On the stable Auslander-Reiten quiver for a symmetric order over a complete discrete valuation ring

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CONTENTS

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1. INTRODUCTION

The notion of almost split sequences was introduced by M. Auslander and I. Reiten in [AR1], and they showed the existence of almost split sequences for Artin algebras. We often use the theory to analyze various additive categories arising from representation theory and prove many important combinatorial and homological properties with the help of the theory, for example see [A2, A3, A4, ARS, ASS, Bu, E, H, Hap, I, I2, I3, I4, I5, I6, I7, IJ, K4, Li2, Y]. Moreover, the theory gives a great impact in other areas such as algebraic geometry and algebraic topology [A5, J]. A combinatorial skeleton of the additive category of indecomposable objects is the Auslander–Reiten quiver, which encapsulates much information on indecomposable objects and irreducible morphisms. Therefore, to determine the shapes of Auslander–Reiten quivers is one of classical problems in representation theory of algebras.

Let $\mathcal{O}$ be a complete discrete valuation ring or a field. An $\mathcal{O}$-algebra A is an $\mathcal{O}$*-order* if A is free of finite rank as an $\mathcal{O}$-module. We put $\mathsf{D} := \mathsf{Hom}_{\mathcal{O}}(-, \mathcal{O})$. Throughout this thesis, modules mean right modules. An $\mathcal{O}$-order A is Gorenstein if $\mathsf{D}(A)$ is a projective A-module. For an $\mathcal{O}$-order A, an A-module M is called an A*-lattice* if M is Cohen–Macaulay as an $\mathcal{O}$-module. We denote by $\mathsf{latt}\text{-}A$ the full subcategory of the module category $\mathsf{mod}\text{-}A$ consisting of A-lattices, where $\mathsf{mod}\text{-}A$ is the category of finitely generated A-modules. Note that, when $\mathcal{O}$ is a field, an $\mathcal{O}$-order is just a finite dimensional $\mathcal{O}$-algebra, and $\mathsf{latt}\text{-}A$ is the module category $\mathsf{mod}\text{-}A$.

Now, let A be an $\mathcal{O}$-order. According to [A3], the category $\mathsf{latt}\text{-}A$ admits almost split sequences if and only if A is an isolated singularity, that is, $\mathsf{gl.dim}(A \otimes_{\mathcal{O}} \mathcal{O}_{\mathsf{p}}) = \mathsf{Kr\text{-}dim}(\mathcal{O}_{\mathsf{p}})$ for all non-maximal $\mathsf{p} \in \mathsf{Spec}(\mathcal{O})$, where $\mathsf{Spec}(\mathcal{O})$ is the spectrum of $\mathcal{O}$. If $\mathcal{O}$ is a field, then an A is always an isolated singularity, and if $\mathcal{O}$ is a complete discrete valuation ring, then A is an isolated singularity if and only if $A \otimes_{\mathcal{O}} \mathcal{K}$ is a semi-simple $\mathcal{K}$-algebra, where $\mathcal{K}$ is the quotient field of $\mathcal{O}$. For an $\mathcal{O}$-order, which is an isolated singularity, one can find some results on the shapes of Auslander–Reiten quivers, for example [ASS, Di1, Di2, Di3, IK, K2, K4, Lu, Ro2, Roy, We, Y].

When A is not an isolated singularity, we have to consider a suitable full subcategory of $\mathsf{latt}\text{-}A$ which admits almost split sequences. It follows from [AR3, Theorem 2.1] that $M \in \mathsf{latt}\text{-}A$ appears at the end term of an almost split sequence if and only if M satisfies the condition ($\natural$):

$$M \otimes_{\mathcal{O}} \mathcal{K} \text{ is projective as an } A \otimes_{\mathcal{O}} \mathcal{K}\text{-module.} \qquad (\natural)$$

Here, the full subcategory of latt-A consisting of A-lattices which satisfy the condition ($\natural$) is denoted by $\mathsf{latt}^{(\natural)}$-$A$. If A is Gorenstein, then the full subcategory $\mathsf{latt}^{(\natural)}$-$A$ admits almost split sequences.

From now on, we consider a Gorenstein $\mathcal{O}$-order A. By the above observation, we can introduce the notion of the stable Auslander–Reiten quiver for $\mathsf{latt}^{(\natural)}$-$A$, which is given by deleting all indecomposable projective-injective modules from the original Auslander–Reiten quiver. Such quivers are stable translation quivers.

Assume that $\mathcal{O}$ is a field. Then any component of the stable Auslander–Reiten quiver, a stable component for short, for $\mathsf{latt}^{(\natural)}$-$A$ ($=$ mod-A) has no loops. Therefore, for a stable component $\mathcal{C}$, there is a directed tree T and a subgroup $G \subset \mathsf{Aut}(\mathbb{Z}T)$ such that $\mathcal{C} \simeq \mathbb{Z}T/G$ as stable translation quivers by the Riedtmann structure theorem [Ri]. Then, the underlying graph of T is called the *tree class* of $\mathcal{C}$. Therefore, in order to determine the shape of $\mathcal{C}$, it is enough to determine the tree class $\overline{T}$ and the group G. Around 1982, P. J. Webb approached this problem when $A = \mathcal{O}G$, where G is a finite group. For a stable component of $\mathcal{O}G$-modules, he constructed a subadditive function (see [HPR]) in order to give candidates for the tree class of the component, and showed that the tree class is either a Dynkin diagram or a Euclidean diagram [We]. His method is very effective to determine the shapes of stable components for a self-injective algebra. Indeed, C. Riedtmann and G. Todorov showed that the tree class of any stable component of a finite dimensional self-injective algebra of finite representation type is one of finite Dynkin diagrams by using Webb's method [Ri2, T].

On the other hand, when $\mathcal{O}$ is a complete discrete valuation ring, stable components admit loops [Wi]. Thus, we have to check that a stable component does not have loops before we apply Webb's method to a component. Moreover, A is of infinite representation type in most cases, and it is difficult to compute almost split sequences. For these reasons, the shapes of stable components of A seem to be largely unknown, and there are only few concrete examples of stable components.

Therefore, the aims of this thesis are the following.

Aims. (1) Give restrictions on the shapes of stable components of a symmetric $\mathcal{O}$-order A (i.e. $A \simeq \mathsf{D}(A)$ as (A, A)-bimodules) when $\mathcal{O}$ is a complete discrete valuation ring.

(2) Give new examples of stable components when A is not an isolated singularity.

Let $\mathcal{O}$ be a complete discrete valuation ring, A a symmetric $\mathcal{O}$-order, $\mathcal{C}$ a stable component of A and τ the AR translation of A. If $\mathcal{C}$ has only finitely many vertices, then A is an isolated singularity (Corollary 3.6.7), and a restriction on the shape of $\mathcal{C}$ had already given by X. Luo [Lu]. Thus, we assume that $\mathcal{C}$ has infinitely many vertices. In order to get candidates for the shape of $\mathcal{C}$, we have to answer the following natural questions.

Questions. Assume that $\mathcal{C}$ has infinitely many vertices.

(1) If loops exist in $\mathcal{C}$, where do loops appear in $\mathcal{C}$?

(2) When does $\mathcal{C}$ admit loops?

(3) If $\mathcal{C}$ has no loops, can we construct a subadditive function on the tree class of $\mathcal{C}$?

Let $\mathcal{C}$ be a stable component having infinitely many vertices. For the question (1), we give a complete answer in our setting. If $\mathcal{C}$ has loops, then $\mathcal{C}$ is a τ-periodic component with the period 1, and the loops appear only on the boundary of $\mathcal{C}$ (Proposition 3.7.5). In particular, the answer of (1) leads to a partial answer to (2), namely $\mathcal{C}$ has no loops when one of the following conditions holds:

(i) $\mathcal{C}$ has a period larger than 1.

(ii) There is a vertex $X \in \mathcal{C}$ such that the middle term of the almost split sequence ending at X, say E_X, has exactly one non-projective direct summand, and E_X does not have X as a direct summand.

(iii) $\mathcal{C}$ is not τ-periodic.

Assume that $\mathcal{C}$ has no loops. Let $\overline{T}$ be the tree class of the component $\mathcal{C}$. If $\mathcal{C}$ is τ-periodic, the tree class $\overline{T}$ always admits the subadditive function which is obtained by averaging the ranks in the same τ-orbit. Hence, the tree class is one of infinite Dynkin diagrams by [HPR].

Let $\mathcal{D} : \mathsf{latt}\text{-}A \to \mathbb{Z}$ be the function defined by

$$\mathcal{D}(X) := \sharp\{\text{non-projective indecomposable direct summands of } X \otimes_{\mathcal{O}} \kappa\},$$

where κ is the residue field of $\mathcal{O}$. As A is symmetric, the AR translation τ is the first syzygy functor (Corollary 3.5.10, [A2], [Hap] and [I5]). It yields that $\mathcal{D}$ is a τ-invariant function. If $A \otimes_{\mathcal{O}} \kappa$ is representation-finite and $\mathcal{C}$ is not τ-periodic, then $\mathcal{D}$ is additive on $\overline{T}$.

Summing up, we have the first main result of this thesis:

Main Theorem 1 (Proposition 3.7.5, Theorems 3.7.6 and 3.7.14)**.** Let $\mathcal{O}$ be a complete discrete valuation ring, κ the residue field and $\mathcal{C}$ a stable component of a symmetric $\mathcal{O}$-order A with infinitely many vertices.

(1) If $\mathcal{C}$ has loops, then $\mathcal{C}$ is τ-periodic. Furthermore, $\mathcal{C} \setminus \{\text{loops}\}$ is of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$. In this case, the loops appear only at the endpoint of $\mathcal{C}$:

$$\mathcal{C} = \begin{tikzcd} \bullet \arrow[loop left] \arrow[loop above, dotted, "\tau"] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop above, dotted, "\tau"] \arrow[r, shift left] & \cdots\cdots \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop above, dotted, "\tau"] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop above, dotted, "\tau"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \end{tikzcd}$$

(2) If $\mathcal{C}$ has no loops, and is τ-periodic, then the tree class of $\mathcal{C}$ is one of infinite Dynkin diagrams.

(3) If $A \otimes_{\mathcal{O}} \kappa$ is of finite representation type, then the tree class of $\mathcal{C}$ is one of infinite Dynkin diagrams or Euclidean diagrams.

We consider the question (3). In order to get new examples of stable components, we focus on a special kind of A-lattices called *Heller lattices*, which is defined to be direct summands of the first syzygy of an indecomposable $A \otimes_{\mathcal{O}} \kappa$-module as an A-module. There are two reasons why we consider Heller lattices. The first reason is that they always belong to $\mathsf{latt}^{(\natural)}$-$A$. Thus, the category $\mathsf{latt}^{(\natural)}$-$A$ admits some stable components containing indecomposable Heller lattices. We call such components *Heller components* of A. Another reason is that Heller lattices of a group algebra play important roles in modular representation theory. For a p-modular system $(\mathcal{K}, \mathcal{O}, \kappa)$ of a finite group G, Heller lattices over $\mathcal{O}G$ were studied by S. Kawata [K3, K4]. It follows from [K3, Theorem 4.4] that Heller lattices over $\mathcal{O}G$ provide us with certain relationship between almost split sequences for latt-$\mathcal{O}G$ and mod-κG, namely he showed that if $0 \to A \to B \to Z_M \to 0$ is the almost split sequence ending at an indecomposable Heller lattice Z_M of an indecomposable κG-module M, then the induced exact sequence

$$0 \to A \otimes_{\mathcal{O}} \kappa \to B \otimes_{\mathcal{O}} \kappa \to Z_M \otimes_{\mathcal{O}} \kappa \to 0$$

is the direct sum of the almost split sequence ending at M and a split sequence (see also [P, Corollary 5.8]). They motivate us to study Heller lattices when A is an arbitrary symmetric $\mathcal{O}$-order.

The second main result is on the shapes of stable components containing Heller lattices when $A = \mathcal{O}[X]/(X^n)$. Since $A \otimes_{\mathcal{O}} \kappa$ is of finite representation type, the tree class of any stable components is one of infinite Dynkin diagrams or Euclidean diagrams by the first main result.

Main Theorem 2 (Proposition 4.2.1, Theorem 4.4.1)**.** Let $\mathcal{O}$ be a complete discrete valuation ring, $A = \mathcal{O}[X]/(X^n)$, for $n \geq 2$. Then, any Heller component is of the form either $\mathbb{Z}A_\infty/\langle \tau^2 \rangle$ or $\mathbb{Z}A_\infty/\langle \tau \rangle$. Moreover, any Heller lattice appears on the boundary of a Heller component.

The last main result is on the shapes of stable components containing Heller lattices when $A = \mathcal{O}[X,Y]/(X^2,Y^2)$. Then $A \otimes_{\mathcal{O}} \kappa$ is of tame representation type.

Main Theorem 3 (Theorems 5.5.1, 5.8.4 and 5.9.5)**.** Let $\mathcal{O}$ be a complete discrete valuation ring and $A = \mathcal{O}[X,Y]/(X^2,Y^2)$. Assume that the residue field κ is algebraically closed. Then, there is a unique non-periodic Heller component $\mathbb{Z}A_\infty$ and infinitely many periodic Heller components whose tree classes are A_∞. Moreover, any Heller lattice appears on the boundary of a Heller component.

This thesis is based on the following three articles:

[AKM] S. Ariki, R. Kase and K. Miyamoto, On Components of stable Auslander–Reiten quivers that contain Heller lattices: the case of truncated polynomial rings, Nagoya math. J., **228** (2017), 72–113, DOI: 10.1017/nmj.2016.53.

[M1] K. Miyamoto, On the non-periodic stable Auslander–Reiten Heller component for the Kronecker algebra over a complete discrete valuation ring, to appear in Osaka J. Math.

[M2] K. Miyamoto, On periodic stable Auslander–Reiten components containing Heller lattices over the symmetric Kronecker algebra, arXiv: 1808.09289.

There are five chapters in this thesis, and the body begins in Chapter 2. The results of [AKM] appear in Chapter 3 and 4, those of [M1] appear in Chapter 3 and 5, those of [M2] appear in Chapter 5. We start with Chapter 2 presenting some fundamentals on orders and lattices, stable translation quivers and finite dimensional algebras over a field. In Chapter 3, we present Auslander–Reiten theory for Gorenstein orders over a complete discrete valuation ring. Moreover, we give a method to construct almost split sequences in Section 3.5, and we prove some properties of stable Auslander–Reiten components for a symmetric order in Section 3.6. These results in Section 3.5 and 3.6 establish a criterion, based on material in [AKM]. In Chapter 4, we determine the shapes of stable components containing Heller lattices over the truncated polynomial rings. The main result appears in [AKM]. In Chapter 5, we determine the shapes of stable components containing Heller lattices over the symmetric Kronecker algebra. The result in the non-periodic case appears in [M1], and the result in the periodic case appears in [M2].

2. PRELIMINARIES

In this thesis, we deal with an algebra over a complete discrete valuation ring. Thus, first, we recall some properties of a complete discrete valuation ring from commutative ring theory in Section 1. In representation theory of algebras, Krull–Schmidt–Azumaya theorem, K–S–A theorem for short, is a fundamental theorem. It is well-known that, for a category of finitely generated modules over a finite dimensional algebra over a field, K–S–A theorem holds, for example see [ASS, Chapter I, 4.10. Unique decomposition theorem]. However, this theorem is not always true when the base ring is an arbitrary commutative ring. Let R be a commutative ring and A an R-algebra. If the base ring R is a complete discrete valuation ring and A is finitely generated as an R-module, K–S–A theorem holds for the category of finitely generated A-modules since the theory of lifting idempotents works in this setting. In Section 2, we recall the theory of lifting idempotents from the standard text [CR]. In Section 3, we introduce orders and lattices. Assume that R is a complete discrete valuation ring. From results in Section 2, the category of A-lattices is a Krull–Schmidt category with enough projectives.

In Section 4, we list Dynkin diagrams and Euclidean diagrams. In Section 5, we introduce valued stable translation quivers. The structure of valued stable translation quivers without loops was studied by Riedtmann, which is well-known as Riedtmann's structure theorem [Ri]. By using this structure theorem, any stable translation quiver without loops $\mathcal{C}$ is of the form $\mathbb{Z}T/G$ for some a directed tree T and an "admissible group" G. Thus, in order to determine the shape of $\mathcal{C}$, it is enough to determine T and G. On the other hand, D. Happel, U. Preiser and C. M. Ringel gave a very nice result on determining T [HPR]. Therefore, we recall the Riedtmann structure theorem and Happel–Preiser–Rigel's results in this section.

In Chapter 4 of this thesis, we will consider the symmetric Kroneker algebra, which is symmetric and "special biserial ". Special biserial algebras over an algebraically closed field are always of tame representation type [WW], and the classification of all indecomposable modules of such an algebra was provided in [BR, WW]. Moreover, there is a combinatorial method of constructing indecomposable modules over such an algebra [Erd, HL]. In Section 6, we introduce symmetric special biserial algebras and explain how to construct indecomposable modules.

Throughout this thesis, we use the following notations. For an algebra Λ, we denote by $\mathsf{mod}\text{-}\Lambda$ the category consisting finitely generated Λ-modules. For $M, N \in \mathsf{mod}\text{-}\Lambda$, we write $\mathsf{Hom}_\Lambda(M, N)$ for the set of A-module homomorphisms from M to N. We also denote by

$\mathsf{proj}\text{-}\Lambda$ the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of projective Λ-modules.

2.1 Complete discrete valuation rings

Let $\mathcal{K}$ be a field. A surjective function $v : \mathcal{K} \setminus \{0\} \to \mathbb{Z}$ is a **discrete valuation** on $\mathcal{K}$ if it satisfies the following two properties for all $x, y \in \mathcal{K} \setminus \{0\}$:

(i) $v(xy) = v(x) + v(y)$

(ii) $v(x + y) \geq \min\{v(x), v(y)\}$

By (i), the discrete valuation v is a group homomorphism. The pair $(\mathcal{K}, v)$ is called a **valuation field**. It is convenient to extend v to the whole of $\mathcal{K}$ by putting $v(0) = \infty$. The set $\mathcal{O} := \{x \in \mathcal{K} \mid v(x) \geq 0\}$ is a local ring, which is called the **discrete valuation ring** of $(\mathcal{K}, v)$. It is easy to see that $\mathcal{K}$ is the quotient field of $\mathcal{O}$. Moreover, if we set $\mathcal{O}^{-1} := \{x^{-1} \mid x \in \mathcal{O} \setminus \{0\}\}$, we have $\mathcal{K} = \mathcal{O} \cup \mathcal{O}^{-1}$ and $\mathcal{O}^{\times} = \mathcal{O} \cap \mathcal{O}^{-1}$. The maximal ideal in $\mathcal{O}$ is given by $P := \{x \in \mathcal{O} \mid v(x) > 0\}$.

We give two typical examples. Let $\mathcal{K} = \mathbb{Q}$ and p a prime number. Then, the map $v_p : \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$ defined by $v_p(x) = a$, where $x = p^a y$ and y is an irreducible fraction whose numerator and denominator are not divisible by p, is a discrete valuation. Then, the valuation ring of v_p is the local ring $\mathbb{Z}_{(p)}$. Another example is that $\mathcal{K} = \mathbf{k}(x)$, where $\mathbf{k}$ is a field and x is an indeterminate. For an irreducible polynomial $f \in \mathbf{k}[x]$, we define v_f in the same manner as the first example. Then, v_f is a discrete valuation.

An integral domain $\mathcal{O}$ is called a **discrete valuation ring** if there is a discrete valuation v on its quotient field $\mathcal{K}$ such that $\mathcal{O}$ is the valuation ring of v.

From now on, $(\mathcal{K}, v)$ is a discrete valuation field, $\mathcal{O}$ is its discrete valuation ring. Let $\varepsilon \in \mathcal{K}$ such that $v(\varepsilon) = 1$. Then, for any $x \in \mathcal{K}$ with $v(x) = i$, the element $x\varepsilon^{-i}$ is invertible in $\mathcal{O}$ since $v(x\varepsilon^{-i}) = 0$. Thus, for any element $x \in \mathcal{K}$, there exist an integer i and an invertible element $u \in \mathcal{O}^{\times}$ such that $x = \varepsilon^i u$. This implies that any non-trivial ideal in $\mathcal{O}$ is of the form (ε^i) for some $i > 0$. In particular, $\mathcal{O}$ is a principal ideal domain.

Let R be a local principal ideal domain and p the maximal ideal. For any non-zero element $x \in R$, there is a non-negative integer i such that $x \in \mathsf{p}^i \setminus \mathsf{p}^{i+1}$ since the intersections of p^k $(k = 1, 2, ...)$ is zero. Thus, we may define $v : R \setminus \{0\} \to \mathbb{Z}$ by $v(x) = i$. Let K be the quotient field of R. For $0 \neq x/y \in K$, we also define $v(x/y) = v(x) - x(y)$. Then, it is easy to see that v is well-defined and (K, v) is a discrete valuation field, and the discrete valuation ring of (K, v) is just R. Therefore, we have the following lemma.

Lemma 2.1.1 ([Mat, Theorem 11.1])**.** Let R be a commutative ring. Then, R is a discrete valuation ring if and only if R is a principal ideal domain and local.

We return to the discrete valuation ring $\mathcal{O}$ and its maximal ideal P. Let M be an $\mathcal{O}$-module. Take a family of submodules $\mathcal{F} = \{P^n M\}_{n=0,1,2,...}$. Obviously, $P^i M \subset P^j M$ for $j < i$. Then, $\mathcal{F}$ makes M into a topological group because one can understand that $\mathcal{F}$

is a system of neighborhoods of 0. The topology is called the **P-adic topology** of M. In particular, $\mathcal{O}$ is a topological ring by the decreasing chain

$$\mathcal{O} \supset P \supset P^2 \supset P^3 \supset \cdots .$$

Then, $\mathcal{O}$ is a Hausdorff space since $\bigcap_{i=1}^{\infty} P^i = 0$. The factor modules M/P^iM is also a topological space by the quotient topology. Then, P^iM is open and closed in M. Indeed, the complement $M \setminus P^iM$ is a union of cosets $x + P^iM$ $(x \in M \setminus P^iM)$. Hence, the topology of M/P^iM is discrete. Consider the natural $\mathcal{O}$-module homomorphisms $f_{i,j} : M/P^jM \to M/P^iM$ for $i < j$. Then the pair $(\{f_{i,j}\}_{i,j}, \{M/P^iM\}_i)$ becomes an inverse system of $\mathcal{O}$-modules. We denote by $\hat{M}$ the inverse limit of $(\{f_{i,j}\}_{i,j}, \{M/P^iM\}_i)$, which is called the **completion** of M. It is easy to see that $\hat{\hat{\mathcal{O}}} \simeq \hat{\mathcal{O}}$. We say that $\mathcal{O}$ is **complete** if $\hat{\mathcal{O}} \simeq \mathcal{O}$ as rings. If M is a finitely generated $\mathcal{O}$-module, we have $M \otimes_{\mathcal{O}} \hat{\mathcal{O}} \simeq \hat{M}$.

Let $d_{\mathcal{K}} : \mathcal{K} \times \mathcal{K} \to \mathbb{R}_{\geq 0}$ be a function defined by $d_{\mathcal{K}}(x, y) := 2^{-v(x-y)}$. Then, the pair $(\mathcal{K}, d_{\mathcal{K}})$ becomes a metric space. By this topology, $\mathcal{K}$ is a topological field, and the induced topology on $\mathcal{O}$ coincides with the P-adic topology of $\mathcal{O}$. Let $\mathsf{C}^{(\mathcal{K})}$ be the set of all Cauchy sequences of $\mathcal{K}$. Then $\mathsf{C}^{(\mathcal{K})}$ admits a commutative ring structure by using the addition and the product of $\mathcal{K}$. Let $\mathcal{N}$ be the ideal in $\mathsf{C}^{(\mathcal{K})}$ consisting of Cauchy sequences which converge on 0. As the ideal $\mathcal{N}$ is maximal, the factor ring $\hat{\mathcal{K}} := \mathsf{C}^{(\mathcal{K})}/\mathcal{N}$ is a field. By the construction of $\hat{\mathcal{K}}$, it is the completion of $\mathcal{K}$ as the metric space $(\mathcal{K}, d_{\mathcal{K}})$. Then, the metric function $d_{\mathcal{K}}$ extends to a metric function $\hat{d}_{\mathcal{K}} : \hat{\mathcal{K}} \times \hat{\mathcal{K}} \to \mathbb{R}_{\geq 0}$, and $(\hat{\mathcal{K}}, \hat{d}_{\mathcal{K}})$ is also a discrete valuation field. The discrete valuation ring of $\hat{\mathcal{K}}$ is isomorphic to the completion $\hat{\mathcal{O}} = \varprojlim \mathcal{O}/P^i$. For the details, see [Mat, Chapter 3, Section 8] and [AM, Chapter 10]. If the complete discrete valuation ring $\hat{\mathcal{O}}$ contains a field, $\hat{\mathcal{O}}$ becomes a formal power series ring over a field with an indeterminate by Cohen's structure theorem [Co].

2.2 Lifting idempotents

Let R be a ring. An element $e \in R$ is an **idempotent** if $e^2 = e$. Two idempotents e and e' are called **orthogonal** if $ee' = e'e = 0$. An idempotent e is said to be **primitive** if e can not be written as a sum $e = e_1 + e_2$, where e_1 and e_2 are non-zero idempotents and orthogonal. Let $R = P_1 \oplus \cdots \oplus P_n$ be a direct sum decomposition of R into R-modules. Then, there are idempotents $e_1 \in P_1, \ldots, e_n \in P_n$ such that $1_R = e_1 + \cdots + e_n$. This implies that $P_i = e_iR$ for each i, and the idempotents $e_1, \ldots, e_n$ are pairwise orthogonal. Conversely, if a set of pairwise orthogonal idempotents $\{e_1, \ldots, e_n\}$ such that $1_R = e_1 + \cdots + e_n$ is given, it gives rise to a direct sum decomposition $R = e_1R \oplus \cdots \oplus e_nR$ as R-modules. Thus, an idempotent $e \in R$ is primitive if and only if eR is indecomposable.

Let M be an R-module and $E = \mathsf{End}_R(M)$. Then, M is an (E, R)-bimodule. For a set of pairwise orthogonal idempotents $e_1, \ldots, e_n$ of E with $1_E = e_1 + \cdots + e_n$, the R-module M is the direct sum $M = e_1M \oplus \cdots \oplus e_nM$ as R-module. Conversely, for a direct sum decomposition of M as R-submodules

$$M = M_1 \oplus \cdots \oplus M_n,$$

there is a set of pairwise orthogonal idempotents $e_1, \ldots, e_n$ of E such that $1_E = e_1 + \cdots + e_n$ and $M_i = e_i M$. Moreover, in this case, $M_i \simeq M_j$ if and only if $Ee_i \simeq Ee_j$ as left E-modules. As a consequence, a non-zero R-module M is indecomposable if and only if $E = \mathsf{End}_R(M)$ has only two idempotents 0 and 1. Thus, for an R-module M, if the endomorphism ring $\mathsf{End}_R(M)$ is local, the R-module M is indecomposable, for example see [AF, 5.10 Proposition]. However, the converse is not always true.

For a ring R, the **Jacobson radical** of R, which is denoted by $\mathsf{rad}(R)$, is defined by

$$\mathsf{rad}(R) = \bigcap_{S\,:\,\text{a simple } A\text{-module}} \mathsf{ann}(S),$$

where $\mathsf{ann}(S)$ is the annihilator of S. An element $x \in R$ lies in the Jacobson radical if and only if $1 - xy$ is unit for all $y \in R$ or, equivalently, $1 - yx$ is unit for all $y \in R$. The Jacobson radical of R is obviously a two-sided ideal. If R is Artinian, the ring R is semi-simple if and only if $\mathsf{rad}(R) = 0$. For $x \in R$, we write $\overline{x}$ for the coset in R/N represented by x. If $e \in R$ be a non-zero idempotent, then $\overline{e}$ is a non-zero idempotent in R/N. Therefore, if $1 = e_1 + \cdots + e_n$ with $e_i e_j = \delta_{i,j} e_i$ and $e_i \neq 0$ is given, this decomposition yields a decomposition $\overline{1} = \overline{e}_1 + \cdots + \overline{e}_n$ of the same kind in R/N.

From now on, we assume that $\mathcal{O}$ is a complete discrete valuation ring, P is the maximal ideal in $\mathcal{O}$, κ the residue field and A is an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module.

Lemma 2.2.1 ([CR, (5.22) Proposition]). Set $\overline{A} = A/AP$, and let $\varphi : A \to \overline{A}$ be the natural projection. Then, the following statements hold.

(1) $\mathsf{rad}(A) = \varphi^{-1}(\mathsf{rad}(\overline{A})) \supset AP$.

(2) The map φ induces an isomorphism $A/\mathsf{rad}(A) \simeq \overline{A}/\mathsf{rad}(\overline{A})$ as κ-algebras.

(3) $A/\mathsf{rad}A$ is semi-simple.

(4) There is a positive integer t such that $(\mathsf{rad}A)^t$ is contained in AP.

Proof. (1) Let M be a simple A-module. We show that $M \cdot (AP) = 0$. As M is simple, $M = mA$ for any $m \neq 0$. Thus, M is a finitely generated R-module, and $MP = 0$ or $MP = M$ since M is simple. If $MP = M$, then $M = 0$ by Nakayama's lemma, a contradiction. Hence, we have $MP = 0$.

(2) Since $\varphi : A \to \overline{A}$ is surjective, the A-module homomorphism φ induces a surjection $A/\mathsf{rad}(A) \to \overline{A}/\mathsf{rad}(\overline{A})$. On the other hand, by (1), there is a surjection $\psi : \overline{A} \to A/\mathsf{rad}(A)$. As $\mathsf{rad}(A/\mathsf{rad}(A)) = 0$, the A-module homomorphism ψ induces the surjection $\overline{A}/\mathsf{rad}(\overline{A}) \to$

$A/\mathsf{rad}(A)$. Because both $\overline{A}/\mathsf{rad}(\overline{A})$ and $A/\mathsf{rad}(A)$ are finite dimensional κ-algebras, these are isomorphic each other.

(3) We notice that $\overline{A}$ is an Artin ring. Thus, $\overline{A}/\mathsf{rad}(\overline{A})$ is semi-simple, and the assertion follows from the statement (2).

(4) As $\overline{A}$ is an Artin ring, the Jacobson radical $\mathsf{rad}(\overline{A})$ is nilpotent. Thus, there is a positive integer t such that $(\mathsf{rad}(\overline{A}))^t = 0$. Now, the claim is clear by (1). $\square$

Lemma 2.2.2 ([CR, (6.5) Proposition])**.** *A is complete with respect to the $\mathsf{rad}(A)$-adic topology.*

Proof. The assertion follows from that $\hat{A} \simeq \hat{\mathcal{O}} \otimes_{\mathcal{O}} A$ and Lemma 2.2.1. $\square$

Lemma 2.2.3 ([CR, (6.6) Proposition])**.** *Let Q and Q' be two finitely generated projective A-modules. Then, $Q \simeq Q'$ if and only if $Q/(Q \cdot (\mathsf{rad}(A))) \simeq Q'/(Q' \cdot (\mathsf{rad}(A)))$.*

Proof. Obviously, $Q \simeq Q'$ implies $Q/(Q \cdot (\mathsf{rad}(A))) \simeq Q'/(Q' \cdot (\mathsf{rad}(A)))$. We show the converse. Set $\tilde{Q} = Q/(Q \cdot (\mathsf{rad}(A))) \simeq Q'/(Q' \cdot (\mathsf{rad}(A)))$. Then the natural projection $Q \to \tilde{Q}$ factors through the natural projection $Q' \to \tilde{Q}$:

$$\begin{tikzcd} & Q \arrow[dl, dashed, "f"'] \arrow[d] \\ Q' \arrow[r] & \tilde{Q} \end{tikzcd}$$

By the above commutative diagram, we have $\mathsf{Coker}(f) = \mathsf{Coker}(f)\mathsf{rad}(A)$. As $\mathsf{Coker}(f)$ is finitely generated, f is surjective by Nakayama's lemma. Since Q' is projective, the A-module homomorphism f splits. By using the same argument after swapping Q and Q', we conclude that f is an isomorphism. $\square$

Proposition 2.2.4 ([CR, (6.7) Theorem on Lifting Idempotents])**.** *The following statements hold.*

(1) *For every idempotent $f \in \overline{A} := A/\mathsf{rad}(A)$, there exists an idempotent $e \in A$ such that $f = \overline{e}$.*

(2) *For two idempotents e_1 and e_2, the A-modules e_1A and e_2A are isomorphic if and only if the $\overline{A}$-modules $\overline{e_1}\overline{A}$ and $\overline{e_2}\overline{A}$ are isomorphic.*

(3) *An idempotent $e \in A$ is primitive if and only if $\overline{e} \in \overline{A}$ is primitive.*

Proof. Consider the identity element in $\mathbb{Z}[X]$:

$$1 = (X + (1 - X))^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} X^{2n-j}(1 - X)^j$$

Let
$$f_n(X) := \sum_{j=0}^{n} \binom{2n}{j} X^{2n-j}(1-X)^j.$$
Then, the polynomial $f_n(X)$ satisfies the following properties:

(a) $f_n(X) \in \mathbb{Z}[X]$.

(b) $f_n(X) \equiv 0 \ \mathsf{mod}\ X^n$ and $f_n(X) \equiv 1 \ \mathsf{mod}\ (X-1)^n$.

(c) $(f_n(X))^2 \equiv f_n(X) \ \mathsf{mod}\ X^n(1-X)^n$.

(d) $f_n(X) \equiv f_{n-1}(X) \ \mathsf{mod}\ X^{n-1}(X-1)^{n-1}$.

(e) $f_1(X) \equiv X \ \mathsf{mod}\ (X - X^2)$.

(1) Let $f \in \overline{A}$ be an idempotent and $a \in A$ such that $\overline{a} = f$. As f is an idempotent, we have $a^2 - a \in \mathsf{rad}(A)$. By (d), we have $f_j(a) \equiv f_{j-1}(a) \ \mathsf{mod}\ (\mathsf{rad}(A))^{j-1}$. Thus, $(f_j(a))$ is a Cauchy sequence. By our assumption that $\mathcal{O}$ is complete and Lemma 2.2.2, the algebra A is also complete with respect to the $\mathsf{rad}(A)$-adic topology. Hence, there is $e \in A$ such that $(f_j(a))$ converges on e. By (c) and (e), we have $\overline{e} = \overline{a} = f$ and $e^2 = e$.

(2) The statement follows from Lemma 2.2.3.

(3) Let e be an idempotent of A, and suppose that $\overline{e}$ is not primitive, so that $\overline{e} = f_1 + f_2$ for some non-zero orthogonal idempotents f_1 and f_2 in $\overline{A}$. By (1), there is an idempotent e_1 such that $\overline{e_1} = f_1$. Take $e_2 = e - e_1$. Then, $e_2^2 = e_2 \neq 0$ and $e_2 e_1 = e_1 e_2 = 0$, a contradiction. □

Therefore, we have the following:

Proposition 2.2.5 ([CR, (6.10) Proposition]). Let M be a finitely generated A-module. Then, M is indecomposable if and only if $\mathsf{End}_A(M)$ is local.

Proof. Assume that M is indecomposable. As $\mathsf{End}_A(M) \subset \mathsf{End}_{\mathcal{O}}(M)$, the endomorphism algebra $\mathsf{End}_A(M)$ is finitely generated as an $\mathcal{O}$-module. It follows from Lemma 2.2.1 and Proposition 2.2.4 that $\mathsf{End}_A(M)/\mathsf{radEnd}_A(M)$ is semi-simple, and lifting idempotents theory works. Thus, M is indecomposable if and only if $\mathsf{End}_A(M)/\mathsf{radEnd}_A(M)$ is a skew-field. □

Theorem 2.2.6 ([CR, (6.12) Krull–Schmidt–Azumaya Theorem]). Let $\mathcal{O}$ be a complete discrete valuation ring, A an $\mathcal{O}$-algebra, which is finitely generated as an $\mathcal{O}$-module, and M a finitely generated A-module. Then, M admits a finite direct sum decomposition into indecomposable submodules. Further. if M has two such decompositions
$$M = \bigoplus_{i=1}^{s} M_i = \bigoplus_{j=1}^{t} N_j,$$
then $r = s$ and there is a permutation $\sigma \in \mathfrak{S}_n$ such that $M_i \simeq N_{\sigma(i)}$ for all i.

Theorem on lifting idempotents and Krull–Schmidt–Azumaya theorem imply that the isomorphism classes in $\mathsf{proj}\text{-}A$ correspond bijectively with those in $\mathsf{proj}\text{-}(A/\mathsf{rad}(A))$, the correspondence being given by mapping the classes of $P \in \mathsf{proj}\text{-}A$ onto the class $P/P\mathsf{rad}(A) \in \mathsf{proj}\text{-}(A/\mathsf{rad}(A))$. Therefore, there is a bijection between the set of isoclasses of the indecomposable projective A-modules and the set of isoclasses of indecomposable projective $A/\mathsf{rad}(A)$-modules.

Let M be a finitely generated A-module and P a finitely generated projective A-module. A surjection $f : P \to M$ is a **projective cover** of M if any $g : N \to P$ such that fg is surjective is surjective.

Lemma 2.2.7 ([CR, (6.20) Proposition])**.** Given two projective covers $f : P \to M$ and $g : Q \to M$, there exists an isomorphism $h : P \to Q$ such that $f = gh$.

Proof. Since P is projective and g is surjective, there is $h : P \to Q$ such that $f = gh$. By the definition of projective covers, h is surjective. Hence, h is a retraction. Let $h' : Q \to P$ such that $hh' = 1_Q$. Since $g : Q \to M$ is a projective cover, h' is also a retraction. Thus, h is an isomorphism with $f = gh$. □

Theorem 2.2.8 ([CR, (6.23) Theorem])**.** Let $\mathcal{O}$ be a complete discrete valuation ring and A an $\mathcal{O}$-algebra, which is finitely generated as an $\mathcal{O}$-module. Then, every $X \in \mathsf{mod}\text{-}A$ has the projective cover. In particular, the category $\mathsf{mod}\text{-}A$ has enough projectives.

Proof. Let $X \in \mathsf{mod}\text{-}A$. Since $\overline{A} := A/\mathsf{rad}(A)$ is semi-simple, $\overline{X} = X/X\mathsf{rad}(A) \in \mathsf{mod}\text{-}\overline{A}$ is a direct sum of indecomposable projective $\overline{A}$-modules $\overline{e_i}\overline{A}$. Hence, there exists a finitely generated projective A-module P such that $\overline{P} \simeq \overline{X}$ as $\overline{A}$-modules, where $\overline{P} = P/P\mathsf{rad}(A)$. As P is projective, we have $f : P \to X$ making the following diagram commute:

$$\begin{tikzcd} P \arrow[r] \arrow[d, dashed, "f"'] & \overline{P} \arrow[d, "\simeq"] \\ X \arrow[r] & \overline{X} \end{tikzcd}$$

The commutative diagram implies that $f(P) + X\mathsf{rad}(A) = X$. By Nakayama's lemma, f is surjective. Then, f gives the projective cover since the map $P \to \overline{P}$ is the projective cover of $\overline{P}$. □

2.3 Orders and lattices

In representation theory of algebras, lattices over orders over a complete discrete valuation ring have been studied extensively.

Definition 2.3.1. Let $\mathcal{O}$ be a complete discrete valuation ring.

(1) An $\mathcal{O}$-algebra A is called an **$\mathcal{O}$-order** if A is free of finite rank as an $\mathcal{O}$-module.

(2) An $\mathcal{O}$-order A is called **Gorenstein** if $\mathsf{Hom}_{\mathcal{O}}(A, \mathcal{O})$ is a projective A-module. If A and $\mathsf{Hom}_A(A, \mathcal{O})$ are isomorphic as (A, A)-bimodules, then A is called **symmetric**.

(3) Let A be an $\mathcal{O}$-order and M an A-module. M is called a **Cohen–Macaulay A-module** or an **A-lattice** if M is free of finite rank as an $\mathcal{O}$-module.

Remark 2.3.2. We follow [I] for the definitions of orders and lattices. This means that the definitions of orders and lattices are different from Auslander's sense [A2, Chapter I, Section 7]. However, if A is a Gorenstein $\mathcal{O}$-order, then A is an $\mathcal{O}$-order in his sense. Moreover, Gorenstein $\mathcal{O}$-orders in Auslander's sense are symmetric $\mathcal{O}$-orders in our sense.

From now on, we use the following conventions in this thesis.

(i) $\mathcal{O}$ is a complete discrete valuation ring with a uniformizer ε.

(ii) κ is the residue field and $\mathcal{K}$ is the quotient field.

(iii) The symbol $\otimes$ means the tensor product taken over the complete discrete valuation ring $\mathcal{O}$.

(iv) A is an $\mathcal{O}$-order and $\overline{A} = A \otimes \kappa = A/\varepsilon A$.

We denote by $\mathsf{latt}\text{-}A$ the full subcategory of $\mathsf{mod}\text{-}A$ consisting of A-lattices. A sequence in $\mathsf{latt}\text{-}A$ is called **exact** if it is exact in $\mathsf{mod}\text{-}A$. We denote by $\underline{\mathsf{latt}}\text{-}A$ the stable module category of $\mathsf{latt}\text{-}A$ by $\mathsf{proj}\text{-}A$. Then, $\mathsf{latt}\text{-}A$ is closed under extensions, and the functor $\mathsf{D} := \mathsf{Hom}_{\mathcal{O}}(-, \mathcal{O})$ induces the duality

$$\mathsf{latt}\text{-}A \overset{\sim}{\longleftrightarrow} \mathsf{latt}\text{-}A^{\mathrm{op}}.$$

We call $I \in \mathsf{latt}\text{-}A$ an **injective** A-lattice if $I \in \mathsf{add}(\mathsf{D}(A^{\mathrm{op}}))$, where $\mathsf{add}(\mathsf{D}(A^{\mathrm{op}}))$ is the full subcategory of $\mathsf{latt}\text{-}A$ consisting of direct summands of finite direct sums of copies of $\mathsf{D}(A^{\mathrm{op}})$. We denote by $\mathsf{inj}\text{-}A$ and $\overline{\mathsf{latt}}\text{-}A$ the category of injective A-lattices and the stable module category of $\mathsf{latt}\text{-}A$ by $\mathsf{inj}\text{-}A$.

By Theorem 2.2.6, the category $\mathsf{latt}\text{-}A$ is a Krull–Schmidt category, that is, any object is isomorphic to a finite direct sum of objects whose endomorphism algebras are local. In brief, any A-lattice admits a unique indecomposable finite direct sum decomposition as an A-module. It follows from Theorem 2.2.8 that the category $\mathsf{latt}\text{-}A$ has enough projectives. Let $\{e_1, \ldots, e_n\}$ be a set of idempotents of A. Theorem 2.2.6 yields that $\{e_1A, \cdots, e_nA\}$ is a complete set of isoclasses of indecomposable projective A-modules if and only if $\{\overline{e_1}\overline{A}, \ldots, \overline{e_n}\overline{A}\}$ is a complete set of isoclasses of indecomposable projective $\overline{A}$-modules since $\varepsilon A \subset \mathsf{rad}(A)$.

An $\mathcal{O}$-order A is called an **isolated singularity** if the algebra $A \otimes \mathcal{K}$ is a semi-simple $\mathcal{K}$-algebra. For example, for a finite group G, the group algebra $\mathcal{O}G$ is an isolated singularity if and only if $\mathsf{char}(\mathcal{K})$ does not divide the order of G or $\mathsf{char}(\mathcal{K}) = 0$ by Maschke's theorem.

In particular, For a p-modular system $(\mathcal{K}, \mathcal{O}, \kappa)$ of a finite group G, the group algebra $\mathcal{O}G$ is an isolated singularity. By [A3], A is an isolated singularity if and only if, for any $X, Y \in \mathsf{latt}\text{-}A$, the set of homomorphisms $\underline{\mathsf{Hom}}_A(X, Y)$ in $\underline{\mathsf{latt}}\text{-}A$ has finite length as an $\mathcal{O}$-module.

We denote by

$$\Omega : \underline{\mathsf{latt}}\text{-}A \longrightarrow \underline{\mathsf{latt}}\text{-}A, \quad \Omega^{-1} : \overline{\mathsf{latt}}\text{-}A \longrightarrow \overline{\mathsf{latt}}\text{-}A$$

the syzygy functor and the cosyzygy functor of A. We also denote by

$$\widetilde{\Omega} : \underline{\mathsf{mod}}\text{-}\overline{A} \longrightarrow \underline{\mathsf{mod}}\text{-}\overline{A}, \quad \widetilde{\Omega}^{-1} : \overline{\mathsf{mod}}\text{-}\overline{A} \longrightarrow \overline{\mathsf{mod}}\text{-}\overline{A}$$

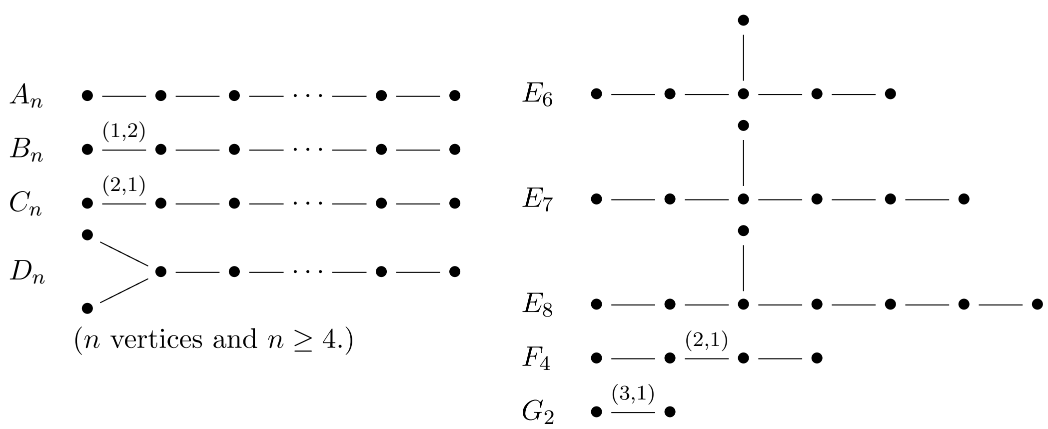
the syzygy functor and the cosyzygy functor of $\overline{A}$. If A is symmetric, the syzygy and cosyzygy functors give category equivalences and quasi-inverse each other.

Lastly, we recall a well-known fact as Miyata's theorem.

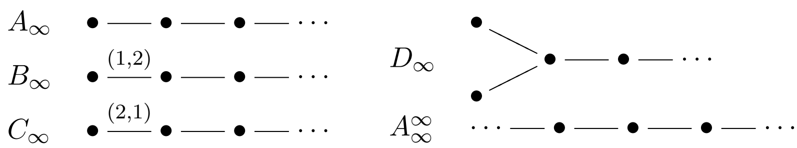
Theorem 2.3.3 ([M, Theorem 1])**.** Let R be a commutative noetherian ring and Λ an R-algebra which is finitely generated as an R-module. Let $\mathbb{E} : 0 \to L \to E \to M \to 0$ be a short exact sequence in $\mathsf{mod}\text{-}\Lambda$. If $E \simeq L \oplus M$ as Λ-modules, then $\mathbb{E}$ splits.

2.4 Dynkin and Euclidean diagrams

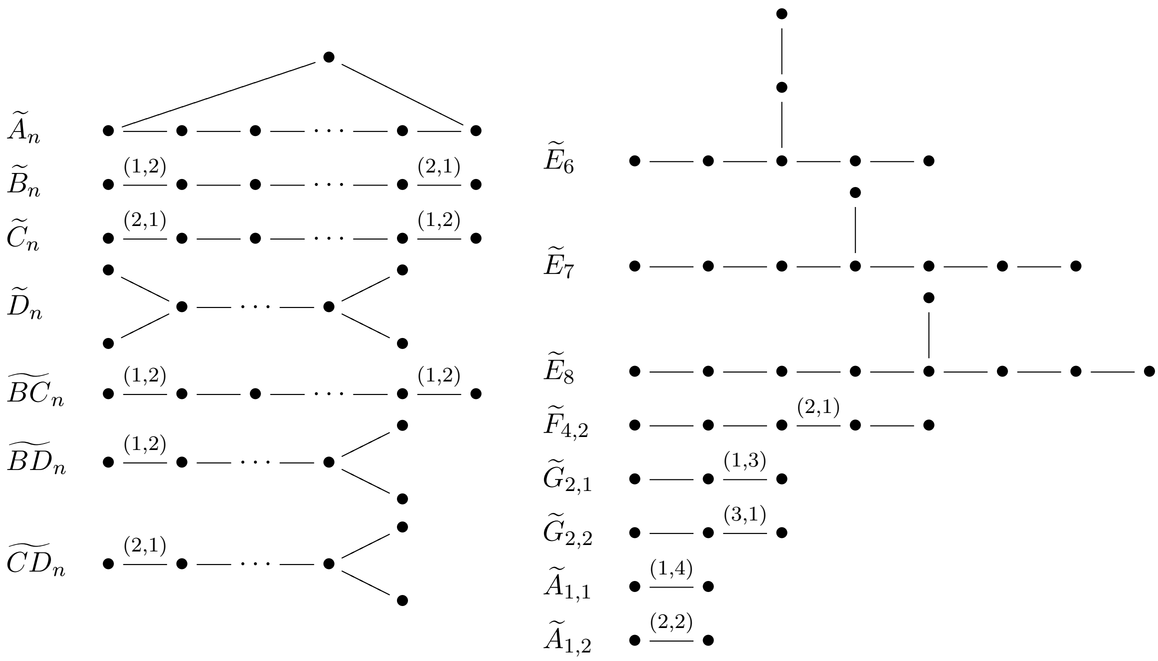
We list Dynkin and Euclidean diagrams. The following labelled undirected graphs are called **finite Dynkin diagrams**.



The following are **infinite Dynkin diagrams**.



The following are **Euclidean diagrams**.



Here, we note that $\widetilde{A}_0$ is a single loop with one vertex and $\widetilde{A}_1$ is the underlying graph of the Kronecker quiver.

2.5 Valued stable translation quivers

In this section, we recall notations on stable translation quivers. A **quiver** $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets Q_0 and Q_1, and two maps $s, t : Q_1 \to Q_0$. Each element of Q_0 and Q_1 is called a vertex and an arrow, respectively. For an arrow $\alpha \in Q_1$, we call $s(\alpha)$ and $t(\alpha)$ the source and the target of α, respectively. We understand that quivers are directed graphs. We write $\overline{Q}$ for the underlying graph of Q. Given two quivers Q and Δ, a **quiver homomorphism** $f : Q \to \Delta$ is a pair of maps $f_0 : Q_0 \to \Delta_0$ and $f_1 : Q_1 \to \Delta_1$ such that $(s \times t) \circ f_1 = (f_0 \times f_0) \circ (s \times t)$. In this section, we assume that quivers have no multiple arrows, that is, the map $(s \times t)$ is injective. Let (Q, v) be a pair of a quiver Q and a map $v : Q_1 \to \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. For an arrow $x \to y$ of Q, we write $v(x \to y) = (d_{xy}, d'_{xy})$, and we understand that there is no arrow from x to y if and only if $d_{xy} = d'_{xy} = 0$. Then, (Q, v) is called a **valued quiver**, and the values of the map v are called **valuations**. For an arrow $\alpha : x \to y$ with $d_{x,y} = d'_{x,y} = 1$, we usually omit to write the valuation of α. For each vertex $x \in Q_0$, we set

$$x^+ = \{y \in Q_0 \mid x \to y \in Q_1\}, \quad x^- = \{y \in Q_0 \mid y \to x \in Q_1\}.$$

Note that a quiver is determined by the set x^+ (or x^-). A quiver Q is **locally finite** if $x^+ \cup x^-$ is a finite set for any $x \in Q_0$, and Q is called **finite** if the sets Q_0 and Q_1 are finite sets. A **translation quiver** is a triple (Q, Q'_0, τ) of a locally finite quiver Q, a subset $Q'_0 \subset Q_0$ and an injective map $\tau : Q'_0 \to Q_0$ satisfying $x^- = (\tau x)^+$. If $Q'_0 = Q_0$ and τ is bijective, the translation quiver is said to be **stable**. Then, we write (Q, τ) for the stable translation quiver, simply. Let $\mathcal{C}$ be a full subquiver of a stable translation quiver (Q, τ). Then, $\mathcal{C}$ is a (connected) **component** if the following three conditions are satisfied.

(i) $\mathcal{C}$ is stable under the quiver automorphism τ.

(ii) $\mathcal{C}$ is a disjoint union of connected components of the underlying undirected graph.

(iii) There is no proper subquiver of $\mathcal{C}$ that satisfies (i) and (ii).

A quiver homomorphism f from a translation quiver (Q, Q'_0, τ) to a translation quiver $(\Delta, \Delta'_0, \tau')$ is a **translation quiver homomorphism** if $f_0 \circ \tau = \tau' \circ f_0$ is satisfied on Q'_0. It is easily seen that τ induces a translation quiver automorphism when (Q, Q'_0, τ) is stable, and we use the same letter τ. In this thesis, we denote by $\mathsf{Aut}_\tau(Q)$ the set of all translation quiver automorphisms of (Q, τ). Let Q and Δ be two stable translation quivers. A surjective translation quiver homomorphism $f : Q \to \Delta$ is a **covering** if $f|_{x^+}$ gives a bijection between x^+ and $(f(x))^+$.

For a stable translation quiver (Q, τ) and a subgroup $G \subset \mathsf{Aut}_\tau(Q)$, we define the translation quiver homomorphism $\pi_G : Q \to Q/G$ by $\pi_G(x) = Gx$ for $x \in Q_0$. A subgroup $G \subset \mathsf{Aut}_\tau(Q)$ is **admissible** if each G-orbit intersects $x^+ \cup \{x\}$ in at most one vertex and $x^- \cup \{x\}$ in at most one vertex, for any $x \in Q_0$. Then, the map π_G is covering.

Definition 2.5.1. A **valued stable translation quiver** is a triple (Q, v, τ) such that

(i) (Q, v) is a valued quiver,

(ii) (Q, τ) is a stable translation quiver,

(iii) $v(\tau y \to x) = (d'_{x,y}, d_{x,y})$ for each arrow $x \to y$.

Given a valued quiver (Q, v), one can construct the valued stable translation quiver $(\mathbb{Z}Q, \tilde{v}, \tau)$ as follows.

- $(\mathbb{Z}Q)_0 := \mathbb{Z} \times Q_0$.
- $(n, x)^+ := \{(n, y) \mid y \in x^+\} \cup \{(n + 1, z) \mid z \in x^-\}$.
- $\tilde{v}((n, x) \to (n, y)) = (d_{x,y}, d'_{x,y})$, $\tilde{v}((n - 1, y) \to (n, x)) = (d'_{x,y}, d_{x,y})$.
- $\tau_0((n, x)) = (n - 1, x)$.

We write it simply $\mathbb{Z}Q$. Note that $\mathbb{Z}Q$ has no loops whenever Q has no loops.

Lemma 2.5.2 ([B, Lemma 4.15.2])**.** Let T be a directed tree and (Q, τ) a stable translation quiver. Given a quiver homomorphism $f : T \to Q$, there is a unique translation quiver homomorphism $\tilde{f} : \mathbb{Z}T \to Q$ such that $\tilde{f}(0, x) = f(x)$.

Proof. For $m \in \mathbb{Z}$ and $x \in T$, let $\tilde{f}(m, x) = \tau^{-m} f(x)$. Then, the morphism $\tilde{f}$ is a translation quiver homomorphism satisfying $\tilde{f}(0, x) = f(x)$. The uniqueness is clear by the definition of translation quiver homomorphisms. $\square$

Lemma 2.5.3 ([B, Lemma 4.15.3])**.** Let T and T' be directed trees. Then, $\mathbb{Z}T \simeq \mathbb{Z}T'$ as stable translation quivers if and only if $\overline{T} \simeq \overline{T'}$.

Proof. Obviously, if $\mathbb{Z}T \simeq \mathbb{Z}T'$ implies that $\overline{T} = \overline{T'}$. Define $f : T \to \mathbb{Z}T'$ as follows. First, we choose a vertex $x \in T$, and let $f(x) = (0, x)$. As T is connected, we may extend this uniquely to a quiver homomorphism $f : T \to \mathbb{Z}T'$ in such a way that f send each $x \in T$ to (n_x, x) for some $n_x \in \mathbb{Z}$. By the above lemma, we have $\tilde{f} : \mathbb{Z}T \to \mathbb{Z}T'$ such that $\tilde{f}(0, x) = (n_x, x)$. Then, $\tilde{f}$ is an isomorphism since the morphism defined by $\mathbb{Z}T' \ni (0, x) \mapsto (-n_x, x) \in \mathbb{Z}T$ is its inverse. $\square$

The following theorem is well-known and it is effective to describe the structure of stable translation quivers [Ri], see also [B, Theorem 4.15.6].

Theorem 2.5.4 (Riedtmann's structure theorm)**.** Let (Q, τ) be a stable translation quiver without loops and $\mathcal{C}$ a connected component of (Q, τ). Then, there exist a directed tree T and an admissible group $G \subseteq \mathsf{Aut}_{\tau_0}(\mathbb{Z}T)$ such that $\mathcal{C} \simeq \mathbb{Z}T/G$ as stable translation quivers. Moreover, $\overline{T}$ is uniquely determined by $\mathcal{C}$, and the admissible group is unique up to conjugation.

In Theorem 2.5.4, the underlying undirected tree $\overline{T}$ is called the **tree class** of $\mathcal{C}$.

Let (Q, τ) be a connected stable translation quiver. A vertex $x \in Q_0$ is called **periodic** if $x = \tau^k x$ for some $k > 0$. If there is a periodic vertex in Q, then all vertices of Q are periodic. Indeed, if x is a periodic vertex in Q, then there is a positive integer and n_x such that $\tau^{n_x} x = x$. Since (Q, τ) is a stable translation quiver, τ^{n_x} induces a bijection on the finite set x^+, and so some power of τ^{n_x} stabilizes x^+ elementwise. Hence, all vertices in x^+ are periodic. It follows that all vertices are periodic. In this case, (Q, τ) is called **periodic**.

Definition 2.5.5. Let I be a set. A **Cartan matrix** on I is a function $C : I \times I \to \mathbb{Z}$ satisfying the following properties.

(i) For all $i \in I$, $C(i, i) = 2$.

(ii) $C(i, j) \leq 0$ for all $i \neq j$, and for each i, we have that $C(i, j) < 0$ for only finitely many $j \in I$.

(iii) $C(i, j) \neq 0$ if and only if $C(j, i) \neq 0$.

Let (Q, v) be a connected valued quiver without loops and two cycles. Then, (Q, v) gives rise to a Cartan matrix on Q_0:

$$C(x,y) = \begin{cases} 2 & \text{if } x = y, \\ -d_{x,y} & \text{if } y \in x^+, \\ -d'_{y,x} & \text{if } y \in x^-, \\ 0 & \text{otherwise.} \end{cases}$$

The above Cartan matrix is denoted by C_Q^v.

Definition 2.5.6. Let C be a Cartan matrix on I. A **subadditive function** for C is a function $\ell : I \to \mathbb{Q}_{>0}$ such that it satisfies

$$\sum_{y \in I} C(x,y)\ell(y) \geq 0$$

for all $x \in I$. A subadditive function ℓ is called **additive** if the equality holds for all $x \in I$. We say that a connected valued quiver Q admits a subadditive function when there exists a subadditive function for a Cartan matrix on Q_0.

Remark 2.5.7. Let (Q, v, τ) be a connected valued stable translation quiver without loops, and let $\overline{T}$ be the tree class of Q. If a function $\ell : Q_0 \to \mathbb{Q}_{>0}$ satisfies $\ell(\tau x) = \ell(x)$ and

$$2\ell(x) \geq \sum_{y \in x^- \cap T} d_{y,x}\ell(y) + \sum_{y \in x^+ \cap T} d'_{x,y}\ell(y),$$

then the restriction $\ell|_T$ is a subadditive function for the Cartan matrix C_T^v on $\overline{T}_0$.

The following theorem is a generalization by D. Happel, U. Preiser and C. M. Ringel of characterizations of Dynkin and Euclidean diagrams by E. B. Vinberg [V] and S. Berman, R. Moody and M. Wonenburger [BMW].

Theorem 2.5.8 ([B, Theorem 4.5.8])**.** Let (Δ, v) be a connected valued quiver without loops. If Δ admits a subadditive function ℓ, then the following statements hold.

(1) The underlying undirected graph $\overline{\Delta}$ is either a finite or infinite Dynkin diagram or a Euclidean diagram.

(2) If ℓ is not additive, then $\overline{\Delta}$ is either a finite Dynkin diagram or A_∞.

(3) If ℓ is additive, then $\overline{\Delta}$ is either an infinite Dynkin diagram or a Euclidean diagram.

(4) If ℓ is unbounded, then $\overline{\Delta}$ is A_∞.

2.6 Indecomposable modules over a special biserial algebra

Throughout this section, $\mathbf{k}$ is an algebraically closed field. Let Q be a quiver, which admits loops and multiple arrows. Set $Q_1^* = \{\alpha^* \mid \alpha \in Q_1\}$. We understand that the symbol α^* is the formal inverse arrow of α, that is, α^* is an arrow such that $s(\alpha^*) = t(\alpha)$, $t(\alpha^*) = s(\alpha)$ and $\alpha^{**} = \alpha$. For a path $w = c_1c_2\cdots c_n$ in Q, we define $s(w) = s(c_1)$, $t(w) = t(c_n)$ and $w^* = c_n^*c_{n-1}^*\cdots c_1^*$. If w is the path with the length 0 at a vertex a, then we understand that w is the trivial path ε_a with $s(\varepsilon_a) = t(\varepsilon_a) = a$ and $\varepsilon_a^* = \varepsilon_a$. A **walk** with length n is a sequence $w = c_1c_2\cdots c_n$ such that each $c_i \in Q_1 \cup Q_1^*$ and $t(c_i) = s(c_{i+1})$ for $i = 1, 2, \ldots, n-1$, and w is called **reduced** if w is either a trivial path or a walk with positive length such that $c_{i+1} \neq c_i^*$ for all $i = 1, 2, \ldots, n-1$. Given a walk w, the source $s(w)$ and the target $t(w)$ are also defined. For two walks $w_1 = c_{11}\cdots c_{1n}$ and $w_2 = c_{21}\cdots c_{2m}$, the product w_1w_2 is defined by

$$w_1w_2 := \begin{cases} c_{11}\cdots c_{1n}c_{21}\cdots c_{2m} & \text{if } t(w_1) = s(w_2), \\ 0 & \text{otherwise.} \end{cases}$$

If w is a walk with $s(w) = t(w)$, then one has also arbitrary powers w^j of w. Assume that $w = c_1c_2\cdots c_n$ is a reduced walk with positive length. The walk w is called a **reduced cycle** if $s(w) = t(w)$ and $c_n \neq c_1^*$. We say that a non-trivial path p is **contained** in w if p or p^* is a subwalk of w. A **relation** in Q with coefficients in $\mathbf{k}$ is a $\mathbf{k}$-linear combination of paths of length at least two having the same source and target.

For a finite quiver Q, the **path algebra**, say $\mathbf{k}Q$, is defined as follows. As a $\mathbf{k}$-vector space,

$$\mathbf{k}Q = \bigoplus_{w:\text{a path in } Q} \mathbf{k}w,$$

and the product in $\mathbf{k}Q$ is defined as the product of walks. Then, there is a direct sum decomposition

$$\mathbf{k}Q = \mathbf{k}Q_0 \oplus \mathbf{k}Q_1 \oplus \mathbf{k}Q_2 \oplus \cdots$$

as $\mathbf{k}$-vector spaces, where, for each $l \geq 0$, the $\mathbf{k}$-vector space $\mathbf{k}Q_l$ is the subspace of $\mathbf{k}Q$ generated by the set of all paths with length l in Q. A two-sided ideal $\mathcal{I}$ in $\mathbf{k}Q$ is **admissible** if there exists a positive integer $n \geq 2$ such that

$$\bigoplus_{l \geq n} \mathbf{k}Q_l \subset \mathcal{I} \subset \bigoplus_{l \geq 2} \mathbf{k}Q_l.$$

If $\mathcal{I}$ is an admissible ideal in $\mathbf{k}Q$, then the factor algebra $\mathbf{k}Q/\mathcal{I}$ is finite dimensional, and the factor algebra $\mathbf{k}Q/\mathcal{I}$ is called a **bound quiver algebra**. Let $\mathbf{k}Q/\mathcal{I}$ be a bound quiver algebra. A path w is called a **zero path** if w belongs to $\mathcal{I}$. A zero path with minimal length is called a **zero relation** of $\mathbf{k}Q/\mathcal{I}$. For non-zero paths p and q from a vertex a to a vertex b, the pair (p, q) is a **binomial relation** of $\mathbf{k}Q/\mathcal{I}$ if there exists $(\lambda, \mu) \in \mathbf{k}^\times \times \mathbf{k}^\times$ such that $\lambda p + \mu q \in \mathcal{I}$.

Let Λ be a basic finite dimensional algebra over $\mathbf{k}$ and $\{e_1, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents of Λ. Then, we define the **Gabriel quiver** of Λ, which is denoted by Q_Λ, as follows:

(a) The set of vertices is $\{1, 2, \ldots, n\}$.

(b) We draw $\mathsf{dim}_\mathbf{k}(e_a(\mathsf{rad}(\Lambda)/(\mathsf{rad}(\Lambda)^2))e_b)$ arrows from a to b.

Note that the Gabriel quiver of Λ does not depend on the choice of a complete set of primitive orthogonal idempotents in Λ. For each arrow $\alpha : a \to b$ in Q_Λ, let $x_\alpha \in \mathsf{rad}(\Lambda)$ such that $\{x_\alpha + \mathsf{rad}(\Lambda)^2 \mid \alpha : a \to b\}$ forms a $\mathbf{k}$-basis of $e_a(\mathsf{rad}(\Lambda)/(\mathsf{rad}(\Lambda)^2))e_b$. Then, the map $\varphi : \mathbf{k}Q_\Lambda \to \Lambda$ defined by

$$a \longmapsto e_a, \quad \alpha \longmapsto x_\alpha \quad (a \in (Q_\Lambda)_0,\ \alpha \in (Q_\Lambda)_1)$$

is a surjective $\mathbf{k}$-algebra homomorphism. It is easy to see that $\mathsf{Ker}(\varphi)$ is an admissible ideal in $\mathbf{k}Q_\Lambda$. Thus, we have:

Theorem 2.6.1 ([ASS, Chapter II, 2.9. Corollary and 3.7. Theorem])**.** Let Λ be an indecomposable finite dimensional algebra over $\mathbf{k}$. Then, there exists a finite connected quiver Q and an admissible ideal $\mathcal{I}$ in $\mathbf{k}Q$ such that Λ is Morita equivalent to $\mathbf{k}Q/\mathcal{I}$. Moreover, the admissible ideal $\mathcal{I}$ is generated by finite relations in Q.

Let Q be a finite quiver. A **k-linear representation** of Q is a system $(M_a, f_\alpha)_{a\in Q_0,\alpha\in Q_1}$ consisting of $\mathbf{k}$-vector spaces M_a and $\mathbf{k}$-linear maps $f_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$. The $\mathbf{k}$-linear representation $M = (M_a, f_\alpha)$ is said to be **finite dimensional** if the sum $\sum_{a\in Q_0} \mathsf{dim}_\mathbf{k}(M_a)$ is finite. For two $\mathbf{k}$-linear representations of Q, say $M = (M_a, f_\alpha)$ and $N = (N_a, g_\alpha)$, a **morphism of representations** $\varphi : M \to N$ is a family $\varphi = (\varphi_a)_{a\in Q_0}$ of $\mathbf{k}$-linear maps $\varphi_a : M_a \to N_a$ $(a \in Q_0)$ that are compatible with the structure maps f_α and g_α, that is, the following square is commutative for all $\alpha : a \to b \in Q_1$:

$$\begin{tikzcd} M_a \arrow[r, "f_\alpha"] \arrow[d, "\varphi_a"'] & M_b \arrow[d, "\varphi_b"] \\ N_a \arrow[r, "g_\alpha"'] & N_b \end{tikzcd}$$

The composition of morphisms of representations is naturally defined.

For a path $w = w_1 \cdots w_t$ with length t, we define the morphism $\varphi_w = \varphi_{w_t} \cdots \varphi_{w_1}$. Then, for a relation $\rho = \sum_{i=1}^n \lambda_i w_i$ in Q, we also define the morphism $\varphi_\rho = \sum_{i=1}^n \lambda_i \varphi_{w_i}$. A $\mathbf{k}$-linear representation $M = (M_a, f_\alpha)$ is said to be **bound by** $\mathcal{I}$ if we have $f_\rho = 0$ for all $\rho \in \mathcal{I}$.

Given a finite quiver Q and an admissible ideal $\mathcal{I}$ in $\mathbf{k}Q$, we define $\mathsf{rep}(Q)$ (resp. $\mathsf{rep}(Q, \mathcal{I})$) to be the $\mathbf{k}$-linear category consisting of finite dimensional k-linear representations of Q (resp. finite dimensional $\mathbf{k}$-linear representations bound by $\mathcal{I}$) and morphisms of representations.

Theorem 2.6.2 ([ASS, Chapter III, 1,6 Theorem])**.** Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents in $\mathbf{k}Q$ (or $\mathbf{k}Q/\mathcal{I}$). Then, there is a $\mathbf{k}$-linear equivalences of categories

$$\mathsf{mod}\text{-}(\mathbf{k}Q) \xleftrightarrow{\sim} \mathsf{rep}(Q), \quad \mathsf{mod}\text{-}(\mathbf{k}Q/\mathcal{I}) \xleftrightarrow{\sim} \mathsf{rep}(Q, \mathcal{I}),$$

which send M to (M_a, f_α), where $M_a = Me_a$,

$$f_\alpha : M_{s(\alpha)} \ni m \longmapsto m\alpha \in M_{t(\alpha)}.$$

We identify these two categories.

Definition 2.6.3. A bound quiver algebra $\Lambda \simeq \mathbf{k}Q/\mathcal{I}$ is called **special biserial** if the following two conditions are satisfied.

(i) For each vertex x of Q, $\sharp x^+ \leq 2$ and $\sharp x^- \leq 2$.

(ii) For each arrow α of Q, there exist at most one arrow β such that $\alpha\beta \notin \mathcal{I}$ and at most one arrow γ such that $\gamma\alpha \notin \mathcal{I}$.

Brauer graph algebras are symmetric special biserial algebras. The converse is also true by K. Erdman and A. Skowroński [ES]. On Brauer graph algebras, see [S]. B. Wald and J. Waschbüsch showed that special biserial algebras are of tame representation type by classifying indecomposable modules over such an algebra into "string modules" and "band modules" [WW]. Moreover, we can construct all finite dimensional indecomposable modules over a special biserial algebra by using a combinatorial method. In this section, we recall the construction of indecomposable modules over a special biserial algebra, see [Erd], [HL] for details.

Let $\Lambda = \mathbf{k}Q/\mathcal{I}$ be a bound quiver algebra. A reduced walk w is said to be a **string path** of Λ if each path contained in w is neither a zero relation nor a maximal subpath of a binomial relation of Λ. A non-trivial reduced cycle is said to be a **band path** of Λ if each of its powers is a string path and it is not a power of a string path with less length.

For each string path w of Λ, the **string module** $M(w)$ is defined as follows. If $w = \varepsilon_a$, then $M(w)$ is the simple Λ-module corresponding to a. For a non-trivial $w = c_1 c_2 \cdots c_n$, $M(w)$ is the $\mathbf{k}$-linear representation $(M(w)_a, M(w)_\alpha)$ given by the following. For $1 \leq i \leq n+1$, we set $\mathbf{k}(i) = \mathbf{k}$. Given a vertex a of Q, we define $M(w)_a = \bigoplus_{i \in \mathcal{W}_a} \mathbf{k}(i)$, where

$$\mathcal{W}_a = \{i \mid s(c_i) = a\} \cup \{n+1 \mid t(c_n) = a\}.$$

For $1 \leq i \leq n$, we define the $\mathbf{k}$-linear map f_{c_i} by

$$f_{c_i} : \begin{cases} \mathbf{k}(i) \longrightarrow \mathbf{k}(i+1), & x \longmapsto x \quad \text{if } c_i \in Q_1, \\ \mathbf{k}(i+1) \longrightarrow \mathbf{k}(i), & x \longmapsto x \quad \text{if } c_i \in Q_1^*. \end{cases}$$

Given an arrow α of Q, we define $M(w)_\alpha$ as the direct sum of the $\mathbf{k}$-linear maps f_{c_i} such that $c_i = \alpha$ or $c_i^* = \alpha$.

Let $w = c_1 c_2 \cdots c_n$ be a band path of Λ and V a finite dimensional indecomposable left $\mathbf{k}[x, x^{-1}]$-module. We construct the **band module** $N(w, V)$ corresponding to w and V as follows. For $1 \leq i \leq n$, we set $V(i) = V$. For $1 \leq i \leq n$, let f'_{c_i} be the $\mathbf{k}$-linear map defined by

$$f'_{c_i} : \begin{cases} V(i) \longrightarrow V(i+1), & v \longmapsto v \quad \text{if } 1 \leq i \leq n-1 \text{ and } c_i \in Q_1, \\ V(i+1) \longrightarrow V(i), & v \longmapsto v \quad \text{if } 1 \leq i \leq n-1 \text{ and } c_i \in Q_1^*, \\ V(n) \longrightarrow V(1), & v \longmapsto xv \quad \text{if } i = n \text{ and } c_n \in Q_1, \\ V(1) \longrightarrow V(n), & v \longmapsto x^{-1}v \quad \text{if } i = n \text{ and } c_n \in Q_1^*. \end{cases}$$

For a vertex a of Q, we define $N(w, V)_a = \bigoplus_{i \in \mathcal{W}'_a} V(i)$, where

$$\mathcal{W}'_a = \{i \mid s(c_i) = a\}.$$

For an arrow α of Q, we define $N(w, V)_\alpha$ as the direct sum of the $\mathbf{k}$-linear maps f'_{c_i} such that $c_i = \alpha$ or $c_i^* = \alpha$.

Theorem 2.6.4 ([WW, (2.3) Proposition])**.** Let Λ be a special biserial algebra. Then, the disjoint union of string modules, band modules and all projective-injective modules corresponding to the binomial relations forms a complete set of isoclasses of finite dimensional indecomposable Λ-modules.

Remark 2.6.5 ([Erd, Chapter II] and [HL])**.** (1) Let w_1 and w_2 be string paths of Λ. Then, the string modules $M(w_1)$ and $M(w_2)$ are isomorphic each other if and only if $w_2 = w_1$ or $w_2 = w_1^*$.

(2) Let $w = c_1 \cdots c_n$ be a band path. A **rotation** of w is a walk of the form $c_{i+1} \cdots c_n c_1 \cdots c_i$. Given two band paths w_1 and w_2, the band modules $N(w_1, V)$ and $N(w_2, V)$ are isomorphic each other if and only if w_2 is a rotation of w_1 or a rotation of w_1^*.

(3) A finite dimensional left $\mathbf{k}[x, x^{-1}]$-module is a finite dimensional $\mathbf{k}$-vector space together with a $\mathbf{k}$-linear automorphism f. If the module is indecomposable, then f is similar to a Jordan block

$$J(\lambda, m) := \begin{pmatrix} \lambda & 1 & \cdots & \cdots & 0 \\ 0 & \lambda & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}$$

for some $\lambda \in \mathbf{k}^\times$ and $m \in \mathbb{Z}_{>0}$.

3. ON THE SHAPES OF STABLE AUSLANDER–REITEN COMPONENTS FOR A SYMMETRIC ORDER

Auslander–Reiten theory for the lattice category latt-A was developed by many authors including M. Auslander and I. Reiten, for example see [A2, A3, A4, AR3, ARS, AS, Bu, I, I3, K4, Ro1, Ro2, RoS, Ru, Y]. In this theory, the existence theorem of almost split sequences is essential. There are two approaches to show the existence of almost split sequence, one is based on an explicit calculation of extension groups [ARS] and the other one is based on the concept of dualizing $\mathcal{O}$-varieties [AR2]. Let A be a Gorenstein $\mathcal{O}$-order. Then, the existence of almost split sequences for latt-A was studied in [A2] and [AR3]. Recall that the category latt-A admits almost split sequences if and only if A is an isolated singularity.

In this chapter, we deal with the case that A is not an isolated singularity. It follows from [AR3, Theorems 2.1 and 2.2] that an A-lattice M appears at the end of an almost split sequence if and only if $M \otimes \mathcal{K}$ is projective as $A \otimes \mathcal{K}$-module. Thus, we introduce the full subcategory of latt-A consisting of A-lattices M such that $M \otimes \mathcal{K}$ is projective in Section 3.4. The subcategory will be denoted by $\mathsf{latt}^{(\natural)}$-$A$ in this thesis, and we will introduce the stable Auslander–Reiten quiver $\Gamma_s(A)$ for $\mathsf{latt}^{(\natural)}$-$A$ when A is a symmetric $\mathcal{O}$-order in Section 3.6.

The first main result of this thesis is on the shapes of stable Auslander–Reiten components of a symmetric $\mathcal{O}$-order, which appears in [AKM] and [M1].

Main Theorem (Proposition 3.7.5, Theorems 3.7.6 and 3.7.14)**.** Let $\mathcal{C}$ be a component of the stable Auslander–Reiten quiver of a symmetric $\mathcal{O}$-order A. Assume that $\Gamma_s(A)$ has infinitely many vertices.

(1) If $\mathcal{C}$ has loops, then $\mathcal{C}$ is τ-periodic. Furthermore, $\mathcal{C} \setminus \{\text{loops}\}$ is of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$. In this case, the loops appear only at the endpoint of $\mathcal{C}$:

$$\mathcal{C} = \circlearrowleft \overset{\tau}{\bullet} \rightleftarrows \overset{\tau}{\bullet} \rightleftarrows \cdots\cdots \rightleftarrows \overset{\tau}{\bullet} \rightleftarrows \overset{\tau}{\bullet} \rightleftarrows \cdots$$

(2) If $\mathcal{C}$ has no loops, and is τ-periodic, then $\mathcal{C}$ is of the form $\mathbb{Z}T/G$, where T is a directed tree whose underlying graph is one of infinite Dynkin diagrams and G is an admissible group.

(3) If $A \otimes \kappa$ is of finite representation type, then the tree class of $\mathcal{C}$ is one of infinite Dynkin diagrams or Euclidean diagrams.

3.1 The ext group and the space of extensions

For an $\mathcal{O}$-order A and A-lattices M, N, we identify often the ext group $\mathsf{Ext}_A^1(M, N)$ and the set of equivalence classes of extensions of M by N for some equivalence relation. In this section, we recall this identification. Note that since $\mathsf{latt}\text{-}A$ is closed under extensions, $\mathsf{latt}\text{-}A$ admits push-outs and pull-backs.

Let $\mathbb{E} : 0 \to N \to E \to M \to 0$ and $\mathbb{E}' : 0 \to N \to E' \to M \to 0$ be two extensions of M by N. We write $\mathbb{E} \sim \mathbb{E}'$ when there is an A-module homomorphism $f : E \to E'$ such that the following diagrams is commutative:

$$\begin{tikzcd}
\mathbb{E}: & 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{E}': & 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

By the five lemma, the A-module homomorphism f is an isomorphism. Thus, the relation $\sim$ is an equivalence relation. We denote by $\mathsf{EXT}_A^1(M, N)$ the set of equivalence classes of the set of extensions of M by N. We write $[\mathbb{E}]$ for the equivalence class represented by $\mathbb{E}$.

For A-lattices M and N, we define $\mathbf{0}_{M,N}$ to be the canonical splittable exact sequence

$$\mathbf{0}_{M,N} : 0 \to N \xrightarrow{\binom{1_N}{0}} N \oplus M \xrightarrow{(0\ 1_M)} M \to 0.$$

Let $\mathbb{E} : 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ and $\mathbb{E}' : 0 \to N \xrightarrow{f'} E' \xrightarrow{g'} M \to 0$ be two extensions of M by N. Then, we define $[\mathbb{E}] + [\mathbb{E}']$ to be the equivalence class of the extension

$$0 \longrightarrow N \xrightarrow{f''} E'' \xrightarrow{g''} M \longrightarrow 0$$

with $E'' = U/V$, where $U = \{(x, x') \in E \oplus E' \mid g(x) = g'(x')\}$, $V = \{(f(x), -f'(x)) \mid x \in N\}$, $f''(x) = (f(x), 0) + V$ and $g''((x, x') + V) = g(x)$. As $\mathsf{latt}\text{-}A$ is closed under extensions, E'' is an A-lattice. The addition $+$ in $\mathsf{EXT}_A^1(M, N)$ is called the **Baer sum**. Then, $\mathsf{EXT}_A^1(M, N)$ is an abelian group with the Baer sum whose the zero element is $[\mathbf{0}_{M,N}]$. Let U and V be two A-lattices. For A-module homomorphisms $u : N \to U$ and $v : V \to M$, we define $u[\mathbb{E}]$ and $[\mathbb{E}]v$ by the lower exact sequence of the push-out along (f, u) and the upper exact sequence of the pull-back along (g, v), respectively.

Let $[\mathbb{E}] = [0 \to N \to E \to M \to 0] \in \mathsf{EXT}_A^1(M, N)$ and $P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \to M$ be the minimal projective presentation of M. Then, we obtain the following commutative diagram since P_0 and P_1 are projective:

$$\begin{tikzcd}
& P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] \arrow[d, dotted, "\xi_1"] & P_0 \arrow[r] \arrow[d, dotted, "\xi_0"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
{[\mathbb{E}]:} & 0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Define two maps

$$\Xi_{M,N} : \mathsf{EXT}^1_A(M,N) \to \mathsf{Ext}^1_A(M,N) = \mathsf{KerHom}_A(p_2,N)/\mathsf{ImHom}_A(p_1,N)$$

and

$$\Sigma_{M,N} : \mathsf{Ext}^1_A(M,N) \to \mathsf{EXT}^1_A(M,N)$$

by the following. The map $\Xi_{M,N}$ is given by $\Xi_{M,N}([\mathbb{E}]) = \xi_1 + \mathsf{ImHom}_A(p_1,N) \in \mathsf{Ext}^1_A(M,N)$. Let $f + \mathsf{ImHom}_A(p_1,N) \in \mathsf{Ext}^1_A(M,N)$. Since $\mathsf{Ker}(p_1) = \mathsf{Im}(p_2)$, we infer that there is $\alpha \in \mathsf{Hom}_A(\mathsf{Im}(p_1),N)$ such that f factors through α. To define $\Sigma_{M,N}$, we consider the push-out diagram along (p_1,α)

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Im}(p_1) \arrow[r, "p_1"] \arrow[d, "\alpha"] & P_0 \arrow[r] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Then, $\Sigma_{M,N}(f + \mathsf{ImHom}_A(p_1,N))$ is to be the lower exact sequence of the above commutative diagram. It is easy to show that $\Xi_{M,N}$ and $\Sigma_{M,N}$ are well-defined and the inverse map of the other. Moreover, these maps are $(\mathsf{End}_A(N), \mathsf{End}_A(M))$-bimodule homomorphisms [SY1, Chapter III, Theorem 3.5 and Proposition 3.8].

3.2 The radical of morphisms

Let A be an $\mathcal{O}$-order. For A-lattices M and N, the **radical** of $\mathsf{Hom}_A(M,N)$ is the $\mathcal{O}$-submodule of $\mathsf{Hom}_A(M,N)$ consisting of $f \in \mathsf{Hom}_A(M,N)$ such that $1_M - gf$ is invertible for any $g \in \mathsf{Hom}_A(N,M)$. The radical of $\mathsf{Hom}_A(M,N)$ is denoted by $\mathsf{rad}(M,N)$. By the definition, we have $\mathsf{rad}(M,M) = \mathsf{radEnd}_A(M)$.

Lemma 3.2.1 ([ASS, Appendix. 3.3 Lemma], [SY1, Chapter III. Lemma 1.1, Proposition 1.2])**.** Let M and N be A-lattices. Then the following statements hold.

(1) The equality

$$\mathsf{rad}(M,N) = \{f \in \mathsf{Hom}_A(M,N) \mid 1_N - fg \text{ is invertible for any } g \in \mathsf{Hom}_A(N,M)\}$$

holds.

(2) The radical $\mathsf{rad}(-,-)$ is an ideal in $\mathsf{latt}\text{-}A$.

Proof. (1) We only show that

$$\mathsf{rad}(M,N) \subset \{f \in \mathsf{Hom}_A(M,N) \mid 1_N - fg \text{ is invertible for any } g \in \mathsf{Hom}_A(N,M)\}.$$

The converse is similar. For any $f \in \mathsf{rad}(M,N)$ and any $g \in \mathsf{Hom}_A(N,M)$, there is $\varphi \in \mathsf{End}_A(M)$ such that $\varphi(1_M - gf) = (1_M - gf)\varphi = 1_M$. Then, $\psi := 1_N + f\varphi g$ is the inverse morphism of $1_N - fg$.

(2) Let $f \in \mathsf{rad}(M,N)$ and $h \in \mathsf{Hom}_A(N,L)$. For any $g \in \mathsf{Hom}_A(L,M)$, the A-module morphism $1_M - ghf$ is invertible by the definition of $\mathsf{rad}(M,N)$. Thus, $hf \in \mathsf{rad}(M,L)$. By the statement (1), for any $h \in \mathsf{Hom}_A(L,M)$, we have $fh \in \mathsf{rad}(L,N)$. □

In particular, for A-lattices $M = M_1 \oplus \cdots \oplus M_n$ and $N = N_1 \oplus \cdots \oplus N_m$, an A-module homomorphism $f = (f_{i,j}) : M \to N$ is in $\mathsf{rad}(M,N)$ if and only if $f_{i,j} : M_i \to N_j$ is in $\mathsf{rad}(M_i, N_j)$ for each i, j.

Lemma 3.2.2 ([SY1, Chapter III. Lemma 1.4])**.** Let M and N be two indecomposable A-lattices. Then, $\mathsf{rad}(M,N)$ is the set of all non-isomorphisms.

Proof. Any $f \in \mathsf{rad}(M,N)$ is a non-isomorphism. We show the converse. Let $0 \neq f \in \mathsf{Hom}_A(M,N)$ be a non-isomorphism and $g \in \mathsf{Hom}_A(N,M)$. Then gf is a non-isomorphism. Indeed, if gf is an isomorphism, $\mathsf{Im}(f)$ is a direct summand of N since f is a section. As N is indecomposable, the morphism f is surjective. Hence f is an isomorphism, a contradiction. Since $\mathsf{End}_A(M)$ is local, the morphism $1_M - gf$ is invertible in $\mathsf{End}_A(M)$. □

Let $m \geq 1$ be a positive integer. For two A-lattices M and N, we define the **m-th power of the radical** $\mathsf{rad}^m(M,N)$ to be the $\mathcal{O}$-submodule of $\mathsf{rad}(M,N)$ consisting of all finite sums of homomorphisms of the form $f_m f_{m-1} \cdots f_2 f_1$ with $f_i \in \mathsf{rad}(M_{i-1}, M_i)$ $(i = 1, \ldots, m)$ for some A-lattices $M = M_0, M_1, \ldots, M_{m-1}, M_m = N$. Clearly, $\mathsf{rad}^m(-,-)$ is an ideal in $\mathsf{latt}\text{-}A$.

3.3 Almost split sequences

In this section, we introduce the notion of almost split sequences for $\mathsf{latt}\text{-}A$.

Definition 3.3.1. (1) Let $f : L \to M$ be a morphism in $\mathsf{latt}\text{-}A$. The morphism f is called **left minimal** if every $h \in \mathsf{End}_A(M)$ with $hf = f$ is an isomorphism, and is called **left almost split** if it is not a section and every $h \in \mathsf{Hom}_A(L,W)$ which is not a section factors through f.

(2) A morphism $g : M \to N$ in $\mathsf{latt}\text{-}A$ is called **right minimal** if every $h \in \mathsf{End}_A(M)$ with $gh = g$ is an isomorphism, and is called **right almost split** if it is not a retraction and every $h \in \mathsf{Hom}_A(W,N)$ which is not a retraction factors through g.

(3) A morphism f is said to be **left minimal almost split** in latt-A if f is both left minimal and left almost split.

Similarly, a **right minimal almost split morphism** in latt-A is defined.

(4) A morphism $f \in \mathsf{Hom}_A(M, N)$ is said to be an **irreducible morphism**, provided that

(i) the morphism f is neither a section nor a retraction,

(ii) if $f = f_2 f_1$, then either f_1 is a section or f_2 is a retraction.

Lemma 3.3.2 ([ASS, Chapter IV. 1.8 Lemma])**.** The following statements hold.

(1) If $f : L \to M$ is left almost split in latt-A, then L is an indecomposable A-lattice.

(2) If $f : M \to N$ is right almost split in latt-A, then N is an indecomposable A-lattice.

Proof. We show only (1). The proof of the statement (2) is similar. Since f is not a section, we have $L \neq 0$. Suppose that $L = L_1 \oplus L_2$ with $L_1 \neq 0 \neq L_2$ as A-lattices. Let $p_i : L \to L_i$ be the canonical projection. Since $\mathsf{Ker}(p_i) = L_i \neq 0$, the morphism p_i is not a section. As f is left almost split, there exists $u_i : M \to L_i$ such that $p_i = u_i f$ for each $i = 1, 2$. Then, we have $\binom{u_1}{u_2} f = 1_L$, a contradiction. $\square$

Lemma 3.3.3 ([ASS, Chapter IV, 1.13 Lemma])**.** Let

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "v"] & N \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram in latt-A, where the rows are non-split exact sequences. Then the following statements hold.

(1) If L is indecomposable and w is an isomorphism, then u is also an isomorphism.

(2) If N is indecomposable and u is an isomorphism, then w is also an isomorphism.

Proof. We only show (1). The proof of (2) is similar. We may assume that $w = 1_N$. Let $[\delta] := [0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0] \in \mathsf{EXT}^1_A(N, L)$. Then, the lower exact sequence is $u[\delta]$. The fact that diagram is commutative means $[\delta] = u[\delta]$. As $[\delta] \neq [\mathbf{0}_{N,L}]$, we have $u \notin \mathsf{radEnd}_A(L)$. Since $\mathsf{End}_A(L)$ is local, the A-module homomorphism u is an isomorphism. $\square$

Corollary 3.3.4. Let $\delta : 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a non-split exact sequence in latt-A. Then the following statements hold.

(1) If L is indecomposable, then g is right minimal.

(2) If N is indecomposable, then f is left minimal.

Lemma 3.3.5 ([ASS, Chapter IV. 1.8 Corollary])**.** The following statements hold.

(1) If $f : L \to M$ is irreducible and injective in $\mathsf{latt}\text{-}A$ and suppose that $\mathsf{Coker}(f)$ is an A-lattice, then $\mathsf{Coker}(f)$ is indecomposable.

(2) If $f : M \to N$ is irreducible and surjective in $\mathsf{latt}\text{-}A$, then $\mathsf{Ker}(f)$ is an indecomposable A-lattice.

Proof. Let f be an irreducible morphism in $\mathsf{latt}\text{-}A$. We show that f is either a proper monomorphism or a proper epimorphism. Since f is neither a section nor a retraction, f is not an isomorphism. Assume that f is not surjective. Consider the following commutative diagram:

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr] & & M \\ & \mathsf{Im}(f) \arrow[ur] & \end{tikzcd}$$

Since the inclusion $\mathsf{Im}(f) \to M$ is not surjective and f is irreducible, the map $L \to \mathsf{Im}(f)$ is a section. In particular, f is injective.

Let $f : L \to M$ be an injective irreducible morphism. By the above arguments, f is not surjective. Hence, the cokernel of f is not zero. By the assumption, $\mathsf{Coker}(f)$ is an A-lattice. Suppose that $\mathsf{Coker}(f) = N_1 \oplus N_2$ with $N_1 \neq 0 \neq N_2$ as an A-lattice. Let $u_i : N_i \to \mathsf{Coker}(f)$ be the canonical inclusion for $i = 1, 2$ and $p : M \to \mathsf{Coker}(f)$ the canonical projection. Then, there is no A-module homomorphism $v_i : M \to N_i$ such that $p = u_i v_i$. Consider the pull-back diagram along (p, u_i):

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f'"] \arrow[d, equal] & V \arrow[r, "g'"] \arrow[d, "w"] & N_i \arrow[r] \arrow[d, "u_i"] & 0 \\ 0 \arrow[r] & L \arrow[r, "f"'] & M \arrow[r, "p"'] & \mathsf{Coker}(f) \arrow[r] & 0 \end{tikzcd}$$

Since f is irreducible, either f' is a section or w is a retraction. If w is a retraction with the right inverse morphism w', we have $p = u_i g' w'$, a contradiction. Thus, f' is a section, and hence g' is a retraction. Let g'' be the right inverse morphism of g'. Take $u'_i := wg''$. Then we have $pu'_i = u_i g' g'' = u_i$. Now, we define an A-module homomorphism $u' : \mathsf{Coker}(f) \to M$ by $u' = (u'_1, u'_2)$. Then, the A-module homomorphism u' is the right inverse morphism of p. Thus, f is a section, which contradicts with the fact that f is irreducible. The proof of (2) is similar. $\square$

Lemma 3.3.6 ([ASS, Chapter IV. 1.10 Theorem])**.** The following statements hold.

(1) If a non-zero A-module homomorphism $f : L \to M$ is left minimal almost split in $\mathsf{latt}\text{-}A$, then f is irreducible.

(2) If a non-zero A-module homomorphism $g : M \to N$ is right minimal almost split in $\mathsf{latt}\text{-}A$, then g is irreducible.

Proof. We show only (1). The proof of the statement (2) is similar. By the definition of left almost split morphisms, f is not a section. If f is a retraction, we have $L \simeq M \oplus \mathsf{Ker}(f)$. By Lemma 3.3.2, the A-lattice M is indecomposable. Since $M \neq 0$, we have $\mathsf{Ker}(f) = 0$ by the indecomposablity of L. Thus, f is an isomorphism, a contradiction. Assume that $f = f_2 f_1$ in $\mathsf{latt}\text{-}A$ and f_1 is not a section. Since f is left almost split, there exists an A-module homomorphism g in $\mathsf{latt}\text{-}A$ such that $f_1 = gf$. As f is left minimal, the A-module homomorphism $f_2 g$ is an isomorphism. Hence, f_2 is a retraction. □

We have the following proposition as the case of finite dimensional algebras.

Proposition 3.3.7 ([A2, Chapter II Proposition 4.4] and [ARS, Chapter V, Proposition 5.9])**.** For a non-split exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in $\mathsf{latt}\text{-}A$, the following statements are equivalent.

(i) f is left minimal almost split in $\mathsf{latt}\text{-}A$ and g is right minimal almost split in $\mathsf{latt}\text{-}A$.

(ii) f is left minimal almost split in $\mathsf{latt}\text{-}A$.

(iii) f is left almost split in $\mathsf{latt}\text{-}A$ and N is indecomposable.

(iv) g is right minimal almost split in $\mathsf{latt}\text{-}A$.

(v) g is right almost split in $\mathsf{latt}\text{-}A$ and L is indecomposable.

Proof. First, we show that the statements (i), (iii) and (v) are equivalent. By Lemma 3.3.2, (i) implies (iii) and (v).

Assume that (iii) holds. By Lemma 3.3.2, the A-lattice L is indecomposable. Let $u : W \to N$ in $\mathsf{latt}\text{-}A$. We claim that if there is no an A-module homomorphism $h : W \to M$ such that $u = gh$, then u is a retraction. Consider the pull-back diagram along (g, u):

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f'"] \arrow[d, equal] & V \arrow[r, "g'"] \arrow[d, "v"] & W \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

If g' is a retraction, there exists $h' : W \to V$ such that $u = gvh'$, which contradicts the choice of u. Thus, f' is not a section. Since f is left almost split in $\mathsf{latt}\text{-}A$, there is an

A-module homomorphism $w : M \to V$ such that $f' = wf$. Now, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "w"] & N \arrow[r] \arrow[d, dotted, "u'"] & 0 \\
0 \arrow[r] & L \arrow[r, "f'"'] & V \arrow[r, "g'"'] & W \arrow[r] & 0
\end{tikzcd}$$

Since N is indecomposable, uu' is an isomorphism by Lemma 3.3.3. Therefore, u is a retraction, hence (v) follows. Similarly, (v) implies (iii).

Assume that (iii) and (v) hold. Then, Corollary 3.3.4 implies that f is left minimal and g is right minimal. Therefore, the statements (i), (iii) and (v) are equivalent.

Clearly, (i) implies (ii) and (iv). We show that (ii) implies (iii). It is enough to show that N is indecomposable. However, the claim follows easily from Lemmas 3.3.5 and 3.3.6. Similarly, (iv) implies that (v). □

An almost split sequence is a special kind of short exact sequences. Among equivalent conditions in Proposition 3.3.7, we choose (v) as the definition of almost split sequences for $\mathsf{latt}\text{-}A$.

Definition 3.3.8 ([Ro2])**.** Let L, M and N be A-lattices. A short exact sequence in $\mathsf{latt}\text{-}A$

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\quad g \quad} N \longrightarrow 0$$

is called an **almost split sequence ending at** N if the following two conditions are satisfied:

(i) The morphism g is right almost split in $\mathsf{latt}\text{-}A$.

(ii) The A-lattice L is indecomposable.

Lemma 3.3.9 ([ARS, Chapter V. Proposition 5.9])**.** For an exact sequence $\mathbb{E} : 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$, the following two conditions are equivalent.

(i) The exact sequence $\mathbb{E}$ is an almost split sequence.

(ii) The A-lattices L and N are indecomposable and the A-module homomorphisms f and g are irreducible.

Proof. It follows from Lemma 3.3.6 and Proposition 3.3.7 that (i) implies (ii). We show the converse. As g is irreducible, it is not a retraction. Let $v : W \to N$ in $\mathsf{latt}\text{-}A$ such that v is not a retraction. We show that the A-module homomorphism v factors through g. We may assume that W is indecomposable. By the proof of Lemma 3.3.5, one of the following statements holds since f is irreducible:

(a) There is $u : W \to M$ in $\mathsf{latt}\text{-}A$ such that $v = gu$.

(b) There is $u' : M \to W$ in $\mathsf{latt}\text{-}A$ such that $g = vu'$.

In the first case, there is noting to prove. Assume the second case. By the definition of irreducible morphisms, u' is a section since v is not a retraction. Thus, we have $W = M \oplus \mathsf{Coker}(u')$ Since $M \neq 0$ and W is indecomposable, we have $\mathsf{Coker}(u') = 0$. Therefore, u' is an isomorphism. But then we have $v = gu^{-1}$. □

Lemma 3.3.10 ([SY1, Chapter III. Lemma 8.2])**.** Let $\mathbb{E} : 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ and $\mathbb{E}' : 0 \to L' \xrightarrow{f'} M' \xrightarrow{g'} N' \to 0$ be almost split sequences for $\mathsf{latt}\text{-}A$. Then the following statements are equivalent.

(i) $\mathbb{E}$ and $\mathbb{E}'$ are isomorphic as short exact sequences.

(ii) L and L' are isomorphic as A-lattices.

(iii) N and N' are isomorphic as A-lattices.

Proof. Clearly, (i) implies (ii) and (iii). We show that (ii) implies (i). Let $u : L \to L'$ be an isomorphism. Since f is left almost split and $f'u$ is not a section, there exists $v : M \to M'$ such that $f'u = vf$. Similarly, there exists $v' : M' \to M$ such that $fu^{-1} = v'f$. Then, v is an isomorphism because f and f' are left minimal. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "v"] & N \arrow[r] \arrow[d, dotted, "w"] & 0 \\
0 \arrow[r] & L' \arrow[r, "f'"'] & M' \arrow[r, "g'"'] & N' \arrow[r] & 0
\end{tikzcd}$$

Since u and v are isomorphisms, so is w. The proof that (iii) implies (i) is similar. □

It follows from Lemma 3.3.10 that an almost split sequence is uniquely determined by the starting term, and is also uniquely determined by the ending term.

Definition 3.3.11. Let A be a Gorenstein $\mathcal{O}$-order and $0 \to L \to M \to N \to 0$ an almost split sequence. We define the **AR translations** τ and τ^{-1} by $\tau(N) = L$ and $\tau^{-1}(L) = M$.

3.4 The existence of almost split sequences

The existence of almost split sequences was first studied by M. Auslander and I. Reiten around 1971 for Artin algebras of finite representation type to analyze "Auslander algebras" [AR2]. Let Λ be a finite dimensional algebra over a field $\mathbf{k}$. In this case, it is well-known that for every indecomposable non-projective finite dimensional Λ-module N, there exists the almost split sequence ending at N. Dually, for every indecomposable non-injective finite dimensional Λ-module L, there exists the almost split sequence starting at N. For example, see [ASS, Chapter IV. 3.1 Theorem] or [SY1, Chapter III. Theorem 8.4].

Let A be a Gorenstein $\mathcal{O}$-order. Recall that, according to [A3], the category latt-A admits almost split sequences if and only if A is an isolated singularity. When A is not an isolated singularity, we have to consider a suitable full subcategory of latt-A which admits almost split sequences. In this case, the existence of almost split sequences had been given in [AR3].

Theorem 3.4.1 ([AR3, Theorems 2.1 and 2.2])**.** Let A be a Gorenstein $\mathcal{O}$-order, M a non-projective indecomposable A-lattice and N a non-injective indecomposable A-lattice. Then, there exists the almost split sequence ending at M if and only if $M \otimes \mathcal{K}$ is projective as an $A \otimes \mathcal{K}$-module. Dually, there exists the almost split sequence starting at N if and only if $N \otimes \mathcal{K}$ is injective as an $A \otimes \mathcal{K}$-module.

We denote by $\mathsf{latt}^{(\natural)}$-$A$ the full subcategory of latt-A consisting of A-lattices M such that $M \otimes \mathcal{K}$ is a projective $A \otimes \mathcal{K}$-module. If $A \otimes \mathcal{K}$ is a self-injective $\mathcal{K}$-algebra, the category $\mathsf{latt}^{(\natural)}$-$A$ admits almost split sequences. Moreover, Theorem 3.4.1 also implies that almost split sequences in $\mathsf{latt}^{(\natural)}$-$A$ are also almost split sequences in latt-A. Obviously, any finitely generated projective A-module belongs to $\mathsf{latt}^{(\natural)}$-$A$. We give other examples of A-lattices in $\mathsf{latt}^{(\natural)}$-$A$.

Definition 3.4.2. Let A be a Gorenstein $\mathcal{O}$-order and M be an indecomposable $\overline{A}$-module. We call each direct summand of $\Omega(M)$ a **Heller lattice** of M.

Remark 3.4.3. The Heller lattice $\Omega(M)$ of an indecomposable $\overline{A}$-module M may not be an indecomposable A-lattice. For instance, we consider a p-modular system $(\mathcal{K}, \mathcal{O}, \kappa)$ of finite p-group G. Let φ be a valuation of the p-modular system. Let κ_G be the trivial κG-module. Then, the Heller lattice of κ_G is $\mathsf{rad}(\mathcal{O}G)$, and it is well-known that $\mathsf{rad}(\mathcal{O}G)$ is decomposable if and only if $|G| = p$ and $\varphi(p) = 1$.

Lemma 3.4.4 ([AKM, Remark 1.12])**.** Any Heller lattice belongs to $\mathsf{latt}^{(\natural)}$-$A$.

Proof. Let M be an indecomposable $\overline{A}$-module. Take the projective cover $P \xrightarrow{p_M} M$ in latt-A. Let Z_M be a Heller lattice of M. By the definition of Heller lattices, Z_M is an A-submodule of P. Hence, we have $Z_M \otimes \mathcal{K} \subset P \otimes \mathcal{K}$. On the other hand, $p_M(\varepsilon P) = 0$ yields that εP is contained in Z_M. Thus, we have $P \otimes \mathcal{K} \subset Z_M \otimes \mathcal{K}$. $\square$

Corollary 3.4.5. For any non-projective indecomposable Heller lattice Z, there exists the almost split sequence ending at Z.

Remark 3.4.6. Heller lattices over group algebras are studied by S. Kawata [K3, K4]. Assume that $(\mathcal{K}, \mathcal{O}, \kappa) \supset (\mathcal{K}', \mathcal{O}', \kappa')$ is an extension of p-modular systems of a finite group G, $\kappa = \kappa'$ are algebraically closed and the ramification index is larger than 1 [P]. Then, any

Heller lattice of $\mathcal{O}G$ is indecomposable, and it appears on the boundary of the Auslander–Reiten quiver. Let M be a non-projective indecomposable κG-module, and Z_M the Heller lattice of M. If we apply $-\otimes_{\mathcal{O}} \kappa$ to the almost split sequence ending at the Heller lattice Z_M, then it is direct sum of the almost split sequence ending at M and a splitable exact sequence. Thus, Heller lattices over a group algebra are the most important lattices in modular representation theory of groups.

3.5 The Construction of almost split sequences

It is natural to ask how to construct almost split sequences. In fact, any almost split sequence is obtained by taking a suitable pull-back diagram [AKM, Proposition 1.15]. In this section, we explain how to construct almost split sequences in $\mathsf{latt}^{(\natural)}\text{-}A$. This construction is a generalization of [Th], and it was explained in [Ta] and [AKM, Appendix].

Set $\mathsf{D}' := \mathsf{Hom}_{\mathcal{O}}(-, \mathcal{K})$ and $\mathsf{D}'' := \mathsf{Hom}_{\mathcal{O}}(-, \mathcal{K}/\mathcal{O})$. We also set $\nu = \mathsf{D}\mathsf{Hom}_A(-, A)$, $\nu' = \mathsf{D}'\mathsf{Hom}_A(-, A)$ and $\nu'' = \mathsf{D}''\mathsf{Hom}_A(-, A)$. The functor $\nu : \mathsf{latt}\text{-}A \to \mathsf{latt}\text{-}A$ is called the **Nakayama functor**, and it induces an equivalence between $\mathsf{proj}\text{-}A$ and $\mathsf{inj}\text{-}A$. In particular, there are bijections between the following three sets [Lu, Proposition 2.8]:

(i) The set of all indecomposable projective A-lattices.

(ii) The set of all indecomposable injective A-lattices.

(iii) The set of all simple modules.

Consider the injective resolution of $\mathcal{O}$ as an $\mathcal{O}$-module:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\iota} \mathcal{K} \xrightarrow{p} \mathcal{K}/\mathcal{O} \longrightarrow 0$$

Lemma 3.5.1. For an A-lattice M, we have the exact sequence $0 \to \nu(M) \to \nu'(M) \to \nu''(M) \to 0$.

Proof. By applying the functor $\mathsf{Hom}_{\mathcal{O}}(X, -)$, where X is an A-lattice, to the sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$, we have

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{O}}(X, \mathcal{O}) \longrightarrow \mathsf{Hom}_{\mathcal{O}}(X, \mathcal{K}) \longrightarrow \mathsf{Hom}_{\mathcal{O}}(X, \mathcal{K}/\mathcal{O}) \longrightarrow \mathsf{Ext}^1_{\mathcal{O}}(X, \mathcal{O}) = 0.$$

In particular, if we take $X = \mathsf{Hom}_A(M, A)$, where M is an A-lattice, we obtain

$$0 \longrightarrow \mathsf{D}(\mathsf{Hom}_A(M, A)) \longrightarrow \mathsf{D}'(\mathsf{Hom}_A(M, A)) \longrightarrow \mathsf{D}''(\mathsf{Hom}_A(M, A)) \longrightarrow 0$$

as required. □

Note that we had also obtained the exact sequence of functors

$$0 \longrightarrow \mathsf{D}(-) \longrightarrow \mathsf{D}'(-) \longrightarrow \mathsf{D}''(-) \longrightarrow 0.$$

Let M be an A-lattice, and let $\lambda : \mathsf{D}((\mathsf{Hom}_A(M, A) \otimes_A -)) \xrightarrow{\sim} \mathsf{Hom}_A(-, \nu(M))$ be the functorial isomorphism. We define λ' and λ'' in the similar manner by replacing ν with ν' and ν'', respectively. We also define the natural transformation

$$\mu_M : \mathsf{Hom}_A(M, A) \otimes_A - \longrightarrow \mathsf{Hom}_A(M, -)$$

by $\mu_M(X)(f \otimes x) = (m \mapsto f(m)x)$. Then, the functor μ_M induces the following three morphisms of functors:

$$\begin{aligned}
\mathsf{D}\mu_M &: \mathsf{D}(\mathsf{Hom}_A(M, -)) \longrightarrow \mathsf{D}(\mathsf{Hom}_A(M, A) \otimes_A -) \\
\mathsf{D}'\mu_M &: \mathsf{D}'(\mathsf{Hom}_A(M, -)) \longrightarrow \mathsf{D}'(\mathsf{Hom}_A(M, A) \otimes_A -) \\
\mathsf{D}''\mu_M &: \mathsf{D}''(\mathsf{Hom}_A(M, -)) \longrightarrow \mathsf{D}''(\mathsf{Hom}_A(M, A) \otimes_A -)
\end{aligned}$$

Lemma 3.5.2. We have the following commutative diagram of functors.

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{DHom}_A(M,-) \arrow[r, "\iota_*"] \arrow[d, "\lambda\circ\mathsf{D}\mu_M"] & \mathsf{D}'\mathsf{Hom}_A(M,-) \arrow[r, "p_*"] \arrow[d, "\lambda'\circ\mathsf{D}'\mu_M"] & \mathsf{D}''\mathsf{Hom}_A(M,-) \arrow[r] \arrow[d, "\lambda''\circ\mathsf{D}''\mu_M"] & 0 \\
0 \arrow[r] & \mathsf{Hom}_A(-,\nu(M)) \arrow[r, "\iota_*"] & \mathsf{Hom}_A(-,\nu'(M)) \arrow[r, "p_*"] & \mathsf{Hom}_A(-,\nu''(M)) &
\end{tikzcd}$$

Here, $p_* = p \circ -$ and $\iota_* = \iota \circ -$.

Proof. For an A-lattice N, it follows from Lemma 3.5.2 and the left exactness of $\mathsf{Hom}_A(N, -)$ that the rows are exact. Thus, it is enough to show that the diagram is commutative. Note that the Hom-tensor adjointness

$$\lambda_N : \mathsf{D}(\mathsf{Hom}_A(M, A) \otimes_A N) \xrightarrow{\sim} \mathsf{Hom}_A(N, \nu(M))$$

is given by $f \mapsto [x \mapsto (y \mapsto f(y \otimes x))]$. By replacing D with D' and D'', we have explicit formulas of λ'_N and λ''_N, respectively. Thus, for $f \in \mathsf{DHom}_A(M, N)$, we have

$$\lambda'_N \circ \mathsf{D}'\mu_M(N) \circ \iota_*(f) = [x \mapsto (y \mapsto \iota f \mu_M(N))(y \otimes x)] = \iota_{*N} \circ \lambda_N \circ \mathsf{D}\mu_M(N)(f).$$

Similarly, the right squire in the diagram is commutative. $\square$

Lemma 3.5.3. Let M be an object in $\mathsf{latt}^{(\natural)}$-$A$. Then, we have a functorial isomorphism

$$\mathsf{Hom}_{A\otimes\mathcal{K}}(M \otimes \mathcal{K}, A \otimes \mathcal{K}) \otimes_A - \simeq \mathsf{Hom}_{A\otimes\mathcal{K}}(M \otimes \mathcal{K}, - \otimes \mathcal{K}).$$

Proof. The statement follows immediately since $M \otimes \mathcal{K}$ is projective. $\square$

Corollary 3.5.4. Let M be an object in $\mathsf{latt}^{(\natural)}$-$A$ and X an A-lattice. Then, $\mathsf{Coker}(\mu_M(X))$ is a torsion $\mathcal{O}$-module.

Proof. Let $q : \mathsf{Coker}(\mu_M(X)) \to \mathcal{K} \otimes \mathsf{Coker}(\mu_M(X))$ be the canonical homomorphism. We show that $\mathsf{Ker}(q) = \mathsf{Coker}(\mu_M(X))$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_A(M,A) \otimes_A X \arrow[r, "\mu_M(X)"] \arrow[d] & \mathsf{Hom}_A(M,X) \arrow[r] \arrow[d] & \mathsf{Coker}(\mu_M(X)) \arrow[r] \arrow[d, "q"] & 0 \\
\mathcal{K} \otimes \mathsf{Hom}_A(M,A) \otimes_A X \arrow[r] \arrow[d, "\simeq"] & \mathcal{K} \otimes \mathsf{Hom}_A(M,X) \arrow[r] \arrow[d, "\simeq"] & \mathcal{K} \otimes \mathsf{Coker}(\mu_M(X)) \arrow[r] \arrow[d, "1"] & 0 \\
\mathsf{Hom}_{A\otimes\mathcal{K}}(M\otimes\mathcal{K}, A\otimes\mathcal{K}) \otimes_A X \arrow[r, "\simeq"] & \mathsf{Hom}_{A\otimes\mathcal{K}}(M\otimes\mathcal{K}, X\otimes\mathcal{K}) \arrow[r] & \mathcal{K} \otimes \mathsf{Coker}(\mu_M(X)) \arrow[r] & 0
\end{tikzcd}$$

Thus, we conclude that $\mathcal{K} \otimes \mathsf{Coker}(\mu_M(X)) = 0$, and $q = 0$. □

Lemma 3.5.5 ([AKM, Lemma A.1])**.** *Let X be an A-lattice and $M \in \mathsf{latt}^{(\natural)}\text{-}A$. Then, the following statements hold.*

(1) $\mathsf{D}'\mu_M(X)$ *is an isomorphism and natural in X.*

(2) $\mathsf{D}\mu_M(X)$ *is a monomorphism and natural in X.*

(3) *If M is a projective A-module, then $\mathsf{D}\mu_M(X)$ is an isomorphism.*

(4) $\mathsf{D}''\mu_M(X)$ *is an epimorphism and natural in X.*

(5) *The sequence*

$$\mathsf{D}\mathsf{Hom}_A(M,X) \xrightarrow{\lambda_X \circ \mathsf{D}\mu_M(X)} \mathsf{Hom}_A(X, \nu(M)) \xrightarrow{p_* \circ (\lambda'_X \circ \mathsf{D}'\mu_M(X))^{-1} \circ \iota_*} \mathsf{D}''\mathsf{Hom}_A(M,X)$$

is exact.

Proof. (1) Since $\mathsf{Coker}(\mu_M(X))$ is a torsion $\mathcal{O}$-module, $\mathsf{D}'\mathsf{Coker}(\mu_M(X)) = 0$. Indeed, if there is a non-zero $\mathcal{O}$-module homomorphism $f \in \mathsf{D}'(\mathsf{Coker}(\mu_M(X)))$, we can take $x \in \mathsf{Coker}(\mu_M(X))$ such that $f(x) \neq 0$. Let $r \in \mathcal{O}$ be a non-zero element of $\mathcal{O}$ such that $rx = 0$. Then, we have $rf(x) = f(rx) = 0$, a contradiction. Thus, we have

$$0 \to \mathsf{D}'\mathsf{Hom}_A(M,X) \xrightarrow{\mathsf{D}'\mu_M(X)} \mathsf{D}'(\mathsf{Hom}_A(M,A) \otimes_A X) \to \mathsf{Ext}^1_{\mathcal{O}}(\mathsf{Coker}(\mu_M(X)), \mathcal{K}).$$

As $\mathcal{K}$ is an injective $\mathcal{O}$-module, we conclude that $\mathsf{D}'\mathsf{Hom}_A(M,X) \simeq \mathsf{D}'(\mathsf{Hom}_A(M,A) \otimes_A X)$.

(2) As $\mathsf{Coker}(\mu_M(X))$ is a torsion $\mathcal{O}$-module, we have also $\mathsf{D}\mathsf{Coker}(\mu_M(X)) = 0$. Hence, (2) follows.

(3) If M is projective, the functor μ_M is an isomorphism. Thus, $\mathsf{D}\mu_M$ is also isomorphism since D induces the duality between $\mathsf{latt}\text{-}A$ and $\mathsf{latt}\text{-}A^{\mathrm{op}}$.

(4) As $\mathcal{K}/\mathcal{O}$ is also an injective $\mathcal{O}$-module, we have $\mathsf{Ext}^1_{\mathcal{O}}(\mathsf{Coker}(\mu_M(X)), \mathcal{K}/\mathcal{O}) = 0$. Thus, the statement (4) holds.

(5) The exactness of the sequence follows from (1) and the commutative diagram in Lemma 3.5.2. □

Lemma 3.5.6 ([AKM, Lemma A.2])**.** Let M be an A-lattice and $p_M : P \to M$ the projective cover. Let $L = \mathsf{D}(\mathsf{Coker}(\mathsf{Hom}_A(p_M, A)))$. Then, the following sequences are exact in $\mathsf{latt}\text{-}A$.

$$0 \longrightarrow L \longrightarrow \nu(P) \xrightarrow{\nu(p_M)} \nu(M) \longrightarrow 0$$
$$0 \longrightarrow \mathsf{DHom}_A(M, -) \xrightarrow{\lambda\circ\mathsf{D}\mu_M(-)} \mathsf{Hom}_A(-, \nu(M)) \longrightarrow \mathsf{Ext}^1_A(-, L) \longrightarrow 0$$

Proof. By applying the functor D to the exact sequence

$$0 \longrightarrow \mathsf{Hom}_A(M, A) \xrightarrow{\mathsf{Hom}_A(p_M,A)} \mathsf{Hom}_A(P, A) \longrightarrow \mathsf{Coker}(\mathsf{Hom}_A(p_M, A)) \longrightarrow 0,$$

we have the exact sequence

$$0 \longrightarrow L \longrightarrow \nu(P) \xrightarrow{\nu(p_M)} \nu(M) \longrightarrow 0.$$

Since ν gives an equivalence $\mathsf{latt}\text{-}A \to \mathsf{latt}\text{-}A$, $\nu(P)$ is an A-lattice, so is L.

For any A-lattice X, we obtain the exact sequence

$$\mathsf{Hom}_A(X, \nu(P)) \longrightarrow \mathsf{Hom}_A(X, \nu(M)) \longrightarrow \mathsf{Ext}^1_A(X, L) \longrightarrow \mathsf{Ext}^1_A(X, \nu(P)) = 0$$

since $\nu(P)$ is an injective A-lattice. Thus, by Lemma 3.5.5, we have the following diagram with exact rows.

$$\begin{tikzcd}
& \mathsf{Hom}_A(X,\nu(P)) \arrow[r, "\nu(p_M)_*:=\mathsf{Hom}_A(X,\nu(p_M))"] & \mathsf{Hom}_A(X,\nu(M)) \arrow[r] \arrow[d, equal] & \mathsf{Ext}^1_A(X,L) \arrow[r] & 0 \\
0 \arrow[r] & \mathsf{DHom}_A(M,X) \arrow[r, "\lambda_X\circ\mathsf{D}\mu_M(X)"'] & \mathsf{Hom}_A(X,\nu(M)) \arrow[r] & \mathsf{D}''\mathsf{Hom}_A(M,X) &
\end{tikzcd}$$

We show that $\nu(p_M)_*$ factors through $\lambda_X \circ \mathsf{D}\mu_M(X)$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Hom}_A(M,A)\otimes_A X \arrow[r, "p_M^*\otimes\mathrm{id}_X"] \arrow[d, "\mu_M(X)"] & \mathsf{Hom}_A(P,A)\otimes_A X \arrow[d, "\mu_P(X)"] \\
\mathsf{Hom}_A(M,X) \arrow[r, "p_M^*:=\mathsf{Hom}_A(p_M,X)"] & \mathsf{Hom}_A(P,X)
\end{tikzcd}$$

By applying D to the above diagram, we obtain the commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_A(X,\nu(M)) & \mathsf{Hom}_A(X,\nu(P)) \arrow[l, "\nu(p_M)_*"'] \\
\mathsf{DHom}_A(M,X) \arrow[u, "\lambda_X\circ\mathsf{D}(\mu_M(X))"] & \mathsf{DHom}_A(P,X), \arrow[l, "\mathsf{D}p_M^*"] \arrow[u, "\lambda_X\circ\mathsf{D}(\mu_P(X))"']
\end{tikzcd}$$

and $\lambda_X \circ \mathsf{D}(\mu_P(X))$ is an isomorphism. Therefore, $\nu(p_M)_*$ factors through $\lambda_X \circ \mathsf{D}\mu_M(X)$. Since $\mathsf{D}p_M^*$ is an epimorphism, the image of $\nu(p_M)_*$ coincides with $\mathsf{Im}(\lambda_X \circ \mathsf{D}(\mu_M(X)))$, and we get the desired exact sequence. $\square$

It follows from Lemma 3.5.6, we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Im}(\nu(p)_*) \arrow[r] \arrow[d, equal] & \mathsf{Hom}_A(X,\nu(M)) \arrow[r] \arrow[d, equal] & \mathsf{Ext}_A^1(X,L) \arrow[r] \arrow[d, dotted, "\exists t_{L,M}^X"] & 0 \\
0 \arrow[r] & \mathsf{Im}(\lambda_X \circ \mathsf{D}(\mu_M(X))) \arrow[r] & \mathsf{Hom}_A(X,\nu(M)) \arrow[r] & \mathsf{D}''\mathsf{Hom}_A(M,X) &
\end{tikzcd}$$

We notice that the A-module homomorphism $t_{L,M}^X : \mathsf{Ext}_A^1(X,L) \to \mathsf{D}''\mathsf{Hom}_A(M,X)$ is injective.

Corollary 3.5.7. Let M be an indecomposable A-lattice and $p_M : P \to M$ the projective cover. Then, $\mathsf{soc}(\mathsf{D}''\mathsf{End}_A(M))$ is a simple $\mathsf{End}_A(M)$-module, and there is an isomorphism

$$\mathsf{soc}(\mathsf{Ext}_A^1(M,L)) \simeq \{f \in \mathsf{D}''(\mathsf{End}_A(M)) \mid f(\mathsf{radEnd}_A(M)) = 0\}.$$

Now, we are ready to give the construction of almost split sequences for $\mathsf{latt}^{(\natural)}$-$A$.

Theorem 3.5.8 ([AKM, Proposition 1.15])**.** Suppose that A is a Gorenstein $\mathcal{O}$-order and M is an indecomposable non-projective A-lattice belonging to $\mathsf{latt}^{(\natural)}$-$A$. Let $p_M : P \to M$ be the projective cover. For $\varphi \in \mathsf{Hom}_A(M,\nu(M))$, we consider the pull-back diagram along $(\nu(p_M),\varphi)$:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & L \arrow[r] & \nu(P) \arrow[r, "\nu(p_M)"] & \nu(M) \arrow[r] & 0
\end{tikzcd}$$

Then the following (1) and (2) are equivalent.

(1) The upper exact sequence $0 \to L \to E \to M \to 0$ is an almost split sequence.

(2) The following three conditions hold.

- (a) φ does not factor through $\nu(p_M)$.
- (b) The A-lattice L is an indecomposable.
- (c) For all $h \in \mathsf{radEnd}_A(M)$, φh factors through $\nu(p_M)$.

Moreover, any almost split sequence is given in this way.

Proof. Assume that the upper exact sequence $0 \to L \to E \to M \to 0$ is an almost split sequence. Suppose that $\varphi = \nu(p_M)s$ for some $s : M \to \nu(P)$. Then, we obtain the following commutative diagram by the universality of the pull-back:

$$\begin{tikzcd}
M \arrow[dr, dotted] \arrow[drr, bend left, "1_M"] \arrow[ddr, bend right, "s"'] & & \\
& E \arrow[r, "g"] \arrow[d] & M \arrow[d, "\varphi"] \\
& \nu(P) \arrow[r, "\nu(p_M)"] & \nu(M)
\end{tikzcd}$$

Thus, g is a retraction, which contradicts with (1). Hence, (a) follows. The condition (b) follows from the definition of almost split sequences. Let $h \in \mathsf{radEnd}_A(M)$. Since M is indecomposable, h is not a retraction. Thus, h factors through the A-module homomorphism $g : E \to M$ since g is right almost split. It implies that φh factors though $\nu(p_M)$. Therefore, (1) implies (2).

We show the converse. Assume that the conditions (a), (b) and (c) hold. By (a), the exact sequence $0 \to L \to E \to M \to 0$ does not split. Since L is indecomposable by the condition (b), it is enough to we show that g is right almost split. As g is not a retraction, we only prove that any A-module homomorphism $h : X \to M$ in latt-A which is not a retraction factors through g. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & F_X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & L \arrow[r] & \nu(P) \arrow[r, "\nu(p_M)"'] & \nu(M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where the first row is the pull-back along (g, h). Let $[\mathbb{E}]$ be the equivalence class in $\mathsf{EXT}^1_A(M, L)$ represented by $0 \to L \to E \to M \to 0$. Then, the equivalence class of the first row in $\mathsf{EXT}^1_A(M, L)$ is $[\mathbb{E}]h$, and h factors through g if and only if $[\mathbb{E}]h = [\mathbf{0}_{X,L}]$. Since M is indecomposable, the condition (c) is equivalent to $[\mathbb{E}]\psi = [\mathbf{0}_{M,L}]$ for any $\psi \in \mathsf{radEnd}_A(M)$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Ext}^1_A(M, L) \arrow[r, "t^M_{L,M}"] \arrow[d, "\mathsf{Ext}^1_A(h,L)"'] & \mathsf{D}''\mathsf{Hom}_A(M, M) \arrow[d, "\mathsf{D}''\mathsf{Hom}_A(M,h)"] \\
0 \arrow[r] & \mathsf{Ext}^1_A(X, L) \arrow[r, "t^X_{L,M}"'] & \mathsf{D}''\mathsf{Hom}_A(M, X).
\end{tikzcd}$$

Let $\xi := t^M_{L,M}(\Xi_{M,L}([\mathbb{E}]))$. Since $h \in \mathsf{rad}(X, M)$, we have $hm \in \mathsf{radEnd}_A(M)$ for all $m \in \mathsf{Hom}_A(M, X)$. Thus,

$$\mathsf{D}''\mathsf{Hom}_A(M, h)(\xi)(m) = \xi(hm) = t^M_{L,M}(\Xi_{M,L}([\mathbb{E}]))(hm) = t^M_{L,M}(\Xi_{M,L}([\mathbb{E}](hm))) = 0$$

holds for all $m \in \mathsf{Hom}_A(M, X)$, and hence we have $\mathsf{Ext}^1_A(h, L)(\Xi_{M,L}(\mathbb{E})) = 0$. This implies that $[\mathbb{E}]h = [\mathbf{0}_{X,L}]$. By the proof of Lemma 3.5.6, any almost split sequence is given in this way since there is a surjection $\mathsf{Hom}_A(M, \nu(M)) \to \mathsf{Ext}^1_A(M, L)$. □

Recall that $\mathcal{O}$ is a complete discrete valuation ring.

Definition 3.5.9. Let M be an A-lattice and $Q \xrightarrow{q} P \xrightarrow{p} M$ the minimal projective resolution of M. Then, we define the **transpose** $\mathsf{Tr}(M)$ of M as the cokernel of the A-module homomorphism $\mathsf{Hom}_A(P, A) \to \mathsf{Hom}_A(Q, A)$.

Corollary 3.5.10. If A is a Gorenstein $\mathcal{O}$-order, then we have a functorial isomorphism $\tau \simeq \Omega\nu$. In particular, if A is symmetric, then there is a functorial isomorphism $\tau \simeq \Omega$.

Proof. Let M be an A-lattice in $\mathsf{latt}^{(\natural)}$-$A$ and let $Q \xrightarrow{q} P \xrightarrow{p} M \to 0$ be the minimal projective presentation of M. Then, it follows from Lemma 3.5.6 that we have the following exact sequence in latt-A:

$$\mathbb{E}: \quad 0 \longrightarrow \mathsf{D}(\mathsf{Coker}(\mathsf{Hom}_A(p, A))) \longrightarrow \nu(P) \xrightarrow{\nu(p)} \nu(M) \longrightarrow 0$$

On the other hand, we have the exact sequence

$$\mathbb{E}': \quad 0 \longrightarrow \mathsf{Coker}(\mathsf{Hom}_A(p, A)) \longrightarrow \mathsf{Hom}_A(Q, A) \longrightarrow \mathsf{Tr}(M) \longrightarrow 0.$$

Since the exact sequence $\mathbb{E}'$ is the projective cover of $\mathsf{Tr}(M)$, we have $\mathsf{D}\Omega\mathsf{Tr}(M) = \tau(M)$. The exact sequence $\mathbb{E}$ implies that $\mathsf{DCoker}(\mathsf{Hom}_A(p, A)) = \Omega(\nu(M))$. Therefore, we have $\tau \simeq \mathsf{D}\Omega\mathsf{Tr} \simeq \Omega\nu$. □

Remark 3.5.11 ([Hap])**.** As A is a Gorenstein $\mathcal{O}$-order, the Nakayama functor $\nu : \mathsf{latt}$-$A \to \mathsf{latt}$-A is an autofunctor, and $\underline{\mathsf{latt}}$-$A$ is a Frobenius category. Hence, $\underline{\mathsf{latt}}$-$A$ is a triangulated category with the shift functor Ω^{-1}. Then, we have a triangulated equivalence $\nu : \underline{\mathsf{latt}}\text{-}A \xrightarrow{\sim} \underline{\mathsf{latt}}\text{-}A$, and the AR translation τ is represented by $\Omega\nu$.

Remark 3.5.12 ([A2, I5])**.** Let R be a noetherian complete local ring with $\mathsf{Kr\text{-}dim}(R) = d$. If an R-algebra Λ is Cohen–Macaulay as an R-module, then Λ is also called an R-order. Assume that Λ is an isolated singularity. We denote by $\mathsf{CM}(\Lambda)$ the category of Cohen–Macaulay Λ-modules, that is, the objects of $\mathsf{CM}(\Lambda)$ are Λ-modules which are Cohen–Macaulay as R-modules. Then, we have a duality

$$\Omega^d\mathsf{Tr} : \underline{\mathsf{CM}}(\Lambda) \longleftrightarrow \underline{\mathsf{CM}}(\Lambda^{\mathrm{op}}),$$

where Ω is the syzygy functor. In this setting, the AR translation τ_d is given by $\mathsf{Hom}_R(-, R)\Omega^d\mathsf{Tr}$. If $d = 0$, then R is a field, and $\mathsf{CM}(\Lambda)$ is just mod-Λ. If $d = 1$, then R is a complete discrete valuation ring, and $\mathsf{CM}(\Lambda)$ is just latt-Λ.

Let A be a symmetric $\mathcal{O}$-order. Since A is symmetric, the AR translation τ is the syzygy founctor on $\underline{\mathsf{latt}}$-$A$. Thus, it is an additive functor and, for an A-lattice M, $\tau(M) = 0$ if and only if M is a projective A-module. Furthermore, since A is symmetric, if M is a non-projective indecomposable A-lattice if and only if $\tau(M)$ is a non-projective indecomposable A-lattice. Let M be a non-projective indecomposable A-lattice in $\mathsf{latt}^{(\natural)}$-$A$. Then, the middle term of an almost split sequence may have projective direct summands.

Theorem 3.5.13 ([A2, Chapter III, Theorem 2.5])**.** Let A be a Gorenstein $\mathcal{O}$-order. Suppose that M is a non-projective indecomposable A-lattice in $\mathsf{latt}^{(\natural)}$-$A$. Let $0 \to \tau M \to E \to M \to 0$ be the almost split sequence ending at M. Then the following are equivalent.

(a) The A-lattice E has a projective indecomposable direct summand.

(b) If $X \to M$ is a surjection in $\mathsf{latt}^{(\natural)}$-$A$ which is not a retraction, then X has a projective indecomposable summand.

(c) $\tau(M)$ is isomorphic to a non-projective direct summand of $\mathsf{rad}P$ for some indecomposable projective A-module P.

3.6 Stable Auslander–Reiten quivers

Throughout this section, we assume that A is a symmetric $\mathcal{O}$-order. The definition of the stable Auslander–Reiten quiver for $\mathsf{latt}^{(\natural)}$-$A$ is given as follows.

Definition 3.6.1. Let A be a symmetric $\mathcal{O}$-order.

(1) The **stable Auslander–Reiten quiver** for $\mathsf{latt}^{(\natural)}$-$A$ is the valued stable translation quiver $(\Gamma_s(A), \tau)$ defined as follows:

- The set of vertices is a complete set of isoclasses of non-projective indecomposable A-lattices in $\mathsf{latt}^{(\natural)}$-$A$.
- We draw a valued arrow $M \xrightarrow{(a,b)} N$ whenever there exist irreducible morphisms $M \to N$, where the valuation (a, b) means:
  (i) a is the multiplicity of M in the middle term of the almost split sequence ending at N.
  (ii) b is the multiplicity of N in the middle term of the almost split sequence starting at M.
- The translation τ is the AR translation.

(2) A component of $\Gamma_s(A)$ containing an indecomposable Heller lattice Z is said to be a **Heller component** of A, and denoted by $\mathcal{HC}(Z)$.

Remark 3.6.2. For a finite dimensional algebra Λ over an algebraically closed field $\mathbf{k}$, it is well-known that the Auslander-Reiten quiver for mod-Λ has no loops, for example see [SY1, Chapter III, Corollary 11.3]. However, in the case of an $\mathcal{O}$-order, there are possibility that $\Gamma_s(A)$ has loops [Wi]. For example, if A is Morita equivalent to either a Bass order or a maximal order, then the Auslander–Reite quiver of such an algebra has loops.

Definition 3.6.3. An $\mathcal{O}$-linear map $d : A \to \mathsf{End}_{\mathcal{O}}(M)$ is called a **derivation** if

$$d(xy) = xd(y) + d(x)y$$

for all $x, y \in A$. We denote by $\mathsf{Der}(A, \mathsf{End}_{\mathcal{O}}(M))$ the $\mathcal{O}$-module of derivations. Note that $\mathsf{Der}(A, \mathsf{End}_{\mathcal{O}}(M))$ is an $\mathcal{O}$-order since A and $\mathsf{End}_{\mathcal{O}}(M)$ are.

Lemma 3.6.4 ([AKM, Lemma 1.24])**.** Let A be an $\mathcal{O}$-order, M an indecomposable A-lattice. Then, there exists an integer s such that $M/\varepsilon^k M$ is an indecomposable $A/\varepsilon^k A$-module, for all $k \geq s$.

Proof. Let k be a positive integer. For $f \in \mathsf{End}_{\mathcal{O}}(M)$ such that $af(m + \varepsilon^k M) = f(am + \varepsilon^k M)$, for $a \in A$ and $m \in M$, we define $D_f \in \mathsf{Hom}_{\mathcal{O}}(A, \mathsf{End}_{\mathcal{O}}(M))$ by

$$D_f(a)(m) = \varepsilon^{-k}(f(am) - af(m))$$

for $a \in A$ and $m \in M$. Then, D_f is a derivation. Indeed,

$$\begin{aligned} D_f(xy)(m) &= \varepsilon^{-k}(f(xym) - xyf(m)) \\ &= \varepsilon^{-k}(xf(ym) - xyf(m)) + \varepsilon^{-k}(f(xym) - xf(ym)) \\ &= xD_f(y)(m) + D_f(x)y(m). \end{aligned}$$

Let $\mathsf{Der}(k)$ be the $\mathcal{O}$-submodule of $\mathsf{Der}(A, \mathsf{End}_{\mathcal{O}}(M))$ which is generated by all such D_f, and we define $\mathsf{Der}(\infty) = \sum_{k \geq 1} \mathsf{Der}(k)$. Since $\mathsf{Der}(A, \mathsf{End}_{\mathcal{O}}(M))$ is a finitely generated $\mathcal{O}$-module, there exists an integer s such that

$$\mathsf{Der}(\infty) = \sum_{k=1}^{s-1} \mathsf{Der}(k).$$

Let $p_k : M \to M/\varepsilon^k M$ be the canonical projection. Then, for any $f \in \mathsf{End}_A(M)$, there exists $\overline{f} \in \mathsf{End}_A(M/\varepsilon^k M)$ such that $\overline{f} p_k = p_k f$. We show that the algebra homomorphism

$$\mathsf{End}_A(M) \ni f \longmapsto \overline{f} \in \mathsf{End}_A(M/\varepsilon^k M)$$

is surjective, for all $k \geq s$. Let $\theta \in \mathsf{End}_A(M/\varepsilon^k M)$, for $k \geq s$. We fix $f \in \mathsf{End}_{\mathcal{O}}(M)$ such that $f(m + \varepsilon^k M) = \theta(m + \varepsilon^k M)$ for $m \in M$. Then, there exist $c_i \in \mathcal{O}$ and $f_i \in \mathsf{End}_{\mathcal{O}}(M)$ that satisfy

$$f_i(m + \varepsilon^{l_i} M) = \theta_i(m + \varepsilon^{l_i} M)$$

for some $1 \leq l_i \leq s-1$ and $\theta_i \in \mathsf{End}_A(M/\varepsilon^{l_i}M)$ such that $D_f = \sum_{i=1}^N c_i D_{f_i}$. More explicitly, we have

$$f(am) - af(m) = \sum_{i=1}^{N} \varepsilon^{k-l_i} c_i(f_i(am) - af_i(m))$$

for $a \in A$ and $m \in M$. It implies that $f - \sum_{i=1}^N \varepsilon^{k-l_i} c_i f_i \in \mathsf{End}_A(M)$. Since it coincides with θ if we reduce modulo ε, we have proved

$$\mathsf{Im}(\mathsf{End}_A(M) \to \mathsf{End}_A(M/\varepsilon^k M)) + \varepsilon \mathsf{End}_A(M/\varepsilon^k M) = \mathsf{End}_A(M/\varepsilon^k M).$$

Thus, Nakayama's lemma implies that $\mathsf{End}_A(M) \to \mathsf{End}_A(M/\varepsilon^k M)$ is surjective, and we have an isomorphism of algebras $\mathsf{End}_A(M)/\varepsilon^k \mathsf{End}_A(M) \simeq \mathsf{End}_A(M/\varepsilon^k M)$. As $\mathcal{O}$ is a complete local ring, the lifting idempotent argument works by Proposition 2.2.4. Hence, $M/\varepsilon^k M$ is decomposable for $k \geq s$ when M is indecomposable. □

The following lemma is well-known as the Harada–Sai lemma.

Lemma 3.6.5 ([ARS, Chapter VI, Corollary 1.3])**.** Let Λ be an Artin algebra, m a positive integer and

$$N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{2^m-2}} N_{2^m-1} \xrightarrow{f_{2^m-1}} N_{2^m}$$

a chain of Λ-module homomorphisms satisfying the following conditions.

(i) The Λ-modules $N_1, \ldots, N_{2^m}$ are indecomposable with $\ell(N_i) \leq m$, where $\ell(M)$ is the length of composition series of M.

(ii) The Λ-module homomorphism $f_i : N_i \to N_{i+1}$ belongs to $\mathsf{rad}(N_i, N_{i+1})$.

Then we have $f_{2^m-1} \cdots f_1 = 0$.

Proof. Let $\{N_i \mid 1 \leq i \leq 2^m\}$ be a collection of indecomposable Λ-modules such that the length of composition series of N_i is less than or equal to m, for all i. We show by induction on n that if

$$N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{2^n-2}} N_{2^n-1} \xrightarrow{f_{2^n-1}} N_{2^n}$$

is a chain of non-zero non-isomorphisms, then we have $\ell(\mathsf{Im}(f_{2^n-1} \cdots f_2 f_1)) \leq m-n$.

When $n = 1$, the statement is clear. Indeed, if $\ell(\mathsf{Im}(f_1)) = m$, then f_1 must be isomorphism. Thus, we have $\ell(\mathsf{Im}(f_1)) \leq m-1$. Assume that $n > 1$. We set $f = f_{2^n-1} \cdots f_2 f_1$, $g = f_{2^n}$ and $h = f_{2^{n+1}-1} \cdots f_{2^n+2} f_{2^n+1}$. By the inductive assumption, we have $\ell(\mathsf{Im}(f)) \leq m-n$ and $\ell(\mathsf{Im}(h)) \leq m-n$. If either $\ell(\mathsf{Im}(f)) < m-n$ or $\ell(\mathsf{Im}(h)) < m-n$, then the statement is clear. Now, we suppose that $\ell(\mathsf{Im}(f)) = m-n$, $\ell(\mathsf{Im}(h)) = m-n$ and $m > n$.

We prove that $\ell(\mathsf{Im}(hgf)) \leq m-n-1$. In order to get a contradiction, we suppose that $\ell(\mathsf{Im}(hgf)) > m-n-1$. As $\ell(\mathsf{Im}(hgf)) \leq \ell(\mathsf{Im}(h)) \leq m-n$, we have $\ell(\mathsf{Im}(hgf)) = m-n$, and the formula

$$\ell(\mathsf{Im}(hgf)) = \ell\left(\frac{\mathsf{Im}(f)}{\mathsf{Im}(f)\cap\mathsf{Ker}(hg)}\right) = \ell(\mathsf{Im}(f)) - \ell(\mathsf{Im}(f)\cap\mathsf{Ker}(hg))$$

yields $\mathsf{Im}(f)\cap\mathsf{Ker}(hg) = 0$. On the other hand, the chain of inclusions $\mathsf{Im}(hgf) \subset \mathsf{Im}(hg) \subset \mathsf{Im}(h)$ implies that $\ell(\mathsf{Im}(hg)) = m-n$, so that

$$\ell(\mathsf{Ker}(hg)) = \ell(N_{2^n+1}) - \ell(\mathsf{Im}(hg)) \leq m-(m-n) = n.$$

Thus, we conclude that $N_{2^n} = \mathsf{Im}(f) \oplus \mathsf{Ker}(hg)$ since $\ell(N_{2^n}) = \ell(\mathsf{Im}(f)) + \ell(\mathsf{Ker}(hg))$. By the indecomposablity of N_{2^n}, the Λ-module homomorphism hg is injective, hence g is injective. Similarly, it follows from $\mathsf{Im}(gf)\cap\mathsf{Ker}(h) = 0$ and the indecomposability of N_{2^n+1} that g is surjective. This is a contradiction. $\square$

The following proposition is a generalization of [ASS, Chapter IV, 5.4 Theorem].

Proposition 3.6.6 ([AKM, Proposition 1.26])**.** Let $\mathcal{C}$ be a component of $\Gamma_s(A)$. Assume that A is indecomposable as an algebra and the number of vertices in $\mathcal{C}$ is finite. Then $\mathcal{C}$ exhausts all non-projective indecomposable A-lattices.

Proof. We add indecomposable projective A-lattices to $\Gamma_s(A)$ to obtain the Auslander–Reiten quiver of A. We show that if $\mathcal{C}$ is a finite component of the Auslander–Reiten quiver then $\mathcal{C}$ exhausts all indecomposable A-lattices.

Assume that M is an indecomposable A-lattice which does not belong to $\mathcal{C}$. It suffices to show that $\mathsf{Hom}_A(M,N) = 0$ and $\mathsf{Hom}_A(N,M) = 0$ for all $N \in \mathcal{C}$. To see that it is sufficient, let P be an indecomposable direct summand of the projective cover of $N \in \mathcal{C}$. Then, P belongs to $\mathcal{C}$ since N belongs to $\mathcal{C}$ and $\mathsf{Hom}_A(P,N) \neq 0$. As A is indecomposable as an algebra, there is no indecomposable projective A-lattice Q with the property that

$$\mathsf{Hom}_A(Q,R) = 0 = \mathsf{Hom}_A(R,Q),$$

for all indecomposable projective A-lattices $R \in \mathcal{C}$. It implies that any direct summand Q of the projective cover of M belongs to $\mathcal{C}$. Then $\mathsf{Hom}_A(Q,M) \neq 0$ implies that $M \in \mathcal{C}$, which contradicts our assumption. Thus, $\mathcal{C}$ exhausts all indecomposable A-lattices.

Assume that there exists a nonzero morphism $f \in \mathsf{Hom}_A(M,N)$. As $M \notin \mathcal{C}$ and $N \in \mathcal{C}$, f is not a retraction. We consider the almost split sequence ending at N, and we denote by $N_1,\ldots,N_r$ the indecomposable direct summands of the middle term of the almost split sequence. Let $g_i^{(1)} : N_i \longrightarrow N$ be irreducible morphisms. Then, there exist $f_i \in \mathsf{Hom}_A(M,N_i)$ such that

$$f = \sum_{i=1}^{r} g_i^{(1)} f_i.$$

If N_i is non-projective, we apply the same procedure to f_i. If N_i is projective, f_i factors through the Heller lattice $\mathsf{rad}N_i$ of the simple $\overline{A}$-module $N_i/\mathsf{rad}(N_i)$. Thus, we apply the procedure after we replace N_i with $\mathsf{rad}N_i$. After repeating n times, we obtain,

$$f = \sum g_i^{(1)} \cdots g_i^{(n)} h_i,$$

such that $g_i^{(j)}$ are morphisms among indecomposable A-lattices in $\mathcal{C}$, h_i are morphisms $M \to X_i$, where X_i are indecomposable A-lattices in $\mathcal{C}$ and they are not isomorphisms.

Since the number of vertices in $\mathcal{C}$ is finite, there exists an integer s such that $X/\varepsilon^s X$ is indecomposable, for all $X \in \mathcal{C}$ by Lemma 3.6.4. Let m be the maximal length of $A/\varepsilon^s A$-modules $X/\varepsilon^s X$, for $X \in \mathcal{C}$. Applying Lemma 3.6.5 to the Artin algebra $A/\varepsilon^s A$ with $n = 2^m - 1$, we obtain

$$\mathsf{Hom}_A(M, N) = \varepsilon^s \mathsf{Hom}_A(M, N),$$

and Nalayama's lemma implies $\mathsf{Hom}_A(M, N) = 0$. The proof of $\mathsf{Hom}_A(N, M) = 0$ is similar. We start with a nonzero morphism $f \in \mathsf{Hom}_A(N, M)$ and consider the almost split sequence starting at N. Let $N_1, \ldots, N_r$ be the indecomposable direct summands of the middle term of the almost split sequence as above, and let

$$g_i^{(1)} : N \longrightarrow N_i$$

be irreducible morphisms. If N_i is projective, then we replace N_i with $\mathsf{rad}N_i$. Then, after repeating the procedure n times, we obtain

$$f = \sum h_i g_i^{(n)} \cdots g_i^{(1)},$$

where h_i are morphisms from indecomposable A-lattices in $\mathcal{C}$ to M. Then, we may deduce $\mathsf{Hom}_A(N, M) = 0$ by the Harada–Sai lemma and Nakayama's lemma as before. □

An $\mathcal{O}$-order A is of **finite CM type** if there are only finitely many isoclasses of indecomposable A-lattices.

Corollary 3.6.7. A symmetric $\mathcal{O}$-order is of finite CM type if and only if the stable Auslander–Reiten component has a finite component. Moreover, if A is of finite CM type, then A is an isolated singularity.

Proof. The first half of the statements immediately follows from Proposition 3.6.6. We show the last half of the statements. If A is of finite CM type, all non-projective indecomposable A-lattices belong to the stable Auslander–Reiten quiver. We apply the arguments in the proof of Proposition 3.6.6 for arbitrary A-lattice M. It implies that $\mathsf{latt}^{(\natural)}\text{-}A = \mathsf{latt}\text{-}A$. □

Remark 3.6.8 ([A3])**.** Corollary 3.6.7 had proven by M. Auslander in [A3], namely if A is a Gorenstein $\mathcal{O}$-order, which is of finite CM type, then A has an isolated singularity.

3.7 On the shape of the stable Auslander–Reiten quiver

To determine the shapes of Auslander–Reiten quivers is one of classical problems in representation theory of algebras. There exist strong restrictions on the shapes of stable Auslander–Reiten quivers for important classes of finite dimensional algebras. In [We], P. J. Webb studied the stable Auslander-Reiten components of group algebras. Let G be a finite group and $\mathbf{k}$ an algebraically closed field with characteristic p such that p divides the order of G. Then, the tree class of any stable component of the group algebra $\mathbf{k}G$ is one of infinite Dynkin diagrams $A_\infty, B_\infty, C_\infty, D_\infty$ or A_∞^∞, or else it is A_n, or one of Euclidean diagrams ([B, Theorem 4.17.4]). Moreover, Erdmann showed that the tree class of any stable component of a wild block of $\mathbf{k}G$ is A_∞ [Erd]. For another example, Riedtmann and Todorov showed that the tree class of any stable component of a finite dimensional self-injective algebra of finite representation type is one of finite Dynkin diagrams [Ri2, T]. However, if the base ring is not a field, then the shapes of (stable) Auslander–Reiten components are mostly unknown. In this section, we give restrictions on the shape of certain stable Auslander–Reiten components [AKM, M1].

Throughout this section, we assume that A is a symmetric $\mathcal{O}$-order and $\mathcal{C}$ is a connected component of $\Gamma_s(A)$. By the definition of the stable Auslande–Reiten quiver, $\mathcal{C}$ is valued stable translation quiver. Assume that $\mathcal{C}$ has no loops. Then, it follows from the Riedtmann structure theorem that there exists a directed tree T and $G \subset \mathsf{Aut}_{\tau_0}(\mathbb{Z}T)$ such that $\mathcal{C}$ is isomorphic to $\mathbb{Z}T/G$. In order to get a restriction on the shape of $\mathcal{C}$, we give candidates for the tree class $\overline{T}$.

First, we consider the case of finite CM type. This case had already given in [Lu]. Assume that $\mathcal{C}$ has finitely many vertices. By Corollary 3.6.7, the $\mathcal{O}$-order A is an isolated singularity. In this case, $\underline{\mathsf{latt}}$-$A$ is a Hom-finite triangulated category. Therefore, the following result follows from [XZ, Theorem 2.3.5].

Theorem 3.7.1 ([Lu, Theorem 2.18])**.** Let A be a symmetric $\mathcal{O}$-order, which is of finite CM type. Then, the tree class of $\Gamma_s(A)$ is one of finite Dynkin diagrams.

From now on, we assume the following.

Assumption 3.7.2. The stable Auslander–Reiten quiver $\Gamma_s(A)$ has infinitely many vertices.

Lemma 3.7.3. Let $\mathcal{C}$ be a component of $\Gamma_s(A)$. Then, $\mathcal{C}$ has infinitely many vertices.

Proof. This follows immediately from Corollary 3.6.7 and Assumption 3.7.2. □

Lemma 3.7.4. Let $\mathcal{C}$ be a periodic component of $\Gamma_s(A)$ without loops and $\overline{T}$ the tree class of $\mathcal{C}$. Then, the function $\mathcal{R} : \mathcal{C}_0 \to \mathbb{Q}_{\geq 0}$ defined by

$$\mathcal{R}(X) := \sum_{i=0}^{n_X - 1} \frac{\mathsf{rank}(\tau^i X)}{n_X},$$

where n_X is a positive integer such that $\tau^{n_X}(X) \simeq X$, is subadditive on T.

Proof. Let $\mathcal{C}$ be a periodic component of $\Gamma_s(A)$ without loops and $\overline{T}$ the tree class of $\mathcal{C}$. For each $X \in T$, the inequality

$$\sum_{Y \in X^-} d_{Y,X} \mathsf{rank}(Y) \leq \mathsf{rank}(X) + \mathsf{rank}(\tau X) \tag{3.1}$$

implies that $\mathcal{R}$ satisfies

$$2\mathcal{R}(X) \geq \sum_{Y \in X^- \cap T} d_{Y,X} \mathcal{R}(Y) + \sum_{Y \in X^+ \cap T} d'_{X,Y} \mathcal{R}(Y) \tag{3.2}$$

for all $X \in T$. This is shown as follows. By the definition of $\mathcal{R}$, it is a τ-invariant function. Let $n = \prod_{Y \to X} n_Y$. Then, we have

$$\begin{aligned}
\sum_{k=0}^{n_X n-1} \left(\sum_{Y \to X} d_{\tau^k Y, \tau^k X} \mathsf{rank}(\tau^k Y) \right) &= \sum_{Y \to X} \sum_{k=0}^{n_X n-1} (d_{Y,X} \mathsf{rank}(\tau^k Y)) \\
&= \sum_{Y \to X} d_{Y,X} \frac{n_X n}{n_Y} \sum_{k=0}^{n_Y - 1} \mathsf{rank}(\tau^k Y) \\
&= \sum_{Y \to X} d_{Y,X} n_X n \mathcal{R}(Y).
\end{aligned}$$

On the other hand, we have

$$\sum_{k=0}^{n_X n-1} (\mathsf{rank}(\tau^k X) + \mathsf{rank}(\tau^{k+1} X)) = 2 \frac{n_X n}{n_X} \sum_{k=0}^{n_X - 1} \mathsf{rank}(\tau^k X) = 2 n_X n \mathcal{R}(X).$$

Thus, the inequality (3.1) yields the inequality (3.2) since $\mathcal{C}$ is a valued stable translation quiver. By Remark 2.5.7, the restriction $\mathcal{R}|_T$ is subadditive. $\square$

Proposition 3.7.5 ([AKM, Lemma 1.23])**.** Let $\mathcal{C}$ be a component of $\Gamma_s(A)$. If $\mathcal{C}$ has loops, then $\mathcal{C}$ is τ-periodic. Furthermore, $\mathcal{C} \setminus \{\text{loops}\}$ is of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$. In this case, there exists exactly one loop and it appears at the endpoint of $\mathcal{C}$ such that the valuation of the loop is trivial.

$$\mathcal{C} = \circlearrowleft \bullet \rightleftarrows \bullet \rightleftarrows \cdots\cdots \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \cdots$$

Proof. First, we show that if $X \in \mathcal{C}$ has a loop, then $X \simeq \tau X$. Suppose that $X \in \mathcal{C}$ has a loop and $X \not\simeq \tau X$. Then the almost split sequence ending at X is of the form

$$0 \to \tau X \to X^{\oplus l_1} \oplus E_X \oplus \tau X^{\oplus l_2} \to X \to 0,$$

where E_X is an A-lattice and $l_1, l_2 \geq 1$. Therefore,

$$\mathsf{rank}(X) + \mathsf{rank}(\tau X) = l_1 \mathsf{rank}(X) + l_2 \mathsf{rank}(\tau X) + \mathsf{rank}(E_X),$$

implies $\mathsf{rank}(E_X) = 0$ and $l_1 = l_2 = 1$. However, it follows from Theorem 2.3.3 that the almost split sequence ending at X splits, a contradiction. Therefore, if X has a loop, then X and τX are isomorphic.

We notice that $\mathcal{C} \setminus \{\text{loops}\}$ is also a valued stable translation quiver, and we may apply the Riedtmann structure theorem. We write $\mathcal{C} \setminus \{\text{loops}\} = \mathbb{Z}T/G$, for a directed tree T and an admissible subgroup G. By Lemma 3.7.4, the function $\mathcal{R}$ is subadditive on T. Since $\mathcal{C}$ has a loop, $\mathcal{R}|_T$ is not additive. Then, Lemma 2.5.8 implies that $\overline{T} = A_\infty$ since $\mathcal{C}$ has infinitely vertices. Thus, we may assume without loss of generality that T is a chain of irreducible maps

$$X_1 \to X_2 \to \cdots \to X_r \to \cdots$$

by Lemma 2.5.3. We assume that X_r has a loop. If $r > 1$ then the almost split sequence starting at X_r is

$$0 \longrightarrow X_r \longrightarrow X_r^{\oplus l} \oplus X_{r+1} \oplus X_{r-1} \oplus P \longrightarrow X_r \longrightarrow 0,$$

where $l \geq 1$ and $P \in \mathsf{proj}\text{-}A$. Since $\mathcal{R}(X_t) \geq 1$ for all $t \geq 1$, we have

$$\mathcal{R}(X_r) \geq (2-l)\mathcal{R}(X_r) \geq \mathcal{R}(X_{r+1}) + \mathcal{R}(X_{r-1}) \geq \mathcal{R}(X_{r+1}) + 1.$$

We show that $\mathcal{R}(X_m) \geq \mathcal{R}(X_{m+1}) + 1$ for $m \geq r$. Suppose that $\mathcal{R}(X_{m-1}) \geq \mathcal{R}(X_m) + 1$ holds. The same argument as above shows $2\mathcal{R}(X_m) \geq \mathcal{R}(X_{m-1}) + \mathcal{R}(X_{m+1})$, and the induction hypothesis implies

$$2\mathcal{R}(X_m) \geq \mathcal{R}(X_{m-1}) + \mathcal{R}(X_{m+1}) \geq \mathcal{R}(X_m) + \mathcal{R}(X_{m+1}) + 1.$$

Hence $\mathcal{R}(X_m) \geq \mathcal{R}(X_{m+1}) + 1$. Thus, there exists a positive integer t such that $\mathcal{R}(X_t) < 0$, which contradicts with $\mathcal{R}(X_t) \geq 1$. Hence $r = 1$, that is, the deleted loops appear only at the endpoint of the homogeneous tube. Then, $l = 1$ by $2 \times \mathsf{rank}(X_1) > l \times \mathsf{rank}(X_1)$. We have proved that the loop is unique and it appears at the endpoint of the homogeneous tube such that the valuation is $(1, 1)$. □

Theorem 3.7.6 ([AKM, Theorem 1.27])**.** Let $\mathcal{C}$ be a τ-periodic component of the stable Auslander–Reiten quiver. If $\mathcal{C}$ has no loops, then $\mathcal{C}$ is of the form $\mathbb{Z}T/G$, where T is a directed tree whose underlying graph is one of infinite Dynkin diagrams and G is an admissible group.

Proof. Since $\mathcal{C}$ has no loops, the statement follows from Theorem 2.5.8 and Lemmas 3.7.3 and 3.7.4. □

Corollary 3.7.7. Let $\mathcal{C}$ be a periodic component of $\Gamma_s(A)$. For $X \in \mathcal{C}$, we denote by E_X the middle term of almost split sequence ending at X. Assume that

(i) there exists a vertex X of $\mathcal{C}$ such that the number of non-projective direct summands of E_X is one, and

(ii) E_X does not have X as a direct summand.

Then, $\mathcal{C}$ dose not have a loop.

Proof. If $\mathcal{C}$ has a loop, $\mathcal{C} \setminus \{\text{the loop}\} \simeq \mathbb{Z}A_\infty/\langle\tau\rangle$, and the loop only appear the boundary of $\mathcal{C}$ by Proposition 3.7.5. By our assumption that E_X has exactly one non-projective indecomposable direct summand, the A-lattice X appears on the boundary of $\mathcal{C}$ and $E_X = X \oplus P$ for some $P \in \mathsf{proj}\text{-}A$. This contradicts with the assumption (ii). $\square$

Corollary 3.7.8. Let $\mathcal{C}$ be a periodic component of $\Gamma_s(A)$. If there exists a vertex X of $\mathcal{C}$ such that the following conditions hold:

(i) The number of non-projective indecomposable direct summands of E_X is 1. We denote by Y the unique non-projective direct summand.

(ii) The number of non-projective indecomposable direct summands of E_Y is 2.

Then, $\mathcal{C}$ is a tube.

Proof. By the assumption (i) and (ii), we have $Y \neq X$. Thus, it follows from Corollary 3.7.7 that $\mathcal{C}$ has no loops. Thus, the tree class $\overline{T}$ of $\mathcal{C}$ is one of infinite Dynkin diagrams. By the assumption (i), $\overline{T} \neq A_\infty^\infty$. By the assumption (ii), $\overline{T} \neq B_\infty$, C_∞, D_∞. Therefore, $\overline{T}$ is A_∞. $\square$

By Proposition 3.7.5, we may assume the following.

Assumption 3.7.9. A stable Auslander–Reiten component of $\Gamma_s(A)$ does not have a loop.

The following proposition is effective to determine the shapes of stable components of A.

Proposition 3.7.10 ([K3, Proposition 4,5])**.** Let A be an $\mathcal{O}$-order and L an indecomposable A-lattice, and let

$$0 \to \tau L \to E \xrightarrow{g} L \to 0$$

be the almost split sequence ending at L. Assume that L is not a direct summand of any Heller lattice. Then, the induced exact sequence

$$0 \to \tau L \otimes \kappa \to E \otimes \kappa \to L \otimes \kappa \to 0$$

splits.

Proof. Let $P \to L \otimes \kappa = L/\varepsilon L$ be the projective cover of $L \otimes \kappa$ as an A-module. Set $Q = \varepsilon P$ and $Z = \Omega(L/\varepsilon L)$. Then, we have $\varepsilon Z \subset Q \subset Z \subset P$ and an isomorphism $\varphi : Q/\varepsilon Z \xrightarrow{\sim} L/\varepsilon L$. Let $i : Q \to Z$ and $\iota : Q/\varepsilon Z \to Z/\varepsilon Z$ be the inclusions, $u : Q \to Q/\varepsilon Z$, $v : L \to L/\varepsilon L$ and $w : Z \to Z/\varepsilon Z$ the projections. Since Q is projective, there exists an A-module homomorphism $\tilde{q} : Q \to L$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & Q \arrow[d, "u"] \arrow[dl, dashed, "\exists \tilde{q}"'] & \\
 & Q/\varepsilon Z \arrow[d, "\varphi"', "\simeq"] & \\
L \arrow[r, "v"] & L/\varepsilon L \arrow[r] & 0
\end{tikzcd}$$

As $v\tilde{q}(\varepsilon Z) = \varphi u(\varepsilon Z) = 0$, we have $\tilde{q}(\varepsilon Z) \subset \varepsilon L$. Thus, we may extend $\tilde{q}$ to an A-module homomorphism $q : Z \to L$. Let $\overline{q} : Z/\varepsilon Z \to L/\varepsilon L$ be the induced morphism from q. Then, the equations

$$\varphi u = v\tilde{q} = vqi = \overline{q}wi = \overline{q}\iota u$$

hold. Thus, φ factors through $\overline{q} : Z/\varepsilon Z \to L/\varepsilon L$. By our assumption that L is not a direct summand of Z, q is not a retraction. Thus, q factors through the morphism g:

$$\begin{tikzcd}
 & & & Z \arrow[d, "q"] \arrow[dl, dashed] & \\
0 \arrow[r] & \tau L \arrow[r] & E \arrow[r, "g"] & L \arrow[r] & 0
\end{tikzcd}$$

Since $1_{L\otimes\kappa} = \overline{q}\iota\varphi^{-1}$, the identity morphism $1_{L\otimes\kappa}$ factors through the induced morphism $\overline{g} : E/\varepsilon E \to L/\varepsilon L$. $\square$

We define the function $\mathcal{D} : \mathsf{latt}^{(\natural)}\text{-}A \to \mathbb{Z}_{\geq 0}$ by the following.

$$\mathcal{D}(X) := \sharp\{\text{non-projective indecomposable direct summands of } X \otimes \kappa\}$$

Lemma 3.7.11 ([M1, Lemma 3.2] and [M2, Lemma 1.16])**.** Suppose that A is a symmetric $\mathcal{O}$-order. Then, for any non-projective A-lattice M, there is an isomorphism $\tau(M) \otimes \kappa \simeq \widetilde{\Omega}(M \otimes \kappa)$. In particular, we have the equality $\mathcal{D}(X) = \mathcal{D}(\tau X)$.

Proof. Let M be an A-lattice and $\pi : P \to M$ the projective cover. Let $Q \otimes \kappa \to M \otimes \kappa$ be the projective cover. Then $\mathsf{rank}\ Q \leq \mathsf{rank}\ P$. On the other hand, it lifts to $Q \to M$ and it is an epimorphism by Nakayama's lemma. Thus, we have $\mathsf{rank}\ Q = \mathsf{rank}\ P$ and $P \otimes \kappa$ is the projective cover of $X \otimes \kappa$. Therefore, we have $\tau(M) \otimes \kappa \simeq \widetilde{\Omega}(M \otimes \kappa)$ as objects in the stable module category $\underline{\mathsf{mod}}\text{-}\overline{A}$. Since the functor $- \otimes \kappa$ is exact on $\mathsf{latt}\text{-}A$, the assertion follows. $\square$

Lemma 3.7.12 ([M1, Lemma 3.3])**.** If a short exact sequence $0 \to \tau L \to E \to L \to 0$ in $\mathsf{latt}^{(\natural)}$-$A$ is the almost split sequence ending at L, then the equality

$$\mathcal{D}(L) + \mathcal{D}(\tau L) = \mathcal{D}(E)$$

holds whenever L is not isomorphic to any direct summand of Heller lattices.

Proof. Let L be an indecomposable A-lattice in $\mathsf{latt}^{(\natural)}$-$A$. Suppose that L is not isomorphic to a Heller lattice. Let $0 \to \tau L \to E \to L \to 0$ be the almost split sequence ending at L. By Proposition 3.7.10, the induced exact sequence

$$0 \to \tau L \otimes \kappa \to E \otimes \kappa \to L \otimes \kappa \to 0$$

splits, which gives the desired conclusion. $\square$

Corollary 3.7.13 ([M2, Lemma 1.19])**.** Let $\mathcal{C}$ be a component of $\Gamma_s(A)$. For an indecomposable Heller lattice $Z \in \mathcal{C}$, let E_Z be the middle term of the almost split sequence ending at Z. If $\mathcal{D}$ satisfies $2\mathcal{D}(Z) \geq \mathcal{D}(E_Z)$ for any indecomposable Heller lattice $Z \in \mathcal{C}$, then $\mathcal{D}$ gives rise to a subadditive function on T, where T is a directed tree such that $\mathcal{C} \simeq \mathbb{Z}T/G$. In particular, $\mathcal{D}|_T$ is additive if and only if the equalities hold for any Z.

Proof. The assertion follows from Lemmas 3.7.11 and 3.7.12. $\square$

Theorem 3.7.14 ([M1, Proposition 5.4])**.** Let $\mathcal{C}$ be a component of the stable Auslander–Reiten quiver of A. Assume either

(i) $\mathcal{C}$ does not contain Heller lattices or

(ii) $\overline{A}$ is of finite representation type.

Then, the tree class of $\mathcal{C}$ is one of infinite Dynkin diagrams or Euclidean diagrams.

Proof. If $\mathcal{C}$ is τ-periodic, the statement had proven by Theorem 3.7.6. Thus, we may assume that $\mathcal{C}$ is not τ-periodic. Since $\mathcal{C}$ has no loops, there exist a directed tree T and an admissible group G such that $\mathcal{C} \simeq \mathbb{Z}T/G$ by Theorem 2.5.4. Suppose that $\mathcal{C}$ does not contain Heller lattices. In this case, the function $\mathcal{D}$ is additive with $\mathcal{D}(X) = \mathcal{D}(\tau X)$, for all $X \in \mathcal{C}$ by Lemmas 3.7.11 and 3.7.12. Thus, the assertion follows from Theorem 2.5.8.

Suppose that $\overline{A}$ is of finite representation type. Since the number of isoclasses of Heller lattices is finite, there exists an integer n_X such that both $\tau^{n_X}X$ and $\tau^{n_X+1}X$ are not Heller lattices for any vertex $X \in \mathcal{C}$. Thus, $\mathcal{D}$ is an additive function on T. $\square$

4. HELLER COMPONENTS: THE CASE OF TRUNCATED POLYNOMIAL RINGS

In this chapter, we determine the shapes of Heller components when $A = \mathcal{O}[X]/(X^n)$. The results in this chapter appear in [AKM]. Since $A \otimes \mathcal{K} = \mathcal{K}[X]/(X^n)$ is not semi-simple, A is not an isolated singularity. In particular, $\mathsf{latt}\text{-}A$ is of infinite representation type, and the stable Auslander–Reiten quiver $\Gamma_s(A)$ has infinitely many vertices. Since $\overline{A}$ is of finite representation type, the tree class of any Heller component of A which has no a loop is one of infinite Dynkin diagrams or Euclidean diagrams by Theorem 3.7.14.

In this chapter, we use the same symbol X as $X + (X^n)$. It is well-known that the stable Auslander–Reiten quiver is given as follows:

$$\begin{tikzcd} M_1 \arrow[loop above, dotted, "\tau"] \arrow[r, shift left] & M_2 \arrow[loop above, dotted, "\tau"] \arrow[l, shift left] \arrow[r, shift left] & \cdots\cdots \arrow[l, shift left] \arrow[r, shift left] & M_{n-2} \arrow[loop above, dotted, "\tau"] \arrow[l, shift left] \arrow[r, shift left] & M_{n-1} \arrow[loop above, dotted, "\tau"] \arrow[l, shift left] \end{tikzcd}$$

Here, the indecomposable $\overline{A}$-module M_i is given as follows. As a κ-vector space, M_i is a $(n-i)$-dimensional κ-vector space

$$M_i = \bigoplus_{k=0}^{n-i-1} \kappa X^{i+k},$$

and the action of X is given in natural way. We denote by Z_i the first syzygy of M_i in $\mathsf{latt}\text{-}A$. Then, Z_i is indecomposable (Lemma 4.1.1).

The second main result of this thesis is the following:

Main Theorem (Proposition 4.2.1, Theorem 4.4.1)**.** Let $\mathcal{O}$ be a complete discrete valuation ring, $A = \mathcal{O}[X]/(X^n)$, for $n \geq 2$. Then, the Heller component containing Z_i and Z_{n-i} is $\mathbb{Z}A_\infty/\langle \tau^2 \rangle$ if $2i \neq n$, and $\mathbb{Z}A_\infty/\langle \tau \rangle$ (i.e. homogeneous tube) if $2i = n$. Moreover, any Heller lattice appears on the boundary of a Heller component.

4.1 Heller lattices

We view M_i as an A-module. Then, the A-module homomorphism $p_{M_i} : A \to M_i$ defined by $1 \mapsto X^i$ is the projective cover of M_i. Therefore, the first syzygy $Z_i := \Omega(M_i)$ is given as follows:

$$Z_i = \mathcal{O}\varepsilon \oplus \mathcal{O}\varepsilon X \oplus \cdots \oplus \mathcal{O}\varepsilon X^{n-i-1} \oplus \mathcal{O}X^{n-i} \oplus \mathcal{O}X^{n-i+1} \oplus \cdots \oplus \mathcal{O}X^{n-1}$$

Lemma 4.1.1 ([AKM, Lemma 2.1])**.** We have the following.

(1) The Heller lattices $Z_1, \dots, Z_{n-1}$ are pairwise non-isomorphic indecomposable A-lattices.

(2) If $\rho \in \mathsf{radEnd}_A(Z_i)$ then $\rho(\varepsilon)$ has the form

$$\rho(\varepsilon) = a_0\varepsilon + \cdots + a_{n-i-1}\varepsilon X^{n-i-1} + a_{n-i}X^{n-i} + \cdots + a_{n-1}X^{n-1},$$

where $a_i \in \mathcal{O}$, for $1 \le i \le n-1$, and $a_0 \in \varepsilon\mathcal{O}$.

Proof. (1) The representing matrix of the action of $X = (x_{i,j})$ on Z_i with respect to the above basis is given by the following matrix

$$x_{i,j} = \begin{cases} \varepsilon & \text{if } i = n-i+1,\ j = n-i, \\ 1 & \text{if } i = j+1,\ j \neq n-i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the endomorphism algebra $\mathsf{End}_A(Z_i)$ is isomorphic to $\{M \in \mathrm{Mat}(n, n, \mathcal{O}) \,|\, MX = XM\}$. The right hand side is contained in the set:

$$\left\{ \begin{pmatrix} a & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & * & \ddots & 0 \\ & & & a \end{pmatrix} \;\middle|\; a \in \mathcal{O} \right\}$$

Hence, $\mathsf{End}_A(Z_i)$ is local. The statement (2) follows immediately from the proof of (1). $\square$

We now consider the following pullback diagram along (p_Z, φ_i):

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Ker}(p_{Z_i}) \arrow[r] \arrow[d, equal] & E_i \arrow[r] \arrow[d] & Z_i \arrow[r] \arrow[d, "\varphi_i"] & 0 \\ 0 \arrow[r] & \mathsf{Ker}(p_{Z_i}) \arrow[r] & A \oplus A \arrow[r, "p_{Z_i}"] & Z_i \arrow[r] & 0 \end{tikzcd}$$

Here, φ_i is the A-module homomorphism defined by $\varphi_i(\varepsilon) = X^{n-1}$ and $p_{Z_i} : A \oplus A \to Z_i$ is defined by $e_1 \longmapsto X^{n-i}$, $e_2 \longmapsto \varepsilon$. Then, p_{Z_i} is the projective cover of Z_i, and the kernel of p_{Z_i} is given by

$$\mathcal{O}(\varepsilon e_1 - X^{n-i}e_2) \oplus \cdots \oplus \mathcal{O}(\varepsilon X^{i-1}e_1 - X^{n-1}e_2) \oplus \mathcal{O}X^i e_1 \oplus \cdots \oplus \mathcal{O}X^{n-1}e_1 \simeq Z_{n-i}.$$

Lemma 4.1.2 ([AKM, Lemma 2.3])**.** The upper exact sequence in the above commutative diagram is the almost split sequence ending at Z_i.

Proof. By Theorem 3.5.8, it is enough to show the following two statements.

(1) The A-module homomorphism φ_i does not factor through p_{Z_i}.

(2) For any $\rho \in \mathsf{radEnd}_A(Z_i)$, $\phi_i\rho$ factors through p_{Z_i}.

Suppose that there is a morphism $\mu = (\mu_1, \mu_2) : Z_i \to A \oplus A$ such that $p_{Z_i}\mu = \varphi_i$. Then we have $X^{n-i}\mu_1(\varepsilon) - \varepsilon\mu_2(\varepsilon) = \varepsilon(\mu_1(X^{n-i}) - \mu_2(\varepsilon)) = X^{n-1}$, a contradiction. Thus, (1) follows.

Next, we show (2). Write $\rho(\varepsilon) = a_0\varepsilon + \cdots + a_{n-i-1}\varepsilon X^{n-i-1} + a_{n-i}X^{n-i} + \cdots + a_{n-1}X^{n-1}$. Then, by Lemma 4.1.1, there exists $a \in \mathcal{O}$ such that $a_0 = \varepsilon a$. We define $\mu \in \mathsf{Hom}_A(Z_i, A \oplus A)$ by $\mu(\varepsilon) = -aX^{n-1}e_2$. Then, it is easy to check that $p_{Z_i}\mu = \varphi_i\rho$ holds. $\square$

By the above lemma, we have the almost split sequence $0 \longrightarrow Z_{n-i} \longrightarrow E_i \longrightarrow Z_i \longrightarrow 0$, where

$$\begin{aligned} E_i =& \mathcal{O}(\varepsilon, X^{n-i}, 0) \oplus \mathcal{O}(\varepsilon X, X^{n-i+1}, 0) \oplus \cdots \oplus \mathcal{O}(\varepsilon X^{i-1}, X^{n-1}, 0) \\ & \oplus \mathcal{O}(X^i, 0, 0) \oplus \mathcal{O}(X^{i+1}, 0, 0) \oplus \cdots \oplus \mathcal{O}(X^{n-1}, 0, 0) \\ & \oplus\ \mathcal{O}(X^{i-1}, 0, \varepsilon) \oplus \mathcal{O}(0, 0, \varepsilon X) \oplus \cdots \oplus \mathcal{O}(0, 0, \varepsilon X^{n-i-1}) \\ & \oplus\ \mathcal{O}(0, 0, X^{n-i}) \oplus \mathcal{O}(0, 0, X^{n-i+1}) \oplus \cdots \oplus \mathcal{O}(0, 0, X^{n-1}). \end{aligned}$$

To simplify the notation, we define $a_0 = b_0 = 0$ and

$$a_k = \begin{cases} (X^{n-k}, 0, 0) & \text{if } 1 \le k \le n-i, \\ (\varepsilon X^{n-k}, X^{2n-k-i}, 0) & \text{if } n-i < k \le n, \end{cases}$$
$$b_k = \begin{cases} (0, 0, X^{n-k}) & \text{if } 1 \le k \le i, \\ (0, 0, \varepsilon X^{n-k}) & \text{if } i < k < n, \\ (X^{i-1}, 0, \varepsilon) & \text{if } k = n. \end{cases}$$

Then, we have

$$Xa_k = \begin{cases} a_{k-1} & \text{if } k \ne n-i+1, \\ \varepsilon a_{k-1} & \text{if } k = n-i+1, \end{cases}$$
$$Xb_k = \begin{cases} b_{k-1} & \text{if } k \ne i+1, n, \\ \varepsilon b_{k-1} & (k = i+1) \\ a_{n-i} + b_{n-1} & \text{if} k = n. \end{cases}$$

and

$$\mathsf{Ker}(X^k) = \bigoplus_{1 \le j \le k} (\mathcal{O}a_j \oplus \mathcal{O}b_j).$$

4.2 The middle terms of almost split sequences ending at Heller lattices

In this section, we show that the middle term E_i of the almost split sequence ending at Z_i is indecomposable, for $2 \le i \le n-1$, and the middle term of the almost split sequence ending at Z_1 has only one non-projective indecomposable direct summand.

Proposition 4.2.1 ([AKM, Proposition 2.4])**.** The following statements hold.

(1) A is an indecomposable direct summand of E_1, and the other direct summand is indecomposable.

(2) For $2 \le i \le n-1$, the A-lattice E_i is indecomposable.

Proof. (1) As $Z_{n-1} = \mathsf{rad}A$, it follows from Theorem 3.5.13 that A is a direct summand of E_1. We also give more explicit computational proof here. Define $x_k, y_k \in E_1$, for $1 \le k \le n$, as follows:

$$x_k = \begin{cases} a_1 + \varepsilon b_1 & \text{if } k = 1, \\ a_k + b_k & \text{if } 2 \le k \le n-1, \\ b_n & \text{if } k = n, \end{cases}$$
$$y_k = \begin{cases} b_k & \text{if } 1 \le k \le n-1, \\ a_n - \varepsilon b_n & \text{if } k = n. \end{cases}$$

Then they form an $\mathcal{O}$-basis of E_1. Moreover, we have $Xx_1 = 0$ and $Xy_1 = 0$,

$$Xx_k = x_{k-1}, \text{ for } 2 \le k \le n, \text{ and } Xy_k = \begin{cases} \varepsilon y_1 & \text{if } k = 2, \\ y_{k-1} & \text{if } 3 \le k \le n-1, \\ -\varepsilon y_{n-1} & \text{if } k = n. \end{cases}$$

Thus, the $\mathcal{O}$-span of $\{x_k \mid 1 \le k \le n\}$ is isomorphic to the indecomposable projective A-lattice A. In particular, A is an indecomposable direct summand of E_1, and the other direct summand is indecomposable, because it becomes $A \otimes \mathcal{K}$ after tensoring with $\mathcal{K}$.

(2) E_{n-1} does not have a projective direct summand by Theorem 3.5.13. As $E_{n-1} \simeq \tau(E_1)$, the A-lattice E_{n-1} is indecomposable. We assume $2 \le i \le n-2$ in the rest of the proof.

Suppose that $E_i = E' \oplus E''$ with $E' \neq 0 \neq E''$ as an A-lattice. Since $E_i \otimes \mathcal{K}$ is projective as an $A \otimes \mathcal{K}$-module and $\mathsf{rank}(E_i) = 2n$, we have $E' \otimes \mathcal{K} \simeq A \otimes \mathcal{K} \simeq E'' \otimes \mathcal{K}$. In particular, $\mathsf{rank}(E') = n = \mathsf{rank}(E'')$. Since

$$0 \to E' \cap \mathsf{Ker}(X^k) \to E' \to \mathsf{Im}(X^k) \to 0$$

and $\mathsf{Im}(X^k)$ is a free $\mathcal{O}$-module, we have the increasing sequence of $\mathcal{O}$-submodules

$$0 \subsetneq \cdots \subsetneq E' \cap \mathsf{Ker}(X^k) \subsetneq E' \cap \mathsf{Ker}(X^{k+1}) \subsetneq \cdots \subsetneq E' \cap \mathsf{Ker}(X^n) = E'$$

such that all the $\mathcal{O}$-submodules are direct summands of E' as $\mathcal{O}$-modules. Thus, we may choose an $\mathcal{O}$-basis $\{e'_k\}_{1\leq k\leq n}$ such that $e'_k \in E' \cap \mathsf{Ker}(X^k) \setminus \mathsf{Ker}(X^{k-1})$. Similarly, we may choose an $\mathcal{O}$-basis $\{e''_k\}_{1\leq k\leq n}$ of E'' such that $e''_k \in E'' \cap \mathsf{Ker}(X^k) \setminus \mathsf{Ker}(X^{k-1})$. Write

$$\begin{array}{rcll} e'_k & = & \alpha_k a_k + \beta_k b_k + A'_k, & \text{for } \alpha_k, \beta_k \in \mathcal{O} \text{ and } A'_k \in \mathsf{Ker}(X^{k-1}), \\ e''_k & = & \gamma_k a_k + \delta_k b_k + A''_k, & \text{for } \gamma_k, \delta_k \in \mathcal{O} \text{ and } A''_k \in \mathsf{Ker}(X^{k-1}). \end{array}$$

Without loss of generality, we may assume $A'_k \in \mathsf{Ker}(X^{k-1}) \cap E''$ and $A''_k \in \mathsf{Ker}(X^{k-1}) \cap E'$. Since $\{e'_k, e''_k\}$ and $\{a_k, b_k\}$ are $\mathcal{O}$-bases of $\mathsf{Ker}(X^k)/\mathsf{Ker}(X^{k-1})$, we have $\alpha_k\delta_k - \beta_k\gamma_k \notin \varepsilon\mathcal{O}$.

As $Xe'_k \in \mathsf{Ker}(X^{k-1}) \cap E'$, there are $f^{(k)}_{k-1}, \ldots, f^{(k)}_1 \in \mathcal{O}$ such that

$$Xe'_k = f^{(k)}_{k-1} e'_{k-1} + \cdots + f^{(k)}_1 e'_1.$$

Similarly, there are $g^{(k)}_{k-1}, \ldots, g^{(k)}_1 \in \mathcal{O}$ such that

$$Xe''_k = g^{(k)}_{k-1} e''_{k-1} + \cdots + g^{(k)}_1 e''_1.$$

The coefficient of a_{k-1} in Xe'_k is given by

$$\begin{cases} \alpha_k & \text{if } k \neq n-i+1, \\ \varepsilon\alpha_k & \text{if } k = n-i+1. \end{cases}$$

Thus, we have

$$f^{(k)}_{k-1}\alpha_{k-1} = \begin{cases} \alpha_k & \text{if } k \neq n-i+1, \\ \varepsilon\alpha_k & \text{if } k = n-i+1. \end{cases}$$

Similarly, we have the following:

$$f^{(k)}_{k-1}\beta_{k-1} = \begin{cases} \beta_k & \text{if } k \neq i+1, \\ \varepsilon\beta_k & \text{if } k = i+1. \end{cases}$$

$$g^{(k)}_{k-1}\gamma_{k-1} = \begin{cases} \gamma_k & \text{if } k \neq n-i+1, \\ \varepsilon\gamma_k & \text{if } k = n-i+1. \end{cases}$$

$$g^{(k)}_{k-1}\delta_{k-1} = \begin{cases} \delta_k & \text{if } k \neq i+1, \\ \varepsilon\delta_k & \text{if } k = i+1. \end{cases}$$

We shall deduce a contradiction in the following three cases and conclude that E_i is indecomposable for $2 \leq i \leq n-2$:

(a) $2 \leq n-i < i$ (b) $2 \leq i = n-i$ (c) $2 \leq i < n-i$

Suppose that we are in (a). We multiply each of e'_k and e''_k by suitable invertible elements to get new $\mathcal{O}$-bases of E' and E'' in order to have the equalities

$$f^{(k)}_{k-1} = \begin{cases} 1 & \text{if } k \neq n-i+1, \\ \varepsilon & \text{if } k = n-i+1, \end{cases} \quad \text{and} \quad g^{(k)}_{k-1} = \begin{cases} 1 & \text{if } k \neq i+1, \\ \varepsilon & \text{if } k = i+1, \end{cases}$$

in the new bases. For $k=1$, we keep the original basis elements e'_1 and e''_1. Suppose that we have already chosen new e'_j and e''_j for $1 \leq j \leq k-1$. If $k \neq n-i+1, i+1$, then it follows from

$$f^{(k)}_{k-1} g^{(k)}_{k-1} (\alpha_{k-1}\delta_{k-1} - \beta_{k-1}\gamma_{k-1}) = \alpha_k\delta_k - \beta_k\gamma_k$$

that $f^{(k)}_{k-1}$ and $g^{(k)}_{k-1}$ are invertible. Thus, multiplying e'_k and e''_k with their inverses respectively, we have $f^{(k)}_{k-1} = 1$, $g^{(k)}_{k-1} = 1$ in the new basis. Note that we have

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} = \cdots\cdot = \begin{pmatrix} \alpha_{n-i} & \beta_{n-i} \\ \gamma_{n-i} & \delta_{n-i} \end{pmatrix}.$$

If $k = n-i+1$, then, by using $i \neq n-i$, we have

$$\begin{aligned} f^{(n-i+1)}_{n-i} g^{(n-i+1)}_{n-i} \alpha_{n-i}\delta_{n-i} &= \varepsilon\alpha_{n-i+1}\delta_{n-i+1}, \\ f^{(n-i+1)}_{n-i} g^{(n-i+1)}_{n-i} \beta_{n-i}\gamma_{n-i} &= \varepsilon\beta_{n-i+1}\gamma_{n-i+1}. \end{aligned}$$

It implies that $f^{(n-i+1)}_{n-i} g^{(n-i+1)}_{n-i} \in \varepsilon\mathcal{O} \setminus \varepsilon^2\mathcal{O}$. Thus, we may assume $f^{(n-i+1)}_{n-i} = \varepsilon$ and $g^{(n-i+1)}_{n-i} = 1$ by swapping E' and E'' if necessary. Then, we have

$$\begin{pmatrix} \alpha_{n-i} & \beta_{n-i} \\ \gamma_{n-i} & \delta_{n-i} \end{pmatrix} = \begin{pmatrix} \alpha_{n-i+1} & \varepsilon^{-1}\beta_{n-i+1} \\ \varepsilon\gamma_{n-i+1} & \delta_{n-i+1} \end{pmatrix} = \cdots = \begin{pmatrix} \alpha_i & \varepsilon^{-1}\beta_i \\ \varepsilon\gamma_i & \delta_i \end{pmatrix}.$$

Finally, if $k = i+1$, then the similar argument shows $f^{(i+1)}_i g^{(i+1)}_i \in \varepsilon\mathcal{O} \setminus \varepsilon^2\mathcal{O}$, and we may assume that $(f^{(i+1)}_i, g^{(i+1)}_i)$ is either $(\varepsilon, 1)$ or $(1, \varepsilon)$. In the former case, we have the equalities

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_i & \varepsilon^{-1}\beta_i \\ \varepsilon\gamma_i & \delta_i \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\alpha_{i+1} & \varepsilon^{-1}\beta_{i+1} \\ \varepsilon\gamma_{i+1} & \varepsilon\delta_{i+1} \end{pmatrix},$$

which implies that $\alpha_{i+1}, \beta_{i+1} \in \varepsilon\mathcal{O}$, a contradiction. Thus, we obtain $f^{(i+1)}_i = 1$, $g^{(i+1)}_i = \varepsilon$. Therefore, we have obtained the desired formula. In particular, we have

$$\begin{gathered} \alpha_{k-1} = \alpha_k, \;\; f^{(k)}_{k-1}\beta_{k-1} = g^{(k)}_{k-1}\beta_k, \;\; g^{(k)}_{k-1}\gamma_{k-1} = f^{(k)}_{k-1}\gamma_k, \;\; \delta_{k-1} = \delta_k, \\ Xa_k = f^{(k)}_{k-1}a_{k-1}, \;\; Xb_k = g^{(k)}_{k-1}b_{k-1} + \delta_{k,n}a_{n-i}. \end{gathered}$$

Suppose that $1 \le k \le n-1$. Then, we have

$$\begin{aligned}
XA'_k &= X(e'_k - \alpha_k a_k - \beta_k b_k) \\
&= Xe'_k - f^{(k)}_{k-1}\alpha_k a_{k-1} - g^{(k)}_{k-1}\beta_k b_{k-1}, \\
f^{(k)}_{k-1}A'_{k-1} &= f^{(k)}_{k-1}(e'_{k-1} - \alpha_{k-1}a_{k-1} - \beta_{k-1}b_{k-1}) \\
&= f^{(k)}_{k-1}e'_{k-1} - f^{(k)}_{k-1}\alpha_k a_{k-1} - g^{(k)}_{k-1}\beta_k b_{k-1}.
\end{aligned}$$

We compute $Xe'_k - f^{(k)}_{k-1}e'_{k-1}$ in two ways:

$$\begin{aligned}
Xe'_k - f^{(k)}_{k-1}e'_{k-1} &= XA'_k - f^{(k)}_{k-1}A'_{k-1} \in E'', \\
Xe'_k - f^{(k)}_{k-1}e'_{k-1} &= f^{(k)}_{k-2}e'_{k-2} + \cdots + f^{(k)}_1 e'_1 \in E'.
\end{aligned}$$

This implies $Xe'_k = f^{(k)}_{k-1}e'_{k-1}$ for $1 \le k \le n-1$.

Next suppose that $k = n$. Then, the similar computation shows

$$\beta_n a_{n-i} + XA'_n - f^{(n)}_{n-1}A'_{n-1} = Xe'_n - f^{(n)}_{n-1}e'_{n-1} = f^{(n)}_{n-2}e'_{n-2} + \cdots + f^{(n)}_1 e'_1.$$

By computing $X^{n-i+1}e'_n - f^{(n)}_{n-1}X^{n-i}e'_{n-1}$ in two ways as before, we obtain

$$X^{n-i+1}A'_n - f^{(n)}_{n-1}X^{n-i}A'_{n-1} = f^{(n)}_{n-2}X^{n-i}e'_{n-2} + \cdots + f^{(n)}_1 X^{n-i}e'_1 = 0.$$

Hence, we have $f^{(n)}_{n-2} = \cdots = f^{(n)}_{n-i+1} = 0$. Now, define

$$z_n = e'_n, \quad z_k = e'_k + X^{n-k-1}(f^{(n)}_{n-i}e'_{n-i} + \cdots + f^{(n)}_1 e'_1)$$

for $1 \le k \le n-1$. Then, $\{z_k \mid 1 \le k \le n\}$ is an $\mathcal{O}$-basis of E', since $X^{n-k-1}(f^{(n)}_{n-i}e'_{n-i} + \cdots + f^{(n)}_1 e'_1)$ belongs to $\mathsf{Ker}(X^{k-1})$. Further, we have $z_k = e'_k$, for $1 \le k \le i-1$. In particular, $z_{n-i} = e'_{n-i}$ by $n-i \le i-1$. Then, we can check that

$$Xz_k = \begin{cases} z_{k-1} & \text{if } k \ne n-i+1, \\ \varepsilon z_{k-1} & \text{if } k = n-i+1. \end{cases}$$

Thus, we conclude that $E' \simeq Z_{n-i}$. Recall that the exact sequence

$$0 \to Z_{n-i} \to E_i \to Z_i \to 0$$

does not split. On the other hand, $E_i \simeq Z_{n-i} \oplus Z_i$ implies that it must split by Theorem 2.3.3. Hence, E_i is indecomposable in (a).

Next assume that we are in (b). Then, For $k \ne i+1$, $f^{(k)}_{k-1}$ and $g^{(k)}_{k-1}$ are invertible as before, and we may choose $f^{(k)}_{k-1} =$ and $g^{(k)}_{k-1} = 1$.

If $k = i+1$, note that

$$f_i^{(i+1)}\alpha_i = \varepsilon\alpha_{i+1},\ \ f_i^{(i+1)}\beta_i = \varepsilon\beta_{i+1},\ \ g_i^{(i+1)}\gamma_i = \varepsilon\gamma_{i+1},\ \ g_i^{(i+1)}\delta_i = \varepsilon\delta_{i+1}.$$

Thus, $\alpha_i, \beta_i \in \varepsilon\mathcal{O}$ if $f_i^{(i+1)}$ is invertible, and $\gamma_i, \delta_i \in \varepsilon\mathcal{O}$ if $g_i^{(i+1)}$ is invertible. But both are impossible. Further, $f_i^{(i+1)}g_i^{(i+1)}(\alpha_i\delta_i - \beta_i\gamma_i) = \varepsilon^2(\alpha_{i+1}\delta_{i+1} - \beta_{i+1}\gamma_{i+1})$ implies $f_i^{(i+1)}g_i^{(i+1)} \in \varepsilon^2\mathcal{O} \setminus \varepsilon^3\mathcal{O}$. Thus, we may choose $f_i^{(i+1)} = \varepsilon$ and $g_i^{(i+1)} = \varepsilon$. Hence, we may assume without loss of generality that

$$f_{k-1}^{(k)} = g_{k-1}^{(k)} = \begin{cases} 1 & \text{if } k \neq i+1, \\ \varepsilon & \text{if } k = i+1, \end{cases}$$

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \cdots\cdot = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} = \begin{pmatrix} \alpha_{i+1} & \beta_{i+1} \\ \gamma_{i+1} & \delta_{i+1} \end{pmatrix} = \cdots\cdot = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$$

and $Xa_k = f_{k-1}^{(k)}a_{k-1}$, $Xb_k = g_{k-1}^{(k)}b_{k-1} + \delta_{k,n}a_i$. For $1 \leq k \leq n-1$, we have

$$XA'_k - f_{k-1}^{(k)}A'_{k-1} = Xe'_k - f_{k-1}^{(k)}e'_{k-1} = f_{k-2}^{(k)}e'_{k-2} + \cdots + f_1^{(k)}e'_1,$$

and the same argument as before shows that

$$Xe'_k = \begin{cases} f_{k-1}^{(k)}e'_{k-1} & \text{if } k \neq n, \\ f_{n-1}^{(n)}e'_{n-1} + f_i^{(n)}e'_i + \cdots + f_1^{(n)}e'_1 & \text{if } k = n. \end{cases}$$

Now, we compute

$$\begin{aligned} &X^{i-1}e'_{n-1} = f_{n-2}^{(n-1)} \cdots f_{n-i}^{(n-i+1)}e'_{n-i} = \varepsilon e'_i, \\ &X^i a_n = f_{n-1}^{(n)} \cdots f_i^{(i+1)}a_i = \varepsilon a_i, \\ &X^i b_n = X^{i-1}(b_{n-1} + a_i) = g_{n-2}^{(n-1)} \cdots g_{n-i}^{(i+1)}b_i + f_{i-1}^{(i)} \cdots f_1^{(2)}a_1 = \varepsilon b_i + a_1. \end{aligned}$$

Thus, we have

$$\begin{aligned} X^i e'_n - X^{i-1}e'_{n-1} &= X^i(\alpha_n a_n + \beta_n b_n + A'_n) - \varepsilon e'_i \\ &= \varepsilon(\alpha_n a_i + \beta_n b_i - e'_i) + X^i A'_n + \beta_n a_1. \end{aligned}$$

If $i+1 \leq k \leq n-1$, then $k - i + 1 \leq n - i = i$ and $X^i e'_k = f_{k-1}^{(k)} \cdots f_{k-i}^{(k-i+1)}e'_{k-i} \in \varepsilon E'$. Thus, $X^i A'_n \in \varepsilon E'$ follows. On the other hand, we have

$$\begin{aligned} X^i e'_n - X^{i-1}e'_{n-1} &= X^{i-1}(Xe'_n - e'_{n-1}) \\ &= X^{i-1}(f_i^{(n)}e'_i + \cdots + f_1^{(n)}e'_1) \\ &= f_i^{(n)}X^{i-1}e'_i \\ &= f_i^{(n)}X^{i-1}(\alpha_i a_i + \beta_i b_i + A'_i) \\ &= f_i^{(n)}(\alpha_i X^{i-1}a_i + \beta_i X^{i-1}b_i) \\ &= f_i^{(n)}(\alpha_i a_1 + \beta_i b_1). \end{aligned}$$

Hence, we obtain $\beta_n a_1 \equiv f_i^{(n)}(\alpha_i a_1 + \beta_i b_1) \bmod \varepsilon\mathcal{O}$. The similar computation using e''_k shows $\delta_n a_1 \equiv f_i^{(n)}(\gamma_i a_1 + \delta_i b_1) \bmod \varepsilon\mathcal{O}$. If $f_i^{(n)}$ was invertible, it would imply $\beta_i, \delta_i \in \varepsilon\mathcal{O}$, which contradicts $\alpha_i\delta_i - \beta_i\gamma_i \in \mathcal{O}^\times$. Thus, $f_i^{(n)} \in \varepsilon\mathcal{O}$ and we have $\beta_n, \delta_n \in \varepsilon\mathcal{O}$, which is again a contradiction. Hence, E_i is indecomposable in (b).

Finally, suppose that we are in (c). Since $E_i \simeq \tau(E_{n-i})$, for $2 \le i \le n-2$, and E_{n-i} is indecomposable by (a), the A-lattice E_i is indecomposable in (c). □

Corollary 4.2.2. The following statements hold.

(1) Any Heller component does not have a loop.

(2) Any Heller lattice appears on the boundary of a Heller component.

Proof. It is enough to show (1). However, the claim follows from Corollary 3.7.7 and Proposition 4.2.1 immediately. □

4.3 The almost split sequence ending at E_i

Recall that the A-lattice E_i is given by

$$\begin{aligned} E_i =& \mathcal{O}(\varepsilon, X^{n-i}, 0) \oplus \mathcal{O}(\varepsilon X, X^{n-i+1}, 0) \oplus \cdots \oplus \mathcal{O}(\varepsilon X^{i-1}, X^{n-1}, 0) \\ &\oplus \mathcal{O}(X^i, 0, 0) \oplus \mathcal{O}(X^{i+1}, 0, 0) \oplus \cdots \oplus \mathcal{O}(X^{n-1}, 0, 0) \\ &\oplus\ \mathcal{O}(X^{i-1}, 0, \varepsilon) \oplus \mathcal{O}(0, 0, \varepsilon X) \oplus \cdots \oplus \mathcal{O}(0, 0, \varepsilon X^{n-i-1}) \\ &\oplus\ \mathcal{O}(0, 0, X^{n-i}) \oplus \mathcal{O}(0, 0, X^{n-i+1}) \oplus \cdots \oplus \mathcal{O}(0, 0, X^{n-1}), \end{aligned}$$

and

$$a_k = \begin{cases} (X^{n-k}, 0, 0) & \text{if } 1 \le k \le n-i, \\ (\varepsilon X^{n-k}, X^{2n-k-i}, 0) & \text{if } n-i < k \le n, \end{cases}$$

$$b_k = \begin{cases} (0, 0, X^{n-k}) & \text{if } 1 \le k \le i, \\ (0, 0, \varepsilon X^{n-k}) & \text{if } i < k < n, \\ (X^{i-1}, 0, \varepsilon) & \text{if } k = n. \end{cases}$$

In this section, we construct the almost split sequence ending at E_i for $2 \le i \le n-2$. For $2 \le i \le n-2$, we define $\pi : A^{\oplus 4} \to E_i$ by

$$\pi(1,0,0,0) = a_n, \quad \pi(0,1,0,0) = b_n, \quad \pi(0,0,1,0) = b_{n-1}, \quad \pi(0,0,0,1) = b_i.$$

Lemma 4.3.1 ([AKM, Lemma 2.5])**.** Let $\pi : A^{\oplus 4} \to E_i$ be as above. Then, the following statements hold.

(1) The A-module homomorphism π is an epimorphism.

(2) There is an isomorphism $\mathsf{Ker}(\pi) \simeq E_{n-i}$ for $2 \leq i \leq n-2$.

Proof. (1) It is easy to check that $a_k, b_k \in \mathsf{Im}(\pi)$ for $1 \leq k \leq n$. Note that E_i is generated by $\{a_n, b_n, b_{n-1}, b_i\}$ as an A-module and $a_{n-i} = Xb_n - b_{n-1}$.

(2) We define an A-module homomorphism $\iota : E_{n-i} \to A^{\oplus 4}$ by

$$\iota(f,g,h) = \left(g, -Xf + \frac{X^{n-i}h}{\varepsilon}, f, -h\right),$$

where $(f,g,h) \in E_{n-i}$. Since $h \in Z_i$, one can write $h = h_0\varepsilon + h_1\varepsilon X + \cdots + h_{i-1}\varepsilon X^{i-1} + h_iX^i + \cdots + h_{n-1}X^{n-1}$, where $h_i \in \mathcal{O}$. Then, we have

$$\frac{X^{n-i}h}{\varepsilon} = h_0X^{n-i} + h_1X^{n-i+1} + \cdots + h_{i-1}X^{n-1}.$$

Note that $(f,g,h) \in A^{\oplus 3}$ belongs to E_{n-i} if and only if $h \in Z_{n-i}$ and $X^if - \varepsilon g = h_0X^{n-1}$. It is clear that ι is a monomorphism and it suffices to show that $\mathsf{Im}(\iota) = \mathsf{Ker}(\pi)$. Since

$$\begin{aligned}
\pi\iota(f,g,h) &= \left(\varepsilon g - X^if + \frac{X^{n-1}h}{\varepsilon}, X^{n-i}g, \varepsilon\left(-Xf + \frac{X^{n-i}h}{\varepsilon}\right) + \varepsilon Xf - X^{n-i}h\right)\\
&= \left(\varepsilon g - X^if + \frac{X^{n-1}h}{\varepsilon}, X^{n-i}g, 0\right)\\
&= (0,0,0),
\end{aligned}$$

we have $\mathsf{Im}(\iota) \subseteq \mathsf{Ker}(\pi)$. Let $(p,q,r,s) \in \mathsf{Ker}(\pi)$. Then we have $\varepsilon p + X^{i-1}q = 0$, $X^{n-i}p = 0$ and $\varepsilon q + \varepsilon Xr + X^{n-i}s = 0$. The third equation shows that the projective cover $A \to M_{n-i}$ given by $f \mapsto X^{n-i}f + \varepsilon A$ sends s to 0. Thus, we have $s \in Z_{n-i}$. Further,

$$X^{n-1}s + \varepsilon(-\varepsilon p + X^ir) = X^{n-1}s + \varepsilon(X^{i-1}q + X^ir) = X^{i-1}(X^{n-i}s + \varepsilon q + Xr) = 0$$

implies $X^ir - \varepsilon p = \dfrac{X^{n-1}(-s)}{\varepsilon}$. Hence, we have $(r,p,-s) \in E_{n-i}$ and

$$\iota(r,p,-s) = \left(p, -Xr - \frac{X^{n-i}s}{\varepsilon}, r, s\right) = (p,q,r,s).$$

Therefore, we have $\mathsf{Ker}(\pi) = \mathsf{Im}(\iota)$, which implies $\mathsf{Ker}(\pi) \simeq E_{n-i}$. $\square$

We consider the following pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & E_{n-i} \arrow[r] \arrow[d, equal] & F_i \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[d, "\phi_i"] & 0 \\
0 \arrow[r] & E_{n-i} \arrow[r, "\iota"] & A^{\oplus 4} \arrow[r, "\pi"] & E_i \arrow[r] & 0,
\end{tikzcd}$$

where ι is the isomorphism $E_{n-i} \simeq \mathsf{Ker}(\pi)$ defined in the proof of Lemma 4.3.1, and

$$\begin{aligned} \phi_i(a_k) &= 0 && \text{for } 1 \le k \le n, \\ \phi_i(b_k) &= 0 && \text{for } 1 \le k \le n-1, \\ \phi_i(b_n) &= b_1 && \text{for } k = n. \end{aligned}$$

Lemma 4.3.2 ([AKM, Lemma 2.6])**.** Suppose that $2 \le i \le n-i$. Let $\rho \in \mathsf{radEnd}_A(E_i)$ such that

$$\rho(a_n) = \alpha a_n + \beta b_n + A, \quad \rho(b_n) = \alpha' a_n + \beta' b_n + B,$$

where $\alpha, \beta, \alpha', \beta' \in \mathcal{O}$ and $A, B \in \mathsf{Ker}(X^{n-1})$. Then we have the following.

(1) $\beta \in \varepsilon\mathcal{O}$, and $\alpha \in \varepsilon\mathcal{O}$ if and only if $\beta' \in \varepsilon\mathcal{O}$.

(2) $\alpha\beta' - \beta\alpha' \in \varepsilon\mathcal{O}$.

Proof. (1) We compute $\rho(\varepsilon X^{n-i} b_n - X^{n-1} a_n)$ in two ways. Since $X^{n-i} b_n = \varepsilon b_i + a_1$ and $X^{n-1} a_n = \varepsilon a_1$, we have $\rho(\varepsilon X^{n-i} b_n - X^{n-1} a_n) = \varepsilon^2 \rho(b_i) \in \varepsilon^2 E_i$. On the other hand, since $X^{n-i} b_n = \varepsilon b_i + a_1$, we have

$$\begin{aligned} \rho(\varepsilon X^{n-i} b_n - X^{n-1} a_n) &= \varepsilon X^{n-i}(\alpha' a_n + \beta' b_n + B) - X^{n-1}(\alpha a_n + \beta b_n + A) \\ &= \varepsilon \alpha' X^{n-i} a_n + \varepsilon^2 \beta' b_i + \varepsilon(\beta' - \alpha) a_1 - \varepsilon \beta b_1 + \varepsilon X^{n-i} B. \end{aligned}$$

Then, $X^{n-i} a_k = \varepsilon a_{k-n+i}$ and $X^{n-i} b_k = \varepsilon b_{k-n+i}$, for $n-i+1 \le k \le n-1$, imply that $\varepsilon X^{n-i} B \in \varepsilon^2 E_i$. Hence, we may divide the both sides by ε. Reducing modulo ε, we have

$$(\beta' - \alpha) a_1 - \beta b_1 \equiv 0 \ \mathsf{mod}\ \varepsilon E_i$$

since $X^{n-i} a_n \equiv 0 \ \mathsf{mod}\ \varepsilon E_i$. Now, the claim is clear.

(2) Since $\rho(a_k), \rho(b_k) \in \mathsf{Ker}(X^k)$, we may write

$$\begin{aligned} \rho(a_k) &= \alpha_k a_k + \beta_k b_k + A_k, \\ \rho(b_k) &= \alpha'_k a_k + \beta'_k b_k + B_k, \end{aligned}$$

where $\alpha_k, \beta_k, \alpha'_k, \beta'_k \in \mathcal{O}$ and $A_k, B_k \in \mathsf{Ker}(X^{k-1})$. We claim that

$$\alpha_k \beta'_k - \beta_k \alpha'_k = \alpha\beta' - \beta\alpha'.$$

To see this, observe that we have the following identities in $E_i/\mathsf{Ker}(X^{k-1})$.

$$\begin{cases} \alpha a_k + \beta b_k \equiv \rho(X^{n-k} a_n) \equiv \rho(a_k) & \mathsf{mod}\ \mathsf{Ker}(X^{k-1}) \ \text{ if } k > n-i, \\ \alpha\varepsilon a_k + \beta b_k \equiv \rho(X^{n-k} a_n) \equiv \varepsilon\rho(a_k) & \mathsf{mod}\ \mathsf{Ker}(X^{k-1}) \ \text{ if } i < k \le n-i, \\ \alpha\varepsilon a_k + \beta\varepsilon b_k \equiv \rho(X^{n-k} a_n) \equiv \varepsilon\rho(a_k) & \mathsf{mod}\ \mathsf{Ker}(X^{k-1}) \ \text{ if } k \le i, \end{cases}$$

$$\begin{cases} \alpha' a_k + \beta' b_k \equiv \rho(X^{n-k}b_n) \equiv \rho(b_k) & \mathsf{mod}\ \mathsf{Ker}(X^{k-1}) \quad \text{if } k > n-i, \\ \alpha' \varepsilon a_k + \beta' b_k \equiv \rho(X^{n-k}b_n) \equiv \rho(b_k) & \mathsf{mod}\ \mathsf{Ker}(X^{k-1}) \quad \text{if } i < k \leq n-i, \\ \alpha' \varepsilon a_k + \beta' \varepsilon b_k \equiv \rho(X^{n-k}b_n) \equiv \varepsilon\rho(b_k) & \mathsf{mod}\ \mathsf{Ker}(X^{k-1}) \quad \text{if } k \leq i. \end{cases}$$

Thus, if we put

$$\begin{aligned} (\overline{a}_k, \overline{b}_k) &= (a_k + \mathsf{Ker}(X^{k-1}), b_k + \mathsf{Ker}(X^{k-1})), \\ (\overline{a}'_k, \overline{b}'_k) &= (\rho(a_k) + \mathsf{Ker}(X^{k-1}), \rho(b_k) + \mathsf{Ker}(X^{k-1})), \end{aligned}$$

then we have

$$(\overline{a}_k, \overline{b}_k) \begin{pmatrix} \alpha_k & \alpha'_k \\ \beta_k & \beta'_k \end{pmatrix} = (\overline{a}'_k, \overline{b}'_k) = (\overline{a}_k, \overline{b}_k) \begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix}$$

or

$$(\overline{a}_k, \overline{b}_k) \begin{pmatrix} \alpha & \alpha'\varepsilon \\ \beta\varepsilon^{-1} & \beta' \end{pmatrix}.$$

Therefore, we have $\alpha_k\beta'_k - \beta_k\alpha'_k = \alpha\beta' - \beta\alpha'$. In particular, if $\alpha\beta' - \beta\alpha' \in \mathcal{O}^\times$, then ρ is surjective, which contradicts with $\rho \in \mathsf{radEnd}_A(E_i)$. $\square$

Lemma 4.3.3 ([AKM, Lemma 2.7])**.** Suppose that $2 \leq i \leq n-i$, and let $\phi_i \in \mathsf{End}_A(E_i)$ be as in the definition of the pullback diagram. Then we have the following.

(1) ϕ_i does not factor through π.

(2) For any $\rho \in \mathsf{radEnd}_A(E_i)$, the A-module homomorphism $\phi_i\rho$ factors through π.

Proof. (1) Suppose that there exists $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) : E_i \to A^{\oplus 4}$ such that $\pi\psi = \phi_i$. Then, we have

$$\begin{aligned} 0 = \pi\psi(a_n) &= (\varepsilon\psi_1(a_n) + X^{i-1}\psi_2(a_n), X^{n-i}\psi_1(a_n), \varepsilon\psi_2(a_n) + \varepsilon X\psi_3(a_n) + X^{n-i}\psi_4(a_n)), \\ b_1 = \pi\psi(b_n) &= (\varepsilon\psi_1(b_n) + X^{i-1}\psi_2(b_n), X^{n-i}\psi_1(b_n), \varepsilon\psi_2(b_n) + \varepsilon X\psi_3(b_n) + X^{n-i}\psi_4(b_n)). \end{aligned}$$

The first equality implies $\psi_4(X^{n-1}a_n) \in \varepsilon^2 A$ by the following computation:

$$\begin{aligned} \psi_4(X^{n-1}a_n) &= X^{i-1}(X^{n-i}\psi_4(a_n)) = -X^{i-1}(\varepsilon\psi_2(a_n) + \varepsilon X\psi_3(a_n)) \\ &= -\varepsilon X^{i-1}\psi_2(a_n) - \varepsilon\psi_3(X^i a_n) = \varepsilon^2\psi_1(a_n) - \varepsilon^2\psi_3(a_{n-i}) \end{aligned}$$

Thus, we conclude $\psi_4(X^{n-i}b_n) \equiv 0 \ \mathsf{mod}\ \varepsilon A$ from

$$\begin{aligned} \varepsilon\psi_4(X^{n-i}b_n) &= \varepsilon\psi_4(X^{n-i-1}a_{n-i} + X^{n-i-1}b_{n-1}) = \varepsilon\psi_4(a_1 + \varepsilon b_i) \\ &= \psi_4(\varepsilon a_1) + \varepsilon^2\psi_4(b_i) = \psi(X^{n-1}a_n) + \varepsilon^2\psi_4(b_i) \in \varepsilon^2 A. \end{aligned}$$

On the other hand, by using $b_1 = (0, 0, X^{n-1})$, the second equality implies

$$\varepsilon\psi_2(b_n) + \varepsilon X\psi_3(b_n) + X^{n-i}\psi_4(b_n) = X^{n-1}.$$

It yields $\psi_4(X^{n-i}b_n) \not\equiv 0 \ \mathsf{mod}\ \varepsilon A$. Hence, we have reached a contradiction.

(2) Let $\rho \in \mathsf{radEnd}_A(E_i)$. We write $\rho(a_n) = \alpha a_n + \beta b_n + A$ and $\rho(b_n) = \alpha' a_n + \beta' b_n + B$, where $\alpha, \beta, \alpha', \beta' \in \mathcal{O}$ and $A, B \in \mathsf{Ker}(X^{n-1})$. Then, $\phi_i\rho(a_n) = \beta b_1$ and $\phi_i\rho(b_n) = \beta' b_1$ hold.

By Lemma 4.3.2 (1), $\beta \in \varepsilon\mathcal{O}$ and if β' was invertible then α would be invertible, which contradicts with Lemma 4.3.2 (2). Thus, $\beta, \beta' \in \varepsilon\mathcal{O}$ follows, and we may define $\psi_2 : E_i \to A$ by

$$(f, g, h) \mapsto \frac{\beta X^{n-1} f}{\varepsilon^2} + \frac{\beta' X^{n-1} h}{\varepsilon^2},$$

where $(f, g, h) \in E_i$. The A-module homomorphism ψ_2 is well-defined. Indeed, we have $\psi_2(a_k) = 0$ and $\psi_2(b_k) = 0$ for $1 \le k \le n-1$ and

$$\psi_2(a_n) = \frac{\beta}{\varepsilon} X^{n-1}, \quad \psi_2(b_n) = \frac{\beta'}{\varepsilon} X^{n-1}.$$

Then $\psi = (0, \psi_2, 0, 0) : E_i \to A^{\oplus 4}$ satisfies $\pi\psi = (X^{i-1}\psi_2, 0, \varepsilon\psi_2) = \phi_i\rho$. □

By Proposition 3.5.8 and Lemma 4.3.3, we obtain the almost split sequence

$$0 \to E_{n-i} \to F_i \to E_i \to 0,$$

where $F_i = \{(p, q, r, s, t) \in A^{\oplus 4} \oplus E_i \mid \pi(p, q, r, s) = \phi_i(t)\}$ for $2 \le i \le n-i$. For $1 \le k \le n$, we define $z_k = (0, 0, 0, 0, a_k) \in F_i$ and $x_k, y_k, w_k \in F_i$ by

$$\begin{aligned}
x_k &= \begin{cases} (0, 0, 0, X^{n-k}, a_k) & \text{if } 1 \le k \le n-i, \\ (0, 0, -X^{2n-i-k-1}, \varepsilon X^{n-k}, a_k) & \text{if } n-i < k \le n. \end{cases} \\
y_k &= \begin{cases} (0, 0, 0, 0, b_k) & \text{if } 1 \le k \le i, \\ (0, 0, 0, X^{n+i-k-1}, b_k + a_{k-i+1}) & \text{if } i < k < n, \\ (0, 0, 0, X^{i-1}, b_n) & \text{if } k = n. \end{cases} \\
w_k &= \begin{cases} (0, -X^{n-k+1}, X^{n-k}, 0, 0) & \text{if } 1 \le k \le i, \\ (X^{n-k+i}, -\varepsilon X^{n-k+1}, \varepsilon X^{n-k}, 0, 0) & \text{if } i < k \le n. \end{cases}
\end{aligned}$$

Note that $(p, q, r, s, t) \in F_i$ if and only if

$$(\varepsilon p + X^{i-1} q, X^{n-i} p, \varepsilon q + \varepsilon X r + X^{n-i} s) = \beta_n b_1,$$

where $t = \sum_{k=1}^{n} (\alpha_k a_k + \beta_k b_k)$.

Lemma 4.3.4 ([AKM, Lemma 2.8]). $\{x_k, y_k, z_k, w_k \mid 1 \le k \le n\}$ is an $\mathcal{O}$-basis of F_i.

Proof. It suffices to show that they generate F_i as an $\mathcal{O}$-module since $\mathsf{rank}\ F_i = 4n$. Let F_i' be the $\mathcal{O}$-submodule generated by $\{x_k, y_k, z_k, w_k \mid 1 \le k \le n\}$. We show first that $(\mathsf{Ker}(\pi), 0) \subseteq F_i'$. Recall that any element of $(\mathsf{Ker}(\pi), 0) = (\mathsf{Im}(\iota), 0)$ is of the form

$$\left(g, -Xf + \frac{X^{n-i}h}{\varepsilon}, f, -h, 0 \right),$$

where $(f,g,h) \in A \oplus A \oplus Z_{n-i}$ and $X^i f - \varepsilon g = X^{n-1}h/\varepsilon$. Thus, $X^{n-i}g = 0$ and g is an $\mathcal{O}$-linear combination of X^{n-k+i} for $i < k \leq n$. Thus, subtracting the corresponding $\mathcal{O}$-linear combination of w_k, for $i < k \leq n$, we may assume $g = 0$. Since

$$h \in Z_{n-i} = \mathcal{O}\varepsilon \oplus \cdots \oplus \mathcal{O}\varepsilon X^{i-1} \oplus \mathcal{O}X^i \oplus \cdots \oplus \mathcal{O}X^{n-1},$$

we may further subtract an $\mathcal{O}$-linear combination of x_k, for $1 \leq k \leq n$, and we may asume $g = h = 0$ without loss of generality. Then, $(0, -Xf, f, 0, 0)$, for $f \in A$ with $X^i f = 0$, is an $\mathcal{O}$-linear combination of w_k, for $1 \leq k \leq i$. Hence, $(\mathsf{Ker}(\pi), 0) \subseteq F_i'$. Next we show that $(0,0,0,0,\mathsf{Ker}(\phi)) \subseteq F'$. But it is clear from $(0,0,0,0,a_k) = z_k$ and

$$(0,0,0,0,b_k) = \begin{cases} y_k & \text{if } 1 \leq k \leq i, \\ y_k - x_{k-i+1} & \text{if } i < k < n. \end{cases}$$

Suppose that $(p,q,r,s,t) \in F_i$. Write $t = \beta b_n + t'$ such that $\beta \in \mathcal{O}$ and $t' \in \mathsf{Ker}(\phi)$. In order to show that $(p,q,r,s,t) \in F_i'$, it is enough to see $(p,q,r,s,\beta b_n) \in F_i'$. Since $\varepsilon q + \varepsilon X r + X^{n-i}s = \beta X^{n-1}$, we have $(p,q,r,s - \beta X^{i-1}) \in \mathsf{Ker}(\pi)$. Therefore, we deduce

$$(p,q,r,s,\beta b_n) = (p,q,r,s-\beta X^{i-1},0) + \beta(0,0,0,X^{i-1},b_n) \in F_i',$$

because $(0,0,0,X^{i-1},b_n) = y_n$. $\square$

Let F_i' be the $\mathcal{O}$-span of $\{x_k, y_k, w_k \mid 1 \leq k \leq n\}$ and F_i'' the $\mathcal{O}$-span of $\{z_k \mid 1 \leq k \leq n\}$. It is easy to compute as follows.

$$\begin{aligned} Xw_k &= \begin{cases} w_{k-1} & \text{if } k \neq i+1, \\ \varepsilon w_i & \text{if } k = i+1. \end{cases} \\ Xx_k &= \begin{cases} x_{k-1} & \text{if } k \neq n-i+1, \\ \varepsilon x_{n-i} - w_1 & \text{if } k = n-i+1. \end{cases} \\ Xy_k &= \begin{cases} y_{k-1} & \text{if } k \neq i+1, \\ \varepsilon y_i + x_1 & \text{if } k = i+1. \end{cases} \\ Xz_k &= \begin{cases} z_{k-1} & \text{if } k \neq n-i+1, \\ \varepsilon z_{n-i} & \text{if } k = n-i+1. \end{cases} \end{aligned}$$

Hence, the direct summands F_i' and F_i'' of $F_i = F_i' \oplus F_i''$ are A-lattices and $F_i'' \simeq Z_{n-i}$.

Lemma 4.3.5 ([AKM, Lemma 2.9])**.** Assume $2 \leq i \leq n-2$. Then the middle term of the almost split sequence ending at E_i is the direct sum of Z_{n-i} and an indecomposable A-lattice.

Proof. Since $\tau(Z_i) \simeq Z_{n-i}$ implies $\tau(E_i) \simeq E_{n-i}$, we may assume $2 \leq i \leq n-i$ without loss of generality. Let F_i' be the A-lattice as above. Then, we have to show that F_i' is an indecomposable A-lattice. Suppose that F_i' is not indecomposable. Then, there exist A-sublattices Z and L such that $F_i' \simeq Z \oplus L$ and $Z \otimes \mathcal{K} \simeq A \otimes \mathcal{K}$. Since

$$\mathsf{Ker}(X^k) \cap F_i' = \bigoplus_{1 \le j \le k} (\mathcal{O}w_j + \mathcal{O}x_j + \mathcal{O}y_j),$$

we may choose an $\mathcal{O}$-basis $\{e_k \mid 1 \le k \le n\}$ of Z such that

$$e_k = \alpha_k w_k + \beta_k x_k + \gamma_k y_k + A_k,$$

where $\alpha_k, \beta_k, \gamma_k \in \mathcal{O}$ with $(\alpha_k, \beta_k, \gamma_k) \notin (\varepsilon\mathcal{O})^{\oplus 3}$ and $A_k \in \mathsf{Ker}(X^{k-1}) \cap L$. Then, we have $\mathsf{Ker}(X^k) \cap Z = \mathcal{O}e_1 \oplus \cdots \oplus \mathcal{O}e_k$ and at least one of $\alpha_k, \beta_k, \gamma_k$ is invertible. Write

$$Xe_k = f_{k-1}^{(k)} e_{k-1} + \cdots + f_1^{(k)} e_1,$$

where $f_1^{(k)}, \ldots, f_{k-1}^{(k)} \in \mathcal{O}$. We first assume that $2 \le i < n-i$. Note that

$$Xe_k = \begin{cases} \alpha_k w_{k-1} + \beta_k x_{k-1} + \gamma_k y_{k-1} + XA_k & \text{if } k \ne i+1, n-i+1, \\ \alpha_{n-i+1} w_{n-i} + \beta_{n-i+1}(\varepsilon x_{n-i} - w_1) + \gamma_{n-i+1} y_{n-i} + XA_{n-i+1} & \text{if } k = n-i+1, \\ \alpha_{i+1}\varepsilon w_i + \beta_{i+1} x_i + \gamma_{i+1}(\varepsilon y_i + x_1) + XA_{i+1} & \text{if } k = i+1. \end{cases}$$

This implies the equation

$$f_{k-1}^{(k)}(\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}) = \begin{cases} (\alpha_k, \beta_k, \gamma_k) & \text{if } k \ne i+1, n-i+1, \\ (\alpha_{n-i+1}, \varepsilon\beta_{n-i+1}, \gamma_{n-i+1}) & \text{if } k = n-i+1, \\ (\varepsilon\alpha_{i+1}, \beta_{i+1}, \varepsilon\gamma_{i+1}) & \text{if } k = i+1. \end{cases}$$

We may assume one of the following two cases occurs.

(1) $f_{k-1}^{(k)} = 1$ $(k \ne n-i+1)$, $f_{n-i}^{(n-i+1)} = \varepsilon$.

(2) $f_{k-1}^{(k)} = 1$ $(k \ne i+1)$, $f_i^{(i+1)} = \varepsilon$.

In fact, since at least one of $\alpha_k, \beta_k, \gamma_k$ is invertible, $f_{k-1}^{(k)}$ is invertible when $k \ne n-i+1, i+1$. By multiplying its inverse to e_k, we obtain $f_1^{(2)} = \cdots = f_{i-1}^{(i)} = 1$ and $(\alpha_1, \beta_1, \gamma_1) = \cdots = (\alpha_i, \beta_i, \gamma_i)$ in the new basis. By the same reason, we have $f_{k-1}^{(k)} \notin \varepsilon^2\mathcal{O}$ for all k. Suppose that both $f_{n-i}^{(n-i+1)}$ and $f_i^{(i+1)}$ are invertible. Then, we may reach

$$(\alpha_i, \beta_i, \gamma_i) = (\varepsilon\alpha_{i+1}, \beta_{i+1}, \varepsilon\gamma_{i+1}) = \cdots = (\varepsilon\alpha_{n-i}, \beta_{n-i}, \varepsilon\gamma_{n-i}) = (\varepsilon\alpha_{n-i+1}, \varepsilon\beta_{n-i+1}, \varepsilon\gamma_{n-i+1}),$$

a contradiction. Suppose that both $f_{n-i}^{(n-i+1)}$ and $f_i^{(i+1)}$ are not invertible. Then,

$$\begin{aligned}(\alpha_i, \beta_i, \gamma_i) &= (\alpha_{i+1}, \varepsilon^{-1}\beta_{i+1}, \gamma_{i+1}) = \cdots = (\alpha_{n-i}, \varepsilon^{-1}\beta_{n-i}, \gamma_{n-i}) \\ &= (\varepsilon^{-1}\alpha_{n-i+1}, \varepsilon^{-1}\beta_{n-i+1}, \varepsilon^{-1}\gamma_{n-i+1}),\end{aligned}$$

which implies that none of $\alpha_{n-i+1}, \beta_{n-i+1}, \gamma_{n-i+1}$ is invertible. Thus, we have proved that we are in the case (1) or the case (2). Suppose that we are in the case (1). Then, we have

$$\begin{aligned} Xe_k - f_{k-1}^{(k)}e_{k-1} &= f_{k-2}^{(k)}e_{k-2} + \cdots + f_1^{(k)}e_1 \\ &= \begin{cases} XA_k - A_{k-1} & \text{if } k \neq n-i+1, i+1, \\ XA_{n-i+1} - \varepsilon A_{n-i} - \beta_{n-i+1}w_1 & \text{if } k = n-i+1, \\ XA_{i+1} - A_i + \gamma_{i+1}x_1 & \text{if } k = i+1. \end{cases} \end{aligned}$$

Since $A_k \in \mathsf{Ker}(X^k) \cap L$, we obtain the equations

$$Xe_k = \begin{cases} e_{k-1} & \text{if } k \neq n-i+1, i+1, \\ \varepsilon e_{n-i} + f_1^{(n-i+1)}e_1 & \text{if } k = n-i+1, \\ e_i + f_1^{(i+1)}e_1 & \text{if } k = i+1, \end{cases}$$

and $XA_{n-i+1} = X^2A_{n-i+2} = \cdots = X^iA_n$. As we are in the case (1),

$$\begin{aligned} (\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2) &= \cdots = (\alpha_i, \beta_i, \gamma_i) \\ &= (\varepsilon\alpha_{i+1}, \beta_{i+1}, \varepsilon\gamma_{i+1}) = \cdots = (\varepsilon\alpha_{n-i}, \beta_{n-i}, \varepsilon\gamma_{n-i}) \\ &= (\alpha_{n-i+1}, \beta_{n-i+1}, \gamma_{n-i+1}) = \cdots = (\alpha_n, \beta_n, \gamma_n) \end{aligned}$$

follows so that we may write

$$e_k = \begin{cases} \varepsilon\alpha w_k + \beta x_k + \varepsilon\gamma y_k + A_k & \text{if } 1 \le k \le i \text{ or } n-i+1 \le k \le n, \\ \alpha w_k + \beta x_k + \gamma y_k + A_k & \text{if } i+1 \le k \le n-i, \end{cases}$$

with $\alpha, \gamma \in \mathcal{O}$ and $\beta \in \mathcal{O}^\times$. Then, $Xe_{n-i+1} = \varepsilon e_{n-i} + f_1^{(n-i+1)}e_1$ implies

$$\varepsilon\alpha w_{n-i} + \beta(\varepsilon x_{n-i} - w_1) + \varepsilon\gamma y_{n-i} + X^iA_n = \varepsilon e_{n-i} + f_1^{(n-i+1)}(\varepsilon\alpha w_1 + \beta x_1 + \varepsilon\gamma y_1).$$

We equate the coefficients of w_1 on both sides. Since contribution from X^iA_n comes from $X^iw_{i+1} = \varepsilon w_1$ only, we conclude that $\beta \in \varepsilon\mathcal{O}$, which contradicts with $\beta \in \mathcal{O}^\times$.

Suppose that we are in the case (2). Then, the same argument as above shows that

$$Xe_k = \begin{cases} e_{k-1} & \text{if } k \neq n-i+1, i+1, \\ e_{n-i} + f_1^{(n-i+1)}e_1 & \text{if } k = n-i+1, \\ \varepsilon e_i + f_1^{(i+1)}e_1 & \text{if } k = i+1. \end{cases}$$

We define an $\mathcal{O}$-basis $\{e''_k\}$ of Z as follows:

(i) $e''_k = e_k$ $(1 \le k \le i)$.

(ii) $e''_{n-i} = e_{n-i} - f_1^{(i+1)}e_{n-2i+1} + f_1^{(n-i+1)}e_1$.

(iii) $e''_{n-1} = e_{n-1} - f_1^{(i+1)} e_{n-i} - f_1^{(i+1)} f_1^{(n-i+1)} e_1$.

(iv) $e''_k = e_k - f_1^{(i+1)} e_{k-i+1}$ $(i+1 \le k \le n,\ k \ne n-i,\, n-1)$.

Then, we have $Z \simeq Z_i$. To summarize, we have proved that if there is a direct summand of rank n then it must be isomorphic to Z_i. As there is an irreducible morphism $Z_i \to E_i$, the A-lattice E_i must be a direct summand of E_{n-i} and we conclude $E_i \simeq E_{n-i}$. Then, there exist $a'_k, b'_k \in E_{n-i}$, for $1 \le k \le n$, such that

$$\begin{aligned} a_n &= \alpha a'_n + \beta b'_n + A, \\ b_n &= \gamma a'_n + \delta b'_n + B, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ with $\alpha\delta - \beta\gamma \in \mathcal{O}^\times$, $A, B \in \mathsf{Ker}(X^{n-1})$, and

$$Xa'_k = \begin{cases} a'_{k-1} & (k \ne n-i+1) \\ \varepsilon a'_{k-1} & (k = n-i+1), \end{cases}$$

$$Xb'_k = \begin{cases} b'_{k-1} & (k \ne i+1, n) \\ \varepsilon b'_{k-1} & (k = i+1) \\ a'_{n-i} + b'_{n-1} & (k = n). \end{cases}$$

We compute $X^{n-i}a_n$ and $X^{n-i}b_n$ as follows.

$$\begin{aligned} \varepsilon a_i &= \varepsilon(\alpha a'_i + \beta b'_i) + \beta a'_1 + X^{n-i}A, \\ \varepsilon b_i &= \varepsilon(\gamma a'_i + \delta b'_i) + \delta a'_1 + X^{n-i}B. \end{aligned}$$

Since $X^{n-i}A, X^{n-i}B \in \varepsilon E_{n-i}$ by $2 \le i < n-i$, we have $\beta, \delta \in \varepsilon\mathcal{O}$, which is a contradiction. Thus, F'_i is indecomposable if $2 \le i < n-i$. It remains to consider $2 \le i = n-i$. We choose an $\mathcal{O}$-basis $\{e_k \mid 1 \le k \le n\}$ of Z and write

$$e_k = \alpha_k w_k + \beta_k x_k + \gamma_k y_k + A_k,$$

as before. Then, we have

$$Xe_k = \begin{cases} \alpha_k w_{k-1} + \beta_k x_{k-1} + \gamma_k y_{k-1} + XA_k & \text{if } k \ne i+1, \\ \alpha_{i+1}\varepsilon w_i + \beta_{i+1}(\varepsilon x_i - w_1) + \gamma_{i+1}(\varepsilon y_i + x_1) + XA_{i+1} & \text{if } k = i+1, \end{cases}$$

and it follows that

$$f_{k-1}^{(k)}(\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}) = \begin{cases} (\alpha_k, \beta_k, \gamma_k) & \text{if } k \ne i+1, \\ (\varepsilon\alpha_{i+1}, \varepsilon\beta_{i+1}, \varepsilon\gamma_{i+1}) & \text{if } k = i+1. \end{cases}$$

Hence, we may assume $f_{k-1}^{(k)} = 1$, for $k \ne i+1$, and $f_i^{(i+1)} = \varepsilon$, without loss of generality. Since $A_k \in \mathsf{Ker}(X^{k-1}) \cap L$, we obtain from the computation of $Xe_k - f_{k-1}^{(k)} e_{k-1}$ that

$$Xe_k = \begin{cases} e_{k-1} & \text{if } k \ne i+1, \\ \varepsilon e_i + f_1^{(i+1)} e_1 & \text{if } k = i+1, \end{cases}$$

and $XA_{i+1} = X^2A_{i+2} = \cdots = X^iA_n$. Let λ, μ and ν be the coefficients of w_{n-i+1}, x_{n-i+1} and y_{n-i+1} in A_n, respectively. Then the coefficients of w_1, x_1, y_1 in XA_{i+1} are $\varepsilon\lambda$, $\varepsilon\mu$, $\varepsilon\nu$. Since $f_1^{(i+1)}e_1 = XA_{i+1} - \varepsilon A_i - \beta_{i+1}w_1 + \gamma_{i+1}x_1$, we have

$$f_1^{(i+1)}\alpha_1 \equiv -\beta_{i+1} \bmod \varepsilon\mathcal{O}, \quad f_1^{(i+1)}\beta_1 \equiv \gamma_{i+1} \bmod \varepsilon\mathcal{O}, \quad f_1^{(i+1)}\gamma_1 \equiv 0 \bmod \varepsilon\mathcal{O}.$$

We may show that $f_1^{(i+1)}$ is not invertible, but whenever it is invertible or not,

$$\gamma_1 = \gamma_2 = \cdots = \gamma_n \text{ and } \beta_1 = \beta_2 = \cdots = \beta_n$$

imply that $\beta_k \equiv 0 \bmod \varepsilon\mathcal{O}$ and $\gamma_k \equiv 0 \bmod \varepsilon\mathcal{O}$, for $1 \le k \le n$. It follows that we may choose an $\mathcal{O}$-basis $\{a'_k, b'_k \mid 1 \le k \le n\}$ of L as

$$\begin{aligned} a'_k &= \lambda'_k w_k + x_k + A'_k, \\ b'_k &= \lambda''_k w_k + y_k + B'_k, \end{aligned}$$

where $\lambda', \lambda'' \in \mathcal{O}$ and $A'_k, B'_k \in \mathsf{Ker}(X^{k-1}) \cap Z$. Write

$$Xa'_k = \sum_{j=1}^{k-1}(g_j^{(k)}a'_j + h_j^{(k)}b'_j).$$

By multiplying $a'_k = \lambda'_k w_k + x_k + A'_k$ with X, we obtain

$$Xa'_k = \begin{cases} \lambda'_k w_{k-1} + x_{k-1} + XA'_k & \text{if } k \neq i+1, \\ \varepsilon\lambda'_{i+1}w_i + \varepsilon x_i - w_1 + XA'_{i+1} & \text{if } k = i+1. \end{cases}$$

Thus, $g_{k-1}^{(k)} = 1$, for $k \neq i+1$, $g_i^{(i+1)} = \varepsilon$, and $h_{k-1}^{(k)} = 0$ for all k. Further, we have

$$Xa'_k - g_{k-1}^{(k)}a'_{k-1} = \begin{cases} XA'_k - A'_{k-1} & \text{if } k \neq i+1, \\ XA'_{i+1} - \varepsilon A'_i - w_1 & \text{if } k = i+1. \end{cases}$$

We obtain $Xa'_k - a'_{k-1} = 0$ if $k \neq i+1$, and if $k = i+1$ then $Xa'_{i+1} - \varepsilon a'_i$ is equal to $g_1^{(i+1)}a'_1 + h_1^{(i+1)}b'_1 = XA'_{i+1} - \varepsilon A'_i - w_1$. Since $XA'_{i+1} = X^2A'_{i+2} = \cdots = X^{n-i}A'_n$, the coefficient of x_1 in XA'_{i+1} is in $\varepsilon\mathcal{O}$. This implies the equation

$$(\lambda'_1 g_1^{(i+1)} + \lambda''_1 h_1^{(i+1)} + 1)w_1 + g_1^{(i+1)}x_1 + h_1^{(i+1)}y_1 \equiv 0 \bmod \varepsilon F'_i.$$

We must have $g_1^{(i+1)}, h_1^{(i+1)} \in \varepsilon\mathcal{O}$, but then $w_1 \equiv 0 \bmod \varepsilon F'_i$, which is impossible. Hence, F'_i is indecomposable if $2 \le n-i = i$. $\square$

4.4 The shapes of Heller components

Now, we determine the shapes of Heller components of the truncated polynomial rings.

Theorem 4.4.1 ([AKM, Theorem 3.1])**.** Let $\mathcal{O}$ be a complete discrete valuation ring, $A = \mathcal{O}[X]/(X^n)$, for $n \geq 2$. Then, the Heller component containing Z_i and Z_{n-i} is $\mathbb{Z}A_\infty/\langle\tau^2\rangle$ if $2i \neq n$, and $\mathbb{Z}A_\infty/\langle\tau\rangle$ if $2i = n$.

Proof. Any Heller component does not admit a loop by Corollary 4.2.2. Let $\mathcal{C}$ be the Heller component containing Z_i and Z_{n-i} and $\overline{T}$ the tree class of $\mathcal{C}$. If $i = 1$ or $i = n-1$, then Proposition 4.2.1 (1) implies that the subadditive function $\mathcal{R}|_T$ is not additive. Thus, the tree class of $\mathcal{C}$ is A_∞. For the remaining cases, the assertion follows from Corollary 3.7.8 and Lemmas 4.2.1 and 4.3.5. □

5. HELLER COMPONENTS: THE CASE OF THE SYMMETRIC KRONECKER ALGEBRA

In this chapter, we determine the shapes of Heller components when A is the symmetric Kronecker algebra $\mathcal{O}[X,Y]/(X^2,Y^2)$. The results in this chapter appear in [M1, M2].

Throughout this chapter, we assume that κ is algebraically closed. Since $A \otimes \mathcal{K} = \mathcal{K}[X,Y]/(X^2,Y^2)$ is not semi-simple, A is not an isolated singularity, and it is of infinite representation type. In particular, the stable Auslander–Reiten quiver $\Gamma_s(A)$ has infinitely many vertices. It is well-known that the stable Auslander–Reiten quiver of $\overline{A}$ is of the form

$$
\begin{tikzcd}[column sep=tiny]
 & & M(-4) \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & M(-2) \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & M(0) \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & M(2) \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & M(4) & \\
\cdots & M(-5) \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & M(-3) \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & M(-1) \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & M(1) \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & M(3) \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
M(\lambda)_1 \arrow[loop above, dotted, no head] \arrow[r, shift left] & M(\lambda)_2 \arrow[loop above, dotted, no head] \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \quad \cdots
\end{tikzcd}
\qquad (\lambda \in \mathbb{P}^1(\kappa) = \kappa \sqcup \{\infty\}),
$$

where $M(0)$ is the simple $\overline{A}$-module and $M(\lambda)_n$ $(n \in \mathbb{Z}_{>0}, \lambda \in \mathbb{P}^1(\kappa))$ is given by

$$
X \mapsto \left(\begin{array}{c|c} \mathbf{0}_n & \mathbf{0}_n \\ \hline \mathbf{1}_n & \mathbf{0}_n \end{array}\right) \quad Y \mapsto \left(\begin{array}{c|c} \mathbf{0}_n & \mathbf{0}_n \\ \hline J(\lambda, n) & \mathbf{0}_n \end{array}\right) \quad (\text{if } \lambda \in \kappa)
$$

$$
X \mapsto \left(\begin{array}{c|c} \mathbf{0}_n & \mathbf{0}_n \\ \hline J(0, n) & \mathbf{0}_n \end{array}\right) \quad Y \mapsto \left(\begin{array}{c|c} \mathbf{0}_n & \mathbf{0}_n \\ \hline \mathbf{1}_n & \mathbf{0}_n \end{array}\right) \quad (\text{if } \lambda = \infty).
$$

Here, we denote by $\mathbf{1}_n$ and $\mathbf{0}_n$ the identity matrix of size n and the zero matrix of size n, respectively. We denote by Z_n and Z_n^λ the first syzygy of $M(n)$ and $M(\lambda)_n$ in $\mathsf{latt}\text{-}A$, respectively. Then, Heller lattices Z_n and Z_n^λ are indecomposable (Proposition 5.1.5).

The last main result of this thesis is the following.

Main Theorem (Theorems 5.5.1, 5.8.4 and 5.9.5)**.** Let $\mathcal{O}$ be a complete discrete valuation ring and $A = \mathcal{O}[X,Y]/(X^2,Y^2)$. Assume that the residue field κ is algebraically closed. Then, the following statements hold.

(1) There is a unique non-periodic Heller component $\mathcal{HC}(Z_0)$, and it is isomorphic to $\mathbb{Z}A_\infty$. Moreover, the Heller lattice Z_n belongs to $\mathcal{HC}(Z_0)$ for all $n \in \mathbb{Z}$.

(2) If the characteristic of κ is 2, then $\mathcal{HC}(Z_n^\lambda) \simeq \mathbb{Z}A_\infty/\langle\tau\rangle$ for all $\lambda \in \mathbb{P}^1(\kappa)$ for all $n \in \mathbb{Z}$.

(3) If the characteristic of κ is not 2, then $\mathcal{HC}(Z_n^\lambda) \simeq \mathbb{Z}A_\infty/\langle\tau\rangle$ if $\lambda = 0$ or ∞, $\mathcal{HC}(Z_n^\lambda) \simeq \mathbb{Z}A_\infty/\langle\tau^2\rangle$ otherwise.

(4) Any Heller lattice appears on the boundary of a Heller component.

Throughout this chapter, we use the symbol X and Y as $X+(X^2,Y^2)$ and $Y+(X^2,Y^2)$, respectively. For a positive integer n, we denote by $e_1,\dots,e_n$ the standard basis of $\mathcal{O}^{\oplus n}$ and we adopt $e_1, Xe_1, Ye_1, XYe_1, \dots, e_n, Xe_n, Ye_n, XYe_n$ as an $\mathcal{O}$-basis of $A^{\oplus n}$. The symmetric Kronecker algebra $\overline{A}$ is the bound quiver algebra over κ defined by the following quiver and relations:

$$\begin{tikzcd} 1 \arrow[loop left, "X"] \arrow[loop right, "Y"] \end{tikzcd} ; \quad X^2 = Y^2 = 0, \quad XY - YX = 0.$$

Since κ is an algebraically closed field, a d-dimensional $\overline{A}$-module M is of the form

$$M = \begin{tikzcd} \kappa^d \arrow[loop left, "M_1"] \arrow[loop right, "M_2"] \end{tikzcd},$$

where M_1 and M_2 are square matrices of size d which commute and square zero by Theorem 2.6.2. To simplify, we denote by (d, M_1, M_2) the $\overline{A}$-module M. Since $\overline{A}$ is a special biserial algebra, we may give all finite dimensional indecomposable modules by Theorem 2.6.4.

We note that the "Kronecker algebra" over a ring R usually means the generalized triangular matrix R-algebra

$$\begin{pmatrix} R & 0 \\ R^2 & R \end{pmatrix}.$$

However, the R-algebra $R[X,Y]/(X^2,Y^2)$ is also called the "Kronecker algebra", see [Erd, Chapter I, Example 4.3]. These two algebras are not isomorphic each other, but there is a functorial relation, which is explained in [G, Section 5], [ARS, Chapter X Section 2] and [SS, Chapter XIX, 1.13 Remark]. In order to distinguish these two Kronecker algebras, we called the algebra $R[X,Y]/(X^2,Y^2)$ the "symmetric" Kronecker algebra.

5.1 Heller lattices

First, we give a complete list of Heller lattices. By Theorem 2.6.4, all finite dimensional indecomposable $\overline{A}$-modules are classified into string modules, band modules and projective-injective modules. We notice that the unique indecomposable projective-injective module $\overline{A}$ is given by

$$\left(4, \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}\right).$$

Now, we present a complete list of the other finite dimensional indecomposable $\overline{A}$-modules, which are denoted by $M(m)$, $M(-m)$, $M(\lambda)_n$, where $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$ and λ lies on the projective line $\mathbb{P}^1(\kappa) = \kappa \sqcup \{\infty\}$.

(i) The string module $M(m) := M((\beta_1^*\beta_2)^m)$ $(m \in \mathbb{Z}_{\geq 0})$ is given by the formula:

$$M(m) = \left(2m+1, \left(\begin{array}{c:c} \mathbf{0}_m & \mathbf{0}_{m+1} \\ \hdashline \mathbf{1}_m & \mathbf{0}_{m+1} \\ 0 \cdots 0 & 0 \cdots 0 \end{array} \right), \left(\begin{array}{c:c} \mathbf{0}_m & \mathbf{0}_{m+1} \\ \hdashline 0 \cdots 0 & 0 \cdots 0 \\ \mathbf{1}_m & \mathbf{0}_{m+1} \end{array} \right) \right)$$

(ii) The string module $M(-m) := M((\beta_1\beta_2^*)^m)$ $(m \in \mathbb{Z}_{\geq 0})$ is given by the formula:

$$M(-m) = \left(2m+1, \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\mathbf{0}_{m+1}} & \mathbf{0}_m \\ \hdashline & 0 & \\ \mathbf{1}_m & \vdots & \mathbf{0}_m \\ & 0 & \end{array} \right), \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\mathbf{0}_{m+1}} & \mathbf{0}_m \\ \hdashline 0 & & \\ \vdots & \mathbf{1}_m & \mathbf{0}_m \\ 0 & & \end{array} \right) \right)$$

(iii) The string module $M(0)_n := M((\beta_1\beta_2^*)^{n-1}\beta_1)$ $(n \in \mathbb{Z}_{>0})$ is given by the formula:

$$M(0)_n = \left(2n, \left(\begin{array}{c:c} \mathbf{0}_n & \mathbf{0}_n \\ \hdashline \mathbf{1}_n & \mathbf{0}_n \end{array} \right), \left(\begin{array}{c:c} \mathbf{0}_n & \mathbf{0}_n \\ \hdashline J(0,n) & \mathbf{0}_n \end{array} \right) \right)$$

(iv) The string module $M(\infty)_n := M(\beta_2(\beta_1^*\beta_2)^{n-1})$ $(n \in \mathbb{Z}_{>0})$ is given by the formula:

$$M(\infty)_n = \left(2n, \left(\begin{array}{c:c} \mathbf{0}_n & \mathbf{0}_n \\ \hdashline J(0,n) & \mathbf{0}_n \end{array} \right), \left(\begin{array}{c:c} \mathbf{0}_n & \mathbf{0}_n \\ \hdashline \mathbf{1}_n & \mathbf{0}_n \end{array} \right) \right)$$

(v) Let V be a finite-dimensional indecomposable left $\kappa[x, x^{-1}]$-module. Assume that V is represented by $x \mapsto J(\lambda, n)$ with respect to a basis of V for some $\lambda \in \kappa^\times$ and $n \in \mathbb{Z}_{>0}$. The band module $M(\lambda)_n := N(\beta_2^*\beta_1, V)$ is given by the formula:

$$M(\lambda)_n = \left(2n, \left(\begin{array}{c:c} \mathbf{0}_n & \mathbf{0}_n \\ \hdashline \mathbf{1}_n & \mathbf{0}_n \end{array} \right), \left(\begin{array}{c:c} \mathbf{0}_n & \mathbf{0}_n \\ \hdashline J(\lambda,n) & \mathbf{0}_n \end{array} \right) \right)$$

Lemma 5.1.1. The set of the $\overline{A}$-modules

$$\{M(m) \mid m \in \mathbb{Z}\} \sqcup \{M(\lambda)_n \mid \lambda \in \mathbb{P}^1(\kappa),\ n \in \mathbb{Z}_{\geq 1}\} \sqcup \{\overline{A}\}$$

forms a complete set of isoclasses of finite dimensional indecomposable modules over $\overline{A}$.

Proof. The assertion follows from Proposition 2.6.4. See also [B, Theorem 4.3.3]. □

For simplicity, we visualize an $\overline{A}$-module as follows:

- Vertices represent basis vectors of the underlying κ-vector spaces.
- Arrows of the form $\longrightarrow$ represent the action of X, and $\dashrightarrow$ represent the action of Y.
- If there is no arrow (resp. dotted arrow) starting at a vertex, then X (resp. Y) annihilates the corresponding basis element.

By using this notation, the indecomposable modules listed above are represented as follows:

1. $\overline{A} = \begin{tikzcd}[row sep=small] & Xe_1 \arrow[dr, dashed] & \\ e_1 \arrow[ur] \arrow[dr, dashed] & & XYe_1 \\ & Ye_1 \arrow[ur] & \end{tikzcd}$.

2. $M(m) = \begin{tikzcd}[row sep=tiny] & v_0 \\ u_1 \arrow[ur] \arrow[r, dashed] & v_1 \\ \vdots & \vdots \\ u_{m-1} \arrow[r, dashed] & v_{m-1} \\ u_m \arrow[ur] \arrow[r, dashed] & v_m \end{tikzcd}$

3. $M(-m) = \begin{tikzcd}[row sep=tiny] u_1 \arrow[r] & v_1 \\ u_2 \arrow[r] \arrow[ur, dashed] & v_2 \\ \vdots & \vdots \\ & v_{m-1} \\ u_m \arrow[r] \arrow[ur, dashed] & v_m \\ u_{m+1} \arrow[ur, dashed] & \end{tikzcd}$

4. $M(0)_n = \begin{tikzcd}[row sep=tiny] u_1 \arrow[r] & v_1 \\ u_2 \arrow[r] \arrow[ur, dashed] & v_2 \\ \vdots & \vdots \\ u_{n-1} \arrow[r] & v_{n-1} \\ u_n \arrow[r] \arrow[ur, dashed] & v_n \end{tikzcd}$

5. $M(\infty)_n = \begin{tikzcd}[row sep=tiny] u_1 \arrow[r, dashed] & v_1 \\ u_2 \arrow[r, dashed] \arrow[ur] & v_2 \\ \vdots & \vdots \\ u_{n-1} \arrow[r, dashed] & v_{n-1} \\ u_n \arrow[r, dashed] \arrow[ur] & v_n \end{tikzcd}$

6. $M(\lambda)_n = \begin{tikzcd}[row sep=small] u_1 \arrow[r] \arrow[r, dashed, bend left=60, "\underline{\lambda}"] & v_1 \\ u_2 \arrow[r] \arrow[ur, dashed] \arrow[r, dashed, bend right=60, "\underline{\lambda}"'] & v_2 \\ \vdots & \vdots \\ u_{n-1} \arrow[r] & v_{n-1} \\ u_n \arrow[r] \arrow[ur, dashed] \arrow[r, dashed, bend right=60, "\underline{\lambda}"'] & v_n \end{tikzcd}$

Here, $\begin{tikzcd}[row sep=small] & v_{i-1} \\ u_i \arrow[ur, dashed] \arrow[r, dashed, bend right=60, "\underline{\lambda}"'] & v_i \end{tikzcd}$ in the picture 6 means $Yu_i = \lambda v_i + v_{i-1}$.

From now on, as a κ-basis of a non-projective indecomposable module over $\overline{A}$, we adopt the above κ-basis.

Remark 5.1.2 ([ARS, ASS, Erd, SY1])**.** Almost split sequences for $\mathsf{mod}\text{-}\overline{A}$ are known to be as follows:

$$\begin{array}{ll} 0 \longrightarrow M(-1) \longrightarrow \overline{A} \oplus M(0) \oplus M(0) \longrightarrow M(1) \longrightarrow 0 & \\ 0 \longrightarrow M(n-1) \longrightarrow M(n) \oplus M(n) \longrightarrow M(n+1) \longrightarrow 0 & (n \neq 0) \\ 0 \longrightarrow M(\lambda)_1 \longrightarrow M(\lambda)_2 \longrightarrow M(\lambda)_1 \longrightarrow 0 & (\lambda \in \mathbb{P}^1(\kappa)) \\ 0 \longrightarrow M(\lambda)_n \longrightarrow M(\lambda)_{n-1} \oplus M(\lambda)_{n+1} \longrightarrow M(\lambda)_n \longrightarrow 0 & (n > 1, \lambda \in \mathbb{P}^1(\kappa)) \end{array}$$

Lemma 5.1.3. For all $m \in \mathbb{Z}$, $\lambda \in \mathbb{P}^1(\kappa)$ and $n \in \mathbb{Z}_{>0}$, there are isomorphisms

$$\widetilde{\Omega}(M(n)) \simeq M(n-1), \quad \widetilde{\Omega}(M(\lambda)_n) \simeq M(-\lambda)_n, \quad \widetilde{\Omega}(M(\infty)_n) \simeq M(\infty)_n.$$

Proof. Since $\overline{A}$ is symmetric, the functor $\widetilde{\Omega}$ on $\underline{\mathsf{mod}}\text{-}\overline{A}$ is an autofunctor. Note that Remark 5.1.2 implies that there are isomorphisms $\widetilde{\Omega}^2(M(l)) \simeq M(l-2)$ in the stable module category $\underline{\mathsf{mod}}\text{-}\overline{A}$ for any l.

First, we show that $\widetilde{\Omega}(M(n)) \simeq M(n-1)$ in $\underline{\mathsf{mod}}\text{-}\overline{A}$ for $n \leq 0$ by induction on n. It is clear for $n = 0$. Assume that the statement holds for $n \leq k \leq 0$. The induction hypothesis $\widetilde{\Omega}(M(n)) \simeq M(n-1)$ implies

$$\widetilde{\Omega}(M(n-1)) \simeq \widetilde{\Omega}^2(M(n)) \simeq M(n-2)$$

in $\underline{\mathsf{mod}}\text{-}\overline{A}$ and the statement is true for $n-1$.

Now, we show that $\widetilde{\Omega}^{-1}(M(n)) \simeq M(n+1)$ in $\underline{\mathsf{mod}}\text{-}\overline{A}$ for $n \geq 0$ by induction on n. It is easy to check that $\widetilde{\Omega}(M(1)) \simeq M(0)$. Thus, the statement is true for $n = 0$. Assume that the statement holds for $1 \leq k \leq n$. The induction hypothesis $\widetilde{\Omega}^{-1}(M(n)) \simeq M(n+1)$ implies

$$\widetilde{\Omega}^{-1}(M(n+1)) \simeq \widetilde{\Omega}^{-2}(M(n)) \simeq M(n+2)$$

in $\underline{\mathsf{mod}}\text{-}\overline{A}$ and the statement is true for $n+1$.

Next, we consider the case of $M(\lambda)_n$. For $\lambda \in \mathbb{P}^1(\kappa)$ and $n > 0$, we define a map $\pi_n^\lambda : (\overline{A})^{\oplus n} \to M(\lambda)_n$ by $\pi_n^\lambda : e_i \mapsto u_i$. Then, π_n^λ is the projective cover of $M(\lambda)_n$ as an $\overline{A}$-module. First, we assume that $\lambda \neq \infty$. In this case, the kernel of π_n^λ is given by

$$\begin{aligned} &\kappa(Ye_1 - \lambda Xe_1) \oplus \kappa XYe_1 \\ &\oplus \kappa(Ye_2 - \lambda Xe_2 - Xe_1) \oplus \kappa(XYe_2) \\ &\oplus \cdots \\ &\oplus \kappa(Ye_n - \lambda Xe_n - Xe_{n-1}) \oplus \kappa XYe_n, \end{aligned}$$

and it is isomorphic to $M(-\lambda)_n$ in $\underline{\mathsf{mod}}\text{-}\overline{A}$. Next, we consider $\lambda = \infty$ case. A κ-basis of the kernel of π_n^∞ is given by

$$\begin{aligned} &\kappa Xe_1 \oplus \kappa XYe_1 \\ &\oplus \kappa(Xe_2 - Ye_1) \oplus \kappa(XYe_2) \\ &\oplus \cdots \\ &\oplus \kappa(Xe_n - Ye_{n-1}) \oplus \kappa XYe_n, \end{aligned}$$

and it is isomorphic to $M(\infty)_n$ in $\underline{\mathsf{mod}}\text{-}\overline{A}$. In the both cases, the isomorphisms are lifted in $\mathsf{mod}\text{-}\overline{A}$ since the kernels have no $\overline{A}$ as a direct summand. □

Let M be a non-projective indecomposable $\overline{A}$-module listed in Lemma 5.1.1. For each m, n and λ, the projective cover of M as an A-module π_M is given by

$$\pi_M : \begin{cases} A^{\oplus m} \longrightarrow M, & e_i \longmapsto u_i \quad \text{if } M \simeq M(m), \quad m > 0, \\ A^{\oplus m+1} \longrightarrow M, & e_i \longmapsto u_i \quad \text{if } M \simeq M(-m), \quad m > 0, \\ A \longrightarrow M, & e_1 \longmapsto u_1 \quad \text{if } M \simeq M(0), \\ A^{\oplus n} \longrightarrow M, & e_i \longmapsto u_i \quad \text{if } M \simeq M(\lambda)_n, \quad n > 0, \, \lambda \in \mathbb{P}^1(\kappa). \end{cases}$$

For $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{P}^1(\kappa)$, we define the Heller A-lattices Z_n and Z_m^λ to be the A-lattices $Z_m := \mathsf{Ker}(\pi_{M(m)})$ and $Z_n^\lambda := \mathsf{Ker}(\pi_{M(\lambda)_n})$. We denote by $\mathbb{B}(m)$ and $\mathbb{B}(\lambda)_n$ the following $\mathcal{O}$-basis of Heller lattices Z_m and Z_n^λ, respectively: For $m > 0$,

$$\begin{aligned} Z_m = {} & \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}\varepsilon X e_1 \oplus \mathcal{O}(Ye_1 - Xe_2) \oplus \mathcal{O}XYe_1 \\ & \oplus \mathcal{O}\varepsilon e_2 \oplus \mathcal{O}\varepsilon X e_2 \oplus \mathcal{O}(Ye_2 - Xe_3) \oplus \mathcal{O}XYe_2 \\ & \oplus \cdots \\ & \oplus \mathcal{O}\varepsilon e_{m-1} \oplus \mathcal{O}\varepsilon X e_{m-1} \oplus \mathcal{O}(Ye_{m-1} - Xe_m) \oplus \mathcal{O}XYe_{m-1} \\ & \oplus \mathcal{O}\varepsilon e_m \oplus \mathcal{O}\varepsilon X e_m \oplus \mathcal{O}\varepsilon Y e_m \oplus \mathcal{O}XYe_m, \\ Z_0 = {} & \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}Xe_1 \oplus \mathcal{O}Ye_1 \oplus \mathcal{O}XYe_1, \\ Z_{-m} = {} & \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}\varepsilon X e_1 \oplus \mathcal{O}Ye_1 \oplus \mathcal{O}XYe_1 \\ & \oplus \mathcal{O}\varepsilon e_2 \oplus \mathcal{O}\varepsilon X e_2 \oplus \mathcal{O}(Ye_2 - Xe_1) \oplus \mathcal{O}XYe_2 \\ & \oplus \cdots \\ & \oplus \mathcal{O}\varepsilon e_m \oplus \mathcal{O}\varepsilon X e_m \oplus \mathcal{O}(Ye_m - Xe_{m-1}) \oplus \mathcal{O}XYe_m \\ & \oplus \mathcal{O}\varepsilon e_{m+1} \oplus \mathcal{O}Xe_{m+1} \oplus \mathcal{O}(Ye_{m+1} - Xe_m) \oplus \mathcal{O}XYe_{m+1}. \end{aligned}$$

For $n > 1$,

$$\begin{aligned} Z_n^\lambda = {} & \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}\varepsilon X e_1 \oplus \mathcal{O}(Ye_1 - \lambda Xe_1) \oplus \mathcal{O}XYe_1 \\ & \oplus \mathcal{O}\varepsilon e_2 \oplus \mathcal{O}\varepsilon X e_2 \oplus \mathcal{O}(Ye_2 - \lambda Xe_2 - Xe_1) \oplus \mathcal{O}XYe_2 \\ & \oplus \cdots \\ & \oplus \mathcal{O}\varepsilon e_n \oplus \mathcal{O}\varepsilon X e_n \oplus \mathcal{O}(Ye_n - \lambda Xe_n - Xe_{n-1}) \oplus \mathcal{O}XYe_n \\ Z_n^\infty = {} & \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}Xe_1 \oplus \mathcal{O}(Ye_1 - Xe_2) \oplus \mathcal{O}XYe_1 \\ & \oplus \mathcal{O}\varepsilon e_2 \oplus \mathcal{O}\varepsilon X e_2 \oplus \mathcal{O}(Ye_2 - Xe_3) \oplus \mathcal{O}XYe_2 \\ & \oplus \cdots \\ & \oplus \mathcal{O}\varepsilon e_{n-1} \oplus \mathcal{O}\varepsilon X e_{n-1} \oplus \mathcal{O}(Ye_{n-1} - Xe_n) \oplus \mathcal{O}XYe_{n-1} \\ & \oplus \mathcal{O}\varepsilon e_n \oplus \mathcal{O}\varepsilon X e_n \oplus \mathcal{O}\varepsilon Y e_n \oplus \mathcal{O}XYe_n, \end{aligned}$$

and

$$Z_1^\lambda = \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}\varepsilon X e_1 \oplus \mathcal{O}(Ye_1 - \lambda X e_1) \oplus \mathcal{O}XYe_1,$$
$$Z_1^\infty = \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}X e_1 \oplus \mathcal{O}\varepsilon Y e_1 \oplus \mathcal{O}XYe_1.$$

To simplify the notations, we use the following symbols:

- For the Heller lattice Z_m ($m \geq 0$), we put

$$\begin{pmatrix} \mathsf{a}_{1,1} & \mathsf{a}_{1,2} & \mathsf{a}_{1,3} & \mathsf{a}_{1,4} \\ \mathsf{a}_{2,1} & \mathsf{a}_{2,2} & \mathsf{a}_{2,3} & \mathsf{a}_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ \mathsf{a}_{m-1,1} & \mathsf{a}_{m-1,2} & \mathsf{a}_{m-1,3} & \mathsf{a}_{m-1,4} \\ \mathsf{a}_{m,1} & \mathsf{a}_{m,2} & \mathsf{a}_{m,3} & \mathsf{a}_{m,4} \end{pmatrix} = \begin{pmatrix} \varepsilon e_1 & \varepsilon X e_1 & (Ye_1 - Xe_2) & XYe_1 \\ \varepsilon e_2 & \varepsilon X e_2 & (Ye_2 - Xe_3) & XYe_2 \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon e_{m-1} & \varepsilon X e_{m-1} & (Ye_{m-1} - Xe_m) & XYe_{m-1} \\ \varepsilon e_m & \varepsilon X e_m & \varepsilon Y e_m & XYe_m \end{pmatrix}$$

when $m > 0$, and if $m = 0$, we put

$$(\mathsf{a}_{1,1}, \mathsf{a}_{1,2}, \mathsf{a}_{1,3}, \mathsf{a}_{1,4}) = (\varepsilon e_1, Xe_1, Ye_1, XYe_1).$$

We understand that $\mathsf{a}_{0,\mathsf{j}} = 0$ for $\mathsf{j} = 1, 2, 3, 4$. Then, X and Y act on Z_m as follows. If $m > 0$, then

$$X\mathsf{a}_{\mathsf{i},\mathsf{j}} = \begin{cases} \mathsf{a}_{\mathsf{i},\mathsf{j}+1} & \text{if } \mathsf{j} = 1, \\ \mathsf{a}_{\mathsf{i},\mathsf{j}+1} & \text{if } \mathsf{i} \neq \mathsf{m} \text{ and } \mathsf{j} = 3, \\ \varepsilon \mathsf{a}_{\mathsf{m},4} & \text{if } \mathsf{i} = \mathsf{m} \text{ and } \mathsf{j} = 3, \\ 0 & \text{otherwise,} \end{cases} \qquad Y\mathsf{a}_{\mathsf{i},\mathsf{j}} = \begin{cases} \varepsilon \mathsf{a}_{\mathsf{i},3} + \mathsf{a}_{\mathsf{i}+1,2} & \text{if } \mathsf{i} \neq \mathsf{m} \text{ and } \mathsf{j} = 1, \\ \varepsilon \mathsf{a}_{4,3} & \text{if } \mathsf{i} = \mathsf{m} \text{ and } \mathsf{j} = 1, \\ \varepsilon \mathsf{a}_{\mathsf{i},4} & \text{if } \mathsf{j} = 2, \\ -\mathsf{a}_{\mathsf{i}+,4} & \text{if } \mathsf{i} \neq \mathsf{m} \text{ and } \mathsf{j} = 3, \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 1$, then

$$X\mathsf{a}_{1,\mathsf{j}} = \begin{cases} \varepsilon \mathsf{a}_{1,2} & \text{if } \mathsf{j} = 1, \\ \mathsf{a}_{1,4} & \text{if } \mathsf{j} = 3, \\ 0 & \text{otherwise,} \end{cases} \qquad Y\mathsf{a}_{1,\mathsf{j}} = \begin{cases} \varepsilon \mathsf{a}_{1,3} & \text{if } \mathsf{j} = 1, \\ \mathsf{a}_{1,4} & \text{if } \mathsf{j} = 2, \\ 0 & \text{otherwise.} \end{cases}$$

- For the Heller lattice Z_{-m} ($m > 0$), we put

$$\begin{pmatrix} \mathsf{b}_{1,1} & \mathsf{b}_{1,2} & \mathsf{b}_{1,3} & \mathsf{b}_{1,4} \\ \mathsf{b}_{2,1} & \mathsf{b}_{2,2} & \mathsf{b}_{2,3} & \mathsf{b}_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ \mathsf{b}_{\mathsf{m},1} & \mathsf{b}_{\mathsf{m},2} & \mathsf{b}_{\mathsf{m},3} & \mathsf{b}_{\mathsf{m},4} \\ \mathsf{b}_{\mathsf{m}+1,1} & \mathsf{b}_{\mathsf{m}+1,2} & \mathsf{b}_{\mathsf{m}+1,3} & \mathsf{b}_{\mathsf{m}+1,4} \end{pmatrix} = \begin{pmatrix} \varepsilon e_1 & \varepsilon X e_1 & Ye_1 & XYe_1 \\ \varepsilon e_2 & \varepsilon X e_2 & (Ye_2 - Xe_1) & XYe_2 \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon e_m & \varepsilon X e_m & (Ye_m - Xe_{m-1}) & XYe_m \\ \varepsilon e_{m+1} & X e_{m+1} & (Ye_{m+1} - Xe_m) & XYe_{m+1} \end{pmatrix}$$

We understand that $\mathsf{b}_{0,\mathsf{j}} = 0$ for $\mathsf{j} = 1,2,3,4$. Then, X and Y act on Z_n^{∞} as follows.

$$X\mathsf{b}_{\mathsf{i},\mathsf{j}} = \begin{cases} \mathsf{b}_{\mathsf{i},2} & \text{if } \mathsf{i} \neq \mathsf{m}+1 \text{ and } \mathsf{j}=1, \\ \varepsilon \mathsf{b}_{\mathsf{m}+1,2} & \text{if } \mathsf{i} = \mathsf{u}+1 \text{ and } \mathsf{j}=1, \\ \mathsf{b}_{\mathsf{i},4} & \text{if } \mathsf{j}=3, \\ 0 & \text{otherwise,} \end{cases} \qquad Y\mathsf{b}_{\mathsf{i},\mathsf{j}} = \begin{cases} \varepsilon \mathsf{b}_{\mathsf{i},3} + \mathsf{b}_{\mathsf{i}-1,2} & \text{if } \mathsf{i} = 1 \text{ and } \mathsf{j}=1, \\ \varepsilon \mathsf{b}_{\mathsf{i},4} & \text{if } \mathsf{i} \neq \mathsf{u}+1 \text{ and } \mathsf{j}=2, \\ \mathsf{b}_{\mathsf{m}+1,4} & \text{if } \mathsf{i} = \mathsf{u}+1 \text{ and } \mathsf{j}=2, \\ -\mathsf{b}_{\mathsf{i}-1,4} & \text{if } \mathsf{j}=3, \\ 0 & \text{otherwise.} \end{cases}$$

- For the Heller lattice Z_n^{λ} ($\lambda \neq \infty$), we put

$$\begin{pmatrix} \mathsf{c}^{\lambda}_{1,1} & \mathsf{c}^{\lambda}_{1,2} & \mathsf{c}^{\lambda}_{1,3} & \mathsf{c}^{\lambda}_{1,4} \\ \mathsf{c}^{\lambda}_{2,1} & \mathsf{c}^{\lambda}_{2,2} & \mathsf{c}^{\lambda}_{2,3} & \mathsf{c}^{\lambda}_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ \mathsf{c}^{\lambda}_{\mathsf{n}-1,1} & \mathsf{c}^{\lambda}_{\mathsf{n}-1,2} & \mathsf{c}^{\lambda}_{\mathsf{n}-1,3} & \mathsf{c}^{\lambda}_{\mathsf{n}-1,4} \\ \mathsf{c}^{\lambda}_{\mathsf{n},1} & \mathsf{c}^{\lambda}_{\mathsf{n},2} & \mathsf{c}^{\lambda}_{\mathsf{n},3} & \mathsf{c}^{\lambda}_{\mathsf{n},4} \end{pmatrix}$$
$$= \begin{pmatrix} \varepsilon e_1 & \varepsilon X e_1 & (Ye_1 - \lambda X e_1) & XYe_1 \\ \varepsilon e_2 & \varepsilon X e_2 & (Ye_2 - \lambda X e_2 - Xe_1) & XYe_2 \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon e_{n-1} & \varepsilon X e_{n-1} & (Ye_{n-1} - \lambda X e_{n-1} - Xe_{n-2}) & XYe_{n-1} \\ \varepsilon e_n & \varepsilon X e_n & (Ye_n - \lambda X e_n - Xe_{n-1}) & XYe_n \end{pmatrix}$$

when $n > 1$, and if $n = 1$, we put

$$(\mathsf{c}^{\lambda}_{1,1}, \mathsf{c}^{\lambda}_{1,2}, \mathsf{c}^{\lambda}_{1,3}, \mathsf{c}^{\lambda}_{1,4}) = (\varepsilon e_1, \varepsilon X e_1, (Ye_1 - \lambda X e_1), XYe_1).$$

Then, X and Y act on Z_n^{λ} as follows. If $n > 1$, then

$$X\mathsf{c}^{\lambda}_{\mathsf{i},\mathsf{j}} = \begin{cases} \mathsf{c}^{\lambda}_{\mathsf{i},\mathsf{j}+1} & \text{if } \mathsf{j} = 1,3, \\ 0 & \text{otherwise,} \end{cases} \qquad Y\mathsf{c}^{\lambda}_{\mathsf{i},\mathsf{j}} = \begin{cases} \varepsilon \mathsf{c}^{\lambda}_{\mathsf{i},3} + \lambda \mathsf{c}^{\lambda}_{\mathsf{i},2} + \mathsf{c}^{\lambda}_{\mathsf{i}-1,2} & \text{if } \mathsf{j}=1, \\ \varepsilon \mathsf{c}^{\lambda}_{\mathsf{i},4} & \text{if } \mathsf{j}=2, \\ -\lambda \mathsf{c}^{\lambda}_{\mathsf{i},4} - \mathsf{c}^{\lambda}_{\mathsf{i}-1,4} & \text{if } \mathsf{j}=3, \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 1$, then

$$X\mathsf{c}^{\lambda}_{1,\mathsf{j}} = \begin{cases} \mathsf{c}^{\lambda}_{1,\mathsf{j}+1} & \text{if } \mathsf{j} = 1,3, \\ 0 & \text{otherwise,} \end{cases} \qquad Y\mathsf{c}^{\lambda}_{1,\mathsf{j}} = \begin{cases} \varepsilon \mathsf{c}^{\lambda}_{1,3} + \lambda \mathsf{c}^{\lambda}_{1,2} & \text{if } \mathsf{j}=1, \\ \varepsilon \mathsf{c}^{\lambda}_{1,4} & \text{if } \mathsf{j}=2, \\ -\lambda \mathsf{c}^{\lambda}_{1,4} & \text{if } \mathsf{j}=3, \\ 0 & \text{otherwise.} \end{cases}$$

- For the Heller lattice Z_n^{∞}, we put

$$\begin{pmatrix} \mathsf{d}_{1,1} & \mathsf{d}_{1,2} & \mathsf{d}_{1,3} & \mathsf{d}_{1,4} \\ \mathsf{d}_{2,1} & \mathsf{d}_{2,2} & \mathsf{d}_{2,3} & \mathsf{d}_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ \mathsf{d}_{\mathsf{n}-1,1} & \mathsf{d}_{\mathsf{n}-1,2} & \mathsf{d}_{\mathsf{n}-1,3} & \mathsf{d}_{\mathsf{n}-1,4} \\ \mathsf{d}_{\mathsf{n},1} & \mathsf{d}_{\mathsf{n},2} & \mathsf{d}_{\mathsf{n},3} & \mathsf{d}_{\mathsf{n},4} \end{pmatrix} = \begin{pmatrix} \varepsilon e_1 & Xe_1 & (Ye_1 - Xe_2) & XYe_1 \\ \varepsilon e_2 & \varepsilon X e_2 & (Ye_2 - Xe_3) & XYe_2 \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon e_{n-1} & \varepsilon X e_{n-1} & (Ye_{n-1} - Xe_n) & XYe_{n-1} \\ \varepsilon e_n & \varepsilon X e_n & \varepsilon Y e_n & XYe_n \end{pmatrix}$$

when $n > 1$, and if $n = 1$, we put

$$(\mathsf{d}_{1,1}, \mathsf{d}_{1,2}, \mathsf{d}_{1,3}, \mathsf{d}_{1,4}) = (\varepsilon e_1, Xe_1, \varepsilon Ye_1, XYe_1).$$

Then, X and Y act on Z_n^∞ as follows. If $n > 1$, then

$$X\mathsf{d}_{\mathsf{i},\mathsf{j}} = \begin{cases} \varepsilon\mathsf{d}_{1,2} & \text{if } \mathsf{i} = \mathsf{j} = 1, \\ \mathsf{d}_{\mathsf{i},2} & \text{if } \mathsf{i} \neq 1,\ \mathsf{j} = 1, \\ \mathsf{d}_{\mathsf{i},4} & \text{if } \mathsf{i} \neq \mathsf{n},\ \mathsf{j} = 3, \\ \varepsilon\mathsf{d}_{\mathsf{n},4} & \text{if } \mathsf{i} = \mathsf{n},\ \mathsf{j} = 3, \\ 0 & \text{otherwise,} \end{cases} \qquad Y\mathsf{d}_{\mathsf{i},\mathsf{j}} = \begin{cases} \varepsilon\mathsf{d}_{\mathsf{i},3} + \mathsf{d}_{\mathsf{i}+1,2} & \text{if } \mathsf{i} \neq \mathsf{n},\ \mathsf{j} = 1, \\ \mathsf{d}_{\mathsf{n},3} & \text{if } \mathsf{i} = \mathsf{n},\ \mathsf{j} = 1, \\ \mathsf{d}_{1,4} & \text{if } \mathsf{i} = 1,\ \mathsf{j} = 2, \\ \varepsilon\mathsf{d}_{\mathsf{i},4} & \text{if } \mathsf{i} \neq 1,\ \mathsf{j} = 2, \\ -\mathsf{d}_{\mathsf{i}+1,4} & \text{if } \mathsf{i} \neq \mathsf{n},\ \mathsf{j} = 3, \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 1$, then

$$X\mathsf{d}_{1,\mathsf{j}} = \begin{cases} \varepsilon\mathsf{d}_{1,\mathsf{j}+1} & \text{if } \mathsf{j} = 1, 3, \\ 0 & \text{otherwise,} \end{cases} \qquad Y\mathsf{d}_{1,\mathsf{j}} = \begin{cases} \mathsf{d}_{1,\mathsf{j}+2} & \text{if } \mathsf{j} = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.1.4 ([M1, Lemma 2.5])**.** Let Z be a Heller lattice over A. Then, the rank of Z as an $\mathcal{O}$-module is divisible by four.

Proof. Let Z be a Heller A-lattice. Then, $Z \otimes \mathcal{K}$ is projective as an $A \otimes \mathcal{K}$-module. On the other hand, the unique projective indecomposable $A \otimes \mathcal{K}$-module is $A \otimes \mathcal{K}$, whose dimension is four. This gives the desired conclusion. □

Proposition 5.1.5 ([M1, Proposition 2.4] and [M2, Proposition 2.8])**.** For $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{P}^1(\kappa)$, the following statements hold.

(1) There are isomorphisms

$$Z_m \otimes \kappa \simeq M(m-1) \oplus M(m), \quad Z_n^\lambda \otimes \kappa \simeq M(\lambda)_n \oplus M(-\lambda)_n,$$

where we set $-\infty = \infty$.

(2) The Heller lattices Z_m and Z_n^λ are indecomposable.

Proof. (1) We show the following five isomorphisms:

(i) $Z_m \otimes \kappa \simeq M(m) \oplus M(m-1)$ for $m > 0$.

(ii) $Z_0 \otimes \kappa \simeq M(0) \oplus M(-1)$.

(iii) $Z_{-m} \otimes \kappa \simeq M(-m) \oplus M(-m-1)$ for $m > 0$.

(iv) $Z_n^\lambda \otimes \kappa \simeq M(\lambda)_n \oplus M(-\lambda)_n$ for $n \geq 1$ and $\lambda \neq \infty$.

(v) $Z_n^\infty \otimes \kappa \simeq M(\infty)_n^{\oplus 2}$ for $n \geq 1$.

(i) For any $m > 0$, we define $\overline{A}$-submodules $Z(m,1)$ and $Z(m,2)$ of $Z_m \otimes \kappa$ by

$$Z(m,1) := \mathsf{Span}_\kappa\{\mathsf{a_{i,1}},\ \mathsf{a_{i,2}}, \mathsf{a_{m,3}} \mid \mathsf{i} = 1,\dots,\mathsf{m}\},$$
$$Z(m,2) := \mathsf{Span}_\kappa\{\mathsf{a_{i,3}},\ \mathsf{a_{j,4}} \mid \mathsf{i} = 1,\dots,\mathsf{m}-1,\ \mathsf{j} = 1,\dots,\mathsf{m}\}.$$

Obviously, $Z_m \otimes \kappa = Z(m,1) \oplus Z(m,2)$. Then, it is easy to see that $Z(m,1) \simeq M(m)$ and $Z(m,2) \simeq M(m-1)$.

(ii) The isomorphism $Z_0 \otimes \kappa \simeq M(0) \oplus M(-1)$ is clear.

(iii) For any $m > 0$, we define $\overline{A}$-submodules $Z(-m,1)$ and $Z(-m,2)$ of $Z_{-m} \otimes \kappa$ by

$$Z(-m,1) := \mathsf{Span}_\kappa\{\mathsf{b_{i,1}},\ \mathsf{b_{j,2}} \mid \mathsf{i} = 1,\dots,\mathsf{m}+1,\ \mathsf{j} = 1,\dots,\mathsf{m}\},$$
$$Z(-m,2) := \mathsf{Span}_\kappa\{\mathsf{b_{m+1,2}},\ \mathsf{b_{i,3}},\ \mathsf{b_{i,4}} \mid \mathsf{i} = 1,\dots,\mathsf{m}+1\}.$$

Obviously, $Z_{-m} \otimes \kappa = Z(-m,1) \oplus Z(-m,2)$. Then, it is easy to see that $Z(-m,1) \simeq M(-m)$ and $Z(-m,2) \simeq M(-m-1)$.

(iv) For any $n > 0$, we define $\overline{A}$-submodules $Z(\lambda,n,1)$ and $Z(\lambda,n,2)$ of $Z_n^\lambda \otimes \kappa$ by

$$Z(\lambda,n,1) := \mathsf{Span}_\kappa\{\mathsf{c}^\lambda_{\mathsf{i,1}},\ \mathsf{c}^\lambda_{\mathsf{i,2}} \mid \mathsf{i} = 1,\dots,\mathsf{n}\},$$
$$Z(\lambda,n,2) := \mathsf{Span}_\kappa\{\mathsf{c}^\lambda_{\mathsf{i,3}},\ \mathsf{c}^\lambda_{\mathsf{i,4}} \mid \mathsf{i} = 1,\dots,\mathsf{n}\}.$$

Then, $Z_n^\lambda \otimes \kappa$ is decomposed into $Z(\lambda,n,1) \oplus Z(\lambda,n,2)$ as $\overline{A}$-modules. Define $\overline{A}$-homomorphisms $f_1^{\lambda,n} : M(\lambda)_n \to Z(\lambda,n,1)$ and $f_2^{\lambda,n} : M(-\lambda)_n \to Z(\lambda,n,2)$ by

$$f_1^{\lambda,n}(u_i) = \mathsf{c}^\lambda_{\mathsf{i,1}},\quad f_1^{\lambda,n}(v_i) = \mathsf{c}^\lambda_{\mathsf{i,2}},\quad f_2^{\lambda,n}(u_i) = (-1)^{\mathsf{i}+1}\mathsf{c}^\lambda_{\mathsf{i,3}},\ \text{and}\ f_2^{\lambda,n}(v_i) = (-1)^{\mathsf{i}+1}\mathsf{c}^\lambda_{\mathsf{i,4}}.$$

As these morphisms are isomorphisms, we have $Z_n^\lambda \otimes \kappa \simeq M(\lambda)_n \oplus M(-\lambda)_n$.

(v) Finally, we show that $Z_n^\infty \otimes \kappa \simeq M(\infty)_n^{\oplus 2}$. For any $n > 0$, we put

$$Z(\infty,n,1) := \mathsf{Span}_\kappa\{\mathsf{d_{i,1}},\ \mathsf{d_{j,2}}, \mathsf{d_{n,3}} \mid \mathsf{i} = 1,\dots,\mathsf{n},\ \mathsf{j} = 2,\dots,\mathsf{n}\},$$
$$Z(\infty,n,2) := \mathsf{Span}_\kappa\left\{\mathsf{d_{1,2}},\ \mathsf{d_{i,3}},\ \mathsf{d_{n,4}},\ \mathsf{d_{j,4}}\ \middle|\ \begin{array}{l}\mathsf{i} = 1,\dots,\mathsf{n}-1,\\ \mathsf{j} = 1,\dots,\mathsf{n}-1\end{array}\right\}.$$

Then, one can show that $Z(\infty,n,1) \simeq Z(\infty,n,2) \simeq M(\infty)_n$ by using similar arguments in the proof of the case of $\lambda \neq \infty$.

(2) First, we prove that the Heller lattice Z_m is indecomposable for any integer m. We obtained an isomorphism $Z_m \otimes \kappa \simeq M(m) \oplus M(m-1)$ by (1). Assume that Z_m is decomposable. We write $Z_m = Z^1 \oplus Z^2$ with $Z^1 \neq 0 \neq Z^2$ as A-lattices. By the Krull–Schmidt–Azumaya theorem, we would obtain two isomorphisms $Z^1 \otimes \kappa \simeq M(m)$ and $Z^2 \otimes \kappa \simeq M(m-1)$. On the other hand, the dimension of $M(m)$ as a κ-vector space is odd, a contradiction with Lemma 5.1.4. Therefore, Z_m is an indecomposable A-lattice.

From now on, we show that the Heller lattice Z_n^λ is indecomposable for any $n > 0$ and $\lambda \in \mathbb{P}^1(\kappa)$. Let $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{\widetilde{Y}}$ be square matrices of size 4 defined by

$$\widetilde{X} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \widetilde{Y} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & -\lambda & 0 \end{pmatrix} \quad \widetilde{\widetilde{Y}} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Then, the representing matrices of the actions of X and Y on Z_n^λ ($\lambda \neq \infty$) with respect to the $\mathcal{O}$-basis $\mathbb{B}_n^\lambda$ are of the form:

$$X = \begin{pmatrix} \widetilde{X} & & & & \\ & \widetilde{X} & & \text{\Large 0} & \\ & & \ddots & & \\ & & & \widetilde{X} & \\ & \text{\Large 0} & & & \widetilde{X} \end{pmatrix} \quad Y = \begin{pmatrix} \widetilde{Y} & \widetilde{\widetilde{Y}} & & & \\ & \widetilde{Y} & \widetilde{\widetilde{Y}} & \text{\Large 0} & \\ & & \ddots & & \\ & & & \widetilde{Y} & \widetilde{\widetilde{Y}} \\ & \text{\Large 0} & & & \widetilde{Y} \end{pmatrix} \in \mathrm{Mat}(4n, 4n, \mathcal{O})$$

Obviously, the Heller lattice Z_1^λ is indecomposable since $Z_1^\lambda \otimes \mathcal{K} \simeq A \otimes \mathcal{K}$. We prove that idempotents of $\mathsf{End}_A(Z_n^\lambda)$ are only $\mathbf{1}_{4n}$ and $\mathbf{0}_{4n}$. Let $M = (m_{i,j})$ be an idempotent of $\mathsf{End}_A(Z_n^\lambda)$. We partition M into n^2 blocks of size 4×4, and denote by $M_{i,j} \in \mathrm{Mat}(4, 4, \mathcal{O})$ the (i, j)-block of M and by $\alpha_{i,j}$ the $(4i-2, 4j-1)$-entry of M. The equalities $MX = XM$ and $MY = YM$ yield that the block $M_{i,j}$ is of the form

$$M_{i,j} = \begin{pmatrix} d_{i,j} & 0 & 0 & 0 \\ m_{4i-2,4j-3} & d_{i,j} & c_{i,j} & 0 \\ m_{4i-1,4j-3} & 0 & d_{i,j} & 0 \\ m_{4i,4j-3} & m_{4i-1,4j-3} & m_{4i,4j-1} & d_{i,j} \end{pmatrix},$$

where

$$d_{i,j} = \begin{cases} m_{1,1} & \text{if } i = j = 1, \\ m_{1,1} + \varepsilon \sum_{k=1}^{j-1} \alpha_{k,k+1} & \text{if } i = j > 1, \\ \varepsilon \sum_{k=1}^{j} \alpha_{i-j-1+k,k} & \text{if } n \geq i > j \geq 1, \\ m_{1,4j-3} & \text{if } n \geq j > i = 1, \\ m_{1,4(j-i)+1} + \varepsilon \sum_{k=1}^{i-1} \alpha_{k,j-i+1+k} & \text{if } n \geq j > i > 1, \end{cases} \tag{5.1}$$

$$c_{i,j} = \begin{cases} 0 & \text{if } i = n, j = 1, \\ \alpha_{i,j} & \text{if } i \neq n, \\ -\sum_{k=1}^{j-1} \alpha_{i-j+k,k} & \text{if } n = i \geq j > 1. \end{cases}$$

Here, we have to choose each element $m_{k,l}$ in $M_{i,j}$ in such a way that the equation $MY = YM$ holds. By comparing the $(1,1)$-entries of M and M^2, we have the equation

$$m_{1,1} = m_{1,1}^2 + \varepsilon \sum_{k=1}^{n-1} m_{1,4k+1} m_{4k-2,3}.$$

We write $\overline{x}$ for the coset in the residue field $\kappa = \mathcal{O}/\varepsilon\mathcal{O}$ represented by $x \in \mathcal{O}$. The above equation implies that $\overline{m_{1,1}}$ is either $\overline{0}$ or $\overline{1}$.

Assume that $\overline{m_{1,1}} = \overline{0}$. Then, the element $d_{i,i}$ belongs to $\varepsilon\mathcal{O}$ for all i by (5.1). By comparing the $(1, 4k+1)$-entries of M and M^2, we have

$$m_{1,4k+1} = m_{1,1}m_{1,4k+1} + \sum_{l=1}^{k} m_{1,4l+1}d_{l+1,l+1} + \varepsilon \sum_{l=k+1}^{n-1} m_{1,4l+1}P(l) \tag{5.2}$$

for some $P(l) \in \mathcal{O}$, and hence $m_{1,4k+1} \in \varepsilon\mathcal{O}$ for all k. From (5.2), $m_{1,4k+1}$ belongs to $\varepsilon^t\mathcal{O}$ for all $t > 0$. It implies that $m_{1,4k+1} = 0$ for all k. Therefore, the first row of M is zero. By comparing the $(5, 5)$-entries of M and M^2, the following equation holds:

$$\varepsilon m_{2,7} = \begin{cases} \varepsilon^2 m_{2,7} & \text{if } n = 2, \\ \varepsilon^2 m_{2,7}^2 + \varepsilon \sum_{k=1}^{n-2} m_{2,4k+7}d_{k+2,2} & \text{if } n > 2. \end{cases}$$

In the case $n = 2$, $m_{2,7} = 0$ because $1 - \varepsilon m_{2,7}$ is invertible. Therefore, we have:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{2,1} & 0 & m_{2,3} & 0 & m_{2,5} & 0 & 0 & 0 \\ m_{3,1} & 0 & 0 & 0 & m_{3,5} & 0 & 0 & 0 \\ m_{4,1} & m_{3,1} & m_{4,3} & 0 & m_{4,5} & m_{3,5} & m_{4,7} & 0 \\ \varepsilon m_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{6,1} & \varepsilon m_{2,3} & 0 & 0 & m_{6,5} & 0 & -m_{2,3} & 0 \\ m_{7,1} & 0 & \varepsilon m_{2,3} & 0 & m_{7,5} & 0 & 0 & 0 \\ m_{8,1} & m_{7,1} & m_{8,3} & \varepsilon m_{2,3} & m_{8,5} & m_{7,5} & m_{8,7} & 0 \end{pmatrix}$$

By $M = M^2$, all elements of M must be 0.

In the other case, first we prove that the $(4k-2)$-th row of M is zero for all $k = 1, 2, \ldots, n$ by induction on k. By comparing the $(2, 4s-1)$-entries of M and M^2, the following equations hold:

$$m_{2,4s-1} = \sum_{l=1}^{n-1} m_{2,4l+3}d_{l+1,s}, \quad s = 1, 2, \ldots, n. \tag{5.3}$$

Since the first row of M is zero, each $d_{l+1,s}$ of the right hand side of (5.3) belongs to $\varepsilon\mathcal{O}$ and so is $m_{2,4s-1}$ for all $s = 1, 2, \ldots, n$. Thus, for $s = 1, 2, \ldots, n$, the element $m_{2,4s-1}$ lies on $\varepsilon^t\mathcal{O}$ for all $t > 0$. It implies that $m_{2,4s-1} = 0$ for all $s = 1, 2, \ldots, n$. Then, the $(2, 4s-3)$-entries of M and M^2 yield

$$m_{2,4s-3} = \sum_{l=1}^{n-2} m_{2,4l+5}d_{l+2,l}, \quad s = 1, 2, \ldots, n.$$

As each $d_{l+2,l}$ belongs to $\varepsilon\mathcal{O}$, so is $m_{2,4s-3}$ for all $s = 1, 2, \ldots, n$. It implies that the element $m_{2,4s-3}$ lies on $\varepsilon^t\mathcal{O}$ for $t > 0$, and hence the second row of M is zero.

Assume that the statement holds for $2 \leq t \leq k-1$, we will show the statement for k. Then, by the induction hypothesis, we have

$$m_{4k-2,4s-1} = \sum_{l=1}^{n-1} m_{4k-2,4l+3} d_{l+1,s}, \quad s = 1, 2, \ldots, n.$$

Thus, we obtain $m_{4k-2,4s-1} = 0$ and

$$m_{4k-2,4s-3} = \sum_{l=1}^{n-2} m_{4k-2,4l+5} d_{l+2,l}, \quad s = 1, 2, \ldots, n$$

by similar arguments to the proof of the case of $k = 1$. It implies that the $(4k-2)$-th row of M is zero for all $k = 2, \ldots, n$.

Since the first and the $(4k-2)$-th row of M are zero for all k, the (i,j)-block of M is of the form

$$M_{i,j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_{4i-1,4j-3} & 0 & 0 & 0 \\ m_{4i,4j-3} & m_{4i-1,4j-3} & m_{4i,4j-1} & 0 \end{pmatrix}.$$

Therefore, we obtain $M = \mathbf{0}_{4n}$ by comparing each entry of M and M^2.

Next we assume that $\overline{m_{1,1}} = \overline{1}$. Then, $\mathbf{1}_{4n} - M$ is an idempotent whose $(1,1)$-entry is belongs to $\varepsilon\mathcal{O}$ and $M = \mathbf{1}_{4n}$ follows. Therefore, the Heller lattice Z_n^λ is indecomposable.

Finally, we show the indecomposablity of Z_n^∞. Let $X_{(a,b)}$, $Y_{(a,b)}$, Y_2 be square matrices of size 4 defined by

$$X_{(a,b)} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{pmatrix} \quad Y_{(a,b)} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix} \quad Y_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where $a,\, b \in \{1, \varepsilon\}$. Then, the representing matrices of the actions of X and Y on Z_n^∞ with respect to the $\mathcal{O}$-basis $\mathbb{B}_n^\infty$ are of the form:

$$X = \begin{pmatrix} X_{(\varepsilon,1)} & & & & \\ & X_{(1,1)} & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & X_{(1,1)} & \\ & & & & X_{(1,\varepsilon)} \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_{(\varepsilon,1)} & & & & \\ Y_2 & Y_{(\varepsilon,\varepsilon)} & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & Y_{(\varepsilon,\varepsilon)} & \\ & & & Y_2 & Y_{(1,\varepsilon)} \end{pmatrix}$$

Then, any endomorphism $M : Z_n^\infty \to Z_n^\infty$ satisfies $MX = XM$ and $MY = YM$. Thus, the endomorphism ring of Z_n^∞ is subset of

$$\left\{ (m_{i,j})_{i,j} \in \mathrm{Mat}(4n, 4n, \mathcal{O}) \;\middle|\; \begin{array}{l} m_{i,i} = m_{i+1,i+1} \text{ for all } 1 \le i \le 4n-1, \\ m_{i,j} = 0 \text{ for } i < j \text{ whenever } (i,j) \ne (2,3), (2,5), (4,5) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad (4,7) \text{ or } (8,9). \end{array} \right\}$$

Let M be an idempotent of the endomorphism ring of Z_n^∞. It follows from the above observation that M must be either the zero matrix or the identity matrix by comparing all entries of M with those of M^2. Therefore, the A-lattice Z_n^∞ is indecomposable. $\square$

Proposition 5.1.6 ([M1, Proposition 2.7] and [M2, Proposition 2.15])**.** For $m \in \mathbb{Z}$, $\lambda \in \mathbb{P}^1(\kappa)$ and $n > 0$, the following statements hold.

(1) There exists an isomorphism $\tau(Z_m) \simeq Z_{m-1}$.

(2) If $\lambda \neq \infty$, there exists an isomorphism $\tau(Z_n^\lambda) \simeq Z_n^{-\lambda}$.

(3) If $\lambda = \infty$, there exists an isomorphism $\tau(Z_n^\infty) \simeq Z_n^\infty$.

In particular, $\Gamma_s(A)$ admits the unique non-periodic Heller component containing Z_0.

Proof. (1) we prove that the indecomposable Heller lattice Z_m is not periodic in $\Gamma_s(A)$. In order to do this, we introduce another $\mathcal{O}$-basis of Z_{-m} for each $m > 0$ as follows;

$$\begin{aligned} Z_{-m} = \, & \mathcal{O}\varepsilon e_1 \oplus \mathcal{O}(Xe_1 - Ye_2) \oplus \mathcal{O}Ye_1 \oplus \mathcal{O}XYe_1 \\ & \oplus \mathcal{O}\varepsilon e_2 \oplus \mathcal{O}(Xe_2 - Ye_3) \oplus \mathcal{O}\varepsilon Ye_2 \oplus \mathcal{O}XYe_2 \\ & \oplus \cdots \\ & \oplus \mathcal{O}\varepsilon e_m \oplus \mathcal{O}(Xe_m - Ye_{m+1}) \oplus \mathcal{O}\varepsilon Ye_m \oplus \mathcal{O}XYe_m \\ & \oplus \mathcal{O}\varepsilon e_{m+1} \oplus \mathcal{O}Xe_{m+1} \oplus \mathcal{O}\varepsilon Ye_{m+1} \oplus \mathcal{O}XYe_{m+1}. \end{aligned}$$

We denote by $\overline{\mathbb{B}}(m)$ this $\mathcal{O}$-basis of Z_{-m}.

We compute $\tau(Z_m)$ in the following five cases.

$$\text{(a) } m = 1, \quad \text{(b) } m > 1, \quad \text{(c) } m = 0, \quad \text{(d) } m = -1, \quad \text{(e) } m < -1.$$

Suppose (a). Since the projective cover of Z_1 is given by

$$\pi_1 : A \oplus A \longrightarrow Z_1, \quad e_1 \longmapsto \mathsf{a}_{1,1}, \quad e_2 \longmapsto \mathsf{a}_{1,4},$$

we have $\tau(Z_1) = \mathcal{O}(-XYe_1 + \varepsilon e_2) \oplus \mathcal{O}Xe_2 \oplus \mathcal{O}Ye_2 \oplus \mathcal{O}XYe_2 \simeq Z_0$.

Suppose (b). Since the projective cover of Z_m is given by

$$\begin{array}{rccl} \pi_m : & A^{\oplus 2m-1} & \longrightarrow & Z_m \\ & e_i & \longmapsto & \begin{cases} \mathsf{a}_{\mathsf{k},1} & \text{if } i = 2\mathsf{k}-1,\ \mathsf{k} = 1,2,3,\ldots,\mathsf{m}, \\ \mathsf{a}_{\mathsf{k},3} & \text{if } i = 2\mathsf{k},\ \mathsf{k} = 1,2,3,\ldots,\mathsf{m}-1, \end{cases} \end{array}$$

we have

$$\begin{aligned}\tau(Z_m) = \bigoplus_{k=1}^{m-2} \Big(& \mathcal{O}(Ye_{2k-1} - Xe_{2k+1} - \varepsilon e_{2k}) \oplus \mathcal{O}(XYe_{2k-1} - \varepsilon Xe_{2k}) \\ & \oplus \mathcal{O}(-Xe_{2k+2} - Ye_{2k}) \oplus \mathcal{O}(-XYe_{2k}) \Big) \\ & \oplus \mathcal{O}(Ye_{2m-3} + Xe_{2m-1} - \varepsilon e_{2m-2}) \oplus \mathcal{O}(XYe_{2m-3} - \varepsilon Xe_{2m-2}) \\ & \oplus \mathcal{O}(XYe_{2m-1} - \varepsilon Ye_{2m-2}) \oplus \mathcal{O}(-XYe_{2m-2}).\end{aligned}$$

We change the above $\mathcal{O}$-basis of $\tau(Z_m)$ by using the invertible matrix $P = (P_{i,j})$ of size $4m$ defined by $P_{i,j} := (-1)^i \delta_{i,j} \mathbf{1}_4$. Then, the representing matrices of the actions of X and Y on $\tau(Z_m)$ with respect to the new ordered $\mathcal{O}$-basis coincide with those on Z_{m-1}. It follows that $\tau(Z_m) \simeq Z_{m-1}$.

Suppose (c). Since the projective cover of Z_0 is given by

$$\pi_0 : A \oplus A \oplus A \longrightarrow Z_0, \quad e_1 \longmapsto \mathsf{a}_{1,1},\ e_2 \longmapsto \mathsf{a}_{1,2},\ e_3 \longmapsto \mathsf{a}_{1,3},$$

we have an isomorphism

$$\begin{aligned}\tau(Z_0) &= \mathcal{O}(-Ye_1 + \varepsilon e_3) \oplus \mathcal{O}(-XYe_1 + \varepsilon Xe_3) \oplus \mathcal{O}Ye_3 \oplus \mathcal{O}XYe_3 \\ &\quad \oplus \mathcal{O}(-Xe_1 + \varepsilon e_2) \oplus \mathcal{O}Xe_2 \oplus \mathcal{O}(Ye_2 - Xe_3) \oplus \mathcal{O}XYe_2 \\ &\simeq Z_{-1}.\end{aligned}$$

Next, we consider the case (d) and (e). The projective cover of Z_{-m} $(m \geq 1)$ is given by

$$\begin{array}{cccc} \pi_{-m}: & A^{\oplus 2m+3} & \longrightarrow & Z_{-m} \\ & e_i & \longmapsto & \begin{cases} \mathsf{b}_{\mathsf{k},1} & \text{if } i = 2\mathsf{k}-1,\ \mathsf{k} = 1, 2, \ldots, \mathsf{m}+1, \\ \mathsf{b}_{\mathsf{k},3} & \text{if } i = 2\mathsf{k},\ \mathsf{k} = 1, 2, \ldots, \mathsf{m}, \\ \mathsf{b}_{\mathsf{m}+1,2} & \text{if } i = 2\mathsf{m}+2, \\ \mathsf{b}_{\mathsf{m}+1,3} & \text{if } i = 2\mathsf{m}+3. \end{cases} \end{array}$$

Thus, an $\mathcal{O}$-basis of $\tau(Z_{-m})$ is given as follows. If $m = 1$, then

$$\begin{aligned}\tau(Z_{-1}) = & \mathcal{O}(\varepsilon e_2 - Ye_1) \oplus \mathcal{O}(Xe_2 + Ye_5) \oplus \mathcal{O}Ye_2 \oplus \mathcal{O}XYe_2 \\ & \oplus \mathcal{O}(Ye_3 - Xe_1 - \varepsilon e_5) \oplus \mathcal{O}(-Ye_4 + Xe_5) \\ & \oplus \mathcal{O}(XYe_1 + \varepsilon Ye_5) \oplus \mathcal{O}XYe_5 \\ & \oplus \mathcal{O}(Xe_3 - \varepsilon e_4) \oplus \mathcal{O}Xe_4 \oplus \mathcal{O}(XYe_3 + \varepsilon Ye_4) \oplus \mathcal{O}XYe_4,\end{aligned}$$

and if $m > 1$, then

$$\begin{aligned}\tau(Z_{-m}) = &\ \mathcal{O}(Ye_1 - \varepsilon e_2) \oplus \mathcal{O}(Ye_4 + Xe_2) \oplus \mathcal{O}Ye_2 \oplus \mathcal{O}XYe_2 \\ &\oplus \bigoplus_{k=1}^{m-2} \Big(\mathcal{O}(Ye_{2k+1} - Xe_{2k-1} - \varepsilon e_{2k+2}) \oplus \mathcal{O}(Ye_{2k+4} + Xe_{2k+2}) \\ &\oplus \mathcal{O}(XYe_{2k-1} + \varepsilon Ye_{2k+2}) \oplus \mathcal{O}XYe_{2k+2} \Big) \\ &\oplus \mathcal{O}(Ye_{2m+1} - Xe_{2m-1} - \varepsilon e_{2m+3}) \oplus \mathcal{O}(-Ye_{2m+2} - Xe_{2m+3}) \\ &\oplus \mathcal{O}(XYe_{2m-1} + \varepsilon Ye_{2m+3}) \oplus \mathcal{O}XYe_{2m+3} \\ &\oplus \mathcal{O}(Xe_{2m+1} - \varepsilon e_{2m+2}) \oplus \mathcal{O}Xe_{2m+2} \\ &\oplus \mathcal{O}(XYe_{2m+1} - \varepsilon Ye_{2m+2}) \oplus \mathcal{O}XYe_{2m+2}.\end{aligned}$$

In the both cases, we have an isomorphisms $\tau(Z_{-m}) \simeq Z_{-m-1}$ for each $m \geq 1$.

(2) The map $\pi_{n,\lambda}$ defined by

$$\begin{array}{rccc} \pi_{n,\lambda} & : & A^{\oplus 2n} & \longrightarrow & Z_n^\lambda \\ & & e_i & \longmapsto & \begin{cases} \mathsf{c}_{\mathsf{k},1}^\lambda & \text{if } i = 2\mathsf{k}-1,\ \mathsf{k} = 1, 2, \dots, \mathsf{n}, \\ \mathsf{c}_{\mathsf{k},3}^\lambda & \text{if } i = 2\mathsf{k},\ \mathsf{k} = 1, 2, \dots, \mathsf{n} \end{cases} \end{array}$$

is the projective cover of Z_n^λ as an A-module. Its kernel $\tau(Z_n^\lambda)$ is given by

$$\begin{aligned}&\mathcal{O}(\varepsilon e_2 - Ye_1 + \lambda Xe_1) \oplus \mathcal{O}(\varepsilon Xe_2 - XYe_1) \oplus \mathcal{O}(Ye_2 + \lambda Xe_2) \oplus \mathcal{O}XYe_2 \\ &\bigoplus_{k=2}^{n} \Big(\mathcal{O}(-1)^{k-1}(\varepsilon e_{2k} - Ye_{2k-1} + \lambda Xe_{2k-1} + Xe_{2k-3}) \oplus \mathcal{O}(-1)^{k-1}(\varepsilon Xe_{2k} - XYe_{2k-1}) \\ &\qquad \oplus \mathcal{O}(-1)^{k-1}(Ye_{2k} + \lambda Xe_{2k} + Xe_{2k-2}) \oplus \mathcal{O}(-1)^{k-1}XYe_{2k} \Big).\end{aligned}$$

Then, the actions X and Y on $\tau(Z_n^\lambda)$ coincide with those on $Z_n^{-\lambda}$.

(3) We define an A-module homomorphism by

$$\begin{array}{rccc} \pi_{n,\infty} & : & A^{\oplus 2n} & \longrightarrow & Z_n^\infty \\ & & e_i & \longmapsto & \begin{cases} \mathsf{d}_{1,1} & \text{if } i = 1, \\ \mathsf{d}_{1,2} & \text{if } i = 2, \\ \mathsf{d}_{\mathsf{k},3} & \text{if } i = 2\mathsf{k}+1,\ \mathsf{k} = 1, 2, \dots, \mathsf{n}-1, \\ \mathsf{d}_{\mathsf{k},1} & \text{if } i = 2\mathsf{k},\ \mathsf{k} = 2, 3, \dots, \mathsf{n}. \end{cases} \end{array}$$

Then, the $\pi_{n,\infty}$ is the projective cover of Z_n^∞, and an $\mathcal{O}$-basis of the kernel of $\pi_{n,\infty}$ is given as follows. If $n = 1$, then the kernel of $\pi_{1,\infty}$ is

$$\mathcal{O}(-Xe_1 + \varepsilon e_2) \oplus \mathcal{O}Xe_2 \oplus \mathcal{O}(-XYe_1 + \varepsilon Ye_2) \oplus \mathcal{O}XYe_2,$$

and it is isomorphic to Z_1^{∞}. If $n=2$, then the kernel of $\pi_{2,\infty}$ is

$$\begin{aligned}&\mathcal{O}(-XYe_1+\varepsilon e_2)\oplus\mathcal{O}Xe_2\oplus\mathcal{O}(-Xe_3+Ye_2)\oplus\mathcal{O}XYe_2\\ &\oplus\mathcal{O}(-Ye_1+Xe_4+\varepsilon e_3)\oplus\mathcal{O}(-XYe_1+\varepsilon Xe_3)\oplus\mathcal{O}(XYe_4+\varepsilon Ye_3)\oplus\mathcal{O}XYe_3,\end{aligned}$$

and it is isomorphic to Z_2^{∞}. Suppose that $n\geq 3$. Then an $\mathcal{O}$-basis of the kernel of $\pi_{n,\infty}$ is given by

$$\begin{aligned}&\mathcal{O}(\varepsilon e_2-Xe_1)\oplus\mathcal{O}Xe_2\oplus\mathcal{O}(Ye_2-Xe_3)\oplus\mathcal{O}XYe_2\\ &\oplus\mathcal{O}(\varepsilon e_3+Xe_4-Ye_1)\oplus\mathcal{O}(\varepsilon Xe_3-XYe_1)\oplus\mathcal{O}(Ye_3+Xe_5)\oplus\mathcal{O}XYe_3\\ &\bigoplus_{k=2}^{n-2}\Big(\mathcal{O}(-1)^{k+1}(\varepsilon e_{2k+1}+Xe_{2(k+1)}-Ye_{2k})\oplus\mathcal{O}(-1)^{k+1}(\varepsilon Xe_{2k+1}-XYe_{2k})\\ &\qquad\qquad\oplus\mathcal{O}(-1)^{k+1}(Ye_{2k+1}+Xe_{2k+3})\oplus\mathcal{O}(-1)^{k+1}XYe_{2k+1}\Big)\\ &\oplus\mathcal{O}(-1)^{n}(\varepsilon e_{2n-1}+Xe_{2n}-Ye_{2(n-1)})\oplus\mathcal{O}(-1)^{n}(\varepsilon Xe_{2n-1}-XYe_{2(n-1)})\\ &\qquad\qquad\oplus\mathcal{O}(-1)^{n}(\varepsilon Ye_{2n-1}+XYe_{2n})\oplus\mathcal{O}(-1)^{n}XYe_{2n-1}.\end{aligned}$$

Then, it is easy to check that the actions X and Y on the kernel of $\pi_{n,\infty}$ coincide with those on Z_n^{∞}. □

5.2 Almost split sequence ending at non-periodic Heller lattices

By Lemmas 3.4.4 and 5.1.5, there are almost split sequences ending at the Heller lattices Z_m and Z_n^{λ} for each $m\in\mathbb{Z}$, $n\in\mathbb{Z}_{>0}$ and $\lambda\in\mathbb{P}^1(\kappa)$. Furthermore, it follows from Proposition 5.1.6, there is the unique non-periodic Heller component $\mathcal{HC}(Z_0)$, which contains Z_m for all $m\in\mathbb{Z}$. In this section, we explain some properties of the middle term of the almost split sequence ending at the Heller lattice Z_m in order to determine the shape of $\mathcal{HC}(Z_0)$. Since the stable Auslander–Reiten quiver $\Gamma_s(A)$ is stable translation quiver, it is enough to consider $n=1$.

Recall that the projective cover of Z_1 is given by

$$\pi_1: A\oplus A\longrightarrow Z_1,\quad e_1\longmapsto \mathsf{a}_{1,1},\quad e_2\longmapsto\mathsf{a}_{1,4}.$$

Then, the representing matrix of π_1 is:

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&\varepsilon&1&0&0&0\end{pmatrix}$$

Let $\psi \in \mathsf{Hom}_A(Z_1, A \oplus A)$, and we write

$$\psi(\varepsilon e) = \sum_{i=1}^{2}(a_{i1}e_i + a_{i2}Xe_i + a_{i3}Ye_i + a_{i4}XYe_i),$$
$$\psi(XYe) = \sum_{i=1}^{2}(b_{i1}e_i + b_{i2}Xe_i + b_{i3}Ye_i + b_{i4}XYe_i).$$

Since $\varepsilon\psi(XYe) = XY\psi(\varepsilon e)$, the representing matrix of ψ is:

$$\begin{pmatrix} \varepsilon b_{14} & 0 & 0 & 0 \\ a_{12} & \varepsilon b_{14} & 0 & 0 \\ a_{13} & 0 & \varepsilon b_{14} & 0 \\ a_{14} & a_{13} & a_{12} & b_{14} \\ \varepsilon b_{24} & 0 & 0 & 0 \\ a_{22} & \varepsilon b_{24} & 0 & 0 \\ a_{23} & 0 & \varepsilon b_{24} & 0 \\ a_{24} & a_{23} & a_{22} & b_{24} \end{pmatrix}$$

Thus, the set of endomorphisms of Z_1 factorizing through π_1 is:

$$\left\{ \begin{pmatrix} \varepsilon\alpha & 0 & 0 & 0 \\ \beta & \varepsilon\alpha & 0 & 0 \\ \gamma & 0 & \varepsilon\alpha & 0 \\ \varepsilon\delta & \varepsilon\gamma & \varepsilon\beta & \varepsilon\alpha \end{pmatrix} \middle| \; \alpha, \beta, \gamma, \delta \in \mathcal{O} \right\}$$

On the other hand, the radical of the endomorphism ring of Z_1 is given by

$$\mathsf{radEnd}_A(Z_1) = \left\{ \begin{pmatrix} \varepsilon a & 0 & 0 & 0 \\ b & \varepsilon a & 0 & 0 \\ c & 0 & \varepsilon a & 0 \\ d & \varepsilon c & \varepsilon b & \varepsilon a \end{pmatrix} \middle| \; a, b, c, d \in \mathcal{O} \right\}.$$

Therefore, we may take an endomorphism φ which satisfies conditions (i) and (iii) in Theorem 3.5.8 as

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and we consider the pullback diagram along (π_1, φ):

$$\begin{tikzcd} 0 \arrow[r] & Z_0 \arrow[r] \arrow[d, equal] & \overline{E}_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & Z_0 \arrow[r] & A \oplus A \arrow[r, "\pi_1"'] & Z_1 \arrow[r] & 0 \end{tikzcd}$$

Since Z_0 is indecomposable, the upper exact sequence is the almost split sequence ending at Z_1. Then, an $\mathcal{O}$-basis of $\overline{E}_1$ is given by

$$\begin{aligned}\overline{E}_1 =&\{(f_1, f_2, x) \in A \oplus A \oplus Z_1 |\ \pi_1(f_1, f_2) = \varphi(x)\}\\ =&\ \mathcal{O}(e_2 + \varepsilon e_3) \oplus \mathcal{O}Xe_2 \oplus \mathcal{O}Ye_2 \oplus \mathcal{O}XYe_2\\ &\oplus \mathcal{O}(XYe_1 + \varepsilon^2 e_3) \oplus \mathcal{O}(\varepsilon Xe_3) \oplus \mathcal{O}(\varepsilon Ye_3) \oplus \mathcal{O}(XYe_3).\end{aligned}$$

Proposition 5.2.1 ([M1, Proposition 2.7])**.** For any integer m, the Heller lattice Z_m appears on the boundary of $\mathcal{HC}(Z_0)$.

Proof. It is enough to show that the A-lattice $\overline{E}_1$ has exactly one non-projective indecomposable direct summand since $\tau(Z_m) \simeq Z_{m-1}$ for all m. Note that, since $\mathsf{rad}(A) = Z_0$, the A-lattice $\overline{E}_1$ has projective direct summands by Theorem 3.5.13. In fact, we have isomorphisms

$$\begin{aligned}\overline{E}_1 \simeq&\ \mathcal{O}(e_2 + \varepsilon e_3) \oplus \mathcal{O}(Xe_2 + \varepsilon Xe_3) \oplus \mathcal{O}(Ye_2 + \varepsilon Ye_3) \oplus \mathcal{O}(XYe_2 + \varepsilon XYe_3)\\ &\oplus \mathcal{O}(\varepsilon^2 e_3) \oplus \mathcal{O}(\varepsilon Xe_3) \oplus \mathcal{O}(\varepsilon Ye_3) \oplus \mathcal{O}(XYe_3)\\ \simeq&\ A \oplus \mathcal{O}\varepsilon^2 e_3 \oplus \mathcal{O}\varepsilon Xe_3 \oplus \mathcal{O}\varepsilon Ye_3 \oplus \mathcal{O}XYe_3.\end{aligned}$$

Let $E_1 = \mathcal{O}\varepsilon^2 e_3 \oplus \mathcal{O}\varepsilon Xe_3 \oplus \mathcal{O}\varepsilon Ye_3 \oplus \mathcal{O}XYe_3$. Then, E_1 is not isomorphic to A. Since $E_1 \otimes \mathcal{K} \simeq A \otimes \mathcal{K}$, the A-lattice E_1 is indecomposable. $\square$

Corollary 5.2.2 ([M1, Proposition 2.12])**.** Let E_m be the middle term of the almost split sequence ending at Z_m.

(1) For any integer $m \neq 1$, the A-lattice E_m is indecomposable.

(2) For any $m \in \mathbb{Z}$, we have an isomorphism $E_m \otimes \kappa \simeq M(m-1)^{\oplus 4}$.

Proof. (1) For any $m \neq 0$, the Heller lattice Z_m is not a direct summand of $\mathsf{rad}(A)$. Thus, the assertion follows from Theorem 3.5.13 and Proposition 5.2.1.

(2) Applying $- \otimes \kappa$ to the A-lattice

$$E_1 = \mathcal{O}\varepsilon^2 e_3 \oplus \mathcal{O}\varepsilon Xe_3 \oplus \mathcal{O}\varepsilon Ye_3 \oplus \mathcal{O}XYe_3,$$

we have $E_1 \otimes \kappa \simeq M(0)^{\oplus 4}$. Thus, Lemma 5.1.3 implies our claim. $\square$

5.3 *Excluding the possibility B_∞, C_∞ and D_∞*

The Heller component $\mathcal{HC}(Z_0)$ does not have a loop. Let $\overline{T}$ be the tree class of $\mathcal{HC}(Z_0)$. It follows Corollary 3.7.13 that the function $\mathcal{D}$ gives a additive function on T. Thus, $\overline{T}$ is one of infinite Dynkin diagrams or Euclidean diagrams. In this section, we exclude the possibility that $T = B_\infty$, C_∞ or D_∞. From now on, we denote by E_m the unique non-projective indecomposable direct summand of the middle term of the almost split sequence ending at Z_m. Let $\overline{F}_m$ be the middle term of the almost split sequence ending at Z_m. The aim of this section is to show that, for any $m \in \mathbb{Z}$, the non-projective indecomposable direct summands of $\overline{F}_m$ are Z_{m-1} and an indecomposable A-lattice F_m. Moreover, for all m, neither Z_m nor E_m are isomorphic to F_n ($n \in \mathbb{Z}$).

It is enough to show the assertion for the case $\overline{F}_1$. We construct the almost split sequence ending at E_1. Since the projective cover of E_1 is given by

$$\pi^{E_1} : A^{\oplus 4} \longrightarrow E_1, \quad e_1 \longmapsto \varepsilon^2 e,\ e_2 \longmapsto \varepsilon X e,\ e_3 \longmapsto \varepsilon Y e,\ e_4 \longmapsto XYe,$$

where e is the identity element of $\mathcal{O}$, the representing matrix of π^{E_1} is the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^2 & 0 & 0 & \varepsilon & 0 & 0 & \varepsilon & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

On the other hand, the radical of $\mathsf{End}_A(E_1)$ is given by

$$\mathsf{radEnd}_A(E_1) = \left\{ \begin{pmatrix} \varepsilon a & 0 & 0 & 0 \\ b & \varepsilon a & 0 & 0 \\ c & 0 & \varepsilon a & 0 \\ d & c & b & \varepsilon a \end{pmatrix} \;\middle|\; a, b, c, d \in \mathcal{O} \right\}.$$

Lemma 5.3.1 ([M1, Lemma 2.17])**.** Any endomorphism of E_1 which factors through π^{E_1} is reprsented by

$$\begin{pmatrix} \varepsilon^2 a & 0 & 0 & 0 \\ \varepsilon^2 b & \varepsilon^2 a & 0 & 0 \\ \varepsilon^2 c & 0 & \varepsilon^2 a & 0 \\ \varepsilon^2 d & \varepsilon^2 c & \varepsilon^2 b & \varepsilon^2 a \end{pmatrix}$$

for some $a, b, c, d \in \mathcal{O}$.

Proof. The proof is straightforward. $\square$

Let $\varphi : E_1 \to E_1$ be the endomorphism defined by $\varphi(\varepsilon^2 e) = \varepsilon XYe$. Note that $\varphi(\varepsilon Xe) = \varphi(\varepsilon Ye) = \varphi(XYe) = 0$. We consider the pullback diagram along (π^{E_1}, φ):

$$\begin{tikzcd} 0 \arrow[r] & E_0 \arrow[r] \arrow[d, equal] & \overline{F}_1 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & E_0 \arrow[r] & A^{\oplus 4} \arrow[r, "\pi^{E_1}"'] & E_1 \arrow[r] & 0 \end{tikzcd} \tag{5.4}$$

Lemma 5.3.2 ([M1, Lemma 2.18])**.** The following statements hold.

(1) φ does not factor through π^{E_1}.

(2) For each $f \in \mathsf{radEnd}_A(E_1)$, $\varphi \circ f$ factors through π^{E_1}.

Proof. (1) If φ factors through π^{E_1}, then it contradicts with Lemma 5.3.1.

(2) Let $f \in \mathsf{radEnd}_A(E_1)$. Assume that $f(\varepsilon^2 e) = \varepsilon a(\varepsilon^2 e)+b(\varepsilon Xe)+c(\varepsilon Ye)+d(XYe)$ for some $a, b, c, d \in \mathcal{O}$. Since $\varepsilon^2 f(XYe) = XYf(\varepsilon^2 e) = \varepsilon^3 aXYe$, we have $f(XYe) = \varepsilon aXYe$, and hence $\varphi \circ f(\varepsilon^2 e) = \varepsilon^2 a(XYe)$. Define $\psi : E_1 \to A^{\oplus 4}$ by $\psi(\varepsilon^2 e) = aXYe_1$. Then, it is easy to check $\varphi \circ f = \pi^{E_1} \circ \psi$. □

By Proposition 3.5.8, the upper short exact sequence in (5.4) is the almost split sequence ending at E_1.

Proposition 5.3.3 ([M1, Proposition 2.16])**.** For any $m \in \mathbb{Z}$, the non-projective indecomposable direct summands of $\overline{F}_m$ are Z_{m-1} and an indecomposable A-lattice F_m. Moreover, for all m, neither Z_m nor E_m are isomorphic to F_n ($n \in \mathbb{Z}$).

Proof. The A-lattice $\overline{F}_1$ is a direct sum of F_1 and F_1', where

$$\begin{aligned} F_1 =& \mathcal{O}(Xe_1 - \varepsilon e_2) \oplus \mathcal{O}Xe_2 \oplus \mathcal{O}(XYe_1 - \varepsilon Ye_2) \oplus \mathcal{O}XYe_2 \\ & \oplus \mathcal{O}(Ye_1 - \varepsilon e_3) \oplus \mathcal{O}(Xe_3 - Ye_2) \oplus \mathcal{O}Ye_3 \oplus \mathcal{O}XYe_3 \\ & \oplus \mathcal{O}(Xe_3 + \varepsilon^2 e) \oplus \mathcal{O}\varepsilon Xe \oplus \mathcal{O}\varepsilon Ye \oplus \mathcal{O}XYe, \\ F_1' =& \mathcal{O}(\varepsilon e_4 + \varepsilon^2 e) \oplus \mathcal{O}(Xe_4 + \varepsilon Xe) \oplus \mathcal{O}(Ye_4 + \varepsilon Ye) \oplus \mathcal{O}(XYe_4 + \varepsilon XYe). \end{aligned}$$

Obviously, the A-lattice F_1' is isomorphic to the Heller lattice Z_0. We show that the A-lattice F_1 is indecomposable. The actions of X and Y on F_1 with respect to the above basis are given by the following matrices:

$$X = \left(\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
-\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \multicolumn{2}{c}{0} & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\
0 & 0 & -\varepsilon & 0 & 0 & -1 & 0 & 0 & & & & \\
 & & & & 0 & 0 & 0 & 0 & & & & \\
 & \multicolumn{2}{c}{0} & & -\varepsilon & 0 & 0 & 0 & & \multicolumn{2}{c}{0} & \\
 & & & & 0 & 0 & 0 & 0 & & & & \\
 & & & & 0 & 0 & 1 & 0 & & & & \\
 & & & & & & & & 0 & 0 & 0 & 0 \\
 & \multicolumn{2}{c}{0} & & & \multicolumn{2}{c}{0} & & \varepsilon & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & \varepsilon & 0
\end{array}\right)$$

$$Y = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & 0 & 0
\end{pmatrix}$$

Let $M = (x_{i,j}) \in \mathsf{End}_A(F_1)$ be an idempotent. By the equalities $MX = XM$ and $MY = YM$, the idempotent M is of the form $M = (M_1 \;\; M_2)$, where M_1 and M_2 are

$$M_1 = \begin{pmatrix}
x_{1,1} & 0 & 0 & 0 & -\varepsilon x_{3,7} & 0 \\
x_{2,1} & x_{1,1} & 0 & 0 & -\varepsilon x_{4,7} & -\varepsilon x_{3,7} \\
x_{3,1} & x_{3,2} & x_{1,1} & 0 & x_{3,5} & x_{3,6} \\
x_{4,1} & x_{4,2} & x_{2,1} & x_{1,1} & x_{4,5} & x_{4,6} \\
-\varepsilon x_{3,2} & 0 & 0 & 0 & x_{1,1} - \varepsilon x_{3,6} & 0 \\
x_{6,1} & -\varepsilon x_{3,2} & 0 & 0 & x_{6,5} & x_{1,1} - \varepsilon x_{3,6} \\
-\varepsilon x_{8,2} & 0 & \varepsilon^2 x_{3,2} & 0 & x_{7,5} & \varepsilon x_{3,2} \\
x_{8,1} & x_{8,2} & x_{8,3} & -\varepsilon x_{3,2} & x_{8,5} & x_{8,6} \\
x_{9,1} & 0 & 0 & 0 & x_{9,5} & 0 \\
x_{10,1} & -x_{9,1} & 0 & 0 & -x_{12,7} & -x_{9,5} \\
x_{11,1} & 0 & \varepsilon x_{9,1} & 0 & x_{11,5} & x_{9,1} \\
x_{12,1} & -x_{11,1} & \varepsilon x_{10,1} & -\varepsilon x_{9,1} & x_{12,5} & x_{12,6}
\end{pmatrix}$$

$$M_2 = \begin{pmatrix}
0 & 0 & \varepsilon x_{3,11} & 0 & 0 & 0 \\
0 & 0 & x_{2,9} & -\varepsilon x_{3,11} & 0 & 0 \\
x_{3,7} & 0 & x_{3,9} & -x_{8,12} & x_{3,11} & 0 \\
x_{4,7} & -\varepsilon x_{3,7} & x_{4,9} & x_{4,10} & x_{4,11} & -x_{3,11} \\
0 & 0 & -\varepsilon x_{8,12} & 0 & 0 & 0 \\
0 & 0 & x_{6,9} & \varepsilon x_{8,12} & 0 & 0 \\
x_{1,1} - \varepsilon x_{3,6} & 0 & \varepsilon x_{8,10} & 0 & \varepsilon x_{8,12} & 0 \\
x_{8,7} & x_{1,1} - \varepsilon x_{3,6} & x_{8,9} & x_{8,10} & x_{8,11} & x_{8,12} \\
0 & 0 & x_{9,9} & 0 & 0 & 0 \\
0 & 0 & x_{10,9} & x_{9,9} & 0 & 0 \\
-x_{9,5} & 0 & x_{11,9} & 0 & x_{9,9} & 0 \\
x_{12,7} & -\varepsilon x_{9,5} & x_{12,9} & x_{11,9} & x_{12,11} & x_{9,9}
\end{pmatrix}$$

such that

$$\begin{cases} x_{2,9} + \varepsilon x_{3,7} - \varepsilon x_{4,11} = 0, \\ \varepsilon x_{3,1} - \varepsilon x_{4,2} + x_{6,1} = 0, \\ \varepsilon x_{3,9} + \varepsilon x_{4,10} + x_{6,9} = 0, \\ x_{6,9} + x_{9,9} = x_{1,1} - \varepsilon x_{3,6} + \varepsilon x_{8,11}, \\ x_{6,1} - x_{8,3} + x_{9,1} = 0, \end{cases} \qquad \begin{cases} x_{6,5} + \varepsilon x_{8,7} + x_{9,5} = 0, \\ x_{7,5} - x_{8,3} + \varepsilon x_{8,6} = 0, \\ x_{9,5} + x_{10,9} - x_{12,11} = 0, \\ -x_{10,1} + x_{11,5} + x_{12,6} = 0. \end{cases}$$

Note that, it follows that we have $x_{6,9} \in \varepsilon\mathcal{O}$ and $x_{9,9} = x_{1,1} - \varepsilon f$ for some $f \in \mathcal{O}$. Since M is an idempotent, the following equality holds:

$$x_{1,1}(1 - x_{1,1}) = \varepsilon x_{3,11}x_{9,1} + \varepsilon^2 x_{3,2}x_{3,7}. \tag{5.5}$$

Assume that $x_{1,1} \equiv 0 \bmod \varepsilon\mathcal{O}$. By the assumption, the element $x_{9,9}$ belongs to $\varepsilon\mathcal{O}$. By comparing the $(9,1)$-entries and $(3,2)$-entries of M and M^2, respectively, we have

$$x_{9,1} = x_{1,1}x_{9,1} + x_{9,1}x_{9,9} - \varepsilon x_{3,2}x_{9,5} \in \varepsilon\mathcal{O}, \tag{5.6}$$

$$x_{3,2} = x_{1,1}x_{3,2} + x_{1,1}x_{3,2} - \varepsilon x_{3,2}x_{3,6} + x_{9,1}x_{8,12}. \tag{5.7}$$

It follows from (5.25) and (5.7) that the equality

$$x_{3,2}(1 - 2x_{1,1} + \varepsilon x_{3,6} + \varepsilon x_{9,5}(1 - x_{1,1} - x_{9,9})^{-1}x_{8,12}) = 0 \tag{5.8}$$

holds. Thus, the elements $x_{3,2}$ and $x_{9,1}$ are zero, and hence $x_{1,1} = 0$. Let $\overline{M}$ be $M \bmod \varepsilon\mathcal{O}$. As $M^2 = M$, it suffices to show that $\overline{M}$ is the zero matrix to conclude that M itself is the zero matrix. Let e_i $(1 \leq i \leq 12)$ be standard row vectors. Then, the span of e_1, e_5, e_9 is stable by $\overline{M}$ and the representing matrix is nilpotent. Thus, $e_i\overline{M} = 0$ holds for $i = 1, 5, 9$. From the equalities

$$e_2\overline{M} = \overline{x_{2,1}}e_1 + \overline{x_{2,9}}e_9, \quad e_6\overline{M} = \overline{x_{6,1}}e_1 - \overline{x_{6,9}}e_9, \text{ and } e_7\overline{M} = \overline{x_{7,5}}e_5,$$

we also obtain $e_i\overline{M} = 0$ for $i = 2, 6, 7$. Then a similar argument shows $e_i\overline{M} = 0$ for $i = 10, 11$, and then for $i = 3, 12$, and finally for $i = 4, 8$.

Assume that $x_{1,1} \equiv 1 \bmod \varepsilon\mathcal{O}$. Then, $\mathbf{1}_{12} - M$ is an idempotent whose $(1,1)$-entry is zero modulo $\varepsilon\mathcal{O}$, and $M = \mathbf{1}_{12}$ follows.

On the other hand, since E_1 is not isomorphic to a Heller lattice, the induced sequence

$$0 \longrightarrow E_0 \otimes \kappa \longrightarrow \overline{F}_1 \otimes \kappa \longrightarrow E_1 \otimes \kappa \longrightarrow 0$$

splits by Proposition 3.7.10. Thus, there is an isomorphism

$$\overline{F}_1 \otimes \kappa \simeq M(0)^{\oplus 4} \oplus M(-1)^{\oplus 4},$$

and hence $F_1 \otimes \kappa \simeq M(0)^{\oplus 3} \oplus M(-1)^{\oplus 3}$ as $F_1' \simeq Z_0$. It follows from Proposition 5.1.5 and Corollary 5.2.2 that F_1 is neither isomorphic to Z_m nor E_m for all m. $\square$

5.4 Valencies of vertices in the non-periodic component

In this section, we observe the number of arrows from each vertex in $\mathcal{HC}(Z_0)$. From Proposition 5.2.1, the Heller lattice Z_n appears on the boundary in $\mathcal{HC}(Z_0)$, and it follows from Proposition 5.3.3 that we have

$$\sharp\{\text{arrows starting at } E_n\} = \sharp\{\text{arrows ending at } E_n\} = 2$$

for all $n \in \mathbb{Z}$. Thus, the component $\mathcal{HC}(Z_0)$ admits the following valued subquiver with trivial valuations:

$$\begin{tikzcd}[column sep=small]
 & & F_{-1} \arrow[dr] & & F_0 \arrow[ll, dotted, "\tau"'] \arrow[dr] & & F_1 \arrow[ll, dotted, "\tau"'] \arrow[dr] & & F_2 \arrow[ll, dotted, "\tau"'] \arrow[dr] & & \\
\cdots & E_{-2} \arrow[ur] \arrow[dr] & & E_{-1} \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr] & & E_0 \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr] & & E_1 \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr] & & E_2 \arrow[ll, dotted, "\tau"'] \arrow[dr] & \cdots \\
 & & Z_{-2} \arrow[ur] & & Z_{-1} \arrow[ll, dotted, "\tau"'] \arrow[ur] & & Z_0 \arrow[ll, dotted, "\tau"'] \arrow[ur] & & Z_1 \arrow[ll, dotted, "\tau"'] \arrow[ur] & & Z_2 \arrow[ll, dotted, "\tau"']
\end{tikzcd}$$

Recall that the function $\mathcal{D} : \mathcal{HC}(Z_0)_0 \to \mathbb{Z}_{\geq 0}$ is defined by

$$\mathcal{D}(X) = \sharp\{\text{non-projective indecomposable direct summands of } X \otimes \kappa\}.$$

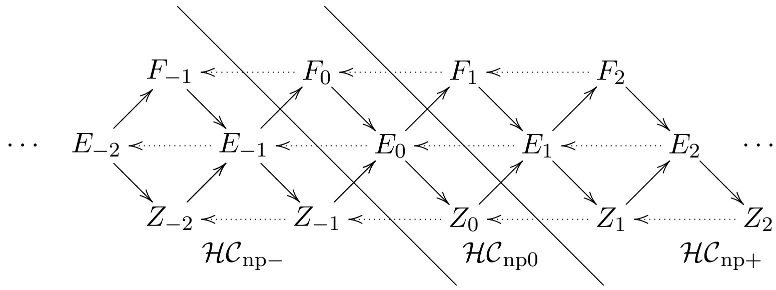
It follows from Corollaries 3.7.13, 5.2.2 and Proposition 5.1.5 (2) that the function $\mathcal{D} : \mathcal{HC}(Z_0) \to \mathbb{Z}_{>0}$ gives an additive function on the tree class $\overline{T}$ of $\mathcal{HC}(Z_0)$. Thus, Theorem 2.5.8 implies that $\overline{T}$ is one of infinite Dynkin diagrams or Euclidean diagrams.

Given a vertex X of $\mathcal{HC}(Z_0)$, we define a non-negative integer $d(X)$ to be the number of arrows from X in $\mathcal{HC}(Z_0)$. In order to exclude some candidates for the tree class T of $\mathcal{HC}(Z_0)$, we introduce a pair of integers $(q(M), H(M))$ for $M \in \mathcal{HC}(Z_0)$ as follows. If M is isomorphic to the Heller lattice Z_n, then $(q(M), H(M)) = (1, n)$. Otherwise, we may choose n such that a composition of irreducible morphisms $f_1 \circ \cdots \circ f_k : Z_n \to M$ has the minimum length, and define $(q(M), H(M)) = (k+1, n+k)$. For an A-lattice M, we also define the equilateral triangle $T(M) \subset \mathcal{HC}(Z_0)$ as follows:

- The vertices of $T(M)$ are M, Z_n and $Z_{H(M)}$.
- The edge $T(M)_1$ is a chain of irreducible morphisms from Z_n to M.
- The edge $T(M)_2$ is a chain of irreducible morphisms from M to $Z_{H(M)}$.
- The edge $T(M)_3$ is a chain of the Auslander–Reiten translation from $Z_{H(M)}$ to Z_n.

The set of vertices of $\mathcal{HC}(Z_0)$ is the disjoint union of the following three sets:

$$\begin{aligned}
\mathcal{HC}_{\mathrm{np}+} &= \{X \in \mathcal{HC}(Z_0) \mid H(X) > 0\},\\
\mathcal{HC}_{\mathrm{np}0} &= \{X \in \mathcal{HC}(Z_0) \mid H(X) = 0\},\\
\mathcal{HC}_{\mathrm{np}-} &= \{X \in \mathcal{HC}(Z_0) \mid H(X) < 0\}.
\end{aligned}$$



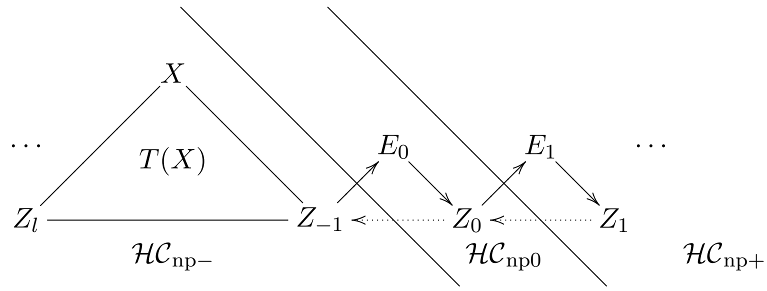
From now on, we assume that $\mathcal{HC}(Z_0) \neq \mathbb{Z}A_\infty$. Then, there exists an A-lattice X such that

(i) the A-lattice X is not isomorphic to Z_m and E_m for all m.

(ii) the triangle $T(X)$ is contained in $\mathcal{HC}_{\mathrm{np}-}$,

(iii) the number of outgoing arrows is two for each A-lattices on the edge $T(X)_1$ except for $Z_{H(X)-q(X)+1}$ and X, and the number of indecomposable direct summands of E_X is not 2, where E_X is the middle term of the almost split sequence ending at X.

(iv) valuations of arrows in the triangle $T(X)$ is trivial.



By the construction of $T(X)$, we have $\mathcal{D}(M) = 2q(M)$ for any $M \in T(X)$. We may assume that $q(X) \geq 3$ and $H(X) = -1$. We set $q(X) = q$. Assume that the almost split sequence ending at X is given by

$$\mathcal{E}(X): \quad 0 \longrightarrow \tau X \longrightarrow \bigoplus_{i=1}^{p} W_i \longrightarrow X \longrightarrow 0,$$

where $W_p \in T(X)$. Then, the neighborhood of X in $\mathcal{HC}(Z_0)$ is given as follows.

$$\begin{tikzcd}
 & \tau W_1 \arrow[ddr] & & W_1 \arrow[ll, dotted, "\tau"'] \arrow[ddr] & \\
 & \tau W_2 \arrow[dr] & & W_2 \arrow[ll, dotted, "\tau"'] \arrow[dr] & \\
 & \vdots & & \vdots & \\
\tau^2 X \arrow[uuur] \arrow[uur] \arrow[dr] \arrow[ddr] & & \tau X \arrow[ll, dotted, "\tau"] \arrow[uuur] \arrow[uur] \arrow[dr] \arrow[ddr] & & X \arrow[ll, dotted, "\tau"] \\
 & \tau W_{p-1} \arrow[ur] & & W_{p-1} \arrow[ll, dotted, "\tau"] \arrow[ur] & \\
 & \tau W_p \arrow[uur] & & W_p \arrow[ll, dotted, "\tau"] \arrow[uur] &
\end{tikzcd} \tag{5.9}$$

Here, we allow the possibility that $W_i \simeq W_k$ for some $i \neq k$ instead of writing the valuation. If $\mathcal{D}(W_i) = s_i$, then the values of $\mathcal{D}$ of (5.9) are as follows:

$$\begin{tikzcd}
 & s_1 \arrow[ddr] & & s_1 \arrow[ll, dotted, "\tau"'] \arrow[ddr] & \\
 & s_2 \arrow[dr] & & s_2 \arrow[ll, dotted, "\tau"'] \arrow[dr] & \\
 & \vdots & & \vdots & \\
2q \arrow[uuur] \arrow[uur] \arrow[dr] \arrow[ddr] & & 2q \arrow[ll, dotted, "\tau"] \arrow[uuur] \arrow[uur] \arrow[dr] \arrow[ddr] & & 2q \arrow[ll, dotted, "\tau"] \\
 & s_{p-1} \arrow[ur] & & s_{p-1} \arrow[ll, dotted, "\tau"] \arrow[ur] & \\
 & 2(q-1) \arrow[uur] & & 2(q-1) \arrow[ll, dotted, "\tau"] \arrow[uur] &
\end{tikzcd} \tag{5.10}$$

Lemma 5.4.1 ([M1, Lemma 3.6])**.** The following statements hold:

(1) The sum of $s_1, s_2, \ldots, s_{p-2}$ and s_{p-1} is $2(q+1)$.

(2) The inequality $s_i \geq q$ is satisfied for any i.

Proof. (1) By Lemma 3.7.12, we have

$$4q = \sum_{i=1}^{p-1} \mathcal{D}(W_i) + \mathcal{D}(W_p) = \sum_{i=1}^{p-1} s_i + 2(q-1).$$

It follows that (1) holds.

(2) Since $\mathcal{D}$ is additve, we obtain that $2s_i \geq 2q$. $\square$

Lemma 5.4.2 ([M1, Lemma 3.7])**.** Suppose that $q < \infty$. Then, $d(X)$ is precisely three.

Proof. Lemma 5.4.1 implies that

$$2(q+1) = \sum_{i=1}^{p-1} s_i \geq (p-1)q.$$

Thus, the inequality $-2 \leq q(3-p)$ holds. Since p and q are positive, we have $p = 1, 2, 3$. If $p = 1$, then $q = -1$ from Lemma 5.4.1 (1), a contradiction. If $p = 2$, then $s_1 = 2(q+1)$, which contradicts with the maximality of q namely, the condition (iii). Therefore, we have $p = 3$. Then, we may assume that the almost split sequence ending at X is of the form

$$0 \longrightarrow \tau X \longrightarrow W_1 \oplus W_2 \oplus Y \longrightarrow X \longrightarrow 0$$

with $Y \in T(X)$. We show that the three non-projective indecomposable A-lattices W_1, W_2 and Y are pairwise non-isomorphic.

Suppose that $Y \simeq W_i$ for some i. Since $Y \in T(X)$, there exist arrows in $T(X)$ such that their valuations are not trivial, a contradiction.

Suppose that $W_1 \simeq W_2$. Then, the neighborhood of X in $\mathcal{HC}(Z_0)$ is the following valued quiver:

$$\begin{tikzcd}
& \tau W_1 \arrow[dr, "{(2,1)}"] & & W_1 \arrow[ll, dotted, "\tau"'] \arrow[dr, "{(2,1)}"] & \\
\tau^2 X \arrow[ur, "{(1,2)}"] \arrow[dr] & & \tau X \arrow[ll, dotted, "\tau"] \arrow[ur, "{(1,2)}"] \arrow[dr] & & X \arrow[ll, dotted, "\tau"] \\
& \tau Y \arrow[ur] & & Y \arrow[ll, dotted, "\tau"'] \arrow[ur] &
\end{tikzcd}$$

Indeed, if we write the value $W_1 \xrightarrow{(a,b)} X$, then clearly $a = 2$ by the assumption. Thus, the almost split sequence ending at X becomes

$$0 \longrightarrow \tau X \longrightarrow W_1^{\oplus 2} \oplus Y \longrightarrow X \longrightarrow 0$$

and we have $\mathcal{D}(W_1) = q + 1$ from Lemma 3.7.12. Suppose that the almost split sequence ending at W_1 is

$$\mathscr{E}(W_1): \quad 0 \longrightarrow \tau W_1 \longrightarrow \tau X^{\oplus b} \oplus U_1 \longrightarrow W_1 \longrightarrow 0,$$

where U_1 is an A-lattice. If $U_1 = 0$, then Lemma 3.7.12 implies that

$$q + 1 = \mathcal{D}(W_1) = qb,$$

hence $q(b-1) = 1$, which contradicts with $q \geq 3$. Thus, $U_1 \neq 0$ and $q(b-1) < 1$. Since $b \geq 1$, we have $b = 1$.

From the almost split sequence $\mathscr{E}(W_1)$, we have $\mathcal{D}(U_1) = 2$, and it implies that U_1 is indecomposable. Therefore, we have $q = 3$ from the inequality

$$4 = \mathcal{D}(U_1) + \mathcal{D}(\tau(U_1)) \geq \mathcal{D}(\tau(W_1)) = q + 1.$$

Note that $\mathcal{HC}(Z_0)$ is the following valued stable translation quiver.

$$
\begin{tikzcd}
 & & U_0 \arrow[dr] & & U_1 \arrow[ll, dotted, "\tau"'] \arrow[dr] & & U_2 \arrow[ll, dotted, "\tau"'] \arrow[dr] & & U_3 \arrow[ll, dotted, "\tau"'] \arrow[dr] & & \\
 & \tau^2 W_1 \arrow[ur] \arrow[dr, "{(2,1)}"'] & & \tau W_1 \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr, "{(2,1)}"'] & & W_1 \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr, "{(2,1)}"'] & & \tau^{-1} W_1 \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr, "{(2,1)}"'] & & \tau^{-2} W_1 \arrow[ll, dotted, "\tau"'] & \\
\cdots\cdots & & F_{-1} \arrow[ur, "{(1,2)}"'] \arrow[dr] & & F_0 \arrow[ll, dotted, "\tau"] \arrow[ur, "{(1,2)}"'] \arrow[dr] & & F_1 \arrow[ll, dotted, "\tau"] \arrow[ur, "{(1,2)}"'] \arrow[dr] & & F_2 \arrow[ll, dotted, "\tau"] \arrow[ur, "{(1,2)}"'] \arrow[dr] & & \cdots\cdots \\
 & E_{-2} \arrow[ur] \arrow[dr] & & E_{-1} \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr] & & E_0 \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr] & & E_1 \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr] & & E_2 \arrow[ll, dotted, "\tau"'] \arrow[dr] & \\
 & & Z_{-2} \arrow[ur] & & Z_{-1} \arrow[ll, dotted, "\tau"'] \arrow[ur] & & Z_0 \arrow[ll, dotted, "\tau"'] \arrow[ur] & & Z_1 \arrow[ll, dotted, "\tau"'] \arrow[ur] & & Z_2 \arrow[ll, dotted, "\tau"']
\end{tikzcd}
\tag{5.11}
$$

It follows from Propositions 3.7.10, 5.1.5 and 5.2.2 that there is an isomorphism

$$\tau W_1^{\oplus 2} \otimes \kappa \simeq M(-3)^{\oplus 3} \oplus M(-2)^{\oplus 2} \oplus M(-1)^{\oplus 3},$$

a contradiction. $\square$

5.5 The shape of the non-periodic Heller component

Now, we determine the shape of the non-periodic Heller component $\mathcal{HC}(Z_0)$.

Theorem 5.5.1 ([M1])**.** Let $\mathcal{O}$ be a complete discrete valuation ring, $A = \mathcal{O}[X,Y]/(X^2,Y^2)$ and $\Gamma_s(A)$ the stable Auslander–Reiten quiver for $\mathsf{latt}^{(\natural)}$-$A$. Assume that the residue field κ is algebraically closed. Then, the component $\mathcal{HC}(Z_0)$ is isomorphic to $\mathbb{Z}A_\infty$.

Proof. Assume that $\overline{T} \neq A_\infty$. It implies from Propositions 5.1.6 and 5.3.3 that $\overline{T}$ is one of $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_{41}$ or $\widetilde{F}_{42}$. On the other hand, Lemma 5.4.2 implies that $\overline{T}$ is neither $\widetilde{F}_{41}$ nor $\widetilde{F}_{42}$.

First, we suppose that $\mathcal{HC}(Z_0) = \mathbb{Z}\widetilde{E}_6$. Then, $\mathcal{HC}(Z_0)$ has the following subquiver with bounds U_n and V_n:

$$\begin{tikzcd}[column sep=small]
 & & & U_1 \arrow[dr] & & U_2 \arrow[dr] & & & \\
 & & W_0 \arrow[ur] \arrow[dr] & V_1 \arrow[dr] & W_1 \arrow[ur] \arrow[dr] & V_2 \arrow[dr] & W_2 \arrow[dr] & & \\
\cdots & F_{-1} \arrow[ur] \arrow[r] \arrow[dr] & W'_0 \arrow[ur] \arrow[r] & F_0 \arrow[ur] \arrow[r] \arrow[dr] & W'_1 \arrow[ur] \arrow[r] & F_1 \arrow[ur] \arrow[r] \arrow[dr] & W'_2 \arrow[r] & F_2 & \cdots \\
 & & E_{-1} \arrow[ur] \arrow[dr] & & E_0 \arrow[ur] \arrow[dr] & & E_1 \arrow[ur] \arrow[dr] & & \\
 & Z_{-2} \arrow[ur] & & Z_{-1} \arrow[ur] & & Z_0 \arrow[ur] & & Z_1 &
\end{tikzcd} \tag{5.12}$$

By writing the ranks as $\mathcal{O}$-modules of vertices in (5.12), we obtain:

$$\begin{tikzcd}[column sep=small]
 & & & \alpha \arrow[dr] & & \alpha' \arrow[dr] & & & \\
 & & x \arrow[ur] \arrow[dr] & \beta \arrow[dr] & x' \arrow[ur] \arrow[dr] & \beta' \arrow[dr] & x'' \arrow[dr] & & \\
\cdots & 36 \arrow[ur] \arrow[r] \arrow[dr] & y \arrow[ur] \arrow[r] & 24 \arrow[ur] \arrow[r] \arrow[dr] & y' \arrow[ur] \arrow[r] & 12 \arrow[ur] \arrow[r] \arrow[dr] & y'' \arrow[r] & \gamma & \cdots \\
 & & 20 \arrow[ur] \arrow[dr] & & 12 \arrow[ur] \arrow[dr] & & 4 \arrow[ur] \arrow[dr] & & \\
 & 12 \arrow[ur] & & 8 \arrow[ur] & & 4 \arrow[ur] & & 4 &
\end{tikzcd} \tag{5.13}$$

Thus, we have the following system of linear equations:

$$\left\{\begin{array}{lcll}
\beta+\beta' & = & y' & \cdots\cdots\cdots(1) \\
\alpha+\alpha' & = & x' & \cdots\cdots\cdots(2) \\
x+y & = & 40 & \cdots\cdots\cdots(3) \\
x'+y' & = & 24 & \cdots\cdots\cdots(4)
\end{array}\right.
\qquad
\left\{\begin{array}{lcll}
x+x' & = & 24+\alpha & \cdots\cdots\cdots(5) \\
y+y' & = & 24+\beta & \cdots\cdots\cdots(6) \\
x'+x'' & = & 12+\alpha' & \cdots\cdots\cdots(7) \\
y'+y'' & = & 12+\beta' & \cdots\cdots\cdots(8)
\end{array}\right.$$

From the equations (1), (2), (5) and (6), we have $x = 24-\alpha'$ and $y = 24-\beta'$. Using these equations and (3), we have $\alpha'+\beta' = 8$. On the other hand, the equations (4),(7), (8) and $\alpha'+\beta' = 8$ imply $x''+y'' = 8$. Thus, we have $\gamma = 0$, a contradiction. Therefore, $\mathcal{HC}(Z_0) \neq \mathbb{Z}\widetilde{E}_6$.

Next we suppose that $\mathcal{HC}(Z_0) = \mathbb{Z}\widetilde{E}_7$. Then, $\mathcal{HC}(Z_0)$ has the following subquiver with upper bounds U_n:

$$
\begin{tikzcd}[column sep=small]
 & & & U_1 \arrow[dr] & & U_2 \arrow[dr] & & & \\
 & & V_0 \arrow[ur] \arrow[dr] & & V_1 \arrow[ur] \arrow[dr] & & V_2 \arrow[dr] & & \\
\cdots & W_{-1} \arrow[ur] \arrow[dr] & & W_0 \arrow[ur] \arrow[dr] & & W_1 \arrow[ur] \arrow[dr] & & W_2 \arrow[dr] & \cdots \\
G_{-2} \arrow[ur] \arrow[r] \arrow[dr] & W'_{-1} \arrow[r] & G_{-1} \arrow[ur] \arrow[r] \arrow[dr] & W'_0 \arrow[r] & G_0 \arrow[ur] \arrow[r] \arrow[dr] & W'_1 \arrow[r] & G_1 \arrow[ur] \arrow[r] \arrow[dr] & W'_2 \arrow[r] & G_2 \\
 & F_{-2} \arrow[ur] & & F_{-1} \arrow[ur] & & F_0 \arrow[ur] & & F_1 \arrow[ur] &
\end{tikzcd}
\tag{5.14}
$$

By writing the ranks as $\mathcal{O}$-modules of vertices in (5.14), we obtain:

$$
\begin{tikzcd}[column sep=small]
 & & & \gamma \arrow[dr] & & \gamma' \arrow[dr] & & & \\
 & & \alpha \arrow[ur] \arrow[dr] & & \alpha' \arrow[ur] \arrow[dr] & & \alpha'' \arrow[dr] & & \\
\cdots & x \arrow[ur] \arrow[dr] & & x' \arrow[ur] \arrow[dr] & & x'' \arrow[ur] \arrow[dr] & & x''' \arrow[dr] & \cdots \\
72 \arrow[ur] \arrow[r] \arrow[dr] & y \arrow[r] & 56 \arrow[ur] \arrow[r] \arrow[dr] & y' \arrow[r] & 40 \arrow[ur] \arrow[r] \arrow[dr] & y'' \arrow[r] & 24 \arrow[ur] \arrow[r] \arrow[dr] & y''' \arrow[r] & 12 \\
 & 48 \arrow[ur] & & 36 \arrow[ur] & & 24 \arrow[ur] & & 12 \arrow[ur] &
\end{tikzcd}
\tag{5.15}
$$

where these unknown letters are the ranks of the corresponding vertices. Thus, we have the following system of linear equations:

$$
\left\{
\begin{array}{lcll}
x+y & = & 80 & \cdots\cdots\cdots (1) \\
x'+y' & = & 60 & \cdots\cdots\cdots (2) \\
x''+y'' & = & 40 & \cdots\cdots\cdots (3) \\
x'''+y''' & = & 24 & \cdots\cdots\cdots (4) \\
x+x' & = & 56+\alpha & \cdots\cdots\cdots (5) \\
x'+x'' & = & 40+\alpha' & \cdots\cdots\cdots (6) \\
x''+x''' & = & 24+\alpha'' & \cdots\cdots\cdots (7)
\end{array}
\right.
\qquad
\left\{
\begin{array}{lcll}
y+y' & = & 56 & \cdots\cdots\cdots (8) \\
y'+y'' & = & 40 & \cdots\cdots\cdots (9) \\
y''+y''' & = & 24 & \cdots\cdots\cdots (10) \\
x'+\gamma & = & \alpha+\alpha' & \cdots\cdots\cdots (11) \\
x''+\gamma' & = & \alpha'+\alpha'' & \cdots\cdots\cdots (12) \\
\gamma+\gamma' & = & \alpha' & \cdots\cdots\cdots (13)
\end{array}
\right.
$$

From the equations (1), (2), (5) and (8), we have $\alpha = 28$. Similarly, the equations $(2), (3), (6)$ and (9) yield $\alpha' = 20$. By adding both sides of the equations (11) and (12), we obtain the equation

$$x' + x'' + \gamma + \gamma' = \alpha + 2\alpha' + \alpha''.$$

From (6) and (13), the left hand side of the above equation is $40 + 2\alpha'$. Then, from (3), (4), (7), (10), we have

$$60 = (x'' + x''') + (y'' + y''') = 64,$$

a contradiction.

Finally, we assume that $\mathcal{HC}(Z_0) = \mathbb{Z}\widetilde{E}_8$. Then, $\mathcal{HC}(Z_0)$ has the following subquiver with upper bounds V_n with $H(K_5) = 5$:

$$\begin{tikzcd}
 & & & K_4 \arrow[dr] & & K_5 \arrow[dr] & & & \\
 & & U_3 \arrow[ur] \arrow[dr] & & U_4 \arrow[ur] \arrow[dr] & & U_5 \arrow[dr] & & \\
\cdots & V_2 \arrow[ur] \arrow[r] \arrow[dr] & U_2' \arrow[r] & V_3 \arrow[ur] \arrow[r] \arrow[dr] & U_3' \arrow[r] & V_4 \arrow[ur] \arrow[r] \arrow[dr] & U_4' \arrow[r] & V_5 & \cdots \\
 & & W_1 \arrow[ur] & & W_2 \arrow[ur] & & W_3 \arrow[ur] & &
\end{tikzcd} \tag{5.16}$$

By writing the ranks as $\mathcal{O}$-modules of vertices in (5.16), we obtain

$$\begin{tikzcd}
 & & & \alpha \arrow[dr] & & \beta \arrow[dr] & & & \\
 & & x \arrow[ur] \arrow[dr] & & x' \arrow[ur] \arrow[dr] & & x'' \arrow[dr] & & \\
\cdots & 48 \arrow[ur] \arrow[r] \arrow[dr] & y \arrow[r] & 44 \arrow[ur] \arrow[r] \arrow[dr] & y' \arrow[r] & 48 \arrow[ur] \arrow[r] \arrow[dr] & y'' \arrow[r] & 60 & \cdots \\
 & & 32 \arrow[ur] & & 32 \arrow[ur] & & 40 \arrow[ur] & &
\end{tikzcd} \tag{5.17}$$

such that these unknown values satisfy the following system of linear equations:

$$\begin{cases} x + y &= 60 \quad \cdots\cdots\cdots (1) \\ x' + y' &= 60 \quad \cdots\cdots\cdots (2) \\ x'' + y'' &= 68 \quad \cdots\cdots\cdots (3) \\ x + x' &= 44 + \alpha \quad \cdots\cdots\cdots (4) \end{cases} \qquad \begin{cases} x' + x'' &= 48 + \beta \quad \cdots\cdots\cdots (5) \\ y + y' &= 44 \quad \cdots\cdots\cdots (6) \\ y' + y'' &= 48 \quad \cdots\cdots\cdots (7) \\ \alpha + \beta &= x' \quad \cdots\cdots\cdots (8) \end{cases}$$

From $(1), (2), (4)$ and (6), we obtain

$$120 = x + x' + y + y' = 88 + \alpha,$$

and hence, $\alpha = 32$. Similarly, using equations $(2), (3), (5)$ and (7), we have $\beta = 32$. The equation (8) implies that $x' = 64$, which contradicts with the equation (2). Thus, the above system of linear equations has no solutions, and we conclude that $\mathcal{HC}(Z_0) \neq \mathbb{Z}\widetilde{E}_8$. Therefore, we have $\mathcal{HC}(Z_0) = \mathbb{Z}A_\infty$. $\square$

5.6 Endomorphims of periodic Heller lattices

Lemma 5.6.1 ([M2, Lemma 3.1 and 4.1])**.** The following statements holds.

(1) An endomorphism $\rho \in \mathsf{End}_A(Z_n^\lambda)$ is determined by $\rho(\mathsf{c}_{1,1}^\lambda), \ldots, \rho(\mathsf{c}_{n,1}^\lambda)$.

(2) An endomorphism $\rho \in \mathsf{End}_A(Z_n^\infty)$ is determined by $\rho(\mathsf{d}_{1,1}), \ldots, \rho(\mathsf{d}_{n,1})$.

Proof. Let $\rho \in \mathsf{End}_A(Z_n^\lambda)$. For any $\mathsf{k} = 1, 2, \ldots, \mathsf{n}$, since ρ is an A-module homomorphism, we have $X\rho(\mathsf{c}_{\mathsf{k},1}^\lambda) = \rho(X\mathsf{c}_{\mathsf{k},1}^\lambda) = \rho(\mathsf{c}_{\mathsf{k},2}^\lambda)$ and $\rho(\mathsf{c}_{\mathsf{k},4}^\lambda) = \varepsilon^{-1}XY\rho(\mathsf{c}_{\mathsf{k},1}^\lambda)$. Assume that $n = 1$. In this case, $Y\rho(\mathsf{c}_{\mathsf{k},1}^\lambda) = \varepsilon\rho(\mathsf{c}_{\mathsf{k},3}^\lambda) - \lambda\rho(\mathsf{c}_{\mathsf{k},2}^\lambda)$ holds. Thus, $\rho \in \mathsf{End}_A(Z_n^\lambda)$ is determined by $\rho(\mathsf{c}_{1,1}^\lambda)$. Now, we assume that $n > 1$. Then, we have

$$\rho(\mathsf{c}_{\mathsf{k},3}^\lambda) = \begin{cases} \varepsilon^{-1}(Y\rho(\mathsf{c}_{1,1}^\lambda) + \lambda\rho(\mathsf{c}_{1,2}^\lambda)) & \mathsf{k} = 1, \\ \varepsilon^{-1}(Y\rho(\mathsf{c}_{\mathsf{k},1}^\lambda) + \lambda\rho(\mathsf{c}_{\mathsf{k},2}^\lambda) + \rho(\mathsf{c}_{\mathsf{k}-1,2}^\lambda)) & \mathsf{k} \neq 1. \end{cases}$$

(2) The proof of (2) is similar. □

Lemma 5.6.2 ([M2, Lemma 3.2 and 4.2])**.** The following statements hold.

(1) Let $\rho \in \mathsf{radEnd}_A(Z_n^\lambda)$. If we write

$$\rho(\mathsf{c}_{\mathsf{k},1}^\lambda) = \sum_{\mathsf{l}=1}^{\mathsf{n}} \mathsf{u}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{c}_{\mathsf{l},1}^\lambda + A(\mathsf{k}), \quad A(\mathsf{k}) \in \mathsf{Span}_{\mathcal{O}}\{\mathsf{c}_{\mathsf{i},\mathsf{j}}^\lambda \mid \mathsf{j} \neq 1\},$$

where $\mathsf{u}_{\mathsf{l},1}^{(\mathsf{k})} \in \mathcal{O}$, then the following statements hold.

(a) $\det(\mathsf{u}_{\mathsf{l},1}^{(\mathsf{k})})_{\mathsf{l},\mathsf{k}} \in \varepsilon\mathcal{O}$.

(b) $\mathsf{u}_{\mathsf{n},1}^{(\mathsf{k})} \in \varepsilon\mathcal{O}$ for all $\mathsf{k} = 1, 2, \ldots, \mathsf{n}$.

(2) Let $\rho \in \mathsf{radEnd}_A(Z_n^\infty)$. If we write

$$\rho(\mathsf{d}_{\mathsf{k},1}) = \sum_{\mathsf{l}=1}^{\mathsf{n}} \mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{l},1} + B(\mathsf{k}), \quad B(\mathsf{k}) \in \mathsf{Span}_{\mathcal{O}}\{\mathsf{d}_{\mathsf{i},\mathsf{j}} \mid \mathsf{j} \neq 1\},$$

where $\mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})} \in \mathcal{O}$, then the following statements hold.

(c) $\det(\mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})})_{\mathsf{l},\mathsf{k}} \in \varepsilon\mathcal{O}$.

(d) $\mathsf{v}_{\mathsf{n},1}^{(\mathsf{k})} \in \varepsilon\mathcal{O}$ for all $\mathsf{k} = 1, 2, \ldots, \mathsf{n}$.

Proof. (1) (a) Let $\rho \in \mathsf{radEnd}_A(Z_n^\lambda)$. Assume that

$$\rho(\mathsf{c}_{\mathsf{k},1}^\lambda) = \sum_{\mathsf{l}=1}^{\mathsf{n}} \mathsf{u}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{c}_{\mathsf{l},1}^\lambda + A(\mathsf{k}) \tag{5.18}$$

as above. We show that if the matrix $\mathsf{C} := (\mathsf{u}_{\mathsf{l},1}^{(\mathsf{k})})_{\mathsf{l},\mathsf{k}}$ is invertible, then ρ is surjective. As $XY\mathsf{c}_{\mathsf{l},1}^\lambda = \varepsilon\mathsf{c}_{\mathsf{l},4}^\lambda$ holds for all $\mathsf{l} = 1, \ldots, \mathsf{n}$, we have

$$(\rho(\mathsf{c}_{1,4}^\lambda), \ldots, \rho(\mathsf{c}_{\mathsf{n},4}^\lambda)) = (\mathsf{c}_{1,4}^\lambda, \ldots, \mathsf{c}_{\mathsf{n},4}^\lambda)\mathsf{C}.$$

Thus, $\mathsf{c}^{\lambda}_{1,4}, \dots, \mathsf{c}^{\lambda}_{n,4}$ are contained in the image of ρ. By (5.18), we have

$$\rho(\mathsf{c}^{\lambda}_{k,2}) = \sum_{l=1}^{n} \mathsf{u}^{(k)}_{l,1} \mathsf{c}^{\lambda}_{l,2} + XA(\mathsf{k}).$$

For each k, since $XA(\mathsf{k})$ belongs to $\mathsf{Span}_{\mathcal{O}}\{\mathsf{c}^{\lambda}_{l,4} \mid \mathsf{l} = 1, \dots, \mathsf{n}\}$, there exists $x(\mathsf{k}) \in \mathsf{Span}_{\mathcal{O}}\{\mathsf{c}^{\lambda}_{l,4} \mid \mathsf{l} = 1, \dots, \mathsf{n}\}$ such that $\rho(x(\mathsf{k})) = XA(\mathsf{k})$. Hence, we have

$$(\rho(\mathsf{c}^{\lambda}_{1,2} - x(1)), \dots, \rho(\mathsf{c}^{\lambda}_{n,2} - x(\mathsf{n}))) = (\mathsf{c}^{\lambda}_{1,2}, \dots, \mathsf{c}^{\lambda}_{n,2})\mathsf{C}.$$

Therefore, $\mathsf{c}^{\lambda}_{1,2}, \dots, \mathsf{c}^{\lambda}_{n,2}$ belong to the image of ρ. Finally, we show that $\mathsf{c}^{\lambda}_{1,3}, \dots, \mathsf{c}^{\lambda}_{n,3}$ belong to the image of ρ. By the equation (5.18), we have

$$\begin{aligned} Y\rho(\mathsf{c}^{\lambda}_{k,1}) &= \mathsf{u}^{(k)}_{1,1}(\varepsilon \mathsf{c}^{\lambda}_{1,3} + \lambda \mathsf{c}^{\lambda}_{1,2}) + \sum_{l=2}^{n} \mathsf{u}^{(k)}_{l,1}(\varepsilon \mathsf{c}^{\lambda}_{l,3} + \lambda \mathsf{c}^{\lambda}_{l,2} + \mathsf{c}^{\lambda}_{l-1,2}) + YA(\mathsf{k}) \\ &= \sum_{l=1}^{n} \varepsilon \mathsf{u}^{(k)}_{l,1} \mathsf{c}^{\lambda}_{l,3} + \sum_{l=1}^{n} \lambda \mathsf{u}^{(k)}_{l,1} \mathsf{c}^{\lambda}_{l,2} + \sum_{l=1}^{n-1} \mathsf{u}^{(k)}_{l+1,1} \mathsf{c}^{\lambda}_{l,2} + YA(\mathsf{k}). \end{aligned}$$

On the other hand, $Y\rho(\mathsf{c}^{\lambda}_{k,1})$ is $\varepsilon\rho(\mathsf{c}^{\lambda}_{k,3}) + \rho(\lambda \mathsf{c}^{\lambda}_{k,2}) + \rho(\lambda \mathsf{c}^{\lambda}_{k-1,1})$. Let $y(\mathsf{k}) \in Z^{\lambda}_{n}$ such that $\rho(y(\mathsf{k})) = \sum_{l=1}^{n} \lambda \mathsf{u}^{(k)}_{l,1} \mathsf{c}^{\lambda}_{l,2} + \sum_{l=1}^{n-1} \mathsf{u}^{(k)}_{l+1,1} \mathsf{c}^{\lambda}_{l,2} + YA(\mathsf{k})$. Then, we have

$$\varepsilon\rho(\mathsf{c}^{\lambda}_{k,3}) = \sum_{l=1}^{n} \varepsilon \mathsf{u}^{(k)}_{l,1} \mathsf{c}^{\lambda}_{l,3} + \rho(-\lambda \mathsf{c}^{\lambda}_{k,2} - \mathsf{c}^{\lambda}_{k-1,2} + y(\mathsf{k})). \tag{5.19}$$

Put $z(\mathsf{k}) = -\lambda \mathsf{c}^{\lambda}_{k,2} - \mathsf{c}^{\lambda}_{k-1,2} + y(\mathsf{k})$. By the construction of $z(\mathsf{k})$, we note that $\rho(z(\mathsf{k}))$ belongs to $\mathsf{Span}_{\mathcal{O}}\{\mathsf{c}^{\lambda}_{i,2}, \mathsf{c}^{\lambda}_{i,4} \mid \mathsf{i} = 1, \dots, \mathsf{n}\}$. Since the restriction of ρ to $\mathsf{Span}_{\mathcal{O}}\{\mathsf{c}^{\lambda}_{i,2}, \mathsf{c}^{\lambda}_{i,4} \mid \mathsf{i} = 1, \dots, \mathsf{n}\}$ is a bijection from $\mathsf{Span}_{\mathcal{O}}\{\mathsf{c}^{\lambda}_{i,2}, \mathsf{c}^{\lambda}_{i,4} \mid \mathsf{i} = 1, \dots, \mathsf{n}\}$ to itself, the equation (5.19) implies that there exists $z'(\mathsf{k}) \in Z^{\lambda}_{n}$ such that $z(\mathsf{k}) = \varepsilon z'(\mathsf{k})$. Then, we have

$$(\rho(\mathsf{c}^{\lambda}_{1,3} - z'(1)), \dots, \rho(\mathsf{c}^{\lambda}_{n,3} - z'(\mathsf{n}))) = (\mathsf{c}^{\lambda}_{1,3}, \dots, \mathsf{c}^{\lambda}_{n,3})\mathsf{C}.$$

This completes the proof of the statement (1).

(b) The statement for $n = 1$ is clear by (1). In order to prove this statement for $n > 1$, we compute $f(Y\mathsf{c}^{\lambda}_{k,1} - \lambda X\mathsf{c}^{\lambda}_{k,1} - X\mathsf{c}^{\lambda}_{k-1,1})$ in two ways. Since $Y\mathsf{c}^{\lambda}_{k,1} = \varepsilon \mathsf{c}^{\lambda}_{k,3} + \lambda \mathsf{c}_{k,2} + \mathsf{c}^{\lambda}_{k-1,2}$ and $\mathsf{c}^{\lambda}_{k,2} = X\mathsf{c}^{\lambda}_{k,1}$, we have

$$\rho(Y\mathsf{c}^{\lambda}_{k,1} - \lambda X\mathsf{c}^{\lambda}_{k,1} - X\mathsf{c}^{\lambda}_{k-1,1}) = \varepsilon f(\mathsf{c}^{\lambda}_{k,3}). \tag{5.20}$$

Now, we assume that $A(\mathsf{k}) = \sum_{\mathsf{l}=1}^{\mathsf{n}}(\mathsf{u}_{\mathsf{l},2}^{(\mathsf{k})}\mathsf{c}_{\mathsf{l},2}^{\lambda} + \mathsf{u}_{\mathsf{l},3}^{(\mathsf{k})}\mathsf{c}_{\mathsf{l},3}^{\lambda} + \mathsf{u}_{\mathsf{l},4}^{(\mathsf{k})}\mathsf{c}_{\mathsf{l},4}^{\lambda})$. For $\mathsf{k} > 1$, the left-hand side of (5.20) is

$$\begin{aligned}&\sum_{\mathsf{l}=1}^{\mathsf{n}-1}(\mathsf{u}_{\mathsf{l}+1,1}^{(\mathsf{k})} - \mathsf{u}_{\mathsf{l},1}^{(\mathsf{k}-1)})\mathsf{c}_{\mathsf{l},2}^{\lambda} - \mathsf{u}_{\mathsf{n},1}^{(\mathsf{k}-1)}\mathsf{c}_{\mathsf{n},2}^{\lambda}\\ &+\varepsilon\sum_{\mathsf{l}=1}^{\mathsf{n}}\mathsf{u}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{c}_{\mathsf{l},3}^{\lambda} + \sum_{\mathsf{l}=1}^{\mathsf{n}-1}(\varepsilon\mathsf{u}_{\mathsf{l},1}^{(\mathsf{k})} - 2\lambda\mathsf{u}_{\mathsf{l},3}^{(\mathsf{k})} - \mathsf{u}_{\mathsf{l},3}^{(\mathsf{k}-1)} - \mathsf{u}_{\mathsf{l}+1,3}^{(\mathsf{k})})\mathsf{c}_{\mathsf{l},4}^{\lambda} + (\varepsilon\mathsf{u}_{\mathsf{n},1}^{(\mathsf{k})} - 2\lambda\mathsf{u}_{\mathsf{n},3}^{(\mathsf{k})} - \mathsf{u}_{\mathsf{n},3}^{(\mathsf{k}-1)})\mathsf{c}_{\mathsf{n},4}^{\lambda}.\end{aligned}$$

Thus, the coefficients $\mathsf{u}_{\mathsf{l}+1,1}^{(\mathsf{k})} - \mathsf{u}_{\mathsf{l},1}^{(\mathsf{k}-1)}$ ($\mathsf{l} = 1, \dots, \mathsf{n}-1$) and $\mathsf{u}_{\mathsf{n},1}^{(\mathsf{k}-1)}$ belong to $\varepsilon\mathcal{O}$. It implies that strictly lower triangular entries of the matrix C belong to $\varepsilon\mathcal{O}$. On the other hand, by the statement (1), we have

$$\det\mathsf{C} \equiv \mathsf{u}_{1,1}^{(1)}\cdots\mathsf{u}_{\mathsf{n},1}^{(\mathsf{n})} + \sum_{e\neq\sigma\in S_n}\mathsf{u}_{1,1}^{((\sigma(1))}\cdots\mathsf{u}_{\mathsf{n},1}^{(\sigma(\mathsf{n}))} \equiv 0 \mod \varepsilon\mathcal{O},$$

where S_n is the symmetric group of degree n and e is its identity element. Hence, $\mathsf{u}_{\mathsf{k},1}^{(\mathsf{k})} \equiv 0$ modulo $\varepsilon\mathcal{O}$ for some k. Since $\mathsf{u}_{\mathsf{k}+1,1}^{(\mathsf{k}+1)} - \mathsf{u}_{\mathsf{k},1}^{(\mathsf{k})} \in \varepsilon\mathcal{O}$ ($\mathsf{k} = 1, \dots, \mathsf{n}-1$), the assertion follows.

(2) (c) We show that any ρ such that the matrix $\mathsf{D} := (\mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})})_{\mathsf{l},\mathsf{k}}$ is invertible is surjective. As $XY\mathsf{d}_{\mathsf{l},4} = \varepsilon\mathsf{d}_{\mathsf{l},4}$ holds for $\mathsf{l} = 1, \dots, \mathsf{n}$, we have

$$(\rho(\mathsf{d}_{1,4}), \dots, \rho(\mathsf{d}_{\mathsf{n},4})) = (\mathsf{d}_{1,4}, \dots, \mathsf{d}_{\mathsf{n},4})\mathsf{D}.$$

Hence, $\mathsf{d}_{1,4}, \dots, \mathsf{d}_{\mathsf{n},4}$ are contained in the image of ρ.

Assume that $n = 1$. By acting X to the both sides of $\rho(\mathsf{d}_{1,1}) = \mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,1} + B(1)$, we have

$$\varepsilon\rho(\mathsf{d}_{1,2}) = \varepsilon\mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,2} + XB(1).$$

Thus, we get $\varepsilon\mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,2} = \varepsilon\rho(\mathsf{d}_{1,2}) - \varepsilon t\mathsf{d}_{1,4}$ for some $t \in \mathcal{O}$ since $XB(1) \in \varepsilon\mathcal{O}\mathsf{d}_{1,4}$. It implies that

$$\mathsf{d}_{1,2} = \rho((\mathsf{v}_{1,1}^{(1)})^{-1}\mathsf{d}_{1,2} - (\mathsf{v}_{1,1}^{(1)})^{-2}t\mathsf{d}_{1,4})).$$

By letting Y act on the both sides of $\rho(\mathsf{d}_{1,1}) = \mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,1} + B(1)$, we have

$$\rho(\mathsf{d}_{1,3}) = \mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,3} + YB(1) = \mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,3} + s\mathsf{d}_{1,4} = \mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,3} + \rho(s\mathsf{v}_{1,1}^{(1)}\mathsf{d}_{1,4})$$

for some $s \in \mathcal{O}$ since $YA(1) \in \mathcal{O}\mathsf{d}_{1,4}$, and hence $\mathsf{d}_{1,3} = \rho((\mathsf{v}_{1,1}^{(1)})^{-1}\mathsf{d}_{1,3} - (a\mathsf{v}_{1,1}^{(1)})^{-2}s\mathsf{d}_{1,4}))$. Therefore, the morphism ρ is surjective.

Next, we assume that $n > 1$. We note that

$$X\rho(\mathsf{d}_{\mathsf{k},1}) = \begin{cases}\varepsilon\rho(\mathsf{d}_{1,2}) & \text{if } \mathsf{k} = 1,\\ \rho(\mathsf{d}_{\mathsf{k},2}) & \text{if } \mathsf{k} \neq 1,\end{cases} \qquad Y\rho(\mathsf{d}_{\mathsf{k},1}) = \begin{cases}\rho(\varepsilon\mathsf{d}_{\mathsf{k},3} + \mathsf{d}_{\mathsf{k}+1,2}) & \text{if } \mathsf{k} \neq \mathsf{n},\\ \rho(\mathsf{d}_{\mathsf{n},3}) & \text{if } \mathsf{k} = \mathsf{n},\end{cases}$$

$$X\left(\sum_{\mathsf{l}=1}^{\mathsf{n}} \mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{l},1} + B(\mathsf{k})\right) = \varepsilon \mathsf{v}_{1,1}^{(\mathsf{k})}\mathsf{d}_{1,2} + \sum_{\mathsf{l}=2}^{\mathsf{n}} \mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{l},2} + XB(\mathsf{k}),$$

$$Y\left(\sum_{\mathsf{l}=1}^{\mathsf{n}} \mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{l},1} + B(\mathsf{k})\right) = \sum_{\mathsf{l}=1}^{\mathsf{n}-1} \mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}(\varepsilon\mathsf{d}_{\mathsf{l},3} + \mathsf{d}_{\mathsf{l}+1,2}) + \mathsf{v}_{\mathsf{n},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{n},3} + YB(\mathsf{k}),$$

and we also note that $XB(\mathsf{k})$ and $YB(\mathsf{k})$ belong to $\mathsf{Span}_{\mathcal{O}}\{\mathsf{d}_{\mathsf{i},4} \mid \mathsf{i} = 1, \dots, \mathsf{n}\}$.

Assume that $\mathsf{k} = 1$. Then, the equality

$$\varepsilon\rho(\mathsf{d}_{1,2}) = \varepsilon \mathsf{v}_{1,1}^{(1)}\mathsf{d}_{\mathsf{l},2} + \sum_{\mathsf{l}=2}^{n} \mathsf{v}_{\mathsf{l},1}^{(1)}\mathsf{d}_{1,2} + XB(1).$$

implies that $\mathsf{v}_{\mathsf{l},1}^{(1)}$ $(\mathsf{l} = 2, 3, \dots, \mathsf{n})$ are in $\varepsilon\mathcal{O}$ and $XB(1) \equiv 0$ modulo $\varepsilon\mathcal{O}$. Thus, there exists $x(1) \in Z_n^\infty$ such that $\varepsilon\rho(x(1)) = XB(1)$. If $\mathsf{k} > 1$, then, for each k, there exists $x(\mathsf{k}) \in Z_n^\infty$ such that $\rho(x(\mathsf{k})) = XB(\mathsf{k})$. Therefore, it is easy to see that

$$(\rho(\mathsf{d}_{1,2} - x(1)), \dots, \rho(\mathsf{d}_{\mathsf{n},2} - x(\mathsf{n}))) = (\mathsf{d}_{1,2}, \dots, \mathsf{d}_{\mathsf{n},2})\begin{pmatrix} \mathsf{v}_{1,1}^{(1)} & \varepsilon\mathsf{v}_{1,1}^{(2)} & & \varepsilon\mathsf{v}_{1,1}^{(\mathsf{n})} \\ \varepsilon^{-1}\mathsf{v}_{2,1}^{(1)} & \mathsf{v}_{2,1}^{(2)} & & \mathsf{v}_{2,1}^{(\mathsf{n})} \\ \vdots & \vdots & \cdots & \vdots \\ \varepsilon^{-1}\mathsf{v}_{1,1}^{(1)} & \mathsf{v}_{\mathsf{n},1}^{(2)} & & \mathsf{v}_{\mathsf{n},1}^{(\mathsf{n})} \end{pmatrix}.$$

Since the determinant of the rightmost matrix in the above equation equals to $\det \mathsf{D}$, each element $\mathsf{d}_{\mathsf{i},2}$ belongs to the image of ρ.

For each $\mathsf{k} = 1, 2, \dots, \mathsf{n}$, let $y(\mathsf{k})$ and $z(\mathsf{k})$ be elements of Z_n^∞ such that $\rho(y(\mathsf{k})) = YB(\mathsf{k})$ and $\rho(z(\mathsf{k})) = \sum_{\mathsf{l}=1}^{\mathsf{n}-1} \mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{l}+1,2}$. Then, we have the equations

$$\varepsilon\rho(\mathsf{d}_{\mathsf{k},3}) = \sum_{\mathsf{l}=1}^{\mathsf{n}-1} \varepsilon\mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{l},3} + \mathsf{v}_{\mathsf{n},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{n},3} + \rho(y(\mathsf{k}) + z(\mathsf{k}) - \mathsf{d}_{\mathsf{k}+1,2}) \quad \text{for } \mathsf{k} = 1, 2, \dots, \mathsf{n} - 1$$

and

$$\rho(\mathsf{d}_{\mathsf{n},3} - y(\mathsf{n}) - z(\mathsf{n})) = \sum_{\mathsf{l}=1}^{\mathsf{n}-1} \varepsilon\mathsf{v}_{\mathsf{l},1}^{(\mathsf{n})}\mathsf{d}_{\mathsf{l},3} + \mathsf{v}_{\mathsf{n},1}^{(\mathsf{n})}\mathsf{d}_{\mathsf{n},3}.$$

As $\rho(y(\mathsf{k})+z(\mathsf{k})-\mathsf{d}_{\mathsf{k}+1,2})$ belongs to $\mathsf{Span}_{\mathcal{O}}\{\mathsf{d}_{\mathsf{i},2}, \mathsf{d}_{\mathsf{i},4} \mid \mathsf{i} = 1, \dots, \mathsf{n}\}$, $\rho(y(\mathsf{k})+z(\mathsf{k})-\mathsf{d}_{\mathsf{k}+1,2}) \equiv 0$ modulo $\varepsilon\mathcal{O}$. Since the restriction of ρ to $\mathsf{Span}_{\mathcal{O}}\{\mathsf{d}_{\mathsf{i},2}, \mathsf{d}_{\mathsf{i},4} \mid \mathsf{i} = 1, \dots, \mathsf{n}\}$ is a bijection from $\mathsf{Span}_{\mathcal{O}}\{\mathsf{d}_{\mathsf{i},2}, \mathsf{d}_{\mathsf{i},4} \mid \mathsf{i} = 1, \dots, \mathsf{n}\}$ to itself, one can define $w(\mathsf{k}) \in Z_n^\infty$ by

$$\rho(w(\mathsf{k})) := \begin{cases} \varepsilon^{-1}\rho(y(\mathsf{k}) + z(\mathsf{k}) - \mathsf{d}_{\mathsf{k}+1,2}) & \text{if } \mathsf{k} \neq \mathsf{n}, \\ \rho(y(\mathsf{k}) + z(\mathsf{k})) & \text{if } \mathsf{k} = \mathsf{n}. \end{cases}$$

This gives the following equation:

$$(\rho(\mathsf{d}_{1,3}-w(1)),\ldots,\rho(\mathsf{d}_{\mathsf{n},3}-w(\mathsf{n}))) = (\mathsf{d}_{1,3},\ldots,\mathsf{d}_{\mathsf{n},3})\begin{pmatrix} \mathsf{v}_{1,1}^{(1)} & & \mathsf{v}_{1,1}^{(\mathsf{n}-1)} & \varepsilon\mathsf{v}_{1,1}^{(\mathsf{n})} \\ \vdots & \cdots & \vdots & \vdots \\ \mathsf{v}_{\mathsf{n}-1,1}^{(1)} & & \mathsf{v}_{\mathsf{n}-1,1}^{(\mathsf{n}-1)} & \varepsilon\mathsf{v}_{\mathsf{n}-1,1}^{(\mathsf{n})} \\ \varepsilon^{-1}\mathsf{v}_{\mathsf{n},1}^{(1)} & & \varepsilon^{-1}\mathsf{v}_{\mathsf{n},1}^{(\mathsf{n}-1)} & \mathsf{v}_{\mathsf{n},1}^{(\mathsf{n})} \end{pmatrix}$$

Since the determinant of the rightmost matrix in the above equation equals to $\det \mathsf{D}$, each element $\mathsf{d}_{\mathsf{i},3}$ belongs to the image of ρ. Therefore, the A-morphism ρ is surjective.

(d) The statement for $n=1$ is clear by (1). In order to prove this statement for $n>1$, we compute $\rho(Y\mathsf{d}_{\mathsf{k},1}-X\mathsf{d}_{\mathsf{k}+1,1})$, for $\mathsf{k}=1,2,\ldots,\mathsf{n}-1$, in two ways. Set $W(\mathsf{k}) = YB(\mathsf{k})-XB(\mathsf{k}+1)$. Since $Y\mathsf{d}_{\mathsf{k},1} = \varepsilon\mathsf{d}_{\mathsf{k},3}+\mathsf{d}_{\mathsf{k}+1,2}$ and $\mathsf{d}_{\mathsf{k},2} = X\mathsf{d}_{\mathsf{k},1}$, we have

$$\rho(Y\mathsf{d}_{\mathsf{k},1}-X\mathsf{d}_{\mathsf{k}+1,1}) = \varepsilon\rho(\mathsf{d}_{\mathsf{k},3}).$$

On the other hand, we have

$$\rho(Y\mathsf{d}_{\mathsf{k},1}-X\mathsf{d}_{\mathsf{k}+1,1}) = -\varepsilon\mathsf{v}_{1,1}^{(\mathsf{k}+1)}\mathsf{d}_{1,2}+\sum_{\mathsf{l}=2}^{\mathsf{n}}(\mathsf{v}_{\mathsf{l}-1,1}^{(\mathsf{k})}-\mathsf{v}_{\mathsf{l},1}^{(\mathsf{k}+1)})\mathsf{d}_{\mathsf{l},2}+\sum_{\mathsf{l}=1}^{\mathsf{n}-1}\varepsilon\mathsf{v}_{\mathsf{l},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{l},3}+\mathsf{v}_{\mathsf{n},1}^{(\mathsf{k})}\mathsf{d}_{\mathsf{n},3}+W(\mathsf{k}).$$

Thus, we have

$$\mathsf{v}_{\mathsf{l}-1,1}^{(\mathsf{k})}-\mathsf{v}_{\mathsf{l},1}^{(\mathsf{k}+1)} \equiv 0 \mod \varepsilon\mathcal{O}, \quad \mathsf{v}_{\mathsf{n},1}^{(\mathsf{k})} \equiv 0 \mod \varepsilon\mathcal{O} \quad \mathsf{l}=2,\ldots,\mathsf{n},\ \mathsf{k}=1,\ldots,\mathsf{n}-1.$$

This means the strictly lower entries of the matrix D belong to $\varepsilon\mathcal{O}$. By the statement (1),

$$\det\mathsf{D} \equiv \mathsf{v}_{1,1}^{(1)}\mathsf{v}_{2,1}^{(2)}\cdots\mathsf{v}_{\mathsf{n},1}^{(\mathsf{n})}+\sum_{e\neq\sigma\in S_n}\mathrm{sgn}(\sigma)\mathsf{v}_{1,1}^{(\sigma(1))}\cdots\mathsf{v}_{\mathsf{n},1}^{(\sigma(\mathsf{n}))} \equiv \mathsf{v}_{1,1}^{(1)}\mathsf{v}_{2,1}^{(2)}\cdots\mathsf{v}_{\mathsf{n},1}^{(\mathsf{n})} \mod \varepsilon\mathcal{O},$$

where S_n is the symmetric group of degree n and e is its identity element. Now, the claim is clear. □

5.7 Almost split sequences ending at periodic Heller lattices

First, we construct the almost split sequence ending at Z_n^λ, where $\lambda\neq\infty$. For each $n>1$, we define an endomorphism $\Phi_n^\lambda : Z_n^\lambda \to Z_n^\lambda$ by

$$\mathsf{c}_{\mathsf{k},1}^\lambda \longmapsto \begin{cases} \mathsf{c}_{\mathsf{n},4}^\lambda & \text{if } \mathsf{k}=\mathsf{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the projective cover of Z_n^λ is given by

$$\begin{array}{rccl} \pi_{n,\lambda} : & A^{\oplus 2n} & \longrightarrow & Z_n^\lambda \\ & e_i & \longmapsto & \begin{cases} \mathsf{c}_{\mathsf{k},1}^\lambda & \text{if } i=2\mathsf{k}-1, \quad \mathsf{k}=1,2,\ldots,\mathsf{n}, \\ \mathsf{c}_{\mathsf{k},3}^\lambda & \text{if } i=2\mathsf{k}, \quad \mathsf{k}=1,2,\ldots,\mathsf{n}. \end{cases} \end{array}$$

Lemma 5.7.1 ([M2, Lemma 3.3])**.** Let Φ_n^λ be the endomorphism of Z_n^λ as above. Then, the following statements hold.

(1) Φ_n^λ does not factor through $\pi_{n,\lambda}$.

(2) For any $\rho \in \mathsf{radEnd}_A(Z_n^\lambda)$, $\Phi_n^\lambda \rho$ factors through $\pi_{n,\lambda}$.

Proof. (1) Suppose that Φ_n^λ factors through the map $\pi_{n,\lambda}$. Let $\psi = (\psi_1, \ldots, \psi_{2n}) : Z_n^\lambda \to A^{\oplus 2n}$ such that $\Phi_n^\lambda = \pi_{n,\lambda}\psi$. Put

$$\psi_k(\mathsf{c}_{\mathsf{i},1}^\lambda) = a_{k,1}^{(\mathsf{i})} + a_{k,2}^{(\mathsf{i})}X + a_{k,3}^{(\mathsf{i})}Y + a_{k,4}^{(\mathsf{i})}XY.$$

By comparing coefficients in $\pi_{n,\lambda}\psi(\mathsf{c}_{\mathsf{k},1}^\lambda)$ with those in $\Phi_n^\lambda(\mathsf{c}_{\mathsf{k},1}^\lambda)$, we have the following equations:

$$\varepsilon a_{2s-1,4}^{(\mathsf{i})} + a_{2s,2}^{(\mathsf{i})} - \lambda a_{2s,3}^{(\mathsf{i})} - a_{2s+2,3}^{(\mathsf{i})} = 0 \quad \text{if } s \neq n, \tag{5.21}$$

$$\varepsilon a_{2n-1,4}^{(\mathsf{i})} + a_{2n,2}^{(\mathsf{i})} - \lambda a_{2n,3}^{(\mathsf{i})} = \begin{cases} 1 & \text{if } \mathsf{i} = \mathsf{n}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.22}$$

On the other hand, as $\psi_k(\mathsf{c}_{\mathsf{i},2}^\lambda) = X\psi_k(\mathsf{c}_{\mathsf{i},1}^\lambda)$, it follows from $\varepsilon\psi_k(\mathsf{c}_{\mathsf{i},3}^\lambda) = Y\psi_k(\mathsf{c}_{\mathsf{i},1}^\lambda) - \lambda\psi_k(\mathsf{c}_{\mathsf{i},2}^\lambda) - \psi_k(\mathsf{c}_{\mathsf{i}-1,2}^\lambda)$ that

$$\varepsilon\psi_k(\mathsf{c}_{\mathsf{i},3}^\lambda) = -(\lambda a_{k,1}^{(\mathsf{i})} - a_{k,1}^{(\mathsf{i}-1)})X + a_{k,1}^{(\mathsf{i})}Y + (a_{k,2}^{(\mathsf{i})} - \lambda a_{k,3}^{(\mathsf{i})} - a_{k,3}^{(\mathsf{i}-1)})XY, \tag{5.23}$$

where $a_{k,3}^{(0)} = 0$, $1 \leq k \leq 2n$ and $1 \leq \mathsf{i} \leq \mathsf{n}$. In order to obtain a contradiction, we show that $a_{2n,2}^{(\mathsf{n})} - \lambda a_{2n,3}^{(\mathsf{n})} \in \varepsilon\mathcal{O}$. By the equation (5.23), this is equivalent to $a_{2n,3}^{(\mathsf{n}-1)} \in \varepsilon\mathcal{O}$. The equation (5.21) implies that $a_{2n,3}^{(\mathsf{n}-1)} \in \varepsilon\mathcal{O}$ if and only if $a_{2n-2,2}^{(\mathsf{n}-1)} - \lambda a_{2n-2,3}^{(\mathsf{n}-1)} \in \varepsilon\mathcal{O}$. By repeating this procedure, we deduce that the claim is equivalent to $a_{2,2}^{(1)} - \lambda a_{2,3}^{(1)} \in \varepsilon\mathcal{O}$. However, $a_{2,2}^{(1)} - \lambda a_{2,3}^{(1)} \in \varepsilon\mathcal{O}$ follows from the equation (5.23). Now, we obtain

$$1 = \varepsilon a_{2n-1,4}^{(\mathsf{n})} + a_{2n,2}^{(\mathsf{n})} - \lambda a_{2n,3}^{(\mathsf{n})} \in \varepsilon\mathcal{O},$$

a contradiction.

(2) Let $\rho \in \mathsf{radEnd}_A(Z_n^\lambda)$. We put

$$\rho(\mathsf{c}_{\mathsf{k},1}^\lambda) = \sum_{\mathsf{i}=1}^{\mathsf{n}} (\mathsf{u}_{\mathsf{i},1}^{(\mathsf{k})}\mathsf{c}_{\mathsf{i},1}^\lambda + \mathsf{u}_{\mathsf{i},2}^{(\mathsf{k})}\mathsf{c}_{\mathsf{i},2}^\lambda + \mathsf{u}_{\mathsf{i},3}^{(\mathsf{k})}\mathsf{c}_{\mathsf{i},3}^\lambda + \mathsf{u}_{\mathsf{i},4}^{(\mathsf{k})}\mathsf{c}_{\mathsf{i},4}^\lambda).$$

Lemma 5.6.2 yields that there exists $\mathsf{f}_{\mathsf{n}.1}^{(\mathsf{k})} \in \mathcal{O}$ such that $\varepsilon\mathsf{f}_{\mathsf{n}.1}^{(\mathsf{k})} = \mathsf{u}_{\mathsf{n},1}^{(\mathsf{k})}$ for each k. We define an A-module homomorphism $\psi : Z_n^\lambda \to A^{\oplus 2n}$ by $\psi(\mathsf{c}_{\mathsf{k},1}^\lambda) = (0, \ldots, 0, \mathsf{f}_{\mathsf{n},1}^{(\mathsf{k})}XY, 0)$. Then, it is easy to check that ψ is well-defined and $\Phi_n^\lambda\rho(\mathsf{c}_{\mathsf{k},1}^\lambda) = \mathsf{u}_{\mathsf{n},1}^{(\mathsf{k})}\mathsf{c}_{\mathsf{n},4}^\lambda = \pi_{n,\lambda}\psi(\mathsf{c}_{\mathsf{k},1}^\lambda)$. ☐

Therefore, we have obtained the following proposition.

Proposition 5.7.2 ([M2, Proposition 3.4])**.** Consider the following pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z_n^{-\lambda} \arrow[r] \arrow[d, equal] & E_n^{\lambda} \arrow[r] \arrow[d] & Z_n^{\lambda} \arrow[r] \arrow[d, "\Phi_n^{\lambda}"] & 0 \\
0 \arrow[r] & Z_n^{-\lambda} \arrow[r] & A^{\oplus 2n} \arrow[r, "\pi_{n,\lambda}"'] & Z_n^{\lambda} \arrow[r] & 0
\end{tikzcd}$$

Then, the upper exact sequence is the almost split sequence ending at Z_n^{λ}.

Proof. The statement follows from Proposition 3.5.8 and Lemma 5.7.1. □

Next, we construct the almost split sequence ending at Z_n^{∞}. Recall that the projective cover of Z_n^{∞} is given by

$$\begin{array}{cccc}
\pi_{n,\infty} : & A^{\oplus 2n} & \longrightarrow & Z_n^{\infty} \\
& e_i & \longmapsto & \begin{cases} \mathsf{d}_{1,1} & \text{if } i = 1, \\ \mathsf{d}_{1,2} & \text{if } i = 2, \\ \mathsf{d}_{\mathsf{k},3} & \text{if } i = 2\mathsf{k}+1,\ \mathsf{k} = 1, 2, \dots, \mathsf{n}-1, \\ \mathsf{d}_{\mathsf{k},1} & \text{if } i = 2\mathsf{k},\ \mathsf{k} = 2, 3, \dots, \mathsf{n}. \end{cases}
\end{array}$$

Now, for each $n \geq 1$, we define an endomorphism $\Phi_n^{\infty} : Z_n^{\infty} \to Z_n^{\infty}$ by

$$\mathsf{d}_{\mathsf{k},1} \longmapsto \begin{cases} \mathsf{d}_{\mathsf{n},4} & \text{if } \mathsf{k} = \mathsf{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, Φ_n^{∞} gives an endomorphism of Z_n^{∞}. First, we construct the almost split sequence ending at Z_1^{∞} by using Φ_1^{∞}.

Lemma 5.7.3 ([M2, Lemma 4.3])**.** Let $\Phi_1^{\infty} : Z_1^{\infty} \to Z_1^{\infty}$ as above. Then, the following statements hold.

(1) Φ_1^{∞} does not factor through $\pi_{1,\infty}$.

(2) For any $\rho \in \mathsf{radEnd}_A(Z_1^{\infty})$, $\Phi_1^{\infty}\rho$ factors through $\pi_{1,\infty}$.

Proof. (1) Suppose that there exists $\psi = (\psi_1, \psi_2) : Z_1^{\infty} \to A \oplus A$ such that $\Phi_1^{\infty} = \pi_{1,\infty}\psi$. Then, we have

$$\mathsf{d}_{1,4} = \Phi_1^{\infty}(\mathsf{d}_{1,1}) = \pi_{1,\infty}\psi(\mathsf{d}_{1,1}) = \psi_{1,\infty}(\mathsf{d}_{1,1})\mathsf{d}_{1,1} + \psi_2(\mathsf{d}_{1,1})\mathsf{d}_{1,2}. \tag{5.24}$$

If we put

$$\psi_1(\mathsf{d}_{1,1}) = a_1 + a_2X + a_3Y + a_4XY, \quad \psi_2(\mathsf{d}_{1,1}) = b_1 + b_2X + b_3Y + b_4XY,$$

where $a_1, \dots, a_4, b_1, \dots, b_4 \in \mathcal{O}$, the rightmost side of (5.24) equals to

$$a_1\mathsf{d}_{1,1} + (\varepsilon a_2 + b_1)\mathsf{d}_{1,2} + a_3\mathsf{d}_{1,3} + (\varepsilon a_4 + b_3)\mathsf{d}_{1,4}.$$

Thus, we have $\psi_2(\mathsf{d}_{1,1}) = -\varepsilon a_2 + b_2X + (1-\varepsilon a_4)Y + b_4XY$. Multiplying X to $\psi_2(\mathsf{d}_{1,1})$, we have

$$\varepsilon\psi_2(\mathsf{d}_{1,2}) = X\psi_2(\mathsf{d}_{1,1}) = -\varepsilon a_2X + (1-\varepsilon a_4)XY,$$

a contradiction.

(2) Let $\rho \in \mathsf{radEnd}_A(Z_1^\infty)$. We write $\rho(\mathsf{d}_{1,1}) = \alpha\mathsf{d}_{1,1} + B(1)$, where $\alpha \in \mathcal{O}$ and $B(1) \in \mathsf{Span}_{\mathcal{O}}\{\mathsf{d}_{1,2}, \mathsf{d}_{1,3}, \mathsf{d}_{1,4}\}$. By Lemma 5.6.2, $\alpha = \varepsilon\alpha'$ for some $\alpha' \in \mathcal{O}$. Define an A-module homomorphism $\psi : Z_1^\infty \to A \oplus A$ by $\psi(\mathsf{d}_{1,1}) = \alpha' XYe_1$. Then, since $\pi_{1,\infty}(\alpha' XYe_1) = \alpha' XY\mathsf{d}_{1,1} = \varepsilon\alpha'\mathsf{d}_{1,4}$, we have $\Phi_1^\infty f(\mathsf{d}_{1,1}) = \alpha\mathsf{d}_{1,4} = \pi_{1,\infty}\psi(\mathsf{d}_{1,1})$. $\square$

From now on, we construct the almost split sequence ending at Z_n^∞ for $n \geq 2$.

Lemma 5.7.4 ([M2, Lemma 4.4])**.** Let $\Phi_n^\infty : Z_n^\infty \to Z_n^\infty$ as above. Then, the following statements hold.

(1) Φ_n^∞ does not factor through $\pi_{n,\infty}$.

(2) For any $\rho \in \mathsf{radEnd}_A(Z_n^\infty)$, $\Phi_n^\infty\rho$ factors through $\pi_{n,\infty}$.

Proof. (1) Suppose that there exists $\psi = (\psi_k)_{k=1,\dots,2n} : Z_n^\infty \to A^{\oplus 2n}$ such that $\Phi_n^\infty = \pi_{n,\infty}\psi$. We put

$$\psi_l(\mathsf{d}_{\mathsf{k},1}) = a_{l,1}^{(\mathsf{k})} + a_{l,2}^{(\mathsf{k})}X + a_{l,3}^{(\mathsf{k})}Y + a_{l,4}^{(\mathsf{k})}XY.$$

Then, we notice that, for all $\mathsf{k} = 1, \dots, \mathsf{n}$ and $l = 1, \dots, 2n$, $a_{l,1}^{(\mathsf{k})}$ belongs to $\varepsilon\mathcal{O}$ since $XY\mathsf{d}_{\mathsf{k},1} = \varepsilon\mathsf{d}_{\mathsf{k},4}$ for all $\mathsf{k} = 1, \dots, \mathsf{n}$. By comparing the coefficient of $\mathsf{d}_{\mathsf{n},4}$ in $\Phi_n^\infty(\mathsf{d}_{\mathsf{n},1})$ with that in $\pi_{n,\infty}\psi(\mathsf{d}_{\mathsf{n},1})$, we have $\varepsilon a_{2n,4}^{(\mathsf{n})} - a_{2n-1,3}^{(\mathsf{n})} = 1$. In order to obtain a contradiction we show that $a_{2n-1,3}^{(\mathsf{n})} \in \varepsilon\mathcal{O}$.

For $\mathsf{s} = 1, \dots, \mathsf{n}$ and $\mathsf{t} = 1, \dots, \mathsf{n}-1$, by comparing the coefficient of $\mathsf{d}_{\mathsf{t},4}$ in $\Phi_n^\infty(\mathsf{d}_{\mathsf{s},1})$ with that in $\pi_{n,\infty}\psi(\mathsf{d}_{\mathsf{s},1})$, we obtain the following equations:

$$\varepsilon a_{1,4}^{(\mathsf{s})} + a_{2,3}^{(\mathsf{s})} + a_{3,2}^{(\mathsf{s})} = 0 \quad \mathsf{t} = 1, \tag{5.25}$$

$$-a_{2t-1,3}^{(\mathsf{s})} + \varepsilon a_{2t,4}^{(\mathsf{s})} + a_{2t+1,3}^{(\mathsf{s})} = 0 \quad \mathsf{t} > 1. \tag{5.26}$$

On the other hand, for $t = 1, \dots, 2n$, the following equations hold:

$$\begin{array}{rclcll} \psi_t(\mathsf{d}_{\mathsf{s},2}) & = & X\psi_t(\mathsf{d}_{\mathsf{s},1}) & = & a_{t,1}^{(\mathsf{s})}X + a_{t,3}^{(\mathsf{s})}XY & \mathsf{s} \neq 1 \quad (*) \\ \varepsilon\psi_t(\mathsf{d}_{\mathsf{s},3}) + \psi_t(\mathsf{d}_{\mathsf{s}+1,2}) & = & Y\psi_t(\mathsf{d}_{\mathsf{s},1}) & = & a_{t,1}^{(\mathsf{s})}Y + a_{t,2}^{(\mathsf{s})}XY & \mathsf{s} \neq \mathsf{n} \quad (**) \end{array}$$

In particular, it follows from $(*)$ that $\psi_{2n-1}(\mathsf{d}_{\mathsf{n},2}) = a_{2n-1,1}^{(\mathsf{n})}X + a_{2n-1,3}^{(\mathsf{n})}XY$ holds. As $a_{2n-1,1}^{(\mathsf{n})} \in \varepsilon\mathcal{O}$, $a_{2n-1,3}^{(\mathsf{n})}$ belongs to $\varepsilon\mathcal{O}$ if and only if $\psi_{2n-1}(\mathsf{d}_{\mathsf{n},2})$ belongs to εA. It is equivalent to $a^{(\mathsf{n-1})}_{2n-1,2} \in \varepsilon\mathcal{O}$ by the equation $(**)$. Then, it follows from the equation (4.4.2) that $a^{(\mathsf{n-1})}_{2n-1,2} \in \varepsilon\mathcal{O}$ if and only if $a^{(\mathsf{n-1})}_{2n-3,3} \in \varepsilon\mathcal{O}$. By repeating this procedure, we deduce that $a^{(\mathsf{n})}_{2n-1,3} \in \varepsilon\mathcal{O}$ if and only if $a^{(1)}_{3,2} \in \varepsilon\mathcal{O}$. Since $\varepsilon\psi_2(\mathsf{d}_{1,2}) = X\psi_2(\mathsf{d}_{1,1}) = a^{(1)}_{2,1}X + a^{(1)}_{2,3}XY$, $a^{(1)}_{2,3}$ belongs to $\varepsilon\mathcal{O}$. It implies that $a^{(1)}_{3,2} \in \varepsilon\mathcal{O}$ by (5.25).

(2) Let $\rho \in \mathsf{radEnd}_A(Z^\infty_n)$. We put

$$\rho(\mathsf{d}_{\mathsf{k},1}) = \sum_{\mathsf{l}=1}^{\mathsf{n}} \mathsf{v}^{(\mathsf{k})}_{\mathsf{l},1}\mathsf{d}_{\mathsf{l},1} + B(\mathsf{k}),$$

where $B(\mathsf{k}) \in \mathsf{Span}_{\mathcal{O}}\{\mathsf{d}_{\mathsf{i},\mathsf{j}} \mid \mathsf{j} \neq 1\}$. By Lemma 5.6.2, there are $\mathsf{e}^{(\mathsf{k})}_{\mathsf{n},1}$ such that $\mathsf{v}^{(\mathsf{k})}_{\mathsf{n},1} = \varepsilon\mathsf{e}^{(\mathsf{k})}_{\mathsf{n},1}$. Define an A-module homomorphism $\psi = (\psi_k)_{k=1,\dots,2n} : Z^\infty_n \to A^{\oplus 2n}$ by $\psi(\mathsf{d}_{\mathsf{k},1}) = (0,\dots,0,\mathsf{e}^{(\mathsf{k})}_{\mathsf{n},1}XY)$. Then, it is easy to check that $\Phi^\infty_n\rho(\mathsf{d}_{\mathsf{k},1}) = \mathsf{v}^{(\mathsf{k})}_{\mathsf{n},1}\mathsf{d}_{\mathsf{n},4} = \pi_{n,\infty}\psi(\mathsf{d}_{\mathsf{k},1})$. □

Summing up, we obtain the following proposition.

Proposition 5.7.5 ([M2, Proposition 4.5])**.** Consider the following pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z^\infty_n \arrow[r] \arrow[d, equal] & E^\infty_n \arrow[r] \arrow[d] & Z^\infty_n \arrow[r] \arrow[d, "\Phi^\infty_n"] & 0 \\
0 \arrow[r] & Z^\infty_n \arrow[r] & A^{\oplus 2n} \arrow[r, "\pi_{n,\infty}"'] & Z^\infty_n \arrow[r] & 0
\end{tikzcd}$$

Then, the upper exact sequence is the almost split sequence ending at Z^∞_n.

Proof. The statement follows from Proposition 3.5.8 and Lemmas 5.7.3 and 5.7.4. □

5.8 *The shape of the Heller component containing Z^λ_n $(\lambda \neq \infty)$*

We denote by E^λ_n the middle term of the almost split sequence ending at Z^λ_n. By Proposition 5.7.2, the A-lattice E^λ_n is of the form

$$E^\lambda_n = \{(x,y) \in A^{\oplus 2n} \oplus Z^\lambda_n \mid \pi_{n,\lambda}(x) = \Phi^\lambda_n(y)\}.$$

Then, an $\mathcal{O}$-basis of the A-lattice E_n^λ is given as follows:

$$\begin{aligned}
E_n^\lambda = {} & \mathcal{O}(\varepsilon e_2 - \lambda X e_1 - Y e_1) \oplus \mathcal{O}(\varepsilon X e_2 - XY e_1) \oplus \mathcal{O}(Y e_2 + \lambda X e_2) \oplus \mathcal{O}(XY e_2) \\
& \bigoplus_{k=2}^{n} \Big(\mathcal{O}(\varepsilon e_{2k} + \lambda X e_{2k-1} - Y e_{2k-1} + X e_{2k-3}) \oplus \mathcal{O}(\varepsilon X e_{2k} - XY e_{2k-1}) \\
& \qquad \oplus \mathcal{O}(Y e_{2k} + \lambda X e_{2k} + X e_{2k-2}) \oplus \mathcal{O}(XY e_{2k}) \Big) \\
& \bigoplus_{\mathsf{k}=1}^{\mathsf{n}-1} \Big(\mathcal{O}\mathsf{a}^\lambda_{\mathsf{k},1} \oplus \mathcal{O}\mathsf{a}^\lambda_{\mathsf{k},2} \oplus \mathsf{b}^\lambda_{\mathsf{k},3} \oplus \mathcal{O}\mathsf{b}^\lambda_{\mathsf{k},4} \Big) \\
& \oplus \mathcal{O}(\mathsf{a}^\lambda_{\mathsf{n},1} - X e_{2n}) \oplus \mathcal{O}\mathsf{a}^\lambda_{\mathsf{n},2} \oplus \mathcal{O}\mathsf{a}^\lambda_{\mathsf{n},3} \oplus \mathcal{O}\mathsf{a}^\lambda_{\mathsf{n},4}
\end{aligned}$$

Lemma 5.8.1 ([M2, Lemma 3.5])**.** The following statements hold.

(1) There is an isomorphism $E_n^\lambda \otimes \kappa \simeq M(\lambda)_{n-1} \oplus M(\lambda)_{n+1} \oplus M(-\lambda)_n^{\oplus 2}$.

(2) We have an isomorphism $\tau(E_n^\lambda) \otimes \kappa \simeq M(-\lambda)_{n+1} \oplus M(-\lambda)_{n+1} M(\lambda)_n^{\oplus 2}$.

(3) E_n^λ is a non-projective indecomposable A-lattice.

Proof. (1) We define $\overline{A}$-submodules of $E_n^\lambda \otimes \kappa$ as follows.

$$E(\lambda, n)_1 := \mathsf{Span}_\kappa \left\{ \begin{array}{l} (\varepsilon e_2 - \lambda X e_1 - Y e_1),\ (\varepsilon X e_2 - XY e_1) \\ (\varepsilon e_{2k} + \lambda X e_{2k-1} - Y e_{2k-1} + X e_{2k-3}), \\ (\varepsilon X e_{2k} - XY e_{2k-1}) \end{array} \;\middle|\; k = 2, \dots, n \right\}$$

$$E(\lambda, n)_2 := \mathsf{Span}_\kappa \left\{ \ \mathsf{a}^\lambda_{\mathsf{k},3},\ \mathsf{a}^\lambda_{\mathsf{k},4} \ \middle|\ \mathsf{k} = 1, \dots, \mathsf{n} \ \right\}$$

$$E(\lambda, n)_3 := \mathsf{Span}_\kappa \left\{ \begin{array}{l} (Y e_2 + \lambda X e_2),\ (XY e_2), \\ (Y e_{2k} + \lambda X e_{2k} + X e_{2k-2} - \mathsf{a}^\lambda_{\mathsf{k}-1,1}), \\ (XY e_{2k} - \mathsf{a}^\lambda_{\mathsf{k}-1,2}), \\ (\mathsf{a}^\lambda_{\mathsf{n},1} - X e_{2n}),\ \mathsf{a}^\lambda_{\mathsf{n},2}, \end{array} \;\middle|\; k = \mathsf{k} = 2, \dots, n \right\}$$

$$E(\lambda, n)_4 := \mathsf{Span}_\kappa \left\{ \ \mathsf{a}^\lambda_{\mathsf{k},1},\ \mathsf{a}^\lambda_{\mathsf{k},2} \ \middle|\ \mathsf{k} = 1, \dots, \mathsf{n} - 1 \ \right\}$$

Then, there are isomorphisms $E_n^\infty \otimes \kappa = E(\lambda, n)_1 \oplus E(\lambda, n)_2 \oplus E(\lambda, n)_3 \oplus E(\lambda, n)_4$ and

$$E(\lambda, n)_1 \simeq E(\lambda, n)_2 \simeq M(-\lambda)_n, \quad E(\lambda, n)_3 \simeq M(\lambda)_{n+1}, \quad E(\lambda, n)_4 \simeq M(\lambda)_{n-1}.$$

(2) This follows from Lemmas 3.7.11, 5.1.3 and the statement (1).

(3) Suppose that E_n^λ is decomposable. We write $E_n^\lambda = E_1 \oplus E_2$ with $E_1 \neq 0 \neq E_2$ as A-lattices. Then, the ranks of the A-lattices E_1 and E_2 are divisible by four. The statement (1) implies that $E_1 \otimes \kappa \simeq M(-\lambda)_n^{\oplus 2}$ and $E_2 \otimes \kappa \simeq M(\lambda)_{n+1} \oplus M(\lambda)_{n-1}$. Assume that n is odd. If $n = 1$, then, E_2 is not isomorphic to any Heller lattice, and it is indecomposable. Let $0 \to \tau E_2 \to Z_n^{-\lambda} \oplus W \to E_2 \to 0$ be the almost split sequence ending at E_2. By Lemma 5.1.3, we have $\tau E_2 \otimes \kappa \simeq \widetilde{\Omega}(M(\lambda)_{n+1}) \simeq M(-\lambda)_{n+1}$. On the other hand, the induced sequence

$$0 \to (\tau E_2 \otimes \kappa) \to (Z_1^{-\lambda} \otimes \kappa) \oplus (W \otimes \kappa) \to (E_2 \otimes \kappa) \to 0$$

splits, which contradicts with Proposition 5.1.5 (3). Now, suppose that $n > 1$. Then, $E_2 \otimes \kappa \simeq M(\lambda)_{n-1} \oplus M(\lambda)_{n+1}$ and E_2 is indecomposable. Indeed, if $E_2 = E_{2,1} \oplus E_{2,2}$ with $E_{2,1} \neq 0 \neq E_{2,2}$ as A-lattices and $E_{2,1} \otimes \kappa \simeq M(\lambda)_{n+1}$, then we have a splitable exact sequence

$$0 \longrightarrow M(-\lambda)_{n+1} \longrightarrow W \oplus M(\lambda)_n \oplus M(-\lambda)_n \longrightarrow M(\lambda)_{n+1} \longrightarrow 0$$

for some $W \in \mathsf{mod}\text{-}\overline{A}$, a contradiction. Thus, E_2 is indecomposable. Then, the indecomposability of E_n^λ follows by the same method as in the proof of $n = 1$.

Assume that n is even. Then, E_2 is indecomposable since the rank of any direct summand of E_n^λ is divisible by four. In this case, we can prove the indecomposability of E_n^λ by using similar arguments. □

Corollary 5.8.2. For any $n > 0$ and $\lambda \in \kappa$, the Heller component $\mathcal{HC}(Z_n^\lambda)$ has no loops.

Lemma 5.8.3 ([M2, Lemma 3.7])**.** Let $\mathcal{C}$ be a component of stable Auslander–Reiten quiver of A. Then, $\mathcal{C}$ has infinitely many vertices.

Proof. The assertion follows from Corollary 3.6.7. □

Now, we are ready to determine the shape of $\mathcal{HC}(Z_n^\lambda)$.

Theorem 5.8.4 ([M2, Theorem 3.8])**.** Let $\mathcal{O}$ be a complete discrete valuation ring, κ its residue field and $A = \mathcal{O}[X,Y]/(X^2,Y^2)$. Assume that κ is algebraically closed and $\lambda \neq \infty$. Then $\mathcal{HC}(Z_n^0) \simeq \mathbb{Z}A_\infty/\langle\tau\rangle$ if $\lambda = 0$, otherwise $\mathcal{HC}(Z_n^\lambda) \simeq \mathbb{Z}A_\infty/\langle\tau^2\rangle$. Moreover, any Heller lattice Z_n^λ appears on the boundary of $\mathcal{HC}(Z_n^\lambda)$.

Proof. Lemma 5.8.1 implies that every Heller lattice Z_n^λ appears on the boundary of $\mathcal{HC}(Z_n^\lambda)(= \mathcal{HC}(Z_n^{-\lambda}))$. It follows from Proposition 3.7.6 and Lemma 5.8.3 that the tree class $\overline{T}$ of $\mathcal{HC}(Z_n^\lambda)$ is one of A_∞, B_∞, C_∞, D_∞ or A_∞^∞.

Let F be the middle term of the almost split sequence ending at E_n^λ. Then, F is the direct sum of $Z_n^{-\lambda}$ and an A-lattice F_n^λ. By Proposition 3.7.10, we have

$$F_n^\lambda \otimes \kappa \simeq M(\lambda)_{n+1} \oplus M(\lambda)_{n-1} \oplus M(-\lambda)_{n+1} \oplus M(-\lambda)_{n-1} \oplus M(\lambda)_n \oplus M(-\lambda)_n.$$

Suppose that F_n^λ is not indecomposable. Then, there is an indecomposable direct summand W of F_n^λ such that the almost split sequence ending at W is of the form $0 \to \tau W \to E_n^{-\lambda} \to W \to 0$. As $\mathsf{rank}(E_n^\lambda) = 8n$, we have $\mathsf{rank}(W) = 4n$. If W is a Heller lattice, then $W \otimes \kappa$ must be isomorphic to $M(\lambda)_n \oplus M(-\lambda)_n$. Then, F_n^λ/W is indecomposable, and it is not a Heller lattice by Proposition 5.1.5. Let $0 \to \tau(F_n^\lambda/W) \to E_n^{-\lambda} \oplus G \to F_n^\lambda/W \to 0$ be the almost split sequence ending at F_n^λ/W. Then, the induced exact sequence

$$0 \to \tau F_n^\lambda/W \otimes \kappa \to E_n^{-\lambda} \otimes \kappa \oplus G \otimes \kappa \to F_n^\lambda/W \otimes \kappa \to 0$$

splits, a contradiction. Thus, W is not a Heller lattice. This implies that the induced exact sequence

$$0 \to \tau W \otimes \kappa \to E_n^{-\lambda} \otimes \kappa \to W \otimes \kappa \to 0$$

splits. However, one can check that this situation does not occur for any W by using Proposition 5.8.1. Therefore, F_n^λ is an indecomposable A-lattice, and $\overline{T} = A_\infty$. □

5.9 The shape of the Heller component containing Z_n^∞

In the last of this chapter, we determine the shape of the Heller component containing the Heller lattice Z_n^∞.

Lemma 5.9.1 ([M2, Lemma 4.6]). (1) An $\mathcal{O}$-basis of E_1^∞ is given by

$$\mathcal{O}(\varepsilon e_2 - Xe_1) \oplus \mathcal{O}Xe_2 \oplus \mathcal{O}(\varepsilon Y e_2 - XYe_1) \oplus \mathcal{O}XYe_2 \oplus \mathcal{O}(\mathsf{b}_{1,1} + Ye_2) \oplus \mathcal{O}\mathsf{b}_{1,2} \oplus \mathcal{O}\mathsf{b}_{1,3} \oplus \mathcal{O}\mathsf{b}_{1,4}.$$

(2) There is an isomorphism $E_1^\infty \otimes \kappa \simeq M(\infty)_1^{\oplus 2} \oplus M(\infty)_2$.

(3) We have an isomorphism $(\tau E_1^\infty) \otimes \kappa \simeq M(\infty)_1^{\oplus 2} \oplus M(\infty)_2$.

(4) E_1^∞ is a non-projective indecomposable A-lattice.

Proof. (1) Straightforward.

(2) We put

$$\begin{aligned}
E(\infty,1)_1 &:= \mathsf{Span}_\kappa\{(\varepsilon e_2 - Xe_1), (\varepsilon Y e_1 - XYe_1)\},\\
E(\infty,1)_2 &:= \mathsf{Span}_\kappa\{\mathsf{b}_{1,2}, \mathsf{b}_{1,4}\},\\
E(\infty,1)_3 &:= \mathsf{Span}_\kappa\{(Xe_2), (XYe_2), (\mathsf{b}_{1,1} + Ye_2), \mathsf{b}_{1,3}\}.
\end{aligned}$$

Then, it is easy to check that $E(\infty,1)_1 \simeq E(\infty,1)_2 \simeq M(\infty)_1$ and $E(\infty,1)_3 \simeq M(\infty)_2$.

(3) This follows from Lemmas 3.7.11, 5.1.3 and the statement (2).

(4) Suppose that E_1^∞ is decomposable. We write $E_1^\infty = E_1 \oplus E_2$ as A-lattices with $E_1 \neq 0 \neq E_2$. Then, the ranks of E_1 and E_2 are divisible by four. Thus, one can assume that $E_1 \otimes \kappa \simeq M(\infty)_1^{\oplus 2}$, $E_2 \otimes \kappa \simeq M(\infty)_2$, and E_1 and E_2 are indecomposable. Then, the A-lattice E_2 is not isomorphic to any Heller lattices by Proposition 5.1.5. Let

$0 \to \tau E_2 \to Z_1^\infty \oplus W \to E_2 \to 0$ be the almost split sequence ending at E_2. By applying $- \otimes \kappa$, the induced sequence

$$0 \to \tau E_2 \otimes \kappa \to Z_1^\infty \otimes \kappa \oplus W \otimes \kappa \to E_2 \otimes \kappa \to 0$$

splits, which contradicts with Proposition 5.1.5 (3). □

By the definition of E_2^λ, we have

$$\begin{aligned} E_2^\infty = & \mathcal{O}(\varepsilon e_2 - Xe_1) \oplus \mathcal{O}(Xe_2) \oplus \mathcal{O}(Xe_3 - Ye_2) \oplus \mathcal{O}(XYe_2) \\ & \oplus \mathcal{O}(\varepsilon e_3 + Xe_4 - Ye_1) \oplus \mathcal{O}(\varepsilon Xe_3 - XYe_1) \oplus \mathcal{O}(\varepsilon Ye_3 + XYe_4) \oplus \mathcal{O}(XYe_3) \\ & \oplus \mathcal{O}\mathsf{b}_{1,1} \oplus \mathcal{O}\mathsf{b}_{1,2} \oplus \mathcal{O}\mathsf{b}_{1,3} \oplus \mathcal{O}\mathsf{b}_{1,4} \\ & \oplus \mathcal{O}(\mathsf{b}_{2,1} - Ye_3) \oplus \mathcal{O}\mathsf{b}_{2,2} \oplus \mathcal{O}\mathsf{b}_{2,3} \oplus \mathcal{O}\mathsf{b}_{2,4}. \end{aligned}$$

Lemma 5.9.2 ([M2, Lemma 4.7])**.** The following statements hold.

(1) There is an isomorphism $E_2^\infty \otimes \kappa \simeq \oplus M(\infty)_2^{\oplus 2} \oplus M(\infty)_1 \oplus M(\infty)_3$.

(2) We have an isomorphism $(\tau^n E_2^\infty) \otimes \kappa \simeq M(\infty)_2^{\oplus 2} \oplus M(\infty)_1 \oplus M(\infty)_3$.

(3) E_2^∞ is a non-projective indecomposable A-lattice.

Proof. (1) We put

$$\begin{aligned} E(\infty,2)_1 &:= \mathsf{Span}_\kappa\{(\varepsilon e_2 - Xe_1), (\varepsilon Xe_3 - XYe_1), (\varepsilon e_3 + Xe_4 - Ye_1), (\varepsilon Ye_3 + XYe_4)\}, \\ E(\infty,2)_2 &:= \mathsf{Span}_\kappa\{\mathsf{b}_{1,2}, \mathsf{b}_{1,3}, \mathsf{b}_{1,4}, \mathsf{b}_{2,4}\}, \\ E(\infty,2)_3 &:= \mathsf{Span}_\kappa\{(Xe_2), (Xe_3 - Ye_2 - \mathsf{b}_{1,1}), (XYe_2), (XYe_3 - \mathsf{b}_{2,2}), (\mathsf{b}_{2,1} - Ye_3), \mathsf{b}_{2,3}\} \\ E(\infty,2)_4 &:= \mathsf{Span}_\kappa\{\mathsf{b}_{1,1}, \mathsf{b}_{2,2}\} \end{aligned}$$

Then, it is easy to check that $E(\infty,2)_1 \simeq E(\infty,2)_2 \simeq M(\infty)_2$, $E(\infty,2)_3 \simeq M(\infty)_3$ and $E(\infty,2)_4 \simeq M(\infty)_1$.

(2) This follows from Lemmas 3.7.11, 5.1.3 and the statement (1).

(3) Suppose that E_2^∞ is decomposable. We write $E_2^\infty \simeq E_1 \oplus E_2$ as A-lattices with $E_1 \neq 0 \neq E_2$. Then, we may assume that $E_1 \otimes \kappa \simeq M(\infty)_2^{\oplus 2}$ and $E_2 \otimes \kappa \simeq M(\infty)_1 \oplus M(\infty)_3$. Note that the A-lattice E_2 is not isomorphic to any Heller lattices and it is indecomposable. Let $0 \to \tau E_2 \to Z_2^\infty \oplus W \to E_2 \to 0$ be the almost split sequence ending at E_2. It follows from Lemma 5.1.3 that $(\tau E_2) \otimes \kappa \simeq \widetilde{\Omega}(M(\infty)_1 \oplus M(\infty)_3) \simeq M(\infty)_1 \oplus M(\infty)_3$. Then, the induced sequence $0 \to \tau E_2 \otimes \kappa \to (Z_2^\infty \otimes \kappa) \oplus (W \otimes \kappa) \to E_2 \otimes \kappa \to 0$ splits, which contradicts with Proposition 5.1.5 (3). □

From now on, we assume that $n > 2$. Then, an $\mathcal{O}$-basis of the A-lattice E_n^∞ is given as follows:

$$\begin{aligned}
E_n^\infty = {} & \mathcal{O}(\varepsilon e_2 - Xe_1) \oplus \mathcal{O}(Xe_2) \oplus \mathcal{O}(Ye_2 - Xe_3) \oplus \mathcal{O}(XYe_2) \\
& \oplus \mathcal{O}(\varepsilon e_3 + Xf_4 - Ye_1) \oplus \mathcal{O}(\varepsilon Xe_3 - XYe_1) \oplus \mathcal{O}(Ye_3 + Xe_5) \oplus \mathcal{O}(XYe_3) \\
& \bigoplus_{k=1}^{n-3} \Big(\mathcal{O}(\varepsilon e_{2k+3} + Xe_{2k+4} - Ye_{2k+2}) \oplus \mathcal{O}(\varepsilon Xe_{2k+3} - XYe_{2k+2}) \\
& \qquad \oplus \mathcal{O}(Ye_{2k+3} + Xe_{2k+5}) \oplus \mathcal{O}(XYe_{2k+3}) \Big) \\
& \oplus \mathcal{O}(\varepsilon e_{2n-1} + Xe_{2n} - Ye_{2n-2}) \oplus \mathcal{O}(\varepsilon Xe_{2n-1} - XYe_{2n-2}) \\
& \oplus \mathcal{O}(\varepsilon Ye_{2n-1} + XYe_{2n}) \oplus \mathcal{O}(XYe_{2n-1}) \\
& \bigoplus_{\mathsf{k=1}}^{\mathsf{n-1}} \Big(\mathcal{O}\mathsf{b}_{\mathsf{k,1}} \oplus \mathcal{O}\mathsf{b}_{\mathsf{k,2}} \oplus \mathsf{b}_{\mathsf{k,3}} \oplus \mathcal{O}\mathsf{b}_{\mathsf{k,4}} \Big) \\
& \oplus \mathcal{O}(\mathsf{b}_{\mathsf{n,1}} - Y_{2n-1}) \oplus \mathcal{O}\mathsf{b}_{\mathsf{n,2}} \oplus \mathcal{O}\mathsf{b}_{\mathsf{n,3}} \oplus \mathcal{O}\mathsf{b}_{\mathsf{n,4}}
\end{aligned}$$

Lemma 5.9.3 ([M2, Lemma 4.8])**.** The following statements hold.

(1) There is an isomorphism $E_n^\infty \otimes \kappa \simeq M(\infty)_n^{\oplus 2} \oplus M(\infty)_{n+1} \oplus M(\infty)_{n-1}$.

(2) We have an isomorphism $(\tau E_n^\infty) \otimes \kappa \simeq M(\infty)_n^{\oplus 2} \oplus M(\infty)_{n-1} \oplus M(\infty)_{n+1}$.

(3) E_n^∞ is a non-projective indecomposable A-lattice.

Proof. (1) The statement is true for $n = 1, 2$ by Lemmas 5.9.1 and 5.9.2. Assume that

$n > 2$. We define $\overline{A}$-submodules of $E_n^\infty \otimes \kappa$ as follows.

$$E(\infty,n)_1 := \mathsf{Span}_\kappa \left\{ \begin{array}{l|l} (\varepsilon e_2 - Xe_1),\ (\varepsilon Xe_3 - XYe_1) & \\ (\varepsilon e_{2k+1} + Xe_{2k+2} - Ye_{2k-1}), & k = 1,\dots,n-1, \\ (\varepsilon Xe_{2l+3} - XYe_{2l+2}), & l = 1,\dots,n-2 \\ (\varepsilon Ye_{2n-1} + XYe_{2n}) & \end{array} \right\}$$

$$E(\infty,n)_2 := \mathsf{Span}_\kappa \left\{ \begin{array}{l|l} \mathsf{b}_{1,2},\ \mathsf{b}_{\mathsf{k},3},\ \mathsf{b}_{\mathsf{l},4} & \mathsf{k} = 1,\dots,\mathsf{n}-1, \\ & \mathsf{l} = 1,\dots,\mathsf{n} \end{array} \right\}$$

$$E(\infty,n)_3 := \mathsf{Span}_\kappa \left\{ \begin{array}{l|l} Xe_2,\ XYe_2, & \\ (Ye_2 - Xe_3 + \mathsf{b}_{1,1}), & \\ (Ye_{2k+1} + Xe_{2k+3} + \mathsf{b}_{\mathsf{k+},1}), & k = \mathsf{k} = 1,\dots,n-2, \\ (XYe_{2l+1} - \mathsf{b}_{\mathsf{l}+1,2}), & l = \mathsf{l} = 1,\dots,n-1 \\ (\mathsf{b}_{\mathsf{n},1} - Ye_{2n-1}),\ \mathsf{b}_{\mathsf{n},3}, & \end{array} \right\}$$

$$E(\infty,n)_4 := \mathsf{Span}_\kappa \left\{ \begin{array}{l|l} \mathsf{b}_{\mathsf{s},1},\ \mathsf{b}_{\mathsf{t},2} & \mathsf{s} = 1,\dots,\mathsf{n}-1, \\ & \mathsf{t} = 2,\dots,\mathsf{n} \end{array} \right\}$$

Then, it is easy to check that

$$\begin{aligned} &E_n^\infty \otimes \kappa = E(\infty,n)_1 \oplus E(\infty,n)_2 \oplus E(\infty,n)_3 \oplus E(\infty,n)_4, \\ &E(\infty,n)_1 \simeq E(\infty,n)_2 \simeq M(\infty)_n, \\ &E(\infty,n)_3 \simeq M(\infty)_{n+1}, \\ &E(\infty,n)_4 \simeq M(\infty)_{n-1}. \end{aligned}$$

(2) This follows from Lemmas 3.7.11, 5.1.3 and the statement (1).

(3) We can prove the indecomposability of E_n^λ by using similar arguments of the proof of the case $\lambda \neq \infty$. □

Corollary 5.9.4. *$\mathcal{HC}(Z_n^\infty) \neq \mathcal{HC}(Z_m^\infty)$ whenever $n \neq m$. Moreover, $\mathcal{HC}(Z_n^\infty)$ has no loops.*

Now, we determine the shape of $\mathcal{HC}(Z_n^\infty)$.

Theorem 5.9.5 ([M2, Theorem 4.10])**.** *Let $\mathcal{O}$ be a complete discrete valuation ring, κ its residue field and $A = \mathcal{O}[X,Y]/(X^2,Y^2)$. Assume that κ is algebraically closed. Then, $\mathcal{HC}(Z_n^\infty) \simeq \mathbb{Z}A_\infty/\langle\tau\rangle$. Moreover, the Heller lattice Z_n^∞ appears on the boundary of $\mathcal{HC}(Z_n^\infty)$.*

Proof. Lemmas 5.9.1, 5.9.2 and 5.9.3 imply that every Heller lattice Z_n^∞ appears on the boundary of $\mathcal{HC}(Z_n^\infty)$. It follows from Proposition 3.7.6 and Lemma 5.8.3 that the tree class $\overline{T}$ of $\mathcal{HC}(Z_n^\infty)$ is one of A_∞, B_∞, C_∞, D_∞ or A_∞^∞.

Let F be the middle term of the almost split sequence ending at E_n^∞. Then, F is the direct sum of Z_n^∞ and an A-lattice F_n^∞. By Proposition 3.7.10, we have

$$F_n^\infty \otimes \kappa \simeq M(\infty)_{n+1}^{\oplus 2} \oplus M(\infty)_{n-1}^{\oplus 2} \oplus M(\infty)_n^{\oplus 2}.$$

Suppose that F_n^∞ is not indecomposable. Then, there is an indecomposable direct summand W of F_n^∞ such that the almost split sequence ending at W is of the form $0 \to \tau W \to E_n^\infty \to W \to 0$. As $\mathsf{rank}(E_n^\infty) = 8n$, we have $\mathsf{rank}(W) = 4n$. If W is a Heller lattice, then $W \otimes \kappa$ must be isomorphic to $M(\infty)_n \oplus M(\infty)_n$. Then, F_n^∞/W is indecomposable, and it is not a Heller lattice by Proposition 5.1.5. Let $0 \to \tau(F_n^\infty/W) \to E_n^\infty \oplus G \to F_n^\infty/W \to 0$ be the almost split sequence ending at F_n^∞/W. Then, the induced exact sequence

$$0 \to \tau F_n^\infty/W \otimes \kappa \to E_n^\infty \otimes \kappa \oplus G \otimes \kappa \to F_n^\infty/W \otimes \kappa \to 0$$

splits, a contradiction. Thus, W is not a Heller lattice. This implies that the induced exact sequence

$$0 \to \tau W \otimes \kappa \to E_n^\infty \otimes \kappa \to W \otimes \kappa \to 0$$

splits. However, one can check that this situation does not occur for any W. Therefore, F_n^∞ is an indecomposable A-lattice, and $\overline{T} = A_\infty$. $\square$

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