Integrality of Drinfeld modular forms of arbitrary rank

Yusuke Sugiyama

Department of Mathematics, Graduate School of Science,
Osaka University

Doctoral Thesis

2019

Abstract

In this thesis, we generalize Gekeler's results on the integrality and congruences among Drinfeld modular forms of rank $r = 2$ ([7]). Let $\mathbb{F}_q[T]$ be the polynomial ring over a finite field $\mathbb{F}_q$ of q-elements. We study the integrality and congruences of Drinfeld modular forms for $\mathrm{GL}_r(\mathbb{F}_q[T])$ for any integer $r \geq 2$. Namely, for any integer $r \geq 2$, we prove that the graded ring of integral Drinfeld modular forms of rank r is generated over $\mathbb{F}_q[T]$ by certain Drinfeld modular forms called the Drinfeld coefficient forms. Lastly, we determine the relations between their reductions modulo nonzero prime ideals of $\mathbb{F}_q[T]$.

Contents

1 Introduction

Let $A = \mathbb{F}_q[T]$ be the polynomial ring over the finite field $\mathbb{F}_q$ of q-elements. We denote by $\mathbb{C}_\infty$ the $(1/T)$-adic completion of an algebraic closure of the field $\mathbb{F}_q((1/T))$ of the formal Laurent series. Let $|\ .\ |$ denote the absolute value on $\mathbb{C}_\infty$ normalized as $|T| = q$. The following analogy is well-known.

Number Field	Function Field
$\mathbb{Z}$	$A = \mathbb{F}_q[T]$
$\mathbb{R}$	$\mathbb{F}_q((1/T))$
$\mathbb{C}$	$\mathbb{C}_\infty$

Here, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of rational integers, the field of real numbers and the field of complex numbers, respectively.

Let $r \geq 2$ be an integer. Drinfeld modular forms of rank r are $\mathbb{C}_\infty$-valued functions on the Drinfeld period domain of rank r satisfying certain conditions. Drinfeld modular forms are regarded as a function field analog of classical modular forms, which are $\mathbb{C}$-valued functions on the complex upper half plane. Basson, Breuer and Pink proved that for each r, the graded ring of Drinfeld modular forms of type zero and rank r is generated over $\mathbb{C}_\infty$ by certain Drinfeld modular forms $\{G_i\}_{i=1}^r$ called the Drinfeld coefficient forms ([3], [4], [5]). In the thesis, we introduce a notion of the integrality of Drinfeld modular forms of any rank r and prove that for any rank $r \geq 2$, the Drinfeld coefficient forms $\{G_i\}_{i=1}^r$ are integral (Theorem 1.1). Moreover, we prove that the graded ring of integral Drinfeld modular forms of type zero and rank r is generated over A by the Drinfeld coefficient forms $\{G_i\}_{i=1}^r$ (Theorem 3.10). In our function field setting, one may define Eisenstein series, which will turn out to be a Drinfeld modular form ([5], [9] or Section 2). The integrality of the Drinfeld coefficient forms enables us to prove that after a certain normalization, the Eisenstein series of a certain weight is also integral (Proposition 4.7). Moreover we prove that the integral Eisenstein series of a certain weight is congruent to 1 modulo a nonzero prime ideal $\mathfrak{p}$ of A (Proposition 4.7). Lastly, we introduce a "modulo $\mathfrak{p}$ reduction map ϵ_r" (Definition 4.3) and give a single generator of the kernel of ϵ_r (Theorem 1.5).

Let us give an overview of some results on the integrality and congruences of classical $\mathbb{C}$-valued modular forms. For a positive even integer k, we denote the holomorphic Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ of weight k by $\mathcal{E}_k$ normalized as the constant term of the Fourier expansion of $\mathcal{E}_k$ is equal to 1. Then it is known that the graded ring of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ whose Fourier coefficients are in $\mathbb{Z}$ coincides with

$$\mathbb{Z}[\mathcal{E}_4, \mathcal{E}_6, \Delta],$$

where $\Delta = 12^{-3}(\mathcal{E}_4^3 - \mathcal{E}_6^2)$ is the Ramanujan's delta function. In 1973, Swinnerton-Dyer determined the "modulo p structure" of modular forms for a prime number p in [15]. Roughly speaking, his result states that any congruence modulo $p > 3$ between the Fourier coefficients of modular forms derives from the following congruence relation

$$\mathcal{E}_{p-1} \equiv 1 \mod p.$$

Here the congruence is defined in terms of the Fourier coefficients, that is, the above congruence relation means that for any positive integer $n > 0$, the n-th Fourier coefficient of $\mathcal{E}_{p-1}$

is congruent to 0 modulo p. More precisely, Swinnerton-Dyer proved that the ring of mod p modular forms is isomorphic to

$$\mathbb{F}_p[X_1, X_2]/(\widetilde{B} - 1),$$

where B is a unique polynomial in two variables with the coefficients in $\mathbb{Z}_p \cap \mathbb{Q}$ such that

$$\mathcal{E}_{p-1} = B(\mathcal{E}_4, \mathcal{E}_6)$$

and $\widetilde{B} \in \mathbb{F}_p[X_1, X_2]$ is the reduction modulo p of B.

Let us next recall a function field analog of the above Swinnerton-Dyer's result obtained by Gekeler for rank 2 Drinfeld modular forms in [7]. Let $\mathfrak{p}$ be a nonzero prime ideal of A generated by a monic irreducible polynomial $P \in A$ of degree $d \geq 1$ and set $|\mathfrak{p}| := |P| = q^d$. Gekeler first proved the integrality of certain Drinfeld modular forms, including normalized Eisenstein series $E^{(|\mathfrak{p}|-1)}$ of weight $|\mathfrak{p}| - 1$. Moreover, Gekeler proved that the ring of rank 2 integral Drinfeld modular forms of type zero is generated over A by the rank 2 Drinfeld coefficient forms G_1 and G_2 as we will see in Section 3 (Theorem 3.11). Here we would like to mention a discrepancy of the notation here and that in [7]. In fact, Gekeler [7] denoted the normalized Eisenstein series of weight $q^i - 1$ by g_i for $i \geq 0$. However, we will use the symbol g_i in a completely different sense (see Example 2.7). Next, Gekeler proved that any congruence modulo $\mathfrak{p}$ between integral Drinfeld modular forms of rank 2 derives from the following congruence relation:

$$E^{(|\mathfrak{p}|-1)} \equiv 1 \mod \mathfrak{p}.$$

Here the congruence is defined in terms of the "u-expansion" coefficients, which we will explain in the next paragraph. More precisely, Gekeler proved that the ring of mod $\mathfrak{p}$ Drinfeld modular forms of rank 2 is isomorphic to

$$\mathbb{F}_{\mathfrak{p}}[X_1, X_2]/(\widetilde{A_d} - 1),$$

where $\mathbb{F}_{\mathfrak{p}}$ is the residue field of $\mathfrak{p}$ and $A_d \in A[X_1, X_2]$ is a polynomial such that

$$E^{(|\mathfrak{p}|-1)} = A_d(G_1, G_2)$$

and $\widetilde{A_d} \in \mathbb{F}_{\mathfrak{p}}[X_1, X_2]$ is the reduction modulo $\mathfrak{p}$ of A_d.

In order to explain the u-expansion of Drinfeld modular forms, let us briefly recall some foundations of Drinfeld modular forms (for more details, see [9], [1], [2] or Section 2 below). Let us roughly recall the definition of Drinfeld modular forms. A *weak* Drinfeld modular form of rank r for $\mathrm{GL}_r(A)$ is a rigid analytically holomorphic $\mathbb{C}_\infty$-valued function on the Drinfeld period domain Ω^r which satisfies a certain automorphy condition for $\mathrm{GL}_r(A)$. It is known that a weak Drinfeld modular form has a local power-series expansion "at infinity" in a certain parameter u, which is considered as an analog of the Fourier expansion of classical $\mathbb{C}$-valued modular forms. More precisely, a weak Drinfeld modular form f of rank r is of the form

$$f = \sum_{n \in \mathbb{Z}} f_n u^n,$$

where f_n is the uniquely determined weak Drinfeld modular form of rank $r - 1$ ([1], [3]). We note that the u-expansion converges on some "small" admissible set of Ω^r but not on all of Ω^r ([1], [3]). A weak Drinfeld modular form f is called a Drinfeld modular form if the u-expansion is of the form

$$f = \sum_{n \geq 0} f_n u^n.$$

Here we give some examples of Drinfeld modular forms in order to state our main result. Our first examples are so called the Drinfeld coefficient forms $\{G_i(\omega)\}_{i=1}^r$, which are defined as the coefficients of the defining polynomial $\varphi_T^{A^r\omega}$ of the Drinfeld module $\varphi^{A^r\omega}$ ([5], [6]). More precisely, for any element $\omega \in \Omega^r$, we denote the defining polynomial of the Drinfeld module associated to the lattice $A^r\omega$ by

$$\varphi_T^{A^r\omega}(X) = \sum_{i=0}^{r} G_i(\omega) X^{q^i}, \ (G_0(\omega) := T \in A).$$

Then each $G_i(\omega)$ turns out to be a Drinfeld modular form of rank r, which is called the i-th Drinfeld coefficient form of rank r ([9], [1], [5]). It is known that the Drinfeld coefficient forms $\{G_i(\omega)\}_{i=1}^r$ generate the ring of Drinfeld modular forms of type 0 ([1], [3], [4], [5]). Our second example is the (Drinfeld-) Eisenstein series $E^{(k)}$ of weight k defined as

$$E^{(k)}(\omega) := \sum_{\lambda \in A^r\omega \setminus \{0\}} \lambda^{-k},$$

which is also a Drinfeld modular form of rank r, weight k and of type 0 for a nonnegative integer k ([1], [5], [9]). We note that Goss first introduced the Drinfeld coefficient forms and the Eisenstein series and he proved that they are *weak* Drinfeld modular forms for $\mathrm{GL}_r(A)$ ([9]).

Now let us go back to the Gekeler's results for $r = 2$ ([7]). He first proved the integrality of the Drinfeld coefficient forms of rank 2 after a suitable normalization of them. Namely, he proved that the u-expansion coefficients of the (normalized) Drinfeld coefficient forms of rank 2 belong to A. We again note that the normalization amounts to taking a representative of $\omega \in \Omega^2 \subset \mathbb{P}^1$ whose second coordinate is fixed as $\omega = (\omega_1, \bar{\pi})$, where $\bar{\pi}$ is the Carlitz period (see Section 2). The integrality of the rank 2 Drinfeld coefficient forms implies that of the normalized Eisenstein series β_d of rank 2, which is a certain normalization of $E^{(q^d-1)}$, since it is known that there exists a polynomial $A_d(X_1, X_2) \in A[X_1, X_2]$ such that $A_d(G_1, G_2) = \beta_d$. Next he determined a "modulo $\mathfrak{p}$ structure" of Drinfeld modular forms of rank 2 for a nonzero prime ideal $\mathfrak{p} \subset A$ of degree $d \geq 1$. For a more precise statement, let us fix some more notation. We set $\mathbb{F}_\mathfrak{p} := A/\mathfrak{p}$. Let us consider the reduction map

$$\epsilon_2 : \mathbb{F}_\mathfrak{p}[X_1, X_2] \to \mathbb{F}_\mathfrak{p}[[u_2]]$$

defined by

$$X_i \mapsto \text{the } u\text{-expansion of } G_i \mod \mathfrak{p}.$$

Note that the map ϵ_2 makes sense since we have the integrality of the Drinfeld coefficient forms of rank 2. Let $\widetilde{A}_d \in \mathbb{F}_\mathfrak{p}[X_1, X_2]$ be the reduction modulo $\mathfrak{p}$ of A_d. Then Gekeler proved

$$\ker(\epsilon_2) = (\widetilde{A}_d - 1).$$

In this thesis, we will prove that the similar assertions hold for Drinfeld modular forms of *arbitrary* rank.

Now let us roughly state our main results. Our first main result is the integrality of the Drinfeld coefficient forms of *arbitrary* rank in the sense of [1] (see Definition 3.5).

Theorem 1.1 ([14] or see Theorem 3.8)**.** *The Drinfeld coefficient forms of arbitrary rank are integral.*

The integrality for $r = 2, 3$ was shown in [7], [1] respectively. Theorem 1.1 follows easily from the following two key propositions. In order to state the propositions, we recall and fix some more notation. Let G_i (resp. g_i) be the i-th coefficient forms of rank r (resp. $r-1$). The r-th coefficient form G_r of rank r is called the discriminant function of rank r. We denote by $A[G_1, \cdots, G_r]$ the ring generated by $\{G_i\}_{i=1}^r$ over A.

Proposition 1.2 (Proposition 3.6)**.** *For a Drinfeld modular form f, write the u-expansion of f as*

$$f(\omega) = \sum_{n \geq 0} f_n u(\omega)^n.$$

Suppose $f \in A[G_1, G_2, \cdots, G_r]$. Then for any $n \geq 0$, we have

$$f_n \in A[g_1, g_2, \cdots, g_{r-1}^{\pm 1}].$$

The next proposition from [1] plays an important role in our proof of the integrality of the coefficient forms.

Proposition 1.3 ([1])**.** *Suppose that for any integer i with $2 \leq i \leq r$, the discriminant functions of rank i are integral. Then the multiplicative inverse of discriminant function of rank r is integral.*

Moreover, we prove the following.

Theorem 1.4 (Theorem 3.10)**.** *The graded ring of integral Drinfeld modular forms of rank r is generated by the Drinfeld coefficient forms $\{G_i\}_{i=1}^r$ over A.*

In Section 4, we will consider congruences of integral Drinfeld modular forms modulo a nonzero prime ideal $\mathfrak{p} \subset A$ of degree $d \geq 1$. In Definition 4.3, we recursively define reduction maps $\{\epsilon_i\}$ and the rings $\left\{\widetilde{A}^0(i)\right\}$ of the "modulo $\mathfrak{p}$ u-expansions". Then we will give a generator of the kernel of the reduction map

$$\epsilon_r : \mathbb{F}_\mathfrak{p}[X_1, X_2, \cdots, X_r] \to \widetilde{A}^0(r-1)[[u_r]],$$

which maps X_i to the u-expansion of $G_i \mod \mathfrak{p}$ (see Definition 4.3). In the same way as the rank 2 case, one can find a polynomial $A_d(X_1, \cdots, X_r) \in A[X_1, \cdots, X_r]$ such that $A_d(G_1, \cdots, G_r) = \beta_d$, where β_d is the normalized Eisenstein series of weight $q^d - 1$ and of rank r. Then we denote the reduction modulo $\mathfrak{p}$ of A_d by $\widetilde{A}_d$. Here is our second main result.

Theorem 1.5 ([14] or see Theorem 4.8)**.** *We have*

$$\ker(\epsilon_r) = (\widetilde{A_d} - 1).$$

Lastly, let us finish this section with a brief overview of a history and recent researches on Drinfeld modular forms which are related to this thesis. Goss first introduced and studied the analytic theory of weak Drinfeld modular forms of arbitrary rank in 1980 ([9]). Algebraic and analytic theory of Drinfeld modular forms of rank 2 were rapidly developed by Goss, Gekeler and many others. Recently, $\mathfrak{p}$-adic properties of Drinfeld modular forms of rank 2 have been studied analytically and geometrically by Vincent and Hattori ([16], [10]). On the other hand, the theory of algebraic Drinfeld modular forms of arbitrary rank was established in [13], [8] and [4]. Moreover, Basson, Breuer and Pink gave some foundations of analytic and algebraic Drinfeld modular forms of *arbitrary* rank ([1], [3], [4], [5]). One may find basic definitions and properties of analytic Drinfeld modular forms of arbitrary rank in [1] and [3]. In [12], Nicole and Rosso have studied geometrically $\mathfrak{p}$-adic properties of Drinfeld modular forms of arbitrary rank. We hope that this thesis gives a first step toward the analytic theory of $\mathfrak{p}$-adic properties of Drinfeld modular forms of arbitrary rank.

Organization of This Thesis

In Section 2, we give an overview of analytic theory of Drinfeld modular forms without proofs. Note that the missing proof may be found in [1], [2], [3], [5] and [6]. In Section 3, we introduce a notion of the integrality of weak Drinfeld modular forms and prove that the Drinfeld coefficient forms of arbitrary rank are integral. Moreover, we will prove the structure theorem of the integral Drinfeld modular forms. In Section 4, we introduce the modulo $\mathfrak{p}$ reduction map ϵ_r and give a single generator of the kernel of ϵ_r.

Acknowledgment

The author especially thanks his supervisor Professor Seidai Yasuda for his numerous suggestions and support. Professor Tadashi Ochiai gave the author many helpful comments on the thesis. The author thanks him very much. The author appreciates that Professor Florian Breuer gave the author a great opportunity to study mathematics for two weeks at the university of Newcastle. The author is grateful for many fruitful comments from Doctor Kohta Gejima when the author started to study this research area. Lastly, the author expresses his gratitude to my parents and Miki Kobayashi for their support.

2 Preliminaries

In this section, we first fix our notation and recall the definition of Drinfeld modular forms. Next we give an overview of analytic Drinfeld modules in order to give some examples of Drinfeld modular forms. For more details of this section, one can refer to [1], [2], [3], [6], [7] and [9].

2.1 Notation

Let $A = \mathbb{F}_q[T]$ be the polynomial ring over the finite field $\mathbb{F}_q$ of q-elements. We denote by $\mathbb{C}_\infty$ the $(1/T)$-adic completion of an algebraic closure of the field $\mathbb{F}_q((1/T))$ of the formal Laurent series. For an integer $r \geq 2$, the Drinfeld period domain of rank r is the rigid analytic space defined by

$$\Omega^r = \Omega^r(\mathbb{C}_\infty) := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \bigcup_H (H(\mathbb{C}_\infty)),$$

where H runs through all $\mathbb{F}_q((1/T))$-rational hyperplanes ([6]). We note that the point set Ω^r has a natural structure as an admissible open subspace of $\mathbb{P}^{r-1}_{\mathbb{C}_\infty}$. We always write and normalize an element $\omega \in \Omega^r$ as a column vector $\omega = {}^t[\omega_1, \omega']$ with $\omega_1 \in \mathbb{C}_\infty$, $\omega' = {}^t[\omega_2, \cdots, \omega_r] \in \Omega^{r-1}$ and $\omega_r = \bar{\pi}$, where $\bar{\pi}$ is a fixed Carlitz period, which satisfies

$$\bar{\pi}^{q-1} = (T^q - T) \sum_{a \in A \setminus \{0\}} a^{1-q}.$$

Next we consider the following action of $\mathrm{GL}_r(A)$ on Ω^r which preserves the above normalization $\omega_r = \bar{\pi}$. For $\gamma \in \mathrm{GL}_r(A)$ and $\omega \in \Omega^r$, we set

$$j(\gamma, \omega) := \bar{\pi}^{-1}(\text{the last entry of } \gamma\omega),$$

where $\gamma\omega$ denotes the usual matrix product. Then the general linear group $\mathrm{GL}_r(A)$ acts on Ω^r as

$$\gamma(\omega) := j(\gamma, \omega)^{-1}\gamma\omega.$$

2.2 Drinfeld modular forms

Now we are ready to define *weak* Drinfeld modular forms for $\mathrm{GL}_r(A)$.

Definition 2.1. *For nonnegative integers $k, m \in \mathbb{Z}$, a function $f : \Omega^r \to \mathbb{C}_\infty$ is called a weak Drinfeld modular form of weight k and type m for $\mathrm{GL}_r(A)$ if f satisfies the following:*

- *f is holomorphic on Ω^r in the rigid analytic sense,*
- *f satisfies the automorphy condition:*

$$f(\gamma(\omega)) = \det(\gamma)^{-m} j(\gamma, \omega)^k f(\omega)$$

for any $\gamma \in \mathrm{GL}_r(A)$ and $\omega \in \Omega^r$.

Remark 2.2. *More generally, one may define weak Drinfeld modular forms for a subgroup Γ of $\mathrm{GL}_r(\mathbb{F}_q(T))$ which is commensurable with $\mathrm{GL}_r(A)$ ([9], [1], [3]). In this thesis, we shall deal only with $\Gamma = \mathrm{GL}_r(A)$. Thus we omit the words "for $\mathrm{GL}_r(A)$" in the rest of this thesis.*

Next we give a quick review of the u-expansions of weak Drinfeld modular forms, which is analogous to the Fourier expansions of classical elliptic modular forms. In order to explain the u-expansion, we need to define a local parameter "at infinity". We say that a subset $\Lambda \subset \mathbb{C}_\infty$ is a lattice if Λ is a finitely generated A-submodule of $\mathbb{C}_\infty$ which has finite intersection with any finite radius disc in $\mathbb{C}_\infty$. The lattice associated to $\omega = {}^t[\omega_1, \cdots, \omega_r] \in \Omega^r$ with $\omega_r = \bar{\pi}$ is given by

$$A^r\omega := \sum_{i=1}^{r} A\omega_i.$$

For a lattice $\Lambda \subset \mathbb{C}_\infty$, we define the exponential function e_Λ associated to the lattice Λ as follows:

$$e_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty, \; z \mapsto z \prod_{\lambda \in \Lambda \setminus \{0\}} (1 - \frac{z}{\lambda}).$$

Definition 2.3 (Local parameter "at infinity"). *For* $\omega = {}^t[\omega_1, \omega'] \in \Omega^r$ *with* $\omega_1 \in \mathbb{C}_\infty, \omega' \in \Omega^{r-1}$, *we set*

$$u(\omega) = u_r(\omega) := e_{A^{r-1}\omega'}(\omega_1)^{-1}.$$

Moreover for $a \in A$, *we set*

$$u_a = u_a(\omega) := e_{A^{r-1}\omega'}(a\omega_1)^{-1} = \varphi_a^{A^{r-1}\omega'}(u^{-1})^{-1}.$$

Note that the dependence of r is implicit in the notation u_a and that we will deal with u_a mainly in Section 3. On the other hand, only in Section 4, we will the notation u_r, which implies the dependence of r. Thus, we hope that the above similar symbols u_r and u_a will make no trouble. Now let us define the u-expansion of weak Drinfeld modular forms.

Theorem 2.4 ([9], [1], [3]). *Let* f *be a weak Drinfeld modular form* $f : \Omega^r \to \mathbb{C}_\infty$ *of weight* k *and type* m. *Then, for each integer* $n \in \mathbb{Z}$, *there exists a unique weak Drinfeld modular form* $f_n : \Omega^{r-1} \to \mathbb{C}_\infty$ *of weight* $k - n$ *and type* m *such that* f *is expressed as an infinite sum*

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n(\omega')u(\omega)^n,$$

which converges on a suitable admissible subset of Ω^r. *The above infinite series expression of* f *is called the* u*-expansion of* f.

Note that Goss showed the above theorem for $r = 2$. In [1] and [3], Basson, Breuer and Pink proved the theorem for any $r \geq 2$, using the local parameter u_r defined by themselves. We are now ready to define Drinfeld modular forms.

Definition 2.5. *For a weak Drinfeld modular form* f, *let us write its* u*-expansion as*

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n(\omega')u(\omega)^n.$$

We say f *is a Drinfeld modular form if* f *is holomorphic "at infinity", that is,* f_n *is identically equal to zero for any* $n < 0$.

Let us give a first example of Drinfeld modular forms.

Example 2.6 (Eisenstein series). *For a non-negative integer* k, *we set*

$$E^{(k)}(\omega) := \sum_{a\in A^r\setminus\{0\}} (a\omega)^{-k},$$

where $a\omega := \sum_{i=1}^r a_i\omega_i$ *with* $a = [a_1, \cdots, a_r] \in A^r, \omega = {}^t[\omega_1, \cdots, \omega_r] \in \Omega^r$. *Then* $E^{(k)}$ *is a Drinfeld modular form of weight* k, *type* 0 *and of rank* r *([1], [3], [9]). Moreover, we set*

$$E^{(0)}(\omega) := -1.$$

We shall often encounter the Eisenstein series of weight $q^k - 1$. *Then we denote the Eisenstein series of weight* $q^k - 1$ *of rank* r *(resp. rank* $r - 1$*) by*

$$\beta_k(\omega) := E^{(q^k-1)}(\omega) \ (resp.\ \ b_k := E^{(q^k-1)}(\omega')).$$

Note that in Section 4, β_k *denotes another normalization of* $E^{(q^k-1)}$ *(Definition 4.5).*

Let us briefly recall analytic Drinfeld modules in order to give another important example of Drinfeld modular forms. Recall that a lattice associated to $\omega = {}^t[\omega_1, \cdots, \omega_r] \in \Omega^r$ with $\omega_r = \bar{\pi}$ is given by

$$A^r\omega = \sum_{i=1}^{r} A\omega_i.$$

Then the Drinfeld module associated to the lattice $A^r\omega$ is an $\mathbb{F}_q$-algebra homomorphism

$$\varphi^{A^r\omega} : A \to \mathrm{End}_{\mathbb{C}_\infty}(\mathbb{G}_a),\ a \mapsto \varphi_a^{A^r\omega}$$

which makes the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^r\omega & \xrightarrow{\text{inclusion}} & \mathbb{C}_\infty & \xrightarrow{e_{A^r\omega}} & \mathbb{C}_\infty & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times a} & & \downarrow{\scriptstyle \times a} & & \downarrow{\scriptstyle \varphi_a^{A^r\omega}} & & \\ 0 & \longrightarrow & A^r\omega & \xrightarrow[\text{inclusion}]{} & \mathbb{C}_\infty & \xrightarrow[e_{A^r\omega}]{} & \mathbb{C}_\infty & \longrightarrow & 0 \end{array}$$

commutative for each $a \in A$, where $\mathbb{G}_a$ is the additive group scheme over $\mathbb{C}_\infty$. It is known that for each $a \in A$, we have

$$\varphi_a^{A^r\omega} : \mathbb{G}_a \to \mathbb{G}_a,\ x \mapsto \sum_{i=0}^{r\deg(a)} G_{i,a}(\omega)x^{q^i}$$

for some $G_{i,a}(\omega) \in \mathbb{C}_\infty$.

Now we are ready to give another example of Drinfeld modular forms.

Example 2.7 (Drinfeld coefficient forms)**.** *For $\omega \in \Omega^r$ and $a \in A = \mathbb{F}_q[T]$, we denote the Drinfeld module associated to the lattice $A^r\omega$ by*

$$\varphi_a^{A^r\omega}(x) = \sum_{i=0}^{r\deg(a)} G_{i,a}(\omega)x^{q^i}.$$

The function $G_{i,a}(\omega)$ is a Drinfeld modular form of weight $q^i - 1$ and type 0 for each $a \in A$ ([1], [9]). We set $G_i(\omega) := G_{i,T}(\omega)$. Then for an integer i with $1 \leq i \leq r$, the Drinfeld modular form G_i is called the i-th coefficient form of rank r. Moreover we denote the Drinfeld coefficient forms of rank $r-1$ by $g_i(\omega')$, where $\omega' \in \Omega^{r-1}$.

In [9], Goss introduced the above two kinds of examples and proved that they are weak Drinfeld modular forms for any $r \geq 2$. Moreover, Goss defined the rank 2 local parameter u_2 and proved that, for $r = 2$, the above two examples are actually Drinfeld modular forms. Recently, Basson, Breuer and Pink defined the local parameter u_r for any $r \geq 2$ and they proved that the above examples are holomorphic "at infinity" ([1], [3], [5]).

We should keep it our mind that we always normalize $\omega = {}^t[\omega_1, \cdots, \omega_r] \in \Omega^r$ as $\omega_r = \bar{\pi}$ since in some papers (e.g. [9], [7], [1]), an element $\omega \in \Omega^r$ is normalized as $\omega_r = 1$.

Note that we may encounter the Eisenstein series $\beta_k(\omega)$ in the logarithm function for the lattice $A^r\omega$ ([9],[7],[1]) as we explain next. Since the differential $e'_{A^r\omega}(z)$ is identically equal to 1, there exists a composition inverse of $e_{A^r\omega}(z)$. We denote the inverse by $\log_{A^r\omega}(z)$. Then we have the following relation between $\log_{A^r\omega}(z)$ and $\beta_k(\omega)$

$$\log_{A^r\omega}(z) = \sum_{k\geq 0}(-\beta_k(\omega))z^{q^k}.$$

We note that the sign of β_k defined in [1], [7] and [9] is different from that of ours.

Recall that the classical $\mathbb{C}$-valued Eisenstein series $\mathcal{E}_4$ and $\mathcal{E}_6$ of weight 4 and 6 appear in the defining equation of the universal elliptic curve over $\mathbb{C}$. Since Drinfeld modules are regarded as an analog of elliptic curves, one may expect that the coefficient forms G_i of rank r are related to the Eisenstein series β_k of rank r. Then we next recall certain important relations in oder to describe the u-expansions of the coefficient forms and the Eisenstein series from [1], [7] and [9].

Proposition 2.8 ([1],[7],[9])**.** *For $0 \leq n$, the coefficient form G_n of rank r is determined recursively as follows:*

$$G_n(\omega) = \beta_1(\omega)G_{n-1}(\omega)^q + \beta_2(\omega)G_{n-2}(\omega)^{q^2} + \cdots\cdots + \beta_{n-1}(\omega)G_1(\omega)^{q^{n-1}} + [n]\beta_n(\omega),$$

where we set $[k] := T^{q^k} - T \in A$ $(k > 0)$, $G_0 := T$ and $G_l := 0$ if $l > r$. Moreover, for a nonnegative integer j, the Eisenstein series of rank r of weight $q^j - 1$ is computed in terms of the rank $r-1$ Eisenstein series as follows:

$$\beta_j(\omega) = b_j(\omega') + \sum_{a\in A^+}(b_0(\omega')u_a^{q^j-1} + b_1(\omega')u_a^{q^j-q} + \cdots + b_{j-1}(\omega')u_a^{q^j-q^{j-1}}),$$

where A^+ denotes the set of monic elements of A.

We will often encounter the following.

Definition 2.9. *For a positive integer i, we set*

$$v_i := \sum_{a\in A^+} u_a^{q^i-1}.$$

Moreover, we set $v_0 := 1$.

Here, we give some examples:

- $b_1(\omega') = \frac{1}{[1]}g_1,\ b_2(\omega') = \frac{1}{[1][2]}([1]g_2 - g_1^{q+1})$,
- $\beta_1(\omega) = \frac{1}{[1]}g_1 - v_1,\ \beta_2(\omega) = \frac{1}{[1][2]}([1]g_2 - g_1^{q+1}) - v_2 + \frac{1}{[1]}g_1v_1^q$,
- $G_1(\omega) = g_1 - [1]v_1,\ G_2(\omega) = g_2 - g_1^qv_1 + g_1v_1^q + [1]^qv_1^{q+1} - [2]v_2$.

Remark 2.10. *For convenience, we set $\beta_0 = b_0 = -1$. Then for $n, j \geq 0$, we may rewrite the above formulas as follows:*

- $\sum_{i=0}^n \beta_i(\omega)G_{n-i}(\omega)^{q^i} = T\beta_n(\omega)$,
- $\beta_j(\omega) = \sum_{i=0}^j v_{n-i}^{q^i}b_i(\omega')$,
- $\sum_{i=0}^n b_i(\omega')g_{n-i}(\omega')^{q^i} = Tb_n(\omega')$,

where we set $G_0 = g_0 = T$ and $G_l = 0$ (resp. $g_l = 0$) if $l > r$ (resp. $l > r-1$).

3 Integrality of Drinfeld Modular Forms

The aim of this section is to prove that the coefficient forms of any rank r are integral (Theorem 1.1). Note that the definition of the integrality of weak Drinfend modular forms is given in Subsection 3.2.

As noted in Section 2, Proposition 1.2, 1.3 play crucial roles in our proof of Theorem 1.1. Since Proposition 1.3 follows easily from an induction on r ([1]), we first focus on Proposition 1.2. At the end of this section, we prove that the coefficient forms of rank r generate the graded ring of integral Drinfeld modular forms of rank r.

3.1 Key Proposition for Theorem 1.1

Recall that the Drinfeld coefficient forms $\{G_i\}_{i=1}^r$ and the Eisenstein series $\{\beta_i\}_{i\geq 0}$ are determined recursively as noted in Proposition 2.8 and Remark 2.10. In order to prove the integrality of the Drinfeld coefficient forms, we need another relation between them (Proposition 3.4). Note that Theorem 1.1 easily follows from the following key Proposition.

Proposition 3.1. *For $0 \leq n \leq r$, we have $G_n \in A[g_1, \cdots, g_{r-1}, v_1, \cdots, v_r]$.*

In order to prove Proposition 3.1, we fix some more notation.

Definition 3.2. *For $0 \leq j \leq n \leq r$, we define elements $A_{n,j}, B_{n,j} \in K[g_1, \cdots, g_{r-1}, v_1, \cdots, v_r]$ as follows:*

$$A_{n,j} = Tv_{n-j}^{q^j} - \sum_{i=j}^{n} v_{i-j}^{g^j} G_{n-i}^{q^i},\ B_{n,j} = \delta_{n,j}T - g_{n-j}^{q^j},$$

where we set $K := \mathbb{F}_q(T)$ and $\delta_{n,j}$ is the Kronecker delta. For instance, we have $A_{n,n} = B_{n,n} = -[n]$.

Remark 3.3. *From Remark 2.10 and Definition 3.2, we have*

$$\sum_{j=0}^{n} A_{n,j} b_j = 0,\ \sum_{j=0}^{n} B_{n,j} b_j = 0.$$

Then the following holds.

Lemma 3.4. *For $0 \leq j \leq n$, we have*

$$A_{n,j} = -[j]v_{n-j}^{q^j} - \sum_{k=0}^{n-j} g_k^{q^j} v_{n-j-k}^{q^{j+k}}$$

Proof. We prove the assertion by induction on n. If $n = 1$, we have $A_{1,0} = -g_1, A_{1,1} = -[1]$ since we have $G_1 = g_1 - [1]v_1$. Suppose that the assertion holds for $n-1$. Then we note that for any $n \geq j \geq 1$, we have a simple relation

$$A_{n,j} = A_{n-1,j-1}^q - [1]v_{n-j}^{q^j}.$$

The induction hypothesis and the above relation imply that the assertion holds for $A_{n,j}$ with $1 \leq j \leq n$. Thus it suffices to prove

$$A_{n,0} = -\sum_{k=0}^{n} g_k v_{n-k}^{q^k}.$$

We put $1 \times (n+1)$ matrices A_n and B_k $(1 \leq k \leq n)$ as follows:

$$A_n := \begin{bmatrix} A_{n,0} & A_{n,1} & A_{n,2} & \ldots & A_{n,n} \end{bmatrix}, \; B_k := \begin{bmatrix} B_{k,0} & B_{k,1} & \ldots & B_{k,k} & 0 & \ldots & 0 \end{bmatrix}.$$

Then we have the following equation on the determinant of an $(n+1) \times (n+1)$ matrix

$$\begin{vmatrix} A_n \\ B_1 \\ \vdots \\ B_n \end{vmatrix} = 0$$

since we have linear relations between $b_0(= -1 \neq 0), b_1, \cdots, b_n$ from Remark 3.3. Recall that we have $B_{k,k} = -[k]$ and set $L_n := \prod_{k=1}^{n}[k]$. Then let us denote the cofactor expansion of the determinant along the first column by

$$(-1)^n L_n A_{n,0} + \sum_{k=1}^{n}(-1)^k B_{k,0} d_k = 0,$$

where d_k is the cofactor of the entry $B_{k,0}(= -g_k)$. Moreover, we have

$$\begin{bmatrix} A_{n,1} & A_{n,2} & \ldots & A_{n,n} \end{bmatrix} = \sum_{k=1}^{n} v_{n-k}^{q^k} \begin{bmatrix} B_{k,1} & \ldots & B_{k,k} & 0 & \ldots & 0 \end{bmatrix}$$

since the assertion holds for $A_{n,j}$ with $1 \leq j \leq n$. This implies that we have an equation on d_k, the cofactor of the entry $B_{k,0}(= -g_k)$;

$$d_k = (-1)^{n-k+1} L_n v_{n-k}^{q^k}.$$

Then the cofactor expansion implies that the assertion holds for $A_{n,0}$. □

Now it is easy to show Proposition 3.1.

Proof. [Proof of Proposition 3.1] Applying Lemma 3.4 for $A_{n,0}$, the assertion follows from an induction on n. □

3.2 Integral Drinfeld Modular Forms

In this subsection, we first prove that the Drinfeld coefficient forms are integral. We again state Proposition 1.2 and 1.3 and give proofs for them. First of all, let us recursively define the integrality of weak Drinfeld modular forms ([1]).

Definition 3.5. *We say a weak Drinfeld modular form of rank 2 is integral if any coefficient of its u-expansion is in A. For $r > 2$, let f be a weak Drinfeld modular form of rank r. Then f is said to be integral if f is of the form*

$$f = \sum_{n \in \mathbb{Z}} f_n u^n$$

with f_n is an integral weak modular form of rank $r-1$ for any $n \in \mathbb{Z}$.

We need the following proposition in oder to prove Theorem 3.8.

Proposition 3.6 (Proposition 1.2)**.** *For a Drinfeld modular form f, write the u-expansion of f as*

$$f(\omega) = \sum_{n \geq 0} f_n(\omega')u(\omega)^n.$$

Suppose $f \in A[G_1, G_2, \cdots, G_r]$. Then, for any $n \geq 0$, we have

$$f_n(\omega') \in A[g_1, g_2, \cdots, g_{r-1}^{\pm 1}].$$

Proof. It suffices to show that the assertion holds for $G_n(\omega)$ $(1 \leq n \leq r)$. By Proposition 3.1, it suffices to check that the coefficients of v_l $(1 \leq l \leq r)$ as a power series in u belong to $A[g_1, g_2, \cdots, g_{r-1}^{\pm 1}]$. Note that as a power series in u , u_a $(a \in A^+)$ is of the form

$$g_{r-1}^{-N} u^{q^{(r-1)\deg(a)}} + o(u^{q^{(r-1)\deg(a)+1}})$$

with the coefficients in $A[g_1, g_2, \cdots, g_{r-1}^{\pm 1}]$ for some positive integer N since we have $u_a = \varphi_a^{A^{(r-1)}\omega'}(u^{-1})^{-1}$. Thus, as a power series in u, only finitely many u_a $(a \in A^+)$ contribute to the coefficient of u^n in v_l for any fixed $n \geq 0$. Therefore, the coefficients of v_l as a power series in u belong to $A[g_1, g_2, \cdots, g_{r-1}^{\pm 1}]$. □

Recall that for any rank $r \geq 2$, the r-th coefficient form G_r is called the discriminant function of rank r. Then the following proposition and its proof are taken from [1, Proposition 3.6.6.(a)].

Proposition 3.7 (Proposition 1.3)**.** *Suppose that for any i with $2 \leq i \leq r$, the rank i discriminant functions are integral modular forms. Then the multiplicative inverse of the rank r discriminant function is an integral weak modular form.*

Proof. It is known that the rank 2 discriminant function is of the form $-u^{q-1} + o(u^q)$ ([7]). Thus the assertion easily follows from an induction on r. □

Now we are ready to prove Theorem 3.8. Gekeler and Basson proved the following theorem for $r = 2$ and $r = 3$, respectively. Note that since they normalize an element $\omega \in \Omega^r$ as $\omega_r = 1$ for $r = 2, 3$, they need a certain normalization which amounts to taking a representative of $\omega \in \Omega^r$ whose last coordinate is fixed as $\omega_r = \bar{\pi}$.

Theorem 3.8 (Theorem 1.1)**.** *For any $r \geq n \geq 2$, the rank r coefficient form $G_n(\omega)$ is an integral modular form.*

Proof. The assertion follows from an induction on r. It holds true when $r = 2$ ([7]). Suppose it holds true for $r - 1$. Then the induction hypothesis and Proposition 1.3 imply that $1/g_{r-1}$ is integral, where g_{r-1} is the rank $r - 1$ discriminant function. Then again the induction hypothesis and Proposition 1.2 complete the proof. □

Lastly we prove that the graded ring of integral Drinfeld modular forms of rank r and of type 0 are generated over A by the coefficient forms $\{G_i\}_{i=1}^r$ of rank r.

Definition 3.9. *For a nonnegative integer k, let M_k^r be the set of Drinfeld modular forms of rank r, weight k and type* 0. *We set*

$$M^r = \bigoplus_{k \geq 0} M_k^r.$$

Similarly, we denote by $M_k^r(A)$ the set of integral Drinfeld modular forms of rank r, weight k and type 0. *We set*

$$M^r(A) = \bigoplus_{k \geq 0} M_k^r(A).$$

Our final aim of this section is to prove the following.

Theorem 3.10. *The graded ring of integral Drinfeld modular forms of rank r and of type* 0 *are generated by the coefficient forms $\{G_i\}_{i=1}^r$ of rank r, that is, we have*

$$M^r(A) = A[G_1, \cdots, G_r].$$

Theorem 3.10 for $r = 2$ has already been shown by Gekeler in [7].

Theorem 3.11 ([7]). *We have*

$$M^2(A) = A[G_1, G_2].$$

Moreover, we need the following to prove Theorem 3.10.

Theorem 3.12 ([3], [4], [5]). *The Drinfeld coefficient forms $\{G_i\}_{i=1}^r$ of rank r are algebraically independent over $\mathbb{C}_\infty$. Moreover, we have*

$$M^r = \mathbb{C}_\infty[G_1, \cdots, G_r].$$

Let us prove Theorem 3.10.

Proof of Theorem 3.10. We prove the inclusion $M^r(A) \subset A[G_1, \cdots, G_r]$ by induction on the rank $r \geq 2$ since the opposite inclusion follows from Theorem 3.8. When $r = 2$, the assertion is nothing but Theorem 3.11. For any positive integer k and any f with

$$f \in M_k^r(A) \subset M^r(A) = \bigoplus_{k \geq 0} M_k^r(A) \subset M^r,$$

it follows from Theorem 3.12 that there exists a polynomial $F_f = F_f(X_1, \cdots, X_r) \in \mathbb{C}_\infty[X_1, \cdots, X_r]$ such that

$$f = F_f(G_1, \cdots, G_r).$$

We need to prove that the polynomial F_f is in $A[X_1, \cdots, X_r]$. Let us write the polynomial F_f as

$$F_f = \sum_{i=0}^{n} F^i(X_1, \cdots, X_{r-1}) X_r^i$$

for some nonnegative integer n and $F^i = F^i(X_1, \cdots, X_{r-1}) \in \mathbb{C}_\infty[X_1, \cdots, X_{r-1}]$. Then it suffices to show that F^i is in $A[X_1, \cdots, X_{r-1}]$ for any i with $0 \leq i \leq n$. In order to prove $F^i \in A[X_1, \cdots, X_{r-1}]$, we again use an induction on i. Thus let us first show $F^0 \in A[X_1, \cdots, X_{r-1}]$. Since the u-expansion of the j-th coefficient form G_j is written as

$$G_j = \begin{cases} g_j + o(u) & (0 \leq j \leq r-1) \\ -g_{r-1}^q u^{q-1} + o(u^q) & (j = r) \end{cases},$$

the constant term of the u-expansion of f is given by $F^0(g_1, \cdots, g_{r-1})$. Thus by definition of the integrality, $F^0(g_1, \cdots, g_{r-1})$ is an integral Drinfeld modular form of rank $r-1$. Then the induction hypothesis on rank r implies that $F^0 \in A[X_1, \cdots, X_{r-1}]$. Next we suppose that $F^l \in A[X_1, \cdots, X_{r-1}]$ for any $0 \leq l \leq i-1$. Then

$$h := \sum_{l=0}^{i-1} F^l(G_1, \cdots, G_{r-1}) G_r^l$$

is an integral Drinfeld modular form. It follows from Proposition 1.3 that

$$(f-h)/G_r^i = F^i(G_1, \cdots, G_{r-1}) + F^{i+1}(G_1, \cdots, G_{r-1}) G_r + \cdots$$

is also integral. Again by looking at the constant term of u-expansion of $(f-h)/G_r^i$, we have $F^i \in A[X_1, \cdots, X_{r-1}]$. □

4 Congruences among Integral Drinfeld Modular Forms

In Section 3, we obtained the structure theorem of integral Drinfeld modular forms. Let $\mathfrak{p} \subset A$ be a nonzero prime ideal of A generated by a monic irreducible polynomial $P(T) \in A[T]$ of degree $d \geq 1$. In this section, we consider the congruences among integral Drinfeld modular forms modulo $\mathfrak{p} \subset A$.

From now on, we denote the rank r local parameter at infinity by $u_r(\omega)$. We give a recursive definition of the congruence between integral weak Drinfeld modular forms modulo the prime ideal $\mathfrak{p}$.

Definition 4.1. *For rank 2 integral weak Drinfeld modular forms*

$$f = \sum_{n \in \mathbb{Z}} f_n u_2(\omega)^n, \ \ h = \sum_{n \in \mathbb{Z}} h_n u_2(\omega)^n,$$

where $f_n, h_n \in A$ and $\omega \in \Omega^2$, we write

$$f \equiv h \mod \mathfrak{p}$$

if

$$f_n \equiv h_n \mod \mathfrak{p}$$

for any $n \in \mathbb{Z}$.

Let

$$f = \sum_{n\in\mathbb{Z}} f_n u_r(\omega)^n, \ h = \sum_{n\in\mathbb{Z}} h_n u_r(\omega)^n$$

be rank $r \geq 3$ integral weak Drinfeld modular forms, where f_n and h_n are rank $r-1$ integral weak Drinfeld modular forms and $\omega \in \Omega^r$. We denote

$$f \equiv h \mod \mathfrak{p}$$

if

$$f_n \equiv h_n \mod \mathfrak{p}$$

for any $n \in \mathbb{Z}$.

Remark 4.2. *Let G_r (resp. g_{r-1}) be the discriminant function of rank $r \geq 2$ (resp. $r-1$). Then the u-expansion of G_r is of the form $-g_{r-1}^q u_r(\omega)^{q-1} + o(u_r(\omega)^q)$ ([1], [7]), where the rank 2 discriminant function is of the form $-u_2(\omega)^{q-1} + o(u_2(\omega)^q)$ ([7]). Thus the discriminant function of any rank is never congruent to zero modulo $\mathfrak{p}$.*

We recursively define the ring $\widetilde{A}(r)$ of "u-expansions modulo $\mathfrak{p}$ " for each $r \geq 2$. Let $\mathbb{F}_\mathfrak{p}$ be the residue field $A/\mathfrak{p}$.

Definition 4.3. *(1) Let ϵ_2 be the $\mathbb{F}_\mathfrak{p}$-algebra homomorphism*

$$\epsilon_2 : \mathbb{F}_\mathfrak{p}[X_1, X_2] \to \mathbb{F}_\mathfrak{p}[[u_2]],$$

which is defined by

$$X_i \mapsto \sum_{n\geq 0} (a_n^{(i)} \mod \mathfrak{p}) u_2^n,$$

where u_2 is an indeterminate and

$$\sum_{n\geq 0} a_n^{(i)} u_2(\omega)^n \ (a_n^{(i)} \in A)$$

is the u-expansion of the i-th Drinfeld coefficient form of rank 2 with $i = 1, 2$.

(2) If an $\mathbb{F}_\mathfrak{p}$-algebra homomorphism ϵ_{j-1} on $\mathbb{F}_\mathfrak{p}[X_1, X_2, \cdots, X_{j-1}]$ is defined for $j \geq 3$, we denote the image of ϵ_{j-1} by $\widetilde{A}(j-1)$. Moreover $\widetilde{A}^0(j-1)$ denotes the localization of the ring $\widetilde{A}(j-1)$ with the multiplicative set $\{\epsilon_{j-1}(X_{j-1})^m\}_{m\geq 0}$. Then ϵ_{r-1} is naturally extended to

$$\epsilon_{j-1}^0 : \mathbb{F}_\mathfrak{p}[X_1, X_2, \cdots, X_{j-1}^{\pm 1}] \to \widetilde{A}^0(j-1).$$

(3) Under the conditions (1) and (2), we define the $\mathbb{F}_\mathfrak{p}$-algebra homomorphism ϵ_r ($r \geq 2$)

$$\epsilon_r : \mathbb{F}_\mathfrak{p}[X_1, X_2, \cdots, X_r] \to \widetilde{A}^0(r-1)[[u_r]] \ (X_i \mapsto \widetilde{G}_i),$$

where u_r is an indeterminate and $\widetilde{G}_i$ is defined as follows. From Proposition 1.2, the u-expansion of the i-th Drinfeld coefficient form G_i of rank r is of the form

$$G_i = \sum_{n\geq 0} P_n^{(i)}(g_1, \cdots, g_{r-1}) u_r(\omega)^n$$

for some $P_n^{(i)} \in A[X_1, \cdots, X_{r-1}^{\pm 1}]$. Let the $\widetilde{P}_n^{(i)} \in \mathbb{F}_{\mathfrak{p}}[X_1, \cdots, X_{r-1}^{\pm 1}]$ denote the reduction modulo $\mathfrak{p}$ of $P_n^{(i)}$. Then, we set

$$\widetilde{G}_i := \sum_{n\geq 0} \epsilon_{r-1}^0(\widetilde{P}_n^{(i)}) u_r^n = \sum_{n\geq 0} \widetilde{P}_n^{(i)}(\epsilon_{r-1}(X_1), \cdots, \epsilon_{r-1}(X_{r-1})) u_r^n.$$

For $r \geq 2$, let G_r be the discriminant function of rank r. We note that Theorem 1.1 and Proposition 1.3 imply that the multiplicative inverse of G_r is an integral weak Drinfeld modular form for any $r \geq 2$. Moreover, we have $G_r = -g_{r-1}^q u_r(\omega)^{q-1} + o(u_r(\omega)^q)$ ([1], [7]). Thus the following makes sense.

Lemma 4.4. *For an integral weak Drinfeld modular form $f \in A[G_1, \cdots, G_r^{\pm 1}]$, we denote by P_f the element of $A[X_1, \cdots, X_r^{\pm 1}]$ such that $P_f(G_1, \cdots, G_r) = f$. Moreover $\widetilde{P}_f \in \mathbb{F}_{\mathfrak{p}}[X_1, \cdots, X_r^{\pm 1}]$ denotes the reduction modulo $\mathfrak{p}$ of the polynomial P_f. Let $f, h \in A[G_1, \cdots, G_r^{\pm 1}]$ be integral weak Drinfeld modular forms. Then the following are equivalent:*

(1) $f \equiv h \mod \mathfrak{p}$,

(2) $\epsilon_r^0(\widetilde{P}_f) = \epsilon_r^0(\widetilde{P}_h)$.

In particular, let $f, h \in A[G_1, \cdots, G_r]$ be integral Drinfeld modular forms. Then

$$f \equiv h \mod \mathfrak{p}$$

if and only if

$$\epsilon_r(\widetilde{P}_f) = \epsilon_r(\widetilde{P}_h).$$

Proof. Let f be an integral weak modular form of rank r. Suppose $f \in A[G_1, \cdots, G_r^{\pm 1}]$. It suffices to show that $f \equiv 0 \mod \mathfrak{p}$ holds if and only if $\epsilon_r^0(\widetilde{P}_f) = 0$. This follows from an induction on r since we can show

$$\epsilon_r^0(\widetilde{P}_f) = \sum_{n\in\mathbb{Z}} \epsilon_{r-1}^0(\widetilde{P}_{f_n}) u_r^n,$$

where $r > 2$ and the u-expansion of f is of the form $f = \sum_{n\in\mathbb{Z}} f_n u_r(\omega)^n$. □

We have another natural way to understand Lemma 4.4 as we explain below. Let f be an integral weak modular form of rank r with the u-expansion

$$f = \sum_{n\in\mathbb{Z}} f_n u_r(\omega)^n.$$

Since f_n ($n \in \mathbb{Z}$) is also an integral weak Drinfeld modular form of rank $r-1$, we may again write

$$f_n = \sum_{j\in\mathbb{Z}} f_{n,j} u_{r-1}(\omega')^j,$$

where $f_{n,j}$ is an integral weak Drinfeld modular form of rank $r-2$. This recursive process says that f can be formally written as

$$f = \sum_I f_I u^I,$$

where $I = (i_2, \cdots, i_r) \in \mathbb{Z}^{r-1}$, $f_I \in A$ and $u^I := u_r(\omega)^{i_r} u_{r-1}(\omega')^{i_{r-1}} \cdots u_2([\omega_{r-1}, \omega_r])^{i_2}$. Let f, h be integral weak Drinfeld modular forms. Suppose $f, h \in A[G_1, \cdots, G_r^{\pm 1}]$ and formally write

$$f = \sum_I f_I u^I, h = \sum_I h_I u^I \ (f_I, h_I \in A).$$

By induction on r, we can show that $f \equiv h \mod \mathfrak{p}$ if and only if $f_I \equiv h_I \mod \mathfrak{p}$ for any I. Thus the following are equivalent:

(1) $f \equiv h \mod \mathfrak{p}$,

(2) $\epsilon_r^0(\widetilde{P}_f) = \epsilon_r^0(\widetilde{P}_h)$,

(3) $f_I \equiv h_I \mod \mathfrak{p}$ for any $I \in \mathbb{Z}^{r-1}$.

One may think the equivalence of the above (1) and (3) is naive and natural. However, the equivalence of (1) and (2) works well in this section.

4.1 Eisenstein Series

In this subsection, we define the normalized Eisenstein series and observe their properties. Recall that for a nonnegative integer k, we set $[k] = T^{q^k} - T$ and $L_n = \prod_{k=1}^n [k]$ ($L_0 := 1$).

Definition 4.5. *From now on, for a non-negative integer k, we set*

$$\beta_k(\omega) := (-1)^{k+1} L_k E^{(q^k-1)}(\omega) \text{ and } b_k(\omega) := (-1)^{k+1} L_k E^{(q^k-1)}(\omega'),$$

where $E^{(q^k-1)}(\omega)$ (resp. $E^{(q^k-1)}(\omega')$) is the Eisenstein series of weight $q^k - 1$ of rank r (resp. rank $r-1$) defined in Example 2.6.

Remark 4.6. *By the normalization of the Eisenstein series, we may rewrite Proposition 2.8 as follows:*

$$(-1)^{n+1} L_{n-1} G_n(\omega) = \sum_{i=1}^{n-1} (-1)^{n-i} \frac{L_{n-1}}{L_i} \beta_i(\omega) G(\omega)_{n-i}^{q^i} + \beta_n(\omega),$$

$$\beta_k(\omega) = b_k(\omega') + [k] \sum_{a\in A^+} \sum_{i=0}^{k-1} (-1)^{k-i} \frac{L_{k-1}}{L_i} b_i(\omega') u_a^{q^k - q^i},$$

where $n, k \geq 0$. Here we set $G_0 = T$ and $G_l = 0$ if $l > r$.

By the above normalization, we can prove the integrality of β_k.

Proposition 4.7. *For a nonnegative integer k, the normalized Eisenstein series β_k is an integral Drinfeld modular form. Moreover let $d \geq 1$ be the degree of the prime ideal $\mathfrak{p}$. Then we have $\beta_d \equiv 1 \mod \mathfrak{p}$.*

Proof. The assertion follows from an induction on the rank $r \geq 2$. In fact, the assertion was obtained in [7] when $r = 2$. For a general rank, the integrality of the normalized Eisenstein series β_k follows from the first displayed equation of Remark 4.6, the induction hypothesis, Theorem 1.1 and Proposition 1.3. The congruence follows from the second displayed equation of Remark 4.6 and the induction hypothesis. □

We note that the first displayed equation of Remark 4.6 implies that there exists a polynomial $A_k \in A[X_1, X_2, \cdots, X_r]$ satisfying

$$A_k(G_1, G_2, \cdots, G_r) = \beta_k(\omega).$$

Let $\widetilde{A}_k \in \mathbb{F}_{\mathfrak{p}}[X_1, X_2, \cdots, X_r]$ be the reduction modulo $\mathfrak{p}$ of A_k. Then Proposition 4.7 and Lemma 4.4 imply that $\widetilde{A}_d - 1 \in \ker(\epsilon_r)$.

Our aim of the rest of the thesis is to prove the following.

Theorem 4.8. $\ker(\epsilon_r) = (\widetilde{A}_d - 1)$

Gekeler proved Theorem 4.8 for $r = 2$ ([7]) by using the Hasse invariant of a Drinfeld module of rank 2. Then let us define the Hasse invariant of rank $r \geq 2$.

4.2 Hasse Invariant

In this subsection, we define the Hasse invariant of a Drinfeld module of arbitrary rank. We first recall that for $\omega \in \Omega^r$, the associated Drinfeld module $\varphi^{A^r\omega}$ satisfies

$$\varphi_T^{A^r\omega}(X) = \sum_{i=0}^{r} G_i(\omega) X^{q^i},$$

where we set $G_0 = T$. Let $\mathfrak{p}$ be a prime ideal of A generated by an irreducible monic polynomial $P(T) \in A$ of degree $d \geq 1$. Then we write

$$\varphi_{P(T)}^{A^r\omega}(X) =: \sum_{i=0}^{rd} H_i X^{q^i}.$$

Thus we have a polynomial $F_i \in A[X_1, X_2, \cdots, X_r]$ such that $H_i = F_i(G_1, \cdots, G_r)$ for each $0 \leq i \leq rd$. We denote the reduction modulo $\mathfrak{p}$ of F_i by $\widetilde{F}_i \in \mathbb{F}_{\mathfrak{p}}[X_1, \cdots, X_r]$.

Definition 4.9. *For each $0 \leq i \leq rd$, we set*

$$\widetilde{H}_i := \epsilon_r(\widetilde{F}_i) \in \widetilde{A}(r) \subset \widetilde{A}^0(r-1)[[u_r]].$$

We say that $\widetilde{H}_d = \epsilon_r(\widetilde{F}_d)$ is the Hasse invariant of rank r. Moreover, we denote the corresponding objects of rank $r-1$ in the above process by the corresponding small letters. For instance, we denote the Hasse invariant of rank $r-1$ by $\widetilde{h}_d = \epsilon_{r-1}(\widetilde{f}_d)$.

We note that by induction on r, we easily see that $\widetilde{A}(r)$ is an integral domain. Let us write the canonical maps as $\mathrm{pr}: A \to \mathbb{F}_{\mathfrak{p}}$ $(a \mapsto \widetilde{a})$ and $\iota : \mathbb{F}_{\mathfrak{p}} \hookrightarrow \mathbb{F}_{\mathfrak{p}}[X_1, \cdots, X_r]$. The composition $(\epsilon_r \circ \iota \circ \mathrm{pr})$ gives A-algebra structure on $\widetilde{A}(r)$. Then we define the Drinfeld module $\widetilde{\varphi}$ over the fraction filed of $\widetilde{A}(r)$ as follows:

$$\widetilde{\varphi}_T(X) := \widetilde{T}X + \sum_{i=1}^{r} \widetilde{G}_i X^{q^i},$$

where $\widetilde{G}_i = \epsilon_r(X_i) \in \widetilde{A}(r)$ is defined in Definition 4.3. Then we have

$$\widetilde{\varphi}_{P(T)}(X) = \sum_{i=0}^{rd} \widetilde{H}_i X^{q^i}.$$

Since the Drinfeld module $\widetilde{\varphi}$ is of characteristic $\mathfrak{p} = (P(T))$, we have

$$\widetilde{H}_i = 0 \ (0 \leq i < d).$$

Next let us prove

$$\widetilde{H}_d = 1 \in \widetilde{A}(r).$$

From Lemma 4.4, it suffices to show $H_d \equiv 1 \mod \mathfrak{p}$. We use the following formula ([1],[7],[9]) in order to describe H_d in terms of the Eisenstein series.

Proposition 4.10. *For a lattice $\Lambda \subset \mathbb{C}_\infty$, let φ^Λ be the Drinfeld modules associated to the lattice Λ. Then for an element $a \in A$, write $\varphi^\Lambda_a = \sum_j g_j(a, \Lambda) X^{q^j}$. Then for $k \geq 1$ and $a \in A$, we have*

$$(a - a^{q^k}) E^{(q^k-1)}(\Lambda) = \sum_{i=0}^{k-1} E^{(q^i-1)}(\Lambda) g_{k-i}(a, \Lambda)^{q^i},$$

where $E^{(q^k-1)}(\Lambda) := \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-(q^k-1)}$ is the Eisenstein series of weight $q^k - 1$ associated to the lattice Λ.

Remark 4.11. *Recall $\mathfrak{p} = (P(T))$ is of degree $d \geq 1$. We have the following congruence relations:*

- $L_i \not\equiv 0 \mod \mathfrak{p}$ *for* $0 \leq i < d$,
- *for* $0 \leq i < d$, $\widetilde{H}_i = 0$, *that is,* $H_i \equiv 0 \mod \mathfrak{p}$,
- $\frac{(-1)^d P(T)}{L_d} \equiv 1, \ \frac{P(T)^{q^d}}{L_d} \equiv 0 \mod \mathfrak{p}$ ([7]).

Note that the congruences make sense since $P(T)$ divides L_d exactly once.

Proposition 4.12. $H_d \equiv 1 \mod \mathfrak{p}$. *Namely we have* $\widetilde{F}_d(X_1, \cdots, X_r) - 1 \in \ker(\epsilon_r)$.

Proof. Recall that we put $\beta_k(\omega) := (-1)^{k+1}L_k E^{(q^k-1)}(\omega)$ in this section. By putting $a := P(T),\ \Lambda := A^r\omega$ and $k := d$ in Proposition 4.10, we have

$$H_d + (-1)^d \frac{P(T)^{q^d}}{L_d}\beta_d = \beta_d \frac{(-1)^d P(T)}{L_d} + \sum_{i=1}^{d-1} (-1)^{i+1} \frac{\beta_i}{L_i} H_{d-i}^{q^i}.$$

Then Remark 4.11 implies $H_d \equiv \beta_d \mod \mathfrak{p}$. Thus the assertion follows from Proposition 4.7. □

We again note that Gekeler proved Proposition 4.12 for $r = 2$ ([7]).

4.3 Irreducibility of $\widetilde{F}_d - 1$

In this subsection, we prove the irreducibility of the polynomial $\widetilde{F}_d - 1 \in \mathbb{F}_{\mathfrak{p}}[X_1, \cdots, X_r]$, which is necessary to show $\widetilde{F}_d - 1$ generates $\ker(\epsilon_r)$. The irreducibility of $\widetilde{F}_d - 1$ follows from the square-freeness of $\widetilde{F}_d$. The argument below is quite similar to that in [7].

We first study the polynomials F_i and f_i. A commutation equation

$$\varphi_T^{A^r\omega} \circ \varphi_{P(T)}^{A^r\omega} = \varphi_{P(T)}^{A^r\omega} \circ \varphi_T^{A^r\omega}$$

implies that the elements $\{H_i\}_{i=0}^{rd}$ are defined recursively as follows.

Proposition 4.13. *$H_0 = P(T)$. For $n \geq 1$, we have*

$$\sum_{i=0}^{n} H_i G_{n-i}^{q^i} = \sum_{i=0}^{n} G_{n-i} H_i^{q^{n-i}},$$

where we set $G_0 = T$ and $G_i = H_j = 0$ for $i > r, j > rd$.

Thus we have the following.

Proposition 4.14. *$F_0 = P(T)$. For $n \geq 1$, we have*

$$\sum_{i=0}^{n} F_i X_{n-i}^{q^i} = \sum_{i=0}^{n} X_{n-i} F_i^{q^{n-i}},$$

where we set $X_i = F_j = 0$ for $i > r$ and $j > rd$.

By the same argument, the similar holds true for f_j.

Proposition 4.15. *$f_0 = P(T)$. For $n \geq 1$, we have*

$$\sum_{i=0}^{n} f_i X_{n-i}^{q^i} = \sum_{i=0}^{n} X_{n-i} f_i^{q^{n-i}},$$

where we set $X_i = f_j = 0$ for $i > r-1$ and $j > (r-1)d$.

Now we can prove the square-freeness of $\widetilde{F}_d \in \mathbb{F}_{\mathfrak{p}}[X_1, \cdots, X_r]$ by induction on rank r.

Proposition 4.16. *The polynomial $\widetilde{F}_d \in \mathbb{F}_\mathfrak{p}[X_1, \cdots, X_r]$ is square-free.*

Proof. The assertion holds when $r = 2$ [7]. For $r > 2$, Proposition 4.14 and Proposition 4.15 imply $F_i(X_1, \cdots, X_{r-1}, 0) = f_i(X_1, \cdots, X_{r-1})$ for $i \geq 0$. Then the induction hypothesis implies that $\widetilde{F}_d(X_1, \cdots, X_{r-1}, 0)$ is square-free. Thus for a square factor $S(X_1, \cdots, X_r)$ of $\widetilde{F}_d(X_1, \cdots, X_r)$, we have $S(X_1, \cdots, X_{r-1}, 0) \in \mathbb{F}_\mathfrak{p}$. We note that the polynomial S must be homogeneous (of course, we assign the weight $q^i - 1$ to X_i for each $1 \leq i \leq r$) since so is $\widetilde{F}_d$. Therefore if $S(X_1, \cdots, X_{r-1}, 0) \in \mathbb{F}_\mathfrak{p}^\times$, then we have $S(X_1, \cdots, X_r) = S(X_1, \cdots, X_{r-1}, 0) \in \mathbb{F}_\mathfrak{p}^\times$. When $S(X_1, \cdots, X_{r-1}, 0) = 0$, X_r divides S and thus X_r divides $\widetilde{F}_d$. This contradicts Proposition 4.12 since we have $G_r = -g_{r-1}^q u_r(\omega)^{q-1} + o(u_r(\omega)^q)$. □

By the same argument in [7], we may prove the irreducibility of $\widetilde{F}_d - 1$.

Corollary 4.17. *The polynomial $\widetilde{F}_d - 1 \in \mathbb{F}_\mathfrak{p}[X_1, \cdots, X_r]$ is irreducible.*

Proof. Suppose that we have a nontrivial factorization $\widetilde{F}_d - 1 = RS$. Writing

$$R = \sum_{i=0}^{m} R_i, S = \sum_{j=0}^{n} S_j$$

as the sums of the homogeneous components, we have

- $R_m S_n = \widetilde{F}_d$
- $R_{m-1}S_n + R_m S_{n-1} = R_{m-2}S_n + R_{m-1}S_{n-1} + R_m S_{n-2} = \cdots = 0$
- $R_0 S_0 = -1$

since $\widetilde{F}_d$ is of homogeneous (of degree $q^d - 1$). Since $m, n \geq 1$ and $\widetilde{F}_d$ is square-free, R_m and S_n have no common factor. This leads to $R_{m-1} = S_{n-1} = 0$ since we have $R_{m-1}S_n + R_m S_{n-1} = 0$. Recursively one has $R_i = S_j = 0$ for any $i < m, j < n$. This contradicts $R_0 S_0 = -1$. □

4.4 Proof of Theorem 1.5

In this subsection, we give an explicit generator of the kernel of ϵ_r (Theorem 1.5, 4.8). Similar to the proof in [7], our proof of Theorem 1.5 consists of the following two steps

Step. 1 $\ker(\epsilon_r) = (\widetilde{F}_d(X_1, \cdots, X_r) - 1)$

Step. 2 $\widetilde{A}_d(X_1, \cdots, X_r) = \widetilde{F}_d(X_1, \cdots, X_r)$.

Then let us first prove that the polynomial $\widetilde{F}_d(X_1, \cdots, X_r) - 1$ generates $\ker(\epsilon_r)$.

Proposition 4.18. *We have*

$$(\widetilde{F}_d(X_1, \cdots, X_r) - 1) = \ker(\epsilon_r).$$

Proof. We first note that $\ker(\epsilon_r)$ is a prime ideal since $\widetilde{A}(r)$ is an integral domain. Let us prove the proposition by induction on r. The assertion holds when $r = 2$ [7]. Then for $r > 2$, let

$$ev : \widetilde{A}^0(r-1)[[u_r]] \to \widetilde{A}^0(r-1)$$

be a morphism defined by

$$u_r \mapsto 0.$$

Since we have

$$G_i = g_i + o(u_r(\omega)),\ G_r = -g_{r-1}^q u_r(\omega)^{q-1} + o(u_r(\omega)^q)$$

for $1 \le i < r$ ([1]), we have Image$(ev \circ \epsilon_r) = \widetilde{A}(r-1)$. The induction hypothesis implies that the ring $\widetilde{A}(r-1)$ is of dimension $r-2$. Thus the ideal $\ker(ev \circ \epsilon_r)$ is a height 2 prime ideal. Then we see that the prime ideal $\ker(\epsilon_r)$ is of height 1 since we have $\ker(\epsilon_r) \subset \ker(ev \circ \epsilon_r)$, $\epsilon_r(X_r) \neq 0$ and $(ev \circ \epsilon_r)(X_r) = 0$. Thus the assertion follows from Proposition 4.12 and Corollary 4.17. □

Lastly it suffices to prove the equation $\widetilde{A}_d(X_1, \cdots, X_r) = \widetilde{F}_d(X_1, \cdots, X_r)$. Let us recall some notation. Since we have

$$(-1)^{n+1} L_{n-1} G_n(\omega) = \sum_{i=1}^{n-1} (-1)^{n-i} \frac{L_{n-1}}{L_i} \beta_i(\omega) G(\omega)_{n-i}^{q^i} + \beta_n(\omega),$$

there exists a polynomial $A_k \in A[X_1, X_2, \cdots, X_r]$ satisfying

$$A_k(G_1, G_2, \cdots, G_r) = \beta_k(\omega).$$

In particular, we see that A_k is monic of degree $(q^k - 1)/(q-1)$ as a polynomial in X_1.

Lemma 4.19. *As a polynomial in X_1, the polynomial F_d is monic of degree $(q^d - 1)/(q-1)$.*

Proof. Recall that from Proposition 4.14, we have $F_0 = P(T)$ and for $n \ge 1$,

$$\sum_{i=0}^{n} F_i X_{n-i}^{q^i} = \sum_{i=0}^{n} X_{n-i} F_i^{q^{n-i}},$$

where we set $X_i = F_j = 0$ for $i > r, j > rd$. It follows from an induction on i that $F_i \in A[X_1, X_2, \cdots, X_r]$ is of degree $(q^i - 1)/(q-1)$ in X_1 for $i \le d$. Then, for $i \le d$, let c_i be the coefficient of $X_1^{(q^i-1)/(q-1)}$ of F_i. Then the elements $\{c_i\}_{i=0}^r$ satisfy

- $c_0 = P(T) \in A$,
- $c_i = [i]^{-1}(c_{i-1}^q - c_{i-1}) \in A$ $(i \ge 1)$.

The above relations imply $c_d = 1$. □

Corollary 4.20. *We have*

$$\widetilde{A}_d(X_1, \cdots, X_r) = \widetilde{F}_d(X_1, \cdots, X_r).$$

Proof. From Proposition 4.7 and Proposition 4.18, we deduce that $\widetilde{F}_d - 1$ divides $\widetilde{A}_d - 1$. Then the assertion follows from the fact that both of F_d and A_d are of monic of degree $(q^d - 1)/(q-1)$ in X_1. □

References

[1] D. Basson, On the coefficients of Drinfeld modular forms of higher rank, PhD Thesis, 2014.

[2] D. Basson, F. Breuer, On certain Drinfeld modular forms of higher rank, J. Théorie des Nombres de Bordeaux. 29 (3) (2017), 827–843.

[3] D. Basson, F. Breuer, R. Pink, Drinfeld modular forms of arbitrary rank, Part I: Analytic Theory, preprint, arXiv:1805.12335

[4] D. Basson, F. Breuer, R. Pink, Drinfeld modular forms of arbitrary rank, Part II: Comparison with Algebraic Theory, preprint, arXiv:1805.12337

[5] D. Basson, F. Breuer, R. Pink, Drinfeld modular forms of arbitrary rank, Part III: Examples, preprint, arXiv:1805.12339

[6] V.G. Drinfeld, Elliptic modules, Mat. Sb. 94 (1974) 594-627 (in Russian); translation in Math. USSR, Sb. 23 (1974) 561–592.

[7] E-U. Gekeler, On the coefficients of Drinfeld modular forms, Invent. Math. 93 (1988), 667–700.

[8] E-U. Gekeler, On Drinfeld modular forms of higher rank, J. Théorie des Nombres de Bordeaux. 29 (3) (2017), 875–902.

[9] D. Goss, π-adic Eisenstein series for function fields. Compos. Math. 41 (1) (1980), 3–38.

[10] S. Hattori, Duality of Drinfeld modules and $\mathfrak{p}$-adic properties of Drinfeld modular forms, preprint, arXiv:1706.07645.

[11] M.M. Kapranov, Cuspidal divisors on the modular varieties of elliptic modules, Izv. Akad. Nauk SSSR Ser. Mat. 51 (3) (1987) 568–583, 688 (in Russian); translation in Math. USSR, Izv. 30 (3) (1988) 533–547.

[12] M.H. Nicole and G. Rosso, Familles de formes modulaires de Drinfeld pour le groupe gnral linaire, preprint, arXiv:1805.08793.

[13] R. Pink, Compactification of Drinfeld modular varieties and Drinfeld modular forms of arbitrary rank, Manuscripta Math. 140 (3-4) (2013) 333–361.

[14] Y. Sugiyama, The integrality and reduction of Drinfeld modular forms of arbitrary rank, Journal of Number Theory 188, (2018), 371–391.

[15] H.P.F. Swinnerton-Dyer, On ℓ-adic representations and congruences for coefficients of modular forms, in: Modular Functions of One Variable III, Proc. Internat. Summer School, Univ. Antwerp, 1972, in: Lecture Notes in Math., vol. 350, Springer, Berlin, 1973, 1–55.

[16] C. Vincent, Drinfeld modular forms modulo $\mathfrak{p}$, Proc. Amer. Math. Soc. 138 (2010), no. 12, 42174229. MR 2680048