L^2 vanishing theorems on Hessian manifolds and geometry of regular convex cones

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Abstract

We show vanishing theorems of L^2-cohomology groups of Kodaira-Nakano type on complete Hessian manifolds by introducing a new operator ∂'_F. On a regular convex cone Ω in $\mathbb{R}^n$ with the Cheng-Yau metric g, we obtain further vanishing theorems of L^2-cohomology groups $L^2H^{p,q}_{\bar{\partial}}(\Omega)$ for $p > q$. Moreover, we show that g gives a harmonic immersion from Ω to $\mathrm{Sym}^+(n)$.

Contents

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0 Introduction

A *flat manifold* (M, D) is a manifold M with a flat affine connection D, where an affine connection is said to be *flat* if the torsion and the curvature vanish identically. A flat affine connection D gives an *affine local coordinate system* $\{x^1, \dots, x^n\}$ satisfying

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0.$$

A Riemannian metric g on a flat manifold (M, D) is said to be a *Hessian metric* if g can be locally expressed in the Hessian form with respect to an affine coordinate system $\{x^1, \dots, x^n\}$ and a *potential function* φ, that is,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.$$

The triplet (M, D, g) is called a *Hessian manifold.* The Hessian structure (D, g) induces a holomorphic coordinate system $\{z^1, \dots, z^n\}$ and a Kähler metric g^T on TM such that

$$z^i = x^i + \sqrt{-1} y^i,$$

$$g^T_{i\bar{j}}(z) = g_{ij}(x),$$

where $\{x^1, \dots, x^n, y^1, \dots, y^n\}$ is a local coordinate system on TM induced by the affine coordinate system $\{x^1, \dots, x^n\}$ and fiber coordinates $\{y^1, \dots, y^n\}$. In this sense, Hessian geometry is a real analogue of Kähler geometry.

A (p, q)-form on a flat manifold (M, D) is a smooth section of $\wedge^{p,q} := \wedge^p T^*M \otimes \wedge^q T^*M$. On the space of (p, q)-forms, a flat connection D induces the Dolbeault-like operator $\bar{\partial} = \sum_i e(\overline{dx^i}) D_{\frac{\partial}{\partial x^i}}$, where $\overline{dx^i} = 1 \otimes dx^i$ and $e(\overline{dx^i}) = \overline{dx^i} \wedge$. For a flat line bundle (F, D^F) over M, the operator $\bar{\partial}$ can be extended on the space of F-valued (p, q)-forms and satisfies $\bar{\partial}^2 = 0$. Then the cohomology group $H^{p,q}_{\bar{\partial}}(M, F)$ is defined with respect to $\bar{\partial}$. On compact Hessian manifolds, Shima proved an analogue of the Kodaira-Nakano vanishing theorem for $H^{p,q}_{\bar{\partial}}(M, F)$ by using the theory of harmonic integrals when there exist a fiber metric h on F and a Riemannian metric g on M such that the *second Koszul forms* $B = -Dd \log h(s, s)$ and $\beta = \frac{1}{2} Dd \log \det[g_{ij}]$ satisfy $B + \beta > 0$, where s is a local frame field on F such that $D^F s = 0$.

THEOREM 2.2.6. *[5] Let (M, D) be an oriented n-dimensional compact flat manifold and (F, D^F) be a flat line bundle over M. Assume there exist a fiber metric h on F and a Riemannian metric g on M such that $B + \beta$ is positive definite, where B and β are the second Koszul forms with respect to h and g, respectively. Then we have*

$$H^{p,q}_{\bar{\partial}}(M, F) = 0, \quad \textit{for } p + q > n.$$

However, many important examples of Hessian manifolds such as regular convex domains (c.f. Theorem 1.2.4 [3]) are noncompact. In Section 3.2, we prove the following theorem which corresponds to Theorem 2.2.6 in the case of complete Hessian manifolds

THEOREM 3.2.3. *Let (M, D, g) be an oriented n-dimensional complete Hessian manifold and (F, D^F) a flat line bundle over M. We denote by h a fiber metric on F. Assume that there exists $\varepsilon > 0$ such that $B + \beta = \varepsilon g$ where B and β are the second Koszul forms with respect to fiber metric h and Hessian metric g respectively. Then for $p + q > n$ and all $v \in L^2(M, F \otimes \wedge^{p,q})$ such that $\bar{\partial} v = 0$, there exists $u \in L^2(M, F \otimes \wedge^{p,q-1})$ such that*

$$\bar{\partial} u = v, \quad \|u\| \le \{\varepsilon(p + q - n)\}^{-\frac{1}{2}} \|v\|.$$

In particular, we have

$$L^2 H^{p,q}_{\bar{\partial}}(M, F) = 0, \quad \textit{for } p + q > n.$$

The L^2-cohomology group $L^2H^{p,q}_{\bar\partial}(M,F)$ is often also written as $H^{p,q}_{(2)}(M,F)$.

Note that we cannot use the harmonic theory for the proof and we need the method of functional analysis as in the case of complete Kähler manifolds. To prove Theorem 3.2.3, we introduce the new operator ∂'_F (c.f. Definition 2.3.1) which is not defined in [5] and we obtain the following as an analogue of Kodaira-Nakano identity.

Theorem 2.3.8. *Let (D,g) be a Hessian structure. Then we have*

$$\bar\Box_F = \Box'_F + [e(\beta+B),\Lambda],$$

where $\bar\Box_F$ and $\Box'_F$ are the Laplacians with respect to $\bar\partial$ and ∂'_F, respectively, and Λ is the adjoint operator with respect to $e(g)$.

An open convex cone Ω in $\mathbb{R}^n$ is said to be *regular* if Ω contains no complete straight lines. In Chapter 4, we give some special properties of regular convex cones with the *Cheng-Yau metrics* (c.f. Theorem 1.2.4 [3]). In Section 4.2, we show vanishing theorems of another type as follows.

Theorem 4.2.3. *Let $(\Omega, D, g = Dd\varphi)$ be a regular convex cone in $\mathbb{R}^n$ with the Cheng-Yau metric. Then for $p > q \geq 1$ and all $v \in L^2(\Omega, \wedge^{p,q})$ such that $\bar\partial v = 0$, there exists $u \in L^2(\Omega, \wedge^{p,q-1})$ such that*

$$\bar\partial u = v, \quad \|u\| \leq (p-q)^{-\frac{1}{2}}\|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\Omega, \wedge^{p,0})$ satisfies $\bar\partial v = 0$, then $v = 0$. In particular, we have

$$L^2H^{p,q}_{\bar\partial}(\Omega) = 0, \quad \text{for } p > q.$$

In the case of a regular convex cone $(\mathbb{R}^n, D, g = -Dd\log(x^1\cdots x^n))$, we have sharp vanishing theorem.

Theorem 4.2.5. *For $p \geq 1$, $q \geq 1$ and $v \in L^2(\mathbb{R}^n_+, \wedge^{p,q})$ such that $\bar\partial v = 0$, there exists $u \in L^2(\mathbb{R}^n_+, \wedge^{p,q-1})$ such that*

$$\bar\partial u = v, \quad \|u\| \leq p^{-\frac{1}{2}}\|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\mathbb{R}^n_+, \wedge^{p,0})$ satisfies $\bar\partial v = 0$, then $v = 0$. In particular, we have

$$L^2H^{p,q}_{\bar\partial}(\mathbb{R}^n_+) = 0, \quad \text{for } p \geq 1 \text{ and } q \geq 0.$$

Finally, we show that the Cheng-Yau metrics on regular convex cones define harmonic immersions into the symmetric space of the positive definite symmetric matrices in Section 4.4.

Theorem 4.4.4. *Let g be the Cheng-Yau metric on a regular convex cone Ω in $\mathbb{R}^n$ and h the Cheng-Yau metric on $Sym^+(n)$. We define a map $F_g : (\Omega, g) \to (Sym^+(n), h)$ by $F_g(x) = [g_{ij}(x)]_{1\leq i,j\leq n}$. We denote by $\nabla^{g,h}$ the connection over $F_g^*TSym^+(n) \otimes T^*\Omega$ induced by the Levi-Civita connections ∇^g and ∇^h. Then*

(1) *F_g is an immersion.*

(2) *F_g is harmonic, that is,*

$$tr_g(\nabla^{g,h}dF_g) = 0.$$

(3) *If Ω is a homogeneous self-dual regular convex cone, then F_g is totally geodesic, that is,*

$$\nabla^{g,h}dF_g = 0.$$

The condition in Theorem 4.4.4 that a regular convex domain is a cone is crucial to obtain a harmonic map (c.f. Example 4.4.5). Furthermore, the condition in Theorem 4.4.4 (3) that a homogeneous regular convex cone is self-dual is also necessary to obtain a totally geodesic map (c.f Exapmle 4.4.6).

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1 Hessian manifolds

In this chapter we give a brief review of Hessian manifolds.

1.1 Hessian manifolds

An affine connection D on a manifold M is said to be *flat* if the torsion tensor T^D and the curvature tensor R^D vanish identically. A manifold M endowed with a flat connection D is called a *flat manifold*, which is denoted by (M, D). On a flat manifold (M, D), there exists a local coordinate system $\{x^1, \ldots, x^n\}$ such that $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$, which is called an *affine coordinate system* with respect to D. The changes between such a local coordinate system are affine transformations.

In this paper, every local coordinate system on flat manifolds is given as an affine coordinate system.

Definition 1.1.1. A Riemannian metric g on a flat manifold (M, D) is said to be a *Hessian metric* if g is locally expressed by

$$g = Dd\varphi,$$

that is,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.$$

Then the pair (D, g) is called a *Hessian structure* on M, and φ is said to be a *potential* of (D, g). A manifold M with a Hessian structure (D, g) is called a *Hessian manifold*, which is denoted by (M, D, g).

Let (M, D) be a flat manifold and TM the tangent bundle over M. We denote by $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ a local coordinate system on TM induced by an affine coordinate system $\{x^1, \ldots, x^n\}$ on M and fiber coordinates $\{y^1, \ldots, y^n\}$. Then a holomorphic coordinate system$\{z^1, \ldots, z^n\}$ on TM is given by

$$z^i = x^i + \sqrt{-1}y^i.$$

For a Riemannian metric g on M we define a Hermitian metric g^T on TM by

$$g^T = \sum_{i,j} g_{ij} dz^i \otimes d\bar{z}^j.$$

It should be remarked that g^T is a Kähler metric if and only if g is a Hessian metric.

Example 1.1.2.

(1) Let (D, g) be the pair consisting of the standard affine connection D and the Euclidean metric on $\mathbb{R}^n$. Then (D, g) is a Hessian structure. Indeed, if we set $\varphi(x) = \frac{1}{2}\sum_j (x^j)^2$, we have

$$\frac{\partial^2 \varphi}{\partial x^i \partial x^j} = \delta_{ij} = g_{ij},$$

where δ_{ij} is the Kronecker delta, that is,

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Moreover, the Kähler metric g^T on $T\mathbb{R}^n \simeq \mathbb{C}^n$ is also the Euclidean metric.

(2) We set $\mathbb{R}_+ = (0, \infty)$. Let D be the standard affine connection, that is, the restriction of the standard affine connection on $\mathbb{R}^n$ to $\mathbb{R}^n_+$. We define a Riemannian metric g on $\mathbb{R}^n_+$ by

$$g_{ij}(x) = \frac{\delta_{ij}}{(x^j)^2}.$$

Then (D, g) is a Hessian structure. Indeed, if we set $\varphi(x) = -\log(x^1 \cdots x^n)$, we have

$$\frac{\partial^2 \varphi}{\partial x^i \partial x^j} = g_{ij}.$$

When $n = 1$, the Kähler metric g^T on $T\mathbb{R}_+ \simeq \mathbb{R}_+ \oplus \sqrt{-1}\,\mathbb{R}$ is the Poincaré metric.

Definition 1.1.3. Let M be a manifold and D a torsion-free affine connection on M. We denote by g a Riemannian metric on M, and by ∇ the Levi-Civita connection of g. We define the *difference tensor* γ of ∇ and D by

$$\gamma = \nabla - D.$$

We denote by $\mathscr{X}(M)$ the space of vector fields on M. Since ∇ and D are torsion-free, it follows that for $X, Y \in \mathscr{X}(M)$

$$\gamma_X Y = \gamma_Y X.$$

It should be remarked that the components $\gamma^i{}_{jk}$ of γ with respect to affine coordinate systems coincide with the Christoffel symbols of ∇.

Definition 1.1.4. Let M be a manifold and D a torsion-free affine connection on M. We denote by g a Riemannian metric on M. We define another affine connection D^* on M as follows:

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D^*_X Z), \quad X, Y, Z \in \mathscr{X}(M)$$

We call D^* the *dual connection* of D with respect to g.

Proposition 1.1.5. *Let (M, D) be a flat manifold and g a Riemannian manifold on M. Then the following conditions are equivalent.*

(1) *(D, g) is a Hessian structure.*

(2) $(D_X g)(Y, Z) = (D_Y g)(X, Z), \quad X, Y, Z \in \mathscr{X}(M) \quad \Big(\Leftrightarrow \dfrac{\partial g_{jk}}{\partial x^i} = \dfrac{\partial g_{ik}}{\partial x^j}\Big)$.

(3) $g(\gamma_X Y, Z) = g(Y, \gamma_X Z), \quad X, Y, Z \in \mathscr{X}(M) \quad (\Leftrightarrow \gamma_{ijk} = \gamma_{jik})$.

(4) $(D_X g)(Y, Z) = 2g(\gamma_X Y, Z), \quad X, Y, Z \in \mathscr{X}(M) \quad \Big(\Leftrightarrow \dfrac{\partial g_{ij}}{\partial x^k} = 2\gamma_{ijk}\Big)$.

(5) $D + D^* = 2\nabla$.

Proof. By the definition of Hessian metrics, (1) implies (2). Since the components $\gamma^i{}_{jk}$ coincide with the Chiristoffel symbols of ∇, we have

$$\gamma_{ijk} = \frac{1}{2}\Big(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\Big).$$

Hence (2), (3) and (4) are equivalent. By the definition of γ and ∇^*, we obtain

$$\begin{aligned} g((D + D^*)_X Y, Z) &= g(D_X Y, Z) + g(D^*_X Y, Z) \\ &= g((\nabla - \gamma)_X Y, Z) + Xg(Y, Z) - g(Y, D_X Z) \\ &= g(\nabla_X Y, Z) - g(\gamma_X Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(Y, D_X Z) \\ &= g(2\nabla_X Y, Z) - g(\gamma_X Y, Z) + g(Y, \gamma_X Z). \end{aligned}$$

This implies that (3) and (5) are equivalent. Finally, we show that (2) implies (1). Let $h_j = \sum_i g_{ij} dx^i$. Then we have

$$dh_j = \sum_{k<i} \Big(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i} \Big) dx^k \wedge dx^i = 0.$$

Hence by Poincarè's lemma, there exists φ_j such that $h_j = d\varphi_j$. If we put $h = \sum_j \varphi_j dx^j$, then $dh = \sum_j h_j \wedge dx^j = 0$. Applying Poincarè's lemma again, there exists φ such that $h = d\varphi$. Therefore we have

$$\frac{\partial^2 \varphi}{\partial x^i \partial x^j} = \frac{\partial \varphi_j}{\partial x^i} = g_{ij}.$$

This completes the proof. □

1.2 Koszul forms on flat manifolds

We introduce Koszul forms which play important roles in Hessian geometry.

Definition 1.2.1. Let (M, D) be a flat manifold and g a Riemannian metric on M. We define a d-closed 1-form α and a symmetric bilinear form β by

$$\alpha = \frac{1}{2} d \log \det[g_{ij}], \quad \beta = D\alpha.$$

Remark that since the changes between affine coordinate systems are affine transformations, α and β are globally well-defined. We call α and β the *first Koszul form* and the *second Koszul form* for (D, g), respectively.

Proposition 1.2.2. *Let (M, D, g) be a Hessian manifold. Then we have the following equations.*

$$\alpha_j := \alpha\Big(\frac{\partial}{\partial x^j}\Big) = \sum_r \gamma^r{}_{rj}, \quad \beta_{ij} := \beta\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \sum_r \frac{\partial \gamma^r{}_{rj}}{\partial x^i}.$$

Proof. From Proposition 1.1.5 (4), we have

$$\alpha_j = \frac{1}{2} \frac{\partial}{\partial x^j} \log \det[g_{kl}] = \frac{1}{2} \sum_{k,l} g^{kl} \frac{\partial g_{kl}}{\partial x^j} = \sum_k \gamma^k{}_{kj},$$
$$\beta_{ij} = \frac{\partial \alpha_j}{\partial x^i} = \sum_k \frac{\partial \gamma^k{}_{kj}}{\partial x^i}.$$

□

Definition 1.2.3. Let (M, D, g) be a Hessian manifold. If there exists $\lambda \in \mathbb{R}$ such that $\beta = \lambda g$, we call g a *Hesse-Einstein metric.*

It should be remarked that a Hessian metric g on M satisfies the Hesse-Einstein condition $\beta = \lambda g$ if and only if the Kähler metric g^T on TM satisfies the Kähler-Einstein condition $\mathrm{Ric} = -\frac{1}{2}\lambda g^T$ [4].

A convex domain in $\mathbb{R}^n$ which contains no full straight lines is called a *regular convex domain.* By the following theorem, on a regular convex domain there exists a complete Hesse-Einstein metric g which satisfies $\beta = g$. It is called the *Cheng-Yau metric.*

Theorem 1.2.4. *[3] On a regular convex domain $\Omega \subset \mathbb{R}^n$, there exists a unique convex function φ such that*

$$\begin{cases} \det\Big[\dfrac{\partial^2 \varphi}{\partial x^i \partial x^j}\Big] = e^{2\varphi} \\ \varphi(x) \to \infty \qquad\qquad (x \to \partial\Omega) \end{cases}$$

and the Hessian metric $g = Dd\varphi$ is complete, where D is the standard affine connection on Ω.

PROPOSITION 1.2.5. *The Cheng-Yau metric g defined by Theorem 1.2.4 is invariant under affine automorphisms of Ω, where an affine automorphism of Ω is restriction of an affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ to Ω which satisfies $A\Omega = \Omega$.*

PROOF. An affine transformation A is denoted by

$$Ax = ((Ax)^1, \dots, (Ax)^n), \quad (Ax)^i = \sum_j a^i_j x^j + b^i.$$

We define a function $\tilde{\varphi}$ on Ω by

$$\tilde{\varphi}(x) = \varphi(Ax) + \log|\det[a^i_j]|.$$

Then we have

$$\tilde{\varphi}(x) \to \infty \quad (x \to \infty).$$

Moreover we obtain

$$\frac{\partial^2 \tilde{\varphi}}{\partial x^i \partial x^j}(x) = \sum_{k,l} a^k_i a^l_j \frac{\partial^2 \varphi}{\partial x^k \partial x^l}(Ax).$$

Hence $\tilde{\varphi}$ is a convex function. Furthermore, it follows that

$$\det\Big[\frac{\partial^2 \tilde{\varphi}}{\partial x^i \partial x^j}(x)\Big] = |\det[a^i_j]|^2 \det\Big[\frac{\partial^2 \varphi}{\partial x^i \partial x^j}(Ax)\Big] = e^{2(\varphi(Ax)+\log|\det[a^i_j]|)} = e^{2\tilde{\varphi}(x)}.$$

Therefore $\tilde{\varphi}$ is also a convex function which satisfies the condition of Theorem 1.2.4. From the uniqueness of the solution we have $\tilde{\varphi} = \varphi$, that is,

$$\varphi(x) = \varphi(Ax) + \log|\det[a^i_j]|.$$

Hence we have

$$g_{ij}(x) = \sum_{k,l} a^k_i a^l_j g_{kl}(Ax).$$

This implies that g is invariant under affine automorphisms. □

COROLLARY 1.2.6. *Let Ω be a regular convex domain in $\mathbb{R}^n$ and x_0 a fixed point in Ω. Assume that $a \in C^\infty(\Omega, GL(n,\mathbb{R}))$ satisfies $x = a(x)x_0$ for all $x \in \Omega$. Then the Cheng-Yau metric $g = Dd\varphi$ on Ω is expressed by*

$$g = -Dd\log|\det a(x)|.$$

PROOF. By the proof of Proposition 1.2.5 we obtain

$$\varphi(x) = \varphi(a(x)x_0) = \varphi(x_0) - \log|\det a(x)|.$$

Hence we have

$$g = Dd\varphi = -Dd\log|\det a(x)|.$$

□

EXAMPLE 1.2.7. Let $(\mathbb{R}^n_+, D, g = Dd\varphi)$ be the same as in Example 1.1.2 (2). Then $\varphi(x) = -\log(x^1 \cdots x^n)$ satisfies the condition of Theorem 1.2.4.

2 (p,q)-forms on flat manifolds

Hereafter, we assume that (M, D) is an oriented flat manifold and g is a Riemannian metric on M. In addition, let F be a real line bundle over M endowed with a flat connection D^F and a fiber metric h. Moreover, we denote by s a local frame field on F such that $D^F s = 0$.

2.1 (p, q)-forms and fundamental operators

We denote by $A^{p,q}(M)$ the space of smooth sections of $\wedge^{p,q} := \wedge^p T^*M \otimes \wedge^q T^*M$. An element in $A^{p,q}(M)$ is called a (p, q)-form. For a p-form ω and a q-form η, $\omega \otimes \eta \in A^{p,q}(M)$ is denoted by $\omega \otimes \bar{\eta}$.

Using an affine coordinate system a (p, q)-form ω is expressed by

$$\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dx^{I_p} \otimes \overline{dx^{J_q}},$$

where

$$I_p = (i_1, \ldots, i_p), \quad 1 \leq i_1 < \cdots < i_p \leq n, \quad J_q = (j_1, \ldots, j_q), \quad 1 \leq j_1 < \cdots < j_q \leq n,$$

$$dx^{I_p} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \quad dx^{J_q} = dx^{j_1} \wedge \cdots \wedge dx^{j_q}.$$

Example 2.1.1. A Riemannian metric g and the second Koszul form β (Definition 1.2.1) are regarded as $(1, 1)$-forms;

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes \overline{dx^j}, \quad \beta = \sum_{i,j} \beta_{ij}\, dx^i \otimes \overline{dx^j}.$$

Definition 2.1.2. We define the exterior product of $\omega \in A^{p,q}(M)$ and $\eta \in A^{r,s}(M)$ by

$$\omega \wedge \eta = \sum_{I_p, J_q, K_r, L_s} \omega_{I_p J_q} \eta_{K_r L_s} dx^{I_p} \wedge dx^{K_r} \otimes \overline{dx^{J_q} \wedge dx^{L_s}},$$

where $\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dx^{I_p} \otimes \overline{dx^{J_q}}$ and $\eta = \sum_{K_r, L_s} \eta_{K_r L_s} dx^{K_r} \otimes \overline{dx^{L_s}}$.

Definition 2.1.3. For $\omega \in A^{r,s}(M)$ we define an exterior product operator $e(\omega) : A^{p,q}(M) \to A^{p+r,q+s}(M)$ by

$$e(\omega)\eta = \omega \wedge \eta.$$

Definition 2.1.4. We denote by $\mathscr{X}(M)$ the set of smooth vector fields on M. For $X \in \mathscr{X}(M)$ we define interior product operators by

$$i(X) : A^{p,q}(M) \to A^{p-1,q}(M), \quad i(X)\omega = \omega(X, \ldots; \ldots),$$
$$\bar{i}(X) : A^{p,q}(M) \to A^{p,q-1}(M), \quad \bar{i}(X)\omega = \omega(\ \ldots; X, \ldots).$$

We denote by $\{E_1, \ldots, E_n\}$ an orthonormal frame field on TM and $\{\theta^1, \ldots, \theta^n\}$ the dual frame field of $E_1, \ldots, E_n$. It should be remarked that

$$\theta^j = \bar{i}(E_j)g, \quad \bar{\theta}^j = i(E_j)g.$$

Definition 2.1.5. We define $L : A^{p,q}(M) \to A^{p+1,q+1}(M)$ and $\Lambda : A^{p,q}(M) \to A^{p-1,q-1}(M)$ by

$$L := e(g) = \sum_j e(\theta^j)e(\bar{\theta}^j), \quad \Lambda := \sum_j i(E_j)\bar{i}(E_j).$$

Proposition 2.1.6. *We have*

$$[L, \Lambda] = p + q - n, \quad \textit{on } A^{p,q}(M).$$

Proof. We obtain

$$\begin{aligned}\Lambda L &= \sum_{j,k} i(E_j)\bar{i}(E_j)e(\theta^k)e(\bar{\theta}^k) = \sum_{j,k} i(E_j)e(\theta^k)\bar{i}(E_j)e(\bar{\theta}^k)\\ &= \sum_{j,k}\{\delta_j^k - e(\theta^k)i(E_j)\}\{\delta_j^k - e(\bar{\theta}^k)\bar{i}(E_j)\}\\ &= n - \sum_j e(\theta^j)i(E_j) - \sum_j e(\bar{\theta^j})\bar{i}(E_j) + L\Lambda.\end{aligned}$$

On $A^{p,q}(M)$ we have $\sum_j e(\theta^j)i(E_j) = p$ and $\sum_j e(\bar{\theta^j})\bar{i}(E_j) = q$. Hence

$$[L, \Lambda] = p + q - n.$$

□

Definition 2.1.7. We define the star operator $\bigstar : A^{p,q}(M) \to A^{n-p,n-q}(M)$ by

$$\omega \wedge \bigstar\eta = \langle \omega, \eta \rangle v_g \otimes \bar{v}_g, \quad \omega, \eta \in A^{p,q}(M),$$

where $\langle\ ,\ \rangle$ is a fiber metric on $\wedge^p T^*M \otimes \wedge^q T^*M$ induced by g and v_g is the volume form of g.

For a multi-index $I_p = (i_1, \dots, i_p)$ such that $1 \le i_1 < \cdots < i_p \le n$, let $I_{n-p} = \{i_{p+1}, \dots, i_n\}$ satisfies $1 \le i_{p+1} < \cdots < i_n \le n$ and the condition that (I_p, I_{n-p}) is a permutation of $(1, \dots, n)$. We denote by $\epsilon_{(I_p, I_{n-p})}$ the sgnature of the permutation (I_p, I_{n-p}). By the definition of the star operator, we have

$$\bigstar(\theta^{I_p} \otimes \bar{\theta}^{J_q}) = \epsilon_{(I_p,I_{n-p})}\epsilon_{(J_q,J_{n-q})}\theta^{I_{n-p}} \otimes \bar{\theta}^{J_{n-q}}.$$

Lemma 2.1.8. (1) *The following identities hold on $A^{p,q}(M)$.*

$$i(X) = (-1)^{p+1}\bigstar^{-1}e(\bar{i}(X)g)\bigstar, \quad \bar{i}(X) = (-1)^{q+1}\bigstar^{-1}e(i(X)g)\bigstar.$$

(2) *The following equations hold for $\omega \in A^{p,q}(M)$, $\eta \in A^{p-1,q}(M), \rho \in A^{p,q-1}(M)$ and $X \in \mathscr{X}(M)$.*

$$\langle i(X)\omega, \eta \rangle = \langle \omega, e(\bar{i}(X)g)\eta \rangle, \quad \langle \bar{i}(X)\omega, \rho \rangle = \langle \omega, e(i(X)g)\rho \rangle,$$

where $\langle\ ,\ \rangle$ is a fiber metric on $\wedge^p T^*M \otimes \wedge^q T^*M$ induced by g.

(3) *The following equation holds for $\omega \in A^{p,q}(M)$ and $\eta \in A^{p-1,q-1}(M)$.*

$$\langle \Lambda\omega, \eta \rangle = \langle \omega, L\eta \rangle.$$

Proof. If $i \notin I_p$, we have $i(E_i)(\theta^{I_p} \otimes \bar{\theta}^{J_q}) = 0$ and $e(\theta^i)(\theta^{I_p} \otimes \bar{\theta}^{J_q}) = 0$. For $i_k \in I_p$ we put $I_{p-1}^k = I_p \setminus \{i_k\}$. If i_k is the l-th index in I_{n-p+1}^k, we have $\epsilon_{(I_{p-1}^k, I_{n-p+1}^k)} = (-1)^{p+k+l+1}\epsilon_{(I_p,I_{n-p})}$. Hence we obtain

$$\begin{aligned}\bigstar i(E_{i_k})(\theta^{I_p} \otimes \overline{\theta^{J_q}}) &= (-1)^{k+1}\bigstar(\theta^{\hat{I}_{p-1}} \otimes \overline{\theta^{J_q}})\\ &= (-1)^{k+1}\epsilon_{(\hat{I}_{p-1},\hat{I}_{n-p+1})}\epsilon_{(J_q,J_{n-q})}\theta^{\hat{I}_{n-p+1}} \otimes \overline{\theta^{J_{n-q}}}\\ &= (-1)^{p+l}\epsilon_{(I_p,I_{n-p})}\epsilon_{(J_q,J_{n-q})}\theta^{\hat{I}_{n-p+1}} \otimes \overline{\theta^{J_{n-q}}}\\ &= (-1)^{p+1}\epsilon_{(I_p,I_{n-p})}\epsilon_{(J_q,J_{n-q})}e(\theta^{i_k})\theta^{I_{n-p}} \otimes \overline{\theta^{J_{n-q}}}\\ &= (-1)^{p+1}e(\theta^{i_k})\bigstar(\theta^{I_p} \otimes \overline{\theta^{J_q}}).\end{aligned}$$

This implies (1). It follows from (1) that

$$\begin{aligned}\langle i(X)\omega,\eta\rangle v_g\otimes\bar v_g &= \eta\wedge\bigstar i(X)\omega\\ &= \eta\wedge(-1)^{p+1}e(\bar i(X)g)\bigstar\omega\\ &= e(\bar i(X)g)\eta\wedge\bigstar\omega\\ &= \langle\omega, e(\bar i(X)g)\eta\rangle v_g\otimes\bar{v_g}.\end{aligned}$$

This implies (2) and we immediately have (3) from (2). □

2.2 Differential operators for (p,q)-forms

We define two operators ∂ and $\bar\partial$ by using the flat connection D.

Definition 2.2.1. We define $\partial: A^{p,q}(M)\to A^{p+1,q}(M)$ and $\bar\partial: A^{p,q}(M)\to A^{p,q+1}(M)$ by

$$\partial=\sum_i e(dx^i)D_{\frac{\partial}{\partial x^i}},\quad \bar\partial=\sum_i e(\overline{dx^i})D_{\frac{\partial}{\partial x^i}}.$$

Since D is flat, we immediately obtain the following lemma.

Lemma 2.2.2. *We have*

$$\partial^2=0,\quad \bar\partial^2=0,\quad \partial\bar\partial=\bar\partial\partial.$$

We denote by $A^{p,q}(M,F)$ the space of F-valued (p,q)-forms. Since the transition functions of $\{s\}$ are constant, ∂ and $\bar\partial$ are extended on $A^{p,q}(M,F)$ by

$$\partial(s\otimes\omega)=s\otimes\partial\omega,$$

$$\bar\partial(s\otimes\omega)=s\otimes\bar\partial\omega.$$

Definition 2.2.3. We denote by $A_0^{p,q}(M,F)$ the space of elements of $A^{p,q}(M,F)$ with compact supports. We define the inner product $(\ ,\ )$ on $A_0^{p,q}(M,F)$ by

$$(\omega,\eta)=\int_M\langle\omega,\eta\rangle\, v_g,$$

where $v_g=\sqrt{\det[g_{ij}]}\,dx^1\wedge\cdots\wedge dx^n$, and $\langle\ ,\ \rangle$ is the metric on $F\otimes\wedge^pT^*M\otimes\wedge^qT^*M$ induced by g and h. We set $\|\omega\|=\sqrt{(\omega,\omega)}$.

Definition 2.2.4. We define $A\in A^{1,0}(M)$ and $B\in A^{1,1}(M)$ by

$$A=-\partial\log h(s,s),\quad B=\bar\partial A.$$

We call A and B the *first Koszul form* and the *second Koszul form* with respect to the fiber metric h, respectively.

Remark that since the transition functions of $\{s\}$ are constant, A and B are globally well-defined.

Example 2.2.5. Let α and β be the first Koszul form and the second Koszul form with respect to the Riemannian metric g, respectively. Then the the first Koszul form A_K and the second Koszul form B_K with respect to the fiber metric g on the canonical bundle $K=\wedge^nT^*M$ are given by

$$A_K=2\alpha,\quad B_K=2\beta.$$

The following theorem is an analogue of the Kodaira-Nakano vanishing theorem.

Theorem 2.2.6. *[5] Let (M, D) be an oriented n-dimensional compact flat manifold and (F, D^F) be a flat line bundle over M. We set*

$$H^{p,q}_{\bar\partial}(M, F) = \frac{Ker\ [\ \bar\partial : A^{p,q}(M, F) \to A^{p,q+1}(M, F)\ ]}{Im\ [\ \bar\partial : A^{p,q-1}(M, F) \to A^{p,q}(M, F)\ ]}.$$

Assume there exist a fiber metric h on F and a Riemannian metric g on M such that $B + \beta$ is positive definite, where B and β are the second Koszul forms with respect to h and g, respectively. Then we have

$$H^{p,q}_{\bar\partial}(M, F) = 0, \quad \textit{for } p + q > n.$$

The star operator $\bigstar$ is extended on $A^{p,q}(M, F)$ by

$$\bigstar(s \otimes \omega) = s \otimes \bigstar\omega.$$

Definition 2.2.7. We define $\delta_F : A^{p,q}(M, F) \to A^{p-1,q}(M, F)$ and $\bar\delta_F : A^{p,q}(M, F) \to A^{p,q-1}(M, F)$ by

$$\delta_F = (-1)^p \bigstar^{-1}\partial\bigstar + i(X_{A+\alpha}), \quad \bar\delta_F = (-1)^q \bigstar^{-1}\bar\partial\bigstar + \bar i(X_{A+\alpha}),$$

where $\bar i(X_{A+\alpha})g = A + \alpha$. The operators will be denoted by δ and $\bar\delta$ if (F, D^F, h) is trivial.

Proposition 2.2.8. *The operators δ_F and $\bar\delta_F$ are the adjoint operators of ∂ and $\bar\partial$ with respect to the inner product $(\ ,\ )$ respectively, that is, for $\omega \in A^{p,q}(M, F)$, $\eta \in A^{p-1,q}_0(M, F)$ and $\rho \in A^{p,q-1}_0(M, F)$ we have*

$$(\delta_F\omega, \eta) = (\omega, \partial\eta), \ (\bar\delta_F\omega, \rho) = (\omega, \bar\partial\rho).$$

Proof. Asumme ω and η are locally expressed by $\omega = s \otimes \omega'$ and $\eta = s \otimes \eta'$. Then we have

$$\begin{aligned}
\langle\omega, \partial\eta\rangle v_g \otimes \bar v_g &= h(s, s)\langle\omega', \partial\eta'\rangle v_g \otimes \bar v_g \\
&= h(s, s)\partial\eta' \wedge \bigstar\omega' \\
&= \partial(h(s, s)\eta' \wedge \bigstar\omega') - (-1)^{p-1}h(s, s)\eta' \wedge \partial\bigstar\omega' - \partial h(s, s) \wedge \eta' \wedge \bigstar\omega' \\
&= \partial(h(s, s)\eta' \wedge \bigstar\omega') + h(s, s)\eta' \wedge (-1)^p\partial\bigstar\omega' + h(s, s)e(A)\eta' \wedge \bigstar\omega' \\
&= \partial(h(s, s)\eta' \wedge \bigstar\omega') + \left(\langle(-1)^p\bigstar^{-1}\partial\bigstar\omega, \eta\rangle + \langle\omega, e(A)\eta\rangle\right) v_g \otimes \bar v_g.
\end{aligned}$$

Let $v^*_g \in A^{0,0}(M, K^*)$ be the dual section of v_g and $\phi \in A^{n-1,0}(M)$ satisfy $\phi \otimes \bar v_g = h(s, s)\eta' \wedge \bigstar\omega'$. Since $\partial v^*_g = -v^*_g \otimes \alpha$ and $\partial(v^*_g \otimes \bar v_g) = 0$, we obtain

$$\begin{aligned}
v^*_g \otimes \partial(h(s, s)\eta' \wedge \bigstar\omega') &= \partial(v^*_g \otimes \phi \otimes \bar v_g) - \partial v^*_g \otimes h(s, s)\eta' \wedge \bigstar\omega' \\
&= v^*_g \otimes d\phi \otimes \bar v_g + v^*_g \otimes \langle\omega, e(\alpha)\eta\rangle v_g \otimes \bar v_g.
\end{aligned}$$

Taking the contraction of v^*_g and $\bar v_g$, we have

$$\begin{aligned}
\langle\omega, \partial\eta\rangle v_g &= d\phi + \left(\langle(-1)^p\bigstar^{-1}\partial\bigstar\omega, \eta\rangle + \langle\omega, e(A + \alpha)\eta\rangle\right) v_g \\
&= d\phi + \langle\left((-1)^p\bigstar^{-1}\partial\bigstar + i(X_{A+\alpha})\right)\omega, \eta\rangle v_g,
\end{aligned}$$

where the last equality follows from Lemma 2.1.8 (2). From Stokes' theorem we obtain

$$(\omega, \partial\eta) = (\delta_F\omega, \eta).$$

$\square$

DEFINITION 2.2.9. We define the connection $\mathscr{D}$ and $\overline{\mathscr{D}}$ on $\wedge^p T^*M \otimes \wedge^q T^*M$ as follows: For $\omega \in A^p(M)$ and $\eta \in A^q(M), X \in \mathscr{X}(M)$

$$\mathscr{D}_X(\omega \otimes \bar{\eta}) = 2\gamma_X \omega \otimes \bar{\eta} + D_X(\omega \otimes \bar{\eta}),$$
$$\overline{\mathscr{D}}_X(\omega \otimes \bar{\eta}) = 2\omega \otimes \overline{\gamma_X \eta} + D_X(\omega \otimes \bar{\eta}),$$

where $\gamma = \nabla - D$ and ∇ is the Levi-Civita connection of g (c.f. Definition 1.1.3).

We obtain the following lemma by definition.

LEMMA 2.2.10. *We have*

$$\mathscr{D}(v_g \otimes \bar{v}_g) = \overline{\mathscr{D}}(v_g \otimes \bar{v}_g) = 0.$$

LEMMA 2.2.11. *The following conditions are equivalent.*

(1) (D, g) *is a Hessian structure.*

(2) $\partial g = 0 \quad (\Leftrightarrow \bar{\partial} g = 0)$.

(3) $\mathscr{D} g = 0 \quad (\Leftrightarrow \overline{\mathscr{D}} g = 0)$.

PROOF. We have

$$\begin{aligned}\partial g = \partial(\sum_{i,j} g_{ij} dx^i \otimes \overline{dx^j}) &= \sum_{k,i,j} \frac{\partial g_{ij}}{\partial x^k} dx^k \wedge dx^i \otimes \overline{dx^j} \\ &= \sum_j (\sum_{k<i} (\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}) dx^k \wedge dx^i) \otimes dx^j,\end{aligned}$$

$$\begin{aligned}(\mathscr{D}_X g)(Y; Z) &= Xg(Y; Z) - g((2\gamma_X + D_X)Y; Z) - g(Y; D_X Z) \\ &= g(\nabla_X Y; Z) + g(Y; \nabla_X Z) - g((\gamma_X + \nabla_X)Y; Z) - g(Y; D_X Z) \\ &= -g(\gamma_X Y; Z) + g(Y; \gamma_X Z).\end{aligned}$$

Hence it follows from Proposition 1.1.5 that the conditions $(1) - (3)$ are equivalent. □

Let D^* be the dual connection of D with respect to g (c.f. Definition 1.1.4). We obtain the following from Proposition 1.1.5.

LEMMA 2.2.12. *Let (D, g) be a Hessian structure. Then we have*

$$\mathscr{D}_X(\omega \otimes \bar{\eta}) = D^*_X \omega \otimes \bar{\eta} + \omega \otimes \overline{D_X \eta},$$
$$\overline{\mathscr{D}}_X(\omega \otimes \bar{\eta}) = D_X \omega \otimes \bar{\eta} + \omega \otimes \overline{D^*_X \eta},$$

for $\omega \in A^p(M)$ and $\eta \in A^q(M), X \in \mathscr{X}(M)$.

LEMMA 2.2.13. *Let (D, g) be a Hessian structure. Then we have*

$$\mathscr{D}\bigstar = \bigstar\overline{\mathscr{D}}.$$

PROOF. For $\omega, \eta \in A^{p,q}(M)$ we have

$$\mathscr{D}_X(\omega \wedge \bigstar\eta) = \mathscr{D}_X \omega \wedge \bigstar\eta + \omega \wedge \mathscr{D}_X \bigstar\eta.$$

From Lemma 2.2.10 and 2.2.12, we also have

$$\begin{aligned}\mathscr{D}_X(\omega \wedge \bigstar\eta) &= \mathscr{D}_X(\langle\omega, \eta\rangle v_g \otimes \bar{v}_g) \\ &= X\langle\omega, \eta\rangle v_g \otimes \bar{v}_g \\ &= (\langle\mathscr{D}_X \omega, \eta\rangle + \langle\omega, \overline{\mathscr{D}}_X \eta\rangle) v_g \otimes \bar{v}_g \\ &= \mathscr{D}_X \omega \wedge \bigstar\eta + \omega \wedge \bigstar\overline{\mathscr{D}}_X \eta.\end{aligned}$$

Hence we obtain $\mathscr{D}\bigstar = \bigstar\overline{\mathscr{D}}$. □

When (D, g) is a Hessian structure, the operators ∂, $\bar{\partial}$, δ_F and $\bar{\delta}_F$ are expressed with $\mathscr{D}$ and $\bar{\mathscr{D}}$.

Proposition 2.2.14. *Let (D, g) be a Hessian structure. Then we have*

$$\partial = \sum_j e(\theta^j)\mathscr{D}_{E_j}, \quad \bar{\partial} = \sum_j e(\bar{\theta}^j)\overline{\mathscr{D}}_{E_j},$$

$$\delta_F = -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j} + i(X_{A+\alpha}), \quad \bar{\delta}_F = -\sum_j \bar{i}(E_j)\mathscr{D}_{E_j} + \bar{i}(X_{A+\alpha}),$$

where $\bar{i}(X_{A+\alpha})g = A + \alpha$.

Proof. Since D^* is torsion-free, on $A^p(M)$ we have

$$\sum_j e(\theta^j)D^*_{E_j} = \sum_j e(\theta^j)D_{E_j} = \sum_j e(dx^j)D_{\frac{\partial}{\partial x^j}} = \partial\ (= d).$$

Hence by Lemma 2.2.12, for $\omega \in A^p(M)$ and $\eta \in A^q(M)$ we obtain

$$\begin{aligned}
\sum_j e(\theta^j)\mathscr{D}_{E_j}(\omega \otimes \bar{\eta}) &= \sum_j e(\theta^j)(D^*_{E_j}\omega \otimes \bar{\eta} + \omega \otimes \overline{D_{E_j}\eta}) \\
&= \sum_j e(\theta^j)(D_{E_j}\omega \otimes \bar{\eta} + \omega \otimes \overline{D_{E_j}\eta}) \\
&= \sum_j e(\theta^j)D_{E_j}(\omega \otimes \bar{\eta}) \\
&= \partial(\omega \otimes \bar{\eta}).
\end{aligned}$$

This implies $\partial = \sum_j e(\theta^j)\mathscr{D}_{E_j}$. Furthermore, by Lemma 2.2.13 and 2.1.8 (1), on $A^{p,q}(M, F)$ we have

$$\begin{aligned}
(-1)^p \bigstar^{-1}\partial\bigstar &= (-1)^p\bigstar^{-1}\sum_j e(\theta^j)\mathscr{D}_{E_j}\bigstar \\
&= \sum_j (-1)^p\bigstar^{-1}e(\theta^j)\bigstar\overline{\mathscr{D}}_{E_j} \\
&= -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j}.
\end{aligned}$$

Hence we obtain $\delta_F = -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j} + i(X_{A+\alpha})$.

□

2.3 The new differential operator ∂'_F

We introduce the operator ∂'_F which is not defined in [5].

Definition 2.3.1. We define the differential operator $\partial'_F : A^{p,q}(M, F) \to A^{p+1,q}(M, F)$ by

$$\partial'_F = \partial - e(A + \alpha).$$

The operator will be denoted by ∂' if (F, D^F, h) is trivial.

Theorem 2.3.2. *We have*

$$(\partial'_F)^2 = 0, \quad \partial'_F\bar{\partial} - \bar{\partial}\partial'_F = e(B + \beta).$$

PROOF. We obtain

$$\begin{aligned}
(\partial'_F)^2 &= (\partial - e(A+\alpha))(\partial - e(A+\alpha)) \\
&= \partial^2 - e(\partial(A+\alpha)) + e(A+\alpha)\partial - e(A+\alpha)\partial + e(A+\alpha)e(A+\alpha) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\partial'_F\bar{\partial} &= (\partial - e(A+\alpha))\bar{\partial} = \partial\bar{\partial} - e(A+\alpha)\bar{\partial}, \\
\bar{\partial}\partial'_F &= \bar{\partial}(\partial - e(A+\alpha)) = \bar{\partial}\partial - e(B+\beta) - e(A+\alpha)\bar{\partial}.
\end{aligned}$$

Hence

$$\partial'_F\bar{\partial} - \bar{\partial}\partial'_F = e(B+\beta).$$

□

DEFINITION 2.3.3. We define $\delta'_F : A^{p,q}(M,F) \to A^{p-1,q}(M,F)$ by

$$\delta'_F = \delta_F - i(X_{A+\alpha}),$$

where $\bar{i}(X)g = A + \alpha$. The operator will be denoted by δ' if (F, D^F, h) is trivial.

We obtain the following Corollaries from Lemma 2.1.8, Proposition 2.2.8 and 2.2.14.

COROLLARY 2.3.4. *The operator δ'_F is the adjoint operator of ∂'_F with respect to the inner product $(\ ,\ )$, that is, for $\omega \in A^{p,q}(M,F)$, $\eta \in A_0^{p-1,q}(M,F)$ we have*

$$(\delta'_F\omega, \eta) = (\omega, \partial'_F\eta).$$

COROLLARY 2.3.5. *Let (D,g) be a Hessian structure. Then we have*

$$\delta'_F = -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j}.$$

The following theorem is an analogue of the Kähler identities.

THEOREM 2.3.6. *Let (D,g) be a Hessian structure. Then we have*

$$\Lambda\partial'_F + \partial'_F\Lambda = -\bar{\delta}_F, \quad \Lambda\bar{\partial} + \bar{\partial}\Lambda = -\delta'_F,$$

$$L\delta'_F + \delta'_F L = -\bar{\partial}, \quad L\bar{\delta}_F + \bar{\delta}_F L = -\partial'_F.$$

PROOF. It follows from Corollary 2.3.5, Proposition 2.2.11 and 2.2.14 that

$$\begin{aligned}
\delta'_F L &= -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j} L = -\sum_j i(E_j) L \overline{\mathscr{D}}_{E_j} \\
&= -\sum_{j,k} i(E_j)e(\theta^k)e(\bar{\theta}^k)\overline{\mathscr{D}}_{E_j} \\
&= -\sum_{j,k} e(\bar{\theta}^k)(\delta^k_j - e(\theta^k)i(E_j))\overline{\mathscr{D}}_{E_j} \\
&= -\sum_j e(\bar{\theta}^j)\overline{\mathscr{D}}_{E_j} + \sum_k e(\bar{\theta}^k)e(\theta^k)\sum_j i(E_j)\overline{\mathscr{D}}_{E_j} \\
&= -\bar{\partial} - L\delta'_F.
\end{aligned}$$

Similarly, we have

$$-\sum_j \bar{i}(E_j)\mathscr{D}_{E_j} L = -\partial + L \sum_j \bar{i}(E_j)\mathscr{D}_{E_j}.$$

Moreover,

$$\begin{aligned}\bar{i}(X_{A+\alpha})L &= \bar{i}(X_{A+\alpha})\sum_k e(\theta^k)e(\bar{\theta}^k)\\ &= \sum_k e(\theta^k)\{(A+\alpha)(E_k) - e(\bar{\theta}^k)\bar{i}(X_{A+\alpha})\}\\ &= e(A+\alpha) - L\bar{i}(X_{A+\alpha}).\end{aligned}$$

Hence it follows from Proposition 2.2.14 that

$$\begin{aligned}\bar{\delta}_F L &= (-\sum_j \bar{i}(E_j)\mathscr{D}_{E_j} + \bar{i}(X_{A+\alpha}))L\\ &= -(\partial - e(A+\alpha)) - L(-\sum_j \bar{i}(E_j)\mathscr{D}_{E_j} + \bar{i}(X_{A+\alpha}))\\ &= -\partial'_F - L\bar{\delta}_F.\end{aligned}$$

We have the other equalities by taking the adjoint operators. □

Definition 2.3.7. We define the Laplacians $\Box'_F$ and $\bar{\Box}$ with respect to ∂'_F and $\bar{\partial}$ by

$$\Box'_F = \partial'_F\delta'_F + \delta'_F\partial'_F, \quad \bar{\Box}_F = \bar{\partial}\bar{\delta}_F + \bar{\delta}_F\bar{\partial}.$$

The Laplacians will be denoted by $\Box'$ and $\bar{\Box}$ if (F, D^F, h) is trivial.

The following theorem is an analogue of the Kodaira-Nakano identity.

Theorem 2.3.8. *Let (D, g) be a Hessian structure. Then we have*

$$\bar{\Box}_F = \Box'_F + [e(\beta + B), \Lambda].$$

Proof. It follows from Theorem 2.3.2 and 2.3.6 that

$$\begin{aligned}\bar{\Box}_F &= \bar{\partial}\bar{\delta}_F + \bar{\delta}_F\bar{\partial} = -\bar{\partial}(\Lambda\partial'_F + \partial'_F\Lambda) - (\Lambda\partial'_F + \partial'_F\Lambda)\bar{\partial}\\ &= (\Lambda\bar{\partial} + \delta'_F)\partial'_F - \bar{\partial}\partial'_F\Lambda - \Lambda\partial'_F\bar{\partial} + \partial'_F(\bar{\partial}\Lambda + \delta'_F)\\ &= \delta'_F\partial'_F + \partial'_F\delta'_F + (\partial'_F\bar{\partial} - \bar{\partial}\partial'_F)\Lambda - \Lambda(\partial'_F\bar{\partial} - \bar{\partial}\partial'_F)\\ &= \Box'_F + [e(B+\beta), \Lambda].\end{aligned}$$

□

3 Vanishing theorems of L^2-cohomology groups

We introduce L^2-cohomology groups on flat manifolds and some vanishing theorems.

3.1 L^2-cohomology groups on flat manifolds

We denote by $L^2(M, F \otimes \wedge^{p,q})$ the completion of $A_0^{p,q}(M, F)$ with respect to the L^2-inner product $(\ ,\ )$ induced by g and h. The space $L^2(M, F \otimes \wedge^{p,q})$ is identified with the space of square-integrable sections of $F \otimes \wedge^{p,q}$.

DEFINITION 3.1.1. For $\omega \in L^2(M, F \otimes \wedge^{p,q})$ we define $\bar{\partial}\omega$ and $\bar{\delta}_F\omega$ as follows:

$$(\bar{\partial}\omega, \eta) = (\omega, \bar{\delta}_F \eta), \quad \text{for } \eta \in A_0^{p,q+1}(M, F),$$

$$(\bar{\delta}_F\omega, \rho) = (\omega, \bar{\partial}\rho), \quad \text{for } \rho \in A_0^{p,q-1}(M, F).$$

In general, we cannot say $\bar{\partial}\omega \in L^2(M, F \otimes \wedge^{p,q+1})$ and $\bar{\delta}_F\omega \in L^2(M, F \otimes \wedge^{p,q-1})$. We set

$$\begin{aligned} W(M, F \otimes \wedge^{p,q}) &= \{\omega \in L^2(M, F \otimes \wedge^{p,q}) \mid \bar{\partial}\omega \in L^2(M, F \otimes \wedge^{p,q+1}), \bar{\delta}_F\omega \in L^2(M, F \otimes \wedge^{p,q-1})\}, \\ D(M, F \otimes \wedge^{p,q}) &= \{\omega \in L^2(M, F \otimes \wedge^{p,q}) \mid \bar{\partial}\omega \in L^2(M, F \otimes \wedge^{p,q+1})\}. \end{aligned}$$

In addition, we define the norm $\| \ \|_W$ on $W(M, F \otimes \wedge^{p,q})$ by

$$\|\omega\|_W = \|\omega\| + \|\bar{\partial}\omega\| + \|\bar{\delta}_F\omega\|, \quad \omega \in W(M, F \otimes \wedge^{p,q}).$$

The space $W(M, F \otimes \wedge^{p,q})$ is complete with respect to $\| \ \|_W$.

PROPOSITION 3.1.2. *[7] If g is complete, the space $A_0^{p,q}(M, F)$ is dense in $W(M, F \otimes \wedge^{p,q})$ with respect to the L^2-norm $\| \ \|_W$.*

DEFINITION 3.1.3. We define the *L^2-cohomology group* of (p, q)-type by

$$L^2 H_{\bar{\partial}}^{p,q}(M, F) = \frac{\operatorname{Ker} [\ \bar{\partial} : D(M, F \otimes \wedge^{p,q}) \to D(M, F \otimes \wedge^{p,q+1})\ ]}{\overline{\operatorname{Im} [\ \bar{\partial} : D(M, F \otimes \wedge^{p,q-1}) \to D(M, F \otimes \wedge^{p,q})\ ]}},$$

where $\overline{\operatorname{Im} [\ \bar{\partial} : D(M, F \otimes \wedge^{p,q-1}) \to D(M, F \otimes \wedge^{p,q})\ ]}$ is the closure of $\operatorname{Im} [\ \bar{\partial} : D(M, F \otimes \wedge^{p,q-1}) \to D(M, F \otimes \wedge^{p,q})\ ]$ with respect to the L^2-norm $\| \ \|$.

3.2 Vanishing theorems of Kodaira-Nakano type

In this section we show vanishing theorems of Kodaira-Nakano type.

LEMMA 3.2.1. *Assume g is a Hessian metric and $B + \beta$ is positive definite. For the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of the matrix $\Big[\sum_k g^{ik}(B + \beta)_{kj}\Big]$, we set $b_q = \sum_{j=1}^{q} \lambda_j$. Then we have*

$$\|\bar{\partial}\omega\|^2 + \|\bar{\delta}_F\omega\|^2 \geq \|b_q^{\frac{1}{2}}\omega\|^2, \quad \textit{for } \omega \in A_0^{n,q}(M, F).$$

PROOF. By Theorem 2.3.8 we obtain

$$\begin{aligned} \|\bar{\partial}\omega\|^2 + \|\bar{\delta}_F\omega\|^2 &= (\bar{\Box}_F\omega, \omega) \\ &= (\Box'_F\omega, \omega) + ([e(B + \beta), \Lambda]\omega, \omega) \\ &\geq ([e(B + \beta), \Lambda]\omega, \omega) \\ &= (e(B + \beta)\Lambda\omega, \omega). \end{aligned}$$

Hence it is sufficient to show $(e(B + \beta)\Lambda\omega, \omega) \geq \|b_q^{\frac{1}{2}}\omega\|^2$.

We take the orthonormal frame field $\{E_1, \ldots, E_n\}$ on TM, where the matrix $[(B + \beta)(E_i, E_j)]$ is diagonal. We set $\mu_j = (B + \beta)(E_j, E_j)$. Using the dual frame field $\{\theta^1, \ldots, \theta^n\}$ of $\{E_1, \ldots, E_n\}$, $\omega \in A_0^{n,q}(M, F)$ is denoted by

$$\omega = \sum_{J_q} \omega_{J_q} \otimes \bar{\theta}^{J_q}, \quad \omega_{J_q} \in A_0^n(M, F).$$

Then

$$\begin{aligned}
e(B+\beta)\Lambda\omega &= \sum_j \mu_j e(\theta^j)e(\bar\theta^j)\sum_k i(E_k)\bar i(E_k)\sum_{J_q}\omega_{J_q}\otimes\bar\theta^{J_q}\\
&= \sum_{j,J_q}\mu_j\omega_{J_q}\otimes e(\bar\theta^j)\bar i(E_j)\bar\theta^{J_q}\\
&= \sum_{J_q}\sum_{j\in J_q}\mu_j\omega_{J_q}\otimes\bar\theta^{J_q}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(e(B+\beta)\Lambda\omega,\omega) &= \int_M\sum_{J_q}\sum_{j\in J_q}\mu_j\langle\omega_{J_q},\omega_{J_q}\rangle v_g\\
&\geq \int_M\sum_{J_q} b_q\langle\omega_{J_q},\omega_{J_q}\rangle v_g = \|b_q^{\frac{1}{2}}\omega\|^2.
\end{aligned}$$

□

THEOREM 3.2.2. *Let (M,D,g) be an oriented n-dimensional complete Hessian manifold and (F,D^F) a flat line bundle over M. We denote by h a fiber metric on F. Assume $B+\beta$ is positive definite, where B and β are the second Koszul forms with respect to fiber metric h and Hessian metric g respectively. For $q\geq 1$ let b_q be the same as in Lemma 3.2.1. Then for all $v\in L^2(M,F\otimes\wedge^{n,q})$ such that $\bar\partial v=0$ and $b_q^{-\frac{1}{2}}v\in L^2(M,F\otimes\wedge^{n,q})$, there exists $u\in L^2(M,F\otimes\wedge^{n,q-1})$ such that*

$$\bar\partial u = v,\quad \|u\|\leq\|b_q^{-\frac{1}{2}}v\|.$$

In particular, if there exists $\varepsilon>0$ such that $B+\beta-\varepsilon g$ is positive semi-definite, we have

$$L^2H^{n,q}_{\bar\partial}(M,F)=0,\quad \textit{for } q\geq 1.$$

PROOF. The theorem can be shown by applying the method as in complex analysis in several variables (c.f Lemma 4.1.1 of [8]) to the case of Hessian manifolds.

We set $\mathrm{Ker}\bar\partial=\{\omega\in L^2(M,F\otimes\wedge^{n,q})\mid\bar\partial\omega=0\}$. Since $\mathrm{Ker}\bar\partial$ is a closed subspace in $L^2(M,F\otimes\wedge^{n,q})$, we have

$$L^2(M,F\otimes\wedge^{n,q})=\mathrm{Ker}\bar\partial\oplus(\mathrm{Ker}\bar\partial)^\perp,$$

where $(\mathrm{Ker}\bar\partial)^\perp$ is the orthogonal complement of $\mathrm{Ker}\bar\partial$. A (n,q)-form $\omega\in L^2(M,F\otimes\wedge^{p,q})$ is expressed by

$$\omega=\omega_1+\omega_2,\quad \omega_1\in\mathrm{Ker}\bar\partial,\quad \omega_2\in(\mathrm{Ker}\bar\partial)^\perp.$$

For $\eta\in A_0^{n,q-1}(M,F)$, we have

$$(\bar\delta_F\omega,\eta)=(\omega,\bar\partial\eta)=0,$$

and so

$$\bar\delta_F\omega_2=0.$$

Since $v\in\mathrm{Ker}\bar\partial$ by assumption, we obtain

$$|(v,\omega)|^2=|(v,\omega_1)|^2=|(b_q^{-\frac{1}{2}}v,b_q^{\frac{1}{2}}\omega_1)|^2\leq\|b_q^{-\frac{1}{2}}v\|^2\|b_q^{\frac{1}{2}}\omega_1\|^2.$$

Assume $\omega\in W(M,F\otimes\wedge^{n,q})$. Then

$$\bar\partial\omega_1=0,\quad \bar\delta_F\omega_1=\bar\delta_F\omega\in L^2(M,F\otimes\wedge^{n,q-1}),$$

and so $\omega_1 \in W(M, F \otimes \wedge^{n,q})$. Hence by Proposition 3.1.2, ω_1 satisfies the inequality in Lemma 3.2.1:

$$\|b_q^{\frac{1}{2}} \omega_1\|^2 \leq \|\bar{\partial}\omega_1\|^2 + \|\bar{\delta}_F \omega_1\|^2 = \|\bar{\delta}_F \omega_1\|^2 = \|\bar{\delta}_F \omega\|^2 < \infty.$$

Therefore for $\omega \in W(M, F \otimes \wedge^{n,q})$ we have

$$|(v, \omega)|^2 \leq \|b_q^{-\frac{1}{2}} v\|^2 \|\bar{\delta}_F \omega\|^2 < \infty.$$

By this inequality a linear functional $\lambda : \bar{\delta}_F W(M, F \otimes \wedge^{n,q}) \ni \bar{\delta}_F \omega \mapsto (v, \omega) \in \mathbb{R}$ is well-defined and the operator norm C is

$$C \leq \|b_q^{-\frac{1}{2}} v\| < \infty.$$

We set $\mathrm{Ker}\bar{\delta}_F = \{\omega \in L^2(M, F \otimes \wedge^{n,q}) \mid \bar{\delta}_F \omega = 0\}$. $\mathrm{Ker}\bar{\delta}_F$ is also a closed subspace in $L^2(M, F \otimes \wedge^{n,q})$ and

$$L^2(M, F \otimes \wedge^{n,q}) = \mathrm{Ker}\bar{\delta}_F \oplus (\mathrm{Ker}\bar{\delta}_F)^{\perp},$$

where $(\mathrm{Ker}\bar{\delta}_F)^{\perp}$ is the orthogonal complement of $\mathrm{Ker}\bar{\delta}_F$. In the same way we have $(\mathrm{Ker}\bar{\delta}_F)^{\perp} \subset \mathrm{Ker}\bar{\partial}$ and for $\hat{\omega} \in (\mathrm{Ker}\bar{\delta}_F)^{\perp} \cap W(M, F \otimes \wedge^{n,q})$,

$$\|b_q^{\frac{1}{2}} \hat{\omega}\|^2 \leq \|\bar{\partial}\hat{\omega}\|^2 + \|\bar{\delta}_F \hat{\omega}\|^2 = \|\bar{\delta}_F \hat{\omega}\|^2.$$

Let $\{\eta_k\} \subset \bar{\delta}_F W(M, F \otimes \wedge^{n,q})$ be a Cauchy sequence with respect to the norm $\| \ \|$ on $L^2(M, F \otimes \wedge^{n,q-1})$. Each η_k is denoted by

$$\eta_k = \bar{\delta}_F \hat{\omega}_k, \quad \hat{\omega}_k \in (\mathrm{Ker}\bar{\delta}_F)^{\perp} \cap W(M, F \otimes \wedge^{n,q}),$$

and by the said inequality $\{\hat{\omega}_k\}$ is also a Cauchy sequence with respect to the norm $\| \ \|$ on $L^2(M, F \otimes \wedge^{n,q})$. This implies $\{\hat{\omega}_k\}$ is a Cauchy sequence with respect to the norm $\| \ \|_W$ on $W(M, F \otimes \wedge^{n,q})$. Hence by completeness of $W(M, F \otimes \wedge^{n,q})$ with respect to $\| \ \|_W$, we have

$$\hat{\omega}_k \to \hat{\omega} \in W(M, F \otimes \wedge^{n,q}) \quad (k \to \infty),$$

and

$$\eta_k \to \bar{\delta}_F \hat{\omega} \quad (k \to \infty).$$

Therefore, $\bar{\delta}_F W(M, F \otimes \wedge^{n,q})$ is a closed space of $L^2(M, F \otimes \wedge^{p,q-1})$ with respect to the norm $\| \ \|$.

From the above, by applying Riesz representation theorem to the linear functional $\lambda : \bar{\delta}_F W(M, F \otimes \wedge^{n,q}) \to \mathbb{R}$, there exists $u \in \bar{\delta}_F W(M, F \otimes \wedge^{n,q})$ such that

$$\begin{cases} \lambda(\eta) = (u, \eta), \quad \eta \in \bar{\delta}_F W(M, F \otimes \wedge^{n,q}) \\ \|u\| = C \leq \|b_q^{-\frac{1}{2}} v\|. \end{cases}$$

By the first equation, for all $\omega \in A_0^{n,q}(M, F)$ we have

$$(v, \omega) = \lambda(\bar{\delta}_F \omega) = (u, \bar{\delta}_F \omega),$$

and so

$$\bar{\partial} u = v.$$

This implies the first assertion.

Suppose there exists $\varepsilon > 0$ such that $B + \beta - \varepsilon g$ is positive semi-definite. Then by the definition of b_q, $b_q \geq \varepsilon q$. Hence for all $v \in L^2(M, F \otimes \wedge^{n,q})$ we obtain

$$\int_M \langle b_q^{-\frac{1}{2}} v, b_q^{-\frac{1}{2}} v \rangle v_g \leq (\varepsilon q)^{-1} \int_M \langle v, v \rangle v_g < \infty,$$

that is,

$$b_q^{-\frac{1}{2}} v \in L^2(M, F \otimes \wedge^{n,q}).$$

This implies the second assertion. $\square$

The following theorem corresponds to Theorem 2.2.6 in the case of complete Hessian manifolds.

THEOREM 3.2.3. *Let (M, D, g) be an oriented n-dimensional complete Hessian manifold and (F, D^F) a flat line bundle over M. We denote by h a fiber metric on F. Assume that there exists $\varepsilon > 0$ such that $B + \beta = \varepsilon g$ where B and β are the second Koszul forms with respect to fiber metric h and Hessian metric g respectively. Then for $p + q > n$ and all $v \in L^2(M, F \otimes \wedge^{p,q})$ such that $\bar{\partial} v = 0$, there exists $u \in L^2(M, F \otimes \wedge^{p,q-1})$ such that*

$$\bar{\partial} u = v, \quad \|u\| \le \{\varepsilon(p + q - n)\}^{-\frac{1}{2}} \|v\|.$$

In particular, we have

$$L^2 H_{\bar{\partial}}^{p,q}(M, F) = 0, \quad \text{for } p + q > n.$$

PROOF. By Proposition 2.1.6, on $A^{p,q}(M, F)$ we have

$$[e(B + \beta), \Lambda] = \varepsilon[L, \Lambda] = \varepsilon(p + q - n).$$

Hence by Theorem 2.3.8, for all $\omega \in A_0^{p,q}(M, F)$ we obtain

$$\|\bar{\partial}\omega\|^2 + \|\bar{\delta}_F \omega\|^2 \ge \varepsilon(p + q - n)\|\omega\|^2.$$

Then the assertions are proved similarly to Theorem 3.2.2. $\square$

COROLLARY 3.2.4. *Let $(\mathbb{R}^n, g)$ be the Euclidean space, D be the canonical affine connection on $\mathbb{R}^n$, and $(F = \mathbb{R}^n \times \mathbb{R}, D^F)$ be the trivial flat line bundle on $\mathbb{R}^n$. In addition, we define a fiber metric h on F by*

$$h(s, s) = e^{-\varphi},$$

where $\varphi(x) = \dfrac{1}{2}\sum_i (x^i)^2$ and $s : \mathbb{R}^n \ni x \mapsto (x, 1) \in F$. Then for $q \ge 1$ and $v \in L^2(\mathbb{R}^n, F \otimes \wedge^{p,q})$ such that $\bar{\partial} v = 0$, there exists $u \in L^2(\mathbb{R}^n, F \otimes \wedge^{p,q-1})$ such that

$$\bar{\partial} u = v, \quad \|u\| \le q^{-\frac{1}{2}} \|v\|.$$

In particular, we have

$$L^2 H_{\bar{\partial}}^{p,q}(\mathbb{R}^n, F) = 0, \quad \text{for } p \ge 0 \text{ and } q \ge 1.$$

PROOF. The Hessian metric $g = Dd\varphi$ is complete and the second Koszul forms with respect to h and g are

$$B = -\partial\bar{\partial} \log h(s, s) = \partial\bar{\partial}\varphi = g, \quad \beta = \frac{1}{2}\partial\bar{\partial} \det[\delta_{ij}] = 0.$$

Hence by Theorem 3.2.3, for $p = n$ we obtain the assertion.

Next, we consider the case of $p = 0$. For $v \in L^2(\mathbb{R}^n, F \otimes \wedge^{0,q})$ we set

$$\hat{v} = dx^1 \wedge \cdots \wedge dx^n \otimes v$$

Then we have $\hat{v} \in L^2(\mathbb{R}^n, F \otimes \wedge^{n,q})$ and $\|\hat{v}\| = \|v\|$. Since $\bar{\partial} v = 0$ and $\bar{\partial}\hat{v} = 0$ are equivalent, by Theorem 3.2.3 there exists $\hat{u} \in L^2(\mathbb{R}^n, F \otimes \wedge^{n,q-1})$ such that $\bar{\partial}\hat{u} = \hat{v}$ and $\|\hat{u}\| \le q^{-\frac{1}{2}} \|\hat{v}\|$. Here $\hat{u}$ can be expressed as

$$\hat{u} = dx^1 \wedge \cdots \wedge dx^n \otimes u, \quad u \in L^{0,q-1}(\mathbb{R}^n, g, F, h),$$

and so

$$dx^1 \wedge \cdots \wedge dx^n \otimes \bar{\partial} u = \bar{\partial} \hat{u} = \hat{v} = dx^1 \wedge \cdots \wedge dx^n \otimes v.$$

Therefore, we have $\bar{\partial} u = v$. Moreover, we obtain

$$\|u\| = \|\hat{u}\| \leq q^{-\frac{1}{2}} \|\hat{v}\| = q^{-\frac{1}{2}} \|v\|.$$

Hence the assertion for $p = 0$ follows.

Finally, for $p \geq 1$, $v \in L^2(\mathbb{R}^n, F \otimes \wedge^{p,q})$ can be expressed as

$$v = \sum_{I_p} dx^{I_p} \otimes v_{I_p}, \quad I_p = (i_1, \ldots, i_p), \quad 1 \leq i_1 < \cdots < i_p \leq n, \quad v_{I_p} \in L^2(\mathbb{R}^n, F \otimes \wedge^{0,q}),$$

and we have

$$\|v\|^2 = \sum_{I_p} \|v_{I_p}\|^2.$$

If $\bar{\partial} v = 0$, for all I_p we obtain $\bar{\partial} v_{I_p} = 0$. Hence by the case of $p = 0$, there exists $\{u_{I_p}\} \subset L^2(\mathbb{R}^n, F \otimes \wedge^{0,q-1})$ such that $\bar{\partial} u_{I_p} = v_{I_p}$ and $\|u_{I_p}\| \leq q^{-\frac{1}{2}} \|v_{I_p}\|$. Here we set

$$u = \sum_{I_p} dx^{I_p} \otimes u_{I_p}.$$

Then we have

$$\bar{\partial} u = \sum_{I_p} dx^{I_p} \otimes \bar{\partial} u_{I_p} = \sum_{I_p} dx^{I_p} \otimes v_{I_p} = v,$$

$$\|u\|^2 = \sum_{I_p} \|u_{I_p}\|^2 \leq \sum_{I_p} q^{-1} \|v_{I_p}\|^2 = q^{-1} \|v\|^2.$$

This completes the proof. □

Corollary 3.2.5. *Let $\Omega \in \mathbb{R}^n$ be a regular convex domain, D be the canonical affine connection on Ω, g be the Cheng-Yau metric defined by Theorem 1.2.4. Then for $p + q > n$ and $v \in L^2(\Omega, \wedge^{p,q})$ such that $\bar{\partial} v = 0$, there exists $u \in L^2(\Omega, \wedge^{p,q-1})$ such that*

$$\bar{\partial} u = v, \quad \|u\| \leq (p + q - n)^{-\frac{1}{2}} \|v\|.$$

In particular, we have

$$L^2 H_{\bar{\partial}}^{p,q}(\Omega) = 0, \quad \textit{for } p + q > n.$$

Proof. Since g is complete and $\beta = g$, the assertion follows from Theorem 3.2.3. □

Let $\Omega \in \mathbb{R}^{n-1}$ be a regular convex domain and we set $V = \{(ty, t) \in \mathbb{R}^n \mid y \in \Omega, t > 0\}$. Let $\tilde{D}$ be the canonical affine connection on V and $\tilde{g}$ be the Cheng-Yau metric on $(V, \tilde{D})$ defined by Theorem 1.2.4. In addition, we define an action $\rho : \mathbb{Z} \to \mathrm{GL}(V)$ by

$$\rho(k)x = e^k x, \quad k \in \mathbb{Z}, \quad x \in V.$$

Then we have $\mathbb{Z} \backslash V \simeq \Omega \times S^1$. Moreover, by Proposition 1.2.5 this action preserves $(\tilde{D}, \tilde{g})$ and so a Hessian structure (D, g) on $\Omega \times S^1$ is defined by projecting $(\tilde{D}, \tilde{g})$ on $\Omega \times S^1$. The Hessian metric g is complete and the second Koszul form with respect to g is equal to g. Hence the following theorem follows from Theorem 3.2.3.

COROLLARY 3.2.6. *Let $(\Omega \times S^1, D, g)$ be as above. Then for $p+q>n$ and $v \in L^2(\Omega \times S^1, \wedge^{p,q})$ such that $\bar\partial v = 0$, there exists $u \in L^2(\Omega \times S^1, \wedge^{p,q-1})$ such that*

$$\bar\partial u = v, \quad \|u\| \le (p+q-n)^{-\frac{1}{2}} \|v\|.$$

In particular, we have

$$L^2 H^{p,q}_{\bar\partial}(\Omega \times S^1) = 0, \quad \text{for } p+q>n.$$

4 Regular convex cones as special Hessian manifolds

A regular convex domain Ω in $\mathbb{R}^n$ is said to be a *regular convex cone* if, for any x in Ω and any positive real number λ, λx belongs to Ω. In this Chapter, we show special properties of the Cheng-Yau metric on regular convex cones.

4.1 The Cheng-Yau metrics on regular convex cones

In this section, we give some important propositions.

PROPOSITION 4.1.1. *Let $(\Omega, D, g = Dd\varphi)$ be a regular convex cone in $\mathbb{R}^n$ with the Cheng-Yau metric. Then we have the following equations.*

(1) $\displaystyle\sum_j x^j \frac{\partial \varphi}{\partial x^j} = -n.$

(2) $\displaystyle grad\,\varphi := \sum_{i,j} g^{ij} \frac{\partial \varphi}{\partial x^i} \frac{\partial}{\partial x^j} = -\sum_j x^j \frac{\partial}{\partial x^j}.$

(3) $\displaystyle\sum_k x^k \gamma_{ijk} = -g_{ij}.$

PROOF. By the proof of Proposition 1.2.5, for $t>0$ and $x \in \Omega$ we have

$$\varphi(tx) = \varphi(x) - n \log t.$$

Then we obtain

$$\sum_j x^j \frac{\partial \varphi}{\partial x^j} = \frac{d}{dt}\Big|_{t=1} \varphi(tx) = -n.$$

Taking the derivative of both sides with respect to x^i we have

$$(*) \quad \frac{\partial \varphi}{\partial x^i} + \sum_j x^j \frac{\partial^2 \varphi}{\partial x^i \partial x^j} = 0.$$

Since $\dfrac{\partial^2 \varphi}{\partial x^i \partial x^j} = g_{ij}$ we obtain

$$\mathrm{grad}\,\varphi = \sum_{i,j} g^{ij} \frac{\partial \varphi}{\partial x^j} \frac{\partial}{\partial x^i} = -\sum_j x^j \frac{\partial}{\partial x^j}.$$

Equation $(*)$ is equivalent to

$$\frac{\partial \varphi}{\partial x^j} + \sum_k x^k g_{jk} = 0.$$

Taking the derivative of both sides with respect to x^i and applying Proposition 1.1.5 we have

$$g_{ij} + g_{ij} + \sum_k 2x^k \gamma_{ijk} = 0,$$

that is,

$$\sum_k x^k \gamma_{ijk} = -g_{ij}.$$

$\square$

PROPOSITION 4.1.2. *Let $g = Dd\varphi$ be the Cheng-Yau metric on a regular convex cone Ω in $\mathbb{R}^n$. Then we have*

$$\nabla\alpha = \nabla d\varphi = 0.$$

PROOF. It follows from Proposition 4.1.1 that

$$\begin{aligned}(\nabla d\varphi)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) &= \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - \sum_k \gamma^k{}_{ij} \frac{\partial \varphi}{\partial x^k} \\ &= g_{ij} - \sum_{k,l} g^{kl} \gamma_{lij} \frac{\partial \varphi}{\partial x^k} \\ &= g_{ij} + \sum_l x^l \gamma_{lij} \\ &= 0.\end{aligned}$$

$\square$

A regular convex cone Ω in $\mathbb{R}^n$ is called *homogeneous* if a subgroup of $\mathrm{GL}(\mathbb{R}^n)$ acts on Ω transitively. In addition, a regular convex cone Ω in $\mathbb{R}^n$ is said to be *self-dual* if there exists a inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^n$ such that the *dual cone* $\Omega^* = \{y \in \mathbb{R}^n \mid \langle x, y \rangle > 0 \text{ for all } x \in \bar{\Omega} \setminus \{0\}\}$ coincides with Ω. It is known that a homogeneous self-dual regular convex cone is a Riemannian symmetric space with respect to the Cheng-Yau metric (c.f. Theorem 4.6 in [4]). Moreover, it is known that a homogeneous self-dual regular convex cone satisfies the following stronger condition than Proposition 4.1.2.

PROPOSITION 4.1.3. *(c.f. Proposition 4.12 in [4]) Let g be the Cheng-Yau metric on a homogeneous self-dual regular convex cone Ω in $\mathbb{R}^n$. Then we have*

$$\nabla\gamma = 0.$$

4.2 L^2-cohomology groups on regular convex cones

In this section we show vanishing theorems for regular convex cones with the Cheng-Yau metrics which differ from Corollary 3.2.5.

We set $H = \sum_j x^j \frac{\partial}{\partial x^j}$ and denote by $\mathscr{L}_H$ Lie differentiation with respect to H.

PROPOSITION 4.2.1. *For $\sigma \in A^p(\Omega)$ we have*

$$\mathscr{L}_H \sigma = D_H \sigma + p\sigma.$$

PROOF. For $X \in \mathscr{X}(\Omega)$ we obtain

$$D_X H = X,$$

and so

$$[H, X] = D_H X - D_X H = D_H X - X.$$

Then for $X_1, \dots, X_p \in \mathscr{X}(\Omega)$ we have

$$\begin{aligned}
&(\mathscr{L}_H\sigma)(X_1, \dots, X_p)\\
=&H\sigma(X_1, \dots, X_p) - \sum_i \sigma(X_1, \dots, [H, X_i], \dots X_p)\\
=&H\sigma(X_1, \dots, X_p) - \sum_i \sigma(X_1, \dots, D_H X_i, \dots, X_p) + p\sigma(X_1, \dots, X_p)\\
=&(D_H\sigma)(X_1, \dots, X_p) + p\sigma(X_1, \dots, X_p).
\end{aligned}$$

□

By Cartan's formula we have the following.

COROLLARY 4.2.2. *For $\omega \in A^{p,q}(\Omega)$ we have*

$$(\partial i(H) + i(H)\partial)\omega = D_H\omega + p\omega,$$

$$(\bar{\partial}\bar{i}(H) + \bar{i}(H)\bar{\partial})\omega = D_H\omega + q\omega.$$

THEOREM 4.2.3. *Let $(\Omega, D, g = Dd\varphi)$ be a regular convex cone in $\mathbb{R}^n$ with the Cheng-Yau metric. Then for $p > q \geq 1$ and all $v \in L^2(\Omega, \wedge^{p,q})$ such that $\bar{\partial}v = 0$, there exists $u \in L^2(\Omega, \wedge^{p,q-1})$ such that*

$$\bar{\partial}u = v, \quad \|u\| \leq (p-q)^{-\frac{1}{2}}\|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\Omega, \wedge^{p,0})$ satisfies $\bar{\partial}v = 0$, then $v = 0$. In particular, we have

$$L^2 H^{p,q}_{\bar{\partial}}(\Omega) = 0, \quad \textit{for } p > q.$$

PROOF. As a corollary of Theorem 2.3.6 we obtain

$$\Lambda\partial + \partial\Lambda = -\bar{\delta} + \bar{i}(X_\alpha), \quad \Lambda\bar{\partial} + \bar{\partial}\Lambda = -\delta + i(X_\alpha).$$

Then we have

$$\begin{aligned}
\bar{\partial}\bar{\delta} &= \bar{\partial}(-\Lambda\partial - \partial\Lambda + \bar{i}(X_\alpha))\\
&= (\Lambda\bar{\partial} + \delta - i(X_\alpha))\partial - \bar{\partial}\partial\Lambda + \bar{\partial}\bar{i}(X_\alpha)\\
&= \delta\partial - i(X_\alpha)\partial + \bar{\partial}\bar{i}(X_\alpha) + \Lambda\bar{\partial}\partial - \bar{\partial}\partial\Lambda,\\
\bar{\delta}\bar{\partial} &= (-\Lambda\partial - \partial\Lambda + \bar{i}(X_\alpha))\bar{\partial}\\
&= -\Lambda\partial\bar{\partial} + \partial(\bar{\partial}\Lambda + \delta - i(X_\alpha)) + \bar{i}(X_\alpha)\bar{\partial}\\
&= \partial\delta - \partial i(X_\alpha) + \bar{i}(X_\alpha)\bar{\partial} - \Lambda\partial\bar{\partial} + \partial\bar{\partial}\Lambda,
\end{aligned}$$

and so

$$\bar{\square} = \square - (\partial i(X_\alpha) + i(X_\alpha)\partial) + (\bar{\partial}\bar{i}(X_\alpha) + \bar{i}(X_\alpha)\bar{\partial}),$$

where $\square = \partial\delta + \delta\partial$. Let φ is the solution of the equation in Theorem 1.2.4. From Proposition 4.1.1 we obtain

$$X_\alpha = \operatorname{grad}\varphi = -H.$$

Hence by Corollary 4.2.2, we have

$$\bar{\square} = \square + p - q.$$

Therefore, for $\omega \in A^{p,q}_0(\Omega)$ we obtain

$$\|\bar{\partial}\omega\|^2 + \|\bar{\delta}\omega\|^2 \geq (p-q)\|\omega\|^2.$$

Then the assertions are proved similarly to Theorem 3.2.2. □

We have the following from Theorem 4.2.3 and Theorem 3.2.5.

COROLLARY 4.2.4. *Let $(\Omega, D, g = Dd\varphi)$ be a regular convex cone in $\mathbb{R}^n$ with the Cheng-Yau metric. Then we have*

$$L^2 H^{p,q}_{\bar\partial}(\Omega) = 0, \quad \textit{for } p + q > n \textit{ or } p > q.$$

The Cheng-Yau metric on a regular convex cone $\mathbb{R}^n_+$ is $g = -Dd\log(x^1 \cdots x^n) = \sum_i (\frac{dx^i}{x^i})^2$. We can apply Corollary 4.2.4 to $(\mathbb{R}^n_+, D, g)$. However, we have a stronger vanishing theorem.

THEOREM 4.2.5. *For $p \geq 1$, $q \geq 1$ and $v \in L^2(\mathbb{R}^n_+, \wedge^{p,q})$ such that $\bar\partial v = 0$, there exists $u \in L^2(\mathbb{R}^n_+, \wedge^{p,q-1})$ such that*

$$\bar\partial u = v, \quad \|u\| \leq p^{-\frac{1}{2}} \|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\mathbb{R}^n_+, \wedge^{p,0})$ satisfies $\bar\partial v = 0$, then $v = 0$. In particular, we have

$$L^2 H^{p,q}_{\bar\partial}(\mathbb{R}^n_+) = 0, \quad \textit{for } p \geq 1 \textit{ and } q \geq 0.$$

Hereafter in this section, we show Theorem 4.2.5. For the canonical coordinate $x = (x^1, \ldots, x^n)$ on $\mathbb{R}^n_+$, we set $t = (t^1, \ldots, t^n) = (\log x^1, \ldots, \log x^n)$.

LEMMA 4.2.6. *The following equations hold.*

(1) $g(\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}) = \delta_{ij}$.

(2) $D_{\frac{\partial}{\partial t^i}} \frac{\partial}{\partial t^j} = \delta_{ij} \frac{\partial}{\partial t^j}, \quad D^*_{\frac{\partial}{\partial t^i}} \frac{\partial}{\partial t^j} = -\delta_{ij} \frac{\partial}{\partial t^j}$.

(3) $D_{\frac{\partial}{\partial t^i}} dt^j = -\delta_i{}^j dt^j, \quad D^*_{\frac{\partial}{\partial t^i}} dt^j = \delta_i{}^j dt^j$.

(4) $\alpha = -\sum_j dt^j, \quad$ *where α is the first Koszul form for (D, g).*

LEMMA 4.2.7. *On $(\mathbb{R}^n_+, D, g)$ we have*

$$\bar\delta = -\sum_j \mathscr{D}_{\frac{\partial}{\partial t^j}} \bar{i}(\frac{\partial}{\partial t^j}).$$

PROOF. By Proposition 2.2.14, Lemma 4.2.6 and 2.2.12 we obtain

$$\begin{aligned}
\bar\delta &= -\sum_j \bar{i}(\frac{\partial}{\partial t^j}) \mathscr{D}_{\frac{\partial}{\partial t^j}} - \bar{i}(\sum_j \frac{\partial}{\partial t^j}) \\
&= -\sum_j \{\bar{i}(\frac{\partial}{\partial t^j}) \mathscr{D}_{\frac{\partial}{\partial t^j}} + \bar{i}(D_{\frac{\partial}{\partial t^j}} \frac{\partial}{\partial t^j})\} \\
&= -\sum_j \mathscr{D}_{\frac{\partial}{\partial t^j}} \bar{i}(\frac{\partial}{\partial t^j}).
\end{aligned}$$

□

PROPOSITION 4.2.8. *Let $\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dt^{I_p} \otimes \overline{dt^{J_q}} \in A^{p,q}(\mathbb{R}^n_+)$. Then we have*

$$\bar\Box \omega = \sum_{I_p, J_q} (\Delta + p)\, \omega_{I_p J_q} dt^{I_p} \otimes \overline{dt^{J_q}},$$

where $\Delta = -\sum_j (\frac{\partial}{\partial t^j})^2$.

PROOF. It is sufficient to show the equation when $\omega = f\, dt^{I_p} \otimes \overline{dt^{J_q}}$. For a multi-index $J_q = (j_1, \ldots, j_q)$, $j_1 < \cdots < j_q$, we define $J_{n-q} = (j_{q+1}, \ldots, j_n)$, $j_{q+1} < \cdots < j_n$, where (J_q, J_{n-q}) is a permutation of $(1, \ldots, n)$. By Lemma 4.2.6 and 4.2.7 we obtain

$$\bar{\partial}\omega = \sum_{i \in J_{n-q}} \frac{\partial f}{\partial t^i}\, dt^{I_p} \otimes \overline{dt^i} \wedge \overline{dt^{J_q}} - \sum_{i \in I_p \cap J_{n-q}} f\, dt^{I_p} \otimes \overline{dt^i} \wedge \overline{dt^{J_q}},$$

$$\begin{aligned}
\bar{\delta}\omega &= -\sum_{j \in J_q} \mathscr{D}_{\frac{\partial}{\partial t^j}} (f\, dt^{I_p} \otimes \bar{i}(\frac{\partial}{\partial t^j})\, \overline{dt^{J_q}}) \\
&= -\sum_{j \in J_q} \frac{\partial f}{\partial t^j}\, dt^{I_p} \otimes \bar{i}(\frac{\partial}{\partial t^j})\, \overline{dt^{J_q}} - \sum_{j \in I_p \cap J_q} f\, dt^{I_p} \otimes \bar{i}(\frac{\partial}{\partial t^j})\, \overline{dt^{J_q}},
\end{aligned}$$

$$\begin{aligned}
\bar{\delta}\bar{\partial}\omega = &- \sum_{i \in J_{n-q}} \sum_{j \in J_q \cup \{i\}} \frac{\partial^2 f}{\partial t^i \partial t^j}\, dt^{I_p} \otimes \bar{i}(\frac{\partial}{\partial t^j})\, (\overline{dt^i} \wedge \overline{dt^{J_q}}) \\
&- \sum_{i \in J_{n-q}} \sum_{j \in I_p \cap (J_q \cup \{i\})} \frac{\partial f}{\partial t^j}\, dt^{I_p} \otimes \bar{i}(\frac{\partial}{\partial t^j})\, (\overline{dt^i} \wedge \overline{dt^{J_q}}) \\
&+ \sum_{i \in I_p \cap J_{n-q}} \sum_{j \in J_q \cup \{i\}} \frac{\partial f}{\partial t^j}\, dt^{I_p} \otimes \bar{i}(\frac{\partial}{\partial t^j})\, (\overline{dt^i} \wedge \overline{dt^{J_q}}) \\
&+ \sum_{i \in I_p \cap J_{n-q}} \sum_{j \in I_p \cap (J_q \cup \{i\})} f\, dt^{I_p} \otimes \bar{i}(\frac{\partial}{\partial t^j})\, (\overline{dt^i} \wedge \overline{dt^{J_q}}),
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}\bar{\delta}\omega = &- \sum_{j \in J_q} \sum_{i \in J_{n-q} \cup \{j\}} \frac{\partial^2 f}{\partial t^i \partial t^j}\, dt^{I_p} \otimes \overline{dt^i} \wedge \bar{i}(\frac{\partial}{\partial t^j})\, \overline{dt^{J_q}} \\
&+ \sum_{j \in J_q} \sum_{i \in I_p \cap (J_{n-q} \cup \{j\})} \frac{\partial f}{\partial t^j}\, dt^{I_p} \otimes \overline{dt^i} \wedge \bar{i}(\frac{\partial}{\partial t^j})\, \overline{dt^{J_q}} \\
&- \sum_{j \in I_p \cap J_q} \sum_{i \in J_{n-q} \cup \{j\}} \frac{\partial f}{\partial t^i}\, dt^{I_p} \otimes \overline{dt^i} \wedge \bar{i}(\frac{\partial}{\partial t^j})\, \overline{dt^{J_q}} \\
&+ \sum_{j \in I_p \cap J_q} \sum_{i \in I_p \cap (J_{n-q} \cup \{j\})} f\, dt^{I_p} \otimes \overline{dt^i} \wedge \bar{i}(\frac{\partial}{\partial t^j})\, \overline{dt^{J_q}}.
\end{aligned}$$

We denote by $(\bar{\delta}\bar{\partial}\omega)_k$ and $(\bar{\partial}\bar{\delta}\omega)_k$ the k-th terms of $\bar{\delta}\bar{\partial}\omega$ and $\bar{\partial}\bar{\delta}\omega$ respectively, where $k = 1, 2, 3, 4$. Then we have

$$\begin{aligned}
(\bar{\delta}\bar{\partial}\omega)_1 + (\bar{\partial}\bar{\delta}\omega)_1 &= -\sum_{j=1}^{n} (\frac{\partial}{\partial t^j})^2 f\, dt^{I_p} \otimes \overline{dt^{J_q}}, \\
(\bar{\delta}\bar{\partial}\omega)_2 + (\bar{\partial}\bar{\delta}\omega)_3 &= -\sum_{j \in I_p} \frac{\partial f}{\partial t^j}\, dt^{I_p} \otimes \overline{dt^{J_q}}, \\
(\bar{\delta}\bar{\partial}\omega)_3 + (\bar{\partial}\bar{\delta}\omega)_2 &= \sum_{j \in I_p} \frac{\partial f}{\partial t^j}\, dt^{I_p} \otimes \overline{dt^{J_q}}, \\
(\bar{\delta}\bar{\partial}\omega)_4 + (\bar{\partial}\bar{\delta}\omega)_4 &= \sum_{j \in I_p} f\, dt^{I_p} \otimes \overline{dt^{J_q}} = pf\, dt^{I_p} \otimes \overline{dt^{J_q}}.
\end{aligned}$$

This completes the proof. □

COROLLARY 4.2.9. *For $\omega \in A_0^{p,q}(\mathbb{R}^n_+)$ we have*

$$\|\bar{\partial}\omega\|^2 + \|\bar{\delta}\omega\|^2 \geq p\|\omega\|^2.$$

PROOF. A (p,q)-form $\omega \in A_0^{p,q}(\mathbb{R}_+^n)$ is expressed by $\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dt^{I_p} \otimes \overline{dt^{J_q}}$. By Lemma 4.2.6 and Proposition 4.2.8 we obtain

$$\begin{aligned}\|\bar\partial\omega\|^2 + \|\bar\delta\omega\|^2 &= (\bar\Box\,\omega,\omega)\\ &= \sum_{I_p,J_q}(\Delta\,\omega_{I_pJ_q},\omega_{I_pJ_q}) + p\|\omega\|^2\\ &\geq p\|\omega\|^2.\end{aligned}$$

□

Using the above, we have Theorem 4.2.5 similarly to Theorem 3.2.2.

4.3 The symmetric space of positive definite real-symmetric matrices

We put $\mathrm{Sym}(n) = \{X \in M(n,\mathbb{R}) \mid {}^tX = X\}$ and $\mathrm{Sym}^+(n) = \{P \in \mathrm{Sym}(n) \mid P \text{ is positive definite.}\}$. In this section, we give a brief summary of the Cheng-Yau metric h on a regular convex cone $\mathrm{Sym}^+(n)$ in $\mathrm{Sym}(n)$ which is identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$. Since $GL(n,\mathbb{R})$ acts on $\mathrm{Sym}^+(n)$ transitively by $A\cdot P := {}^tAPA$ for $A \in GL(n,\mathbb{R})$ and $P \in \mathrm{Sym}^+(n)$, $\mathrm{Sym}^+(n)$ is homogeneous. Further, $\mathrm{Sym}^+(n)$ is self-dual with respect to an inner product $\langle P,Q\rangle := \mathrm{tr}\,(PQ)$ (c.f. Example 4.1 in [4]).

We denote by $P = (P_{ij})_{1\leq i\leq j\leq n}$ the standard coordinate of $\mathrm{Sym}^+(n)$. The Cheng-Yau metric h on $\mathrm{Sym}^+(n)$ is expressed by

$$h = -\frac{n+1}{2} Dd\log\det P.$$

LEMMA 4.3.1. *For all X_1, X_2, $X_3 \in Sym(n)$ and $\sigma \in S_3$, we have*

$$tr\,(X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)}) = \ tr\,(X_1X_2X_3).$$

PROOF. We have

$$\begin{aligned}\mathrm{tr}\,(X_1X_2X_3) &= \ \mathrm{tr}\,{}^t(X_1X_2X_3)\\ &= \ \mathrm{tr}\,({}^tX_3{}^tX_2{}^tX_1)\\ &= \ \mathrm{tr}\,(X_3X_2X_1).\end{aligned}$$

The other equations follow from the commutativity of the trace. □

PROPOSITION 4.3.2. *Let $P \in Sym^+(n)$. We identify $X = [X_{ij}]_{1\leq i,j\leq n} \in Sym(n)$ with $\sum_{i\leq j} X_{ij}\Big(\frac{\partial}{\partial P_{ij}}\Big)_P \in T_P Sym^+(n)$. Then we have*

$$\begin{aligned}h(X,Y) &= \frac{n+1}{2}\,tr\,(P^{-1}XP^{-1}Y),\\ (Dh)(X,Y,Z) &= -(n+1)\,tr\,(P^{-1}XP^{-1}YP^{-1}Z),\\ \gamma(X,Y) := \gamma_X Y &= -\frac{1}{2}(XP^{-1}Y + YP^{-1}X).\end{aligned}$$

PROOF. For $X \in \mathrm{Sym}(n)$ we define a vector field $\tilde X = \sum_{i\leq j} X_{ij}\frac{\partial}{\partial P_{ij}} \in \mathscr{X}(\mathrm{Sym}^+(n))$. Since $D_{\tilde X}\tilde Y = 0$

for all $X, Y \in \mathrm{Sym}(n)$, we have

$$
\begin{aligned}
h(X,Y) &= -\frac{n+1}{2}(D_{\tilde{Y}} d\log\det P)_P(X) \\
&= -\frac{n+1}{2}(\tilde{Y}\tilde{X}\log\det P)_P \\
&= -\frac{n+1}{2}(\tilde{Y}\,\mathrm{tr}\,(P^{-1}\tilde{X}))_P \\
&= \frac{n+1}{2}\,\mathrm{tr}\,(P^{-1}YP^{-1}X) \\
&= \frac{n+1}{2}\,\mathrm{tr}\,(P^{-1}XP^{-1}Y), \\
(Dh)(X,Y,Z) &= \frac{n+1}{2}(D_{\tilde{Z}}h)_P(X,Y) \\
&= \frac{n+1}{2}(\tilde{Z}h(\tilde{X},\tilde{Y}))_P \\
&= -\frac{n+1}{2}\,\mathrm{tr}\,(P^{-1}ZP^{-1}XP^{-1}Y + P^{-1}XP^{-1}ZP^{-1}Y).
\end{aligned}
$$

Since $P \in \mathrm{Sym}^+(n)$, there exists $P^{\frac{1}{2}} \in \mathrm{Sym}^+(n)$ such that $(P^{\frac{1}{2}})^2 = P$. Hence it follows from Lemma 4.3.1 that

$$
\begin{aligned}
\mathrm{tr}\,(P^{-1}ZP^{-1}XP^{-1}Y) &= \mathrm{tr}\,(P^{-\frac{1}{2}}ZP^{-1}XP^{-1}P^{-1}YP^{-\frac{1}{2}}) \\
&= \mathrm{tr}\,((P^{-\frac{1}{2}}ZP^{-\frac{1}{2}})(P^{-\frac{1}{2}}XP^{-\frac{1}{2}})(P^{-\frac{1}{2}}YP^{-\frac{1}{2}})) \\
&= \mathrm{tr}\,((P^{-\frac{1}{2}}XP^{-\frac{1}{2}})(P^{-\frac{1}{2}}YP^{-\frac{1}{2}})(P^{-\frac{1}{2}}ZP^{-\frac{1}{2}})) \\
&= \mathrm{tr}\,(P^{-1}XP^{-1}YP^{-1}Z).
\end{aligned}
$$

Therefore we have

$$
(Dh)(X,Y,Z) = -(n+1)\,\mathrm{tr}\,(P^{-1}XP^{-1}YP^{-1}Z).
$$

Moreover, we obtain

$$
\begin{aligned}
h\Big(-\frac{1}{2}(XP^{-1}Y + YP^{-1}X), Z\Big) &= -\frac{n+1}{4}\,\mathrm{tr}\,(P^{-1}XP^{-1}YP^{-1}Z + P^{-1}YP^{-1}XP^{-1}Z) \\
&= -\frac{n+1}{2}\,\mathrm{tr}\,(P^{-1}XP^{-1}YP^{-1}Z) \\
&= \frac{1}{2}(Dh)(X,Y,Z) \\
&= h(\gamma(X,Y),Z),
\end{aligned}
$$

where the last equality follows from Proposition 1.1.5. Hence we have

$$
\gamma(X,Y) = -\frac{1}{2}(XP^{-1}Y + YP^{-1}X).
$$

□

4.4 Harmonic immersions into $\mathrm{Sym}^+(n)$

Let g be a Riemannian metric on a domain Ω in $\mathbb{R}^n$ and h the Cheng-Yau metric on $\mathrm{Sym}^+(n)$. In this section, we consider the map $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$ such that $F_g(x) = [g_{ij}]_{1\le i,j\le n}$. Since $T\mathrm{Sym}^+(n) \cong \mathrm{Sym}^+(n) \times \mathrm{Sym}(n)$, the space of all C^∞-sections of $F_g^* T\mathrm{Sym}^+(n)$ can be identified with $C^\infty(\Omega, \mathrm{Sym}(n))$. In particular, $(\nabla^{g,h} dF_g)\Big(\dfrac{\partial}{\partial x^k}, \dfrac{\partial}{\partial x^l}\Big)$ and $\mathrm{tr}_g(\nabla^{g,h} dF_g)$ belong to $C^\infty(\Omega, \mathrm{Sym}(n))$, where $\nabla^{g,h}$ is the connection over $F_g^* T\mathrm{Sym}^+(n) \otimes T^*\Omega$ induced by the Levi-Civita connections ∇^g and ∇^h.

PROPOSITION 4.4.1. *We have*

$$(\nabla^{g,h}dF_g)\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)=\frac{\partial^2 F_g}{\partial x^k\partial x^l}-\frac{1}{2}\Big(\frac{\partial F_g}{\partial x^k}F_g^{-1}\frac{\partial F_g}{\partial x^l}+\frac{\partial F_g}{\partial x_l}F_g^{-1}\frac{\partial F_g}{\partial x_k}\Big)-\sum_r(\gamma_g)^r{}_{kl}\frac{\partial F_g}{\partial x^r},$$
$$tr_g(\nabla^{g,h}dF_g))=\sum_{k,l}g^{kl}\Big(\frac{\partial^2 F_g}{\partial x^k\partial x^l}-\frac{\partial F_g}{\partial x^k}F_g^{-1}\frac{\partial F_g}{\partial x^l}-\sum_r(\gamma_g)^r{}_{kl}\frac{\partial F_g}{\partial x^r}\Big).$$

PROOF. We obtain

$$\begin{aligned}(\nabla^{g,h}dF_g)\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)&=\nabla^h_{\frac{\partial}{\partial x^k}}dF_g\Big(\frac{\partial}{\partial x^l}\Big)-dF_g\Big(\nabla^g_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\Big)\\&=\frac{\partial^2 F_g}{\partial x^k\partial x^l}+(\gamma_h)_{F_g}\Big(\frac{\partial F_g}{\partial x^k},\frac{\partial F_g}{\partial x_l}\Big)-dF_g\Big(\sum_r(\gamma_g)^r{}_{kl}\frac{\partial}{\partial x^r}\Big)\\&=\frac{\partial^2 F_g}{\partial x^k\partial x^l}-\frac{1}{2}\Big(\frac{\partial F_g}{\partial x^k}F_g^{-1}\frac{\partial F_g}{\partial x^l}+\frac{\partial F_g}{\partial x_l}F_g^{-1}\frac{\partial F_g}{\partial x_k}\Big)-\sum_r(\gamma_g)^r{}_{kl}\frac{\partial F_g}{\partial x^r},\end{aligned}$$

where the last equality follows from Proposition 4.3.2 in the case of $X=\dfrac{\partial F_g}{\partial x^k}$, $Y=\dfrac{\partial F_g}{\partial x^l}$ and $P=F_g$. We also have

$$\begin{aligned}\mathrm{tr}_g(\nabla^{g,h}dF_g)&=\sum_{k,l}g^{kl}(\nabla^{g,h}dF_g)\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)\\&=\sum_{k,l}g^{kl}\Big(\frac{\partial^2 F_g}{\partial x^k\partial x^l}-\frac{\partial F_g}{\partial x^k}F_g^{-1}\frac{\partial F_g}{\partial x^l}-\sum_r(\gamma_g)^r{}_{kl}\frac{\partial F_g}{\partial x^r}\Big).\end{aligned}$$

□

We denote by $(\nabla^{g,h}dF_g)_{ij}\Big(\dfrac{\partial}{\partial x^k},\dfrac{\partial}{\partial x^l}\Big)$ and $\mathrm{tr}_g(\nabla^{g,h}dF_g)_{ij}$ the (i,j)-components of $(\nabla^{g,h}dF_g)\Big(\dfrac{\partial}{\partial x^k},\dfrac{\partial}{\partial x^l}\Big)$ and $\mathrm{tr}_g(\nabla^{g,h}dF_g)$, respectively.

THEOREM 4.4.2. *If g is a Hessian metric, then we have*

$$(\nabla^{g,h}dF_g)_{ij}\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)=2g\Big((\nabla^g_{\frac{\partial}{\partial x^i}}\gamma_g)\Big(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\Big),\frac{\partial}{\partial x^l}\Big),$$
$$tr_g(\nabla^{g,h}dF_g)_{ij}=2\left((\beta_g)_{ij}-(\alpha_g)^r(\gamma_g)_{rij}\right)=2(\nabla^g\alpha_g)\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big).$$

In particular, $(\nabla^{g,h}dF_g)_{ij}\Big(\dfrac{\partial}{\partial x^k},\dfrac{\partial}{\partial x^l}\Big)$ *is symmetric with respect to i,j,k,l.*

PROOF. From Proposition 4.4.1 and Proposition 1.1.5 we obtain

$$\begin{aligned}((\nabla^{g,h}dF_g)_{ij}\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)&=\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}-\frac{1}{2}\sum_{r,s}\Big(\frac{\partial g_{ir}}{\partial x^k}g^{rs}\frac{\partial g_{sj}}{\partial x^l}+\frac{\partial g_{ir}}{\partial x_l}g^{rs}\frac{\partial g_{sj}}{\partial x_k}\Big)-\sum_r(\gamma_g)^r{}_{kl}\frac{\partial g_{ij}}{\partial x^r}\\&=2\frac{\partial(\gamma_g)_{jkl}}{\partial x^i}-2\sum_{r,s}g^{rs}\left((\gamma_g)_{rik}(\gamma_g)_{sjl}+(\gamma_g)_{ilr}(\gamma_g)_{jks}\right)-2\sum_r(\gamma_g)^r{}_{kl}(\gamma_g)_{rij}\\&=2\Big(\frac{\partial(\gamma_g)_{jkl}}{\partial x^i}-\sum_r\left((\gamma_g)_{jrl}(\gamma_g)^r{}_{ik}+(\gamma_g)_{jkr}(\gamma_g)^r{}_{il}+(\gamma_g)_{rkl}(\gamma_g)^r{}_{ij}\right)\Big)\\&=2g\Big((\nabla^g_{\frac{\partial}{\partial x^i}}\gamma_g)\Big(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\Big),\frac{\partial}{\partial x^l}\Big).\end{aligned}$$

It follows from Proposition 1.2.2 that

$$\begin{aligned}
\mathrm{tr}_g(\nabla^{g,h}dF_g)_{ij} &= \sum_{k,l} g^{kl}\Big(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \sum_{r,s}\frac{\partial g_{ir}}{\partial x^k} g^{rs}\frac{\partial g_{sj}}{\partial x^l} - \sum_r (\gamma_g)^r{}_{kl}\frac{\partial g_{ij}}{\partial x^r}\Big)\\
&= \sum_{k,l} g^{kl}\Big(\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \sum_{r,s}\frac{\partial g_{kr}}{\partial x^i} g^{rs}\frac{\partial g_{sl}}{\partial x^j}\Big) - \sum_r (\alpha_g)^r\frac{\partial g_{ij}}{\partial x^r}\\
&= 2\Big((\beta_g)_{ij} - \sum_r (\alpha_g)^r(\gamma_g)_{rij}\Big)\\
&= 2(\nabla^g\alpha_g)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)
\end{aligned}$$

$\square$

Corollary 4.4.3. *If g is a Hessian metric on a domain Ω in $\mathbb{R}^n$ and the map $F_g : (\Omega, g) \to (Sym^+(n), h)$ is harmonic, then $tr_g\beta_g$ is nonnegative constant.*

Proof. Since $\sum_{i,j} g^{ij}\,\mathrm{tr}_g(\nabla^{g,h}dF_g)_{ij} = 0$, it follows from Theorem 4.4.2 that

$$\begin{aligned}
0 &= \sum_{i,j} g^{ij}((\beta_g)_{ij} - \sum_r (\alpha_g)^r(\gamma_g)_{rij})\\
&= \mathrm{tr}_g\beta_g - \sum_r (\alpha_g)^r(\alpha_g)_r.
\end{aligned}$$

Moreover, since $\nabla^g\alpha_g = 0$ by Theorem 4.4.2, $\sum_r (\alpha_g)^r(\alpha_g)_r$ is nonnegative constant. $\square$

Theorem 4.4.4. *Let g be the Cheng-Yau metric on a regular convex cone Ω in $\mathbb{R}^n$ and h the Cheng-Yau metric on $Sym^+(n)$. We define a map $F_g : (\Omega, g) \to (Sym^+(n), h)$ by $F_g(x) = [g_{ij}(x)]_{1\le i,j\le n}$. We denote by $\nabla^{g,h}$ the connection over $F_g^*TSym^+(n) \otimes T^*\Omega$ induced by the Levi-Civita connections ∇^g and ∇^h. Then*

(1) *F_g is an immersion.*

(2) *F_g is harmonic, that is,*

$$tr_g(\nabla^{g,h}dF_g) = 0.$$

(3) *If Ω is a homogeneous self-dual regular convex cone, then F_g is totally geodesic, that is,*

$$\nabla^{g,h}dF_g = 0.$$

Proof. Let $a = \sum_k a^k\Big(\frac{\partial}{\partial x^k}\Big)_x \in \mathrm{Ker}(dF_g)_x$, that is, $\sum_k \frac{\partial g_{ij}}{\partial x^k}a^k = 0$ for all $1 \le i,j \le n$. It follows from Proposition 4.1.1 that

$$0 = \sum_{i,k} x^i\frac{\partial g_{ij}}{\partial x^k}a^k = -2\sum_k g_{jk}a^k$$

Hence $a = 0$. This implies that F_g is an immersion. It follows from Theorem 4.4.2 and Proposition 4.1.2 that $\mathrm{tr}_g(\nabla^{g,h}dF_g) = 2\nabla^g\alpha_g = 0$. If Ω is a homogeneous self-dual regular convex cone, we have $\nabla^{g,h}dF_g = 2\nabla^g\gamma_g = 0$ from Theorem 4.4.2 and Proposition 4.1.3. $\square$

The condition in Theorem 4.4.4 that a regular convex domain is a cone is crucial to obtain a harmonic map. The following example is a regular convex domain which does not give a harmonic map.

EXAMPLE 4.4.5. Let $\Omega = \{x = (x^1, x^2) \in \mathbb{R}^2 \mid x^2 - \frac{1}{2}(x^1)^2 > 0\}$. The regular convex domain Ω with the Cheng-Yau metric g is known as an example of Hessian manifolds of constant Hessian sectional curvature (c.f. Proposition 3.8 in [4]). The solution of the equation in Theorem 1.2.4 is

$$\varphi(x) = -\frac{3}{2}\log(x^2 - \frac{1}{2}(x^1)^2) + \log\frac{3}{2}.$$

We have

$$\begin{aligned}
d\varphi &= \frac{3}{2}\frac{1}{x^2 - (\frac{1}{2}(x^1)^2)}(x^1 dx^1 - dx^2),\\
F_g &= \frac{3}{2}\frac{1}{(x^2 - (\frac{1}{2}(x^1)^2))^2}\begin{bmatrix} x^2 + \frac{1}{2}(x^1)^2 & -x^1 \\ -x^1 & 1 \end{bmatrix},\\
F_g^{-1} &= \frac{2}{3}(x^2 - \frac{1}{2}(x^1)^2)\begin{bmatrix} 1 & x^1 \\ x^1 & x^2 + \frac{1}{2}(x^1)^2 \end{bmatrix},\\
\operatorname{grad}\varphi &= \begin{bmatrix} \dfrac{\partial}{\partial x^1} & \dfrac{\partial}{\partial x^2} \end{bmatrix} F_g^{-1} \begin{bmatrix} \dfrac{\partial\varphi}{\partial x^1} \\ \dfrac{\partial\varphi}{\partial x^2} \end{bmatrix} = -(x^2 - \frac{1}{2}(x^1)^2)\frac{\partial}{\partial x^2}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\operatorname{tr}_g(\nabla^g \alpha_g) &= \operatorname{tr}_g\beta_g - \sum_r (\alpha_g)^r(\alpha_g)_r\\
&= \operatorname{tr}_g g - d\varphi(\operatorname{grad}\varphi)\\
&= 2 - \frac{3}{2}\\
&= \frac{1}{2} \neq 0.
\end{aligned}$$

Therefore $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$ is not harmonic from Theorem 4.4.2.

The condition in Theorem 4.4.4 (3) that a homogeneous regular convex cone is self-dual is also necessary to obtain a totally geodesic map. The following example is a homogeneous regular convex cone which does not give a totally geodesic map.

EXAMPLE 4.4.6. We define a 5-dimensional vector space V and a regular convex cone Ω in V by

$$V = \left\{ v = \begin{bmatrix} v^1 & v^2 & v^4 \\ v^2 & v^3 & 0 \\ v^4 & 0 & v^5 \end{bmatrix} \in \mathrm{Sym}(3) \right\},$$

$$\Omega = \left\{ x = \begin{bmatrix} x^1 & x^2 & x^4 \\ x^2 & x^3 & 0 \\ x^4 & 0 & x^5 \end{bmatrix} \in V \mid x' := \begin{bmatrix} x^1 & x^2 \\ x^2 & x^3 \end{bmatrix}, x'' := \begin{bmatrix} x^1 & x^4 \\ x^4 & x^5 \end{bmatrix} \in \mathrm{Sym}^+(2) \right\}.$$

The regular convex cone Ω is called the *Vinberg cone*, which is known as an example of non-self-dual homogeneous regular convex cones [6]. Let

$$G = \left\{ A = (A', A'') = \left(\begin{bmatrix} a & 0 \\ b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ b_2 & c_2 \end{bmatrix} \right) \in GL(2, \mathbb{R})^2 \mid a, c_1, c_2 > 0 \right\}.$$

Then we can define a group representation $\rho : G \to GL(V)$ by

$$(\rho(A)v)' = A'v'\,{}^tA', \quad (\rho(A)v)'' = A''v''\,{}^tA'' \quad \text{for } v \in V \text{ and } A = (A', A'') \in G.$$

We obtain $\rho(A)x \in \Omega$ for all $x \in \Omega$ and all $A \in G$, that is, $\rho(G)$ acts on Ω. We define $B : \Omega \to G$ by

$$B(x) = \left(\begin{bmatrix} \sqrt{x^1} & 0 \\ \frac{x^2}{\sqrt{x^1}} & \sqrt{\frac{x^1x^3-(x^2)^2}{x^1}} \end{bmatrix}, \begin{bmatrix} \sqrt{x^1} & 0 \\ \frac{x^4}{\sqrt{x^1}} & \sqrt{\frac{x^1x^5-(x^4)^2}{x^1}} \end{bmatrix} \right).$$

Then we have

$$\rho(B(x)) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x^1 & x^2 & x^4 \\ x^2 & x^3 & 0 \\ x^4 & 0 & x^5 \end{bmatrix} = x.$$

Hence $\rho(G)$ acts on Ω transitively. Let $\tilde{B}(x) \in \mathrm{GL}(5, \mathbb{R})$ be the matrix representation of $\rho(B(x)) \in \mathrm{GL}(V)$ with respect to the standard basis

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Then we obtain

$$\tilde{B}(x) = \begin{bmatrix} x^1 & 0 & 0 & 0 & 0 \\ x^2 & \sqrt{x^1x^3 - (x^2)^2} & 0 & 0 & 0 \\ \frac{(x^2)^2}{x^1} & \frac{2x^2\sqrt{x^1x^3-(x^2)^2}}{x^1} & \frac{x^1x^3-(x^2)^2}{x^1} & 0 & 0 \\ x^4 & 0 & 0 & \sqrt{x^1x^5 - (x^4)^2} & 0 \\ \frac{(x^4)^2}{x^1} & 0 & 0 & \frac{2x^4\sqrt{x^1x^5-(x^4)^2}}{x^1} & \frac{x^1x^5-(x^4)^2}{x^1} \end{bmatrix}$$

Therefore it follows from Corollary 1.2.6 that the Cheng-Yau metric g on Ω is expressed by

$$\begin{aligned} g &= -Dd \log |\det \tilde{B}(x)| \\ &= -Dd \log\Big(\frac{(x^1x^3 - (x^2)^2)^{\frac{3}{2}} (x^1x^5 - (x^4)^2)^{\frac{3}{2}}}{(x^1)} \Big) \\ &= -Dd\Big(\frac{3}{2} \log(x^1x^3 - (x^2)^2) + \frac{3}{2} \log(x^1x^5 - (x^4)^2) - \log x^1 \Big), \end{aligned}$$

that is,

$$F_g = \begin{bmatrix} \frac{3(x^3)^2}{2(x^1x^3-(x^2)^2)^2} + \frac{3(x^5)^2}{2(x^1x^5-(x^4)^2)^2} - \frac{1}{(x^1)^2} & \frac{-3x^2x^3}{(x^1x^3-(x^2)^2)^2} & \frac{3(x^2)^2}{2(x^1x^3-(x^2)^2)^2} & \frac{-3x^4x^5}{(x^1x^5-(x^4)^2)^2} & \frac{3(x^4)^2}{2(x^1x^5-(x^4)^2)^2} \\ \frac{-3x^2x^3}{(x^1x^3-(x^2)^2)^2} & \frac{3(x^1x^3+(x^2)^2)}{(x^1x^3-(x^2)^2)^2} & \frac{-3x^1x^2}{(x^1x^3-(x^2)^2)^2} & 0 & 0 \\ \frac{3(x^2)^2}{(x^1x^5-(x^4)^2)^2} & \frac{-3x^1x^2}{(x^1x^3-(x^2)^2)^2} & \frac{3(x^1)^2}{2(x^1x^3-(x^2)^2)^2} & 0 & 0 \\ \frac{-3x^4x^5}{(x^1x^5-(x^4)^2)^2} & 0 & 0 & \frac{3(x^1x^5+(x^4)^2)}{(x^1x^5-(x^4)^2)^2} & \frac{-3x^1x^4}{(x^1x^5-(x^4)^2)^2} \\ \frac{3(x^4)^2}{2(x^1x^5-(x^4)^2)^2} & 0 & 0 & \frac{-3x^1x^4}{(x^1x^5-(x^4)^2)^2} & \frac{3(x^1)^2}{2(x^1x^5-(x^4)^2)^2} \end{bmatrix}.$$

Let x_0 be the unit 3×3 matrix in Ω, that is, $x_0^1 = x_0^3 = x_0^5 = 1$ and $x_0^2 = x_0^4 = 0$. Since $(\gamma_g)_{22r} = \dfrac{1}{2} \dfrac{\partial g_{22}}{\partial x^r}$, we obtain

$$\begin{aligned} (\gamma_g)_{221}(x_0) &= (\gamma_g)_{223}(x_0) = -\frac{3}{2}, \\ (\gamma_g)_{222}(x_0) &= (\gamma_g)_{224}(x_0) = (\gamma_g)_{225}(x_0) = 0, \\ \frac{\partial (\gamma_g)_{222}}{\partial x^2}(x_0) &= 9. \end{aligned}$$

We also have

$$F_g(x_0) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}.$$

Hence

$$\begin{aligned} g_{x_0}\left((\nabla^g_{\frac{\partial}{\partial x^2}}\gamma_g)\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}\right), \frac{\partial}{\partial x^2}\right) &= \frac{\partial(\gamma_g)_{222}}{\partial x^2}(x_0) - 3\sum_r (\gamma_g)_{22r}(x_0)(\gamma_g)^r{}_{22}(x_0) \\ &= \frac{\partial(\gamma_g)_{222}}{\partial x^2}(x_0) - 3\Big(g^{11}(x_0)(\gamma_g)_{221}(x_0)^2 + g^{33}(\gamma_g)_{223}(x_0)^2\Big) \\ &= 9 - 3\Big(\frac{1}{2}\left(-\frac{3}{2}\right)^2 + \frac{2}{3}\Big(-\frac{3}{2}\Big)^2\Big) \\ &= \frac{9}{8} \neq 0. \end{aligned}$$

Therefore $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$ is not totally geodesic from Theorem 4.4.2.

References

[1] S. Akagawa, Vanishing theorems of L^2-cohomology groups on Hessian manifolds, J. Math. Soc. Japan 71 (2019), no. 1, 65-89.

[2] S. Akagawa, The Cheng-Yau metrics on regular convex cones as harmonic immersions into the symmetric space of positive definite real symmetric matrices, preprint 2018.

[3] S. Y. Cheng and S. T. Yau, The real Monge-Ampére equation and affine flat structures, Proc. the 1980 Beijing symposium of differential geometry and defferential equations, Science Press, Beijing, China, Gordon and Breach, Science Publishers Inc., New York, 1982, 339–370.

[4] H. Shima, The geometry of Hessian structures, World Scientific, Singapore, 2007.

[5] H. Shima, Vanishing theorems for compact Hessian manifolds. Ann. Inst. Fourier 36 (1986), no. 3, 183-205.

[6] Vinberg, É.B., Homogeneous cones, Soviet Math. Dokl. 1 (1960) , 787–790.

[7] J.-P. Demailly, L^2 estimates for the $\bar{\partial}$ operator on complex manifolds, Notes de cours, Ecole d'été de Mathématiques (Analyse Complexe), Institut Fourier, Grenoble, Juin 1996.

[8] L. Hörmander, An introduction to complex analysis in several variables. Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990. xii+254 pp.

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