overline{x \mid \cdots \mid 1 \mid 2 \mid \cdots},$$

where  $r \geq 1$  and  $x$  is either  $1'$  or  $1$ . The entry  $2$  in the above configuration is unprimed because  $k^{(2)} = 1$  and  $s_2 = +$ . Hence,  $\tilde{e}_1^{\pm P} T = \mathbf{0}$  so that  $\tilde{e}_1^F \mathbf{A} = \mathbf{0}$ . In the following, we assume that  $\mathbf{a}_1 = \emptyset$  or  $s_2 \neq +$ .

(i):  $\mathbf{a}_1 = \emptyset$ .

(i-1):  $|\mathbf{a}_2| = 1$ . In this case,  $T_{1,1}$  is either  $2'$  or  $2$ . If  $T_{1,1} = 2'$ , then  $s_2 = -$  and  $T'_{1,1} = 1'$ , i.e.,  $s'_1 = -$ . On the other hand, if  $T_{1,1} = 2$ , then  $s_2 = +$  and  $T'_{1,1} = 1$ , i.e.,  $s'_1 = +$ . In both cases,  $s'_1 = s_2$ .

(i-2):  $|\mathbf{a}_2| \geq 2$ . As in (i-1), we have  $s'_1 = s_2$ . Since  $1 = x_1^{(2)} = \cdots = x_s^{(2)}$  and  $y_1^{(2)} < \cdots < y_s^{(2)}$ ,  $k^{(2)} = 1$ . We claim that  $s'_2 = +$ . In the case when  $s_2 = +$ , the configuration of the first row of  $T$  is

$$\overline{2 \mid 2 \mid \cdots}.$$

If  $s'_2 = -$ , then the configuration of the first row of  $T'$  is

$$\overline{1 \mid 2' \mid \cdots}.$$

Note that  $s'_1 = +$  so  $T'_{1,1} = 1$ . This contradicts the previous configuration because the configuration of the first row of  $\tilde{e}_1^{\pm P} T$  is

$$\boxed{1 \mid 2 \mid \cdots}.$$

In the case when  $s_2 = -$ , the configuration of the first row of  $T$  is

$$\boxed{2' \mid 2 \mid \cdots}.$$

If  $s'_2 = -$ , then the configuration of the first row of  $T'$  is

$$\boxed{1' \mid 2' \mid \cdots}.$$

Note that  $s'_1 = -$  so  $T'_{1,1} = 1'$ . This contradicts the previous configuration because the configuration of the first row of  $\tilde{e}_1^{\pm P} T$  is

$$\boxed{1' \mid 2 \mid \cdots}.$$

In both cases, we have that  $s'_2 = +$ .

(ii):  $\mathbf{a}_1 \neq \emptyset$ . In this case,  $s_2 = -$ . It is clear that  $s'_1 = s_1$  and  $s'_2 = 0$  when  $|\mathbf{a}_2| = 1$ . When  $|\mathbf{a}_2| \geq 2$ , it is also clear that  $s'_1 = s_1$ . We claim that  $s'_2 = +$ . Since  $s_2 = -$  and  $k^{(2)} = 1$ , the configuration of the first row of  $T$  is

$$\boxed{x \mid 1 \mid \cdots \mid \overbrace{1 \mid 2' \mid 2 \mid \cdots}^{r+1}},$$

where  $r \geq 1$  and  $x$  is either  $1'$  or  $1$ . If  $s'_2 = -$ , then the configuration of the first row of  $T'$  is

$$\boxed{x \mid 1 \mid \cdots \mid \overbrace{1 \mid 2' \mid \cdots}^{r+1}}.$$

This contradicts the previous configuration because the configuration of the first row of  $\tilde{e}_1^{\pm P} T$  is

$$\boxed{x \mid 1 \mid \cdots \mid \overbrace{1 \mid 2 \mid \cdots}^{r+1}}.$$

Consequently, we have that  $s'_2 = +$ .

The verification of the rule of  $\tilde{e}_1^F$  is now complete.

Next, let us determine the action of  $\tilde{f}_1^F$ . Similar arguments lead to

$$\tilde{f}_1^F \mathbf{A} = (s'_1 \mathbf{a}'_1)(s'_2 \mathbf{a}'_2)(s_3 \mathbf{a}_2) \cdots (s_m \mathbf{a}_m),$$

where  $\mathbf{a}'_1 = u_1^{(1)} \cdots u_{r-1}^{(1)}$  and  $\mathbf{a}'_2 = u_r^{(1)} u_1^{(2)} \cdots u_s^{(2)}$  if  $\tilde{f}_1^F \mathbf{A} \neq \mathbf{0}$ . In particular,  $\tilde{f}_1^F \mathbf{A} = \mathbf{0}$  if  $\mathbf{a}_1 = \emptyset$ . Furthermore,  $\mathbf{a}'_2$  must be unimodal to be  $\tilde{f}_1^F \mathbf{A} \neq \mathbf{0}$ .

We are left with the task of determining signs  $s'_1$  and  $s'_2$ . Let  $\mathbf{A}' = (s'_1 \mathbf{a}'_1)(s'_2 \mathbf{a}'_2) \cdots \in U_m^{B\pm}(w)$  be a signed unimodal factorization such that  $K'_B(\mathbf{A}') = (P, T')$  for  $T' = \tilde{f}_1^{\pm P} T$ .

(f-1):  $\mathbf{a}_1 \mathbf{a}_2$  is not unimodal. In this case,  $\mathbf{a}_1 \neq \emptyset$  and  $\mathbf{a}_2 \neq \emptyset$ . Here, the inequality  $x_r^{(1)} < x_1^{(2)}$  holds. Otherwise,

$$x_1^{(1)} = \cdots = x_r^{(1)} = x_1^{(2)} = \cdots = x_s^{(2)} = 1$$

and

$$y_1^{(1)} < \cdots < y_r^{(1)} < y_1^{(2)} \cdots < y_s^{(2)}$$

so that  $\mathbf{a}_1 \mathbf{a}_2$  would be unimodal, which contradicts the assumption of (f-1). Since  $x_r^{(1)} < x_1^{(2)}$ , the configuration of  $T$  is

$$\begin{array}{|c|c|c|} \hline x & 1 & \cdots \\ \hline & y & \\ \hline \end{array} \quad \begin{array}{c} \cdots \\ \vdots \end{array}$$

where  $x$  is either  $1'$  or  $1$  and  $y$  is either  $2'$  or  $2$ . If the entry  $z$  ( $z$  is either  $2'$  or  $2$ ) appears in the first row of  $T$ , then such an entry appears after  $y$  in the course of the type  $B$  primed Kraškievich insertion of  $\mathbf{a}_2$ . The position is  $(1, i)$  with  $i \geq 3$  so that  $k^{(2)} = 1$ . Hence, if the entry  $2'$  or  $2$  appears in the first row of  $T$ , it must be  $2$  (not  $2'$ ). Let  $(1, i)$  be the position of the rightmost  $1$  in the first row of  $T$  ( $i \geq 2$ ), i.e.,  $(x_r^{(1)}, y_r^{(1)}) = (1, i)$  and  $T_{1,i+1} = 2$  ( $i = r$ ).

$$\begin{array}{|c|c|c|c|c|} \hline x & 1 & \cdots & \overset{i}{1} & \cdots \\ \hline \end{array}.$$

By Lemma 3.2 with Eq. (4.3), the configuration of the first row of  $T'$  is

$$\begin{array}{|c|c|c|c|c|} \hline x & 1 & \cdots & \overset{i}{2'} & \cdots \\ \hline \end{array}.$$

Since entries  $1'$  or  $1$  in the first row of  $T$  except at  $(1, i)$  do not change,  $s'_1 = s_1$ . If  $k^{(2)'} = 1$ , then  $(1 =)x_1^{(2)'} = \dots = x_s^{(2)'}$  and  $(y_r^{(1)} =)y_1^{(2)'} \dots < y_s^{(2)'}$ , which implies that  $\mathbf{a}_1 \mathbf{a}_2 = \mathbf{a}'_1 \mathbf{a}'_2$  is unimodal, contradicting the assumption of (f-1). Hence,  $k^{(2)'} \neq 1$ . Let  $k = k^{(2)'} \geq 2$ . Since the entries of  $T$  and those of  $T'$  at the same positions except at  $(x_r^{(1)}, y_r^{(1)})$  are unchanged, the entry at the position  $(x_k^{(2)'}, y_k^{(2)'}) = (x_{k-1}^{(2)}, y_{k-1}^{(2)})$  of  $T$  is the same as that at the position  $(x_k^{(2)'}, y_k^{(2)'})$  of  $T'$  so that  $s'_2 = s_2$ .

(f-2):  $\mathbf{a}_1 \mathbf{a}_2$  is unimodal. Since

$$x_1^{(1)} = \dots = x_r^{(1)} = x_1^{(2)} = \dots = x_s^{(2)} = 1$$

and

$$y_1^{(1)} < \dots < y_r^{(1)} < y_1^{(2)} \dots < y_s^{(2)},$$

we have  $k^{(2)} = 1$  and  $k^{(2)'} = 1$ . If  $\mathbf{a}_2 \neq \emptyset$  and  $s_2 = -$ , then the configuration of the first row of  $T$  is

$$\boxed{x \mid 1 \mid \dots \mid 1 \mid 2' \mid \dots},$$

where  $i \geq 1$  and  $x = 1'$  or  $1$ . Hence,  $\tilde{f}_1^{\pm P} T = \mathbf{0}$  so that  $\tilde{f}_1^F \mathbf{A} = \mathbf{0}$ . In the following, we assume that  $\mathbf{a}_2 = \emptyset$  or  $s_2 \neq -$ . When  $|\mathbf{a}_1| = 1$ ,  $T_{1,1} = 1'$  or  $1$ . If  $T_{1,1} = 1'$ , then  $s_1 = -$  and  $T'_{1,1} = 2'$ , i.e.,  $s'_2 = -$ . On the other hand, if  $T_{1,1} = 1$ , then  $s_1 = +$  and  $T'_{1,1} = 2$ , i.e.,  $s'_2 = +$ . In both cases,  $s'_2 = s_1$ . When  $|\mathbf{a}_1| \geq 2$ , it is clear that  $s'_1 = s_1$ . The configuration of the first row of  $T$  is

$$\boxed{x \mid 1 \mid \dots \mid 1 \mid \dots},$$

where  $i \geq 2$ . Under the action of  $\tilde{f}_1^{\pm P}$ ,  $T_{1,i} = 1$  is changed to  $2'$ . Since  $k^{(2)'} = 1$ , we have that  $s'_2 = -$ .

The verification of the rule of  $\tilde{f}_1^F$  is now complete.

By Theorem 4.2 and by the construction of type  $B$  primed Kraśkiewicz insertion, each connected component of  $U_m^{B\pm}(w)$  consists of the signed unimodal factorizations of type  $B$  Coxeter-Knuth related reduced words.

□

For example, we have  $R_B(32\bar{1}) = \{0121, 0212, 2012\}$ . Since  $0121 \stackrel{B}{\sim} 0212$  by (2) of Definition 4.1 with  $a = 0$  and  $b = 1$ , the signed unimodal factorizations (with  $m$  factors) of reduced words  $0121$  and  $0212$  form a connected

$\mathfrak{q}(m)$ -crystal for a fixed prime type. Figure 4.1 shows an example of such a  $\mathfrak{q}(3)$ -crystal structure. The insertion tableau  $P \in \text{SDT}_w^B(\lambda)$  in  $K'_B$  (Eq. (4.1)) is

$$\begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline & 0 & \\ \hline \end{array},$$

i.e.,  $\lambda = (3, 1)$ . The crystal structure of  $\text{PT}_3^\pm(\lambda)$  corresponding to Fig. 4.1 is the same as in Fig. 3.1, i.e., the prime type is  $\emptyset$ . Another choice of the prime type gives another  $\mathfrak{q}(3)$ -crystal structure of the signed unimodal factorizations of 0121 and 0212. The unimodal factorizations of the reduced word 2012 also form a  $\mathfrak{q}(m)$ -crystal. Figure 4.2 shows an example of such a  $\mathfrak{q}(3)$ -crystal structure (only the higher weight part). The insertion tableau  $P \in \text{SDT}_w^B(\lambda)$  in  $K'_B$  (Eq. (4.1)) is

$$\begin{array}{|c|c|c|c|} \hline 2 & 0 & 1 & 2 \\ \hline \end{array},$$

i.e.,  $\lambda = (4)$  and the prime type is  $\emptyset$ .

**Example 4.7** *In the crystal graph of Fig. 4.1, there are 12  $\tilde{f}_1^F$  edges listed below:*

- (1)  $(+02)(+12)(\ ) \xrightarrow{\bar{1}} (+0)(+212)(\ )$
- (2)  $(+02)(+1)(+2) \xrightarrow{\bar{1}} (+0)(+21)(+2)$
- (3)  $(+02)(\ )(+12) \xrightarrow{\bar{1}} (+0)(-2)(+12)$
- (4)  $(+0)(+2)(+12) \xrightarrow{\bar{1}} (\ )(+02)(+12)$
- (5)  $(+0)(\ )(+212) \xrightarrow{\bar{1}} (\ )(+0)(+212)$
- (6)  $(+012)(+1)(\ ) \xrightarrow{\bar{1}} (+01)(+21)(\ )$
- (7)  $(+012)(\ )(+1) \xrightarrow{\bar{1}} (+01)(-2)(+1)$
- (8)  $(+01)(+2)(+1) \xrightarrow{\bar{1}} (+0)(-12)(+1)$
- (9)  $(+0)(+12)(+1) \xrightarrow{\bar{1}} (\ )(+012)(+1)$

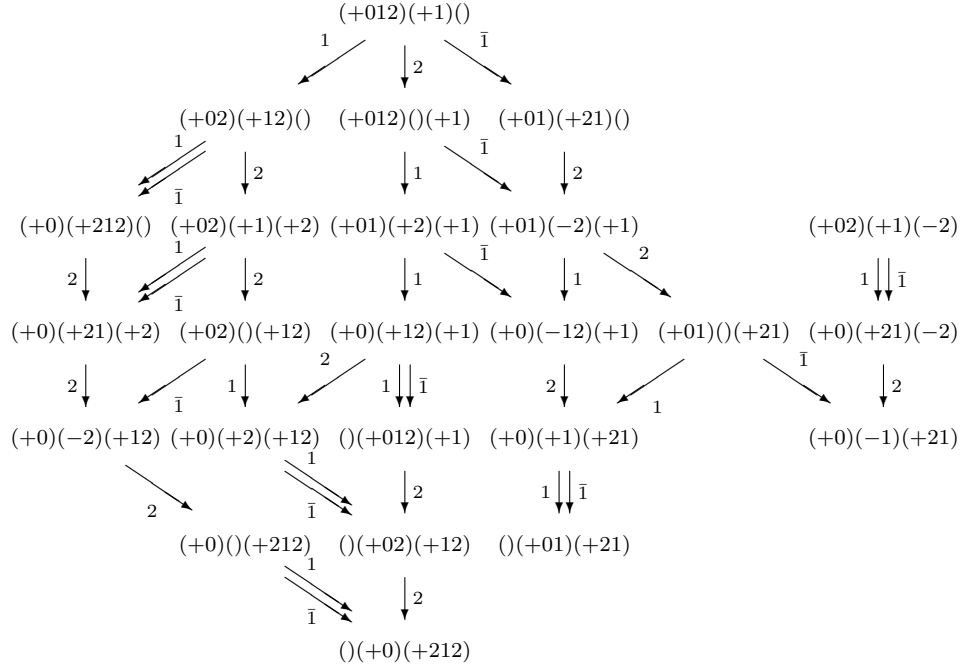


Figure 4.1: An example of  $\mathfrak{q}(3)$ -crystal structure of signed unimodal factorizations of the type  $B$  Coxeter-Knuth related reduced words 0121 and 0212.

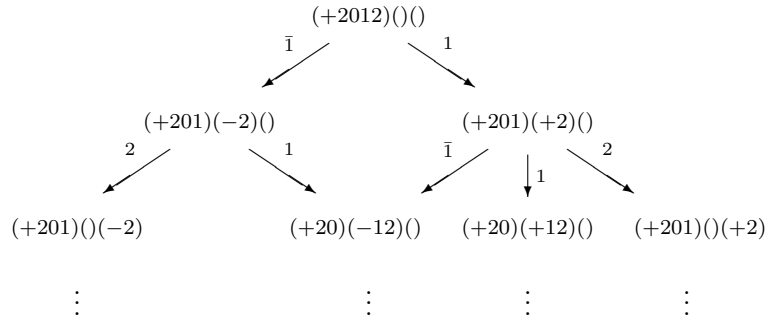


Figure 4.2: An example of  $\mathfrak{q}(3)$ -crystal structure of signed unimodal factorizations of the reduced word 2012. The total number of vertices is 33.

$$(10) \ (+0)(+1)(+21) \xrightarrow{\bar{1}} ()(+01)(+21)$$

$$(11) \ (+01)()(+21) \xrightarrow{\bar{1}} (+0)(-1)(+21)$$

$$(12) \ (+02)(+1)(-2) \xrightarrow{\bar{1}} (+0)(+21)(-2)$$

*Among them, (1), (2), (6), and (12) follow the rule of Case (f-1) of Theorem 4.4, (4), (5), (9), and (10) correspond to Case (f-2) with  $|\mathbf{a}_1| = 1$ , and (3), (7), (8), and (11) correspond to Case (f-2) with  $|\mathbf{a}_1| \geq 2$ .*

# Chapter 5

## $\mathfrak{q}$ -crystal structure on signed unimodal factorizations of flattened words of type $D$

In this chapter, we give a  $\mathfrak{q}(m)$ -crystal structure on signed unimodal factorizations with  $m$  factors of flattened words of type  $D$ . Flattened words of type  $D$ , which were introduced by Billey and Haiman [3], play the part of reduced words of type  $B$  in the type  $D$  setting. The arguments and results of this chapter are parallel to those of Chapter 4.

### 5.1 The type $D$ Kraśkiewicz insertion

Let  $W_n^D$  be the Coxeter group of type  $D_n$ . Its elements can be represented as signed permutations with even number of signs. The element  $w \in W_n^D$  is generated by simple reflections,

$$\begin{aligned} s_0 &= \bar{2}\bar{1}3 \cdots n, \\ s_i &= 12 \cdots i - 1 \ i + 1 \ i \ i + 2 \cdots n \quad (i = 1, 2, \dots, n-1), \end{aligned}$$

where we use the one line notation for signed permutations [17]. The definition of reduced words in type  $D$  is the same as that in type  $B$ . The set of all reduced words of the element  $w \in W_n^D$  is denoted by  $R_D(w)$ . Now, let us define flattened words.

**Definition 5.1** *Let  $\mathbf{a} = a_1 a_2 \cdots a_l$  be a reduced word in  $D_n$ . Convert all*

the 0's that appear in  $\mathbf{a}$  to 1's and denote this new word by  $\check{\mathbf{a}}$ . We call  $\check{\mathbf{a}}$  a flattened word. We denote the set of flattened words of  $w \in W_n^D$  by  $\check{R}_D(w)$ .

The elements of  $\check{R}_D(w)$  are classified by the type  $D$  Coxeter-Knuth relations [23].

**Definition 5.2** Let  $\mathbf{a}, \mathbf{b} \in \check{R}_D(w)$  for  $w \in W_n^D$ . Suppose that  $\mathbf{a}$  and  $\mathbf{b}$  are written, respectively, in the forms  $\mathbf{a} = \mathbf{cxd}$  and  $\mathbf{b} = \mathbf{cyd}$ , where  $\mathbf{x}$  and  $\mathbf{y}$  are the subwords of four adjacent letters of  $\mathbf{a}$  and  $\mathbf{b}$ , respectively. If  $\mathbf{x} \sim \mathbf{y}$  or  $\mathbf{y} \sim \mathbf{x}$  is one of the relations listed below, we say that  $\mathbf{a}$  is elementary type  $D$  Coxeter-Knuth related to  $\mathbf{b}$ :

- (1)  $1121 \sim 1212$
- (2)  $ab(b+1)b \sim a(b+1)b(b+1)$  with  $a < b$
- (3)  $ba(b+1)b \sim b(b+1)ab$  with  $a < b$
- (4)  $a(a+1)ba \sim a(a+1)ab$  with  $a+1 < b$
- (5)  $(a+1)ab(a+1) \sim (a+1)ba(a+1)$  with  $a+1 < b$
- (6)  $abdc \sim adbc$
- (7)  $acdb \sim acbd$
- (8)  $adcb \sim dacb$
- (9)  $badc \sim bdac$

and their reverses, where  $a < b < c < d$  or  $1 = a = b < c < d$  unless otherwise stated.

Let  $\mathbf{a}, \mathbf{b} \in \check{R}_D(w)$ . If there exists a sequence of flattened words,

$$\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k = \mathbf{b}$$

such that  $\mathbf{a}_i$  is elementary type  $D$  Coxeter-Knuth related to  $\mathbf{a}_{i+1}$ , we say that  $\mathbf{a}$  and  $\mathbf{b}$  are type  $D$  Coxeter-Knuth related and write  $\mathbf{a} \stackrel{D}{\sim} \mathbf{b}$ .

**Example 5.1** We have  $\check{R}_D = \{12113, 21213, 21231, 12131, 12311\}$ . Since  $1211 \sim 2121$  by the reverse of (1) of Definition 5.2 and  $1213 \sim 1231$  by (4) of Definition 5.2 with  $a = 1$  and  $b = 3$ ,  $12113 \stackrel{D}{\sim} 21213 \stackrel{D}{\sim} 21231$  and  $12131 \stackrel{D}{\sim} 12311$ . However,  $21231 \not\stackrel{D}{\sim} 12131$ .

**Definition 5.3** A flattened word  $\mathbf{c} = c_1 c_2 \cdots c_l$  is called *unimodal* if there exists  $k$  such that

- (1)  $c_1 > c_2 > \cdots > c_{k-1}$
- (2)  $c_k < c_{k+1} < \cdots < c_l$
- (3)  $c_{k-1} > c_k$  or  $c_{k-1} = c_k = 1$

We call  $\mathbf{c}_\downarrow = c_1 c_2 \cdots c_k$  the *decreasing part* of  $\mathbf{c}$  and  $\mathbf{c}_\uparrow = c_{k+1} c_{k+2} \cdots c_l$  the *increasing part* of  $\mathbf{c}$ .

**Definition 5.4** Let  $P$  be a shifted tableau of shape  $\lambda$ . If

- (1)  $P_{l(\lambda)} P_{l(\lambda)-1} \cdots P_1$  is a flattened word of  $w \in W_n^D$  and
- (2)  $P_i$  is a unimodal word of maximum length in  $P_{i+1} P_i$  for  $i = 1, \dots, l(\lambda) - 1$ ,

then  $P$  is called a *type D standard decomposition tableau* of  $w$ . We denote the set of such tableaux by  $\text{SDT}_w^D(\lambda)$ .

Let  $\mathbf{a} = a_1 a_2 \cdots a_m \in \check{R}(w)$  for  $w \in W_n^D$ . We recursively construct a sequence of pairs of tableaux,

$$(\emptyset, \emptyset) = (P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \dots, (P^{(m)}, Q^{(m)}) = (P, Q).$$

The algorithm to obtain  $(P^{(i)}, Q^{(i)})$  from  $(P^{(i-1)}, Q^{(i-1)})$  is as follows:

- (1) Let  $a = a_i$  and  $R$  be the first row of  $P^{(i-1)}$ .
- (2) Insert  $a$  into  $R$  as follows:
  - (i) If  $Ra$  is unimodal ( $R$  is possibly empty), append  $\boxed{a}$  to  $R$  and let  $P^{(i)}$  be this new tableau. To get  $Q^{(i)}$ , we add  $\boxed{i}$  to  $Q^{(i-1)}$  such that  $P^{(i)}$  and  $Q^{(i)}$  have the same shifted shape. Stop.
  - (ii) If  $Ra$  is not unimodal, let  $b$  be the smallest element in  $R_\uparrow$ , the increasing part of  $R$ , larger than or equal to  $a$ .
    - (ii-1):  $a = 1$  and  $R$  contains 2112 as a subsequence. Leave  $R$  unchanged and go to (2) with  $a = 1$  and  $R$  being the next row.

- (ii-2):  $a = 1$  and  $R$  contains 212 as a subsequence. Change the last 2 into a 1 and go to (2) with  $a = 1$  and  $R$  being the next row.
- (ii-3):  $a = 1$  and  $R$  contains 112 but not 2112 as a subsequence. Change the first 1 into a 2 and go to (2) with  $a = 1$  and  $R$  being the next row.
- (ii-4):  $b \neq a$ . Put  $a$  in  $b$ 's position and let  $c = b$ .
- (ii-5):  $b = a$ . Leave  $R_{\uparrow}$  unchanged and let  $c = a + 1$ .
- We insert  $c$  into  $R_{\downarrow}$ , the decreasing part of  $R$ . Let  $d$  be the largest element in  $R_{\downarrow}$  which is smaller than or equal to  $c$ . If this number is 1 and there are two 1's in  $R_{\downarrow}$ , let  $d$  be the left 1.
- (ii-6):  $d \neq c$ . Put  $c$  in  $d$ 's place and let  $a' = d$ .
- (ii-7):  $d = c$ . Leave  $R_{\downarrow}$  unchanged and let  $a' = c - 1$ .

(3) Repeat (2) with  $a = a'$  and  $R$  being the next row.

The tableau  $P$  (resp.  $Q$ ) is called the insertion (resp. recording) tableau. The algorithm above is called the Kraśkiewicz insertion of type  $D$  [23].

**Example 5.2** For a flattened word  $112132 \in \check{R}_D(34\bar{1}\bar{2})$ , we have

$$P = \begin{array}{|c|c|c|c|} \hline 3 & 1 & 2 & 3 \\ \hline 1 & 2 & & \\ \hline \end{array} \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & & \\ \hline \end{array}.$$

**Theorem 5.1** ([23]) *The Kraśkiewicz insertion of type  $D$  gives a bijection between  $\check{R}_D(w)$  and pairs of tableaux  $(P, Q)$  where  $P \in \text{SDT}_w^D(\lambda)$  and  $Q \in \text{ST}_n(\lambda)$  for  $w \in W_n^D$ .*

$$K_D : \check{R}(w) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathcal{P}^+} [\text{SDT}_w^D(\lambda) \times \text{ST}_n(\lambda)].$$

**Theorem 5.2** *Two flattened words in  $\check{R}_D(w)$  are type  $D$  Coxeter-Knuth related if they have the same insertion tableau.*

*Proof.* The assertion is the immediate consequence of Theorem 4.12 of [23].  $\square$

**Example 5.3** We continue Example 5.1. Flattened words 12131 and 12311 in  $\check{R}_D(\bar{2}\bar{3}41)$  have the insertion tableau

$$\begin{array}{|c|c|c|} \hline 3 & 1 & 1 \\ \hline 1 & 2 & \\ \hline \end{array},$$

whereas flattened words 12113, 21213, and 21231 have the insertion tableau

$$\begin{array}{|c|c|c|c|} \hline 2 & 1 & 1 & 3 \\ \hline 1 & & & \\ \hline \end{array}.$$

## 5.2 q-crystal structure on signed unimodal factorizations of flattened words of type $D$

For a flattened word  $\mathbf{c} \in \check{R}_D(w)$ , a *signed unimodal factorization* of  $\mathbf{c}$  is defined similarly as in Section 4.2. We denote by  $U_m^{D\pm}(w)$  the set of all signed unimodal factorizations with  $m$  factors of flattened words of  $w \in W_n^D$ .

**Lemma 5.1** ([23]) *Let  $\mathbf{c} = c_1 c_2 \dots c_m \in \check{R}_D(w)$  and  $K_D(\mathbf{c}) = (P, Q)$ . The subword  $c_i c_{i+1} \dots c_j$  is unimodal if and only if the entries  $i, i+1, \dots, j$  form a vee in  $Q$ .*

Now, the algorithm of the primed Kraśkiewicz insertion of type  $B$  ( $K'_B$ ) can be translated into that of type  $D$  ( $K'_D$ ) by changing the type  $B$  Kraśkiewicz insertion to the type  $D$  one, where the role of Lemma 4.1 is replaced by that of Lemma 5.1. For  $\mathbf{C} \in U_m^{D\pm}(w)$ , we write  $K'_D(\mathbf{C}) = (P, T)$ , where  $P \in \text{SDT}_w^D(\lambda)$  and  $T \in \text{PT}_m^\pm(\lambda)$  for  $w \in W_n^D$  and some  $\lambda \in \mathcal{P}^+$ .

**Example 5.4** *For a signed unimodal factorization*

$$\mathbf{C} = (+112)(-1)(+32) \in U_3^{D\pm}(34\bar{1}\bar{2})$$

of the flattened word in Example 5.2, we have

$$P = \begin{array}{|c|c|c|c|} \hline 3 & 1 & 2 & 3 \\ \hline 1 & 2 & & \\ \hline \end{array} \quad T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3' \\ \hline 2' & 3 & & \\ \hline \end{array}.$$

**Theorem 5.3** *The primed Kraśkiewicz insertion of type  $D$  gives a bijection between  $U_m^{D\pm}(w)$  and pairs of tableaux  $(P, T)$  where  $P \in \text{SDT}_w^D(\lambda)$  and  $T \in \text{PT}_m^\pm(\lambda)$  for  $w \in W_n^D$ .*

$$K'_D : U_m^{D\pm}(w) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathcal{P}^+} [\text{SDT}_w^D(\lambda) \times \text{PT}_m^\pm(\lambda)]. \quad (5.1)$$

*Proof.* The proof is the same as that of Theorem 3.7 in [14].  $\square$

**Theorem 5.4** *The set of all signed unimodal factorizations  $U_m^{D\pm}(w)$  with  $m$  factors of flattened words of  $w \in W_n^D$  admits a  $\mathfrak{q}(m)$ -crystal structure. Each connected component of  $U_m^{D\pm}(w)$  consists of the signed unimodal factorizations of type  $D$  Coxeter-Knuth related flattened words.*

*The even Kashiwara operators on  $U_m^{D\pm}(w)$  are given by*

$$\begin{aligned}\tilde{e}_i^F &= K_D'^{-1} \circ (\text{Id}, \tilde{e}_i^{\pm P}) \circ K_D', \\ \tilde{f}_i^F &= K_D'^{-1} \circ (\text{Id}, \tilde{f}_i^{\pm P}) \circ K_D',\end{aligned}$$

*( $i = 1, \dots, m-1$ ) and the action of odd Kashiwara operators are given by the rules described in Theorem 4.4.*

*Proof.* The proof is the same as that of Theorem 4.4. Theorem 5.2 (Lemma 5.1) plays the role of Theorem 4.2 (Lemma 4.1) in the proof of Theorem 4.4. By the construction of type  $D$  primed Kraśkiewicz insertion, it is obvious that the actions of odd Kashiwara operators  $\tilde{e}_1^F$  and  $\tilde{f}_1^F$  on  $\mathbf{C} \in U_m^{D\pm}(w)$  are exactly the same as those on  $\mathbf{A} \in U_m^{C\pm}(w)$ .  $\square$

For example, we have  $\check{R}(4\bar{2}\bar{1}3) = \{1321, 3121\}$ . Since  $1321 \stackrel{D}{\sim} 3121$  by (8) of Definition 5.2 with  $a = b = 1$ ,  $c = 2$ , and  $d = 3$ , the signed unimodal factorizations (with  $m$  factors) of flattened words 1321 and 3121 form a connected  $\mathfrak{q}(m)$ -crystal for a fixed prime type. Figure 5.1 shows an example of such a  $\mathfrak{q}(3)$ -crystal structure. The insertion tableau  $P \in \text{SDT}_w^D(\lambda)$  in  $K_D'$  (Eq. (5.1)) is

$$\begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline & 1 & \\ \hline \end{array},$$

i.e.,  $\lambda = (3, 1)$ . The crystal structure of  $\text{PT}_3^\pm(\lambda)$  corresponding to Fig. 5.1 is the same as Fig. 3.1, i.e., the prime type is  $\emptyset$ .

**Example 5.5** *In the crystal graph of Fig. 5.1, there are 12  $\tilde{f}_1^F$  edges listed below:*

$$\begin{aligned}(1) \quad & (+13)(+21)(\ ) \xrightarrow{\bar{1}} (+1)(+321)(\ ) \\ (2) \quad & (+13)(+2)(+1) \xrightarrow{\bar{1}} (+1)(+32)(+1)\end{aligned}$$

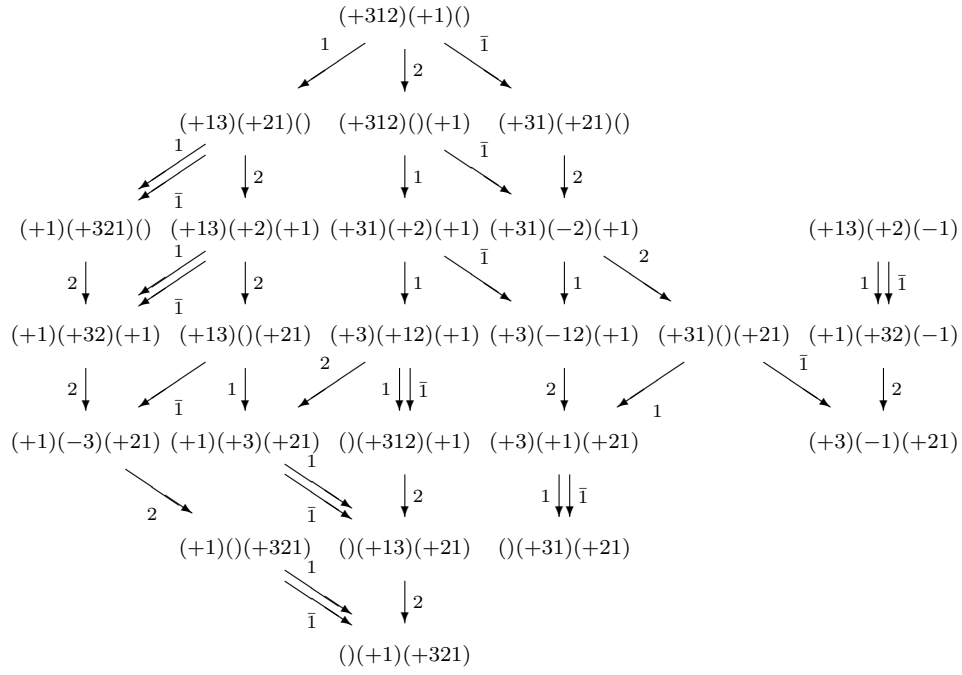


Figure 5.1: An example of  $\mathfrak{q}(3)$ -crystal structure of signed unimodal factorizations of the type  $D$  Coxeter-Knuth related flattened words 1321 and 3121.

$$(3) \ (+13)()(+21) \xrightarrow{\bar{1}} (+1)(-3)(+21)$$

$$(4) \ (+1)(+3)(+21) \xrightarrow{\bar{1}} ()(+13)(+21)$$

$$(5) \ (+1)()(+321) \xrightarrow{\bar{1}} ()(+1)(+321)$$

$$(6) \ (+312)(+1)() \xrightarrow{\bar{1}} (+31)(+21)()$$

$$(7) \ (+312)()(+1) \xrightarrow{\bar{1}} (+31)(-2)(+1)$$

$$(8) \ (+31)(+2)(+1) \xrightarrow{\bar{1}} (+3)(-12)(+1)$$

$$(9) \ (+3)(+12)(+1) \xrightarrow{\bar{1}} ()(+312)(+1)$$

$$(10) \ (+3)(+1)(+21) \xrightarrow{\bar{1}} ()(+31)(+21)$$

$$(11) \ (+31)()(+21) \xrightarrow{\bar{1}} (+3)(-1)(+21)$$

$$(12) \ (+13)(+2)(-1) \xrightarrow{\bar{1}} (+1)(+32)(-1)$$

Among them, (1), (2), (6), and (12) follow the rule of Case (f-1) of Theorem 4.4, (4), (5), (9), and (10) correspond to Case (f-2) with  $|\mathbf{a}_1| = 1$ , and (3), (7), (8), and (11) correspond to Case (f-2) with  $|\mathbf{a}_1| \geq 2$ .

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