Lower bound estimates for the lifespan of small solutions to nonlinear Schrödinger equations

(非線形シュレディンガー方程式の小さな解のライフスパンの下界評価)

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Chapter 1

Introduction

This thesis is concerned with the large-time behavior of solutions to the Cauchy problem with small initial data for nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^{p-1}u, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \varepsilon\varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.0.1}$$

where $\varepsilon > 0$ is a small parameter, $p > 1$, $\lambda \in \mathbb{C}$. $\varphi = \varphi(x)$ is a $\mathbb{C}$-valued known function which belongs to suitable weighted Sobolev space $H^{s,\sigma}(\mathbb{R})$. $u = u(t,x)$ is a $\mathbb{C}$-valued unknown function. The equation (1.0.1) appears in nonlinear optical fiber (see [1]). Since the local existence in $H^{s,\sigma}(\mathbb{R})$ is well-known (see e.g., [3] and the references cited therein), we are interested in large-time behavior of solutions to (1.0.1).

First of all, let us consider the case of $\lambda = 0$ (which we refer to as *free* in what follows). In this case, it is well-known that the following estimates hold:

$$\|u(t)\|_{L^2(\mathbb{R})} \leq C\varepsilon, \quad \|u(t)\|_{L^\infty(\mathbb{R})} \leq \frac{C\varepsilon}{(1+t)^{1/2}} \quad (t > 0).$$

Next, let us assume that these properties are still valid in the nonlinear case (1.0.1). Then we would obtain

$$\int_0^t \|\lambda |u(s)|^{p-1}u(s)\|_{L^2(\mathbb{R})} ds \leq C\varepsilon^p \int_0^t (1+s)^{-(p-1)/2} ds \leq C\varepsilon^p$$

when $p > 3$, i.e., $(p-1)/2 > 1$. From this observation and the smallness of ε, one may expect that the effect of the nonlinearity is almost negligible and (1.0.1) could be regarded as a perturbation of the free Schrödinger equations when $p > 3$. According to Tsutsumi–Yajima [76], there exists a unique global solution to (1.0.1), and the global solution behaves like a free solution as $t \to \infty$. On the other hand when $1 < p \leq 3$, the situation changes dramatically, as pointed out by Strauss [71], Barab [2], Ozawa [62], Hayashi–Naumkin [28], and so on. Note that this threshold becomes $p = 1 + 2/d$ in the d-dimensional settings. In the critical case $p = 3$, the standard perturbative approach is valid only for $t \lesssim \exp(o(\varepsilon^{-2}))$ in general. This comes from the above heuristic arguement, that is,

$$\int_0^t \|\lambda|u(s)|^2 u(s)\|_{L^2(\mathbb{R})} ds \leq C\varepsilon \cdot \varepsilon^2 \log(1 + t).$$

Similarly, in the subcritical case $1 < p < 3$, the standard perturbative approach is valid only for $t \lesssim o(\varepsilon^{-2(p-1)/(3-p)})$, since

$$\int_0^t \|\lambda|u(s)|^{p-1} u(s)\|_{L^2(\mathbb{R})} ds \leq C\varepsilon \cdot \varepsilon^{p-1}(1 + t)^{(3-p)/2}.$$

So, our problem is to make clear how the nonlinearity affects the behavior of the solutions for $t \gtrsim \exp(O(\varepsilon^{-2}))$ when $p = 3$, and for $t \gtrsim O(\varepsilon^{-2(p-1)/(3-p)})$ when $1 < p < 3$, respectively.

The purpose of this thesis is to develop the understanding for large-time behavior of solutions to nonlinear Schrödinger equations in terms of the detailed lifespan estimates. Our motivation comes from the important works due to John [40] and Hörmander [35] which deal with quasilinear wave equations in three space dimensions. Remember that the detailed lifespan estimates obtained in [40] and [35] have close connection with the so-called null condition introduced by Klainerman [50] and Christodoulou [6]. What we intend here is to reveal analogous structure in nonlinear Schrödinger equations.

This thesis is organized as follows. Chapter 2 deals with the cubic derivative nonlinear Schrödinger equations in $\mathbb{R}$. We provide a detailed lower bound estimate for the lifespan of the solution, which can be computed explicitly from the initial data and the nonlinear term. This is an extension and a refinement of the previous work by Sunagawa [73]. This part is the joint work [66] with Hideaki Sunagawa. Chapter 3 is devoted to the lifespan of solutions to subcritical nonlinear Schrödinger equations in $\mathbb{R}^d$ for $d = 1, 2, 3$. We provide a detailed lower bound estimate for the lifespan of the solution, which can be computed explicitly from the initial data and the nonlinear term. This is an extension, a refinement and a generalization of the previous work by Sasaki [69]. This part is the joint work [67] with Hideaki Sunagawa and Shunsuke Yasuda. Finally, in Chapter 4, we study a two-component system of cubic nonlinear Schrödinger equations in $\mathbb{R}$. Based on the author's paper [65], we provide a detailed lower bound estimate for the lifespan of the solution to the system, which can be computed explicitly from the initial data, the masses and the nonlinear term.

Before closing this chapter, we introduce some notation. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d)$ denotes the Lebesgue space on $\mathbb{R}^d$ and $\|\cdot\|_{L^p(\mathbb{R}^d)}$ denotes the L^p norm of $\mathbb{R}^d$. We denote by $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$ the L^2 inner product of $\mathbb{R}^d$. For $m \in \mathbb{N}$ and $1 \le p \le \infty$, we denote by $W^{m,p}(\mathbb{R}^d)$ the $L^p(\mathbb{R}^d)$-based Sobolev space of order m

$$W^{m,p}(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) \;\middle|\; \partial_x^\alpha f \in L^p(\mathbb{R}^d)\ (\alpha \in (\mathbb{N} \cup \{0\})^d,\ |\alpha| \le m) \right\}$$

equipped with the norm

$$\|f\|_{W^{m,p}(\mathbb{R}^d)} := \sum_{|\alpha| \le m} \|\partial_x^\alpha f\|_{L^p(\mathbb{R}^d)}.$$

For s, $\sigma \ge 0$, we denote by $H^{s,\sigma}(\mathbb{R}^d)$ the weighted Sobolev space

$$H^{s,\sigma}(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; (1 + |x|^2)^{\sigma/2}(1 - \Delta)^{s/2} f \in L^2(\mathbb{R}^d) \right\}$$

equipped with the norm

$$\|f\|_{H^{s,\sigma}(\mathbb{R}^d)} := \|(1 + |x|^2)^{\sigma/2}(1 - \Delta)^{s/2} f\|_{L^2(\mathbb{R}^d)}.$$

We write $H^s(\mathbb{R}^d) = H^{s,0}(\mathbb{R}^d)$ for simplicity. We write $C(I; X)$ for the space of continuous functions from an interval I of $\mathbb{R}$ to a Banach space X. The Fourier transform of ϕ is defined by

$$\mathcal{F}\phi(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \phi(x)\, dx \quad (\xi \in \mathbb{R}^d)$$

and the inverse Fourier transform of ϕ is defined by

$$\mathcal{F}^{-1}\phi(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \phi(\xi)\, d\xi \quad (x \in \mathbb{R}^d).$$

We denote several positive constants by C, which may vary from one line to another.

Chapter 2

The lifespan of small solutions to cubic derivative nonlinear Schrödinger equations in one space dimension

2.1 Introduction

This chapter is based on the joint work [66] with Hideaki Sunagawa. Throughout this chapter, we focus on the following initial value problem:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = N(u, \partial_x u), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = \varepsilon\varphi(x), & x \in \mathbb{R}, \end{cases} \tag{2.1.1}$$

where $i = \sqrt{-1}$, $u = u(t, x)$ is a $\mathbb{C}$-valued unknown function, $\varepsilon > 0$ is a small parameter which is responsible for the size of the initial data, and $\varphi = \varphi(x)$ is a $\mathbb{C}$-valued known function which belongs to $H^3 \cap H^{2,1}(\mathbb{R})$. $N = N(u, \partial_x u)$ is the nonlinear term which is always assumed to be a cubic homogeneous polynomial in $(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ with complex coefficients.

Let us recall some known results briefly. The most well-studied case is the gauge-invariant case, that is the case where the nonlinear term N satisfies

$$N(e^{i\theta}z, e^{i\theta}\zeta) = e^{i\theta}N(z, \zeta), \qquad (z, \zeta) \in \mathbb{C} \times \mathbb{C}, \quad \theta \in \mathbb{R}. \tag{2.1.2}$$

There are a lot of works devoted to large-time behavior of the solution to (2.1.1) under (2.1.2) (see e.g., [75], [44], [63], [16], [28], [27], [70], [55], [26] and the references cited therein). On the other hand, if (2.1.2) is violated, the situation becomes delicate due to the appearance of oscillation structure. It is pointed out in [19] (see also [18], [74], [59]) that contribution of non-gauge-invariant terms may be regarded as a short-range perturbation if at least one derivative of u is included, whereas, as studied in [20], [21], [22], [23], [60], [24], etc., it turns out that contribution of non-gauge-invariant cubic terms without derivative is quite difficult to handle. In what follows, let us assume that the nonlinear term N satisfies

$$N(e^{i\theta}, 0) = e^{i\theta} N(1, 0), \qquad \theta \in \mathbb{R}, \tag{2.1.3}$$

to exclude the worst terms u^3, $\overline{u}^2 u$ and $\overline{u}^3$ (see Section 2.6 for explicit representation of N satisfying (2.1.3)). We also define $\nu : \mathbb{R} \to \mathbb{C}$ by

$$\nu(\xi) := \frac{1}{2\pi i} \oint_{|z|=1} N(z, i\xi z) \frac{dz}{z^2}, \qquad \xi \in \mathbb{R}.$$

Roughly speaking, this contour integral extracts the contribution of the gauge-invariant part in the nonlinear term N. Remark that $\nu(\xi)$ coincides with $N(1, i\xi)$ in the gauge-invariant case (see also (2.6.7) below). Typical previous results on global existence and large-time asymptotic behavior of solutions to (2.1.1) under (2.1.3) can be summarized in terms of $\nu(\xi)$ as follows:

(i) If $\operatorname{Im} \nu(\xi) \leq 0$ for all $\xi \in \mathbb{R}$, then the solution exists globally in time for sufficiently small ε. Moreover the solution satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \frac{C\varepsilon}{\sqrt{1+t}}, \quad t \geq 0,$$

where the constant C is independent of ε ([19], [26], etc.).

(ii) If $\operatorname{Im} \nu(\xi) = 0$ for all $\xi \in \mathbb{R}$, then the solution has a logarithmic oscillating factor in the asymptotic profile, i.e., it holds that

$$u(t, x) = \frac{1}{\sqrt{it}} \tilde{\alpha}(x/t) \exp\left(i\frac{x^2}{2t} - i|\tilde{\alpha}(x/t)|^2 \operatorname{Re} \nu(x/t) \log t \right) + o(t^{-1/2})$$

as $t \to +\infty$ uniformly in $x \in \mathbb{R}$, where $\tilde{\alpha}$ is a suitable $\mathbb{C}$-valued function on $\mathbb{R}$ satisfying $\|\tilde{\alpha}\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon$. In particular, the solution is asymptotically free if and only if $\nu(\xi)$ vanishes identically on $\mathbb{R}$ ([75], [44], [16], [28], [27], [19], etc.).

(iii) If $\sup_{\xi \in \mathbb{R}} \operatorname{Im} \nu(\xi) < 0$, then the solution gains an additional logarithmic time-decay:

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \frac{C\varepsilon}{\sqrt{(1+t)\{1 + \varepsilon^2 \log(2+t)\}}}, \quad t \geq 0,$$

where the constant C is independent of ε ([70], [26], [52], etc.).

Now, let us turn our attentions to the remaining case: $\operatorname{Im}\nu(\xi_0) > 0$ for some $\xi_0 \in \mathbb{R}$. To the author's knowledge, there is no global existence result in that case, and many interesting problems are left unsolved especially when we focus on the issue of small data blow-up. In the previous paper by Sunagawa [73], lower bounds for the lifespan T_ε of the solution to (2.1.1) are considered in detail under the assumption (2.1.2). It is proved in [73] that

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{2 \sup_{\xi \in \mathbb{R}} \left(|\mathcal{F}\varphi(\xi)|^2 \operatorname{Im} N(1, i\xi) \right)} =: \tau_0. \tag{2.1.4}$$

He proved it by constructing an approximate solution u_a which blows up at the time $t = \exp(\tau_0/\varepsilon^2)$ and getting an a priori estimate not for the solution u itself but for the difference $u - u_a$. What is important in (2.1.4) is that this is quite analogous to the famous results due to John [40] and Hörmander [35] which concern quasilinear wave equations in three space dimensions (see [36], [57], [8] for analogous results on the Klein-Gordon equation, and also [56], [41], [9], [55], [72], [78], [38], etc. for related works of them). Remember that the detailed lifespan estimates obtained in [40] and [35] are fairly sharp and have close connection with the so-called *null condition* introduced by Klainerman [50] and Christodoulou [6]. However, the approach exploited in [73] has the following two drawbacks:

- it heavily relies on the gauge-invariance (2.1.2),
- it requires higher regularity and faster decay as $|x| \to \infty$ for φ than those for $u(t, \cdot)$.

The purpose of this chapter is to improve these two points. To state the main result, let us define $\tilde{\tau}_0 \in (0, +\infty]$ by

$$\tilde{\tau}_0 = \frac{1}{2 \sup_{\xi \in \mathbb{R}} \left(|\mathcal{F}\varphi(\xi)|^2 \operatorname{Im} \nu(\xi) \right)}, \tag{2.1.5}$$

where we associate $1/0 = +\infty$. Remark that the right-hand side of (2.1.5) is always positive if $\varphi \in H^{2,1}(\mathbb{R})$, because $\operatorname{Im}\nu(\xi) = O(|\xi|^3)$ and $|\mathcal{F}\varphi(\xi)|^2 = O(|\xi|^{-4})$ as $|\xi| \to \infty$. In particular, we can easily check that $\tilde{\tau}_0 = +\infty$ if $\operatorname{Im}\nu(\xi) \leq 0$ for all $\xi \in \mathbb{R}$. We also note that $\tilde{\tau}_0$ coincides with τ_0 in the gauge-invariant case. The main result of this chapter is as follows:

Theorem 2.1.1. *Assume that $\varphi \in H^3 \cap H^{2,1}(\mathbb{R})$. Suppose that the nonlinear term N satisfies* (2.1.3). *Let T_ε be the supremum of $T > 0$ such that* (2.1.1) *admits a unique solution in $C([0, T); H^3 \cap H^{2,1}(\mathbb{R}))$. Then we have*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \tilde{\tau}_0,$$

where $\tilde{\tau}_0 \in (0, +\infty]$ *is given by* (2.1.5).

Remark 2.1.2. The above theorem concerns only the forward Cauchy problem (i.e., for $t > 0$). For the backward Cauchy problem, we can obtain a simliar lower bound $\tilde{\tau}_0'$ which can be written explicitly in terms of N and φ. Indeed, if $u(t,x)$ is a solution to (2.1.1) for $t < 0$, then $\overline{u(-t,x)}$ is also a solution to the Schrödinger equation with another cubic derivative nonlinearity for $t > 0$. However, it should be noted that $\tilde{\tau}_0'$ does not coincide with $\tilde{\tau}_0$ in general. For example, when $N = -i|u|^2 u$ and $\varphi \not\equiv 0$, we can check that $\tilde{\tau}_0 = +\infty$ and $\tilde{\tau}_0' < +\infty$, whence the small data global existence is expected only for the positive time direction. On the other hand, if $\operatorname{Im} \nu(\xi) = 0$ for all $\xi \in \mathbb{R}$, then we have $\tilde{\tau}_0 = \tilde{\tau}_0' = +\infty$. In fact, the solution exists globally in both time directions in that case.

We close this section with the contents of this chapter: Section 2.2 is devoted to a lemma on some ordinary differential equation. In Section 2.3, we recall basic properties of the operators $\mathcal{J}$ and $\mathcal{Z}$, as well as the smoothing property of the linear Schrödinger equations. After that, we will get an a priori estimate in Section 2.4, and Theorem 2.1.1 will be proved in Section 2.5. The proof of technical lemmas will be given in Section 2.6.

2.2 A lemma on ODE

In this section we introduce a lemma on some ordinary differential equation, keeping in mind an application to (2.4.10) below.

Let κ, $\theta_0 : \mathbb{R} \to \mathbb{C}$ be continuous functions satisfying

$$\sup_{\xi\in\mathbb{R}} |\kappa(\xi)| < \infty, \quad \sup_{\xi\in\mathbb{R}} |\theta_0(\xi)| < \infty, \quad \sup_{\xi\in\mathbb{R}} \big(|\theta_0(\xi)|^2 \operatorname{Im} \kappa(\xi)\big) \geq 0.$$

We set $C_1 = \sup_{\xi\in\mathbb{R}} |\kappa(\xi)|$ and define $\tau_1 \in (0, +\infty]$ by

$$\tau_1 = \frac{1}{2 \sup_{\xi\in\mathbb{R}} \big(|\theta_0(\xi)|^2 \operatorname{Im} \kappa(\xi)\big)},$$

where $1/0$ is understood as $+\infty$. Let $\beta_0 : [1, T) \times \mathbb{R} \to \mathbb{C}$ be a solution to

$$\begin{cases} i\partial_t \beta_0(t,\xi) = \dfrac{\kappa(\xi)}{t} |\beta_0(t,\xi)|^2 \beta_0(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ \beta_0(1,\xi) = \varepsilon \theta_0(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{2.2.1}$$

where $\varepsilon > 0$ is a parameter. Multipling the equation (2.2.1) by $\overline{\beta_0}$, and taking the imaginary part of the result, we have

$$\begin{cases} \partial_t \left(|\beta_0(t,\xi)|^2\right) = \dfrac{2 \operatorname{Im} \kappa(\xi)}{t} |\beta_0(t,\xi)|^4, & t > 1,\ \xi \in \mathbb{R}, \\ \beta_0(1,\xi) = \varepsilon \theta_0(\xi), & \xi \in \mathbb{R}. \end{cases}$$

Solving this ordinary differential equation by separation of variables, we have

$$|\beta_0(t,\xi)|^2 = \frac{\varepsilon^2|\theta_0(\xi)|^2}{1-2\varepsilon^2|\theta_0(\xi)|^2 \operatorname{Im}\kappa(\xi)\log t},$$

as long as the denominator is strictly positive. In view of this expression, we can see that

$$\sup_{(t,\xi)\in[1,e^{\sigma/\varepsilon^2}]\times\mathbb{R}} |\beta_0(t,\xi)| \le C_2\varepsilon \tag{2.2.2}$$

for $\sigma \in (0,\tau_1)$, where

$$C_2 = \frac{1}{\sqrt{1-\sigma/\tau_1}} \sup_{\xi\in\mathbb{R}} |\theta_0(\xi)| \quad (< +\infty),$$

while

$$\sup_{\xi\in\mathbb{R}} |\beta_0(t,\xi)| \to +\infty \quad \text{as} \quad t \to \exp(\tau_1/\varepsilon^2) - 0$$

if $\tau_1 < \infty$.

Next we consider a perturbation of (2.2.1). For this purpose, let $T > 1$ and let $\theta_1 : \mathbb{R} \to \mathbb{C}$, $\rho : [1,T)\times\mathbb{R} \to \mathbb{C}$ be continuous functions satisfying

$$\sup_{\xi\in\mathbb{R}} |\theta_1(\xi)| \le C_3\varepsilon^{1+\delta}, \qquad \sup_{(t,\xi)\in[1,T)\times\mathbb{R}} t^{1+\mu}|\rho(t,\xi)| \le C_4\varepsilon^{1+\delta}$$

with some positive constants C_3, C_4, δ and μ. Let $\beta : [1,T)\times\mathbb{R} \to \mathbb{C}$ be a solution to

$$\begin{cases} i\partial_t\beta(t,\xi) = \dfrac{\kappa(\xi)}{t}|\beta(t,\xi)|^2\beta(t,\xi) + \rho(t,\xi), & (t,\xi)\in(1,T)\times\mathbb{R}, \\ \beta(1,\xi) = \varepsilon\theta_0(\xi) + \theta_1(\xi), & \xi\in\mathbb{R}. \end{cases}$$

The following lemma asserts that an estimate similar to (2.2.2) remains valid if (2.2.1) is perturbed by ρ and θ_1:

Lemma 2.2.1. *Let $\sigma \in (0,\tau_1)$. We set $T_* = \min\{T, e^{\sigma/\varepsilon^2}\}$. For $\varepsilon \in (0, M^{-1/\delta}]$, we have*

$$\sup_{(t,\xi)\in[1,T_*)\times\mathbb{R}} |\beta(t,\xi)| \le C_2\varepsilon + M\varepsilon^{1+\delta},$$

where

$$M = \left(2C_3 + \frac{C_4}{\mu}\right) e^{C_1(1+3C_2+3C_2^2)\sigma}.$$

Proof. We put $w(t,\xi) = \beta(t,\xi) - \beta_0(t,\xi)$ and

$$T_{**} = \sup\Big\{\tilde{T} \in [1, T_*) \ \Big|\ \sup_{(t,\xi)\in[1,\tilde{T})\times\mathbb{R}} |w(t,\xi)| \le M\varepsilon^{1+\delta}\Big\}.$$

Note that $T_{**} > 1$, because of the estimate

$$|w(1,\xi)| = |\theta_1(\xi)| \le C_3\varepsilon^{1+\delta} \le \frac{M}{2}\varepsilon^{1+\delta}$$

and the continuity of w. Since w satisfies

$$i\partial_t w = \frac{\kappa(\xi)}{t}\Big(|w+\beta_0|^2(w+\beta_0) - |\beta_0|^2\beta_0\Big) + \rho,$$

we see that

$$\begin{aligned}
\partial_t\left(|w|^2\right) &= 2\,\mathrm{Im}\Big(\overline{w}\cdot i\partial_t w\Big)\\
&\le \frac{2}{t}C_1\Big(M^2\varepsilon^{2+2\delta} + 3C_2M\varepsilon^{2+\delta} + 3C_2^2\varepsilon^2\Big)|w|^2 + |w||\rho|\\
&\le \frac{2}{t}\tilde{C}\varepsilon^2|w|^2 + \frac{C_4\varepsilon^{1+\delta}}{t^{1+\mu}}|w|
\end{aligned}$$

for $t \in [1, T_{**})$, where $\tilde{C} = C_1(1+3C_2+3C_2^2)$. By the Gronwall-type argument, we obtain

$$\begin{aligned}
|w(t,\xi)| &\le \left(|\theta_1(\xi)| + \int_1^t \frac{C_4\varepsilon^{1+\delta}}{2s^{1+\mu+\tilde{C}\varepsilon^2}}ds\right)e^{\tilde{C}\varepsilon^2\log t}\\
&\le \left(C_3\varepsilon^{1+\delta} + \frac{C_4\varepsilon^{1+\delta}}{2(\mu+\tilde{C}\varepsilon^2)}\right)e^{\tilde{C}\sigma}\\
&\le \frac{M}{2}\varepsilon^{1+\delta}
\end{aligned}$$

for $(t,\xi) \in [1, T_{**})\times\mathbb{R}$. This contradicts the definition of T_{**} if $T_{**} < T_*$. Therefore we conclude $T_{**} = T_*$. In other words, we have

$$\sup_{(t,\xi)\in[1,T_*)\times\mathbb{R}} |w(t,\xi)| \le M\varepsilon^{1+\delta},$$

whence

$$|\beta(t,\xi)| \le |\beta_0(t,\xi)| + |w(t,\xi)| \le C_2\varepsilon + M\varepsilon^{1+\delta}$$

for $(t,\xi) \in [1, T_*)\times\mathbb{R}$. This completes the proof. ∎

2.3 Preliminaries related to the Schrödinger operator

This section is devoted to preliminaries related to the operator $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$.

2.3.1 The operators $\mathcal{J}$ and $\mathcal{Z}$

We introduce $\mathcal{J} = x + it\partial_x$ and $\mathcal{Z} = x\partial_x + 2t\partial_t$, which have good compatibility with the operator $\mathcal{L}$. Then the following relations hold

$$[\partial_x, \mathcal{J}] = 1, \quad [\mathcal{L}, \mathcal{J}] = 0, \quad [\partial_x, \mathcal{Z}] = \partial_x, \quad [\mathcal{L}, \mathcal{Z}] = 2\mathcal{L},$$

where $[\cdot, \cdot]$ stands for the commutator of two linear operators. Another important relation is

$$\mathcal{J}\partial_x = \mathcal{Z} + 2it\mathcal{L}, \tag{2.3.1}$$

which will play the key role in our analysis. Next we set

$$(\mathcal{U}(t)\phi)(x) = e^{i\frac{t}{2}\partial_x^2}\phi(x) = \frac{1}{\sqrt{2\pi i t}} \int_{\mathbb{R}} e^{i\frac{(x-y)^2}{2t}} \phi(y)dy$$

for $t > 0$. We will occasionally abbreviate $\mathcal{U}(t)$ to $\mathcal{U}$ if it causes no confusion. The following lemma is well-known.

Lemma 2.3.1. *We have*

$$\|v\|_{L^\infty(\mathbb{R})} \leq t^{-1/2} \left\|\mathcal{F}\mathcal{U}^{-1}v\right\|_{L^\infty(\mathbb{R})} + Ct^{-3/4}(\|v\|_{L^2(\mathbb{R})} + \|\mathcal{J}v\|_{L^2(\mathbb{R})})$$

for $t > 0$.

We skip the proof of this lemma (see the series of papers by Hayashi and Naumkin [16]–[24]).

2.3.2 A smoothing property

In this subsection, we recall a smoothing property of the linear Schrödinger equations, which will be used effectively in Step 3 of §2.4.1. As is well-known, the standard energy method causes a derivative loss when the nonlinear term involves the derivatives of the unknown function. To overcome this difficulty we make use of smoothing effect. Among various kinds of smoothing properties, we will follow the approach of [25], whose original idea is due to Doi [10] (see also [5] and the references cited therein for the history and more information of this subject). Let $\mathcal{H}$ be the Hilbert transform, that is,

$$\mathcal{H}v(x) := \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{v(y)}{x-y}\,dy.$$

With a non-negative weight function $\Phi(x)$, let us define the operator $\mathcal{S}_\Phi$ by

$$\mathcal{S}_\Phi v(x) := \left\{\cosh\left(\int_{-\infty}^{x} \Phi(y)\,dy\right)\right\} v(x) - i\left\{\sinh\left(\int_{-\infty}^{x} \Phi(y)\,dy\right)\right\}\mathcal{H}v(x).$$

Note that $\mathcal{S}_\Phi$ is L^2-automorphism and both $\|\mathcal{S}_\Phi\|_{L^2\to L^2}$ and $\|\mathcal{S}_\Phi^{-1}\|_{L^2\to L^2}$ are dominated by $C\exp(\|\Phi\|_{L^1(\mathbb{R})})$. Roughly speaking, the operator $\mathcal{S}_\Phi$ is chosen so that

$$[\mathcal{L},\mathcal{S}_\Phi] = -i\Phi\mathcal{S}_\Phi|\partial_x| + \text{`harmless terms'},$$

and the first term in the right-hand side enables us to gain the half-derivative $|\partial_x|^{1/2}$. More precisely, we have the following two lemmas:

Lemma 2.3.2 ([25])**.** *Let $v = v(t,x)$ and $\psi = \psi(t,x)$ be $\mathbb{C}$-valued smooth functions. We set $\Phi(t,x) = \eta(|\psi|^2 + |\psi_x|^2)$ with $\eta \geq 1$. Then there exists the constant C, which is independent of η, such that*

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{S}_\Phi v(t)\|^2_{L^2(\mathbb{R})} + \left\|\sqrt{\Phi(t)}\mathcal{S}_\Phi|\partial_x|^{1/2}v(t)\right\|^2_{L^2(\mathbb{R})}\\
&\leq Ce^{C\eta\|\psi(t)\|^2_{H^1(\mathbb{R})}}\left(\eta\|\psi(t)\|^2_{W^{2,\infty}} + \eta^3\|\psi(t)\|^6_{W^{1,\infty}} + \eta\|\psi(t)\|_{H^1}\|\mathcal{L}\psi(t)\|_{H^1}\right)\|v(t)\|^2_{L^2(\mathbb{R})}\\
&\quad + 2\left|\langle\mathcal{S}_\Phi v(t), \mathcal{S}_\Phi\mathcal{L}v(t)\rangle_{L^2(\mathbb{R})}\right|.
\end{aligned}$$

Lemma 2.3.3 ([25])**.** *Let $v = v(x)$ and $\psi = \psi(x)$ be $\mathbb{C}$-valued smooth functions. Suppose that q_1 and q_2 are quadratic homogeneous polynomials in $(\psi, \overline{\psi}, \psi_x, \overline{\psi_x})$. We set $\Phi(x) = \eta(|\psi|^2 + |\psi_x|^2)$ with $\eta \geq 1$. Then there exists the constant C, which is independent of η, such that*

$$\begin{aligned}
&\left|\langle\mathcal{S}_\Phi v, \mathcal{S}_\Phi q_1(\psi,\psi_x)\partial_x v\rangle_{L^2(\mathbb{R})}\right| + \left|\left\langle\mathcal{S}_\Phi v, \mathcal{S}_\Phi q_2(\psi,\psi_x)\overline{\partial_x v}\right\rangle_{L^2(\mathbb{R})}\right|\\
&\leq C\eta^{-1}e^{C\eta\|\psi\|^2_{H^1(\mathbb{R})}}\left\|\sqrt{\Phi}\mathcal{S}_\Phi|\partial_x|^{1/2}v\right\|^2_{L^2(\mathbb{R})}\\
&\quad + Ce^{C\eta\|\psi\|^2_{H^1(\mathbb{R})}}\left(1 + \eta^2\|\psi\|^4_{H^1(\mathbb{R})} + \eta^2\|\psi\|^4_{W^{1,\infty}(\mathbb{R})}\right)\|\psi\|^2_{W^{2,\infty}(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

For the proof, see Section 2 in [25] (see also the appendix of [52]).

2.4 A priori estimate

Throughout this section, we fix $\sigma \in (0, \tilde{\tau}_0)$ and $T \in (0, e^{\sigma/\varepsilon^2}]$, where $\tilde{\tau}_0$ is defined by (2.1.5). Let $u \in C([0,T); H^3 \cap H^{2,1}(\mathbb{R}))$ be a solution to (2.1.1) for $t \in [0,T)$, and we set $\alpha(t,\xi) = \mathcal{F}\big[\mathcal{U}(t)^{-1}u(t,\cdot)\big](\xi)$, where $\mathcal{U}(t)$ is given in Section 2.3. We also put

$$E(T) = \sup_{t\in[0,T)} \Big((1+t)^{-\gamma}(\|u(t)\|_{H^3(\mathbb{R})} + \|\mathcal{J}u(t)\|_{H^2(\mathbb{R})}) + \sup_{\xi\in\mathbb{R}}(\langle\xi\rangle^2|\alpha(t,\xi)|) \Big)$$

with $\gamma \in (0, 1/12)$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. The goal of this section is to prove the following:

Lemma 2.4.1. *Assume that the nonlinear term N satisfies* (2.1.3)*. Let σ, T and γ be as above. Then there exist positive constants ε_0 and K, not depending on T, such that*

$$E(T) \le \varepsilon^{2/3} \tag{2.4.1}$$

implies

$$E(T) \le K\varepsilon,$$

provided that $\varepsilon \in (0, \varepsilon_0]$.

We divide the proof of this lemma into two subsections. We remark that many parts of the proof below are similar to that of Section 3 in [19], although we need modifications to fit for our purpose.

2.4.1 L^2-estimates

In this part, we consider the bound for $\|u(t)\|_{H^3(\mathbb{R})} + \|\mathcal{J}u(t)\|_{H^2(\mathbb{R})}$. By virtue of the inequality

$$\begin{aligned}&\|u(t)\|_{H^3(\mathbb{R})} + \|\mathcal{J}u(t)\|_{H^2(\mathbb{R})}\\ &\le C\big(\|u(t)\|_{L^2(\mathbb{R})} + \|\partial_x^3 u(t)\|_{L^2(\mathbb{R})} + \|\mathcal{J}u(t)\|_{L^2(\mathbb{R})} + \|\partial_x\mathcal{J}\partial_x u(t)\|_{L^2(\mathbb{R})}\big),\end{aligned} \tag{2.4.2}$$

it suffices to show that each term in the right-hand side can be dominated by $C\varepsilon(1+t)^\gamma$. We are going to estimate these four terms by separate ways.

Step 1: Estimate for $\|u(t)\|_{L^2(\mathbb{R})}$. First we remark that the assumption (2.4.1) yields

$$\|u(t)\|_{W^{2,\infty}(\mathbb{R})} \le \frac{C\varepsilon^{2/3}}{(1+t)^{1/2}}$$

for $t \in [0,T)$. Indeed, the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (2.4.1) yield

$$\|u(t)\|_{W^{2,\infty}(\mathbb{R})} \le C\|u(t)\|_{H^3(\mathbb{R})} \le C\varepsilon^{2/3}$$

for $t \le 1$, while it follows from Lemma 2.3.1 that

$$\|u(t)\|_{W^{2,\infty}(\mathbb{R})} \le \frac{C}{t^{1/2}} \sup_{\xi\in\mathbb{R}} \big(\langle\xi\rangle^2|\alpha(t,\xi)|\big) + \frac{C}{t^{3/4}}\big(\|u(t)\|_{H^2(\mathbb{R})} + \|\mathcal{J}u(t)\|_{H^2(\mathbb{R})}\big) \le \frac{C\varepsilon^{2/3}}{t^{1/2}}$$

for $t \in [1, T)$. Now, by the standard energy method, we have

$$\frac{d}{dt}\|u(t)\|_{L^2(\mathbb{R})} \le \|N(u,u_x)\|_{L^2(\mathbb{R})} \le C\|u(t)\|^2_{W^{1,\infty}(\mathbb{R})}\|u(t)\|_{H^1(\mathbb{R})} \le \frac{C\varepsilon^2}{(1+t)^{1-\gamma}},$$

whence

$$\begin{aligned}\|u(t)\|_{L^2(\mathbb{R})} &\le \|u(0)\|_{L^2(\mathbb{R})} + \int_0^t \frac{C\varepsilon^2}{(1+\tau)^{1-\gamma}}d\tau \\ &\le C\varepsilon + C\varepsilon^2(1+t)^\gamma \\ &\le C\varepsilon(1+t)^\gamma \end{aligned} \tag{2.4.3}$$

for $t \in [0, T)$.

Step 2: Estimate for $\|\mathcal{J}u(t)\|_{L^2(\mathbb{R})}$. If $t \le 1$, there is no difficulty because we do not have to pay attentions to possible growth in t. Indeed, since

$$\|\mathcal{J}N(u(t),u_x(t))\|_{L^2(\mathbb{R})} \le C(1+t)\|u\|^2_{W^{1,\infty}(\mathbb{R})}(\|\mathcal{J}u\|_{H^1(\mathbb{R})} + \|u\|_{H^1(\mathbb{R})}) \le C\varepsilon^2,$$

we have

$$\|\mathcal{J}u(t)\|_{L^2(\mathbb{R})} \le C\|u(0)\|_{H^{0,1}(\mathbb{R})} + \int_0^1 \|\mathcal{J}N(u(\tau),u_x(\tau))\|_{L^2(\mathbb{R})}d\tau \le C\varepsilon + C\varepsilon^2$$

for $t \le 1$. To consider the remaining case $t \ge 1$, let us first recall a remarkable lemma due to Hayashi–Naumkin [19]:

Lemma 2.4.2. *Assume that the nonlinear term N satisfies* (2.1.3). *Then the following decomposition holds:*

$$\mathcal{J}N(u,u_x) = \mathcal{L}(tP) + Q,$$

where P is a cubic homogeneous polynomial in $(u, \overline{u}, u_x, \overline{u_x})$, and Q satisfies

$$\begin{aligned}\|Q\|_{L^2(\mathbb{R})} &\le C\|u\|^2_{W^{2,\infty}(\mathbb{R})}(\|u\|_{H^1(\mathbb{R})} + \|\mathcal{J}u\|_{H^2(\mathbb{R})} + \|\mathcal{Z}u\|_{H^1(\mathbb{R})}) \\ &\quad + \frac{C}{t}\|u\|_{W^{2,\infty}(\mathbb{R})}(\|\mathcal{J}u\|_{H^2(\mathbb{R})} + \|u\|_{H^1(\mathbb{R})})^2\end{aligned}$$

for $t \ge 1$.

We will give a sketch of the proof in Section 2.6. Now we are going to apply this lemma. Let $t \in [1, T)$. Since the above decomposition and the commutative relation $[\mathcal{L}, \mathcal{J}] = 0$ allow us to rewrite the original equation as

$$\mathcal{L}(\mathcal{J}u - tP) = Q,$$

the standard energy method yields

$$\|\mathcal{J}u(t) - tP\|_{L^2(\mathbb{R})} \leq C(\varepsilon + \varepsilon^2) + \int_1^t \|Q(\tau)\|_{L^2(\mathbb{R})} d\tau.$$

By the relation (2.3.1), we have

$$\begin{aligned}
\|\mathcal{Z}u\|_{H^1(\mathbb{R})} &\leq \|\mathcal{J}\partial_x u\|_{H^1(\mathbb{R})} + 2t\|N(u, u_x)\|_{H^1(\mathbb{R})} \\
&\leq C\varepsilon^{2/3}(1+t)^{\gamma} + C\varepsilon^2 t(1+t)^{-1+\gamma} \\
&\leq C\varepsilon^{2/3}(1+t)^{\gamma},
\end{aligned}$$

which leads to

$$\|Q(t)\|_{L^2(\mathbb{R})} \leq \frac{C\varepsilon^2}{(1+t)^{1-\gamma}}.$$

Also we have

$$\|P\|_{L^2(\mathbb{R})} \leq C\|u\|^2_{W^{1,\infty}(\mathbb{R})}\|u\|_{H^1(\mathbb{R})} \leq \frac{C\varepsilon^2}{(1+t)^{1-\gamma}}.$$

Summing up these, we have

$$\|\mathcal{J}u(t)\|_{L^2(\mathbb{R})} \leq t\|P\|_{L^2(\mathbb{R})} + \|\mathcal{J}u(t) - tP\|_{L^2(\mathbb{R})} \leq C\varepsilon(1+t)^{\gamma} \tag{2.4.4}$$

for $t \in [1, T)$.

Step 3: Estimate for $\|\partial_x^3 u(t)\|_{L^2(\mathbb{R})}$. As we have mentioned in Section 2.3, the standard energy method causes a derivative loss. So we make use of smoothing effect. We apply Lemma 2.3.2 with $v = \partial_x^3 u$, $\psi = u$ and $\eta = \varepsilon^{-2/3}$. Then we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{S}_\Phi \partial_x^3 u(t)\|^2_{L^2(\mathbb{R})} + \left\|\sqrt{\Phi}\mathcal{S}_\Phi |\partial_x|^{1/2}\partial_x^3 u\right\|^2_{L^2(\mathbb{R})} \\
&\leq CB(t)\|\partial_x^3 u\|^2_{L^2(\mathbb{R})} + 2\left|\left\langle \mathcal{S}_\Phi \partial_x^3 u, \mathcal{S}_\Phi \partial_x^3 N(u, u_x)\right\rangle_{L^2(\mathbb{R})}\right|,
\end{aligned}$$

where

$$\begin{aligned}
B(t) = e^{C\varepsilon^{-2/3}\|u\|^2_{H^1(\mathbb{R})}}\Big(&\varepsilon^{-2/3}\|u(t)\|^2_{W^{2,\infty}(\mathbb{R})} + \varepsilon^{-2}\|u(t)\|^6_{W^{1,\infty}(\mathbb{R})} \\
&+ \varepsilon^{-2/3}\|u(t)\|_{H^1(\mathbb{R})}\|N(u, u_x)\|_{H^1(\mathbb{R})}\Big).
\end{aligned}$$

Since (2.4.1) yields

$$\|u(t)\|_{H^1(\mathbb{R})} \le C\|\alpha(t)\|_{H^{0,1}(\mathbb{R})} \le C\left(\int_{\mathbb{R}} \frac{d\xi}{\langle\xi\rangle^2}\right)^{1/2} \sup_{\xi\in\mathbb{R}}\bigl(\langle\xi\rangle^2|\alpha(t,\xi)|\bigr) \le C\varepsilon^{2/3},$$

we see that $B(t)$ can be dominated by $C\varepsilon^{2/3}(1+t)^{-1}$. Also we observe that the usual Leibniz rule leads to

$$\partial_x^3 N(u,u_x) = q_1(u,u_x)\partial_x(\partial_x^3 u) + q_2(u,u_x)\partial_x(\overline{\partial_x^3 u}) + \rho_1,$$

where q_1, q_2 are defined by

$$q_1(z,\zeta) = \frac{\partial N}{\partial \zeta}(z,\zeta), \qquad q_2(z,\zeta) = \frac{\partial N}{\partial \bar{\zeta}}(z,\zeta), \tag{2.4.5}$$

and ρ_1 satisfies

$$\|\rho_1\|_{L^2(\mathbb{R})} \le C\|u\|^2_{W^{2,\infty}(\mathbb{R})}\|u\|_{H^3(\mathbb{R})}.$$

By Lemma 2.3.3, we have

$$\begin{aligned}
&\Bigl|\bigl\langle \mathcal{S}_\Phi\partial_x^3 u, \mathcal{S}_\Phi\partial_x^3 N(u,u_x)\bigr\rangle_{L^2(\mathbb{R})}\Bigr| \\
&\le \bigl|\langle \mathcal{S}_\Phi\partial_x^3 u, \mathcal{S}_\Phi q_1(u,u_x)\partial_x(\partial_x^3 u)\rangle_{L^2(\mathbb{R})}\bigr| + \bigl|\langle \mathcal{S}_\Phi\partial_x^3 u, \mathcal{S}_\Phi q_2(u,u_x)\partial_x(\overline{\partial_x^3 u})\rangle_{L^2(\mathbb{R})}\bigr| \\
&\quad + C\|\mathcal{S}_\Phi\partial_x^3 u\|_{L^2(\mathbb{R})}\|\mathcal{S}_\Phi\rho_1\|_{L^2(\mathbb{R})} \\
&\le C\varepsilon^{2/3}e^{C\varepsilon^{-2/3}\|u\|^2_{H^1(\mathbb{R})}}\Bigl\|\sqrt{\Phi}\mathcal{S}_\Phi|\partial_x|^{1/2}\partial_x^3 u\Bigr\|^2_{L^2(\mathbb{R})} \\
&\quad + Ce^{C\varepsilon^{-2/3}\|u\|^2_{H^1(\mathbb{R})}}\Bigl(1+\varepsilon^{-4/3}\|u\|^4_{H^1(\mathbb{R})} + \varepsilon^{-4/3}\|u\|^4_{W^{1,\infty}(\mathbb{R})}\Bigr)\|u\|^2_{W^{2,\infty}(\mathbb{R})}\|\partial_x^3 u\|^2_{L^2(\mathbb{R})} \\
&\quad + Ce^{C\varepsilon^{-2/3}\|u\|^2_{H^1(\mathbb{R})}}\|u\|^2_{W^{2,\infty}(\mathbb{R})}\|u\|^2_{H^3(\mathbb{R})} \\
&\le C_0\varepsilon^{2/3}\Bigl\|\sqrt{\Phi}\mathcal{S}_\Phi|\partial_x|^{1/2}\partial_x^3 u\Bigr\|^2_{L^2(\mathbb{R})} + \frac{C\varepsilon^{8/3}}{(1+t)^{1-2\gamma}}
\end{aligned}$$

with some positive constant C_0 not depending on ε. Piecing the above estimates all together, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\mathcal{S}_\Phi\partial_x^3 u(t)\|^2_{L^2(\mathbb{R})} &\le \bigl(2C_0\varepsilon^{2/3}-1\bigr)\Bigl\|\sqrt{\Phi}\mathcal{S}_\Phi|\partial_x|^{1/2}\partial_x^3 u\Bigr\|^2_{L^2(\mathbb{R})} + \frac{C(\varepsilon^2+\varepsilon^{8/3})}{(1+t)^{1-2\gamma}} \\
&\le \frac{C\varepsilon^2}{(1+t)^{1-2\gamma}},
\end{aligned}$$

provided that $\varepsilon \le (2C_0)^{-3/2}$. Integrating with respect to t, we have

$$\|\mathcal{S}_\Phi\partial_x^3 u(t)\|^2_{L^2(\mathbb{R})} \le Ce^{C\varepsilon^{2/3}}\varepsilon^2 + C\varepsilon^2(1+t)^{2\gamma} \le C\varepsilon^2(1+t)^{2\gamma},$$

whence

$$\|\partial_x^3 u(t)\|_{L^2(\mathbb{R})} \le e^{C\varepsilon^{-2/3}\|u(t)\|_{H^1(\mathbb{R})}^2}\|\mathcal{S}_\Phi \partial_x^3 u(t)\|_{L^2(\mathbb{R})} \le C\varepsilon(1+t)^\gamma \tag{2.4.6}$$

for $t \in [0, T)$.

Step 4: Estimate for $\|\partial_x \mathcal{J}\partial_x u(t)\|_{L^2(\mathbb{R})}$. We also make use of Lemmas 2.3.2 and 2.3.3 in this step. By using the commutation relation $[\mathcal{L}, \partial_x \mathcal{Z}] = 2\partial_x \mathcal{L}$ and the Leibniz rule for $\mathcal{Z}$, we have

$$\mathcal{L}\partial_x \mathcal{Z}u = q_1(u, u_x)\partial_x(\partial_x \mathcal{Z}u) + q_2(u, u_x)\partial_x(\overline{\partial_x \mathcal{Z}u}) + \rho_2,$$

where q_1, q_2 are given by (2.4.5), and ρ_2 satisfies

$$\|\rho_2\|_{L^2(\mathbb{R})} \le C\|u\|_{W^{2,\infty}(\mathbb{R})}^2(\|u\|_{H^2(\mathbb{R})} + \|\mathcal{Z}u\|_{H^1(\mathbb{R})}).$$

Since the relation (2.3.1) leads to

$$\|\mathcal{Z}u\|_{H^1(\mathbb{R})} \le \|\mathcal{J}\partial_x u\|_{H^1(\mathbb{R})} + 2t\|N(u, u_x)\|_{H^1(\mathbb{R})} \le C\varepsilon^{2/3}(1+t)^\gamma,$$

we see that

$$\|\rho_2\|_{L^2(\mathbb{R})} \le \frac{C\varepsilon^2}{(1+t)^{1-\gamma}}.$$

Thus, in the same way as Step 3, we have

$$\frac{d}{dt}\|\mathcal{S}_\Phi \partial_x \mathcal{Z}u(t)\|_{L^2(\mathbb{R})}^2 \le \frac{C\varepsilon^2}{(1+t)^{1-2\gamma}},$$

which yields

$$\|\partial_x \mathcal{Z}u(t)\|_{L^2(\mathbb{R})} \le C\varepsilon(1+t)^\gamma.$$

Finally, by using the relation (2.3.1) again, we obtain

$$\begin{aligned}
\|\partial_x \mathcal{J}\partial_x u(t)\|_{L^2(\mathbb{R})} &\le \|\partial_x \mathcal{Z}u\|_{L^2(\mathbb{R})} + 2t\|\partial_x N(u, u_x)\|_{L^2(\mathbb{R})} \\
&\le C\varepsilon(1+t)^\gamma + 2t\frac{C\varepsilon^2}{(1+t)^{1-\gamma}} \\
&\le C\varepsilon(1+t)^\gamma
\end{aligned} \tag{2.4.7}$$

for $t \in [0, T)$.

Final step. Substituting (2.4.3), (2.4.4), (2.4.6) and (2.4.7) into (2.4.2), we arrive at the desired estimate

$$\|u(t)\|_{H^3(\mathbb{R})} + \|\mathcal{J}u(t)\|_{H^2(\mathbb{R})} \le C\varepsilon(1+t)^\gamma$$

for $t \in [0, T)$.

2.4.2 Estimates for α

In this part, we will show $\langle\xi\rangle^2|\alpha(t,\xi)| \le C\varepsilon$ for $(t,\xi) \in [0,T) \times \mathbb{R}$ under the assumption (2.4.1). If $t \le 1$, the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ yields this estimate immediately. Hence we may assume $T > 1$ and $t \in [1,T)$ in what follows.

Now let us introduce a useful lemma, which is due to Hayashi–Naumkin [19], though the expression is slightly different. We write $\alpha_\omega(t,\xi) = \alpha(t,\xi/\omega)$ for $\omega \in \mathbb{R}\backslash\{0\}$.

Lemma 2.4.3. *Assume that the nonlinear term N satisfies* (2.1.3). *Then, for $l \in \{0,1,2\}$, the following decomposition holds:*

$$\begin{aligned}\mathcal{F}[\mathcal{U}(t)^{-1}\partial_x^l N(u,u_x)](\xi) &= \frac{(i\xi)^l\nu(\xi)}{t}|\alpha|^2\alpha + \frac{\xi e^{i\frac{t}{3}\xi^2}}{t}\mu_{1,l}(\xi)\alpha_3^3 \\ &\quad + \frac{\xi e^{i\frac{2t}{3}\xi^2}}{t}\mu_{2,l}(\xi)\big(\overline{\alpha_{-3}}\big)^3 + \frac{\xi e^{it\xi^2}}{t}\mu_{3,l}(\xi)|\alpha_{-1}|^2\overline{\alpha_{-1}} + R_l, \quad (2.4.8)\end{aligned}$$

where $\mu_{1,l}(\xi)$, $\mu_{2,l}(\xi)$, $\mu_{3,l}(\xi)$ are polynomials in ξ of order at most $2+l$, and $R_l(t,\xi)$ satisfies

$$\sum_{l=0}^{2}\|R_l(t)\|_{L^\infty(\mathbb{R})} \le \frac{C}{t^{5/4}}\big(\|u(t)\|_{H^3(\mathbb{R})} + \|\mathcal{J}u(t)\|_{H^2(\mathbb{R})}\big)^3$$

for $t \ge 1$.

The proof of this lemma will be given in Section 2.6. It follows from this lemma that

$$\begin{aligned}\langle\xi\rangle^2 i\partial_t\alpha &= \mathcal{F}\mathcal{U}^{-1}(1-\partial_x^2)\mathcal{L}u \\ &= \mathcal{F}\mathcal{U}^{-1}N(u,u_x) - \mathcal{F}\mathcal{U}^{-1}\partial_x^2 N(u,u_x) \\ &= \frac{\langle\xi\rangle^2\nu(\xi)}{t}|\alpha|^2\alpha + V + R_0 - R_2, \end{aligned} \quad (2.4.9)$$

where

$$V(t,\xi) = \frac{\xi e^{it\xi^2/3}}{t}p_1(\xi)\alpha_3^3 + \frac{\xi e^{2it\xi^2/3}}{t}p_2(\xi)\overline{\alpha_{-3}}^3 + \frac{\xi e^{it\xi^2}}{t}p_3(\xi)|\alpha_{-1}|^2\overline{\alpha_{-1}}$$

with $p_k(\xi) = \mu_{k,0}(\xi) - \mu_{k,2}(\xi)$ $(k=1,2,3)$. We deduce from (2.4.1) and (2.4.9) that

$$|\partial_t\alpha(t,\xi)| \le \frac{C\varepsilon^2}{\langle\xi\rangle^2 t}, \qquad (t,\xi) \in [1,T) \times \mathbb{R}.$$

Also, by using the identity

$$\frac{\xi e^{i\omega t\xi^2}}{t}A(t,\xi) = i\partial_t\left(\frac{-i\xi e^{i\omega t\xi^2}}{1+i\omega t\xi^2}A(t,\xi)\right) - te^{i\omega t\xi^2}\partial_t\left(\frac{\xi A(t,\xi)}{t(1+i\omega t\xi^2)}\right),$$

we see that V can be rewritten as $i\partial_t V_1 + V_2$ with suitable V_1, V_2 satisfying

$$|V_1(t,\xi)| \le \frac{C\varepsilon^2}{t^{1/2}}, \quad |V_2(t,\xi)| \le \frac{C\varepsilon^2}{t^{3/2}}.$$

Note that

$$\sup_{\xi\in\mathbb{R}} \left| \frac{\xi}{1+i\omega t\xi^2} \right| \le \frac{C}{t^{1/2}}$$

if $\omega \in \mathbb{R}\backslash\{0\}$. Now we set $\beta(t,\xi) = \langle\xi\rangle^2\alpha(t,\xi) - V_1(t,\xi)$ and $\kappa(\xi) = \langle\xi\rangle^{-4}\nu(\xi)$. Then we have

$$i\partial_t\beta(t,\xi) = \frac{\kappa(\xi)}{t}|\beta(t,\xi)|^2\beta(t,\xi) + \rho(t,\xi), \tag{2.4.10}$$

where

$$\rho(t,\xi) = \frac{\kappa(\xi)}{t}\left(\left|\langle\xi\rangle^2\alpha\right|^2 \langle\xi\rangle^2\alpha - |\beta|^2\beta \right) + V_2(t,\xi) + R_0(t,\xi) - R_2(t,\xi).$$

Remark that ρ can be regarded as a remainder because we have

$$|\rho(t,\xi)| \le \frac{C}{t}\cdot\left(C\varepsilon^{2/3}\right)^2\cdot\frac{C\varepsilon^2}{t^{1/2}} + \frac{C\varepsilon^2}{t^{3/2}} + \frac{C}{t^{5/4}}\cdot\left(C\varepsilon^{2/3}t^{\gamma}\right)^3 \le \frac{C\varepsilon^2}{t^{1+\mu}}$$

with $\mu = 1/4 - 3\gamma > 0$. Moreover we have

$$\begin{aligned}
&|\beta(1,\xi) - \varepsilon\langle\xi\rangle^2\mathcal{F}\varphi(\xi)| \\
&\le C\left\|(1-\partial_x^2)\left(\mathcal{U}(1)^{-1}u(1,\cdot) - \varepsilon\varphi\right)\right\|_{L^2(\mathbb{R})}^{1/2}\left\|(1-\partial_x^2)\left(\mathcal{U}(1)^{-1}u(1,\cdot) - \varepsilon\varphi\right)\right\|_{H^{0,1}(\mathbb{R})}^{1/2} \\
&\quad + \sup_{\xi\in\mathbb{R}}|V_1(1,\xi)| \\
&\le C\left(\int_0^1 \left\|N(u(t),u_x(t))\right\|_{H^2(\mathbb{R})}dt\right)^{1/2}\varepsilon^{1/2} + C\varepsilon^2 \\
&\le C\varepsilon^2,
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality $\|\phi\|_{L^\infty} \le C\|\phi\|_{L^2}^{1/2}\|\partial_x\phi\|_{L^2}^{1/2}$. Therefore we can apply Lemma 2.2.1 with $\theta_0(\xi) = \langle\xi\rangle^2\mathcal{F}\varphi(\xi)$ and $\tau_1 = \tilde{\tau}_0$ to obtain $|\beta(t,\xi)| \le C\varepsilon$, whence

$$\langle\xi\rangle^2|\alpha(t,\xi)| \le |\beta(t,\xi)| + |V_1(t,\xi)| \le C\varepsilon$$

for $(t,\xi) \in [1,T)\times\mathbb{R}$, as desired. □

2.5 Proof of Theorem 2.1.1

Now we prove Theorem 2.1.1. First we state a standard local existence result without a proof. Let $t_0 \geq 0$ be fixed, and consider the initial value problem

$$\begin{cases} \mathcal{L}u = N(u, u_x), & t > t_0,\ x \in \mathbb{R}, \\ u(t_0, x) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{2.5.1}$$

Lemma 2.5.1. *Let the nonlinear term N be a cubic homogeneous polynomial in $(u, \overline{u}, u_x, \overline{u_x})$. Let $\psi \in H^3 \cap H^{2,1}(\mathbb{R})$. Then there exists $T_0 = T_0(\|\psi\|_{H^3(\mathbb{R})}) > 0$, independent of t_0, such that* (2.5.1) *has a unique solution $u \in C\left([t_0, t_0 + T_0); H^3 \cap H^{2,1}(\mathbb{R})\right)$.*

See [45], [30], [4], [44], [25], etc., for more details on local existence theorems.

Proof of Theorem 2.1.1. Let T_ε be the lifespan defined in the statement of Theorem 2.1.1. We remark that Lemma 2.5.1 with $t_0 = 0$ and $\psi = \varepsilon\varphi$ implies $T_\varepsilon > 0$. Next we set

$$T^* = \sup\{T \in [0, T_\varepsilon) \mid E(T) \leq \varepsilon^{2/3}\}.$$

Note that $T^* > 0$ if ε is suitably small, because of the estimate $E(0) \leq C\varepsilon \leq (1/2)\varepsilon^{2/3}$ and the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$. Now, we take $\sigma \in (0, \tilde{\tau}_0)$ and assume $T^* \leq e^{\sigma/\varepsilon^2}$. Then Lemma 2.4.1 with $T = T^*$ yields

$$E(T^*) \leq K\varepsilon \leq \frac{1}{2}\varepsilon^{2/3}$$

if $\varepsilon \leq \min\{\varepsilon_0, (2K)^{-3}\}$. By the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$, we can choose $\Delta > 0$ such that

$$E(T^* + \Delta) \leq \varepsilon^{2/3}.$$

This contradicts the definition of T^*. Therefore we must have $T^* \geq e^{\sigma/\varepsilon^2}$ if ε is suitably small. Consequently, we have

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \sigma.$$

Since $\sigma \in (0, \tilde{\tau}_0)$ is arbitrary, we arrive at the desired conclusion. □

2.6 Proof of Lemmas 2.4.2 and 2.4.3

In this section, we will prove Lemmas 2.4.2 and 2.4.3 along the idea of [19].

2.6.1 Proof of Lemma 2.4.2

At first we will prove Lemma 2.4.2. We observe that the nonlinear term N satisfying (2.1.3) can be written as $N = F + G$, where

$$\begin{aligned} F = {}& a_1 u^2 u_x + a_2 u u_x^2 + a_3 u_x^3 + b_1 \overline{u^2 u_x} + b_2 \overline{u u_x^2} + b_3 \overline{u_x^3} \\ &+ c_1 \overline{u^2} u_x + c_2 |u|^2 \overline{u_x} + c_3 u \overline{u_x^2} + c_4 |u_x|^2 \overline{u} + c_5 |u_x|^2 \overline{u_x} \end{aligned} \tag{2.6.1}$$

and

$$G = \lambda_1 |u|^2 u + \lambda_2 |u|^2 u_x + \lambda_3 u^2 \overline{u_x} + \lambda_4 |u_x|^2 u + \lambda_5 \overline{u} u_x^2 + \lambda_6 |u_x|^2 u_x \tag{2.6.2}$$

with $a_j,\ b_j,\ c_j,\ \lambda_j \in \mathbb{C}$. Note that G is gauge-invariant, while F is not. By using the identities

$$\phi \partial_x \psi = (\partial_x \phi)\psi + \frac{1}{it}\big(\phi \mathcal{J}\psi - (\mathcal{J}\phi)\psi\big) \tag{2.6.3}$$

and

$$\phi \partial_x \overline{\psi} = -(\partial_x \phi)\overline{\psi} + \frac{1}{it}\big((\mathcal{J}\phi)\overline{\psi} - \phi \overline{\mathcal{J}\psi}\big), \tag{2.6.4}$$

we see that F can be rewritten as $\partial_x F_1 + \frac{1}{it} F_2$, where

$$\begin{aligned} F_1 = {}& \frac{a_1}{3} u^3 + \frac{a_2}{3} u^2 u_x + \frac{a_3}{3} u_x^2 u + \frac{b_1}{3} \overline{u^3} + \frac{b_2}{3} \overline{u^2 u_x} + \frac{b_3}{3} \overline{u u_x^2} \\ &+ (c_2 - c_1)|u|^2 \overline{u} + c_3 |u|^2 \overline{u_x} + c_4 \overline{u}^2 u_x + c_5 |u_x|^2 \overline{u} \end{aligned}$$

and

$$\begin{aligned} F_2 = {}& \frac{a_2}{3} u\big(u \mathcal{J} u_x - u_x \mathcal{J} u\big) - \frac{2a_3}{3} u_x \big(u \mathcal{J} u_x - u_x \mathcal{J} u\big) \\ &- \frac{b_2}{3} \overline{u\big(u \mathcal{J} u_x - u_x \mathcal{J} u\big)} + \frac{2b_3}{3} \overline{u_x \big(u \mathcal{J} u_x - u_x \mathcal{J} u\big)} \\ &+ (c_2 - 2c_1)\overline{u}\big(u \overline{\mathcal{J} u} - \overline{u} \mathcal{J} u\big) - c_3 \overline{u}\big(\overline{u}_x \mathcal{J} u - u \overline{\mathcal{J} u_x}\big) \\ &- c_4 \overline{u}\big(\overline{u} \mathcal{J} u_x - u_x \overline{\mathcal{J} u}\big) - c_5 \overline{u}\big(\overline{u}_x \mathcal{J} u_x - u_x \overline{\mathcal{J} u_x}\big). \end{aligned}$$

We deduce from the relation (2.3.1) that

$$\mathcal{J} N(u, u_x) = (\mathcal{Z} + 2it\mathcal{L}) F_1 + \frac{1}{it} \mathcal{J} F_2 + \mathcal{J} G = \mathcal{L}(tP) + Q,$$

where $P = 2iF_1$ and $Q = (\mathcal{Z} + 2)F_1 + \frac{1}{it}\mathcal{J} F_2 + \mathcal{J} G$. By the Leibniz rule for $\mathcal{Z}$, we have

$$\|(\mathcal{Z} + 2)F_1\|_{L^2} \le C \|u\|_{W^{1,\infty}}^2 \big(\|\mathcal{Z} u\|_{H^1} + \|u\|_{H^1}\big).$$

On the other hand, since G is gauge-invariant, we can use the identity

$$\mathcal{J}(f_1 f_2 \overline{f_3}) = (\mathcal{J} f_1) f_2 \overline{f_3} + f_1 (\mathcal{J} f_2) \overline{f_3} - f_1 f_2 \overline{\mathcal{J} f_3}$$

to obtain

$$\|\mathcal{J} G\|_{L^2} \le C \|u\|_{W^{1,\infty}}^2 \big(\|\mathcal{J} u\|_{H^1} + \|u\|_{L^2}\big).$$

In order to get the L^2-bound for $\frac{1}{it}\mathcal{J} F_2$, we apply the identities

$$\mathcal{J}(f_1 f_2 f_3) = it\big\{\partial_x (f_1 f_2 f_3) - (\partial_x f_1) f_2 f_3\big\} + (\mathcal{J} f_1) f_2 f_3,$$

$$\mathcal{J}(\overline{f_1 f_2 f_3}) = it\big\{\partial_x (\overline{f_1 f_2 f_3}) + (\overline{\partial_x f_1}) \overline{f_2 f_3}\big\} + (\overline{\mathcal{J} f_1}) \overline{f_2 f_3}$$

to each term of F_2 multiplied by $\mathcal{J}$, and use the inequality

$$\|\mathcal{J} \partial_x \phi\|_{L^\infty} \le C\big(\|\mathcal{J}\phi\|_{H^2} + \|\phi\|_{H^1}\big).$$

Then we have

$$\|\mathcal{J} F_2\|_{L^2} \le Ct \|u\|_{W^{2,\infty}}^2 \big(\|\mathcal{J} u\|_{H^2} + \|u\|_{H^1}\big) + C\|u\|_{W^{2,\infty}} \big(\|\mathcal{J} u\|_{H^2} + \|u\|_{H^1}\big)^2.$$

Piecing them together, we arrive at the desired decomposition. $\square$

2.6.2 Proof of Lemma 2.4.3

Before proceeding to the proof of Lemma 2.4.3, we introduce some notations. We put

$$\big(\mathcal{M}(t)\phi\big)(x) = e^{i\frac{x^2}{2t}}\phi(x), \quad \big(\mathcal{D}(t)\phi\big)(x) = \frac{1}{\sqrt{it}}\phi\left(\frac{x}{t}\right), \quad \mathcal{W}(t)\phi = \mathcal{F}\mathcal{M}(t)\mathcal{F}^{-1}\phi,$$

so that $\mathcal{U}(t)$ is decomposed into $\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{W}(t)\mathcal{F}$. Note that

$$\|(\mathcal{W}(t) - 1)\phi\|_{L^\infty} + \|(\mathcal{W}(t)^{-1} - 1)\phi\|_{L^\infty} \le Ct^{-1/4}\|\phi\|_{H^1}, \tag{2.6.5}$$

which comes from the inequalities $|e^{i\theta} - 1| \le C|\theta|^{1/2}$ and $\|\phi\|_{L^\infty} \le C\|\phi\|_{L^2}^{1/2}\|\partial_x\phi\|_{L^2}^{1/2}$. In what follows, we will occasionally omit "(t)" from $\mathcal{M}(t)$, $\mathcal{D}(t)$, $\mathcal{W}(t)$ if it causes no confusion, and we will write $\mathcal{D}_\omega = \mathcal{D}(\omega)$ for $\omega \in \mathbb{R}\backslash\{0\}$.

Lemma 2.6.1. *We have*

$$\|\mathcal{F}\mathcal{U}^{-1}[f_1 f_2 f_3]\|_{L^\infty} + \|\mathcal{F}\mathcal{U}^{-1}[f_1 f_2 \overline{f_3}]\|_{L^\infty} \le \frac{C}{t^{1/2}}\|f_1\|_{L^2}\|f_2\|_{L^2}\big(\|f_3\|_{L^2} + \|\mathcal{J} f_3\|_{L^2}\big).$$

Proof. From the relation $\mathcal{F}\mathcal{U}^{-1} = \mathcal{W}^{-1}\mathcal{D}^{-1}\mathcal{M}^{-1}$ and the estimate $\|\mathcal{W}^{-1}\phi\|_{L^\infty} \leq Ct^{1/2}\|\phi\|_{L^1}$, it follows that

$$\begin{aligned}
\|\mathcal{F}\mathcal{U}^{-1}[f_1 f_2 f_3]\|_{L^\infty} &\leq Ct^{1/2}\|\mathcal{D}^{-1}\mathcal{M}^{-1}[f_1 f_2 f_3]\|_{L^1} \\
&= Ct^{1/2} \cdot t^{-1/2}\|f_1 f_2 f_3\|_{L^1} \\
&\leq C\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^\infty} \\
&\leq Ct^{-1/2}\|f_1\|_{L^2}\|f_2\|_{L^2}\big(\|f_3\|_{L^2} + \|\mathcal{J}f_3\|_{L^2}\big).
\end{aligned}$$

We have used the inequality $\|f\|_{L^\infty} \leq Ct^{-1/2}\|f\|_{L^2}^{1/2}\|\mathcal{J}f\|_{L^2}^{1/2}$ in the last line. The estimate for $\|\mathcal{F}\mathcal{U}^{-1}(f_1 f_2 \overline{f_3})\|_{L^\infty}$ can be shown in the same way. □

Next we set $(\mathcal{E}^\omega(t)f)(y) = e^{i\omega\frac{ty^2}{2}}f(y)$ and $\mathcal{A}_\omega(t) = \mathcal{V}(t)^{-1}\mathcal{E}^{\omega-1}(t) - \mathcal{E}^{\frac{\omega-1}{\omega}}(t)\mathcal{D}_\omega$.

Lemma 2.6.2. *For $\omega \in \mathbb{R}\backslash\{0\}$, we have*

$$\|\mathcal{A}_\omega(t)f\|_{L^\infty} \leq Ct^{-1/4}\|f\|_{H^1}.$$

Proof. It follows from the relation $\mathcal{W}(t)^{-1} = \mathcal{U}(\frac{1}{t})$ that

$$\begin{aligned}
\mathcal{W}(t)^{-1}\mathcal{E}^{\omega-1}(t)f(\xi) &= \sqrt{\frac{t}{2\pi i}}\int_{\mathbb{R}} e^{i\frac{t}{2}(\xi-y)^2}e^{i\frac{\omega-1}{2}ty^2}f(y)\,dy \\
&= e^{i\frac{\omega-1}{2\omega}t\xi^2}\sqrt{\frac{t}{2\pi i}}\int_{\mathbb{R}} e^{i\frac{\omega t}{2}\left(\frac{\xi}{\omega}-y\right)^2}f(y)\,dy \\
&= \mathcal{E}^{\frac{\omega-1}{\omega}}(t)\mathcal{D}_\omega\mathcal{W}(-\omega t)f(\xi).
\end{aligned}$$

Hence we deduce from (2.6.5) that

$$\begin{aligned}
\|\mathcal{A}_\omega(t)f\|_{L^\infty} &= \|\mathcal{E}^{\frac{\omega-1}{\omega}}(t)\mathcal{D}_\omega(\mathcal{W}(-\omega t) - 1)f\|_{L^\infty} \\
&\leq C\|(\mathcal{W}(-\omega t) - 1)f\|_{L^\infty} \\
&\leq Ct^{-1/4}\|f\|_{H^1}.
\end{aligned}$$

□

Now we are going to prove Lemma 2.4.3. For simplicity of exposition, we consider only the case where

$$N(u, u_x) = \lambda|u_x|^2u_x + au_x^3 + b\overline{u_x^3} + c|u_x|^2\overline{u_x}. \tag{2.6.6}$$

General cubic terms N satisfying (2.1.3) (or, equivalently, $N = F + G$ with (2.6.1) and (2.6.2)) can be treated in the same way. Note that

$$\nu(\xi) = i\xi^3 \int_0^{2\pi} \left(\lambda e^{i\theta} - ae^{3i\theta} + be^{-3i\theta} - ce^{-i\theta}\right) e^{-i\theta} \frac{d\theta}{2\pi} = i\lambda\xi^3$$

if N is given by (2.6.6), whereas

$$\nu(\xi) = \lambda_1 + i(\lambda_2 - \lambda_3)\xi + (\lambda_4 - \lambda_5)\xi^2 + i\lambda_6\xi^3 \tag{2.6.7}$$

if $N = F + G$ with (2.6.1), (2.6.2).

First we consider the case of $l = 0$. We put $\alpha^{(s)} = (i\xi)^s\alpha$ so that $\partial_x^s u = \mathcal{M}\mathcal{D}\mathcal{W}\alpha^{(s)}$. We also set $\left(\mathcal{M}^\omega(t)f\right)(y) = e^{i\omega\frac{y^2}{2t}} f(y)$. Then it follows that

$$\begin{aligned}
N(u, u_x) &= \lambda\mathcal{M}|\mathcal{D}\mathcal{W}\alpha^{(1)}|^2\mathcal{D}\mathcal{W}\alpha^{(1)} + a\mathcal{M}^3(\mathcal{D}\mathcal{W}\alpha^{(1)})^3 \\
&\quad + b\mathcal{M}^{-3}(\overline{\mathcal{D}\mathcal{W}\alpha^{(1)}})^3 + c\mathcal{M}^{-1}|\mathcal{D}\mathcal{W}\alpha^{(1)}|^2\overline{\mathcal{D}\mathcal{W}\alpha^{(1)}} \\
&= \frac{\lambda}{t}\mathcal{M}\mathcal{D}\left[|\mathcal{V}\alpha^{(1)}|^2\mathcal{V}\alpha^{(1)}\right] + \frac{a}{t}\mathcal{M}^3\mathcal{D}\left[(\mathcal{W}\alpha^{(1)})^3\right] \\
&\quad + \frac{b}{t}\mathcal{M}^{-3}\mathcal{D}\left[(\overline{\mathcal{W}\alpha^{(1)}})^3\right] + \frac{c}{t}\mathcal{M}^{-1}\mathcal{D}\left[|\mathcal{W}\alpha^{(1)}|^2\overline{\mathcal{W}\alpha^{(1)}}\right].
\end{aligned}$$

By the relation $\mathcal{F}\mathcal{U}^{-1}\mathcal{M}^\omega\mathcal{D} = \mathcal{W}^{-1}\mathcal{E}^{\omega-1}$, we have

$$\begin{aligned}
\mathcal{F}\mathcal{U}^{-1}N(u, u_x) &= \frac{\lambda}{t}\mathcal{W}^{-1}\left[|\mathcal{W}\alpha^{(1)}|^2\mathcal{W}\alpha^{(1)}\right] + \frac{a}{t}\mathcal{W}^{-1}\mathcal{E}^2\left[(\mathcal{W}\alpha^{(1)})^3\right] \\
&\quad + \frac{b}{t}\mathcal{W}^{-1}\mathcal{E}^{-4}\left[(\overline{\mathcal{W}\alpha^{(1)}})^3\right] + \frac{c}{t}\mathcal{W}^{-1}\mathcal{E}^{-2}\left[|\mathcal{W}\alpha^{(1)}|^2\overline{\mathcal{W}\alpha^{(1)}}\right] \\
&= \frac{\lambda}{t}i\xi^3|\alpha|^2\alpha + \frac{a}{t}\mathcal{E}^{\frac{2}{3}}\mathcal{D}_3\left[-i\xi^3\alpha^3\right] \\
&\quad + \frac{b}{t}\mathcal{E}^{\frac{4}{3}}\mathcal{D}_{-3}\left[i\xi^3\overline{\alpha}^3\right] + \frac{c}{t}\mathcal{E}^2\mathcal{D}_{-1}\left[-i\xi^3|\alpha|^2\overline{\alpha}\right] + \frac{\Omega_0}{t},
\end{aligned}$$

where

$$\begin{aligned}
\Omega_0 = &-\lambda\Big(|\alpha^{(1)}|^2\alpha^{(1)} - |\mathcal{W}\alpha^{(1)}|^2\mathcal{W}\alpha^{(1)}\Big) + \lambda\left(\mathcal{W}^{-1} - 1\right)\left[|\mathcal{W}\alpha^{(1)}|^2\mathcal{V}\alpha^{(1)}\right] \\
&- a\mathcal{E}^{\frac{2}{3}}(t)\mathcal{D}_3\left[(\alpha^{(1)})^3 - (\mathcal{W}\alpha^{(1)})^3\right] + a\mathcal{A}_3(t)\left[(\mathcal{W}\alpha^{(1)})^3\right] \\
&- b\mathcal{E}^{\frac{4}{3}}(t)\mathcal{D}_{-3}\left[(\overline{\alpha^{(1)}})^3 - (\overline{\mathcal{W}\alpha^{(1)}})^3\right] + b\mathcal{A}_{-3}(t)\left[(\overline{\mathcal{W}\alpha^{(1)}})^3\right] \\
&- c\mathcal{E}^2(t)\mathcal{D}_{-1}\left[|\alpha^{(1)}|^2\overline{\alpha^{(1)}} - |\mathcal{W}\alpha^{(1)}|^2\overline{\mathcal{W}\alpha^{(1)}}\right] + c\mathcal{A}_{-1}(t)\left[|\mathcal{W}\alpha^{(1)}|^2\overline{\mathcal{W}\alpha^{(1)}}\right].
\end{aligned}$$

By virtue of (2.6.5) and Lemma 2.6.2, we see that

$$\|\Omega_0\|_{L^\infty} \le \frac{C}{t^{1/4}}(\|u\|_{H^2} + \|\mathcal{J}u\|_{H^1})^3.$$

Therefore we obtain (2.4.8) with $l = 0$ by putting $\mu_{1,0}(\xi) = -i\frac{a}{27\sqrt{3}}\xi^2$, $\mu_{2,0}(\xi) = \frac{-b}{27\sqrt{3}}\xi^2$, $\mu_{3,0}(\xi) = c\xi^2$.

Next we consider the case of $l = 1$. It follow from the identity (2.6.4) that

$$\partial_x N(u, u_x) = \lambda|u_x|^2 u_{xx} + 3au_x^2 u_{xx} + 3b\overline{u_x^2 u_{xx}} + c|u_x|^2\overline{u_{xx}} + \frac{1}{it}r_1,$$

where $r_1 = (\lambda u_x + c\overline{u}_x)\big((\mathcal{J}u_x)\overline{u_x} - u_x\overline{\mathcal{J}u_x}\big)$. By Lemma 2.6.1, we obtain

$$\|\mathcal{F}\mathcal{U}^{-1}r_1\|_{L^\infty} \le \frac{C}{t^{1/2}}\big(\|u\|_{H^1} + \|\mathcal{J}u\|_{H^1}\big)^3.$$

We also set $h_1 = \lambda|u_x|^2 u_{xx} + 3au_x^2 u_{xx} + 3b\overline{u_x^2 u_{xx}} + c|u_x|^2\overline{u_{xx}}$ so that

$$\mathcal{F}\mathcal{U}^{-1}\partial_x N(u, u_x) = \mathcal{F}\mathcal{U}^{-1}h_1 + \frac{1}{it}\mathcal{F}\mathcal{U}^{-1}r_1.$$

Then, as in the previous case, we have

$$\begin{aligned}
&\mathcal{F}\mathcal{U}^{-1}h_1 \\
&= -\frac{\lambda}{t}\xi^4|\alpha|^2\alpha + \frac{3a}{t}\mathcal{E}^{\frac{2}{3}}\mathcal{D}_3\Big[\xi^4\alpha^3\Big] + \frac{3b}{t}\mathcal{E}^{\frac{4}{3}}\mathcal{D}_{-3}\Big[\xi^4\overline{\alpha}^3\Big] + \frac{c}{t}\mathcal{E}^2\mathcal{D}_{-1}\left[-\xi^4|\alpha|^2\overline{\alpha}\right] + \frac{\Omega_1}{t},
\end{aligned}$$

where

$$\begin{aligned}
\Omega_1 =& -\lambda\Big(|\alpha^{(1)}|^2\alpha^{(2)} - |\mathcal{W}\alpha^{(1)}|^2\mathcal{W}\alpha^{(2)}\Big) + \lambda\Big(\mathcal{W}^{-1} - 1\Big)\Big[|\mathcal{W}\alpha^{(1)}|^2\mathcal{V}\alpha^{(2)}\Big] \\
&- 3a\mathcal{E}^{\frac{2}{3}}(t)\mathcal{D}_3\Big[(\alpha^{(1)})^2\alpha^{(2)} - (\mathcal{W}\alpha^{(1)})^2\mathcal{W}\alpha^{(1)}\Big] + 3a\mathcal{A}_3(t)\Big[(\mathcal{W}\alpha^{(1)})^2\mathcal{W}\alpha^{(2)}\Big] \\
&- 3b\mathcal{E}^{\frac{4}{3}}(t)\mathcal{D}_{-3}\Big[\overline{(\alpha^{(1)})^2\alpha^{(2)}} - \overline{(\mathcal{W}\alpha^{(1)})^2\mathcal{W}\alpha^{(2)}}\Big] + 3b\mathcal{A}_{-3}(t)\Big[\overline{(\mathcal{W}\alpha^{(1)})^2\mathcal{W}\alpha^{(2)}}\Big] \\
&+ c\mathcal{E}^2(t)\mathcal{D}_{-1}\Big[|\alpha^{(1)}|^2\overline{\alpha^{(2)}} - |\mathcal{W}\alpha^{(1)}|^2\overline{\mathcal{W}\alpha^{(2)}}\Big] + c\mathcal{A}_{-1}(t)\Big[|\mathcal{W}\alpha^{(1)}|^2\overline{\mathcal{W}\alpha^{(2)}}\Big].
\end{aligned}$$

By (2.6.5) and Lemma 2.6.2, we have

$$\|\Omega_1\|_{L^\infty} \le \frac{C}{t^{1/4}}(\|u\|_{H^3} + \|\mathcal{J}u\|_{H^2})^3.$$

Therefore, by setting $\mu_{1,1}(\xi) = \frac{a}{27\sqrt{3}}\xi^3$, $\mu_{2,1}(\xi) = \frac{b}{27\sqrt{3}i}\xi^3$, $\mu_{3,1}(\xi) = -ic\xi^3$, we obtain (2.4.8) with $l = 1$.

Finally we consider the case of $l = 2$. From the identities (2.6.3) and (2.6.4), it follows that

$$\begin{aligned}
\partial_x^2 N(u, u_x) &= \lambda u_x \cdot \overline{u_x} \partial_x (u_{xx}) + c\overline{u_x} \cdot u_x \partial_x (\overline{u_{xx}}) + (\lambda u_{xx} + c\overline{u_{xx}}) \partial_x \big(|u_x|^2\big) \\
&\quad + 3a u_x \big(2u_{xx}^2 + u_x \partial_x (u_{xx})\big) + 3b\overline{u_x \big(2u_{xx}^2 + u_x \partial_x (u_{xx})\big)} + \frac{1}{it} \partial_x r_1 \\
&= h_2 + \frac{1}{it} r_2,
\end{aligned}$$

where $h_2 = -\lambda |u_{xx}|^2 u_x + 9a u_{xx}^2 u_x + 9b\overline{u_{xx}^2 u_x} - c|u_{xx}|^2 \overline{u_x}$ and

$$\begin{aligned}
r_2 = {}& \lambda u_x \big((\mathcal{J} u_{xx}) \overline{u_x} - u_{xx} \overline{\mathcal{J} u_x}\big) + c\overline{u}_x \big((\mathcal{J} u_x) \overline{u_{xx}} - u_x \overline{\mathcal{J} u_{xx}}\big) \\
& + (\lambda u_{xx} + c\overline{u_{xx}}) \big((\mathcal{J} u_x) \overline{u_x} - u_x \overline{\mathcal{J} u_x}\big) + 3a u_x \big(u_x \mathcal{J} u_{xx} - (\mathcal{J} u_x) u_{xx}\big) \\
& - 3b\overline{u_x \big(u_x \mathcal{J} u_{xx} - (\mathcal{J} u_x) u_{xx}\big)} + \partial_x r_1.
\end{aligned}$$

We deduce as before that

$$\begin{aligned}
& \mathcal{F}\mathcal{U}^{-1} h_2 \\
& = -\frac{\lambda}{t} i\xi^5 |\alpha|^2 \alpha + \frac{9a}{t} \mathcal{E}^{\frac{2}{3}} \mathcal{D}_3 \Big[i\xi^5 \alpha^3\Big] + \frac{9b}{t} \mathcal{E}^{\frac{4}{3}} \mathcal{D}_{-3} \Big[-i\xi^5 \overline{\alpha}^3\Big] - \frac{c}{t} \mathcal{E}^2 \mathcal{D}_{-1} \Big[-i\xi^5 |\alpha|^2 \overline{\alpha}\Big] + \frac{\Omega_2}{t}
\end{aligned}$$

with

$$\|\Omega_2\|_{L^\infty} \le \frac{C}{t^{1/4}} (\|u\|_{H^3} + \|\mathcal{J} u\|_{H^2})^3.$$

We also have

$$\|\mathcal{F}\mathcal{U}^{-1} r_2\|_{L^\infty} \le \frac{C}{t^{1/2}} \big(\|u\|_{H^2} + \|\mathcal{J} u\|_{H^2}\big)^3$$

by virtue of Lemma 2.6.1. Now we set $\mu_{1,2}(\xi) = \frac{ai}{27\sqrt{3}} \xi^4$, $\mu_{2,2}(\xi) = \frac{b}{27\sqrt{3}} \xi^4$, $\mu_{3,2}(\xi) = -c\xi^4$. Then we arrive at (2.4.8) with $l = 2$. This completes the proof of Lemma 2.4.3. □

Chapter 3

The lifespan of small solutions to subcritical nonlinear Schrödinger equations in dimension $d \leq 3$

3.1 Introduction

This chapter is based on the joint work [67] with Hideaki Sunagawa and Shunsuke Yasuda. We consider the following initial value problem:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\partial_x^2 u = \lambda |u|^{p-1}u, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \varepsilon\varphi(x), & x \in \mathbb{R}, \end{cases} \tag{3.1.1}$$

where $i = \sqrt{-1}$, $u = u(t,x)$ is a $\mathbb{C}$-valued unknown function, $\lambda \in \mathbb{C}$ and $p > 1$. $\varphi = \varphi(x)$ is a prescribed $\mathbb{C}$-valued function which belongs to a suitable weighted Sobolev space, and $\varepsilon > 0$ is a small parameter which is responsible for the size of the initial data. We are interested in the lifespan T_ε for the solution $u(t,x)$ to (3.1.1) in the case of $p < 3$ and $\operatorname{Im}\lambda > 0$. Before going into details, let us summarize the backgrounds briefly.

First we consider the simpler case $p > 3$. In this case, small data global existence for (3.1.1) is well-known. Moreover, the solution behaves like the free solution in the large time (see [76]). On the other hand when $1 < p \leq 3$, the situation changes dramatically, as pointed out by Strauss [71], Barab [2], Ozawa [62], Hayashi–Naumkin [28], and so on. Note that this threshold becomes $p = 1 + 2/d$ in the d-dimensional settings.

Next let us turn our attention to the critical case $p = 3$. In [28], it has been shown that the solution to (3.1.1) with $p = 3$ and $\lambda \in \mathbb{R}$ behaves like

$$u(t,x) = \frac{1}{\sqrt{it}}\tilde{\alpha}(x/t)e^{i\{x^2/(2t) - \lambda|\tilde{\alpha}(x/t)|^2 \log t\}} + o(t^{-1/2}) \quad \text{in } L^\infty(\mathbb{R}_x)$$

as $t \to \infty$ with a suitable $\mathbb{C}$-valued function $\tilde{\alpha}$ satisfying $\|\tilde{\alpha}\|_{L^\infty(\mathbb{R})} \le C\varepsilon$. An important consequence of this asymptotic expression is that the solution decays like $O(t^{-1/2})$ in $L^\infty(\mathbb{R}_x)$, while it does not behave like the free solution unless $\lambda = 0$. In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension. This result has been extended in [12] to the case where p is less than and sufficiently close to 3. When $\lambda \in \mathbb{C}$, the situation changes slightly. Indeed, it has been verified in [70] that the small data solution to (4.1.1) decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \to \infty$ if $p = 3$ and $\operatorname{Im}\lambda < 0$. This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect (see also [72] for a closely related result for the Klein-Gordon equation). The above-mentioned result has been extended in [48], [49], [14], [39], etc., to the subcritical case $p < 3$ and $\operatorname{Im}\lambda < 0$. However, it should be noted that these results essentially rely on the a priori L^2-bound for the solution u coming from the conservation law

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})}^2 - 2\operatorname{Im}\lambda \int_0^t \|u(\tau,\cdot)\|_{L^{p+1}(\mathbb{R})}^{p+1} d\tau = \|u(0,\cdot)\|_{L^2(\mathbb{R})}^2,$$

which is valid only when $\operatorname{Im}\lambda \le 0$. In what follows, we focus on the remaining case $p \le 3$ and $\operatorname{Im}\lambda > 0$. This is the worst situation for small data global existence because the nonlinearity must be considered as a long-range perturbation and the a priori L^2-bound for u is violated. To the author's best knowledge, there is no positive result in that case. As for the lifespan T_ε, the standard perturbative argument yields a lower estimate in the form

$$T_\varepsilon \ge \begin{cases} e^{C/\varepsilon^2} & \text{(when } p = 3\text{)} \\ C\varepsilon^{-2(p-1)/(3-p)} & \text{(when } 1 < p < 3\text{)} \end{cases}$$

with some $C > 0$, provided that ε is suitably small (see Chapter 1, as well as Section 3.3 below for more detail). In other words, we have

$$\liminf_{\varepsilon \to +0} \int_1^{T_\varepsilon} \left(\frac{\varepsilon}{t^{1/2}}\right)^{p-1} dt > 0.$$

However, this estimate does not tell us the dependence of T_ε on $\operatorname{Im}\lambda$. So we are led to the question: *how does T_ε depend on* $\operatorname{Im}\lambda$*?* In the cubic case ($p = 3$), the following more precise estimate for T_ε has been derived in [73] and [66]:

$$\liminf_{\varepsilon \to +0} (\varepsilon^2 \log T_\varepsilon) \ge \frac{1}{2\operatorname{Im}\lambda \sup_{\xi\in\mathbb{R}} |\mathcal{F}\varphi(\xi)|^2},$$

as we have seen in Chapter 2. This gives an answer to the question raised above for the cubic case. In fact, more general cubic nonlinear terms depending also on $\partial_x u$ have been treated in [73] and [66] (see also [58] for a related work). When $p < 3$ and $\operatorname{Im}\lambda > 0$, the situation is the most delicate and quite little is known so far. To the author's knowledge, there is only one result which concerns the dependence of T_ε on $\operatorname{Im}\lambda$ in the case of $p < 3$:

Proposition 3.1.1 (Sasaki [69])**.** *Assume* $2 \le p < 3$, $\operatorname{Im}\lambda > 0$ *and* $(1+x^2)\varphi \in \Sigma$. *Let* T_ε *be the supremum of* $T > 0$ *such that* (3.1.1) *admits a unique solution* $u \in C([0,T);\Sigma)$. *Then we have*

$$\liminf_{\varepsilon\to+0} \left(\varepsilon^{2(p-1)/(3-p)} T_\varepsilon\right) \ge \left(\frac{3-p}{2(p-1)\operatorname{Im}\lambda \sup\limits_{\xi\in\mathbb{R}} |\mathcal{F}\varphi(\xi)|^{p-1}} \right)^{2/(3-p)},$$

where $\Sigma = \{f \in L^2(\mathbb{R}) \mid \|f\|_\Sigma < \infty\}$ *with* $\|f\|_\Sigma = \|f\|_{L^2(\mathbb{R})} + \|\partial_x f\|_{L^2(\mathbb{R})} + \|xf\|_{L^2(\mathbb{R})}$.

However, the approach exploited in [69] has the following two drawbacks:

- the detailed lifespan estimate is unknown in the case of $1 < p < 2$,
- it requires faster decay as $|x| \to \infty$ for φ than that for $u(t,\cdot)$.

The purpose of this chapter is to improve these two points and to give a higher dimensional generalization. In what follows, we consider a d-dimensional generalization of (3.1.1). For the notational convenience, we write the power p of the nonlinearity as $p = 1 + 2\theta/d$ so that the condition $1 < p < 1 + 2/d$ is interpreted as $0 < \theta < 1$. Then we are led to the following initial value problem:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = \lambda |u|^{2\theta/d} u, & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = \varepsilon\varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.1.2}$$

where $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_d)^2$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. To state the main result, let us introduce some notation. We define $\Sigma^s := H^s \cap H^{0,s}(\mathbb{R}^d)$ with the norm $\|f\|_{\Sigma^s} := \|f\|_{H^s(\mathbb{R}^d)} + \|f\|_{H^{0,s}(\mathbb{R}^d)}$. We set $\mathcal{U}(t) := \exp(\frac{it}{2}\Delta)$ so that the solution v to the free Schrödinger equation

$$i\partial_t v + \frac{1}{2}\Delta v = 0, \quad v(0,x) = \phi(x)$$

can be written as $v(t) = \mathcal{U}(t)\phi$. The main result is as follows.

Theorem 3.1.2. *Let* $1 \le d \le 3$, $0 < \theta < 1$ *and* $\lambda \in \mathbb{C}$ *with* $\operatorname{Im}\lambda > 0$. *Assume*

$$d/2 < s < \min\{2, 1 + 2\theta/d\} \tag{3.1.3}$$

and $\varphi \in \Sigma^s$. Let T_ε be the supremum of $T > 0$ such that (3.1.2) *admits a unique solution u satisfying $\mathcal{U}(\cdot)^{-1}u \in C([0,T); \Sigma^s)$. Then we have*

$$\liminf_{\varepsilon \to +0} \left(\varepsilon^{2\theta/d} T_\varepsilon^{\,1-\theta}\right) \geq \frac{(1-\theta)d}{2\theta \operatorname{Im} \lambda \sup_{\xi \in \mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^{2\theta/d}}. \tag{3.1.4}$$

Remark 3.1.3. The assumption (3.1.3) is never satisfied when $d \geq 4$. That is the reason why Theorem 3.1.2 is available only for $d \leq 3$. When $d = 1$ or 2, (3.1.3) is satisfied for any $0 < \theta < 1$. In particular, our result can be viewed as an extension of Proposition 3.1.1 because it corresponds to the case of $d = 1$, $1/2 \leq \theta < 1$ and $s = 1$ in Theorem 3.1.2. On the other hand, when $d = 3$, (3.1.3) is satisfied only if $\theta > 3/4$ (or, equivalently, $3/2 < p < 5/3$ with $p = 1 + 2\theta/3$). The author does not know whether the same assertion holds true or not when $d \geq 4$ or $d = 3$ with $\theta \leq 3/4$.

Remark 3.1.4. The author does not know whether (3.1.4) is optimal or not. An example of the blowing-up solution to (3.1.1) with arbitrarily small $\varepsilon > 0$ has been given by Kita [47] under a particular choice of φ and some additional restrictions on λ and p. However, it seems difficult to specify the lifespan for the blowing-up solution given in [47].

Now, let us explain the differences between the approach of [69] and ours. The method of [69] consists of two steps: the first step is to construct a suitable approximate solution u_a which blows up at the expected time, and the second step is to get an a priori estimate not for the solution u itself but for their difference $u - u_a$ (see also [73] for the cubic case). Drawbacks of this approach come from the first step. In fact, according to Remark 1.3 in [69], this approach can not be used in the case $1 < p < 2$. Remark that this implies the method of [69] is not suitable for d-dimensional settings when $d \geq 2$, because our main interest is the case of $p < 1 + 2/d$. Also, in view of Proposition 3.1 in [69], the additional decay assumption on φ as $|x| \to \infty$ (i.e., higher regularity for $\mathcal{F}\varphi$) seems essential for the method of [69]. On the other hand, our approach presented below does not rely on approximate solutions at all. Instead, we will reduce the original PDE (3.1.2) to a simpler ordinary differential equation satisfied by $A(t,\xi) = \mathcal{F}\big[\mathcal{U}(t)^{-1}u(t,\cdot)\big](\xi)$ up to a harmless remainder term R (see (3.5.1) below). An ODE lemma prepared in Section 3.4 below will allow us to get an a priori bound for u directly. Similar idea has been used in [66] for one-dimensional cubic derivative nonlinear Schrödinger equations, but we must be more careful because we are considering the situation in which the degree of the nonlinearity is lower.

We close this section with the contents of this chapter. In the next section, we state basic lemmas which will be useful in the subsequent sections. In Section 3.3, we will derive a rough lower estimate for T_ε, that is, $\liminf_{\varepsilon \to +0}(\varepsilon^{2\theta/d} T_\varepsilon^{1-\theta}) > 0$. Section 3.4 is devoted to an

ODE lemma which plays an important role in getting an a priori bound for the solution. After that, Theorem 3.1.2 will be proved in Section 3.5 by means of the so-called bootstrap argument. Finally, in Section 3.6, we discuss the critical case $\theta = 1$.

3.2 Basic lemmas

In this section, we introduce several lemmas that will be useful in the subsequent sections.

Lemma 3.2.1. *Let $s > d/2$. There exists a constant C such that*

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{(1+t)^{d/2}} \|\mathcal{U}(t)^{-1}\phi\|_{\Sigma^s}$$

for $t \ge 0$.

Proof. We start with the standard Gagliardo-Nirenberg-Sobolev inequality:

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \le C\|\phi\|_{L^2(\mathbb{R}^d)}^{1-d/2s} \|(-\Delta)^{s/2}\phi\|_{L^2(\mathbb{R}^d)}^{d/2s}. \tag{3.2.1}$$

We also introduce $\mathcal{M}(t) = \exp(\frac{i|x|^2}{2t})$ for $t > 0$. Then we can check that

$$\mathcal{U}(t)|x|^s\mathcal{U}(t)^{-1}\phi = \mathcal{M}(t)(-t^2\Delta)^{s/2}\mathcal{M}(t)^{-1}\phi$$

(see e.g., [30]), from which it follows that

$$\begin{aligned} t^{d/2}\|\phi\|_{L^\infty(\mathbb{R}^d)} &= t^{d/2}\|\mathcal{M}(t)^{-1}\phi\|_{L^\infty(\mathbb{R}^d)} \\ &\le C\|\mathcal{M}(t)^{-1}\phi\|_{L^2(\mathbb{R}^d)}^{1-d/2s} \|(-t^2\Delta)^{s/2}\mathcal{M}(t)^{-1}\phi\|_{L^2(\mathbb{R}^d)}^{d/2s} \\ &\le C\|\phi\|_{L^2(\mathbb{R}^d)}^{1-d/2s} \||x|^s\mathcal{U}(t)^{-1}\phi\|_{L^2(\mathbb{R}^d)}^{d/2s} \end{aligned}$$

for $t > 0$. Combining the two inequalities above, we obtain

$$\begin{aligned} (1+t)^{d/2}\|\phi\|_{L^\infty(\mathbb{R}^d)} &\le \frac{C(1+t)^{d/2}}{(1+t^{d/2})} \|\phi\|_{L^2(\mathbb{R}^d)}^{1-d/2s} \big(\|(-\Delta)^{s/2}\phi\|_{L^2(\mathbb{R}^d)}^{d/2s} + \big\||x|^s\mathcal{U}(t)^{-1}\phi\big\|_{L^2(\mathbb{R}^d)}^{d/2s}\big) \\ &\le C\left(\|\phi\|_{H^s(\mathbb{R}^d)} + \|\mathcal{U}(t)^{-1}\phi\|_{H^{0,s}(\mathbb{R}^d)}\right) \\ &= C\|\mathcal{U}(t)^{-1}\phi\|_{\Sigma^s}. \end{aligned}$$

□

Lemma 3.2.2. *Let $\gamma \in (0,1]$ and $s > d/2 + 2\gamma$. There exists a constant C such that*

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{t^{d/2}}\|\mathcal{F}\mathcal{U}(t)^{-1}\phi\|_{L^\infty(\mathbb{R}^d)} + \frac{C}{t^{d/2+\gamma}}\|\mathcal{U}(t)^{-1}\phi\|_{H^{0,s}(\mathbb{R}^d)}$$

for $t \ge 1$.

See Lemma 2.2 in [28] for the proof.

Next we define $G_p : \mathbb{C} \to \mathbb{C}$ with $p > 1$ by $G_p(z) = |z|^{p-1}z$ for $z \in \mathbb{C}$. Note that the nonlinear term in (3.1.2) can be written by $\lambda G_{1+2\theta/d}(u)$ with $0 < \theta < 1$ and $\operatorname{Im}\lambda > 0$. The following lemmas are concerned with estimates for G_p:

Lemma 3.2.3. *For $z, w \in \mathbb{C}$, we have*

$$\big|G_p(z) - G_p(w)\big| \le p(|z| + |w|)^{p-1}|z - w|.$$

Proof. Without loss of generality, we may assume $|z| > |w|$. For $\nu > 0$, we observe the relations

$$|z|^\nu - |w|^\nu = (|z| - |w|)\int_0^1 \nu\big(t|z| + (1-t)|w|\big)^{\nu-1}\,dt$$

and

$$\sup_{t\in[0,1]}\big(t|z| + (1-t)|w|\big)^{\nu-1}|w| \le \left\{\begin{array}{ll} (|z|+|w|)^{\nu-1}|w| & (\text{if } \nu \ge 1) \\ |w|^\nu & (\text{if } \nu < 1) \end{array}\right\} \le (|z|+|w|)^\nu.$$

Then we have

$$\big|(|z|^\nu - |w|^\nu)w\big| \le \big||z| - |w|\big| \cdot \nu\big(|z| + |w|\big)^\nu \le \nu\big(|z| + |w|\big)^\nu|z - w|.$$

We apply the above inequality with $\nu = p - 1$ to obtain

$$\big|G_p(z) - G_p(w)\big| \le \big|(|z|^{p-1} - |w|^{p-1})w\big| + |z|^{p-1}|z - w| \le p\big(|z| + |w|\big)^{p-1}|z - w|.$$

□

Lemma 3.2.4. *Let $0 \le s < \min\{2, p\}$. There exists a constant C such that*

$$\|G_p(\phi)\|_{H^s(\mathbb{R}^d)} \le C\|\phi\|^{p-1}_{L^\infty(\mathbb{R}^d)}\|\phi\|_{H^s(\mathbb{R}^d)}$$

and

$$\|\mathcal{U}(t)^{-1}G_p(\phi)\|_{H^{0,s}(\mathbb{R}^d)} \le C\|\phi\|^{p-1}_{L^\infty(\mathbb{R}^d)}\|\mathcal{U}(t)^{-1}\phi\|_{H^{0,s}(\mathbb{R}^d)}$$

for $t \ge 0$.

For the proof, see Lemma 3.4 in [11], Lemma 2.3 in [28], etc.

Corollary 3.2.5. *Let $d/2 < s < \min\{2, p\}$. There exists a constant C such that*

$$\|\mathcal{U}(t)^{-1}G_p(\phi)\|_{\Sigma^s} \le \frac{C}{(1+t)^{d(p-1)/2}}\|\mathcal{U}(t)^{-1}\phi\|^p_{\Sigma^s}$$

for $t \ge 0$.

Proof. By Lemmas 3.2.4 and 3.2.1, we have

$$\begin{aligned}
\|\mathcal{U}(t)^{-1}G_p(\phi)\|_{\Sigma^s} &= \|G_p(\phi)\|_{H^s(\mathbb{R}^d)} + \big\|\mathcal{U}(t)^{-1}G_p(\phi)\big\|_{H^{0,s}(\mathbb{R}^d)} \\
&\le C\|\phi\|_{L^\infty(\mathbb{R}^d)}^{p-1}\big(\|\phi\|_{H^s(\mathbb{R}^d)} + \|\mathcal{U}(t)^{-1}\phi\|_{H^{0,s}(\mathbb{R}^d)}\big) \\
&\le \frac{C}{(1+t)^{d(p-1)/2}}\|\mathcal{U}(t)^{-1}\phi\|_{\Sigma^s}^p.
\end{aligned}$$

□

Lemma 3.2.6. *Let $\gamma \in (0, 1/2]$ and $d/2 + 2\gamma < s < \min\{2, p\}$. Then there exists a constant C such that*

$$\left\|\mathcal{F}\mathcal{U}(t)^{-1}G_p(\phi) - \frac{1}{t^{d(p-1)/2}}G_p\big(\mathcal{F}\mathcal{U}(t)^{-1}\phi\big)\right\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{t^{d(p-1)/2+\gamma}}\|\mathcal{U}(t)^{-1}\phi\|_{H^{0,s}(\mathbb{R}^d)}^p$$

for $t \ge 1$.

This lemma can be shown in almost the same way as the derivation of (3.16) and (3.17) in [28] (see also Lemma 2.2 in [48]), so we skip the proof.

3.3 A rough lower estimate for the lifespan

In what follows, we write $N(u) = \lambda|u|^{2\theta/d}u = \lambda G_{1+2\theta/d}(u)$ and $\Phi = \|\varphi\|_{\Sigma^s}$, where s satisfies (3.1.3). The goal of this section is to derive a rough lower estimate for T_ε. The argument of this section is quite standard and any new idea is not needed, so we shall be brief.

Proposition 3.3.1. *Let T_ε be the lifespan defined in the statement of Theorem 3.1.2. There exists $D_0 > 0$ such that $T_\varepsilon \ge D_0\varepsilon^{-2\theta/(1-\theta)d}$. Moreover the solution u satisfies*

$$\|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s} \le 2\Phi\varepsilon \tag{3.3.1}$$

for $t \le D_0\varepsilon^{-2\theta/(1-\theta)d}$.

Proof. Since the local existence in Σ^s is well-known (see e.g., [3] and the references cited therein), what we have to do is to see the solution $u(t)$ stays bounded as long as t is less than the expected value.

Let $T > 0$ and let $u(t)$ be the solution to (3.1.2) in the time interval $[0, T)$. We set

$$E(T) = \sup_{t\in[0,T)} \|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s}.$$

Then, it follows from Corollary 3.2.5 that

$$\left\| \mathcal{U}(t)^{-1} N(u) \right\|_{\Sigma^s} \le \frac{CE(T)^{2\theta/d+1}}{(1+t)^\theta}$$

for $t < T$. Therefore the standard energy integral method leads to

$$\begin{aligned} E(T) &\le \|u(0)\|_{\Sigma^s} + C \int_0^T \left\| \mathcal{U}(t)^{-1} N(u) \right\|_{\Sigma^s} dt \\ &\le \varepsilon \|\varphi\|_{\Sigma^s} + CE(T)^{2\theta/d+1} \int_0^T \frac{dt}{(1+t)^\theta} \\ &\le \Phi\varepsilon + C_* E(T)^{2\theta/d+1} T^{1-\theta}, \end{aligned}$$

where the constant C_* is independent of ε and T. With this C_*, we choose $D_0 > 0$ so that

$$C_* 3^{1+2\theta/d} \Phi^{2\theta/d} {D_0}^{1-\theta} \le 1.$$

Now we prove Proposition 3.3.1 via the bootstrap arguement. We assume a weak bound $E(T) \le 3\Phi\varepsilon$. Then the above estimate yields the stronger bound

$$E(T) \le \Phi\varepsilon + C_* (3\Phi\varepsilon)^{2\theta/d+1} (D_0 \varepsilon^{-2\theta/d(1-\theta)})^{1-\theta} \le 2\Phi\varepsilon$$

if $T \le D_0 \varepsilon^{-2\theta/d(1-\theta)}$. This shows that the solution $u(t)$ can exist as long as $t \le D_0 \varepsilon^{-2\theta/d(1-\theta)}$. In other words, we have $T_\varepsilon \ge D_0 \varepsilon^{-2\theta/d(1-\theta)}$. We also have the desired estimate (3.3.1). □

Remark 3.3.2. In the proof of Proposition 3.3.1, we do not use any information on the sign of $\operatorname{Im} \lambda$. We need something more to clarify the dependence of T_ε on $\operatorname{Im} \lambda$, that is our main purpose of the present work.

3.4 An ODE Lemma

In this section, we introduce an ODE lemma which will be used effectively in Section 3.5 and Section 3.8. The argument in this section is a modification of that of §2 in [66] to fit for the present purpose.

We suppose $0 < a < 1$, $b > 0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda > 0$, so that we will choose $a = \theta$ and $b = \frac{2\theta}{d}$ in Section 3.5. Let $\psi_0 : \mathbb{R}^d \to \mathbb{C}$ be a continuous function satisfying

$$\Psi_0 := \sup_{\xi \in \mathbb{R}^d} |\psi_0(\xi)| < \infty.$$

We set $q = \frac{b}{2(1-a)}$ and define $\tau_1 > 0$ by

$$\frac{1}{\tau_1} := \left(2q \operatorname{Im} \lambda {\Psi_0}^b \right)^{1/(1-a)}.$$

For fixed $t_* > 0$, let $\eta_0 : [t_*, T) \times \mathbb{R}^d \to \mathbb{C}$ be a solution to

$$\begin{cases} i\partial_t \eta_0 = \dfrac{\lambda}{t^a}|\eta_0|^b \eta_0, & t > t_*,\ \xi \in \mathbb{R}^d, \\ \eta_0(t_*, \xi) = \varepsilon\psi_0(\xi), & \xi \in \mathbb{R}^d, \end{cases} \tag{3.4.1}$$

where $\varepsilon > 0$ is a parameter. It is immediate to check that

$$|\eta_0(t,\xi)|^b = \frac{(\varepsilon|\psi_0(\xi)|)^b}{1 + 2q \operatorname{Im}\lambda |\psi_0(\xi)|^b \varepsilon^b t_*^{1-a} - 2q \operatorname{Im}\lambda |\psi_0(\xi)|^b \varepsilon^b t^{1-a}},$$

as long as the denominator is strictly positive. In view of this expression, we see that

$$\sup_{(t,\xi)\in[t_*, \sigma\varepsilon^{-2q}]\times\mathbb{R}^d} |\eta_0(t,\xi)| \le C_0 \varepsilon \tag{3.4.2}$$

for $\sigma \in (0, \tau_1)$, where

$$C_0 = \frac{\Psi_0}{\left(1 - (\sigma/\tau_1)^{1-a}\right)^{1/b}}.$$

Next we consider a perturbation of (4.3.1). Let $T > t_*$ and let $\psi_1 : \mathbb{R}^d \to \mathbb{C}$, $\rho : [t_*, T) \times \mathbb{R}^d \to \mathbb{C}$ be continuous functions satisfying

$$|\psi_1(\xi)| \le C_1 \varepsilon^{1+\delta}$$

and

$$|\rho(t,\xi)| \le \frac{C_2 \varepsilon^{1+b+\delta}}{t^a}$$

with some positive constants C_1, C_2 and δ. Let $\eta : [t_*, T) \times \mathbb{R}^d \to \mathbb{C}$ be a solution to

$$\begin{cases} i\partial_t \eta = \dfrac{\lambda}{t^a}|\eta|^b \eta + \rho, & t \in (t_*, T),\ \xi \in \mathbb{R}^d, \\ \eta(t_*, \xi) = \varepsilon\psi_0(\xi) + \psi_1(\xi), & \xi \in \mathbb{R}^d. \end{cases}$$

The following lemma asserts that an estimate similar to (3.4.2) remains valid if (4.3.1) is perturbed by ρ and ψ_1:

Lemma 3.4.1. *Let $\sigma \in (0, \tau_1)$ and let $\eta(t,\xi)$ be as above. We set $T_* = \min\{T, \sigma\varepsilon^{-2q}\}$ for $0 < \varepsilon \le \min\{1, M^{-1/\delta}\}$. We have*

$$|\eta(t,\xi)| \le C_0\varepsilon + M\varepsilon^{1+\delta} \le (C_0 + 1)\varepsilon$$

for $(t,\xi) \in [t_*, T_*) \times \mathbb{R}^d$, where

$$M = 2\left(C_1^2 + \frac{C_2^2}{2C_3}\right)^{1/2} \exp\left(\frac{C_3 \sigma^{1-a}}{2(1-a)}\right)$$

with

$$C_3 = 2|\lambda|(b+1)(2C_0+1)^b + \frac{1}{2}.$$

Proof. We set $w = \eta - \eta_0$ and

$$T_{**} = \sup\Big\{\tilde{T} \in [t_*, T_*) \ \Big|\ \sup_{(t,\xi)\in[t_*,\tilde{T})\times\mathbb{R}^d} |w(t,\xi)| \le M\varepsilon^{1+\delta}\Big\}.$$

We observe that

$$i\partial_t w = \frac{\lambda}{t^a}\Big(|\eta_0 + w|^b(\eta_0 + w) - |\eta_0|^b\eta_0\Big) + \rho, \quad w(t_*, \xi) = \psi_1(\xi).$$

We also note that $T_{**} > t_*$, because of the estimate

$$|w(t_*, \xi)| = |\psi_1(\xi)| \le C_1\varepsilon^{1+\delta} \le \frac{M}{2}\varepsilon^{1+\delta}$$

and the continuity of w. Now we set

$$f(t,\xi) = |w(t,\xi)|^2 + \frac{C_2^2}{2C_3}\varepsilon^{2+2\delta}.$$

Then it follows from Lemma 3.2.3 that

$$\begin{aligned}
\partial_t f(t,\xi) &= 2\,\mathrm{Im}\Big(i\partial_t w \cdot \overline{w}\Big) \\
&\le \frac{2|\lambda|}{t^a}(b+1)\Big(2|\eta_0| + |w|\Big)^b |w|^2 + |\rho||w| \\
&\le \frac{2|\lambda|(b+1)}{t^a}\Big(2C_0\varepsilon + M\varepsilon^{1+\delta}\Big)^b |w|^2 + |w| \cdot \frac{C_2\varepsilon^{1+b+\delta}}{t^a} \\
&\le \frac{\varepsilon^b}{t^a}\Big\{\Big(C_3 - \frac{1}{2}\Big)|w|^2 + |w| \cdot C_2\varepsilon^{1+\delta}\Big\} \\
&\le \frac{\varepsilon^b}{t^a}\left(C_3|w|^2 + \frac{C_2^2}{2}\varepsilon^{2+2\delta}\right) \\
&= \frac{C_3\varepsilon^b}{t^a} f(t,\xi)
\end{aligned}$$

for $t \in (t_*, T_{**})$, as well as

$$f(t_*,\xi) \le (C_1\varepsilon^{1+\delta})^2 + \frac{C_2^2}{2C_3}\varepsilon^{2+2\delta} \le \left(C_1^2 + \frac{C_2^2}{2C_3}\right)\varepsilon^{2+2\delta}.$$

These lead to

$$\begin{aligned}
f(t,\xi) &\le f(t_*,\xi)\exp\left(\int_{t_*}^{\sigma\varepsilon^{-2q}} \frac{C_3\varepsilon^b}{\tau^a}\,d\tau\right) \\
&\le \left(C_1^2 + \frac{C_2^2}{2C_3}\right)\varepsilon^{2+2\delta}\exp\left(\frac{C_3\sigma^{1-a}}{1-a}\varepsilon^{b-2q(1-a)}\right) \\
&\le \left(\frac{M}{2}\varepsilon^{1+\delta}\right)^2,
\end{aligned}$$

whence

$$|w(t,\xi)| \le \sqrt{f(t,\xi)} \le \frac{M}{2}\varepsilon^{1+\delta}$$

for $(t,\xi) \in [t_*, T_{**}) \times \mathbb{R}^d$. This contradicts the definition of T_{**} if $T_{**} < T_*$. Therefore we conclude that $T_{**} = T_*$. In other words, we have

$$\sup_{(t,\xi)\in[t_*,T_*)\times\mathbb{R}^d} |w(t,\xi)| \le \sqrt{f(t,\xi)} \le M\varepsilon^{1+\delta}.$$

Going back to the definition of w, we have

$$|\eta(t,\xi)| \le |\eta_0(t,\xi)| + |w(t,\xi)| \le C_0\varepsilon + M\varepsilon^{1+\delta}$$

for $(t,\xi) \in [t_*, T_*) \times \mathbb{R}^d$, as desired. $\square$

Next we consider the case of $a = 1$ and $b = 2/d$. We suppose $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$. Let $\varphi_0 : \mathbb{R}^d \to \mathbb{C}$ be a continuous function satisfying

$$\sup_{\xi\in\mathbb{R}^d} |\varphi_0(\xi)| < \infty.$$

We define $\tau_2 > 0$ by

$$\tau_2 := \frac{d}{2\operatorname{Im}\lambda \sup\limits_{\xi\in\mathbb{R}^d} |\varphi_0(\xi)|^{2/d}}.$$

Let $\beta_0 : [1,T) \times \mathbb{R}^d \to \mathbb{C}$ be a solution to

$$\begin{cases} i\partial_t\beta_0 = \dfrac{\lambda}{t}|\beta_0|^{2/d}\beta_0, & t > 1,\ \xi \in \mathbb{R}^d, \\ \beta_0(1,\xi) = \varepsilon\varphi_0(\xi), & \xi \in \mathbb{R}^d, \end{cases} \tag{3.4.3}$$

where $\varepsilon > 0$ is a parameter. Then it is easy to see that

$$|\beta_0(t,\xi)|^{2/d} = \frac{(\varepsilon|\varphi_0(\xi)|)^{2/d}}{1 - \dfrac{2}{d}\operatorname{Im}\lambda|\varphi_0(\xi)|^{2/d}\varepsilon^{2/d}\log t},$$

as long as the denominator is strictly positive. In view of this expression, we can see that

$$\sup_{(t,\xi)\in[1,e^{\sigma/\varepsilon^{2/d}}]\times\mathbb{R}^d} |\beta_0(t,\xi)| \le D_0\varepsilon \tag{3.4.4}$$

for $\sigma \in (0, \tau_2)$, where

$$D_0 = \frac{1}{(1-(\sigma/\tau_2))^{d/2}} \sup_{\xi\in\mathbb{R}^d} |\varphi_0(\xi)|.$$

Next we consider a perturbation of (3.4.3). Let $T > 1$ and let $\varphi_1 : \mathbb{R}^d \to \mathbb{C}$, $\rho : [1, T) \times \mathbb{R}^d \to \mathbb{C}$ be continuous functions satisfying

$$\sup_{\xi \in \mathbb{R}^d} |\varphi_1(\xi)| \leq D_1 \varepsilon^{1+\delta}, \qquad \sup_{(t,\xi) \in [1,T) \times \mathbb{R}^d} t^{1+\mu} |\rho(t, \xi)| \leq D_2 \varepsilon^{1+\delta}$$

with some positive constants D_1, D_2, δ and μ. Let $\beta : [1, T) \times \mathbb{R}^d \to \mathbb{C}$ be a solution to

$$\begin{cases} i\partial_t \beta = \dfrac{\lambda}{t} |\beta|^{2/d} \beta + \rho, & t \in (1, T), \ \xi \in \mathbb{R}^d, \\ \beta(1, \xi) = \varepsilon \varphi_0(\xi) + \varphi_1(\xi), & \xi \in \mathbb{R}^d. \end{cases}$$

The following lemma asserts that an estimate similar to (3.4.2) remains valid if (3.4.3) is perturbed by ρ and φ_1:

Lemma 3.4.2. *Let $\sigma \in (0, \tau_2)$ and let $\beta(t, \xi)$ be as above. We set $T_* = \min\{T, e^{\sigma/\varepsilon^{2/d}}\}$ for $0 < \varepsilon \leq \min\{1, M^{-1/\delta}\}$. We have*

$$|\beta(t, \xi)| \leq D_0 \varepsilon + M \varepsilon^{1+\delta} \leq (D_0 + 1)\varepsilon$$

for $(t, \xi) \in [1, T_*) \times \mathbb{R}^d$, where

$$M = \left(2D_1 + \frac{D_2}{\mu} \right) e^{D_3 \sigma}$$

with

$$D_3 = |\lambda|(1 + 2/d)(2C_0 + 1)^{2/d}.$$

Proof. We set $w = \beta - \beta_0$ and

$$T_{**} = \sup \Big\{ \tilde{T} \in [1, T_*) \ \Big| \ \sup_{(t,\xi) \in [1,\tilde{T}) \times \mathbb{R}^d} |w(t, \xi)| \leq M \varepsilon^{1+\delta} \Big\}.$$

Note that $T_{**} > 1$, because of the estimate

$$|w(1, \xi)| = |\varphi_1(\xi)| \leq D_1 \varepsilon^{1+\delta} \leq \frac{M}{2} \varepsilon^{1+\delta}$$

and the continuity of w. We observe that

$$i\partial_t w = \frac{\lambda}{t} \Big(|\beta_0 + w|^{2/d} (\beta_0 + w) - |\beta_0|^{2/d} \beta_0 \Big) + \rho.$$

Then it follows from Lemma 3.2.3 that

$$\begin{aligned}
\partial_t \left(|w|^2\right) &= 2\operatorname{Im}\Big(i\partial_t w\cdot\overline{w}\Big)\\
&\le \frac{2|\lambda|}{t}(1+2/d)\Big(2|\beta_0|+|w|\Big)^{2/d}|w|^2+|\rho||w|\\
&\le \frac{2|\lambda|}{t}(1+2/d)\Big(2D_0\varepsilon+M\varepsilon^{1+\delta}\Big)^{2/d}|w|^2+|\rho||w|\\
&\le \frac{2}{t}D_3\varepsilon^{2/d}|w|^2+\frac{D_2\varepsilon^{1+\delta}}{t^{1+\mu}}|w|
\end{aligned}$$

for $t\in(1,T_{**})$. By the Gronwall-type arguement, we obtain

$$\begin{aligned}
|w(t,\xi)| &\le \left(|\varphi_1(\xi)|+\int_1^t\frac{D_2\varepsilon^{1+\delta}}{2s^{1+\mu+D_3\varepsilon^{2/d}}}ds\right)e^{D_3\varepsilon^{2/d}\log t}\\
&\le \left(D_1\varepsilon^{1+\delta}+\frac{D_2\varepsilon^{1+\delta}}{2(\mu+D_3\varepsilon^{2/d})}\right)e^{D_3\sigma}\\
&\le \frac{M}{2}\varepsilon^{1+\delta}
\end{aligned}$$

for $(t,\xi)\in[1,T_{**})\times\mathbb{R}^d$. This contradicts the definition of T_{**} if $T_{**}<T_*$. Therefore we conclude that $T_{**}=T_*$. In other words, we have

$$\sup_{(t,\xi)\in[1,T_*)\times\mathbb{R}^d}|w(t,\xi)|\le M\varepsilon^{1+\delta},$$

whence

$$|\beta(t,\xi)|\le|\beta_0(t,\xi)|+|w(t,\xi)|\le D_0\varepsilon+M\varepsilon^{1+\delta}$$

for $(t,\xi)\in[1,T_*)\times\mathbb{R}^d$, as desired. □

3.5 Bootstrap argument in the large time

Now we are ready to pursue the behavior of the solution $u(t)$ of (3.1.2) for $t\gtrsim o(\varepsilon^{-2\theta/d(1-\theta)})$. For this purpose, we set $t_*=\varepsilon^{-\theta/(1-\theta)d}$, and let ε be small enough to satisfy $\varepsilon^{\theta/(1-\theta)d}<D_0$. Then, since $t_*\le D_0\varepsilon^{-2\theta/(1-\theta)d}$, Proposition 3.3.1 gives us $E(t_*)\le 2\Phi\varepsilon$. Next we set

$$\tau_0:=\left(\frac{(1-\theta)d}{2\theta\operatorname{Im}\lambda\sup\limits_{\xi\in\mathbb{R}^d}|\mathcal{F}\varphi(\xi)|^{2\theta/d}}\right)^{1/(1-\theta)}$$

and fix $\sigma \in (0, \tau_0)$, $T \in (t_*, \sigma\varepsilon^{-2\theta/d(1-\theta)}]$. Note that the right-hand side in (3.1.4) is equal to $\tau_0^{1-\theta}$, and that $\tau_0 = \tau_1$, when we choose $a = \theta$ and $b = \frac{2\theta}{d}$. For the solution $u(t)$ in the interval $t \in [0, T)$, we put

$$E(T) = \sup_{t\in[0,T)} \|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s}$$

as in the proof of Proposition 3.3.1. The following lemma is the main step toward Theorem 3.1.2.

Lemma 3.5.1. *Let σ and T be as above. Then there exist constants $\varepsilon_0 > 0$ and $K > 4\Phi$, which are independent of T, such that the estimate $E(T) \le K\varepsilon$ implies the better estimate $E(T) \le K\varepsilon/2$ if $\varepsilon \in (0, \varepsilon_0]$.*

Proof. It suffices to consider $t \in [t_*, T)$, because we already know that $E(t_*) \le 2\Phi\varepsilon$. For $t \in [t_*, T)$, we set $A(t,\xi) = \mathcal{F}\big[\mathcal{U}(t)^{-1}u(t,\cdot)\big](\xi)$ and

$$R(t,\xi) = \mathcal{F}\big[\mathcal{U}(t)^{-1}N(u(t,\cdot))\big](\xi) - t^{-\theta}N(A(t,\xi)),$$

so that

$$i\partial_t A = \mathcal{F}\mathcal{U}(t)^{-1}\left(i\partial_t + \frac{1}{2}\Delta\right)u = \mathcal{F}\mathcal{U}(t)^{-1}N(u) = \frac{\lambda}{t^\theta}|A|^{2\theta/d}A + R. \tag{3.5.1}$$

Next we take $\gamma = (2s-d)/8 \in (0, 1/2]$. Note that $s - d/2 = 4\gamma > 2\gamma$. Since R can be written as

$$R(t,\xi) = \lambda\Big(\mathcal{F}\mathcal{U}(t)^{-1}G_{1+2\theta/d}(u) - t^{-\theta}G_{1+2\theta/d}(\mathcal{F}\mathcal{U}(t)^{-1}u)\Big),$$

Lemma 3.2.6 yields

$$|R(t,\xi)| \le \frac{C}{t^{\theta+\gamma}}E(T)^{2\theta/d+1} \le \frac{C\varepsilon^{1+2\theta/d}}{t^\theta}K^{1+2\theta/d}t_*^{-\gamma} \le \frac{C\varepsilon^{1+2\theta/d+\gamma\theta/2d(1-\theta)}}{t^\theta}$$

if $E(T) \le K\varepsilon$ and $K^{1+2\theta/d}\varepsilon^{\gamma\theta/2d(1-\theta)} \le 1$. Moreover, when we put $\psi(\xi) = A(t_*,\xi) - \varepsilon\mathcal{F}\varphi(\xi)$, we have

$$\begin{aligned}
|\psi(\xi)| &\le C\|\mathcal{U}(t_*)^{-1}u(t_*,\cdot) - \varepsilon\varphi\|_{L^2(\mathbb{R}^d)}^{1-d/2s}\|\mathcal{U}(t_*)^{-1}u(t_*,\cdot) - \varepsilon\varphi\|_{H^{0,s}(\mathbb{R}^d)}^{d/2s} \\
&\le C\left(\int_0^{t_*}\|\mathcal{U}(t)^{-1}N(u)\|_{L^2(\mathbb{R}^d)}\,dt\right)^{1-d/2s}(C\varepsilon)^{d/2s} \\
&\le C\left(\int_0^{\varepsilon^{-\theta/(1-\theta)d}}\frac{(2\Phi\varepsilon)^{1+2\theta/d}}{(1+t)^\theta}\,dt\right)^{1-d/2s}(C\varepsilon)^{d/2s} \\
&\le C\varepsilon^{1+3\theta(1/d-1/2s)},
\end{aligned}$$

where we have used the Gagliardo-Nirenberg-Sobolev inequality (3.2.1), Lemma 3.2.4, Lemma 3.2.1 and Proposition 3.3.1. Therefore we can apply Lemma 3.4.1 with $\eta = A$, $a = \theta$, $b = 2\theta/d$, $\delta = \min\{3\theta(1/d - 1/2s), \gamma\theta/2d(1-\theta)\}$, $\psi_0 = \mathcal{F}\varphi$, $\psi_1 = \psi$ and $\rho = R$ to obtain

$$|A(t,\xi)| \le (C_0 + 1)\varepsilon$$

for $(t,\xi) \in [t_*, T) \times \mathbb{R}^d$, where

$$C_0 = \frac{1}{(1 - (\sigma/\tau_0)^{1-\theta})^{d/2\theta}} \sup_{\xi\in\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|.$$

Note that C_0 is independent of ε, K and T. By this estimate and Lemma 3.2.2, we have

$$\begin{aligned}
\|u(t)\|_{L^\infty(\mathbb{R}^d)} &\le t^{-d/2}\|A(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} + Ct^{-d/2-\gamma}\|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s} \\
&\le t^{-d/2}\Big((C_0+1)\varepsilon + CK\varepsilon t_*^{-\gamma}\Big) \\
&\le t^{-d/2}\Big(C\varepsilon + CK\varepsilon^{1+\gamma\theta/d(1-\theta)}\Big) \\
&\le C\varepsilon t^{-d/2},
\end{aligned}$$

if $K\varepsilon^{\gamma\theta/d(1-\theta)} \le 1$. By the standard energy inequality combined with Lemma 3.2.4, we obtain

$$\begin{aligned}
\sup_{t_*\le t<T} \|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s} &\le \|\mathcal{U}(t_*)^{-1}u(t_*)\|_{\Sigma^s} \exp\left(\int_{t_*}^{T} C\|u(t)\|_{L^\infty(\mathbb{R}^d)}^{2\theta/d} dt\right) \\
&\le 2\Phi\varepsilon \exp\left(C\varepsilon^{2\theta/d}\int_0^{\sigma\varepsilon^{-2\theta/d(1-\theta)}} \frac{dt}{t^\theta}\right) \\
&\le \big(2\Phi e^{C_\star}\big)\varepsilon
\end{aligned}$$

for $t \in [t_*, T)$, where the constant $C_\star$ is independent of ε, K and T. Now we set $K = 4\Phi e^{C_\star}$. Then we arrive at the desired estimate $E(T) \le K\varepsilon/2$. □

Proof of Theorem 3.1.2. Let T_ε be the lifespan defined in the statement of Theorem 3.1.2. We fix $\sigma \in (0, \tau_0)$ and set

$$T^* = \sup\big\{t \in [0, T_\varepsilon) \;\big|\; E(t) \le K\varepsilon\big\},$$

where K is given in Lemma 3.5.1. Now we assume $T^* \le \sigma\varepsilon^{-2\theta/d(1-\theta)}$. Then, Lemma 3.5.1 with $T = T^*$ implies $E(T^*) \le K\varepsilon/2$ if $\varepsilon \le \varepsilon_0$. By the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$, we can choose $\tilde{\delta} > 0$ such that $E(T^* + \tilde{\delta}) \le K\varepsilon$, which contradicts the definition of T^*. Therefore we must have $T^* \ge \sigma\varepsilon^{-2\theta/d(1-\theta)}$ if $\varepsilon \le \varepsilon_0$. As a consequence, we obtain

$$\liminf_{\varepsilon\to+0} \varepsilon^{2\theta/d} T_\varepsilon^{\,1-\theta} \ge \sigma^{1-\theta}.$$

Since $\sigma \in (0, \tau_0)$ is arbitrary, we arrive at the desired estimate (3.1.4). □

3.6 The critical case

We consider the critical case $\theta = 1$, that is,

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = \lambda |u|^{2/d} u, & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = \varepsilon\varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.6.1}$$

where $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$. As mentioned in Section 3.1, one dimensional case ($d = 1$) has been covered in the previous works [73] and [66]. Minor modifications of the method in the previous sections allow us to treat the case of $d = 2$, 3.

Theorem 3.6.1. *Let $1 \le d \le 3$ and $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$. Assume $\varphi \in \Sigma^s$ with s satisfying* (3.1.3)*. Let T_ε be the supremum of $T > 0$ such that* (3.6.1) *admits a unique solution u satisfying $\mathcal{U}(\cdot)^{-1}u \in C([0,T);\Sigma^s)$. Then we have*

$$\liminf_{\varepsilon\to+0} \left(\varepsilon^{2/d}\log T_\varepsilon\right) \ge \frac{d}{2\operatorname{Im}\lambda \sup\limits_{\xi\in\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^{2/d}}.$$

Since the proof of Theorem 3.6.1 is almost the same as that for Theorem 3.1.2, we only prove an a priori estimate for the solution to (3.6.1).

Throughout this section, we fix $\sigma \in (0, \tau_2)$ and $T \in (0, e^{\sigma/\varepsilon^{(2/d)}}]$, where

$$\tau_2 := \frac{d}{2\operatorname{Im}\lambda \sup\limits_{\xi\in\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^{2/d}}.$$

Let u be a solution to (3.6.1) satisfying $\mathcal{U}(\cdot)^{-1}u \in C([0,T);\Sigma^s)$. We set

$$\alpha(t,\xi) := \mathcal{F}\big[\mathcal{U}(t)^{-1}u(t,\cdot)\big](\xi).$$

We also define

$$E(T) := \sup_{(t,\xi)\in[0,T)\times\mathbb{R}^d} (|\alpha(t,\xi)|) + \sup_{0\le t<T} \left[(1+t)^{-\gamma}(\|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s})\right]$$

with $\gamma = (2s-d)/8 \in (0, 1/2]$. Our goal here is to prove the following:

Lemma 3.6.2. *Let σ, T and γ be as above. Then there exist positive constants ε_0 and $K > \tilde{C} + 1$, where $\tilde{C}$ is defined by* (3.6.4) *below, not depending on T, such that*

$$E(T) \le K\varepsilon \tag{3.6.2}$$

implies the stronger estimate

$$E(T) \le \frac{K}{2}\varepsilon,$$

provided that $\varepsilon \in (0, \varepsilon_0]$.

We divide the proof of this lemma into two subsections.

3.6.1 L^2-estimates

In the first part, we consider the bound for $\|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s}$. We first remark that Lemma 3.2.2 and the assumption (3.6.2) lead to

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C\varepsilon}{t^{d/2}}$$

for $t \geq 1$. Note that $s-d/2 = 4\gamma > 2\gamma$. Indeed the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ yields

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C\|u(t)\|_{H^s(\mathbb{R}^d)} \leq C\varepsilon$$

for $t \leq 1$. From these, we have

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C\varepsilon}{(1+t)^{d/2}} \tag{3.6.3}$$

for $t \in [0, T)$. It follows from Lemma 3.2.4, (3.6.3) and the assumption (3.6.2) that

$$\left\|\mathcal{U}(t)^{-1}N(u(t))\right\|_{\Sigma^s} \leq C\|u(t)\|_{L^\infty(\mathbb{R}^d)}^{1+2/d}\|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s} \leq \frac{C\varepsilon^{1+2/d}}{(1+t)^{1-\gamma}}$$

for $t < T$. Therefore the standard energy integral method leads to

$$\begin{aligned}
\|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s} &\leq \|u(0)\|_{\Sigma^s} + C\int_0^t \left\|\mathcal{U}(\tau)^{-1}N(u(\tau))\right\|_{\Sigma^s} d\tau \\
&\leq \varepsilon\|\varphi\|_{\Sigma^s} + C\varepsilon^{1+2/d}\int_0^t \frac{d\tau}{(1+\tau)^{1-\gamma}} \\
&\leq C_*\varepsilon(1+t)^\gamma,
\end{aligned}$$

where the positive constant C_* is independent of ε and T.

3.6.2 Estimates for α

In this part, we will show $|\alpha(t,\xi)| \leq C\varepsilon$ for $(t,\xi) \in [0,T) \times \mathbb{R}^d$ under the assumption (3.6.2). When $0 \leq t \leq 1$, the desired estimate follows immediately from the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and (3.6.2). Hence we have only to consider the case of $T > 1$ and $t \in [1, T)$. We have

$$i\partial_t\alpha = \mathcal{F}\mathcal{U}(t)^{-1}\left(i\partial_t + \frac{1}{2}\Delta\right)u = \mathcal{F}\mathcal{U}(t)^{-1}N(u) = \frac{\lambda}{t}|\alpha|^{2/d}\alpha + R_1,$$

where

$$\begin{aligned}R_1(t,\xi) &= \mathcal{F}\big[\mathcal{U}(t)^{-1}N(u(t,\cdot))\big](\xi) - t^{-1}N(\alpha(t,\xi))\\ &= \lambda\Big(\mathcal{F}\mathcal{U}(t)^{-1}G_{1+2/d}(u) - t^{-1}G_{1+2/d}(\mathcal{F}\mathcal{U}(t)^{-1}u)\Big).\end{aligned}$$

The proof of Lemma 3.2.6 yields

$$|R_1(t,\xi)| \le \frac{C\varepsilon^{2/d+1}}{t^{1+\gamma}}.$$

Moreover we have

$$\begin{aligned}&|\alpha(1,\xi) - \varepsilon\mathcal{F}\varphi(\xi)|\\ &\le C\|\mathcal{U}(1)^{-1}u(1,\cdot) - \varepsilon\varphi\|_{L^2(\mathbb{R}^d)}^{1-d/2s}\|\mathcal{U}(1)^{-1}u(1,\cdot) - \varepsilon\varphi\|_{H^{0,s}(\mathbb{R}^d)}^{d/2s}\\ &\le C\left(\int_0^1 \|\mathcal{U}(\tau)^{-1}N(u(\tau))\|_{L^2(\mathbb{R}^d)}\, d\tau\right)^{1-d/2s}(C\varepsilon)^{d/2s}\\ &\le C\left(\int_0^1 \frac{\varepsilon^{1+2/d}}{(1+\tau)^{1-\gamma}}\, d\tau\right)^{1-d/2s}(C\varepsilon)^{d/2s}\\ &\le C\varepsilon^{1+3(1/d-1/2s)},\end{aligned}$$

where we have used the Gagliardo-Nirenberg-Sobolev inequality (3.2.1), Lemma 3.2.4, Lemma 3.2.1 and Proposition 3.3.1.

Therefore we can apply Lemma 3.4.2 with $\beta = \alpha$, $\delta = \min\{3(1/d - 1/2s), 2/d\}$, $\varphi_0 = \mathcal{F}\varphi$, $\varphi_1 = \alpha(1) - \varepsilon\mathcal{F}\varphi$ and $\rho = R_1$ to obtain

$$|\alpha(t,\xi)| \le (\tilde{C}+1)\varepsilon$$

for $(t,\xi) \in [1,T) \times \mathbb{R}^d$, where

$$\tilde{C} = \frac{1}{(1-(\sigma/\tau_2))^{d/2}} \sup_{\xi\in\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|. \tag{3.6.4}$$

Now we set $K = 2\max\{C_*, \tilde{C}+1\}$. Then we arrive at the desired estimate $E(T) \le K\varepsilon/2$. □

Chapter 4

The lifespan of small solutions to a system of cubic nonlinear Schrödinger equations in one space dimension

4.1 Introduction

This chapter is based on the author's work [65]. We consider the following initial value problem:

$$\begin{cases} \mathcal{L}_{m_j} u_j = F_j(u), & t > 0,\ x \in \mathbb{R}, \\ u_j(0,x) = \varepsilon \varphi_j(x), & x \in \mathbb{R} \end{cases} \tag{4.1.1}$$

for $j = 1, \ldots, N$, where $\mathcal{L}_m = i\partial_t + \frac{1}{2m}\partial_x^2$, $i = \sqrt{-1}$, $m_j \in \mathbb{R}\backslash\{0\}$ and $u = (u_j(t,x))_{1\le j\le N}$ is a $\mathbb{C}^N$-valued unknown function. The nonlinear term $F = (F_j)_{1\le j\le N}$ is assumed to be a cubic homogeneous polynomial in $(u, \overline{u})$. Also we assume that the system (4.1.1) satisfies the so-called *gauge invariance*:

$$F_j(e^{im_1\theta} z_1, \ldots, e^{im_N\theta} z_N) = e^{im_j\theta} F_j(z_1, \ldots, z_N)$$

for $j = 1, \ldots, N$ and $\theta \in \mathbb{R}$, $z = (z_j)_{1\le j\le N} \in \mathbb{C}^N$. $\varepsilon > 0$ is a small parameter which is responsible for the size of the initial data, and $\varphi = (\varphi_j(x))_{1\le j\le N}$ is a $\mathbb{C}^N$-valued known function which belongs to $(H^1 \cap H^{0,1}(\mathbb{R}))^N$. We are interested in large-time behavior of the small amplitude solution for (4.1.1).

Let us recall the backgrounds briefly. We begin with the single case ($N = 1$):

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u, \quad t > 0,\ x \in \mathbb{R} \tag{4.1.2}$$

with $\lambda \in \mathbb{R}$. According to Hayashi–Naumkin [28], the solution to (4.1.2) with small initial data exists globally in time and the global solution behaves like

$$u(t,x) = \frac{1}{\sqrt{it}}\tilde{\alpha}(x/t)\exp\left(i\frac{x^2}{2t} - i\lambda|\tilde{\alpha}(x/t)|^2\log t\right) + o(t^{-1/2})$$

as $t \to +\infty$ uniformly in $x \in \mathbb{R}$, where $\tilde{\alpha}$ is a suitable $\mathbb{C}$-valued function on $\mathbb{R}$ satisfying $\|\tilde{\alpha}\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon$. An important consequence of this asymptotic expression is that the solution decays like $O(t^{-1/2})$ in $L^\infty(\mathbb{R}_x)$, while it does not behave like the free solution unless $\lambda = 0$. In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension. If $\lambda \in \mathbb{C}$ in (4.1.2), another kind of long-range effect can be observed. Shimomura [70] showed that the small data solution to (4.1.2) exists globally in time and decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \to \infty$ if $\operatorname{Im}\lambda < 0$. This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect. If $\operatorname{Im}\lambda > 0$, Sunagawa [73] and Sagawa–Sunagawa [66] have derived the following more precise estimate for the lifespan T_ε of the solution to (4.1.2) with initial data $u(0,x) = \varepsilon\phi(x)$:

$$\liminf_{\varepsilon\to+0}(\varepsilon^2\log T_\varepsilon) \geq \frac{1}{2\operatorname{Im}\lambda\sup\limits_{\xi\in\mathbb{R}}|\mathcal{F}\phi(\xi)|^2}, \tag{4.1.3}$$

as we have mentioned in previous chapters. This estimate tells us the dependence of T_ε on $\operatorname{Im}\lambda$. Roughly speaking, the estimate (4.1.3) is derived from the ordinary differential equation

$$\begin{cases} i\partial_t f(t,\xi) = \frac{\lambda}{t}|f(t,\xi)|^2 f(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ f(1,\xi) = \varepsilon\mathcal{F}\phi(\xi), & \xi \in \mathbb{R}. \end{cases}$$

This ordinary differential equation can be solved explicitly as follows:

$$|f(t,\xi)|^2 = \frac{\varepsilon^2|\mathcal{F}\phi(\xi)|^2}{1 - 2\operatorname{Im}\lambda|\mathcal{F}\phi(\xi)|^2\varepsilon^2\log t},$$

as long as the denominator is strictly positive. Hence the solution $f(t,\xi)$ blows up at

$$\varepsilon^2\log t = \frac{1}{2\operatorname{Im}\lambda\sup\limits_{\xi\in\mathbb{R}}|\mathcal{F}\phi(\xi)|^2}.$$

This observation implies that the small data solution $u(t,x)$ of (4.1.2) with $\operatorname{Im}\lambda > 0$ may blow up in finite time. An example of the blowing-up solution to (4.1.2) with arbitrarily small $\varepsilon > 0$ has been given by Kita [47] under a particular choice of ϕ when $\operatorname{Im}\lambda > 0$.

However, it seems difficult to specify the lifespan for the blowing-up solution given in [47], and the optimality of (4.1.3) is left to be unknown. Next let us turn our attentions to the system case ($N \geq 2$). An interesting feature in the system case is that the behavior of solutions are affected by the combinations of the masses as well as the structure of the nonlinearity (see e.g., [7], [14], [15], [31], [34], [37], [43], [46], [51], [52], [53], [61], [64], [68], [77], etc.). In [52], several structural conditions on F have been introduced under which small data global existence holds, and time-decay properties of the global solutions have been investigated. As a result, we come up with the following question: *what happens if the structural conditions on F given in* [52] *are violated*? However, it seems difficult to treat the general N-component system (4.1.1). As the first step we consider the following two-component system ($N = 2$):

$$\begin{cases} \mathcal{L}_{m_1} u_1(t,x) = \lambda |u_2(t,x)|^2 u_1(t,x), & t > 0,\ x \in \mathbb{R}, \\ \mathcal{L}_{m_2} u_2(t,x) = \mu |u_1(t,x)|^2 u_2(t,x), & t > 0,\ x \in \mathbb{R}, \\ u_1(0,x) = \varepsilon\varphi(x),\ u_2(0,x) = \varepsilon\psi(x), & x \in \mathbb{R} \end{cases} \tag{4.1.4}$$

with $m_1,\ m_2 \in \mathbb{R}\backslash\{0\}$, $\lambda,\ \mu \in \mathbb{C}$ and $\varphi,\ \psi \in H^1 \cap H^{0,1}(\mathbb{R})$. This kind of the two-component nonlinear Schrödinger system appears in physics (see [32] and [33]). The approach of Li–Sunagawa [52] implies small data global existence and boundedness of the solution $u = (u_1, u_2)$ for (4.1.4) under the either of the following three conditions:

- $\operatorname{Im}\lambda < 0$,
- $\operatorname{Im}\mu < 0$,
- $\operatorname{Im}\lambda = \operatorname{Im}\mu = 0$.

According to [52], large-time behavior of the solution for (4.1.4) deeply relates to the following system of ordinary differential equations:

$$\begin{cases} i\partial_t A_1(t,\xi) = \frac{\lambda}{t} |A_2(t,\xi)|^2 A_1(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ i\partial_t A_2(t,\xi) = \frac{\mu}{t} |A_1(t,\xi)|^2 A_2(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ A_1(1,\xi) = \varepsilon\mathcal{F}_{m_1}\varphi(\xi),\ A_2(1,\xi) = \varepsilon\mathcal{F}_{m_2}\psi(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{4.1.5}$$

where $\mathcal{F}_m$ denotes the scaled Fourier transform which will be defined in the next section. We note that global existence and boundedness of the solution $A = (A_1, A_2)$ to the reduced system (4.1.5) holds in this case. We check it for $\operatorname{Im}\lambda < 0$ (the same is true for the other cases). Multiplying the equations of system (4.1.5) by $\overline{A_1}$ and $\overline{A_2}$ respectively, and taking the imaginary part of the result, we have

$$\begin{cases} \partial_t(|A_1(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda}{t} |A_1(t,\xi)|^2 |A_2(t,\xi)|^2, & t > 1,\ \xi \in \mathbb{R}, \\ \partial_t(|A_2(t,\xi)|^2) = \frac{2\operatorname{Im}\mu}{t} |A_1(t,\xi)|^2 |A_2(t,\xi)|^2, & t > 1,\ \xi \in \mathbb{R}, \\ A_1(1,\xi) = \varepsilon\mathcal{F}_{m_1}\varphi(\xi),\ A_2(1,\xi) = \varepsilon\mathcal{F}_{m_2}\psi(\xi), & \xi \in \mathbb{R}. \end{cases}$$

Therefore we see

$$\partial_t(|\operatorname{Im}\mu||A_1(t,\xi)|^2 - \operatorname{Im}\lambda|A_2(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda(|\operatorname{Im}\mu| - \operatorname{Im}\mu)}{t}|A_1|^2|A_2|^2 \le 0$$

for $\operatorname{Im}\mu \neq 0$, and

$$\partial_t(|A_1(t,\xi)|^2 + |A_2(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda}{t}|A_1|^2|A_2|^2 \le 0$$

for $\operatorname{Im}\mu = 0$. Hence we obtain

$$|A_1(t,\xi)|^2 + |A_2(t,\xi)|^2 \le C\varepsilon^2(|\mathcal{F}_{m_1}\varphi(\xi)|^2 + |\mathcal{F}_{m_2}\psi(\xi)|^2)$$

for $t \ge 1$ and some constant $C > 0$. This observation yields global existence and boundedness of the solution $u = (u_1, u_2)$ to the original system (4.1.4) (see [52] for details). However, the remaining cases are left unsolved so far, that is,

- $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$,
- $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu = 0$,
- $\operatorname{Im}\lambda = 0$ and $\operatorname{Im}\mu > 0$.

The aim of this chapter is to clarify large-time behavior of the solution to (4.1.4) with $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$. Since the solution of the reduced system (4.1.5) blows up at finite time when $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$ (see Section 4.3 for details), it could be natural to expect that the lifespan of the solution to the original system (4.1.4) is characterized by the blow-up time of the solution to the reduced system (4.1.5). We will justify the half of this expectation.

To state the main result, let us define $\tau_0 \in (0, +\infty]$ by

$$\tau_0 := \frac{1}{2}\inf_{\xi\in\mathbb{R}}\left\{\frac{\log(\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2) - \log(\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2)}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2 - \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2}\right\}. \tag{4.1.6}$$

We remark that if $\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2 = \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi^*)|^2$ at some $\xi^* \in \mathbb{R}$, then we define

$$\frac{\log(\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2) - \log(\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi^*)|^2)}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2 - \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi^*)|^2} = \frac{1}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2}.$$

We also remark that the right-hand side of (4.1.6) is always positive. Because of $|\mathcal{F}_{m_1}\varphi(\xi)| < +\infty$, $|\mathcal{F}_{m_2}\psi(\xi)| < +\infty$ and mean value theorem, we have

$$\begin{aligned}&\frac{\log(\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2) - \log(\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2)}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2 - \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2}\\ &\ge \min\left\{\inf_{\xi\in\mathbb{R}}\left(\frac{1}{2\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2}\right),\ \inf_{\xi\in\mathbb{R}}\left(\frac{1}{2\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2}\right)\right\} > 0.\end{aligned}$$

The main result of this chapter is as follows:

Theorem 4.1.1. *Assume that* φ, $\psi \in H^1 \cap H^{0,1}(\mathbb{R})$*, and that* λ, $\mu \in \mathbb{C}$ *with* $\operatorname{Im}\lambda > 0$ *and* $\operatorname{Im}\mu > 0$*. Let* T_ε *be the supremum of* $T > 0$ *such that* (4.1.4) *admits a unique solution* $u = (u_1, u_2) \in \left(C([0,T); H^1 \cap H^{0,1}(\mathbb{R}))\right)^2$*. Then we have*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \tau_0, \tag{4.1.7}$$

where $\tau_0 \in (0, +\infty]$ *is given by* (4.1.6).

Remark 4.1.2. From this result, we clarify that the lower bound estimate for the lifespan of the solution to (4.1.4) holds when $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$. Moreover the lower bound estimate (4.1.7) is different from single case one (4.1.3) in general. It is caused by the initial data and the structure of the nonlinearities on the system (4.1.4) (see Section 4.3 for details). Therefore Theorem 4.1.1 tells us another kind of large-time behavior of solutions which does not correspond to the single case and heavily depends on the initial data and the structure of the nonlinearities on the system. This is new knowledge on the system case. However the author does not know whether (4.1.7) is optimal or not.

Remark 4.1.3. As for the remaining cases, that is,

- $\operatorname{Im}\lambda > 0$ *and* $\operatorname{Im}\mu = 0$,
- $\operatorname{Im}\lambda = 0$ *and* $\operatorname{Im}\mu > 0$,

the solution of the reduced system (4.1.5) grow up at $t \to +\infty$. Therefore it is natural to expect that the solution of the original system (4.1.4) also grow up at $t \to +\infty$. However the author does not know whether this expectation is true or not. Even the small data global existence is not trivial at all, and what we can show by the present approach is only $\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon = +\infty$.

We close this section with the contents of this chapter. In the next section, we state preliminaries. Section 4.3 is devoted to a lemma on some system of ordinary differential equations. In this section, we derive the Riccati-type differential equation from the reduced system (4.1.5). This is the new ingredient of the proof. After that, we will get an a priori estimate in Section 4.4, and Theorem 4.1.1 will be proved in Section 4.5.

4.2 Preliminaries

In this section, we summarize basic facts related to the Schrödinger operator $\mathcal{L}_m = i\partial_t + \frac{1}{2m}\partial_x^2$. We set $\mathcal{J}_m(t) = x + \frac{it}{m}\partial_x$. It is well-known that this operator has good compatibility with $\mathcal{L}_m$ as follows:

$$[\mathcal{L}_m, \mathcal{J}_m(t)] = 0, \ [\partial_x, \mathcal{J}_m(t)] = 1,$$

where $[\cdot, \cdot]$ stands for the commutator of two linear operators. Next we set the free Schrödinger evolution operator

$$(\mathcal{U}_m(t)\phi)(x) := e^{i\frac{t}{2m}\partial_x^2}\phi(x) = \sqrt{\frac{|m|}{2\pi t}}e^{-i\frac{\pi}{4}\mathrm{sgn}(m)}\int_{\mathbb{R}} e^{im\frac{(x-y)^2}{2t}}\phi(y)dy$$

for $m \in \mathbb{R}\backslash\{0\}$ and $t > 0$. We also introduce the scaled Fourier transform $\mathcal{F}_m$ by

$$(\mathcal{F}_m\phi)(\xi) := |m|^{1/2}e^{-i\frac{\pi}{4}\mathrm{sgn}(m)}\mathcal{F}\phi(m\xi) = \sqrt{\frac{|m|}{2\pi}}e^{-i\frac{\pi}{4}\mathrm{sgn}(m)}\int_{\mathbb{R}} e^{-imy\xi}\phi(y)dy,$$

as well as auxiliary operators

$$(\mathcal{M}_m(t)\phi)(x) := e^{im\frac{x^2}{2t}}\phi(x),\ (\mathcal{D}(t)\phi)(x) := \frac{1}{\sqrt{t}}\phi\left(\frac{x}{t}\right),$$

$$\mathcal{W}_m(t)\phi := \mathcal{F}_m\mathcal{M}_m(t)\mathcal{F}_m^{-1}\phi,$$

so that $\mathcal{U}_m(t)$ can be decomposed into

$$\mathcal{U}_m(t) = \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{F}_m\mathcal{M}_m(t) = \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{W}_m(t)\mathcal{F}_m.$$

In what follows, we will occasionally omit "(t)" from $\mathcal{J}_m(t)$, $\mathcal{U}_m(t)$, $\mathcal{M}_m(t)$, $\mathcal{D}(t)$ and $\mathcal{W}_m(t)$, if it causes no confusion.

Lemma 4.2.1. *Let m, μ_1, μ_2, μ_3 be non-zero real constants satisfying $m = \mu_1 + \mu_2 + \mu_3$. For smooth $\mathbb{C}$-valued functions f_1, f_2 and f_3, we have*

$$\mathcal{J}_m(f_1 f_2 \overline{f_3}) = \frac{\mu_1}{m}(\mathcal{J}_{\mu_1}f_1)f_2\overline{f_3} + \frac{\mu_2}{m}f_1(\mathcal{J}_{\mu_2}f_2)\overline{f_3} + \frac{\mu_3}{m}f_1 f_2(\overline{\mathcal{J}_{-\mu_3}f_3}).$$

Lemma 4.2.2. *Let m be a non-zero real constant. We have*

$$\|\phi - \mathcal{M}_m\mathcal{D}\mathcal{F}_m\mathcal{U}_m^{-1}\phi\|_{L^\infty} \le Ct^{-3/4}(\|\phi\|_{L^2} + \|\mathcal{J}_m\phi\|_{L^2})$$

and

$$\|\phi\|_{L^\infty} \le t^{-1/2}\|\mathcal{F}_m\mathcal{U}_m^{-1}\phi\|_{L^\infty} + Ct^{-3/4}(\|\phi\|_{L^2} + \|\mathcal{J}_m\phi\|_{L^2})$$

for $t \ge 1$.

Lemma 4.2.3. *Let m be a non-zero real constant. For smooth $\mathbb{C}$-valued functions f_1, f_2 and f_3, we have*

$$\|\mathcal{F}_m\mathcal{U}_m^{-1}(f_1 f_2 f_3)\|_{L^\infty} \le C\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^\infty}.$$

We skip the proof of these lemmas (see e.g., §3 of [52] and its references cited therein).

4.3 A technical lemma

In this section, we introduce a lemma on some system of ordinary differential equations which will be used effectively in the next section. Throughout this section, we always assume that λ, $\mu \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$. Let φ_0, $\psi_0 : \mathbb{R} \to \mathbb{C}$ be continuous functions satisfying

$$\sup_{\xi\in\mathbb{R}} |\varphi_0(\xi)| < \infty, \ \sup_{\xi\in\mathbb{R}} |\psi_0(\xi)| < \infty.$$

We define $\tau_1 \in (0, \infty]$ by

$$\tau_1 := \frac{1}{2} \inf_{\xi\in\mathbb{R}} \left\{ \frac{\log(\operatorname{Im}\mu|\varphi_0(\xi)|^2) - \log(\operatorname{Im}\lambda|\psi_0(\xi)|^2)}{\operatorname{Im}\mu|\varphi_0(\xi)|^2 - \operatorname{Im}\lambda|\psi_0(\xi)|^2} \right\}.$$

We remark that if $\operatorname{Im}\mu|\varphi_0(\xi^*)|^2 = \operatorname{Im}\lambda|\psi_0(\xi^*)|^2$ at some $\xi^* \in \mathbb{R}$, then we define

$$\frac{\log(\operatorname{Im}\mu|\varphi_0(\xi^*)|^2) - \log(\operatorname{Im}\lambda|\psi_0(\xi^*)|^2)}{\operatorname{Im}\mu|\varphi_0(\xi^*)|^2 - \operatorname{Im}\lambda|\psi_0(\xi^*)|^2} = \frac{1}{\operatorname{Im}\mu|\varphi_0(\xi^*)|^2}.$$

Let $(\alpha_0(t,\xi), \beta_0(t,\xi))$ be a solution to

$$\begin{cases} i\partial_t\alpha_0(t,\xi) = \frac{\lambda}{t}|\beta_0(t,\xi)|^2\alpha_0(t,\xi), & t > 1, \ \xi \in \mathbb{R}, \\ i\partial_t\beta_0(t,\xi) = \frac{\mu}{t}|\alpha_0(t,\xi)|^2\beta_0(t,\xi), & t > 1, \ \xi \in \mathbb{R}, \\ \alpha_0(1,\xi) = \varepsilon\varphi_0(\xi), \beta_0(1,\xi) = \varepsilon\psi_0(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{4.3.1}$$

where $\varepsilon > 0$ is a parameter. If $\varphi_0(\xi^*) = 0$ or $\psi_0(\xi^*) = 0$ at some $\xi^* \in \mathbb{R}$, then we can immediately solve the system (4.3.1) to find that $|\alpha_0(t,\xi^*)|^2 + |\beta_0(t,\xi^*)|^2 \le C\varepsilon^2$. In what follows, we consider (4.3.1) at $\xi \in \mathbb{R}$ with $\varphi_0(\xi) \neq 0$ and $\psi_0(\xi) \neq 0$. At first we consider the case of $\xi \in \mathbb{R}$ with $\operatorname{Im}\mu|\varphi_0(\xi)|^2 > \operatorname{Im}\lambda|\psi_0(\xi)|^2$. Multiplying the equations of system (4.3.1) by $\overline{\alpha_0}$ and $\overline{\beta_0}$ respectively, and taking the imaginary part of the result, we have

$$\begin{cases} \partial_t(|\alpha_0(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda}{t}|\alpha_0(t,\xi)|^2|\beta_0(t,\xi)|^2, & t > 1, \ \xi \in \mathbb{R}, \\ \partial_t(|\beta_0(t,\xi)|^2) = \frac{2\operatorname{Im}\mu}{t}|\alpha_0(t,\xi)|^2|\beta_0(t,\xi)|^2, & t > 1, \ \xi \in \mathbb{R}, \\ \alpha_0(1,\xi) = \varepsilon\varphi_0(\xi), \beta_0(1,\xi) = \varepsilon\psi_0(\xi), & \xi \in \mathbb{R}. \end{cases}$$

Therefore we see

$$\partial_t(\operatorname{Im}\mu|\alpha_0(t,\xi)|^2 - \operatorname{Im}\lambda|\beta_0(t,\xi)|^2) = 0,$$

so that

$$\begin{aligned} \operatorname{Im}\mu|\alpha_0(t,\xi)|^2 - \operatorname{Im}\lambda|\beta_0(t,\xi)|^2 &= \varepsilon^2(\operatorname{Im}\mu|\varphi_0(\xi)|^2 - \operatorname{Im}\lambda|\psi_0(\xi)|^2) \\ &=: \varepsilon^2 G(\xi), \end{aligned} \tag{4.3.2}$$

to obtain the Riccati-type differential equation

$$\partial_t(|\beta_0(t,\xi)|^2) = \frac{2}{t}|\beta_0(t,\xi)|^2\{\operatorname{Im}\lambda|\beta_0(t,\xi)|^2 + \varepsilon^2 G(\xi)\}.$$

Solving this Riccati-type equation, and applying the result to (4.3.2), we have

$$\begin{aligned}
|\alpha_0(t,\xi)|^2 &= \varepsilon^2\left(\frac{\operatorname{Im}\lambda}{\operatorname{Im}\mu}|\psi_0(\xi)|^2\frac{G(\xi)}{\operatorname{Im}\mu|\varphi_0(\xi)|^2 t^{-2\varepsilon^2 G(\xi)} - \operatorname{Im}\lambda|\psi_0(\xi)|^2} + \frac{G(\xi)}{\operatorname{Im}\mu}\right),\\
|\beta_0(t,\xi)|^2 &= \varepsilon^2|\psi_0(\xi)|^2\frac{G(\xi)}{\operatorname{Im}\mu|\varphi_0(\xi)|^2 t^{-2\varepsilon^2 G(\xi)} - \operatorname{Im}\lambda|\psi_0(\xi)|^2},
\end{aligned}$$

as long as the denominators are strictly positive. Similarly if $\xi \in \mathbb{R}$ with $\operatorname{Im}\mu|\varphi_0(\xi)|^2 < \operatorname{Im}\lambda|\psi_0(\xi)|^2$, we can see that

$$\begin{aligned}
|\alpha_0(t,\xi)|^2 &= \varepsilon^2|\varphi_0(\xi)|^2\frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda|\psi_0(\xi)|^2 t^{-2\varepsilon^2\tilde{G}(\xi)} - \operatorname{Im}\mu|\varphi_0(\xi)|^2},\\
|\beta_0(t,\xi)|^2 &= \varepsilon^2\left(\frac{\operatorname{Im}\mu}{\operatorname{Im}\lambda}|\varphi_0(\xi)|^2\frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda|\psi_0(\xi)|^2 t^{-2\varepsilon^2\tilde{G}(\xi)} - \operatorname{Im}\mu|\varphi_0(\xi)|^2} + \frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda}\right),
\end{aligned}$$

where $\tilde{G}(\xi) := \operatorname{Im}\lambda|\psi_0(\xi)|^2 - \operatorname{Im}\mu|\varphi_0(\xi)|^2$. At last, we consider the remaining case $\xi \in \mathbb{R}$ with $\operatorname{Im}\mu|\varphi_0(\xi)|^2 = \operatorname{Im}\lambda|\psi_0(\xi)|^2$. From (4.3.2), we can see that $\operatorname{Im}\mu|\alpha_0(t,\xi)|^2 = \operatorname{Im}\lambda|\beta_0(t,\xi)|^2$ to obtain

$$\partial_t(|\beta_0(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda}{t}|\beta_0(t,\xi)|^4.$$

Solving this equation, we have

$$|\alpha_0(t,\xi)|^2 = \frac{\varepsilon^2|\varphi_0(\xi)|^2}{1 - 2\varepsilon^2|\psi_0(\xi)|^2\operatorname{Im}\lambda\log t},\ |\beta_0(t,\xi)|^2 = \frac{\varepsilon^2|\psi_0(\xi)|^2}{1 - 2\varepsilon^2|\psi_0(\xi)|^2\operatorname{Im}\lambda\log t}.$$

Note that the solution $(\alpha_0(t,\xi),\ \beta_0(t,\xi))$ blows up at the time $t = e^{\tau_1/\varepsilon^2}$, which comes from the minimum time that the denominators $\operatorname{Im}\mu|\varphi_0(\xi)|^2 t^{-2\varepsilon^2 G(\xi)} - \operatorname{Im}\lambda|\psi_0(\xi)|^2 = 0$ at some $\xi \in \mathbb{R}$. This is the reason why τ_0 appears in the lower bound estimate (4.1.7). Therefore we see that

$$\sup_{(t,\xi)\in[1,e^{\sigma/\varepsilon^2}]\times\mathbb{R}}(|\alpha_0(t,\xi)|^2 + |\beta_0(t,\xi)|^2) \le C_1^2\varepsilon^2 \tag{4.3.3}$$

for $\sigma \in (0,\tau_1)$, where

$$C_1 = \sqrt{\max\{A, B, D\}}$$

and

$$A = \sup_{\substack{\xi\in\mathbb{R},\\ \frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}>1}} \left[(|\varphi_0(\xi)|^2 + |\psi_0(\xi)|^2) \frac{\frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2} - 1}{\left(\frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}\right)^{1-\frac{\sigma}{\tau_1}} - 1} + \frac{G(\xi)}{\operatorname{Im}\mu} \right],$$

$$B = \sup_{\substack{\xi\in\mathbb{R},\\ \frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}<1}} \left[(|\varphi_0(\xi)|^2 + |\psi_0(\xi)|^2) \frac{\frac{\operatorname{Im}\lambda|\psi_0(\xi)|^2}{\operatorname{Im}\mu|\varphi_0(\xi)|^2} - 1}{\left(\frac{\operatorname{Im}\lambda|\psi_0(\xi)|^2}{\operatorname{Im}\mu|\varphi_0(\xi)|^2}\right)^{1-\frac{\sigma}{\tau_1}} - 1} + \frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda} \right],$$

$$D = \sup_{\substack{\xi\in\mathbb{R},\\ \frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}=1}} \left[\frac{1}{1-\frac{\sigma}{\tau_1}} (|\varphi_0(\xi)|^2 + |\psi_0(\xi)|^2) \right].$$

Next we consider a perturbation of (4.3.1). Let $T > 1$ and let φ_1, $\psi_1 : \mathbb{R} \to \mathbb{C}$, ρ, $\nu : [1,T) \times \mathbb{R} \to \mathbb{C}$ be continuous functions satisfying

$$\sup_{\xi\in\mathbb{R}} (|\varphi_1(\xi)| + |\psi_1(\xi)|) \le C_2\varepsilon^{1+\delta}, \qquad \sup_{(t,\xi)\in[1,T)\times\mathbb{R}} t^{1+\omega}(|\rho(t,\xi)| + |\nu(t,\xi)|) \le C_3\varepsilon^{1+\delta}$$

with some positive constants C_2, C_3, δ and ω. Let $(\alpha_1(t,\xi), \beta_1(t,\xi))$ be the solution to

$$\begin{cases} i\partial_t\alpha_1(t,\xi) = \frac{\lambda}{t}|\beta_1(t,\xi)|^2\alpha_1(t,\xi) + \rho(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ i\partial_t\beta_1(t,\xi) = \frac{\mu}{t}|\alpha_1(t,\xi)|^2\beta_1(t,\xi) + \nu(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ \alpha_1(1,\xi) = \varepsilon\varphi_0(\xi) + \varphi_1(\xi), \beta_1(1,\xi) = \varepsilon\psi_0(\xi) + \psi_1(\xi), & \xi \in \mathbb{R}, \end{cases}$$

The following lemma asserts that an estimate similar to (4.3.3) remains valid if (4.3.1) is perturbed by ρ, ν and φ_1, ψ_1:

Lemma 4.3.1. *Let $\sigma \in (0,\tau_1)$. We set $T_* = \min\{T, e^{\sigma/\varepsilon^2}\}$. For $\varepsilon \in (0, M^{-1/\delta}]$, we have*

$$\sup_{(t,\xi)\in[1,T_*)\times\mathbb{R}} (|\alpha_1(t,\xi)| + |\beta_1(t,\xi)|) \le \sqrt{2}C_1\varepsilon + M\varepsilon^{1+\delta},$$

where

$$M = \left(2C_2 + \frac{C_3}{\omega}\right) e^{\frac{\operatorname{Im}\lambda + \operatorname{Im}\mu}{2}(1+3C_1+4C_1^2)\sigma}.$$

Proof. We put $w(t,\xi) = \alpha_1(t,\xi) - \alpha_0(t,\xi)$, $z(t,\xi) = \beta_1(t,\xi) - \beta_0(t,\xi)$ and

$$T_{**} = \sup\left\{\tilde{T} \in [1,T_*) \;\middle|\; \sup_{(t,\xi)\in[1,\tilde{T})\times\mathbb{R}} (|w(t,\xi)| + |z(t,\xi)|) \le M\varepsilon^{1+\delta}\right\}.$$

Note that $T_{**} > 1$, because of the estimate

$$|w(1,\xi)| + |z(1,\xi)| = |\varphi_1(\xi)| + |\psi_1(\xi)| \leq C_2\varepsilon^{1+\delta} \leq \frac{M}{2}\varepsilon^{1+\delta}$$

and the continuity of w and z. Since w satisfies

$$i\partial_t w = \frac{\lambda}{t}\Big(|z+\beta_0|^2(w+\alpha_0) - |\beta_0|^2\alpha_0\Big) + \rho,$$

we see that

$$\begin{aligned}\partial_t\left(|w|^2\right) &= 2\,\mathrm{Im}\Big(\overline{w}\cdot i\partial_t w\Big)\\ &\leq \frac{2\,\mathrm{Im}\,\lambda}{t}\left\{\Big(M^2\varepsilon^{2+2\delta} + 3C_1M\varepsilon^{2+\delta} + 2C_1^2\varepsilon^2\Big)|w||z| + C_1^2\varepsilon^2|w|^2\right\} + |w||\rho|\end{aligned}$$

for $t \in [1, T_{**})$. Similarly we see that

$$\partial_t\left(|z|^2\right) \leq \frac{2\,\mathrm{Im}\,\mu}{t}\left\{\Big(M^2\varepsilon^{2+2\delta} + 3C_1M\varepsilon^{2+\delta} + 2C_1^2\varepsilon^2\Big)|w||z| + C_1^2\varepsilon^2|z|^2\right\} + |z||\nu|$$

for $t \in [1, T_{**})$. From these, we have

$$\partial_t(|w|^2 + |z|^2) \leq \frac{2}{t}\tilde{C}\varepsilon^2(|w|^2+|z|^2) + \frac{C_3\varepsilon^{1+\delta}}{t^{1+\omega}}(|w|^2+|z|^2)^{1/2},$$

where $\tilde{C} = \frac{\mathrm{Im}\,\lambda + \mathrm{Im}\,\mu}{2}(1 + 3C_1 + 4C_1^2)$. By the Gronwall-type argument, we obtain

$$\begin{aligned}(|w|^2+|z|^2)^{1/2} &\leq \left((|\varphi_1(\xi)|^2 + |\psi_1(\xi)|^2)^{1/2} + \int_1^t \frac{C_3\varepsilon^{1+\delta}}{2s^{1+\omega+\tilde{C}\varepsilon^2}}ds\right)e^{\tilde{C}\varepsilon^2\log t}\\ &\leq \left(C_2\varepsilon^{1+\delta} + \frac{C_3\varepsilon^{1+\delta}}{2(\omega+\tilde{C}\varepsilon^2)}\right)e^{\tilde{C}\sigma}\\ &\leq \frac{M}{2}\varepsilon^{1+\delta}\end{aligned}$$

for $(t,\xi) \in [1, T_{**}) \times \mathbb{R}$. Hence

$$|w(t,\xi)| + |z(t,\xi)| \leq \sqrt{2}(|w|^2+|z|^2)^{1/2} \leq \frac{M}{\sqrt{2}}\varepsilon^{1+\delta}.$$

This contradicts the definition of T_{**} if $T_{**} < T_*$. Therefore we conclude $T_{**} = T_*$. In other words, we have

$$\sup_{(t,\xi)\in[1,T_*)\times\mathbb{R}} |w(t,\xi)| + |z(t,\xi)| \leq M\varepsilon^{1+\delta},$$

whence

$$|\alpha_1(t,\xi)| + |\beta_1(t,\xi)| \leq |\alpha_0(t,\xi)| + |\beta_0(t,\xi)| + |w(t,\xi)| + |z(t,\xi)| \leq \sqrt{2}C_1\varepsilon + M\varepsilon^{1+\delta}$$

for $(t,\xi) \in [1, T_*) \times \mathbb{R}$. This completes the proof. □

4.4 A priori estimate

This section is devoted to getting an a priori estimate for the solution to (4.1.1). Throughout this section, we fix $\sigma \in (0, \tau_0)$ and $T \in (0, e^{\sigma/\varepsilon^2}]$, where τ_0 is defined by (4.1.6). Let $u = (u_1, u_2) \in (C([0,T); H^1 \cap H^{0,1}(\mathbb{R})))^2$ be a solution to (4.1.1) for $t \in [0,T)$. We set

$$\alpha(t,\xi) := \mathcal{F}_{m_1}\big[\mathcal{U}_{m_1}(t)^{-1}u_1(t,\cdot)\big](\xi),\ \ \beta(t,\xi) := \mathcal{F}_{m_2}\big[\mathcal{U}_{m_2}(t)^{-1}u_2(t,\cdot)\big](\xi).$$

We also define

$$\begin{aligned} E(T) :=& \sup_{(t,\xi)\in[0,T)\times\mathbb{R}} (|\alpha(t,\xi)| + |\beta(t,\xi)|) \\ &+ \sup_{0\le t<T} \big[(1+t)^{-\gamma}(\|u_1(t)\|_{H^1} + \|u_2(t)\|_{H^1} + \|\mathcal{J}_{m_1}u_1(t)\|_{L^2} + \|\mathcal{J}_{m_2}u_2(t)\|_{L^2})\big] \end{aligned}$$

with $\gamma \in (0, 1/12)$. The goal of this section is to prove the following:

Lemma 4.4.1. *Let σ, T and γ be as above. Then there exist positive constants ε_0 and K, not depending on T, such that*

$$E(T) \le \varepsilon^{2/3} \tag{4.4.1}$$

implies the stronger estimate

$$E(T) \le K\varepsilon,$$

provided that $\varepsilon \in (0, \varepsilon_0]$.

We divide the proof of this lemma into two subsections. We remark that many parts of the proof below are similar to that of Section 3 in [52], although we need modifications to fit for our purpose.

4.4.1 L^2-estimates

In the first part, we consider the bound for $\|u_1(t)\|_{H^1}$, $\|u_2(t)\|_{H^1}$, $\|\mathcal{J}_{m_1}u_1(t)\|_{L^2}$ and $\|\mathcal{J}_{m_2}u_2(t)\|_{L^2}$. We first remark that Lemma 4.2.2 and the assumption (4.4.1) lead to

$$\|u_1(t)\|_{L^\infty} + \|u_2(t)\|_{L^\infty} \le \frac{C\varepsilon^{2/3}}{t^{1/2}}$$

for $t \ge 1$. Indeed the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ yields

$$\|u_1(t)\|_{L^\infty} + \|u_2(t)\|_{L^\infty} \le C(\|u_1(t)\|_{H^1} + \|u_2(t)\|_{H^1}) \le C\varepsilon^{2/3}$$

for $t \le 1$. From these, we have

$$\|u_1(t)\|_{L^\infty} + \|u_2(t)\|_{L^\infty} \le \frac{C\varepsilon^{2/3}}{\sqrt{1+t}}$$

for $t \in [0, T)$. Now we see from the standard energy method that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_1(t)\|_{H^1} + \|u_2(t)\|_{H^1}\right) \\
&\le |\lambda|(2\|u_1(t)\|_{L^\infty}\|u_2(t)\|_{L^\infty}\|u_2(t)\|_{H^1} + \|u_2(t)\|_{L^\infty}^2\|u_1(t)\|_{H^1}) \\
&\quad + |\mu|(2\|u_1(t)\|_{L^\infty}\|u_2(t)\|_{L^\infty}\|u_1(t)\|_{H^1} + \|u_1(t)\|_{L^\infty}^2\|u_2(t)\|_{H^1}) \\
&\le \frac{C\varepsilon^2}{(1+t)^{1-\gamma}},
\end{aligned}$$

whence

$$\|u_1(t)\|_{H^1} + \|u_2(t)\|_{H^1} \le \varepsilon(\|\varphi\|_{H^1} + \|\psi\|_{H^1}) + \int_0^t \frac{C\varepsilon^2}{(1+s)^{1-\gamma}} ds \le C\varepsilon(1+t)^\gamma. \qquad (4.4.2)$$

Next we deduce from Lemma 4.2.1 that

$$\mathcal{J}_{m_1}(|u_2|^2 u_1) = \frac{m_2}{m_1}(\mathcal{J}_{m_2}u_2)\overline{u_2}u_1 - \frac{m_2}{m_1}(\overline{\mathcal{J}_{m_2}u_2})u_2u_1 + |u_2|^2(\mathcal{J}_{m_1}u_1).$$

We also remember the commutation relation $[\mathcal{L}_{m_1}, \mathcal{J}_{m_1}] = 0$. From these, it follows that

$$\mathcal{L}_{m_1}\mathcal{J}_{m_1}u_1 = \lambda\left(\frac{m_2}{m_1}(\mathcal{J}_{m_2}u_2)\overline{u_2}u_1 - \frac{m_2}{m_1}(\overline{\mathcal{J}_{m_2}u_2})u_2u_1 + |u_2|^2(\mathcal{J}_{m_1}u_1)\right).$$

Therefore the standard energy method leads to

$$\|\mathcal{J}_{m_1}u_1\|_{L^2} \le \varepsilon\|x\varphi\|_{L^2} + \int_0^t \frac{C\varepsilon^2}{(1+s)^{1-\gamma}} ds \le C\varepsilon(1+t)^\gamma. \qquad (4.4.3)$$

In the same way, we have

$$\|\mathcal{J}_{m_2}u_2\|_{L^2} \le C\varepsilon(1+t)^\gamma. \qquad (4.4.4)$$

Collecting (4.4.2), (4.4.3) and (4.4.4), we arrive at the desired estimate

$$\|u_1(t)\|_{H^1} + \|u_2(t)\|_{H^1} + \|\mathcal{J}_{m_1}u_1(t)\|_{L^2} + \|\mathcal{J}_{m_2}u_2(t)\|_{L^2} \le C\varepsilon(1+t)^\gamma$$

for $t \in [0, T)$.

4.4.2 Estimates for α and β

In this part, we will show $|\alpha(t,\xi)| + |\beta(t,\xi)| \le C\varepsilon$ for $(t,\xi) \in [0,T) \times \mathbb{R}$ under the assumption (4.4.1). When $0 \le t \le 1$, the desired estimate follows immediately from the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (4.4.1). Hence we have only to consider the case of $T > 1$ and $t \in [1,T)$. We have

$$\begin{aligned} i\partial_t \alpha(t,\xi) &= \mathcal{F}_{m_1}\mathcal{U}_{m_1}^{-1}[\mathcal{L}_{m_1}u_1] = \mathcal{F}_{m_1}\mathcal{U}_{m_1}^{-1}[\lambda|u_2|^2u_1] \\ &= \frac{\lambda}{t}|\beta(t,\xi)|^2\alpha(t,\xi) + \rho_1(t,\xi), \end{aligned}$$

where

$$\rho_1(t,\xi) = \frac{\lambda}{t}\mathcal{W}_{m_1}^{-1}\left[|\mathcal{W}_{m_2}\beta|^2\mathcal{W}_{m_1}\alpha\right] - \frac{\lambda}{t}|\beta|^2\alpha.$$

In the same way, we have

$$\begin{aligned} i\partial_t \beta(t,\xi) &= \mathcal{F}_{m_2}\mathcal{U}_{m_2}^{-1}[\mathcal{L}_{m_2}u_2] = \mathcal{F}_{m_2}\mathcal{U}_{m_2}^{-1}[\mu|u_1|^2u_2] \\ &= \frac{\mu}{t}|\alpha(t,\xi)|^2\beta(t,\xi) + \rho_2(t,\xi), \end{aligned}$$

where

$$\rho_2(t,\xi) = \frac{\mu}{t}\mathcal{W}_{m_2}^{-1}\left[|\mathcal{W}_{m_1}\alpha|^2\mathcal{W}_{m_2}\beta\right] - \frac{\mu}{t}|\alpha|^2\beta.$$

Note that the inequality $\|(\mathcal{W}_m - 1)\phi\|_{L^\infty} + \|(\mathcal{W}_m^{-1} - 1)\phi\|_{L^\infty} \le Ct^{-1/4}\|\phi\|_{H^1}$ and Lemma 4.2.3 lead to

$$|\rho_1(t,\xi)| + |\rho_2(t,\xi)| \le \frac{C\varepsilon^2}{t^{1+\omega}}$$

with $\omega = 1/4 - 3\gamma > 0$. Moreover we have

$$\begin{aligned} &|\alpha(1,\xi) - \varepsilon\mathcal{F}_{m_1}\varphi(\xi)| \\ &\le C\|u_1(1,\cdot) - \mathcal{U}_{m_1}(1)\varepsilon\varphi\|_{L^2}^{1/2}\|\mathcal{J}_{m_1}(1)\{u_1(1,\cdot) - \mathcal{U}_{m_1}(1)\varepsilon\varphi\}\|_{L^2}^{1/2} \\ &= C\|u_1(1,\cdot) - \mathcal{U}_{m_1}(1)\varepsilon\varphi\|_{L^2}^{1/2}\|\mathcal{J}_{m_1}(1)u_1(1,\cdot) - \mathcal{U}_{m_1}(1)x\varepsilon\varphi\|_{L^2}^{1/2} \\ &\le C\left(\int_0^1 \|\lambda|u_2(s)|^2u_1(s)\|_{L^2}ds\right)^{1/2}\varepsilon^{1/2} \\ &\le C\varepsilon^2, \end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality $\|\phi\|_{L^\infty} \le C\|\phi\|_{L^2}^{1/2}\|\partial_x\phi\|_{L^2}^{1/2}$ and the relation $\mathcal{J}_m(t) = \mathcal{U}_m(t)x\mathcal{U}_m(t)^{-1}$. In the same way, we have

$$|\beta(1,\xi) - \varepsilon\mathcal{F}_{m_2}\psi(\xi)| \le C\varepsilon^2.$$

Therefore we can apply Lemma 4.3.1 with $\varphi_0(\xi) = \mathcal{F}_{m_1}\varphi(\xi)$, $\psi_0(\xi) = \mathcal{F}_{m_2}\psi(\xi)$, $\delta = 1$, $\omega = 1/4 - 3\gamma > 0$ and $\tau_1 = \tau_0$ to obtain

$$|\alpha(t,\xi)| + |\beta(t,\xi)| \le C\varepsilon.$$

4.5 Proof of Theorem 4.1.1

Now we prove Theorem 4.1.1. At first, local existence of the solution to (4.1.4) is proved in a standard way applying the contraction mapping principle (see [3]). Let T_ε be the lifespan defined in the statement of Theorem 4.1.1. Next we set

$$T^* = \sup\{T \in [0, T_\varepsilon) \mid E(T) \le \varepsilon^{2/3}\}.$$

Note that $T^* > 0$ if ε is suitably small, because of the estimate $E(0) \le C\varepsilon \le \frac{1}{2}\varepsilon^{2/3}$ and the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$. Now, we take $\sigma \in (0, \tau_0)$ and assume $T^* \le e^{\sigma/\varepsilon^2}$. Then Lemma 4.4.1 with $T = T^*$ yields

$$E(T^*) \le K\varepsilon \le \frac{1}{2}\varepsilon^{2/3}$$

if $\varepsilon \le \min\{\varepsilon_0, (2K)^{-3}\}$. By the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$, we can choose $\Delta > 0$ such that

$$E(T^* + \Delta) \le \varepsilon^{2/3}.$$

This contradicts the definition of T^*. Therefore we must have $T^* \ge e^{\sigma/\varepsilon^2}$ if ε is suitably small. As a consequence, we obtain

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \ge \sigma.$$

Since $\sigma \in (0, \tau_0)$ is arbitrary, we arrive at the desired conclusion. □

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