Analysis of Temperature Distribution with Radial Symmetrical Cooling Terminal around Moving Heat Source†

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Nomenclature: The following nomenclature will be used throughout this paper;

- q : heat input, [cal/s]
- q' : heat input per unit length ($=q/h$), [cal/s·cm]
- h : thickness of plate, [cm]
- v : speed of source, [cm/s]
- t : time, [s]
- T : temperature, [°C]
- T_0 : temperature at cooling terminal, [°C]
- T_f : reference temperature, [°C]
- K : heat conductivity, [cal/s·cm·°C]
- $1/2\lambda$: thermal diffusivity, [cm^2/s]
- R : radius of cooling terminal, [cm]
- n : non-dimensional heat input ($=\lambda vq/2\pi K(T_f-T_0)$)
- θ : normalized temperature ($=(T-T_0)/(T_f-T_0)$)
- Λ : non-dimensional radius of cooling terminal ($=\lambda vR$)
- ρ : normalized radius ($=r/R$) ($0 \leqq \rho \leqq 1$)

Assumptions: Mathematical analyses were conducted under the following assumptions;

1) The physical properties of the plate, i.e., K and λ are independent of the temperature and the position.
2) The moving speed v and the rate of heat input q are constant.
3) Heat losses from the surface by convection and radiation are neglected.
4) Cooling terminal of constant temperature T_0 is placed over the equidistant plane from the moving source.

Point Heat Source: The basic differential equation of heat conduction is expressed as the following form in the fixed co-ordinate (x, y, z).

$$2\lambda\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} \tag{1}$$

Supposing that a point heat source q is supplied on the surface of semi-infinite plate and moves along the x-axis with the constant speed of v, one can rewrite Equation (1) into the moving co-ordinate system (ξ, y, z) whose origin is taken at the heat source.

$$\frac{\partial^2 T}{\partial \xi^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=2\lambda\left(-v\frac{\partial T}{\partial \xi}+\frac{\partial T}{\partial t}\right) \tag{2}$$

where, $\xi = x - vt$.

Taking

$$T - T_0 = exp.\ (-\lambda v\xi)\cdot\Phi(\xi, y, z) \quad (3),$$

Equation (2) under the quasi-stationary state becomes

$$\frac{\partial^2 \Phi}{\partial \xi^2}+\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 \Phi}{\partial z^2}-(\lambda v)^2\Phi=0 \quad (4),$$

or in polar co-ordinate (r, α, β),

$$\frac{d^2(r\Phi)}{dr^2}-(\lambda v)^2(r\Phi)=0 \quad (5).$$

Therefore, the temperature is solved from the general solution of Equation (4) and Equation (3).

$$T - T_0 = \frac{1}{r}exp.(-\lambda vr\cos\alpha\cos\beta)\ \mathrm{x}$$
$$[C_1\ exp.\ (\lambda vr)+C_2\ exp.\ (-\lambda vr)] \tag{6}$$

The boundary conditions are

$$\begin{cases} T = T_0 \text{ at } r = R \\ -2\pi rK\,(\partial T/\partial r)\to q \text{ as } r\to 0 \end{cases} \quad (7).$$

Therefore, the solution under question becomes

$$T - T_0 = \frac{q}{2\pi K}\frac{1}{r}exp.\ [-\lambda vr\,(1+\cos\alpha\cos\beta)]\ \mathrm{x}$$
$$\frac{1-exp.\ [-2\lambda v\,(R-r)]}{1-exp.\ [-2\lambda vR]} \quad (8),$$

or in non-dimensional form[1])

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$$\frac{\theta}{n} = \frac{1}{\rho\Lambda} exp. \ [-\rho\Lambda\,(1 + cos\ \alpha\ cos\ \beta)]\ \mathrm{x}$$
$$\frac{1 - exp.\ [-2\Lambda\,(1 - \rho)]}{1 - exp.\ [-2\Lambda]} \qquad (9).$$

Here, the term, i.e.,

$$\gamma \equiv \frac{1 - exp.\ [-2\Lambda(1 - \rho)]}{1 - exp.\ [-2\Lambda]} \qquad (10)$$

represents the cooling factor that depends on the radius Λ. It is evident that

$$\lim_{\Lambda\to\infty} \lambda = 1 \quad \text{for} \quad 0 \leqq \rho < 1,$$

and Equation (9) coincides completely with Rosenthal's solution[2] of

$$\frac{\theta}{n} = \frac{1}{\lambda v r} exp.\ [-\lambda v r\,(1 + cos\ a\ cos\,\beta)] \qquad (11),$$

since $\rho\Lambda = (r/R)\,(\lambda v R) = \lambda v r$.

In **Figures** 1 and 2 are shown the behaviors of γ against Λ and the temperature distribution behind a source along the moving axis (ξ–axis) for several Λ–values. In these calculations, T_0 was taken to zero.

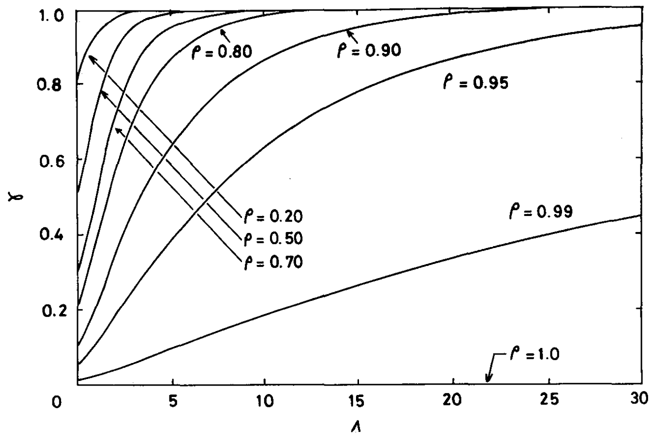

Fig. 1 Behavior of cooling factor against radius of cooling terminal (Three dimensional case)

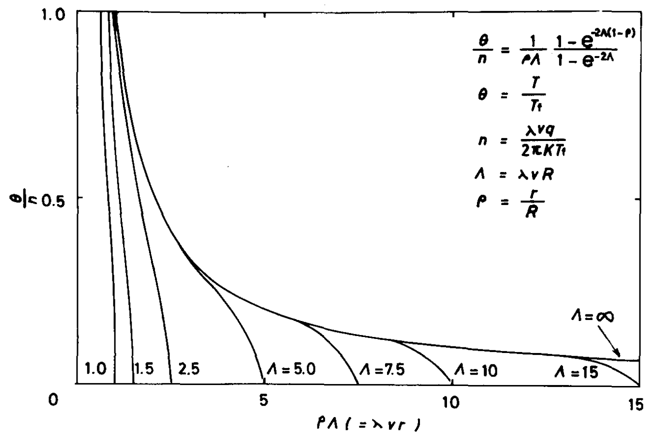

Fig. 2 Temperature distribution behind point heat source along moving axis when cooling terminal is placed around a source

Linear Heat Source: In two dimensional heat flow, there is no heat flux in the z–direction. Thus $\partial T/\partial z = 0$ and Equation (4) is expressed in cylindrical co-ordinate (r, α) as the form of

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} - (\lambda v)^2\,\Phi = 0 \qquad (12).$$

The solution of temperature is, therefore, given by

$$T - T_0 = \frac{q}{2\pi h K} exp.\ [-\lambda v r\ cos\ \alpha] \cdot K_0(\lambda v r)\ \mathrm{x}$$
$$[1 - \frac{K_0\,(\lambda v R) \cdot I_0\ (\lambda v r)}{I_0\ (\lambda v R) \cdot K_0\,(\lambda v r)}] \qquad (13),$$

under the boundary conditions of

$$\begin{cases} T = T_0 \ \text{ at } r = R \\ -2\pi r K\,(\frac{\partial T}{\partial r}) \to q' = q/h \quad \text{as } r \to 0 \end{cases} \qquad (14).$$

Equation (13) can be also written in non-dimensional form as

$$\frac{\theta}{n} = \frac{1}{\eta\Lambda}\ exp.\ [-\rho\Lambda\,cos\,\alpha] \cdot K_0\ (\rho\Lambda)\ \mathrm{x}$$
$$[1 - \frac{K_0(\Lambda)}{I_0\ (\Lambda)} \cdot \frac{I_0\ (\rho\Lambda)}{K_0\ (\rho\Lambda)}] \qquad (15).$$

Here, the non-dimensional plate thickness η is defined by $\eta = (\lambda v h)/\Lambda = (\lambda v h)/(\lambda v R) = h/R$.

The cooling term of Equation (15) is

$$\gamma' = 1 - \frac{K_0(\Lambda)}{I_0(\Lambda)} \cdot \frac{I_0\ (\rho\Lambda)}{K_0\ (\rho\Lambda)} \qquad (16).$$

Figure 3 shows the change of γ' with Λ, in which the three dimensional cooling term, i.e., γ of Equation (10) is also represented by dotted lines. As seen in the figure, behaviors of γ and γ' are very similar except in smaller values of Λ, and they coincide at higher Λ-values. It is, therefore, possible to express Equation (15) as the following form in higher values of Λ.

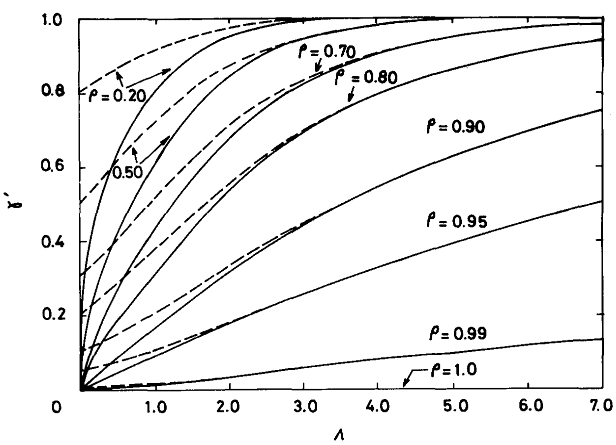

Fig. 3 Behavior of cooling factor against radius of cooling terminal (Solid line for linear source and dotted line for point source)

$$\frac{\theta}{n} = \frac{1}{\eta\Lambda} exp. \ [-\rho\Lambda \cos\alpha] \cdot K_0 \ (\rho\Lambda) \ \mathrm{x}$$

$$[\frac{1 - exp. \ [-2\Lambda \ (1 - \rho \ )]}{1 - exp. \ (-2\Lambda)}] \ \text{for} \ \Lambda > 4 \quad (17).$$

It is also evident that Equation (15) coincides with the solution by Rosenthal in two-dimensional heat flow at very high values of Λ.

References

1) N. Christensen et al.: "Distribution of Temperature in Arc Welding", British Welding J., No. 12, Vol. 54 (1965),
2) D. Rosenthal: "The Theory of Moving Sources of Heat and Its Application to Metal Treatment", Transactions of A.S.M.E., Nov., 1946, pp. 849-866.