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## ON HYPOELLIPTIC OPERATORS WITH MULTIPLE CHARACTERISTICS OF ODD ORDER

YOSHINORI MORIMOTO

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**Introduction.** In the recent paper [4], Hörmander has clarified Egorov's work [3] on sub-elliptic operators, by improving several points. The purpose of the present paper is to show that the method in [4] is applicable to certain pseudodifferential operators with multiple characteristics of odd order. Rubinstein [13], Wenston [15], [16], Popivanov [12], Menikoff [6] and Popivanov-Popov [17] independently treated some class of differential (or pseudodifferential) operators with double (or multiple, see [17]) characteristics satisfying the conditions similar to those given by Nirenberg-Treves [10] for operators of principal type.

It should be noted that, roughly speaking, operators considered in those papers can be reduced micro-locally to  $D_{x_1} + ix_1^k D_{x_2}$  ( $k$  integer), which was studied by Mizohata [8]. On the other hand, the operator considered in the present paper can not be reduced only to Mizohata type everywhere in the sense of micro-local. At some point it will be reduced even to Egorov type  $D_{x_1} + i(x_1^s D_{x_2} + x_1^a x_2^b |D_x|)$ , where  $|D_x|$  denotes the square root of  $D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2$  and  $s, a, b$  are integers.

The plan of this paper is as follows. In Section 1 we state the assumptions and result. In Section 2 we reduce the proof of main theorem to "sub-elliptic estimate" for a localized operator whose symbol has a parameter  $0 < \lambda \leq 1$  (see (2.37) and (3.4)). To prove this estimate, in Section 3 we show that we can use the same method as in [4]. The most part of Section 3 is devoted to show that the symbol of the localized operator satisfies inequalities similar to those in [4, Section 4]. In final section we prove the non-hypoellipticity of some operator in order to show the importance of the notion of modified-null-bicharacteristic curve, which is introduced in Section 1.

### 1. Assumptions and result

We say that  $p(x, \xi) \in \mathcal{G}^\infty(R_x^n \times R_\xi^n)$  belongs to  $\bar{S}^m$  when  $p(x, \xi)$  is positively homogeneous of degree  $m$  in  $|\xi| \geq 1/2$ . (Clearly  $\bar{S}^m$  is the subset of  $S^m = S_{1,0}^m$ . We refer the definition of  $S_{1,0}^m$  to Kumano-go [5, p. 50].) For a conic set  $U \subset R_x^n \times R_\xi^n$  and  $q(x, \xi) \in C^\infty(U)$  with positive homogeneous of degree  $m$  in  $|\xi| \geq 1/2$

we write  $q(x, \xi) \in \bar{S}^m(U)$ .

Let  $p_1(x, \xi)$  belong to  $\bar{S}^1$  and be real principal type, that is, be real valued and satisfy

$$(1.1) \quad d_{x\xi} p_1(x, \xi) \neq 0 \quad \text{on } \Gamma = p_1^{-1}(0) \cap \{|\xi| \geq 1/2\}.$$

Let  $l$  be an odd integer  $\geq 3$  and let  $a(x, \xi) \in \bar{S}^{l-1}$  be complex valued and satisfy

$$(1.2) \quad \text{Re } a \neq 0 \quad \text{on } \Gamma$$

and moreover

$$(1.3) \quad H_{p_1} \text{Re } a = 0 \quad \text{on } \Gamma,$$

where  $H_{p_1}$  denotes the Hamilton vector field of  $p_1$ .

Then under certain conditions among  $p_1$ ,  $\text{Re } a$  and  $\text{Im } a$  we shall discuss the hypoellipticity for a pseudodifferential operator  $L$  of order  $l$  which has the form

$$(1.4) \quad \begin{aligned} L &= P(x, D_x) + A(x, D_x) \quad \text{in } R_x^n, \\ \sigma(P) &= (p_1(x, \xi))^l, \quad \sigma(A) = a(x, \xi). \end{aligned}$$

Here  $\sigma(P)$  denotes the symbol of pseudodifferential operator  $P(x, D_x)$ .

First we assume that

$$(1.5) \quad \text{for any } (x_0, \xi_0) \in \Gamma \text{ there exist a conic neighborhood } U \text{ of } (x_0, \xi_0) \text{ and } q_0(x, \xi) \in \bar{S}^0(U) \text{ such that}$$

$$(1.6) \quad H_{p_1} q_0 = 1 \quad \text{in } U$$

and for  $j=1, \dots, l-2$

$$(1.7) \quad H_{q_0}^j a = 0 \quad \text{on } \Gamma \cap U.$$

To state the second condition corresponding to (A) in Egorov [3], or ( $\Psi$ ) in Nirenberg-Treves [10], we define the modified-null-bicharacteristic curve of  $p_1$  through  $(x_0, \xi_0) \in \Gamma$  by the curve

$$(1.8) \quad [-T, T] \ni t \mapsto (x(t), \xi(t)),$$

where  $(x(t), \xi(t))$  is the solution to

$$(1.9) \quad \begin{aligned} dx/dt &= d_\xi(p_1 + {}^l\sqrt{\text{Re } a}) \\ d\xi/dt &= -d_x(p_1 + {}^l\sqrt{\text{Re } a}), \quad (x(0), \xi(0)) = (x_0, \xi_0), \end{aligned}$$

and  ${}^l\sqrt{\text{Re } a}$  denotes a unique real  $l$  power root of  $\text{Re } a$ . It follows from (1.3) that if  $(x_0, \xi_0) \in \Gamma$ , then  $(x(t), \xi(t)) \in \Gamma$ . The right hand side of (1.9) are not homogeneous in  $\xi$ , so that the behavior of modified-null-bicharacteristic curve is not so. But we can define it on  $[-T, T]$  for some  $T > 0$  uniformly if  $(x_0, \xi_0) \in \Gamma$  varies in a compact conic set ( $|\xi_0| > 1$ ), because  $d_\xi {}^l\sqrt{\text{Re } a}$  and  $|\xi|^{-1} d_x {}^l\sqrt{\text{Re } a}$

are  $\mathcal{O}(|\xi|^{-1/l})$ . Second condition is that

$$(1.10) \quad \left\{ \begin{array}{l} \text{for any } (x_0, \xi_0) \in \Gamma \text{ with } |\xi_0| \text{ sufficiently large,} \\ \text{if } \text{Im } a(x(t_0), \xi(t_0)) > 0 \text{ for some } t_0 \in [-T, T], \\ \text{then } \text{Im } a(x(t), \xi(t)) \geq 0 \text{ for all } t \in (t_0, T], \\ \text{where } (x(t), \xi(t)) \text{ is the modified-null-bicharacteristic} \\ \text{curve of } p_1 \text{ through } (x_0, \xi_0). \end{array} \right.$$

For a multi-index  $I=(i_1, \dots, i_k)$  whose components  $i_j$  are 1 or 2, we use the following notations:  $|I|=k, b(I)=$ “the number of  $j$  such that  $i_j=2$ ”,  $\mathcal{J}=\{I; b(I) \leq l-1\}$ ,  $\mu(I)=(l-1)|I|/(l-b(I))$ . For a  $\mu > 0$  we put  $\mathcal{J}_\mu=\{I \in \mathcal{J}; \mu(I) \leq \mu\}$ . Set  $Q_0=\{\mu(I); I \in \mathcal{J}\}$ . Then we can write  $Q_0=\{\mu_j; j=1, 2, \dots\}$  with an appropriate increasing sequence of rational numbers. For any  $(x, \xi) \in \Gamma, \mu(x, \xi)$  denotes the rational  $\mu_j \in Q_0$  such that  $p_I(x, \xi) \neq 0$  for some  $I \in \mathcal{J}_{\mu_j}$  and  $p_I(x, \xi)=0$  for any  $I \in \mathcal{J}_{\mu_{j-1}}$ , where

$$(1.11) \quad p_I(x, \xi) = H_{i_1} H_{i_2} \dots H_{i_{k-2}} p_{i_k}(x, \xi),$$

$H_1=H_{p_1}, H_2=H_{p_2}$  and  $p_2(x, \xi)=\text{Im } a(x, \xi)$ . The third condition corresponding to (B) in Egorov [3] is that

$$(1.12) \quad \left\{ \begin{array}{l} \text{for all } (x, \xi) \in \Gamma \text{ there exists some } \mu \in Q_0 \text{ such} \\ \text{that } \mu(x, \xi) \leq \mu < \infty . \end{array} \right.$$

In what follows we denote the norm in Sobolev space  $H^s$  by  $\|\cdot\|_s$ . We write  $\|\cdot\|=\|\cdot\|_0$ .

**Theorem 1.1.** *Let (1.1)–(1.3) hold and let  $L(x, D_x)$  in (1.4) satisfy (1.5)–(1.7), (1.10) and (1.12). Then for any compact set  $K$  of  $R^n$  there exists a constant  $C_K$  such that*

$$(1.13) \quad \|u\|_{l-2+\sigma'} + \|P'u\|_\sigma \leq C_K(\|Lu\| + \|u\|), \quad u \in C_0^\infty(K),$$

where  $\sigma(P') = (p_1(x, \xi))^{l-1}, \sigma = (l-1)/l\mu, \sigma' = \sigma + 1/l$ .

REMARK 1. It follows from (1.13) that  $P$  is hypoelliptic. See Oleinik-Radkevich [11] and Morimoto [9, Theorem 2.2]. Theorem 2.2 of [9] is stated only for differential operators but its proof is also applicable to pseudo-differential operators.

REMARK 2. In differential operators we have the following example:

$$D_{x_1}^5 + D_{x_2}^4 + D_{x_3}^4 + i(x_1^2 x_2 (D_{x_2}^4 + D_{x_3}^4) + x_1^3 D_{x_2}^2 (D_{x_2}^2 + D_{x_3}^2)) \text{ in } R_x^3.$$

All conditions of Theorem 1.1 are satisfied. Specially, the condition (1.10) is satisfied at  $(0, \xi_0)$  with  $\xi_0=(0, 0, \xi_{03})$  as follows; the sign of  $\text{Im } a$  changes from –

to  $+$  along the modified-null-bicharacteristic curve of  $p_1$  through  $(0, \xi_0)$ . At the point  $(0, \xi_0)$  this example must be reduced to Egorov type. Note that Egorov's operator of principal type is not the differential operator but the pseudodifferential operator. (See the introduction of [3].) For this example we have  $\sigma=1/25$ .

REMARK 3. The condition (1.10) is delicate and necessary in general for the hypoellipticity of  $P$ . Indeed, the operator with replaced  $\text{Re } a = \xi_2^4 + \xi_3^4$  in the above example by  $(\xi_2 - \xi_3)^4 + \xi_3^4$  satisfies all conditions except (1.10), which is violated at  $(0, \xi_0)$  with  $\xi_0 = (0, 0, \xi_{03})$  and  $\xi_{03} > 0$ . Furthermore, we have

**Proposition 1.2.** *Differential operator  $\tilde{L}$*

$$(1.14) \quad \tilde{L} = D_{x_1}^5 + (D_{x_2} - D_{x_4})^4 + D_{x_3}^4 \\ + i(x_1^2 x_2 (D_{x_2}^4 + D_{x_3}^4) + x_1^3 D_{x_2}^2 (D_{x_2}^2 + D_{x_3}^2)) \text{ in } R^3,$$

is not hypoelliptic at the origin.

## 2. Reduction to localized operator

Let  $h(x) \in C_0^\infty(R_x^n)$  be 1 for  $|x| < 1/2$  and vanish for  $|x| > 1$ . Set  $h_\varepsilon(x) = h(x/\varepsilon)$  for a small  $\varepsilon > 0$ . For a  $f(x, \xi) \in \bar{S}^m$  and  $\gamma = (x_0, \bar{\xi}_0) \in R^n \times S^{n-1}$  we introduce a pseudodifferential operator  $F_{\gamma, \varepsilon, \lambda}$  with a parameter  $0 < \lambda \leq 1$  and a small  $\varepsilon > 0$  as follows:

$$(2.1) \quad F_{\gamma, \varepsilon, \lambda}(y, D_y)v \\ = \lambda^{-2m} \int e^{iy\eta} h_\varepsilon(\lambda y) f(x_0 + \lambda y, \bar{\xi}_0 + \lambda \eta) h_\varepsilon(\lambda \eta) \hat{v}(\eta) d\eta, \\ v \in S_y, \quad \hat{v}(\eta) = (2\pi)^{-n} v(\eta),$$

where  $\hat{v}$  denotes the Fourier transform of  $v$ . Obviously, for a fixed  $\varepsilon > 0$ ,  $\{\lambda^{2m} \sigma(F_{\gamma, \varepsilon, \lambda})(y, \eta); 0 < \lambda \leq 1\}$  is a bounded set of  $S_{0,0}^0$ . Furthermore we obtain for a sufficiently small  $\varepsilon > 0$

$$(2.2) \quad (F_{\gamma, \varepsilon, \lambda} v)(\lambda^{-1}(x - x_0)) \\ = e^{-i\lambda^{-2} x \cdot \bar{\xi}_0} h_\varepsilon(x - x_0) f(x, D_x) h_\varepsilon(\lambda^2 D_x - \bar{\xi}_0) u(x),$$

where  $\hat{u}(\xi) = \lambda^n \hat{v}(\lambda(\bar{\xi} - \lambda^{-2} \bar{\xi}_0)) \exp(i x_0(\lambda^{-2} \bar{\xi}_0 - \xi))$ .

**Lemma 2.1.** *For any compact set  $K \subset R^n$  there exists a constant  $C_K$  such that (1.13) holds if and only if for any  $\gamma = (x_0, \bar{\xi}_0) \in R^n \times S^{n-1}$  one can find positives  $\varepsilon_i = \varepsilon_{i, \gamma}$  ( $i = 1, 2, 3$ ,  $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$ ) and a constant  $C_\gamma$  so that for any  $0 < \lambda \leq 1$  the following estimate holds;*

$$(2.3) \quad \lambda^{-2(l-2+\sigma')} \|H_{\varepsilon_1, \lambda} v\| + \lambda^{-2\sigma} \|P'_{\gamma, \varepsilon_1, \lambda} v\| \\ \leq C_\gamma (\|P_{\gamma, \varepsilon_2, \lambda} + A_{\gamma, \varepsilon_2, \lambda}\| v\| + \|P'_{\gamma, \varepsilon_3, \lambda} v\| \\ + \lambda^{-2(l-2+\sigma)} \|v\|), \quad v \in S_y,$$

where  $H_{\varepsilon,\lambda}, P_{\gamma,\varepsilon,\lambda}, P'_{\gamma,\varepsilon,\lambda}$  and  $A_{\gamma,\varepsilon,\lambda}$  are defined by (2.1) with  $f$  replaced by  $1, p_1^1, p_1^{1-1}$  and  $a$ , respectively.

Proof. In view of (2.2) it is not difficult to see the necessity of (2.3). We only show the sufficiency. The proof is the same way as in [4, p. 143], except the appearance of the second term in left hand side of (1.13) or (2.3). Note that for any  $\bar{\xi}_0(|\bar{\xi}_0|=1)$  and any small  $\varepsilon>0$  and any real  $s$

$$(2.4) \quad \begin{aligned} C^{-1} \|h_\varepsilon(\lambda D_y)v\| &\leq \|(\bar{\xi}_0 + \lambda D_y)^s h_\varepsilon(\lambda D_y)v\| \\ &\leq C \|h_\varepsilon(\lambda D_y)v\|, \quad v \in S_y, \end{aligned}$$

holds for some  $C=C_{s,\varepsilon}$  since  $C^{-1} \leq |\bar{\xi}_0 + \xi|^s \leq C$  on  $\text{supp } h_\varepsilon(\xi)$ . Substituting  $\hat{v}(\eta) = h_{\varepsilon_0}(\lambda \eta) \hat{u}(\lambda^{-1}\eta + \lambda^{-2}\bar{\xi}_0) \exp(ix_0(\lambda^{-1}\eta + \lambda^{-2}\bar{\xi}_0))$  for  $u \in C_0^\infty(K)$  and some  $\varepsilon_0 > \varepsilon_3$  into (2.3), we obtain by means of (2.2) and (2.4)

$$\begin{aligned} &\|h_1(x-x_0)h_1(\lambda^2 D_x - \bar{\xi}_0) |D_x|^{l-2+\sigma'} u\| \\ &\quad + \|h_1(x-x_0)P'(x, D_x) |D_x|^\sigma h_1(\lambda^2 D_x - \bar{\xi}_0) u\| \\ &\leq C (\|h_2(x-x_0)(P(x, D_x) + A(x, D_x))h_2(\lambda^2 D_x - \bar{\xi}_0) u\| \\ &\quad + \|h_3(x-x_0)P'(x, D_x)h_3(\lambda^2 D_x - \bar{\xi}_0) u\| \\ &\quad + \|h_0(\lambda^2 D_x - \bar{\xi}_0) |D_x|^{l-2+\sigma} u\|), \end{aligned}$$

where  $h_j = h_{\varepsilon_j}$ . Here we used the fact that, for a fixed  $\varepsilon > 0, \{\lambda^{-2}[h_\varepsilon(\lambda y), h_\varepsilon(\lambda D_y)]; 0 < \lambda \leq 1\}$  is a bounded set of  $S_{0,0}^0$ . Since  $[P, h_2(\lambda^2 D_x - \bar{\xi}_0)]$  can be estimated by the second term of the left hand side, the proof is completed from the following proposition and usual finite covering argument over  $K \times S^{n-1}$ .

**Proposition 2.2.** *Let  $h(\xi) \in C_0^\infty(R^n)$  be 1 in a neighborhood of 0 and let  $\bar{\xi}_0$  belong to  $S^{n-1}$ . Then one can find some  $\psi_j(\xi) \in \bar{S}^0 (j=1,2)$  such that*

(2.5)  $\psi_j(\bar{\xi}_0) \neq 0, \text{supp } \psi_j \subset \text{some conic neighborhood of } \bar{\xi}_0 \text{ and for any } N \text{ we have for some constant } C > 0$

$$\begin{aligned} &C^{-1} \|\psi_1(D_x)u\|^2 \\ &\leq \int_0^1 \|h(\lambda^2 D_x - \bar{\xi}_0)u\|^2 / \lambda \, d\lambda + \|u\|_{-N} \\ &\leq C (\|\psi_2(D_x)u\|^2 + \|u\|_{-N}), \quad u \in \mathcal{S}_x \end{aligned}$$

Proof. Put  $r = |\xi|, \theta = \xi/|\xi|$ . Then

$$\begin{aligned} &\int_0^1 \|h(\lambda^2 D_x - \bar{\xi}_0)u\|^2 / \lambda \, d\lambda \\ &= \int d\theta \int_0^1 d\lambda \int_0^\infty h(\lambda^2 r \theta - \bar{\xi}_0)^2 |\hat{u}(r\theta)|^2 / \lambda \, dr. \end{aligned}$$

It is easy to see that  $\text{supp } h(\lambda^2 r \theta - \bar{\xi}_0)$  is evaluated from above and below by

$$\{(\theta, r, \lambda); \theta \in \text{supp } \psi_j \cap S^{n-1} \text{ and } C_j^{-1} \leq r\lambda^2 \leq C_j\}$$

for some  $\psi_j \in \bar{S}^0$  satisfying (2.5) and some  $C_j > 0$  ( $j=1, 2$ ). Therefore the integral is bounded by constant times

$$\int \psi_j^2(\theta) d\theta \int_{1/C}^\infty |\hat{u}(r\theta)|^2 dr \int_{(Cr)^{-1/2}}^{(Cr)^{1/2}} d\lambda,$$

where we used  $(r/C)^{1/2} < 1/\lambda < (Cr)^{1/2}$  and  $C=C_j$ . This gives the desired estimate.

REMARK. The content of this proposition is briefly stated in [4, p. 143].

Since (2.3) is valid for  $\gamma \notin \Gamma$ , in view of Lemma 2.1 we now fix a  $\gamma \in \Gamma$ . Let a function  $f_\lambda(y, \eta) \in \mathcal{B}^\infty(R_y^n \times R_\eta^n)$  with a parameter  $0 < \lambda \leq 1$  satisfy

$$(2.6) \quad |\partial_y^\alpha \partial_\eta^\beta f_\lambda(y, \eta)| \leq C_{\alpha\beta} \lambda^{|\alpha+\beta|}$$

for any  $\alpha, \beta$ , where  $C_{\alpha\beta}$  is a constant independent of  $\lambda$ . Define a pseudodifferential operator  $F_\lambda(y, D_y)$  by

$$(2.7) \quad F_\lambda v = \int e^{iy\eta} f_\lambda(y, \eta) \hat{v}(\eta) d\eta, \quad v \in \mathcal{S}.$$

If  $f_\lambda(y, \eta)$  equals  $f(\lambda y, \lambda \eta)$  for some  $f(x, \xi) \in \mathcal{B}^\infty(R_x^n \times R_\xi^n)$ , then we say that the operator  $F_\lambda(y, D_y)$  has an original symbol  $f(x, \xi)$ . Under this notation, (2.3) for a fixed  $\gamma$  becomes

$$(2.8) \quad \begin{aligned} & \lambda^{-2(l-2+\sigma')} \|H_{\varepsilon_1, \lambda} v\| + \lambda^{-2(l-1+\sigma)} \|P'_{\varepsilon_1, \lambda} v\| \\ & \leq C (\|(\lambda^{-2l} P_{\varepsilon_2, \lambda} + \lambda^{-2(l-1)} A_{\varepsilon_2, \lambda}) v\| \\ & \quad + \lambda^{-2(l-1)} \|P'_{\varepsilon_3, \lambda} v\| + \lambda^{-2(l-2+\sigma)} \|v\|), \quad v \in \mathcal{S}, \end{aligned}$$

where the original symbols of  $H_{\varepsilon, \lambda}$ ,  $P_{\varepsilon, \lambda}$ ,  $P'_{\varepsilon, \lambda}$  and  $A_{\varepsilon, \lambda}$  are  $h_\varepsilon(x, \xi) = h_\varepsilon(x)h_\varepsilon(\xi)$ ,  $h_\varepsilon(p_{1, \gamma})^l$ ,  $h_\varepsilon(p_{1, \gamma})^{l-1}$  and  $h_\varepsilon a_\gamma$  respectively. Here  $p_{1, \gamma}(x, \xi) = p_1(x + x_0, \xi + \xi_0)$ ,  $a_\gamma(x, \xi) = a(x + x_0, \xi + \xi_0)$ .

**Lemma 2.3.** *If (2.8) is valid and  $\mathcal{X}$  is a  $C^\infty$  canonical transformation keeping 0 fixed which is defined near 0, then (2.8) remains valid with some other  $\varepsilon_j$  and  $C$  if  $P_{\varepsilon, \lambda}$ ,  $A_{\varepsilon, \lambda}$  and  $P'_{\varepsilon, \lambda}$  are replaced by  $\tilde{P}_{\varepsilon, \lambda}$ ,  $\tilde{A}_{\varepsilon, \lambda}$  and  $\tilde{P}'_{\varepsilon, \lambda}$ , respectively, whose original symbols are  $h_\varepsilon(p_{1, \gamma} \circ \mathcal{X})^l$ ,  $h_\varepsilon(a_\gamma \circ \mathcal{X})$  and  $h_\varepsilon(p_{1, \gamma} \circ \mathcal{X})^{l-1}$ , respectively.*

As pointed out in [4, the proof of Lemma 3.2] it suffices to prove the lemma when  $\mathcal{X}$  has a generating function  $S(x, \xi)$ , that is,  $\mathcal{X}; (x, d_x S(x, \xi)) \mapsto (d_\xi S(x, \xi), \xi)$ . The proof is based on several propositions on Fourier integral operators with phase function  $S_\lambda(y, \eta) = \lambda^{-2} S(\lambda y, \lambda \eta)$ .

**DEFINITION 2.4.** For any  $f_\lambda(y, \eta) \in \mathcal{B}^\infty$  with (2.6) and for any  $S(x, \xi) \in C^\infty(R_x^n \times R_\xi^n)$  satisfying

$$(2.9) \quad \det \partial_x \partial_{\xi} S(0, 0) \neq 0, \quad (d_x S(0, 0), d_{\xi} S(0, 0)) = (0, 0)$$

we define the Fourier integral operator  $F_{S_{\lambda}}(y, D_y)$  with a parameter  $0 < \lambda \leq 1$  by

$$(2.10) \quad F_{S_{\lambda}} v = \int e^{i S_{\lambda}(y, \eta)} k_{\lambda}(y, \eta) f_{\lambda}(y, \eta) \hat{\nu}(\eta) d\eta, \quad v \in \mathcal{S}$$

where  $S_{\lambda}(y, \eta) = \lambda^{-2} S(\lambda y, \lambda \eta)$  and  $k_{\lambda}(y, \eta) = k(\lambda y, \lambda \eta)$ . Here we assume that  $k(x, \xi) \in C^{\infty}$  is 1 in a neighborhood of 0 and  $\det \partial_x \partial_{\xi} S(x, \xi) \neq 0$  on  $\text{supp } k$ . We define the conjugate Fourier integral operator  $F_{S_{\lambda}^*}$  with a parameter  $0 < \lambda \leq 1$  by

$$(2.11) \quad F_{S_{\lambda}^*}(y, D_y) v = \iint e^{i(y\eta - S_{\lambda}(\tilde{y}, \eta))} k_{\lambda}(\tilde{y}, \eta) f_{\lambda}(\tilde{y}, \eta) v(\tilde{y}) d\tilde{y} d\eta, \quad v \in \mathcal{S},$$

We call  $f_{\lambda}(y, \eta)$  the symbol of  $F_{S_{\lambda}}(y, D_y)$  ( $F_{S_{\lambda}^*}(y, D_y)$ ), and moreover if  $f_{\lambda}(y, \eta) = f(\lambda y, \lambda \eta)$  for some  $f \in \mathcal{B}^{\infty}$ , then we call  $f(x, \xi)$  the original symbol of  $F_{S_{\lambda}}(F_{S_{\lambda}^*})$ . We write  $F_{S_{\lambda}} = I_{S_{\lambda}}, F_{S_{\lambda}^*} = I_{S_{\lambda}^*}$  if  $f = 1$ .

Put

$$(2.12) \quad \begin{aligned} \tilde{d}_{\xi} S(x, \tilde{\xi}, \xi) &= \int_0^1 d_{\xi} S(x, \tilde{\xi} + \theta(\xi - \tilde{\xi})) d\theta \\ \tilde{d}_x S(\tilde{x}, x, \xi) &= \int_0^1 d_x S(\tilde{x} + \theta(x - \tilde{x}), \xi) d\theta \end{aligned}$$

Put  $\tilde{x} = \tilde{d}_{\xi} S(x, \tilde{\xi}, \xi)$  and  $\tilde{\xi} = \tilde{d}_x S(\tilde{x}, x, \xi)$ . Then the inverses

$$(2.13) \quad x = \phi(\tilde{x}; \tilde{\xi}, \xi) \quad \text{and} \quad \xi = \psi(\tilde{\xi}; \tilde{x}, x)$$

exist, respectively, in a neighborhood of 0 on account of (2.9) if  $\text{supp } k$  is sufficiently small.

**Proposition 2.5.** *If  $\text{supp } k$  in (2.10) and (2.11) is sufficiently small, then  $I_{S_{\lambda}} I_{S_{\lambda}^*}$  and  $I_{S_{\lambda}^*} I_{S_{\lambda}}$  are pseudodifferential operators whose symbols are*

$$(2.14) \quad \int e^{-i\tilde{y}\tilde{\eta}} r_{\lambda}(y, \eta + \tilde{\eta}, y + \tilde{y}) d\tilde{y} d\tilde{\eta}$$

and

$$(2.15) \quad \iint e^{-i\tilde{y}\tilde{\eta}} r_{\lambda}^*(\eta + \tilde{\eta}, y + \tilde{y}, \eta) d\tilde{y} d\tilde{\eta}$$

respectively, where  $r_{\lambda}(y, \tilde{\eta}, \tilde{y})$  and  $r_{\lambda}^*(\eta, \tilde{y}, \tilde{\eta})$  are given by

$$(2.16) \quad (k(x, \xi) k(\tilde{x}, \tilde{\xi}) | \det \int_0^1 \partial_x \partial_{\xi} S(\tilde{x} + \theta(x - \tilde{x}), \xi) d\theta |^{-1}) \quad \xi = \psi(\tilde{\xi}; \tilde{x}, x)$$

and

$$(2.17) \quad (k(x, \xi) k(x, \tilde{\xi}) | \det \int_0^1 \partial_x \partial_{\xi} S(x, \tilde{\xi} + \theta(\xi - \tilde{\xi})) d\theta |^{-1}) \quad x = \psi(\tilde{x}; \tilde{\xi}, \xi),$$

respectively. Here  $(x, \tilde{x}, \xi, \tilde{\xi}) = (\lambda(y, \tilde{y}, \eta, \tilde{\eta}))$ .



Proof is directly calculated by means of the change of variable;  $\tilde{\xi} = \tilde{d}_x S(x, x, \xi)$  and  $\tilde{x} = \tilde{d}_\xi S(x, \xi, \xi)$  respectively.

**Corollary 2.6.** *The operators  $F_{S_\lambda}$  and  $F_{S_\lambda}^*$  for any  $f_\lambda$  with (2.6) are  $L_2$ -bounded uniformly with respect to  $0 < \lambda \leq 1$ . If  $\varepsilon$  is small enough, then  $I_{S_\lambda} I_{S_\lambda}^*$  and  $I_{S_\lambda}^* I_{S_\lambda}$  are elliptic on  $\text{supp } h_\varepsilon(\lambda y, \lambda \eta)$ , that is, the estimates*

$$(2.18) \quad \|H_{\varepsilon, \lambda} v\| \leq C_1 (\|I_{S_\lambda} I_{S_\lambda}^* v\| + \lambda^2 \|v\|),$$

$$(2.19) \quad \|H_{\varepsilon, \lambda} v\| \leq C_2 (\|I_{S_\lambda}^* I_{S_\lambda} v\| + \lambda^2 \|v\|), \quad v \in \mathcal{S},$$

hold for some constants  $C_1$  and  $C_2$ .

Proof. The symbols of  $F_{S_\lambda}^* F_{S_\lambda}$  and  $F_{S_\lambda} F_{S_\lambda}^*$  are given by the versions of (2.14) and (2.15), respectively, which belong to a bounded set of  $S_{0,0}^0$  uniformly on account of (2.6). The boundedness of  $F_{S_\lambda}^* F_{S_\lambda}$  and  $F_{S_\lambda} F_{S_\lambda}^*$  show the first statement. Note that for any  $p(x, \tilde{\xi}, \tilde{x}, \xi) \in \mathcal{B}^\infty(\mathbb{R}^{4n})$

$$(2.20) \quad O_s - \iint e^{-i\tilde{x}\tilde{\xi}} p(x, \tilde{\xi}, \tilde{x}, \xi) d\tilde{x} d\tilde{\xi} = \sum_{|\alpha| \leq N} p_\alpha(x, 0, 0, \xi) / \alpha! + N \sum_{|\beta| = N} \int_0^1 (1-\theta)^{N-1} \\ O_s - \iint e^{-i\tilde{x}\tilde{\xi}} p_\beta(x, \theta\tilde{\xi}, \tilde{x}, \xi) d\tilde{x} d\tilde{\xi} d\theta / \beta!$$

holds for any positive integer  $N$ , where  $p_\alpha(x, \tilde{\xi}, \tilde{x}, \xi) = \partial_{\tilde{\xi}}^\alpha \partial_{\tilde{x}}^\alpha p(x, \tilde{\xi}, \tilde{x}, \xi)$ . Here  $O_s - \iint$  denotes the oscillatory integral (see [5, p. 42]). Applications of (2.20) to (2.14) and (2.15) yield (2.18) and (2.19), respectively.

**Proposition 2.7.** *Let  $F_\lambda(y, D_y)$  be the pseudodifferential operator with original symbol  $f(x, \xi) \in \mathcal{B}^\infty$ . Let  $G_{S_\lambda}$  and  $\tilde{G}_{S_\lambda}$  be Fourier integral operators whose original symbols are  $f(x, d_x S(x, \xi))$  and  $f(d_\xi S(x, \xi), \xi)$ , respectively. Then  $\lambda^{-2}(F_\lambda I_{S_\lambda} - G_{S_\lambda})$  and  $\lambda^{-2}(I_{S_\lambda} F_\lambda - \tilde{G}_{S_\lambda})$  are  $L_2$ -bounded operators uniformly with respect to  $0 < \lambda \leq 1$ .*

Proof. It is easy to check that the equations

$$F_\lambda I_{S_\lambda} v = \int e^{iS_\lambda(y, \eta)} r_\lambda(y, \eta) \hat{v}(\eta) d\eta$$

and

$$I_{S_\lambda} F_\lambda v = \int e^{iS_\lambda(y, \eta)} \tilde{r}_\lambda(y, \eta) \hat{v}(\eta) d\eta,$$

for

$$(2.21) \quad r_\lambda(y, \eta) = O_s - \iint e^{-i\tilde{y}\tilde{\eta}} f(x, \tilde{\xi} + \tilde{d}_x S(x, x + \tilde{x}, \xi)) k(x + \tilde{x}, \xi) d\tilde{y} d\tilde{\eta}$$

and

$$(2.22) \quad \tilde{r}_\lambda(y, \eta) = O_s - \iint e^{-i\tilde{y}\tilde{\eta}} f(\tilde{x} + \tilde{d}_\xi S(x, \xi + \tilde{\xi}, \xi), \xi) k(x, \xi + \tilde{\xi}) d\tilde{y} d\tilde{\eta},$$

respectively, where  $(x, \tilde{x}, \xi, \tilde{\xi}) = \lambda(y, \tilde{y}, \eta, \tilde{\eta})$ . Applications of (2.20) to (2.21) and (2.22) complete the proof.

**Corollary 2.8.** For  $f(x, \xi) \in C_0^\infty$  with support contained a sufficiently small neighborhood of 0, set  $\tilde{f}(x, \xi) = (f \circ \mathcal{X})(x, \xi)$ , where  $\mathcal{X}$  is defined by

$$(x, d_x S(x, \xi)) \rightarrow (d_\xi S(x, \xi), \xi).$$

Then  $\lambda^{-2}(F_\lambda I_{S_\lambda} - I_{S_\lambda} \tilde{F}_\lambda)$  is  $L_2$ -bounded uniformly for  $0 < \lambda \leq 1$ . Furthermore, if  $f(x, \xi) = (p_1(x, \xi))' h_\varepsilon(x, \xi)$  then for  $\tilde{f} = (p_1' h_\varepsilon) \circ \mathcal{X}$ , the estimate

$$(2.23) \quad \|(F_\lambda I_{S_\lambda} - I_{S_\lambda} \tilde{F}_\lambda)v\| \leq C(\lambda^2 \|F_\lambda' I_{S_\lambda} v\| + \lambda^2 \|I_{S_\lambda} \tilde{F}_\lambda' v\|), \quad v \in \mathcal{S},$$

holds for some constant  $C$ , where the original symbols of  $F_\lambda'$  and  $\tilde{F}_\lambda'$  are  $(p_1)^{t-1} h_{\varepsilon'}$  and  $((p_1)^{t-1} h_{\varepsilon'}) \circ \mathcal{X}$  for some  $\varepsilon' > \varepsilon$ .

Proof. The first part follows from Proposition 2.7. The second part is obtained by checking the second terms of the expansions of (2.21) and (2.22).

REMARK. It is clear that  $\lambda^{-2}(I_{S_\lambda}^* F_\lambda - F_\lambda I_{S_\lambda}^*)$  is  $L_2$ -bounded operator uniformly for  $0 < \lambda \leq 1$ . Indeed, this follows from  $\|F_\lambda I_{S_\lambda} - I_{S_\lambda} \tilde{F}_\lambda\| = \|I_{S_\lambda}^* F_\lambda^* - \tilde{F}_\lambda^* I_{S_\lambda}^*\|$  and the fact that  $\lambda^{-2}(F_\lambda - F_\lambda^*)$  and  $\lambda^{-2}(\tilde{F}_\lambda - \tilde{F}_\lambda^*)$  are  $L_2$ -bounded.

Proof of Lemma 2.3. Taking  $I_{S_\lambda} v$  as  $v$  in (2.8) and noting Corollary 2.8, we obtain (2.8) for operators transformed, by using the fact that for any small  $\varepsilon > 0$  the estimate

$$C^{-1} \|H_{\varepsilon', \lambda} v\| \leq \|I_{S_\lambda} H_{\varepsilon, \lambda} v\| + \lambda^2 \|v\| \leq C(\|H_{\varepsilon'', \lambda} v\| + \lambda^2 \|v\|), \quad v \in \mathcal{S},$$

holds for some  $0 < \varepsilon' < \varepsilon < \varepsilon''$  and some constant  $C$ , which follows from (2.19).

Now we take a canonical transformation  $\mathcal{X}$  such that  $p_{1, \gamma} \circ \mathcal{X} = \xi_1$  and  $q_{0, \gamma} \circ \mathcal{X} = x_1$ , where  $q_{0, \gamma}$  is defined from  $q_0$  given in (1.5) by the same way as in  $p_{1, \gamma}$ . Darboux theorem (see [7, Proposition 3.1]) shows that (1.6) guarantees the existence of such a  $\mathcal{X}$ . Application of Lemma 2.3 gives

**Lemma 2.9.** The estimate (2.8) is valid if for some  $\varepsilon$  and  $\varepsilon'$  ( $\varepsilon' > \varepsilon > 0$ ) the estimate

$$(2.24) \quad \begin{aligned} & \lambda^{-2(t-2+\sigma')} \|h_\varepsilon(\lambda y', \lambda D_{y'}) v\| + \lambda^{-2\sigma} \|D_{y_1}^{t-1} h_\varepsilon(\lambda y', \lambda D_{y'}) v\| \\ & \leq C(\|(D_{y_1}^t + \lambda^{-2(t-1)} \tilde{A}_{\varepsilon', \lambda}(y, D_{y'}))v\| \\ & \quad + \|D_{y_1}^{t-1} v\| + \lambda^{-2(t-2+\sigma)} \|v\|), \\ & \text{if } v \in \mathcal{S} \text{ vanishes for } |y_1| > \varepsilon, \end{aligned}$$

holds for some  $C$ , where the symbol of  $\tilde{A}_{\varepsilon, \lambda}(y, D_{y'})$  is  $\tilde{a}(y_1, \lambda y', \lambda \eta') h_\varepsilon(\lambda y', \lambda \eta')$ . Here  $\tilde{a}(x, \xi') = (a_\gamma \circ \mathcal{X})(x, 0, \xi')$ .

Proof. Condition (1.7) leads us to the

$$(2.25) \quad a_{\gamma \circ \mathcal{X}}(x, \xi) = a_{\gamma \circ \mathcal{X}}(x, 0, \xi') + b(x, \xi) \xi_1^{l-1}$$

for some  $b$  near origin since  $H_{q_0}$  is transformed to  $\partial_{\xi_1}$ . Therefore, substituting  $h_{\varepsilon}(y_1, \lambda^2 D_{y_1}) h_{\varepsilon'}(\lambda y', \lambda D_{y'}) v$  ( $\varepsilon < \varepsilon' < \varepsilon'$ ) for  $v$  of (2.24) with replaced  $\tilde{a}$  by  $a_{\gamma \circ \mathcal{X}}(x, \xi)$  and changing variable  $(y_1, \eta_1)$  into  $(\lambda y_1, \lambda^{-1} \eta_1)$  we get the estimate (2.8) canonically transformed.

**Proposition 2.10.** *Let  $g_{\lambda}(y, \eta) \in C^{\infty}$  with a parameter  $0 < \lambda \leq 1$  satisfy for any  $\alpha, \beta$*

$$(2.26) \quad |\partial_{\eta}^{\alpha} \partial_y^{\beta} g_{\lambda}| \leq C_{\alpha\beta} (|\eta_1|^2 + \lambda^{-4\delta})^{(m-|\alpha_1|)/2} \lambda^{|\alpha'| + \beta'}$$

where  $0 < \delta \leq 1$  and  $m$  integer. Assume that, for some  $h_{\lambda}(y', \eta') = h(\lambda y', \lambda \eta')$ ,  $g_{\lambda}$  satisfies for some  $c_0 > 0$

$$(2.27) \quad |g_{\lambda}| \geq c_0 (|\eta_1|^2 + \lambda^{-4\delta})^{m/2} \text{ on } \{|y_1| < 1\} \times \text{supp } h_{\lambda}.$$

Then we get

$$(2.28) \quad \begin{aligned} & \|D_{y_1}^m h_{\lambda}(y', D_{y'}) v\| + \lambda^{-2\delta m} \|h_{\lambda}(y', D_{y'}) v\| \\ & \leq C (\|G_{\lambda}(y, D_y) v\| + \|v\|), \quad \text{if } v \in \mathcal{S} \text{ vanishes } |y_1| > 1. \end{aligned}$$

Proof is omitted. (for example, see [5, p. 77]).

Applying this proposition with  $m=l$  and  $\delta=(l-1)/l$  to  $\eta_1^l + \lambda^{-2(l-1)} \tilde{a}(y_1, \lambda y', \lambda \eta') h_{\varepsilon}(\lambda y', \lambda \eta')$ , we obtain (2.24) if  $\text{Im } \tilde{a}(0) \neq 0$ . From now on we assume  $\text{Im } \tilde{a}(0) = 0$ . Let  $\omega(x, \xi')$  be a  $l$  power root of  $(\tilde{a} h_{\varepsilon})(x, \xi')$  such that  $\omega(0)$  is real (since  $\text{Re } \tilde{a}(0) \neq 0$  by (1.2)).

Then we obtain the factorization

$$\xi_1^l + \tilde{a} h_{\varepsilon} = (\xi_1 + \omega) \sum_{j=1}^{l-1} (-\omega)^{j-1} \xi_1^{l-j-1}$$

Set  $\omega_{\lambda}(y, \eta') = \omega(y_1, \lambda y', \lambda \eta')$  and set  $l_{2,\lambda}(y, \eta) = \sum_{j=1}^{l-1} (-\lambda^{-2\delta} \omega_{\lambda})^{j-1} \eta_1^{l-j}$  ( $\delta=(l-1)/l$ ).

Since  $l_{2,\lambda}(y, \eta)$  satisfies (2.27) with  $m=l-1$  and  $\delta=(l-1)/l$ , we get (2.24) if we show for some  $C$

$$(2.29) \quad \begin{aligned} & \lambda^{-2\sigma} \|h_{\varepsilon,\lambda}(y', D_{y'}) v\| \leq C (\|(D_{y_1} + \lambda^{-2\delta} \omega_{\lambda}(y, D_{y'})) v\| + \|v\|), \\ & \text{if } v \in \mathcal{S} \text{ vanishes } |y_1| > \varepsilon. \end{aligned}$$

For brevity we denote  $\tilde{a}(x, \xi') h_{\varepsilon'}(x', \xi')$  by  $\tilde{a}(x, \xi')$  in what follows. Note that  $\text{Re } \tilde{a}$  is independent of  $x_1$  on account of (1.3) because  $H_{p_1}$  and  $\Gamma$  were transformed to  $\partial_{x_1}$  and  $\xi_1=0$ , respectively, by the  $\mathcal{X}$ . Using the expansion  $(1+z)^{1/l} = 1 + z/l + O(z)^2$ , we obtain

$$(2.30) \quad \begin{aligned} \omega(x, \xi') &= (\operatorname{Re} \tilde{a})^{l/l}(1 + \operatorname{Im} \tilde{a} / \operatorname{Re} \tilde{a})^{l/l} \\ &= r(x', \xi') + iq(x, \xi') + \mathcal{O}(|x| + |\xi'|)q(x, \xi') \end{aligned}$$

if we set  $r = (\operatorname{Re} \tilde{a})^{l/l}$ ,  $q = \operatorname{Im} \tilde{a} / (\operatorname{Re} \tilde{a})^{(l-1)/l}$ . Hence (2.29) follows if we show that for some  $C$

$$(2.31) \quad \begin{aligned} &\lambda^{-2\sigma} \|h_{\varepsilon, \lambda}(y', D_{y'})v\| + \lambda^{-2\delta} \|q_{\lambda}(y, D_y)h_{\varepsilon, \lambda}(y', D_{y'})v\| \\ &\leq C(\|D_{y_1} + \lambda^{-2\delta}(r_{\lambda}(y', D_{y'}) + iq_{\lambda}(y, D_{y'}))v\| + \|v\|), \\ &\text{if } v \in \mathcal{S} \text{ vanishes for } |y_1| > \varepsilon, \end{aligned}$$

where  $q_{\lambda}(y, \eta') = q(y_1, \lambda y', \lambda \eta')$  and  $r_{\lambda}(y', \eta') = r(\lambda y', \lambda \eta')$ . Indeed, if we take  $h_{\varepsilon'}(x', \xi')$  such that  $h_{\varepsilon}(x', \xi') = 1$  on  $\operatorname{supp} h_{\varepsilon'}$  and substitute  $h_{\varepsilon', \lambda}v$  into (2.31), then we get (2.29) with replaced  $v$  by  $h_{\varepsilon', \lambda}v$  since the part corresponding to third term of the left hand side of (2.30) can be estimated by the second term of the right hand side of (2.31) when  $\varepsilon$  is small enough. (Note that  $\varepsilon'' < \varepsilon$ ).

Let  $\Phi(x, \xi')$  be the solution to

$$(2.32) \quad \partial_{x_1} \Phi + r(x', d_{x'} \Phi) = 0, \quad \Phi(0, x', \xi') = x' \xi'$$

Without loss of generality we assume that the  $\Phi$  exists on  $\{|x_1| < \varepsilon\} \times \operatorname{supp} h_{\varepsilon}(x', \xi')$ . Put  $\Phi_{\lambda}(y, \eta') = \lambda^{-2} \Phi(\lambda^{2/l} y_1, y', \lambda \eta')$ . Then  $\Phi_{\lambda}$  of course exists on  $\{|y_1| < \varepsilon\} \times \operatorname{supp} h_{\varepsilon, \lambda}(y', \eta')$ . If we regard  $y_1$  as a parameter, in the same way as in (2.10) and (2.11), we can define the Fourier integral operator and the conjugate Fourier integral operator with phase function  $\Phi_{\lambda}(y, \eta')$  and the symbol  $f_{\lambda}(y, \eta')$  satisfying

$$(2.6') \quad |\partial_y^{\alpha} \partial_{\eta'}^{\beta'} f_{\lambda}| \leq C_{\alpha \beta'} \lambda^{|\alpha' + \beta'|}$$

by

$$(2.33) \quad F_{\Phi_{\lambda}} v(y) = \int e^{i\Phi_{\lambda}(y, \eta')} k_{\lambda}(y', \eta') f_{\lambda}(y, \eta') \check{v}(y_1, \eta') d\eta'$$

and

$$(2.34) \quad \begin{aligned} F_{\Phi_{\lambda}}^* v(y) &= \iint e^{i(y' \eta' - \Phi_{\lambda}(y_1, \tilde{y}', \eta'))} k_{\lambda}(\tilde{y}', \eta') \\ &\quad f_{\lambda}(y_1, \tilde{y}', \eta') v(y_1, \tilde{y}') d\tilde{y}' d\eta', \end{aligned}$$

when  $v \in \mathcal{S}$  vanishes for  $|y_1| > \varepsilon$ . Here  $\check{v}(y_1, \eta')$  denotes the Fourier transform of  $v(y)$  with respect to  $y'$ . Set

$$(2.35) \quad \Psi_{\lambda}(y, \eta) = y_1 \eta_1 + \Phi_{\lambda}(y, \eta').$$

Let  $\mathcal{X}_{\lambda}$  denote a canonical transformation with generating function  $\Psi_{\lambda}$ , that is,  $\mathcal{X}_{\lambda}; (y, d_y \Psi_{\lambda}(y, \eta)) \mapsto (d_{\eta} \Psi_{\lambda}(y, \eta), \eta)$ . Note that  $\mathcal{X}_{\lambda}$  and  $\mathcal{X}_{\lambda}^{-1}$  are defined for  $\{|y_1| < \varepsilon\} \times \operatorname{supp} h_{\varepsilon, \lambda}(y', \eta')$  if  $\varepsilon$  is small enough. Set  $\tilde{q}_{\lambda}(y, \eta') = q_{\lambda} \circ \mathcal{X}_{\lambda}(y, \eta')$ . It is easy to check that

$$(2.36) \quad \tilde{q}_{\lambda}(y, \eta') = q(y_1, \psi_{\lambda}(y, \eta'), d_{x'} \Phi(\lambda^{2/l} y_1, \psi_{\lambda}(y, \eta'), \lambda \eta'))$$

where  $\psi_\lambda(y, \eta') = \psi(\lambda y'; \lambda^{2/l} y_1, \lambda \eta')$  and  $\psi(x'; x_1, \xi')$  is defined as the inverse of  $x' = d_{\xi'} \Phi(x_1, \cdot, \xi')$ .

**Lemma 2.11.** *Assume that for some  $\varepsilon_1 > 0$  and some constant  $C_1$  the estimate*

$$(2.37) \quad \begin{aligned} & \lambda^{-2\sigma} \|h_{\varepsilon_1, \lambda}(y', D_{y'})v\| + \lambda^{-2\delta} \|\tilde{q}_\lambda(y, D_{y'})h_{\varepsilon_1, \lambda}(y', D_{y'})v\| \\ & \leq C_1 (\|(D_{y_1} + i\lambda^{-2\delta} \tilde{q}_\lambda(y, D_{y'}))v\| + \|v\|) \end{aligned}$$

holds if  $v \in \mathcal{S}$  vanishes for  $|y_1| > \varepsilon_1$ . Then (2.3) holds for some  $\varepsilon > 0$  and  $C$ .

*Proof.* Fix  $y_1$  as a parameter and let  $\Phi_\lambda(y_1, y', \eta')$  correspond to  $S_\lambda(y, \eta)$  in Definition 2.4. By the remark after Corollary 2.8, we see that the  $\lambda^{-2}(I_{\Phi_\lambda}^* q_\lambda - \tilde{q}_\lambda I_{\Phi_\lambda}^*)$  as an operator on  $L_2(R_{y'}^{n-1})$  has a uniform bound with respect to  $|y_1| < \varepsilon$  and  $0 < \lambda \leq 1$ , therefore it has a uniform bound as an operator on  $L_2([- \varepsilon, \varepsilon] \times R_{y'}^{n-1})$  by integrating with respect to  $y_1$ .

Note that

$$\begin{aligned} D_{y_1} I_{\Phi_\lambda}^* v &= I_{\Phi_\lambda}^* D_{y_1} v - (\partial_{y_1} \Phi_\lambda)_{\Phi_\lambda}^* v \\ &= I_{\Phi_\lambda}^* D_{y_1} v + \lambda^{-2\delta} T_{\Phi_\lambda}^* v, \\ & \text{if } v \in \mathcal{S} \text{ vanishes for } |y_1| > \varepsilon, \end{aligned}$$

where  $t_\lambda(y, \eta') = r_\lambda(y', d_{y'} \Phi_\lambda(y, \eta'))$ . In fact this follows from  $\partial_{y_1} \Phi_\lambda(y, \eta') = -\lambda^{-2\delta} r_\lambda(y', d_{y'} \Phi_\lambda(y, \eta'))$ . The adjoint form of Proposition 2.7 shows that the second term equals  $I_{\Phi_\lambda}^* \lambda^{-2\delta} R_\lambda$  modulo  $L_2$ -bounded operator. Hence, substitution  $I_{\Phi_\lambda}^* v$  for  $v$  of (2.37) gives (2.31) because for  $\tilde{h}_{\varepsilon, \lambda}(y_1, y', \eta')$  defined from  $h_\varepsilon(x', \xi')$  in the same way as (2.36), there exists some  $\varepsilon'$  such that  $\tilde{h}_{\varepsilon, \lambda} = 1$  on  $\text{supp } h_\varepsilon(\lambda y', \lambda \eta') \times \{|y_1| < \varepsilon\}$ , provided that  $0 < \lambda \leq \lambda_0$  for some sufficiently small  $\lambda_0$ .

In the rest of this section we investigate the properties of  $\tilde{q}_\lambda(y, \eta')$  derived from the assumptions. It follows from (2.36) that for any  $\alpha, \beta'$  and some  $C_{\alpha\beta'}$  independent of  $\lambda$

$$(2.38) \quad \begin{aligned} & |\partial_{y'}^\alpha \partial_{\eta'}^{\beta'} \tilde{q}_\lambda(y, \eta')| \leq C_{\alpha\beta'} \lambda^{|\alpha'| + \beta'|} \\ & \text{on } \Omega_\varepsilon = \{|y_1| < \varepsilon, \quad |y'| + |\eta'| < \varepsilon \lambda^{-1}\}. \end{aligned}$$

The second property is that

$$(2.39) \quad \begin{aligned} & \tilde{q}_\lambda(y, \eta') \text{ does not change sign from } + \text{ to } - \text{ for} \\ & y_1 \text{ increasing if } \lambda \text{ is small enough.} \end{aligned}$$

This follows from (1.10). In fact, since it follows from (2.25) that

$$d_{x, \xi}(\text{Re } a_{\gamma \circ \mathcal{X}})^{1/l}(x, 0, \xi') = d_{x, \xi}(\text{Re } a_{\gamma \circ \mathcal{X}})^{1/l}(x, 0, \xi')$$

we obtain the property (2.39) because the modified-null-bicharacteristic curve is invariant under canonical transformations  $\mathcal{X}$  and  $\mathcal{X}_\lambda$ . The invariance of

Poisson brackets for canonical transformations and (2.25) give the following;

$$(2.40) \quad \left\{ \begin{array}{l} \text{for any } (y, \eta') \in \Omega_\varepsilon = \{|y_1| < \varepsilon, |y'| + |\eta'| < \varepsilon\lambda^{-1}\} \\ \text{there exists a } I \in \mathcal{I}_\mu \text{ such that } \tilde{p}_{I,\lambda}(y, \eta') \neq 0 \text{ if} \\ \lambda \text{ is small enough.} \end{array} \right.$$

Here  $\tilde{p}_{I,\lambda}(y, \eta')$  is defined in the same way as  $p_I(x, \xi)$  of (1.11) with  $p_1 = \eta_1$  and  $p_2 = \lambda^{-2\delta} \tilde{q}_\lambda(y, \eta')$ . Indeed, note that (1.12) is invariant under canonical transformations and in changing  $p_i (i=1, 2)$  to  $f_i p_i$  for non-vanishing functions  $f_i (i=1, 2)$ . By means of (2.25) and the definition of  $\mathcal{I}$ , we see that for any  $(y, \eta') \in \Omega_\varepsilon$  there exist a  $I \in \mathcal{I}_\mu$  and  $c_I > 0$  such that

$$|p_{I,\lambda}^0(y, \eta')| \geq c_I \lambda^{-2+2b(I)/l},$$

where  $p_{I,\lambda}^0$  is defined by setting  $p_1 = \eta_1$  and  $p_2 = \lambda^{-2\delta} q_\lambda(y, \eta')$ . If  $p_{I,\lambda}$  denotes  $p_{I,\lambda}^0$  with replaced  $\eta_1$  by  $\eta_1 + \lambda^{-2\delta} r_\lambda(y', \eta')$ , we obtain

$$|p_{I,\lambda} - p_{I,\lambda}^0| \leq C_I \lambda^{-2\delta+2b(I)/l}$$

for some constant  $C_I$  determined by the derivatives of  $r$  and  $q$ . Consequently, the invariance of Poisson brackets under  $\mathcal{X}_\lambda$  gives (2.40).

### 3. Proof of Theorem 1.1

In order to prove Theorem 1.1, as observed in the preceding section, it suffices to show (2.37). For the sake of simplicity, we denote  $\lambda^{-2\delta} \tilde{q}_\lambda(y, \eta')$  by  $q(x, \xi')$ . Suppose that  $\lambda$  is small enough. By means of a constant scale change in the variables, we may assume from (2.38)–(2.40) that  $q(x, \xi')$  satisfies the following conditions: For any  $\alpha, \beta$  and some constant  $C_{\alpha\beta}$  (independent of  $\lambda$ )

$$(3.1) \quad |D_\xi^\alpha D_x^\beta q(x, \xi')| \leq C_{\alpha\beta} \lambda^{|\alpha'| + \beta' - 2\delta} \quad \text{in } \Omega = \{|x_1| < 1, |(x', \xi')| < \lambda^{-1}\}$$

$$(3.2) \quad q(x, \xi') \text{ does not change sign from } + \text{ to } - \text{ for } x_1 \text{ increasing.}$$

$$(3.3) \quad \text{For any } (x, \xi') \in \Omega \text{ there exist some } \mu \in Q_0 \text{ and some } c_0 > 0 \text{ such that}$$

$$\sum_{I \in \mathcal{I}_\mu} \lambda^{2-2b(I)/l} |p_I(x, \xi')| \geq c_0 > 0$$

provided that  $p_1 = \xi_1, p_2 = q(x, \xi')$ . Then (2.37) is stated as follows; for some  $C$

$$(3.4) \quad \begin{aligned} &\lambda^{-2\sigma} \|h(\lambda x', \lambda D')u\| + \|D_{x_1} h(\lambda x', \lambda D')u\| \\ &\leq C (\|(D_{x_1} + iq(x, D'))u\| + \|u\|), \\ &\quad \text{if } u \in \mathcal{S} \text{ vanishes for } |x_1| > 1/2, \end{aligned}$$

where  $h$  is  $C^\infty$  with support in a ball of radius  $1/2$ .

The proof of (3.4) for  $q(x, \xi)$  with (3.1)–(3.3) is the same as in showing [4, (6.1) and (6.30)] except the difference of “weight”. (See [4, (4.1) and (4.3)].) To prove [4, (6.1) and (6.30)] it was important to obtain inequalities for  $q$  in [4, Section 4]. So we sketch the argument corresponding to [4, Section 4]. Put

$$(3.5) \quad M(x, \xi') = \max_{I \in \mathcal{J}_\mu} |\hat{p}_I(x, \xi')/\rho|^{1/I}.$$

Here  $\rho$  is a large parameter, whose role is the same as in [4, Section 4] (see [4, p. 149]). By (3.1) and (3.3) we have

$$(3.6) \quad C_1 \lambda^{-2\delta/\mu} \rho^{-(l-1)/\mu} \leq M(x, \xi') \leq C_2 \lambda^{-2\delta}/\rho.$$

Here and in what follows the constants are independent of  $\lambda$  and  $\rho$ .

The definition of  $M=M(0)$  means in particular that

$$(3.7) \quad |D_{x_1}^j q(0)| < \rho M^{j+1}, \quad j \leq \mu-1,$$

and where (3.1) is valid we have by (3.6)

$$(3.8) \quad |D_{x_1}^j q| \leq \mathcal{O}(\lambda^c) \rho M^{l\mu+1}$$

since  $\rho^{l-2}/M^{l\mu+1-\mu} \ll 1$  if  $\lambda$  is small enough, where  $c$  is some positive. If we set

$$F(t, y', \eta') = q(t/M, y'(\rho M)^{1/2\delta}, \eta'(\rho M)^{1/2\delta})/\rho M,$$

then the application of [4, Lemma 7.1] to  $F(t, y', \eta')$  shows that

$$(3.9) \quad |D_{\xi_1}^\alpha D_{\xi_2}^\beta (\tilde{q}(x, \xi') - \xi_2 \partial \tilde{q}(x_1, 0)/\partial \xi_2)| \leq C_{\alpha\beta} \rho M^{\beta_1+1} (\rho M)^{-|\alpha'+\beta'|/2\delta}$$

if  $|x_1 M| < 1$ ,  $|(x', \xi')| < C(\rho M)^{1/2\delta}$ ,

where  $\tilde{q}$  is determined from  $q$  by a symplectic orthogonal transformation.

Let  $\varepsilon = \lambda^\kappa$  with  $0 < \kappa < 1/\mu$  (which is different from  $\varepsilon$  in the preceding section). Then

$$(3.10) \quad \varepsilon^2 (\rho M)^{1/\delta} \gg 1.$$

As in [4], using this  $\varepsilon$  we consider the following two cases.

Case I. Assume that

$$(3.11) \quad |d_{x'_\xi} D_{x_1}^j q(0)| \leq \varepsilon \rho M^{j+1}, \quad j < \mu'.$$

where  $\mu' = (l-2)(\mu-1)/2(l-1)$ . In view of (3.1), (3.6) and (3.10) we get for some  $c > 0$

$$(3.11)' \quad |d_{x'_\xi} D_{x_1}^j q(x, \xi')| \leq \mathcal{O}(\lambda^c) \varepsilon \rho M^{l\mu'+1}, \quad j > \mu', (x, \xi') \in \Omega.$$

Then it is easy to check that the argument corresponding to Case I in [4, Section

4] follows with  $k$  and  $k/2$  replaced by  $[\mu - 1]$  and  $\mu'$ .

Case II. Assume now that (3.11) is not fulfilled. Choose  $s < \mu'$  so that with  $q^{(j)} = \partial^j q / \partial x_1^j$

$$(3.12) \quad d_{x_1 \xi'} q^{(s)}(0) = a$$

$$(3.13) \quad d_{x_1 \xi'} q^{(j)}(0) \leq a M^{j-s} \quad \text{for } j < \mu'$$

Then (3.1) and the fact that (3.11) is not valid give

$$(3.14) \quad \varepsilon \rho M^{s+1} < a \leq C \lambda^{1-2\delta}.$$

In view of (3.11)' we have then for every  $j$

$$(3.13)' \quad |D_{x_1}^j \tilde{q}(x_1, 0) / \partial \xi_2| \leq C_j a M^{j-s} = C_j \rho M^{j+1} A_2, \quad |x_1 M| < 1,$$

where  $A_2 = a / \rho M^{s+1}$ . The equality of (3.13)' holds when  $j = s$ . From (3.9) we can therefore obtain an estimate of the form [4, (4.10)] with  $B_2 = A_3 = B_3 = \dots = (\rho M)^{-1/2\delta}$  and  $K = M$ . However,  $A_2 B_2 = a / M^{s+1} (\rho M)^{1/2\delta}$  so [4, Lemma 4.1] is not applicable if  $a > M^{s+1} (\rho M)^{1/2\delta}$ . In that case we shall replace the orthogonal symplectic transformation which led from  $q$  to  $\tilde{q}$  by a non-linear canonical transformation.

Thus assume for the moment that (3.12), (3.13) and

$$(3.14)' \quad M^{s+1} (\rho M)^{1/2\delta} < a \leq C \lambda^{1-2\delta}$$

are fulfilled. Let  $b = a \lambda^{2(\delta-1)}$ . Then the function

$$Q(x', \xi') = (q^{(s)}(0, bx', b\xi') - q^{(s)}(0, 0, 0)) / ab$$

is in a bounded subset of  $C^\infty(U)$  ( $U = \{|(x', \xi')| < C^{-1}\}$ ) since  $|(bx', b\xi')| < \lambda^{-1}$ . Hence there exists some canonical transformation  $\mathcal{X}$  belonging to a bounded set in  $C^\infty$  for  $0 < \lambda \leq 1$  such that

$$Q \circ \mathcal{X}(x', \xi') = \xi_2$$

in a neighborhood of 0. If we put

$$\begin{aligned} \mathcal{X}_b(x', \xi') &= b\mathcal{X}(b^{-1}x', b^{-1}\xi') \\ \tilde{q}(x, \xi') &= q(x_1, \mathcal{X}_b(x', \xi')), \end{aligned}$$

then we obtain

$$(3.15) \quad \begin{aligned} \tilde{q}^{(s)}(x, \xi') &= a\xi_2 + \tilde{q}^{(s)}(0, 0) \\ &\text{when } x_1 = 0, |(x', \xi')| < cb. \end{aligned}$$

By the same way as in [4] we get



$$(3.16) \quad |a^j(\partial/\partial x_2)^j(\partial/\partial x_1)^i \tilde{q}(0)| \leq C_{ij} M^{j(s+1)} \rho M^{i+1}$$

for any  $i, j$  satisfying

$$j \leq l-2 \quad \text{and} \quad (i+1+j(s+1))(l-1)/(l-j-1) \leq \mu .$$

If we introduce

$$(3.17) \quad B_2 = M^{s+1}/a ,$$

noting incidentally that  $\rho A_2 B_2 = 1$  as required in [4, Lemma 4.1], this means that we have bounds for the derivatives of  $\tilde{q}(x_1/M, x_2/B_2, 0)/\rho M$  at 0.

As stated in [4, p. 156], when we derive (3.16), we can replace the canonical transformation  $\mathcal{X}_b(x', \xi')$  by another  $\tilde{\mathcal{X}}_b(x', \xi')$  which is linear in all variables except  $x_2$ , provided that the integral curve of the Hamilton field of  $\tilde{q}^{(s)}(0, x', \xi')$  by new  $\tilde{\mathcal{X}}_b$  is also the  $x_2$  axis  $x_2 = at, x_3 = \dots = \xi_n = 0$ . We denote  $\tilde{\mathcal{X}}_b$  by  $\mathcal{X}_b$  in what follows.

By the analogous calculation as in showing [4, (4.23) and (4.23)'] we obtain when  $|x_1 M| < 1, |(x', \xi')| < cb$

$$(3.18) \quad |D_\xi^\alpha D_x^\beta \tilde{q}(x, \xi')| \leq C_{\alpha\beta} a M^\beta b^{1-|\alpha'+\beta'|} \quad \text{if } |\alpha'+\beta'| \neq 0$$

$$(3.18)' \quad |D_\xi^\alpha D_x^\beta \tilde{q}(x, \xi')| \leq C'_{\alpha\beta} \lambda^{-2\delta} b^{-|\alpha'+\beta'|} \quad \text{for any } \alpha', \beta' .$$

In particular (3.18)' is a much better estimate than (3.16) if  $j \geq l-1$  or  $(i+1+j(s+1))(l-1)/(l-j-1) > \mu$ , and it is not only valid at 0. Hence (3.16) leads us to uniform bounds for  $\tilde{q}(x_1/M, x_2/B_2, 0)/\rho M$  and all of its derivatives when  $|x_1| < 1$  and  $|x_2| < 1$ .

Since  $b \gg (\rho M)^{1/2\delta}$  we can apply [4, Lemma 7.1] to

$$F(t, x_2, y) = (M\rho)^{-1} \tilde{q}(t/M, x_2/B_2, x''(\rho M)^{1/2\delta}, \xi'(\rho M)^{1/2\delta})$$

where  $x'' = (x_3, \dots, x_n)$  and  $y = (x'', \xi')$ . Therefore, by the same way as in getting [4, (4.25)] we obtain

$$(3.19) \quad |D_\xi^\alpha D_x^\beta (\tilde{q}(x, \xi') - \xi_2 \partial \tilde{q}(x_1, x_2, 0) / \partial \xi_2) \leq C M^{\beta_1+1} \rho B_2^{\beta_2} (\rho M)^{-|\alpha'+\beta'|/2\delta}$$

when  $|x_1| M < 1, |(x'', \xi')| < (\rho M)^{1/2\delta}, |x_2| < 1/B_2 .$

As in [4], we obtain [4, (4.26)] and [4, (4.27)] with replaced the right hand side by  $C b^{-\beta_2} \lambda^{2/l}$ . Hence it follows from (3.12) that  $(B_2/M) \partial \tilde{q}(x_1/M, 0) / \partial \xi_2$  is essentially a normalized polynomial in  $x_1$  of degree  $[\mu']$  and  $\partial \tilde{q}(x_1/M, x_2, 0) / \partial \xi_2$  is almost independent of  $x_2$ .

From (3.19) and these inequalities we obtain with  $B_1 = M, A_j = B_j = (\rho M)^{-1/2\delta}$  when  $j > 2$

$$(3.20) \quad |D_{\xi}^{\alpha} D_x^{\beta} \tilde{q}(x, \xi')| \leq C_{\alpha\beta} \rho M A^{\alpha} B^{\beta}, \quad \text{if } |x_1 M| < 1, |x_2 B_2| < 1, \\ |\xi_2 A_2| < N, |(x'', \xi'')| < (\rho M)^{1/2\delta},$$

where  $N$  is a fixed but arbitrary constant. If we write  $\tilde{M}(x, \xi') = M(x_1, \mathcal{X}_b(x', \xi'))$ , it follows in view of [4, Lemma 4.1], where we take  $A_1 = 1/M$  and  $K = M$ , that

$$(3.21) \quad \tilde{M}(x, \xi') \leq C_N M \quad \text{if } |x_1 M| < 1, |x_2 B_2| < 1, \\ |\xi_2 A_2| < N, |(x'', \xi'')| < (\rho M)^{1/2\delta}.$$

when (3.14) is fulfilled but not (3.14)' we get the same conclusion with  $B_2$  replaced by  $(\rho M)^{-1/2\delta}$  and  $\mathcal{X}_b$  equal to orthogonal symplectic transformation such that  $\tilde{q}(x, \xi') = q(x_1, \mathcal{X}_b(x', \xi'))$ .

The argument corresponding to the rest of [4, Section 4] can be done by the same way if we let (3.1)–(3.3), (3.16), (3.18), (3.18)', (3.19), (3.20), (3.21),  $[\mu - 1]$ ,  $\mu'$ ,  $\mathcal{X}_b$  and  $(\rho M)^{1/2\delta}$  correspond to [4, (4.1)–(4.3), (4.21), (4.23), (4.23)', (4.25), (4.28), (4.29),  $k, k/2, \mathcal{X}_a$  and  $\sqrt{\rho M}$ ] respectively.

Because we have got the result corresponding to [4, Section 4] we can easily prove (3.4) by the same way as in [4, Section 6], if we take the above correspondence. The detail is left to the reader.

#### 4. Proof of Proposition 1.2

The method of the proof is a version of [2] (, see also [15]). Suppose that  $\tilde{L}$  is hypoelliptic at the origin. Then, as well-known, there exist a positive integer  $s$ , a constant  $C$  and some neighborhood  $U$  of 0 such that

$$(4.1) \quad \|u\| \leq C \|\Lambda^s \tilde{L}u\|, \quad u \in C_0^\infty(U),$$

where the symbol of  $\Lambda$  is  $(|\xi_1|^{10} + |\xi'|^8 + 1)$ . Hence for any large  $N$  there exists a  $C_N$  such that

$$(4.1)' \quad \|u\| \leq C_N (\|\Lambda^s \tilde{L}u\| + \| |x|^N u \| + \| |x|^N \Lambda^s \tilde{L}u \|), \quad u \in \mathcal{S}.$$

If we take the canonical change of variables  $(x, \xi)$  into

$$(4.2) \quad (\lambda^2 x_1, \lambda^9 x', \lambda^{-2\tau-2}(\xi_{01} + \lambda^{2\tau} \xi_1), \lambda^{-2\tau-9}(\xi'_0 + \lambda^{2\tau} \xi')), \\ (0 < \lambda \leq 1, \tau = 13),$$

with a fixed  $\xi_0 = (\xi_{01}, 0, \xi_{03})$  satisfying

$$\xi_{01}^5 + 2\xi_{03}^4 = 0 \quad \text{and} \quad \xi_{03} > 0,$$

then it follows from (4.1)' that

$$(4.3) \quad \|u\| \leq C'_N (\lambda^{-M} \|L_\lambda u\| + \lambda^{2N-M} \|R_0(x, \lambda^{2\tau} D_x)u\|), \quad u \in \mathcal{S},$$

where

$$\begin{aligned}
L_\lambda &= L_1(\lambda^{2r}D_x) + i\lambda^\tau x_1^2 x_2 \xi_{03}^4 + \sum_{j=1}^2 \lambda^{\tau(2-j)} R_j(x, \lambda^{2r}D_x), \\
L_1(\xi) &= 5\xi_{01}^4 \xi_1 + 4\xi_{03}^3 (2\xi_3 - \xi_2), \\
R_j(x, \xi) &= \sum_{\substack{j \leq |\alpha| \\ |\alpha| + \beta \leq N_j}} a_{\alpha\beta j}(\lambda) x^\beta \xi^\alpha \quad (j = 0, 1, 2) \\
(a_{\alpha\beta j}(\lambda) &= \mathcal{O}(\lambda^c), c \geq 0).
\end{aligned}$$

Here  $M$  and  $N_j (j=0, 1, 2)$  are some integers depending on  $s$ . Since  $\xi_{03} > 0$  we can take  $w(x)$  such that

$$\begin{aligned}
(4.4) \quad & L_1(\partial_x w) + ix_1^2 x_2 \xi_{03}^4 = 0 \\
& |\operatorname{Im} w| \geq c_0 |x|^4, \quad c_0 > 0.
\end{aligned}$$

Put  $u_\lambda(x) = (\exp i\lambda^{-\tau} w(x)) \sum_{j=0}^{N_j} v_j(x, \lambda) \lambda^{\tau j}$ , where  $v_j$  will be determined later such that they are polynomials of  $x$  whose coefficients have the same property as  $a_{\alpha\beta j}$ . Then we have

$$\begin{aligned}
L_\lambda u_\lambda &= \exp i\lambda^{-\tau} w \sum_{j=2}^{N_j} (L_1(D_x) + A(x, \partial_x w)) v_{j-2} + F_j(x) \lambda^{\tau j}, \\
(F_2 &= 0).
\end{aligned}$$

Here  $A(x, \xi) = \sum_{|\alpha|=1} a_{\alpha\beta 1} x^\beta \xi^\alpha$  and  $F_j(x)$  is a linear combination of the functions  $v_0, \dots, v_{j-3}$  and their derivatives. By means of Cauchy-Kowalewska theorem, solve the transport equations

$$(L_1(D_x) + A(x, \partial_x w)) v_j + F_{j+2}(x) = \mathcal{O}(|x|^{N-j})$$

under the condition  $v_0(0, \lambda) = 1$  and  $v_j(0, \lambda) = 0 (j \geq 1)$ , successively. Then the analytic solution obtained, which is defined in a certain neighborhood  $\omega$  of the origin, is to be multiplied by a cut off function  $\phi \in C_0^\infty(\omega)$  which equals 1 in another neighborhood of the origin. The multiplication does not affect (4.5) because  $(1-\phi)v_j = \mathcal{O}(|x|^N)$ . Substituting  $u_\lambda(x) \in \mathcal{S}$  into (4.3) and changing variables  $x$  into  $\lambda^{\tau/4} x$  give us the contradiction if  $2N > M$  and if  $\lambda$  tends to 0.

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Department of Engineering Mathematics  
 Faculty of Engineering  
 Nagoya University  
 Furo-cho, Chikusa-ku,  
 Nagoya 464, Japan

