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Doctoral Dissertation

# Kaon-nucleon interactions and $\Lambda(1405)$ in the Skyrme model

# Skyrme模型を用いたK中間子核子間相互作用及び$\Lambda(1405)$の研究

Takashi Ezoe

Graduate School of Science,
Osaka University

Supervisor: Professor Atsushi Hosaka

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**Abstract**

We study kaon-nucleon systems and the $\Lambda\,(1405)$ as a $\bar{K}N$ Feshbach resonance in the Skyrme model. For the kaon-nucleon systems, we use an approach based on the bound state approach proposed by Callan and Klebanov, but the kaon is introduced around the rotating hedgehog soliton. This corresponds to the variation after projection, changing the order of collective quantization in $1/N_c$ expansion scheme. However, we consider that the method is more suited to investigate weakly interacting kaon-nucleon systems including loosely $\bar{K}N$ bound states such as $\Lambda\,(1405)$.

We have found that there exists one $\bar{K}N$ bound state with the binding energy of order ten MeV and that the $\bar{K}N$ potential has an attraction pocket in the middle range and a repulsive component in the short range. We also investigate kaon-nucleon scattering states and examine phase shifts. The phase shift indicates that the $\bar{K}N$ interaction is attractive, while the $KN$ one is weakly repulsive. We have shown that the difference in the $\bar{K}N$ attraction and $KN$ repulsion is simply explained by the $\omega$-meson exchange.

As an extension of our approach, we have evaluated the decay width of the $\bar{K}N$ Feshbach resonance by combining the Callan-Klebanov's and our bound state approaches. The obtained width is few MeV, which means that the $\bar{K}N$ Feshbach resonance appears as narrow resonance in our method.

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Takashi Ezoe

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# Chapter 1

# Introduction

## 1.1 Introduction

The $\Lambda(1405)$ resonance is considered to be one of the candidates of the exotic hadrons, whose properties can not be easily explained by a simple quark model. The $\Lambda(1405)$ is a negative parity $(P = -1)$ hadron resonance with spin $J = 1/2$, isospin $I = 0$, and strangeness $S = -1$ [1]. It is located just below the $\bar{K}N$ threshold and above the $\pi\Sigma$ one. Theoretically, Dalitz and Tuan predicted the existence of the $\Lambda(1405)$ as a $\bar{K}N$ Feshbach resonance in 1959 [2, 3]. In 1961, Alston, *et al.* reported an evidence of the $\Lambda(1405)$ resonance by an experiment [4].

There are long-standing discussions for the structure of the $\Lambda(1405)$ resonance. It is well-known that the $\Lambda(1405)$ is difficult to be described by a simple three-quark state [5]. In a quark model, a strange baryon is predicted to be heavier than a non-strange baryon due to the mass-hierarchy among up, down, and strange-quarks. However, the mass of the $\Lambda(1405)$ is smaller than that of the counterpart in the nucleon sector, $N(1535)$. Furthermore, considering the mass difference in the $\Lambda(1405)$ $\left(J^P = 1/2^-\right)$ and $\Lambda(1520)$ $\left(J^P = 3/2^-\right)$ states, which are considered to be a spin-orbit partner, we find that it is much larger than that in the nucleon sector, $N(1535)$ and $N(1520)$. According to these fact, $\Lambda(1405)$ is not consider to de a three-quark state, but to be a meson-baryon quasi-bound state.

Actually, the $\Lambda(1405)$ resonance is described as a $\bar{K}N$ quasi-bound state embedded in $\pi\Sigma$ continuum in coupled channel approaches with a vector meson exchange potential [6]. Recently, there have been many works for the $\Lambda(1405)$ in a couple channel framework with the chiral symmetry in the low-energy quantum chromodynamics (QCD), which is called the chiral unitary approach [7, 8, 9]. In this approach, they combine the low-energy effective interaction based on the chiral symmetry and the unitarity condition for the scattering amplitude. There have been many works for the $\Lambda(1405)$ resonance. However, its detailed properties are yet under debate. One of the possible reasons for this is that the $\bar{K}N$ interaction is not understood very well.

Several $\bar{K}N$ interactions have been investigated by a phenomenological approach [10, 11] and chiral theories [12, 13, 14]. Akaishi and Yamazaki have constructed the $\bar{K}N$ potential in a phenomenological approach [10, 11]. Their potential is a strong attraction and it contains several model parameters. The chiral approach is based on the low-energy theorem of the spontaneously broken chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry, that is called the Weinberg-Tomozawa interaction [15, 16]. It gives the correct scattering amplitude. However, as we mentioned, it needs unitarization for resonance states which need to introduce a parameter to regularize divergences in theoretical calculations. It is associated with the point-like nature of the interaction. Furthermore, in the chiral approach, the potential is not required if the observables can be calculated from the amplitude. We consider that the interaction as a potential form is more important when we investigate few-body nuclear systems with the anti-kaon, which is called kaonic nuclei.

## 1.2 Purpose of the thesis

In this work, we have investigated the properties of the $\Lambda\,(1405)$ resonance in the Skyrme model [17, 18, 19]. To do that, we have first constructed a new method to derive the kaon-nucleon potential. Our approach is based on the bound state approach in the Skyrme model [20, 21], which is proposed by Callan and Klebanov to investigate hyperons. In the Skyrme model, the nucleon emerge as a soliton of a pion non-linear field theory, which is called the Hedgehog soliton. Therefore, the nucleon appears an object with the finite size in the three-dimensional configuration space. Due to the finite structures of the nucleon as a soliton, the resulting interaction reflects nucleon structures. In the Skyrme model, there are several parameters, but they are determined by the properties of the nucleon and delta-particle. In this sense, our approach is free from parameters.

As we mentioned, our approach is based on the Callan-Klebanov's (CK) approach, where kaons are introduced as fluctuations around the Hedgehog soliton. Their original bound state approach obeys the $1/N_c$ expansion scheme for the semi-classical quantization of the kaon fluctuation and Hedgehog soliton. Because of the strong attraction from the Wess-Zumino term [22, 23, 24], bound states exist in the $\bar{K}$-Hedgehog systems. After collective quantization of the $\bar{K}$-Hedgehog bound system, the quantum number of anti-kaon transmutates due to the strong correlation between the spin and isospin of the background Hedgehog configuration. Furthermore, the parity of the kaon flips due to the existence the background Hedgehog [25, 26]. As a result, in the obtained quantized $\bar{K}$-Hedgehog system, the anti-kaon is regarded as an s-quark. We consider that their approach is not suited to investigation of kaon-nucleon systems.

In the present work, we show an alternative approach for physical kaon-nucleon systems [27, 28]. We first quantized the Hedgehog soliton by the isospin rotation to generate the physical nucleon and then the physical kaon is introduced as a fluctuation around it. However, our approach does not follow the precise $1/N_c$ counting rule. This is because the contribution of the rotation is the higher order than the kaon fluctuation.

The difference in the CK and our approach is explained by the coupling of the kaon and Hedgehog soliton. The CK approach is based on the strong coupling limit of them, while ours is based on the weak coupling limit.

Using our approach, we have investigated the $\bar{K}N\left(J^P = 1/2^-, I = 0\right)$ bound state and its interaction [27, 28]. Furthermore, we have discussed the s-wave $KN$ and $\bar{K}N$ scattering states [29]. Related to the scattering states, we have a comment. There are several works for the pion-nucleon and kaon-nucleon scatterings in the Skyrme model [30, 31, 32, 33]. However, the pion and kaon are introduced as fluctuations around the Hedgehog soliton in these works, which corresponds to the CK approach.

Second, we have investigated the $\Lambda\,(1405)$ resonance as an extension of our approach. It is well-known that $\bar{K}N$-$\pi\Sigma$ coupling is important for the discussions of $\Lambda\,(1405)$. It was investigated in Ref. [34], but the CK approach is employed in the work.

In this work, we have regarded the $\Lambda\,(1405)$ resonance as a $\bar{K}N$ Feshbach resonance as a first attempt. To evaluate the decay width of $\bar{K}N \to \pi\Sigma$ process, we introduce an effective Lagrangian and interpret the matrix element of the Lagrangian as a coupling constant. When we calculate the matrix element, we have combined the CK and our approaches. For the $\pi\Sigma$ channel, we employ the CK approach to generate $\Sigma$. Contrary, we apply our approach for the $\bar{K}N$ channel.

## 1.3 Construction of the thesis

We organize the thesis as follows. In the next chapter, we explain the SU(2) Skyrme model [17, 18, 19]. In Chapter 3, we extend the SU(2) Skyrme model to the SU(3) one and introduce the bound state approach in the Skyrme model [20, 21] which is starting point our study. Then, we show a new method to describe the kaon-nucleon system in the Skyrme model and investigate the $\bar{K}N$ bound state and s-wave kaon-nucleon scattering state. In Chapter 4, we consider the $\Lambda\,(1405)$ resonance in the Skyrme model. In the present work, we consider the $\bar{K}N$ Feshbach resonance as a first attempt. In the end, we summarized this thesis and show our future plan.

# Chapter 2

# The Skyrme model

The Skyrme model [17, 18, 19] is one of the effective models in the low energy QCD. One of the features in this model is to describe baryons as a mesonic soliton called the Hedgehog soliton. As 't Hooft and Witten have discussed [35, 36, 37], the low-energy QCD is written as an effective theory of mesons in the large $N_c$ limit. The Skyrme model is one of the effective mesonic theories and it is constructed by the pion. In this section, we briefly explain an outline of the SU(2) Skyrme model. To do that, we first show the model Lagrangian of the Skyrme model. And then, we introduce the Hedgehog ansatz to describe the baryon. Finally, we show the static properties of the baryons in the Skyrme model.

## 2.1 Model Lagrangian

Let us start with the following Skyrme Lagrangian [17, 18, 19],

$$\mathcal{L}_{Skyrme} = \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2}\mathrm{tr}\left[\left(\partial_\mu U\right) U^\dagger, \left(\partial_\nu U\right) U^\dagger\right]^2, \tag{2.1}$$

where $U$ is the static SU(2)-valued chiral field,

$$U(\boldsymbol{x}) = \exp\left[i\frac{2}{F_\pi}\tau_a \pi_a(\boldsymbol{x})\right], \qquad a = 1, 2, 3, \tag{2.2}$$

where $\tau_a$ $(a = 1, 2, 3)$ are the Pauli's matrices.

The first term in Eq. (2.1) is the Lagrangian of the non-linear $\sigma$ model. It contains the kinetic term of the meson in the leading order in powers of the pion field,

$$\frac{1}{16}F_\pi^2 \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \mathcal{O}\left(\pi^4\right). \tag{2.3}$$

The second one is a higher derivative term which is so-called the Skyrme term [17, 18, 19], which is important to stabilize the system as we will see.

The Lagrangian Eq. (2.1) has several parameters: the first one is the pion decay constant $F_\pi$, whose experimental value is 186 MeV. The second is a dimensionless parameter $e$, which is called the Skyrme parameter. It is related to the size of the soliton. In this study, we consider chiral limit for the up and down sector for simplicity.

To demonstrate the stability by the Skyrme term, we first consider a finite soliton configuration, $U(\boldsymbol{x})$. The energy of the system with the configuration is obtained by a volume-integral of the Lagrangian,

$$\begin{aligned} E &\equiv \int d^3x \mathcal{L}_{Skyrme} \\ &= \int d^3x \left\{ \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2} \mathrm{tr}\left[\left(\partial_\mu U\right) U^\dagger, \left(\partial_\nu U\right) U^\dagger\right]^2 \right\} \\ &\equiv E^{(2)} + E^{(4)}, \end{aligned} \tag{2.4}$$

where $E^{(2)}$ and $E^{(4)}$ are, respectively, the energies from the second-derivative and fourth-derivative terms.

Next, we rescale the space coordinate in $U$ with a scale parameter $\lambda$,

$$U(\boldsymbol{x}) \to U(\lambda \boldsymbol{x}). \tag{2.5}$$

After rescaling, the energy is given by,

$$\begin{aligned} E^{(\lambda)} &= \int d^3x \left\{ \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu U(\lambda\boldsymbol{x}) \partial^\mu U^\dagger(\lambda\boldsymbol{x})\right) \right. \\ &\qquad \left. + \frac{1}{32e^2} \mathrm{tr}\left[\left(\partial_\mu U(\lambda\boldsymbol{x})\right) U^\dagger(\lambda\boldsymbol{x}), \left(\partial_\nu U(\lambda\boldsymbol{x})\right) U^\dagger(\lambda\boldsymbol{x})\right]^2 \right\} \\ &= \int d^3(\lambda x) \frac{1}{\lambda^3} \left\{ \lambda^2 \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu^{(\lambda)} U(\lambda\boldsymbol{x}) \partial^{\mu(\lambda)} U^\dagger(\lambda\boldsymbol{x})\right) \right. \\ &\qquad \left. + \lambda^4 \frac{1}{32e^2} \mathrm{tr}\left[\left(\partial_\mu^{(\lambda)} U(\lambda\boldsymbol{x})\right) U^\dagger(\lambda\boldsymbol{x}), \left(\partial_\nu^{(\lambda)} U(\lambda\boldsymbol{x})\right) U^\dagger(\lambda\boldsymbol{x})\right]^2 \right\} \\ &= \frac{1}{\lambda} E^{(2)} + \lambda E^{(4)}, \end{aligned} \tag{2.6}$$

where $\partial_\mu^{(\lambda)} \equiv \partial / \partial(\lambda x^\mu)$. Eq. (2.6) shows that $\frac{1}{\lambda} E^{(2)}$ decreases as the scale increases while $\lambda E^{(4)}$ increases as $\lambda$ becomes large. Thanks to the Skyrme term, the system becomes stable. As a result, the total energy $E^{(2)} + E^{(4)}$ has a minimum point, where $E^{(2)}$ is equal to $E^{(4)}$.

## 2.2 Hedgehog ansatz

In this subsection, we introduce a static ansatz to describe the SU(2) baryons, the nucleon and the delta-particle. This anstaz is called the Hedgehog ansatz which is constructed by the SU(2) Nambu-Goldstone bosons, that is the pions, and the ansatz is given by,

$$U_H\left(\boldsymbol{x}\right)=\exp\left[iF\left(r\right)\boldsymbol{\tau}\cdot\hat{r}\right]=\cos F\left(r\right)+i\left(\boldsymbol{\tau}\cdot\hat{r}\right)\sin F\left(r\right), \tag{2.7}$$

where $F\left(r\right)$ is the soliton profile function and $\boldsymbol{\tau}$ is the Pauli's matrices. In Eq (2.7), the pion isospin vector is parallel to the pion radial vector, which is definition of the Hedgehog structure.

We first show the properties of the soliton profile function. By substituting the Hedgehog ansatz Eq. (2.7) for the SU(2) Skyrme Lagrangian,

$$\mathcal{L}=\frac{1}{16}F_\pi^2\mathrm{tr}\left(\partial_\mu U\partial^\mu U^\dagger\right)+\frac{1}{32e^2}\mathrm{tr}\left[\left(\partial_\mu U\right)U^\dagger,\left(\partial_\nu U\right)U^\dagger\right]^2, \tag{2.8}$$

we obtain the equation of motion for the soliton profile function by the variational principle [38],

$$\begin{aligned}\left(\frac{1}{4}y^2+2s^2\right)F''\left(y\right)+\frac{1}{2}yF'\left(y\right)+\sin\left(2F\right)\left(F'\left(y\right)\right)^2&\\-\frac{1}{4}\sin\left(2F\left(y\right)\right)+\frac{s^2}{y^2}\sin\left(2F\left(y\right)\right)=0,&\end{aligned} \tag{2.9}$$

where $s\equiv\sin\left(F\left(y\right)/2\right)$ and we introduce a dimensionless radial distance, $y$, which is, so-called, the standard unit [39],

$$y=eF_\pi r. \tag{2.10}$$

Numerically solving the equation of motion Eq. (2.9) under the following boundary condition,

$$\begin{cases}F\left(y=0\right)=\pi\\F\left(y=\infty\right)=0,\end{cases} \tag{2.11}$$

we obtain the soliton profile function, which is shown in Fig. 2.1. For the numerical calculation, we use a parameter set proposed by Adkins, Nappi, and Witten, $F_\pi=186$ MeV and $e=5.45$, which reproduces the masses of the nucleon and delta [38].

We mention here the meaning of the boundary condition Eq. (2.11). To do that, we show the baryon number current, $B_\mu$, which is derived from the Wess-Zumino term [22, 23, 24] as the Noether's current [40] associated with the U(1) vector transformation,

$$B^\mu=\frac{N_c}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[L_\nu L_\alpha L_\beta\right], \tag{2.12}$$

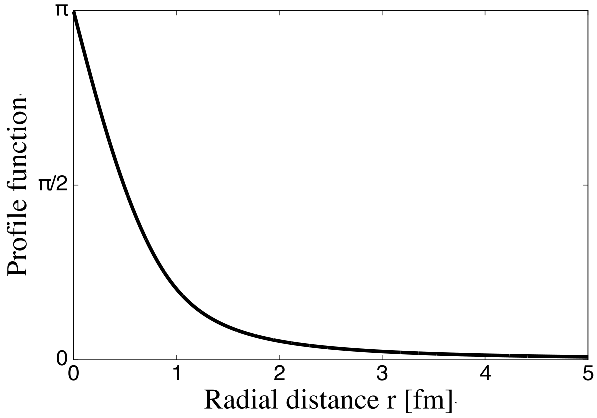


Figure 2.1: Profile function $F(r)$ appearing in the Hedgehog ansatz Eq. (2.7). Parameters for the numerical calculation are $F_\pi = 186$ MeV and $e = 5.45$.

where $L_\mu = U^\dagger \partial_\mu U$. The derivation of the baryon number current from the Wess-Zumino action is shown in Appendix B. Substituting the Hedgehog ansatz $U_H$ for $U$, we obtain,

$$B^0 = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F' \tag{2.13}$$

where $B^0$ is a baryon number charge density. Taking a volume-integral of $B^0$, we obtain the baryon number, $N_B$,

$$\begin{aligned}\int d^3x B^0 &= \int d^3x \left(-\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F'\right) \\ &= \int_0^\infty dr \left(-\frac{2}{\pi}\sin^2 F F'\right) \\ &= \left[\frac{1}{2\pi}\sin\left(2F(r)\right) - \frac{1}{\pi}F(r)\right]_0^\infty = N_B. \end{aligned} \tag{2.14}$$

From the topological point of view, $B^0$ is a topological charge density for the Hedgehog configuration. Therefore, the volume-integral of $B^0$ is identified with a winding number, which is an integer, $N_B = 0, 1, 2, \cdots$. Indeed, we obtain $N_B = 1$, that is a single baryon state, with the boundary condition Eq. (2.11) for the profile function. In other words, the boundary condition Eq. (2.11) ensures that the hedgehog soliton has baryon number one after quantization.

Finally we show important properties of the Hedgehog ansatz. We can easily check that the Skyrme Lagrangian Eq. (2.1) in invariant under the spin and isospin rotations,

- Spin rotation

$$\begin{aligned} U_H(\boldsymbol{x}) &= \exp\left[iF(r)\,\boldsymbol{\tau}\cdot\hat{r}\right] \\ &\to \exp\left[iF(r)\,\tau_a R_{ab}\hat{r}_b\right], \end{aligned} \tag{2.15}$$

where $R_{ab}$ is a spatial rotation matrix.

- Isospin rotation

$$\begin{aligned} U_H(\boldsymbol{x}) &= \exp\left[iF(r)\,\boldsymbol{\tau}\cdot\hat{r}\right] \\ &\to A\exp\left[iF(r)\,\boldsymbol{\tau}\cdot\hat{r}\right]A^{\dagger}, \end{aligned} \tag{2.16}$$

where $A$ is an isospin rotation matrix.

Furthermore, the Hedgehog ansatz, itself, is invariant under the simultaneous rotation in the configuration and isospin spaces, because the Hedgehog structure, $\boldsymbol{\tau}\cdot\hat{r}$, correlates the spin and isospin spaces,

$$\begin{aligned} U_H(\boldsymbol{x}) &= \exp\left[iF(r)\,\boldsymbol{\tau}\cdot\hat{r}\right] \\ &\to A\exp\left[iF(r)\,\tau_a R_{ab}\hat{r}_b\right]A^{\dagger} = \exp\left[iF(r)\,\boldsymbol{\tau}\cdot\hat{r}\right]. \end{aligned} \tag{2.17}$$

This means that the spin ($\boldsymbol{J}$) and isospin ($\boldsymbol{I}$) are not good quantum numbers but the grand-spin ($\boldsymbol{K}=\boldsymbol{J}+\boldsymbol{I}$) is a good quantum number for the Hedgehog ansatz,

$$\begin{aligned} \left[\boldsymbol{K}, U_H\right] &= \left[\boldsymbol{x}\times(-i\nabla), U_H\right] + \left[\frac{\boldsymbol{\tau}}{2}, U_H\right] \\ &= i\sin F\left\{\left[\boldsymbol{x}\times(-i\nabla), \boldsymbol{\tau}\cdot\hat{r}\right] + \left[\frac{\boldsymbol{\tau}}{2}, \boldsymbol{\tau}\cdot\hat{r}\right]\right\} \quad (\because \text{Eq. (2.7)}) \\ &= i\sin F\left\{-i\epsilon_{abc}\hat{r}_b\tau_c + i\epsilon_{abc}\hat{r}_b\tau_c\right\} = 0 \end{aligned} \tag{2.18}$$

where $\boldsymbol{x}\times(-i\nabla)$ and $\frac{\boldsymbol{\tau}}{2}$ are generators for spin and isospin rotations, respectively. We call this simultaneous rotation symmetry the Hedgehog symmetry. We can assume that the Hedgehog ansatz describes baryons with $K=0$, that is $I=J$, after quantization from above discussion.

## 2.3 Collective quantization

In the previous section, we have introduced the Hedgehog ansatz and show its properties. However, since the ansatz is a classical object, it does not have any quantum numbers. Therefore we quantize it to describe physical particles by the semi-classical quantization scheme. In this scheme, we introduce the zero-energy modes around the classical hedgehog configuration, which is important excitation in the low energy region. They physically correspond to the spin and isospin rotations for the Hedgehog ansatz. However, as we mentioned in the end of the previous section, it is enough to consider either spin or isospin rotation for the Hedgehog ansatz thanks to the Hedgehog structure. Therefore we consider the isospin rotation and introduce zero-energy excitation around the Hedgehog ansatz,

$$\begin{aligned} U_H(\boldsymbol{x}) &= \exp\left[iF(r)\,\boldsymbol{\tau}\cdot\hat{r}\right] \\ &\to A(t)\exp\left[iF(r)\,\boldsymbol{\tau}\cdot\hat{r}\right]A^{\dagger}(t), \end{aligned} \tag{2.19}$$

where $A(t)$ is a SU(2) time-dependent isospin rotation matrix,

$$A(t) = a_0(t) + i\boldsymbol{\tau} \cdot \boldsymbol{a}(t), \quad \sum_{\mu=0}^{3} a_\mu^2 = 1. \tag{2.20}$$

We regard $A(t)$ as a quantum-mechanical degrees of freedom.

Substituting the isospin rotating Hedgehog ansatz Eq. (2.19) for the Skyrme Lagrangian Eq. (2.1), we obtain,

$$\begin{aligned}
L &\equiv \int d^3x \mathcal{L}_{Skyrme}\left(U_H = AU_HA^\dagger\right) \\
&= -M_{sol} + \Lambda \mathrm{tr}\left[\dot{A}\dot{A}^\dagger\right] \quad \left(\dot{A} \equiv \frac{dA(t)}{dt}\right) \\
&= -M_{sol} + 2\Lambda \sum_{\mu=0}^{3} \left(\dot{a}_\mu\right)^2 \quad \left(\dot{a} \equiv \frac{da(t)}{dt}\right) \quad (\because \text{Eq. (2.20)}),
\end{aligned} \tag{2.21}$$

where $M_{sol}$ is the classical soliton mass which is obtained by substituting the Hedgehog ansatz Eq. (2.7) for the Skyrme Lagrangian Eq. (2.1),

$$\begin{aligned}
M_{sol} &= \int d^3x \mathcal{L}_{Skyrme} \\
&= 4\pi \int_0^\infty dr\ r^2 \left\{\frac{1}{8}F_\pi^2\left[F'^2 + 2\frac{s^2}{r^2}\right] + \frac{1}{2e^2}\frac{s^2}{r^2}\left[\frac{s^2}{r^2} + 2\left(F'\right)^2\right]\right\},
\end{aligned} \tag{2.22}$$

and $\Lambda$ is moment of inertia of the soliton which is given by,

$$\Lambda = \frac{2\pi}{3}F_\pi^2 \int dr\ r^2 \sin^2 F \left[1 + \frac{4}{\left(eF_\pi\right)^2}\left(\left(F'\right)^2 + \frac{\sin^2 F}{r^2}\right)\right]. \tag{2.23}$$

To understand the meaning of $\Lambda$, we write the SU(2) isospin rotation matrix as follows,

$$A(t) = \exp\left[i\boldsymbol{\Omega} \cdot \frac{\boldsymbol{\tau}}{2}t\right], \tag{2.24}$$

where $\boldsymbol{\Omega}$ corresponds to the angular velocity and $\frac{\boldsymbol{\tau}}{2}$ is a generator. Using this, we obtain from Eq. (2.21),

$$L = -M_{sol} + \frac{\Lambda}{2}\boldsymbol{\Omega}^2, \tag{2.25}$$

which means that the second term is a rotation energy and $\Lambda$ is moment of inertia.

To perform the quantization, we next introduce the conjugate momenta,

$$\pi_\mu \equiv \frac{\partial L}{\partial \dot{a}_\mu} = 4\Lambda \dot{a}_\mu. \tag{2.26}$$

Then, we can derive the Hamiltonian,

$$
\begin{aligned}
H &\equiv \pi_\mu \dot{a}_\mu - L \\
&= M_{sol} + 2\Lambda \left(\dot{a}_\mu\right)^2 \\
&= M_{sol} + \frac{1}{2\Lambda}\sum_{\mu=0}^{3}\left(\pi_\mu\right)^2 .
\end{aligned}
\tag{2.27}
$$

Performing the canonical quantization scheme, $\pi_\mu = -i\partial/\partial a_\mu$, we obtain the quantized Hamiltonian,

$$
H = M_{sol} + \frac{1}{2\Lambda}\sum_{\mu=0}^{3}\left(-\frac{\partial^2}{\partial a_\mu^2}\right). \tag{2.28}
$$

The second term in Eq. (2.28) is the four-dimensional Laplacian. Therefore, the eigen function of the Hamiltonian is given by polynomials of $a_\mu$ [38] (equivalently the Wigner's D-functions [41]).

Next, we consider the wave function in the SU(2) isospin space spanned by $a_\mu$. We show here an important fact that the rotating Hedgehog ansatz, $AU_HA^\dagger$, is invariant under the following transformation,

$$
A \to -A. \tag{2.29}
$$

According to Ref. [42], we have two choices to quantize the Hedgehog soliton,

$$
\begin{cases}
\psi\left(A\right) = +\psi\left(-A\right) & \text{for boson wave functions} \\
\psi\left(A\right) = -\psi\left(-A\right) & \text{for fermion wave functions.}
\end{cases}
\tag{2.30}
$$

Here we use the lower condition for the nucleon. In other word, we construct the nucleon wave functions as odd-order polynomials. The normalized nucleon wave functions are give by [38],

$$
\begin{cases}
|p\uparrow\rangle = \frac{1}{\pi}\left(a_1 + ia_2\right) \\
|p\downarrow\rangle = -\frac{i}{\pi}\left(a_0 - ia_3\right) \\
|n\uparrow\rangle = \frac{i}{\pi}\left(a_0 + ia_3\right) \\
|n\downarrow\rangle = -\frac{1}{\pi}\left(a_1 - ia_2\right),
\end{cases}
\tag{2.31}
$$

where $\uparrow$ ($\downarrow$) stands for the third direction of spin.

We can easily check the quantum number of the above wave function by using the spin ($\hat{\boldsymbol{J}}$) and isospin($\hat{\boldsymbol{I}}$) operators obtained as the Noether's charges [38, 43] whose derivation is shown in Appendix A,

$$
\begin{cases}
\hat{J}_i = \frac{i}{2}\left(a_i\frac{\partial}{\partial a_0} - a_0\frac{\partial}{\partial a_i} - \epsilon_{ijk}a_j\frac{\partial}{\partial a_k}\right) \\
\hat{I}_i = \frac{i}{2}\left(a_0\frac{\partial}{\partial a_i} - a_i\frac{\partial}{\partial a_0} - \epsilon_{ijk}a_j\frac{\partial}{\partial a_k}\right),
\end{cases}
\tag{2.32}
$$

with

$$\boldsymbol{J}^2 = \boldsymbol{I}^2 = \frac{1}{4}\sum_{\mu=0}^{3}\left(-\frac{\partial^2}{\partial {a_\mu}^2}\right). \tag{2.33}$$

Furthermore, using the quantized Hamiltonian Eq. (2.28) and the nucleon wave functions Eq. (2.31), we obtain the mass formula for the nucleon,

$$M_N = M_{sol} + \frac{1}{2\Lambda}\frac{3}{4}. \tag{2.34}$$

# Chapter 3

# Kaon-nucleon systems in the Skyrme model

In this chapter, we discuss the kaon-nucleon systems in the Skyrme model. To do that, we first extend the SU(2) Skyrme model to the SU(3) one in order to incorporate the strangeness degree of freedom. In Sec. 3.2, we explain the bound state approach in the Skyrme model proposed by Callan and Klebanov [20, 21]. They have constructed an ansatz which is the so-called bound state ansatz (approach) for the study of the properties of hyperons. In Sec. 3.3, we introduce an alternative ansatz which we consider suitable for physical kaon-nucleon systems. In the following two sections, we investigate the kaon-nucleon bound state and scattering states. Finally, we summarize the result of the kaon-nucleon systems.

## 3.1 Model Lagrangian

Let us start with the SU(3) Skyrme action,

$$\Gamma = \int d^4x \left\{ \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2} \mathrm{tr}\left[\left(\partial_\mu U\right) U^\dagger, \left(\partial_\nu U\right) U^\dagger\right]^2 + L_{SB} \right\} + \Gamma_{WZ}, \tag{3.1}$$

where $U$ is the SU(3)-valued chiral field,

$$U = \exp\left[i\frac{2}{F_\pi}\lambda_a\phi_a\right], \qquad a = 1, 2, 3, \cdots, 8, \tag{3.2}$$

$$\phi = \frac{1}{\sqrt{2}}\sum_{a=1}^{8}\lambda_a\phi_a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \tag{3.3}$$

and $\lambda_a$ $(a = 1, 2, \cdots 8)$ are the Gell-Mann matrices.

The first and second terms in Eq. (3.1) are the original Skyrme actions [17, 18, 19] and the third term is the symmetry breaking term due to finite masses of the SU(3) pseudo-scalar mesons [44, 45],

$$L_{SB} = \frac{1}{48} F_\pi^2 \left(m_\pi^2 + 2m_K^2\right) \mathrm{tr}\left(U + U^\dagger - 2\right) + \frac{\sqrt{3}}{24} F_\pi^2 \left(m_\pi^2 - m_K^2\right) \mathrm{tr}\left[\lambda_8 \left(U + U^\dagger\right)\right]. \tag{3.4}$$

The last term in Eq. (3.1) is the Wess-Zumino (WZ) action [22, 23, 24], which is the SU(3) chiral anomaly,

$$\Gamma_{WZ} = -\frac{iN_c}{240\pi^2} \int d^5x\ \varepsilon^{\mu\nu\alpha\beta\gamma} \mathrm{tr}\left[\left(U^\dagger \partial_\mu U\right)\left(U^\dagger \partial_\nu U\right)\left(U^\dagger \partial_\alpha U\right)\left(U^\dagger \partial_\beta U\right)\left(U^\dagger \partial_\gamma U\right)\right], \tag{3.5}$$

where $N_c$ is the number of colors, $N_c = 3$. The Wess-Zumino term, $\Gamma_{WZ}$, is defined in the fifth-dimensional integral. The boundary of the fifth-dimensional space corresponds to the space-time four-dimensional space.

Because in the present study, the $\eta$-meson is irrelevant to describe the hyperon and kaon-nucleon systems, we need two decay constants $F_\pi$ and $F_K$. Phenomenologically, they differ only by about 20% [46]. Therefore, we will use a common value of them, typically their average. Therefore, the parameters of the action Eq. (3.1) are as follows: the first one is the pion decay constant, $F_\pi$, whose experimental value is 186 MeV. The second is a dimensionless parameter $e$, called the Skyrme parameter, which is related to the size of the soliton. The last ones are the masses of the quarks. In this study, we consider chiral limit for the up and down sector for simplicity, while we treat the strange quark as a massive particle. This means that we treat the pion as a massless particle while the kaon as a massive one.

## 3.2 The bound state approach

In this section, we explain the bound state approach in the Skyeme model proposed by Callan and Klebanov [20, 21], which has become the starting point of our discussions. First, we explain how to extend the SU(2) Skyrme model to the SU(3) one. There is a problem which comes from how to treat the SU(3) flavor symmetry breaking when we extend to the SU(3) sector. To overcome the problem, there exist two major approaches.

- *Collective coordinate quantization method* [44, 45]

  This approach is based on the weakly breaking of the SU(3) flavor symmetry. Therefore, we divide the Lagrangian into the two parts: symmetry part and symmetry breaking one. The former is quantized by the collective quantization scheme, which is naive-extension of the SU(2) Skyrme model. The latter is treated in a perturbative way.

- *Bound state approach* [20, 21]

  In this approach, we assume that the SU(3) flavor symmetry is strongly broken. Therefore, we need a special treatment for the strangeness degrees of freedom. To do that, we divide the chiral field, $U$, constructed from the SU(3) Nambu-Goldstone bosons into the two parts: the SU(2) part and the other one. The former is nothing but the pion part (the Hedgehog soliton) and treated in the same way as the ordinary SU(2) Skyrme model. The latter is the kaon part which is treated as a small fluctuation (vibration in the strangeness direction) around the Hedgehog soliton.

In this study, we employ the bound state approach for the kaon-nucleon systems and $\Lambda\,(1405)$. In the rest of this section, we briefly explain the original bound state approach proposed by Callan and Klebanov [20, 21].

### 3.2.1 Callan-Klebanov ansatz

We first introduce the ansatz proposed by Callan and Klebanov (CK ansatz, $U_{CK}$) [20, 21], which can describe the hyperons,

$$U_{CK} = \sqrt{N_H} U_K \sqrt{N_H}, \tag{3.6}$$

where $\sqrt{N_H}$ is essentially the Hedgehog soliton with the soliton profile $F\,(r)$,

$$N_H = \begin{pmatrix} U_H & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_H = \xi^2 = \exp\left[iF\,(r)\,\boldsymbol{\tau}\cdot\hat{r}\right], \tag{3.7}$$

and $U_K$ is the kaon field defined by,

$$U_K = \exp\left[\frac{2\sqrt{2}i}{F_\pi}\begin{pmatrix} 0 & K \\ K^\dagger & 0 \end{pmatrix}\right], \qquad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \tag{3.8}$$

Here we use the pion decay constant $F_\pi$ instead of the kaon one in $U_K$ as we mentioned in the previous section.

Callan and Klebanov followed the $1/N_c$ expansion scheme when they constructed the ansatz. The Hedgehog soliton is in the leading order of $N_c$ and then the kaon is introduced as a fluctuation, which is in the next-to-leading order, ${N_c}^0$, around the Hedgehog soliton. Finally, the kaon-Hedgehog system is rotated in the isospin space and the rotation energy is of order $1/N_c$.

### 3.2.2 Kaon-Hedgehog bound state

In order to discuss the hyperons, they first have considered the bound state of the kaon and Hedgehog soliton, and then they have quantized the bound state with an isospin rotation matrix.

From now on, we review their approach. To do that, we first substitute their ansatz Eq (3.6) for the SU(3) Skyrme action Eq. (3.1) and then expand $U_K$ up to second order of the kaon field, $K$. After some calculations, we obtain,

$$L = L_{Skyrme} + L_{KH}, \tag{3.9}$$

where $L_{Skyrme}$ is the classical Skyrme Lagrangian and $L_{KH}$ is the kaon-Hedgehog interaction Lagrangian given by,

$$\begin{aligned} L_{KH} &= (D_\mu K)^\dagger D^\mu K - K^\dagger a_\mu^\dagger a^\mu K - m_K^2 K^\dagger K \\ &+\frac{1}{(eF_\pi)^2}\left\{-\frac{1}{8}K^\dagger K \mathrm{tr}\left[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger\right]^2 - 2\left(D_\mu K\right)^\dagger D_\nu K \mathrm{tr}\left(a^\mu a^\nu\right)\right. \\ &\left. \qquad -\frac{1}{2}\left(D_\mu K\right)^\dagger D^\mu K \mathrm{tr}\left(\partial_\nu U^\dagger \partial^\nu U\right) + 6\left(D_\nu K\right)^\dagger \left[a^\nu, a^\mu\right] D_\mu K\right\} \\ &+\frac{3i}{F_\pi^2} B^\mu \left[\left(D_\mu K\right)^\dagger K - K^\dagger \left(D_\mu K\right)\right]. \end{aligned} \tag{3.10}$$

In Eq. (3.10), $m_K$ is the mass of the kaon, the covariant derivative $D_\mu$ is defined as

$$D_\mu = \partial_\mu + v_\mu, \tag{3.11}$$

and the vector and axial vector currents are

$$v_\mu = \frac{1}{2}\left(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger\right), \tag{3.12}$$

$$a_\mu = \frac{1}{2}\left(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger\right). \tag{3.13}$$

Finally, the last term of Eq. (3.10) is derived from the Wess-Zumino action with the baryon number current $B^\mu$ [38],

$$B^\mu = -\frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}\left[\left(U^\dagger \partial_\nu U\right)\left(U^\dagger \partial_\alpha U\right)\left(U^\dagger \partial_\beta U\right)\right]. \tag{3.14}$$

Next, we consider the stationary energy eigen state of the kaon,

$$K(\boldsymbol{r}, t) = K(\boldsymbol{r}) e^{-iEt}, \tag{3.15}$$

where $E$ is the total mass of the kaon which includes its rest mass. If there exists a solution with $E < m_K$, it corresponds to the kaon-Hedgehog bound state. For the spatial part of the kaon field, we expand it into partial waves. However, due to the Hedgehog symmetry as we mentioned in Sec. 2.2, we do not use the spherical harmonics but the vector harmonics, $\mathcal{Y}_{TLT_z}(\Omega)$,

$$K(\boldsymbol{r}) = \sum_{T,L,T_z} C_{TLT_z}\mathcal{Y}_{TLT_z}(\Omega)k_{TL}(r), \tag{3.16}$$

where $T$ and $L$ are the grand-spin and angular momentum of the kaon, respectively, and $T_z$ is the third component of $T$. Using the spherical harmonics, $Y_{Lm}(\Omega)$, the vector harmonics is written as,

$$\mathcal{Y}_{TLT_z}(\theta,\phi) = \sum_m (L\ m\ 1/2\ T_z - m|T\ T_z)Y_{Lm}(\Omega)\,\chi_{T_z-m}, \tag{3.17}$$

where $\chi_{T_z-m}$ $(T_z - m = \pm 1/2)$ is a two component isospinor,

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.18}$$

Substituting Eqs. (3.15) and (3.16), we obtain,

$$-L_{KH} = h(r)k^{*}_{TL}{}'k'_{TL} + k^{*}_{TL}k_{TL}\left(m_K^2 - f(r)E^2 + V_{eff}(r)\right), \tag{3.19}$$

where,

$$f(r) = 1 + \frac{1}{(eF_\pi)^2}\left\{2s(r) + d(r)\right\}, \quad h(r) = 1 + \frac{2}{(eF_\pi)^2}s(r), \tag{3.20}$$

$$d(r) = \left(\frac{dF(r)}{dr}\right)^2 = (F')^2, \quad s(r) = \left(\sin^2 F/r\right)^2, \quad c(r) = \sin^2\frac{F}{2} \tag{3.21}$$

$$\begin{aligned} V_{eff}(r) \quad = \quad & -\frac{1}{4}(d+2s) - \frac{2s}{(eF_\pi)^2}(s+2d) + \frac{l(l+1) + 2c^2 + 4c\boldsymbol{I}\cdot\boldsymbol{L}}{r^2} \\ & + \frac{1}{(eF_\pi)^2}\frac{d+s}{r^2}\left(l(l+1) + 2c^2 + 4c\boldsymbol{I}\cdot\boldsymbol{L}\right) \\ & + \frac{1}{(eF_\pi)^2}\frac{6}{r^2}\left\{s\left(c^2 + 2c\boldsymbol{I}\cdot\boldsymbol{L} - \boldsymbol{I}\cdot\boldsymbol{L}\right) + \frac{d}{dr}\left((c + \boldsymbol{I}\cdot\boldsymbol{L})F'\sin F\right)\right\} \\ & + \frac{N_c E}{\pi^2 F_\pi^2}\frac{\sin^2 F}{r^2}\frac{dF}{dr}, \end{aligned} \tag{3.22}$$

and $\boldsymbol{I} = \boldsymbol{\tau}/2$ is the isospin operator of the kaon. We have performed the angular integral in this calculation. In the above equations, the terms with the coefficient $1/(eF_\pi)^2$ come from the Skyrme term and the last term in the kaon-Hedgehog interaction one Eq. (3.22) comes from the Wess-Zumino action.

We can easily obtain the equation of motion for each partial wave from the Lagrangian Eq. (3.19) via the variational principle,

$$-\frac{1}{r^2}\frac{d}{dr}\left(h(r)r^2\frac{dk}{dr}\right) - E^2 f(r)k + \left(m_K^2 + V_{eff}(r)\right)k = 0. \tag{3.23}$$

The first term corresponds to the kinetic energy of the kaon, the second is related to the eigen energy, the third and last terms are, respectively, the mass and interaction ones.

The equation of motion Eq. (3.23) looks like the Klein-Gordon (K.G.) equation. However, due to the existence of the radial-dependent function, $h(r)$ and $f(r)$, in the first and second terms, Eq. (3.23) is slightly different from the ordinary K.G. equation. This is because, in the CK approach, the kaon is moving in the background field of the hedgehog configuration. Therefore, if the kaon is far away from the Hedgehog soliton ($r \to \infty$), the equation of motion Eq. (3.23) reduces to the ordinary K.G. equation, which is the same situation as our approach which we will discuss in the next section.

To solve the equation of motion for the bound state, we investigate the short-range behavior of the wave function. To do that, we consider the profile function at short distances [38]

$$F(r \sim 0) = \pi - ar. \tag{3.24}$$

Using Eq. (3.24), the equation of motion at short distances reduces to,

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dk}{dr}\right) + \frac{l_{eff}(l_{eff}+1)}{r^2}k = 0, \tag{3.25}$$

where we define an effective angular momentum $l_{eff}$ as

$$l_{eff}(l_{eff}+1) = l(l+1) + 4\boldsymbol{I}\cdot\boldsymbol{L} + 2. \tag{3.26}$$

From Eq. (3.25), we find that there exists a repulsive centrifugal force by the effective angular momentum, $l_{eff}$, in this system. The boundary condition for the bound state wave function is given by,

$$k(r \sim 0) \propto r^{l_{eff}} \tag{3.27}$$

Table 3.1: values of the effective angular momentum $l_{eff}$ for various kaon partial waves.

| $l$ | $T = I + L$ | $\boldsymbol{I}\cdot\boldsymbol{L}$ | $l_{eff}(l_{eff}+1)$ | $l_{eff}$ |
|---|---|---|---|---|
| 0 | 1/2 | 0 | 2 | 1 |
| 1 | 3/2 | 1/2 | 6 | 2 |
| 1 | 1/2 | $-1$ | 0 | 0 |
| 2 | 5/2 | 1 | 12 | 3 |
| 2 | 3/2 | $-3/2$ | 2 | 1 |
| 3 | 7/2 | 3/2 | 20 | 4 |
| 3 | 5/2 | $-2$ | 6 | 2 |

In Tab. 3.1, we show the explicit values of $l_{eff}$. Interestingly, the lowest mode dose not appear in the s-wave ($l = 0$) but in the p-wave ($l = 1$) in this system. This is one of the nontrivial features in the bound state approach.

Now, we numerically solve the equation of motion Eq. (3.23) for the low-lying bound states ($l = 1, L_{eff} = 0$ and $l = 0, L_{eff} = 1$). We show the normalized radial wave functions and the bound state properties for the s- and p-waves kaon-Hedgehog system in Fig. 3.1 and Tab. 3.2. For the numerical calculation, we use the same parameter set as the Callan and Klebanov did: $F_\pi = 129$ MeV, $e = 5.45$, and the mass of the kaon, $m_K = 495$ MeV [20, 21].

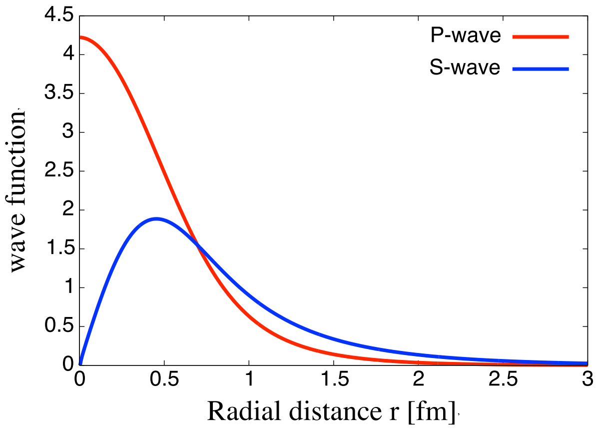


Figure 3.1: Normalized radial wave functions for the s- and p-waves kaon with $F_\pi = 129$ MeV and $e = 5.45$. They are in units of $1/\mathrm{fm}^{3/2}$.

Table 3.2: Bound state properties in the CK approach.

| $l$ | $l_{eff}$ | B.E. [MeV] | $\langle r_K^2 \rangle^{1/2}$ | Physical state |
|---|---|---|---|---|
| 0 | 1 | 67.7 | 1.04 | $\Lambda\,(1405)$ |
| 1 | 0 | 218.3 | 0.67 | $\Lambda\,(1116)$ |

In Fig. 3.1, we use the following normalization condition [1] for both wave functions,

$$\int d^3x \left|Y_0^0 k\,(r)\right|^2 = \int drr^2 \left|k\,(r)\right|^2 = 1. \tag{3.28}$$

From Fig. 3.1, we can see the s-wave kaon behaves as a p-wave while the p-wave one does as an s-wave. In Tab. 3.2, we show the binding energy (B.E.), the root

[1] In chapter 4, we will introduce another normalization, which is originally introduced by Callan and Klebanov. However, the bound state properties discussed in this chapter do not depend on the normalization scheme.

mean square radius, and corresponding physical state for each s- and p-wave kaon[2]. The p-wave bound state corresponds to the ground state $\Lambda$ and the s-wave one to the $\Lambda(1405)$ in the approach. We find that the kaon radii, $\langle r_K^2 \rangle^{1/2}$, are about 0.7 fm for the p-wave and 1.0 fm for the s-wave in the CK approach. These small radii seem to be consistent with their interpretation of the kaon-hedgehog systems as a strong binding system. We will discuss their result again with comparing with our approach in Sec. 3.6.

### 3.2.3 Kaon-Hedgehog potential

Next, we consider the kaon-Hedgehog interaction in the CK approach. We derive it from the equation of motion Eq. (3.23). However, in Eq. (3.23), the interaction is defined in the K.G.-like equation. Therefore, it carries dimension MeV$^2$. To derive the potential in units of MeV, we rewrite Eq. (3.23) into the Schödinger-like equation with the potential, $U(r)$, in units of MeV,

$$-\frac{1}{m_K+E}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dk_{TL}(r)}{dr}\right)+U(r)\,k_{TL}(r)=\varepsilon k_{TL}(r)\,, \tag{3.29}$$

and

$$\begin{aligned} U(r) &= -\frac{1}{m_K+E}\left[\frac{h(r)-1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{dh(r)}{dr}\frac{d}{dr}\right] \\ &\quad -\frac{(f(r)-1)\,E^2}{m_K+E}+\frac{V_{eff}(r)}{m_K+E}, \end{aligned} \tag{3.30}$$

where $h(r)$, $f(r)$, and $V_{eff}(r)$ are given in the previous subsection, and $\varepsilon$ is defined by

$$E=m_K+\varepsilon. \tag{3.31}$$

As we have shown in Eq. (3.30), the obtained potential $U(r)$ is nonlocal due to the spatial derivative operator. Therefore, we define the following equivalent quantitiy with the kaon partial wave function,

$$\tilde{U}(r)=\frac{U(r)\,k_{TL}(r)}{k_{TL}(r)}. \tag{3.32}$$

[2]We have performed numerical calculations for the radius in the CK approach.

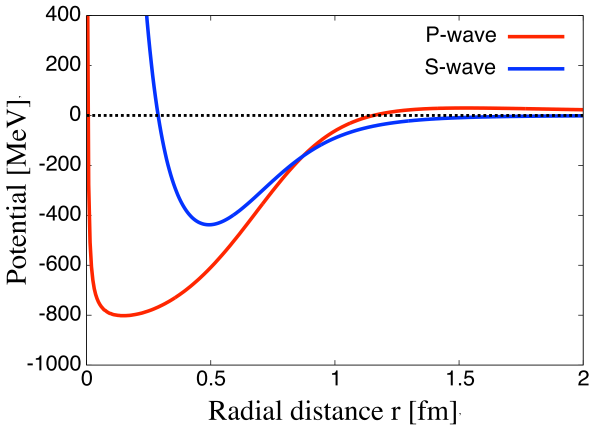


Figure 3.2: Kaon-Hedgehog potentials for the s- and p-wave kaon-Hedgehog bound states.

In Fig. 3.2, we plot the potentials Eq. (3.32) for the s-wave and p-wave kaon-Hedgehog bound states. For the s-wave, we see a repulsive component at short distances. This comes from the centrifugal-like component due to the effective angular momentum, $l_{eff}$. On the other hand, for the p-wave, the potential has a strong attraction near the origin which makes the strong bound state in the p-wave bound state. However, near the origin, the potential goes to infinity. Analytically, the p-wave potential has no centrifugal component because the effective angular momentum is zero as we shown in Tab. 3.1. This is just a numerical problem for solving the bound state wave function for small $r$.

### 3.2.4 Collective quantization as hyperons

Finally, we briefly explain how the hyperons are generated in the CK approach. The kaon-Hedgehog system is in the zeroth order of $N_c$ and it is a classical system. To obtain the quantum numbers, we must introduce the $1/N_c$ contributions. To do that, we quantize the system by the collective quantization scheme,

$$U_{CK} \to A\left(t\right) \xi\left(\boldsymbol{r}\right) U_K\left(\boldsymbol{r}, t\right) \xi\left(\boldsymbol{r}\right) A^{\dagger}\left(t\right), \tag{3.33}$$

which corresponds to,

$$\begin{cases} \xi\left(\boldsymbol{r}\right) \to \tilde{\xi}\left(\boldsymbol{r}, t\right) \equiv A\left(t\right) \xi\left(\boldsymbol{r}\right) A^{\dagger}\left(t\right) \\ K\left(\boldsymbol{r}, t\right) \to \tilde{K}\left(\boldsymbol{r}, t\right) \equiv A\left(t\right) K\left(\boldsymbol{r}, t\right), \end{cases} \tag{3.34}$$

where $A\left(t\right)$ is a time-dependent SU(2) isospin rotation matrix. In Eq. (3.34), $\tilde{K}$ is the kaon field observed in the body-fixed frame of the Hedgehog soliton, whereas $K$ is the one in the laboratory frame of the soliton.

After quantization, a nontrivial phenomena occurs in their approach [20, 21]: due to the background field of the Hedgehog soliton, the kaon quantum number transmutates. To show it explicitly, we consider an isospin and a spatial rotations of the rotating system.

- Isospin rotation

  When $D(\theta)$ is an isospin rotation matrix, $\tilde{\xi}$ and $\tilde{K}$ transform as,

$$\begin{cases} \tilde{\xi}(\boldsymbol{r},t) \to D(\theta)\,A(t)\,\xi(\boldsymbol{r})\,A^{\dagger}(t)\,D^{\dagger}(\theta) \\ \tilde{K}(\boldsymbol{r},t) \to D(\theta)\,A(t)\,K(\boldsymbol{r},t)\,, \end{cases} \tag{3.35}$$

  which means,

$$\begin{cases} A(t) \to D(\theta)\,A(t) \\ K(\boldsymbol{r},t) \to K(\boldsymbol{r},t)\,. \end{cases} \tag{3.36}$$

  From Eq. (3.36), we find that the isospin rotator, $A(t)$, change after the isospin rotation while the kaon field in the Laboratory frame, $K(\boldsymbol{r},t)$, does not change after rotaion. This means that the isospin rotator carries the isospin of the kaon field in the Laboratory frame in the CK approach. In other words, $K(\boldsymbol{r},t)$ dose not have the isospin quantum number after collective quantization.

- Spatial rotation

  Under the spatial rotation, the tilded variables transform as,

$$\begin{cases} \tilde{\xi}(\boldsymbol{r},t) \to A(t)\,R(\phi)\,\xi(\boldsymbol{r})\,R^{\dagger}(\phi)\,A^{\dagger}(t) \\ \tilde{K}(\boldsymbol{r},t) \to A(t)\,R(\phi)\,K(\boldsymbol{r},t)\,, \end{cases} \tag{3.37}$$

  where $R(\phi)$ is a spatial rotation matrix,

$$R(\phi) = e^{i\boldsymbol{\phi}\cdot\boldsymbol{J}}. \tag{3.38}$$

  Furthermore using the Hedgehog symmetry mentioned in Sec. 2.2, we can write,

$$\begin{cases} \tilde{\xi}(\boldsymbol{r},t) \to A(t)\,e^{i\boldsymbol{\phi}\cdot\boldsymbol{J}}\xi(\boldsymbol{r})\,e^{-i\boldsymbol{\phi}\cdot\boldsymbol{J}}A^{\dagger}(t) = A(t)\,e^{-i\boldsymbol{\phi}\cdot\boldsymbol{I}}\xi(\boldsymbol{r})\,e^{i\boldsymbol{\phi}\cdot\boldsymbol{I}}A^{\dagger}(t) \\ \tilde{K}(\boldsymbol{r},t) \to A(t)\,e^{i\boldsymbol{\phi}\cdot\boldsymbol{J}}K(\boldsymbol{r},t) = A(t)\,e^{-i\boldsymbol{\phi}\cdot\boldsymbol{I}}e^{i\boldsymbol{\phi}\cdot\boldsymbol{T}}K(\boldsymbol{r},t)\,. \end{cases} \tag{3.39}$$

  The above equations mean that under spatial rotation $A(t)$ and $K(\boldsymbol{r},t)$ transform as,

$$\begin{cases} A(t) \to A(t)\,e^{-i\boldsymbol{\phi}\cdot\boldsymbol{I}} \\ K(\boldsymbol{r},t) \to e^{i\boldsymbol{\phi}\cdot\boldsymbol{T}}K(\boldsymbol{r},t)\,, \end{cases} \tag{3.40}$$

which shows that the spatial rotation operator acting on the kaon field, $K(\boldsymbol{r}, t)$, is $e^{i\boldsymbol{\phi}\cdot\boldsymbol{T}}$, which means that the spin operator of the kaon field is the grand spin, $\boldsymbol{T}$.

Let us summarize above discussion. After collective quantization, the kaon has the isospin ($I$) and spin quantum ($J$) numbers, $(I, J) = (0, T)$. From Tab. 3.1, we find that the kaon has $(I, J) = (0, 1/2)$ for the lowest kaon-Hedgehog bound state. This quantum number is nothing but that of an s-quark. On the other hand, the Hedgehog soliton is quantized as a $I = J$ state. To generate the hyperons, we integrally quantize the Hedgehog soliton ($I = J = 0, 1, 2, \cdots$).

As a result, the anti-kaon ($s\bar{u}$ or $s\bar{d}$) behaves as an s-quark while the kaon ( $\bar{s}u$ or $\bar{s}d$) does as an $\bar{s}$-quark. At the same time, the Hedgehog soliton is not quantized as a nucleon but a di-quark. Therefore, we obtain the hyperons as a bound state of the s-quark and di-quark.

## 3.3 Alternative ansatz for kaon-nucleon systems

In the previous section, we have introduced the CK ansatz for the hyperons but it is not suited to the kaon-nucleon systems. Therefore, we have constructed an alternative ansatz ($U_{EH}$) for the physical kaon-nucleon systems [27, 28]

$$U_{EH} = A(t)\sqrt{N_H}A^\dagger(t)\,U_K A(t)\sqrt{N_H}A^\dagger(t)\,. \tag{3.41}$$

where $N_H$ is and $U_K$ is defined by Eq. (3.7) and Eq. (3.8) in the previous section, respectively, and $A(t)$ is a time-dependent isospin rotation matrix. This ansatz is constructed as follows: we first generate the nucleon by rotating the Hedgehog soliton and then we introduce the kaon fluctuation around the nucleon.

At a single glance, our ansatz is very similar to the Callan-Klebanov one as shown below.

- Callan-Klebanov (CK) ansatz

$$U_{CK} = \sqrt{N_H}U_K\sqrt{N_H} \rightarrow A(t)\sqrt{N_H}U_K\sqrt{N_H}A^\dagger(t) \tag{3.42}$$

- Ezoe-Hosaka (EH) ansatz

$$U_{EH} = A(t)\sqrt{N_H}A^\dagger(t)\,U_K A(t)\sqrt{N_H}A^\dagger(t) \tag{3.43}$$

The difference in them is where $A$ and $A^\dagger$ appear. In the CK approach, we rotate the hole system. On the other hand, we rotate only the hedgehog soliton in our approach. However we consider that this tiny difference makes a big difference in physics.

Here, we would like to summarize the difference in the CK and our ansatzes. In the CK ansatz, the kaon is introduced as the fluctuation around the Hedgehog soliton and then we quantize the kaon and Hedgehog soliton system by isospin rotation. Their quantization method Eq. (3.42) is based on the picture that the kaon is strongly bound to the hedgehog baryon. As a result, the hyperons are generated as a bound state of a strange quark and a di-quarks.

On the other hand, in our ansatz, the kaon is introduced as the fluctuation around the rotating Hedgehog soliton. What is important here is that the Hedgehog soliton is first rotated in our ansatz Eq. (3.41). As a result, we obtain the kaon and rotating Hedgehog soliton systems, which is eventually the kaon-nucleon systems. This is based on the picture that the kaon is weakly bound to the nucleon as expected to hadronic molecules. In short, we can explain the difference in the two approach as follows: the CK approach corresponds to the projection after variation whereas ours to the variation after projection in the many-body physics [47].

Furthermore, let us consider the difference in the two approaches from the point of the $1/N_c$ expansion. As we mentioned in the previous section, the CK approach follows the $1/N_c$ expansion scheme. We would like to explain again. The classical Hedgehog soliton is the leading order on $N_c$. The kaon fluctuation is introduced around the Hedgehog soliton, which is zeroth order of $N_c$. Finally, we quantized the kaon-Hedgehog system by the isospin rotation and the rotation energy is of order $1/N_c$.

Contrary, in the EH approach, we first generate the nucleon by rotating the Hedgehog in the isospin space and then we introduce the kaon fluctuation around the nucleon. Therefore, our approach violates the $1/N_c$ expansion.

## 3.4 Equation of motion

We have introduced the ansatz for the kaon-nucleon systems in the previous section. Now, let us derive the equation of motion for the kaon. To do that, we first substitute our ansatz Eq. (3.41) for the SU(3) Skyrme action Eq. (3.1) and then expand $U_K$ up to second order of the kaon field, $K$, which corresponds to the harmonic oscillator approximation. After hard exercise, we obtain,

$$L = L_{SU(2)} + L_{KN} \tag{3.44}$$

$$L_{SU(2)} = \frac{1}{16}F_\pi^2 \mathrm{tr}\left(\partial_\mu \tilde{U}_H \partial^\mu \tilde{U}_H^\dagger\right) + \frac{1}{32e^2}\mathrm{tr}\left[\left(\partial_\mu \tilde{U}_H\right)\tilde{U}_H^\dagger, \left(\partial_\nu \tilde{U} v\right)\tilde{U}_H^\dagger\right]^2 \tag{3.45}$$

$$
\begin{aligned}
L_{KN} &= (D_\mu K)^\dagger D^\mu K - K^\dagger a_\mu^\dagger a^\mu K - m_K^2 K^\dagger K \\
&+\frac{1}{(eF_\pi)^2}\left\{-\frac{1}{8}K^\dagger K \mathrm{tr}\left[\partial_\mu \tilde{U}\tilde{U}^\dagger, \partial_\nu \tilde{U}\tilde{U}^\dagger\right]^2 - 2\left(D_\mu K\right)^\dagger D_\nu K \mathrm{tr}\left(a^\mu a^\nu\right)\right. \\
&\qquad \left. -\frac{1}{2}\left(D_\mu K\right)^\dagger D^\mu K \mathrm{tr}\left(\partial_\nu \tilde{U}^\dagger \partial^\nu \tilde{U}\right) + 6\left(D_\nu K\right)^\dagger \left[a^\nu, a^\mu\right] D_\mu K\right\} \\
&+\frac{3i}{F_\pi^2}B^\mu \left[\left(D_\mu K\right)^\dagger K - K^\dagger \left(D_\mu K\right)\right],
\end{aligned}
\tag{3.46}
$$

where the first term Eq. (3.45) is the quantized SU(2) Skyrme Lagrangian which describes the properties of the nucleon and delta-particle [38] and the second one Eq. (3.46) is the kaon-nucleon effective Lagrangian.

In Eq. (3.46), the covariant derivative is defined as

$$
D_\mu = \partial_\mu + v_\mu, \tag{3.47}
$$

and the vector and axial vector currents are

$$
v_\mu = \frac{1}{2}\left(\tilde{\xi}^\dagger \partial_\mu \tilde{\xi} + \tilde{\xi}\partial_\mu \tilde{\xi}^\dagger\right), \tag{3.48}
$$

$$
a_\mu = \frac{1}{2}\left(\tilde{\xi}^\dagger \partial_\mu \tilde{\xi} - \tilde{\xi}\partial_\mu \tilde{\xi}^\dagger\right). \tag{3.49}
$$

Finally, the last term of Eq. (3.46) is derived from the Wess-Zumino term with the baryonic current $B^\mu$ [38],

$$
B^\mu = -\frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}\left[\left(\tilde{U}^\dagger \partial_\nu \tilde{U}\right)\left(\tilde{U}^\dagger \partial_\alpha \tilde{U}\right)\left(\tilde{U}^\dagger \partial_\beta \tilde{U}\right)\right]. \tag{3.50}
$$

In the above equations, the tilded quantities are the rotating Hedgehog soliton as our ansatz requires;

$$
\tilde{U} = A(t)\,\xi^2 A^\dagger(t), \qquad \tilde{\xi} = A(t)\,\xi A^\dagger(t). \tag{3.51}
$$

Next, we decompose the kaon field into the two part: isospin and spatial parts,

$$
K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = \psi_I K(\boldsymbol{r}, t) \to \psi_I K(\boldsymbol{r})\, e^{-iEt}, \tag{3.52}
$$

where $\psi_I$ is the two-component isospinor. Furthermore, we expand the spatial component $K(\boldsymbol{r})$ by the spherical harmonics,

$$
K(\boldsymbol{r}) = \sum_{lm\alpha} C_{lm\alpha} Y_l^m(\Omega)\, k_l^\alpha(r), \tag{3.53}
$$

where $l$ is the orbital angular momentum of the kaon, $m$ is the third component of $l$, $\alpha$ stands for other quantum numbers, and $\Omega$ for angles, $\theta$ and $\phi$.

Finally taking a variation with respect to the kaon wave function, we obtain the equation of motion for each partial wave, $k_l^\alpha(r)$.

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 h(r)\frac{dk_l^\alpha(r)}{dr}\right) - E^2 f(r)\, k_l^\alpha(r) + \left(m_K^2 + V(r)\right) k_l^\alpha(r) = 0, \quad (3.54)$$

where $E$ is the total energy of the kaon including its rest mass, $h(r)$, $f(r)$, and $V(r)$ are radial dependent functions,

$$h(r) = 1 + \frac{1}{(eF_\pi)^2}\frac{2}{r^2}\sin^2 F, \quad (3.55)$$

$$f(r) = 1 + \frac{1}{(eF_\pi)^2}\left(\frac{2}{r^2}\sin^2 F + F'^2\right), \quad (3.56)$$

$$V(r) = V_0^c(r) + V_\tau^c(r)\, I_{KN} + V_0^{LS}(r)\, J_{KN} + V_\tau^{LS}(r)\, J_{KN} I_{KN}, \quad (3.57)$$

and

$$I_{KN} = \boldsymbol{I}^K \cdot \boldsymbol{I}^N, \qquad J_{KN} = \boldsymbol{L}^K \cdot \boldsymbol{J}^N. \quad (3.58)$$

In Eq. (3.58), the nucleon spin and isospin operators, $\boldsymbol{J}^N$ and $\boldsymbol{I}^N$, are given by [43],

$$\boldsymbol{J}^N = i\Lambda \mathrm{tr}\left[\boldsymbol{\tau}\dot{A}^\dagger(t)\, A(t)\right], \quad (3.59)$$

$$\boldsymbol{I}^N = i\Lambda \mathrm{tr}\left[\boldsymbol{\tau}\dot{A}(t)\, A^\dagger(t)\right], \quad (3.60)$$

where $\dot{A}(t)$ is the time derivative of $A(t)$, $\boldsymbol{\tau}$ is the $2 \times 2$ Pauli matrices, and $\Lambda$ is the soliton moment of inertia which is given by [38]

$$\Lambda = \frac{2\pi}{3}F_\pi^2 \int dr\; r^2 \sin^2 F\left[1 + \frac{4}{(eF_\pi)^2}\left(F'^2 + \frac{\sin^2 F}{r^2}\right)\right]. \quad (3.61)$$

The kaon isospin operator, $\boldsymbol{I}^K$, is given by the $2 \times 2$ Pauli matrices

$$\boldsymbol{I}^K = \frac{\boldsymbol{\tau}}{2}. \quad (3.62)$$

Lastly, $\boldsymbol{L}^K$ in Eq. (3.58) is the orbital angular momentum operator for the kaon

$$\boldsymbol{L}^K = \boldsymbol{r} \times \boldsymbol{p}^K. \quad (3.63)$$

In the equation of motion Eq. (3.54), the first term corresponds to the kinetic term of the kaon, the second one is related to the eigen energy, $\varepsilon = E - m_K$, and the third and last terms are mass and kaon-nucleon interaction terms, respectively, whose explicit form is shown in Appendix C.

Using our ansatz Eq. (3.41), the resulting Lagrangian and interaction term, $V(r)$, in Eq. (3.54) contain the rotation matrix $A(t)$ in several places. This means that, in these equations, terms with different order of $1/N_c$ exist simultaneously, indicating the violation of $1/N_c$ expansion. This, however, is the feature of the present approach which we consider suited to the study of the physical kaon and nucleon interaction. We emphasize that the interaction Eq. (3.57) has four components; the isospin independent and dependent central forces, $V_0^c$ and $V_\tau^c$, respectively, and similarly for the spin-orbit (LS) forces $V_0^{LS}$ and $V_\tau^{LS}$, which completes the general structure of the potential between the isospinor-pseudoscalar kaon and isospinor-spinor nucleon.

## 3.5 Kaon-nucleon bound state

In this section, we consider the kaon-nucleon bound state. By solving the equation of motion Eq. (3.54) for various channels, we have found one bound state in $\bar{K}N\left(J^P = 1/2^-, I = 0\right)$ channel, where it is considered that the $\Lambda(1405)$ appears. In our approach, there are three parameters: the pion decay constant, $F_\pi$, the Skyrme parameter, $e$, and the mass of the kaon, $m_K$. We keep $m_K$ at 495 MeV and consider three parameter sets for $F_\pi$ and $e$ shown in Tab. 3.3.

Table 3.3: Parameter sets for numerical calculations.

| | $F_\pi$ [MeV] | $e$ |
|---|---|---|
| Set A | 205 | 4.67 |
| Set B | 186 | 4.82 |
| Set C | 129 | 5.45 |

These parameter sets reproduce the same moment of inertial, which means that they reproduce the experimental mass difference in the nucleon and $\Delta$. We would like to make more detailed explanation below.

1. Set A:

   We employ a slightly large pion decay constant (in our notation, the physical value of the pion decay constant is 186 MeV). This is motivated by the fact that the kaon decay constant, $F_K$, is larger than the pion one ($F_K = 221$ MeV) [46]. In this study, we consider the physical systems with the pion and kaon. Therefore, we choose the 10%-large value ($186 \times 1.1 = 205$ MeV) as the pion decay constant in order to effectively take into account the difference in the pion and kaon decay constants. The Skyrme parameter, $e$, is determined to reproduce the mass difference in $N$ and $\Delta$ with $F_\pi = 205$ MeV.

2. Set B:

   The Skyrme parameter $e$ is determined to reproduce the experimental mass difference between the nucleon and $\Delta$-particle when we take the pion decay constant at its physical value, 186 MeV.

3. Set C:

   This is proposed by Adkins, Nappi, and Witten [38], which reproduces the physical masses of the nucleon and $\Delta$.

### 3.5.1 Wave function and bound state properties

We first consider the bound state properties for each parameter sets. As Callan and Klebanov discussed [20, 21], the bound state properties are different in the kaon ($K$) and anti-kaon ($\bar{K}$) for the SU(3) Skyrme model. This is due to the Wess-Zumino term which physically corresponds to the $\omega$-meson exchange [48]: it is attractive for the $\bar{K}$ whereas repulsive for the $K$. Therefore, bound states exist for the $\bar{K}N$ systems and we have numerically confirm it.

Table 3.4: Parameter sets and binding energies for $\bar{K}N\,(I=0)$ channel.

| | $F_\pi$ [MeV] | $e$ | B.E. [MeV] |
|---|---|---|---|
| Set A | 205 | 4.67 | 20.0 |
| Set B | 186 | 4.82 | 31.4 |
| Set C | 129 | 5.45 | 79.5 |

Table 3.5: Parameter sets and binding energies for $\bar{K}N\,(I=1)$ channel.

| | $F_\pi$ [MeV] | $e$ | B.E. [MeV] |
|---|---|---|---|
| Set A | 205 | 4.67 | — |
| Set B | 186 | 4.82 | almost 0 |
| Set C | 129 | 5.45 | 33.0 |

We have numerically solved the equation of motion Eq. (3.54) for various $\bar{K}N$ channels with the three parameter sets discussed above. As a result, we have found one bound state in the $\bar{K}N\left(J^P=1/2^-\right)$ channel and we summarize the result in Tabs. 3.4 and 3.5. For the set A, one bound state exists in $J^P=1/2^-, I=0$ channel and the total mass of the $\bar{K}N$ bound state is close to the one of $\Lambda\,(1405)$

For the sets B and C, we have found one bound state in both $I=0$ and 1 channels. Let us consider the $I=0$ bound states for the set B and C. The binding energies are 31.4 MeV and 79.5 MeV for the set B and C, respectively. These bound states should be identified with $\Lambda\,(1405)$, but their binding energies are large, especially for the set C.

Contrary, for $I = 1$ channel, there is no bound state for the set A as we mentioned. For the sets B and C, one bound state exists. However the binding is very weak for the set B: its binding energy is almost zero (but finite). For the set C, the binding energy is 33 MeV. This bound state may be identified with a $\Sigma$-hyperon. There are several $\Sigma$ resonances but with weak significance [1] Considering the mass difference in the $\bar{K}N$ bound states with $I = 0$ and 1, the latter might be $\Sigma\,(1480)$.

From Tabs. 3.4 and 3.5, we can see that the binding energy becomes larger as the pion decay constant gets smaller. This is similar to what is expcted in the Weinberg-Tomozawa (WT) interaction [15, 16], which is proportional to $1/F_\pi^2$. We consider the leading contribution to the interaction obtained from our approach is the WT (type) interaction. It seems important to use the experimental value of $F_\pi$ in order to numerically reproduce the properties of the $\bar{K}N$ systems.

Now we study more detailed properties of the bound state. In Fig. 3.3, we show the normalized radial wave function for the bound state anti-kaon for three parameter sets. The normalization condition is as follows:

$$\int d^3x \left|Y_0^0 k\,(r)\right|^2 = \int drr^2 \left|k\,(r)\right|^2 = 1. \tag{3.64}$$

They are numerically obtained from the equation of motion Eq. (3.54).

From Fig. 3.3, we can find that the wave functions vanish at the origin even though we consider the s-wave kaon-nucleon bound state. This is because of the existence of the repulsive core which we will mention in the next subsection.

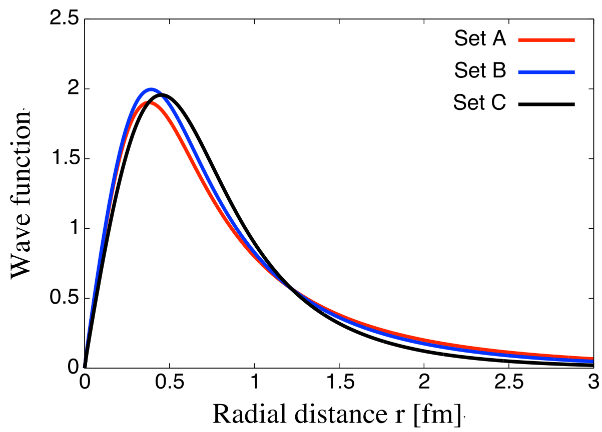


Figure 3.3: Normalized radial wave functions for the $\bar{K}N\left(J^P = 1/2^-, I = 0\right)$ for the parameter sets A, B and C. They are in units of $1/\mathrm{fm}^{3/2}$.

Furthermore, in Tab. 3.6, we show the parameter sets and bounding energies (B.E.), and root mean square radii for the baryon number distribution of the nucleon and kaon wave function. The former radius is defined by,

$$\langle r_N^2 \rangle = \int d^3 x r^2 B^0(r) = \int_0^\infty dr r^2 \rho_B(r), \tag{3.65}$$

where $B^0(r)$ and $\rho_B(r)$ are, respectively, the baryon charge and baryon charge density given by [38],

$$B^0(r) = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F' \tag{3.66}$$

and

$$\rho_B = 4\pi r^2 B^0(r) = -\frac{2}{\pi}\sin^2 F F'. \tag{3.67}$$

The latter is,

$$\langle r_K^2 \rangle = \int d^3 x r^2 \left| Y_0^0 k(r) \right|^2 = \int_0^\infty dr r^4 k^2(r), \quad Y_0^0 = \frac{1}{\sqrt{4\pi}}. \tag{3.68}$$

Table 3.6: Parameter sets and bound state properties.

| | $F_\pi$ [MeV] | $e$ | B.E. [MeV] | $\langle r_N^2 \rangle^{1/2}$ [fm] | $\langle r_K^2 \rangle^{1/2}$ [fm] |
|---|---|---|---|---|---|
| Set A | 205 | 4.67 | 20.0 | 0.44 | 1.34 |
| Set B | 186 | 4.82 | 31.4 | 0.46 | 1.17 |
| Set C | 129 | 5.45 | 79.5 | 0.59 | 0.99 |

From Tab. 3.6, we easily find that the baryon number radius, which corresponds to the nucleon core size, is about 0.5 fm, while the anti-kaon wave function extends up to 1 fm or more. This indicates that the $\bar{K}N$ bound state is a weakly binding object like as a hadronic molecule.

Looking at Tab. 3.6 a bit more carefully, we observe that as the pion decay constant increases, the baryon number radius decreases whereas the anti-kaon distribution increases. This comes from different reasons. The former comes from the parameter dependence of the Hedgehog soliton. As we have shown in Eqs. (3.65), (3.66), and (3.67), the baryon number radius is determined by the soliton profile. Naively thinking, the Hedgehog soliton is regarded as the nucleon in the Skyrme model. Therefore, as the Hedgehog soliton extends more, the nucleon size, that is the baryon number radius, becomes large. We numerically solve the equation of motion for the soliton profile, Eq. (2.9), for the three parameter sets and show their shapes in Fig. 3.4. Actually, the soliton profile function ex-

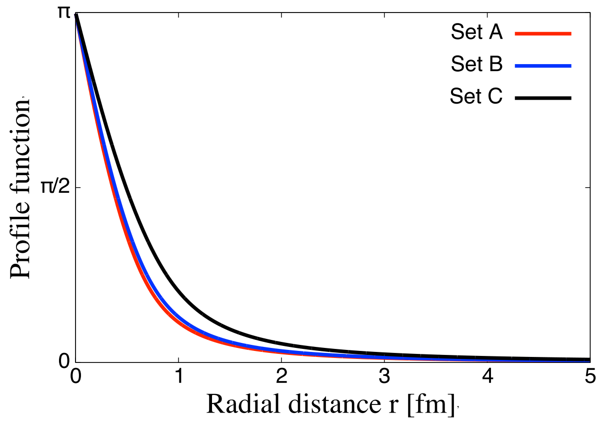


Figure 3.4: Profile functions for the three parameter sets A, B, and C.

tends in the order of set A, B, and C, which is consistent with the ordering of the baryon number radii in Tab. 3.6.

We can understand the parameter dependence of the profile function in terms of the standard unit introduced in Eq. (2.10). In the SU(2) Skyrme, there are two parameter, $F_\pi$ and $e$. They are scaled-out by the standard unit, where the radial distance is, we would like to show again, expressed by,

$$y = eF_\pi r. \tag{3.69}$$

By using this, soliton profiles for various parameter sets are related by a simple scale transformation to each other. The soliton profiles as functions of the physical radial distance $r$ for the three parameter sets A, B, and C, which are obtained from the standard profile function with the scaling rule Eq. (3.69). From Fig. 3.4, we find that the profile function for the set C is the most extended among the three parameter sets; soliton size is inversely proportional to $eF_\pi$.

On the other hand, the $F_\pi$-dependence of the anti-kaon distribution is explained by the bound state properties. The wave function less extends for the larger binding energy, which is seen in Tab. 3.6. We consider that this behavior comes from the WT interaction, which is inversely proportional to $F_\pi^2$ [15, 16].

### 3.5.2 $\bar{K}N$ potential

In this subsection, we discuss the kaon-nucleon potential. As we have shown in the equation of motion Eq. (3.54), the interaction term $V(r)$ is defined in the Klein-Gordon like equation. Therefore, it carries the dimension MeV$^2$. We consider that it is convenient to define a potential in units of MeV in order to discuss few-body systems. To do that, we first rewrite the equation of motion into the Schrödinger like equation, and define the potential $U(r)$ as follows, which is in units of MeV,

$$-\frac{1}{m_K+E}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dk_l^\alpha(r)}{dr}\right)+U(r)\,k_l^\alpha(r)=\varepsilon k_l^\alpha(r)\,, \tag{3.70}$$

and

$$\begin{aligned} U(r) &= -\frac{1}{m_K+E}\left[\frac{h(r)-1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{dh(r)}{dr}\frac{d}{dr}\right] \\ &\quad -\frac{(f(r)-1)\,E^2}{m_K+E}+\frac{V(r)}{m_K+E}, \end{aligned} \tag{3.71}$$

where $E$ is the total energy of the kaon, which is introduced in Sec. 3.4, and $\varepsilon$ is introduced in Eq. (3.31),

$$\varepsilon = E - m_K. \tag{3.72}$$

The $r$-dependent functions, $h(r)$, $f(r)$, and $V(r)$, are, respectively, given by Eqs. (3.55), (3.56), and (3.57).

The obtained potential $U(r)$ has several properties: First, it is nonlocal and depends on the energy of the kaon. The nonlocality comes form the spatial derivative operators. Second, it contains four components, isospin dependent and independent central forces, and the similar spin-orbit forces as we mentioned in Sec. 3.4. Finally, at short distances, this potential behaves as an attractive or a repulsive force proportional to $1/r^2$ depending on the total isospin and total spin. Since the potential contains the spatial derivative operators, we define the following quantity as we have done in the Sec. 3.2.3, what we call equivalent local potential,

$$\tilde{U}(r)=\frac{U(r)\,k_l^\alpha(r)}{k_l^\alpha(r)}. \tag{3.73}$$

In this study, we have numerically calculated it with the partial wave function, especially with the bound state wave function. Therefore, exactly speaking, the potential derived here is for the $\bar{K}N\left(I=0, J^P=1/2^-\right)$ bound state. However, we can in principle derive the potentials for all states.

The resulting potential is shown in Fig. 3.5 for the three parameter sets A, B, and C. From this figure, we find that the minimum of the potential moves from inside to outside in the ordering of A, B, and C. This property is explained by the parameter dependence of the profile function shown in Fig. 3.4 and is consistent with the shapes of the kaon wave functions explained in Sec. 3.5.1.

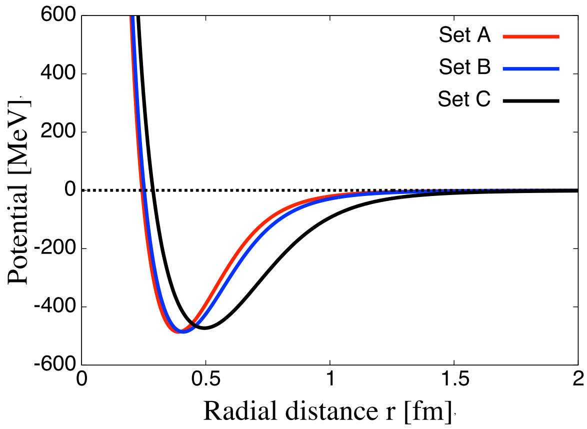


Figure 3.5: The equivalent local potential $\tilde{U}\,(r)$ for the $\bar{K}N$ bound state.

In Fig. 3.5, we find an attractive pocket in the middle range. This pocket is dominantly made by the Wess-Zumino action, which physically corresponds to the $\omega$-meson exchange. At short distances, we see a repulsive component. We will discuss it in the next section.

## 3.6 Comparison with the CK approach

In this section we compare our results with those of the Callan-Klebanov. In our approach, the lowest bound state exist in the s-wave kaon-nucleon channel. On the other hand, in the CK approach, it does in the p-wave kaon-Hedgehog channel. We summarize several results for the set C in Tab. 3.7.

Table 3.7: Comparisons between the EH and CK approaches

| Ezoe-Hosaka approach | | | Callan-Klebanov approach | | | | Physical state |
|---|---|---|---|---|---|---|---|
| $l$ | B.E. [MeV] | $\langle r_K^2\rangle^{1/2}$ [fm] | $l$ | $l_{eff}$ | B.E. [MeV] | $\langle r_K^2\rangle^{1/2}$ [fm] | |
| 0 | 79.5 | 0.99 | 0 | 1 | 67.7 | 1.05 | $\Lambda\,(1405)$ |
| — | — | — | 1 | 0 | 218.3 | 0.67 | $\Lambda\,(1116)$ |

From Tab. 3.7, we find that the kaon radius, $\langle r_K^2\rangle^{1/2}$, is about 0.7 fm for the lowest bound state (the p-wave one) in the CK approach, which is slightly smaller than that of our present approach. This result for the small radius seems consistent with their interpretation of the kaon hedgehog system as the strange quark and di-quark system for hyperons.

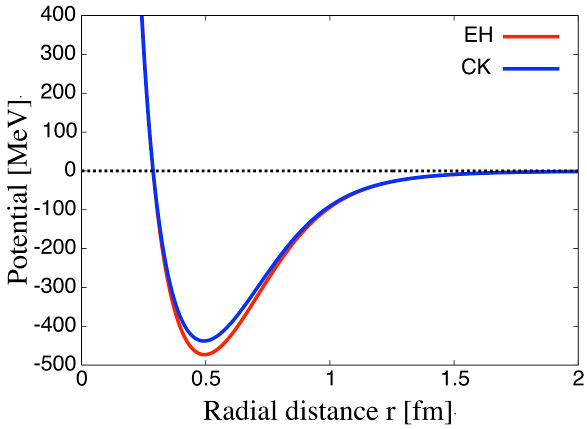


Figure 3.6: The s-wave $\bar{K}N$ potential in the EH approach and s-wave kaon-Hedgehog potential in the CK one.

In Fig. 3.6, we show the potentials for the s-wave bound states for the EH and CK approaches, which correspond to the interaction in the $\Lambda\,(1405)$. However, its structure is different in the EH and CK approaches: in the EH approach, the $\Lambda\,(1405)$ is a weekly binding object of the anti-kaon and nucleon, while, in the CK approach, it is a first excitation state of the ground state $\Lambda$. From Fig. 3.6, we find that these potentials look very similar. There is an attractive pocket in the middle range and is a repulsive core at short distances. In the CK approach, the repulsive core comes from the effective angular momentum, $l_{eff}$ as we mentioned in Sec. 3.2.3. We consider from this fact that the presence of the centrifugal-like potential in the CK approach is related to the exsistence of the repulsive core in our approach.

## 3.7 Kaon-nucleon scattering states

In the end of this chapter, we consider the kaon-nucleon scattering state [29]. As we shown in Eq. (3.54), we have already obtained the equation of motion. Therefore, it is possible to discuss any kaon-nucleon channels. However, we would like to concentrate on the lowest spin-parity channels in this study. We have calculated s-wave kaon-nucleon scattering states for the all parameter sets. However, for the realistic situations of the kaon-nucleon systems, it turns out that the use of the physical pion decay constant is important. Therefore, we will show the results for the sets A and B in the following discussion.

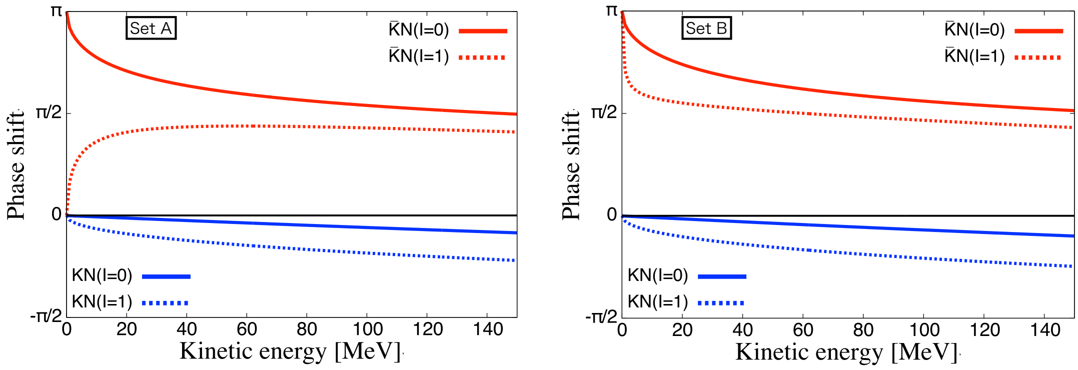


Figure 3.7: Phase shifts for the kaon-nucleon scattering states with $J^P = 1/2^-$ for the parameter sets A (left) and B (right)

First, we show the scattering phase for the various s-wave channels as functions of the kinetic energy, $\varepsilon$, defined in Eq. (3.72) in Fig. 3.7. For the set A (left panel), the phase shift of $\bar{K}N\,(I=0)$ channel starts from $\pi$ at zero kinetic energy due to the Levinson's theorem [49]. For the $I=1$ channel, the phase shift indicates that the interaction in attractive but it is not enough strong to form a bound state. On the other hand, for the $KN$ channels, the phase shifts show that the interaction is repulsive but that for the $I=1$ channel is more strong.

For the set B, the $\bar{K}N$ bound state exists in both $I=0$ and $I=1$ channels as we discussed in Sec. 3.5.1. However, for the $I=1$ channel, the bound is very shallow, which is consistent with the phase shift in Fig. 3.7. The $I=1$ bound state disappears by using the slightly larger pion decay constant. It may be meaningful to fine-tune the parameter set, but we will not do because our approach contains only kaon-nucleon channels. Physically, the $\pi\Sigma$ channels are very important to make further discussion of the kaon-nucleon system. In Chapter. 4, we will include the $\pi\Sigma$ channel as a extension of this approach.

From Fig. 3.7, we find that the strength of the attraction in the $\bar{K}N$ channels and the repulsion in the $KN$ channels for the set A are weaker than that for the set B. This reflects that the obtained potential is approximately proportional to $1/F_\pi^2$ as the WT thorem [15, 16].

To complete the discussion up to here, we consider the phase shift for the $\bar{K}N\,(I=0)$ channel for the all parameter sets. In Fig. 3.8, the phase shift indicates that the attraction between the anti-kaon and nucleon becomes weaker in the order of C, B, and A, which is consistent with the bound state properties shown in Tab. 3.6.

Finally, let us consider the scattering length, $a$, for the $\bar{K}N\left(J^P = 1/2^-\right)$ scattering state, which is defined by,

$$a = -\lim_{k\to 0}\frac{\tan\delta\left(k\right)}{k}, \tag{3.74}$$

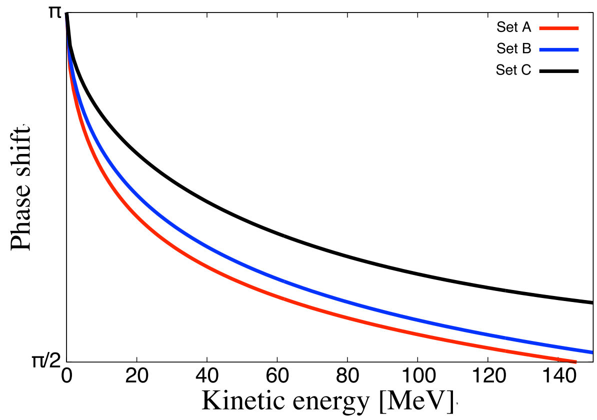


Figure 3.8: Phase shift for the $\bar{K}N\left(J^P = 1/2^-, I = 0\right)$ bound state for the three parameter sets, A, B, and C.

where $k$ is the wave number and $\delta(k)$ is the phase shift. Using this equation, we have derived the scattering lengths for the isospin 0 and 1 channel which denoted $a_{I=0}$ and $a_{I=1}$, respectively. The results are summarized in Tab. 3.8.

Table 3.8: Scattering length for the $\bar{K}N\left(J^P = 1/2^-\right)$ scattering state.

| | Set A | Set B |
|---|---|---|
| $a_{I=0}$ | 1.58 fm | 1.32 fm |
| $a_{I=1}$ | -3.41 fm | 8.22 fm |

From this table, we find that $a_{I=0}$ for the set A is longer than that for the set B. This is because the $\bar{K}N$ attraction for the set B is stronger than for the set A. For the isospin 1 case, the situation is the same as the isospin 0 case. As we summarized in Tab. 3.5, there is no bound state in $\bar{K}N\left(J^P = 1/2^-, I = 1\right)$ channel for the set A, whereas one bound state with a few binding energy for the set B. Therefore, the sign of $a_{I=1}$ is different in the set A and B, and the absolute value of $a_{I=1}$ is smaller for the set A then for the set B.

All scattering lengths summarized in Tab. 3.8 are real. However, those obtained from theoretical calculations and experiments are complex, for example [50, 51, 13]. The reason is that we do not consider the coupled channel effect, which is important for more realistic discussion. Because of this, in the present study, we will not make further quantitative discussions here.

We have also shown the scattering lengths for the $KN$ channels in Tab. 3.9. All scattering lengths shown in this table indicate that the $KN$ interaction is weakly repulsive, which is consistent with the properties of the $KN$ interaction shown in

Fig. 3.7. We find that the scattering lengths for the set B is slightly longer than those for the set A. The reason is the same as the case of $\bar{K}N$ scattering length.

Table 3.9: Scattering length for the $KN$ $\left(J^P = 1/2^-\right)$ scattering state.

| | Set A | Set B |
|---|---|---|
| $a_{I=0}$ | 0.028 fm | 0.029 fm |
| $a_{I=1}$ | 0.46 fm | 0.53 fm |

The scattering lengths shown in Tab. 3.9 are real because it is considered that the $KN$ channel does not couple to the other channel. Actually, the empirical $KN$ scattering lengths are from 0.03 fm to 0.1 fm for the $I = 0$ channel and about 0.3 fm for the $I = 1$ channel [52, 53, 54]. We consider that our result is in good agreement with these results.

# Chapter 4

# $\Lambda(1405)$ as a $\bar{K}N$ Feshbach resonance

In the previous chapter, we have shown that there exists one bound state in $\bar{K}N\left(J^P = 1/2^-, I = 0\right)$ channel with the binding energy of order ten MeV. Following the result, we extend our approach to the $\Lambda(1405)$ resonance. As we have mentioned, the $\Lambda(1405)$ is considered to be a resonance state of the $\bar{K}N$ and $\pi\Sigma$ channels. Therefore, we need to solve a coupled channel equation to discuss the $\Lambda(1405)$. However, in this study, we regard the $\Lambda(1405)$ as a $\bar{K}N$ Feshbach resonance and investigate it in the Skyrme model. In this chapter, we will first derive a formula for the decay width of the $\Lambda(1405)$. Next, we explain the formalism in the Skyrme model and then we show numerical results.

## 4.1 Formula for the decay width

In this section, we consider the $\bar{K}N$ Feshbach resonance from the point of view of an effective field theory. Let us start with the following effective Lagrangian,

$$\mathcal{L}_{eff} = g\bar{\psi}^a_\Sigma \pi^a \psi_{\Lambda^*} + (h.c.)\,, \tag{4.1}$$

where $g$ is a dimensionless coupling constant for the $\Lambda(1405)$-$\pi\Sigma$ vertex, $a$ is the isospin indices, $a = 1, 2, 3$, and $(h.c.)$ stands for the Hermitian conjugate of the first term. The field operators for $\Sigma$, $\pi$, and $\Lambda(1405)$ stand for $\bar{\psi}_\Sigma$, $\pi$, and $\psi_{\Lambda^*}$, respectively. For our convenience, we abbreviate $\Lambda(1405)$ to $\Lambda^*$ in the Lagrangian. The Feynman diagram for the $\Lambda(1405) \to \pi\Sigma$ process is shown in Fig. 4.1.

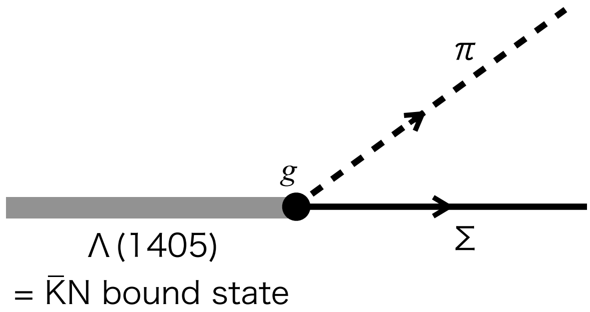


Figure 4.1: The Feynman diagram for the $\bar{K}N$ Feshbach resonance.

Using the Lagrangian Eq. (4.1), we first consider the following matrix element with the initial $\Lambda\,(1405)$ and final $\pi\Sigma$ states,

$$\begin{aligned}\langle\pi\Sigma|\,\mathcal{L}_{eff}\,|\Lambda\,(1405)\rangle &= \langle\pi\Sigma|\,g\bar{\psi}^a_\Sigma\pi^a\psi_{\Lambda^*}\,|\Lambda\,(1405)\rangle \\ &= g\,\langle\pi|\,\pi^a\,|0\rangle\,\langle\Sigma|\,\bar{\psi}^a_\Sigma\,|0\rangle\,\langle 0|\,\psi_{\Lambda^*}\,|\Lambda\,(1405)\rangle\,. \end{aligned}\tag{4.2}$$

Furthermore, we derive the invariant amplitude from the matrix element,

$$\begin{aligned}\mathcal{M}\,(\Lambda\,(1405)\to\pi\Sigma) &= \int d^4x\,\langle\pi\Sigma|\,\mathcal{L}_{eff}\,|\Lambda\,(1405)\rangle \\ &= g\bar{u}_s\,(\boldsymbol{p}_\Sigma)\,u_r\,(\boldsymbol{p}_{\Lambda^*})\,(2\pi)^4\,\delta^{(4)}\,(p_{\Lambda^*}-p_\pi-p_\Sigma)\,, \end{aligned}\tag{4.3}$$

where $\bar{u}_s\,(\boldsymbol{p}_\Sigma)$ and $u_r\,(\boldsymbol{p}_{\Lambda^*})$ are two-component spinors for $\Sigma$ and $\Lambda\,(1405)$, respectively, and $p$ $(\boldsymbol{p})$ is the four-momentum (three-momentum) for the corresponding particles.

Finally, we obtain the decay width for $\Lambda\,(1405)\to\pi\Sigma$ process by using the amplitude,

$$\begin{aligned}\Gamma_{\Lambda^*\to\pi\Sigma} &= 3\times\frac{1}{2}\sum_r\sum_s\int\frac{1}{2m_\Lambda}\left(\prod_{f=\pi,\Sigma}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right) \\ &\quad\times g^2|\bar{u}_s\,(\boldsymbol{p}_\Sigma)\,u_r\,(\boldsymbol{p}_{\Lambda^*})\,|^2\,(2\pi)^4\,\delta^{(4)}\,(p_{\Lambda^*}-p_\pi-p_\Sigma) \\ &= g^2\frac{3}{\pi}|\boldsymbol{p}|\frac{E_\Sigma+m_\Sigma}{4\,(E_\Sigma+E_\pi)}, \end{aligned}\tag{4.4}$$

where the factor 3 in the first line comes from the fact that $\Lambda\,(1405)$ can decay into $\pi^0\Sigma^0$, $\pi^+\Sigma^-$, and $\pi^-\Sigma^+$, and $|\boldsymbol{p}|$ is the momentum transfer, which is determined by the energy-momentum conservation. The mass and energy are denoted by $m$ and $E$, respectively,

$$\begin{cases} E_\pi = \sqrt{{m_\pi}^2+\boldsymbol{p}^2} \\ E_\Sigma = \sqrt{{m_\Sigma}^2+\boldsymbol{p}^2}. \end{cases}\tag{4.5}$$

In this approach, we determine the coupling constant $g$ to reproduce the experimental data. The detailed derivation of Eq. (4.4) is shown in Appendix D.

## 4.2 $\bar{K}N$ Feshbach resonance in the Skyrme model

As we discussed in the previous section, we need decay width as an input to determine the coupling constant in the field theoretical approach and the decay width is observed by experiments. Contrary, our approach shown in this section does not need the decay width as the input because it is possible to derive the decay width as a theoretical prediction. The key in our approach is that the matrix element is interpreted as the coupling constant $g$. Once we obtain the coupling constant as the matrix element, the decay width can be calculated through the formula Eq. (4.4).

In this section, we explain how to derive the coupling constant for the $\Lambda\,(1405)$-$\pi\Sigma$ vertex. To do that, we need to evaluate the following matrix element,

$$\langle\pi\Sigma|\,\mathcal{L}_{int}\,|\Lambda\,(1405)\rangle\,, \tag{4.6}$$

where $\mathcal{L}_{int}$ is the interaction Lagrangian. The Feynman diagram corresponding to this matrix element has already been shown in Fig. 4.1. To calculate the matrix element, we first introduce the interaction Lagrangian for the $\Lambda\,(1405)$-$\pi\Sigma$ vertex and then we construct the initial and final states. The initial $\Lambda\,(1405)$ appears as the $\bar{K}N$ bound state in our approach, while the final $\Sigma$ is constructed in the CK approach. Finally, using the wave functions, we evaluate the matrix element.

We summarize the necessary quantities to calculate the matrix element below and explain one by one in the following subsections.

1. Interaction Lagrangian
2. Initial and final states
3. Wave functions for the particles

### 4.2.1 Interaction Lagrangian

First, we construct the interaction Lagrangian for the $\Lambda\,(1405)$-$\pi\Sigma$ vertex. To do that, we employ a current-current interaction in this study,

$$\mathcal{L}_{int} = \frac{2}{F_\pi}\partial^\mu\pi^a J^{5,a}_\mu, \tag{4.7}$$

where $\partial^\mu\pi^a$ is the pion axial current and $J^a_\mu$ is the baryon axial current and they couple to the $\Lambda\,(1405)$-$\pi\Sigma$ vertex. Taking a matrix element of the interaction Lagrangian with the initial $\Lambda\,(1405)$ and final $\pi\Sigma$ states, we obtain,

$$\langle\pi\Sigma|\,\mathcal{L}_{int}\,|\Lambda\,(1405)\rangle = \frac{2}{F_\pi}\,\langle\pi|\,\partial^\mu\pi^a\,|0\rangle\,\langle\Sigma|\,J^{5,a}_\mu\,|\Lambda\,(1405)\rangle\,. \tag{4.8}$$

In this study, we combine the CK and our approaches to describe the $\Lambda(1405)$ as we have shown in Fig. 4.2; the $\Lambda(1405)$ is realized as the $\bar{K}N$ (quasi-)bound state in our approach and the $\Sigma$ is done as the bound state of the s-quark and di-quark in the CK approach. Therefore, the matrix element Eq. (4.8) is rewritten as follows,

$$\langle \pi\Sigma | \mathcal{L}_{int} | \Lambda(1405) \rangle = \frac{2}{F_\pi} \langle \pi | \partial^\mu \pi^a | 0 \rangle \langle sd | J_\mu^{5,a} | \bar{K}N \rangle, \tag{4.9}$$

where $s$ and $d$ stand for the s-quark and di-quark, respectively.

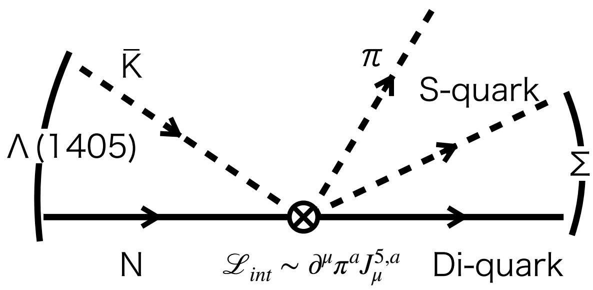


Figure 4.2: The $\Lambda(1405)$-$\pi\Sigma$ vertex in this study.

In Eq. (4.9), the first term on the right hand side is the one pion matrix element and is computed trivially. But the second one is not. Therefore, we concentrate on the second term of the right hand side in the rest of this subsection. To do that, we derive the axial current $J_\mu^{5,a}$, which is obtained from the SU(3) Skyrme Lagrangian as a Noether's current [40].

We show the SU(3) Skyrme action again,

$$\Gamma = \int d^4x \left\{ \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2} \mathrm{tr}\left[ (\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger \right]^2 + L_{SB} \right\} + \Gamma_{WZ}. \tag{4.10}$$

Under the axial transformation, the variable $U$ transforms as follows,

$$U \to g_A U g_A. \tag{4.11}$$

In Eq. (4.11), $g_A$ is an axial transformation operator,

$$g_A = e^{i\boldsymbol{\theta}\cdot\boldsymbol{T}/2} \equiv e^{i\phi}, \tag{4.12}$$

where $\boldsymbol{T} = T^a\,(a = 1, 2, 3, \cdots, 8)$ is the Gell-Mann matrices and $\boldsymbol{\theta}$ is a rotation angle.

Using Eqs. (4.10) and (4.11), we obtain the axial current $J_\mu^{5,a}$ form the Noether's theorem,

$$
\begin{aligned}
J^{5,\mu,a} = & \frac{iF_\pi^2}{16}\mathrm{tr}\left[T^a\left(R^\mu - L^\mu\right)\right] \\
& + \frac{i}{16e^2}\mathrm{tr}\left[T^a\left\{\left[R^\nu,\left[R_\nu,R^\mu\right]\right] - \left[L^\nu,\left[L_\nu,L^\mu\right]\right]\right\}\right] \\
& - \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{T^a}{2}\left(L_\nu L_\alpha L_\beta + R_\nu R_\alpha R_\beta\right)\right],
\end{aligned}
\tag{4.13}
$$

where $R_\mu$ and $L_\mu$ are defined by

$$
\begin{cases}
R_\mu = U\partial_\mu U^\dagger \\
L_\mu = U^\dagger \partial_\mu U.
\end{cases}
\tag{4.14}
$$

The first term of Eq. (4.13) comes from the second derivative term, the second one from the Skyrme term, and the last from the Wess-Zumino action. The derivation of Eq. (4.13) is shown in Appendix E.

The obtained axial current has time and spatial components. In this study, we consider the most dominant component of the axial current as a first attempt. To do that, we consider the non-relativistic limit of the axial current Eq. (4.13). Following the discussion in Appendix F, the time component of the axial current is dominant due to the parity conservation between the $\bar{K}N$ and $\Sigma$.

Now, we consider the variable $U$. It is given by the the Callan-Klebanov ansatz Eq. (3.6) in the classical level,

$$
U = \sqrt{N_H}U_K\sqrt{N_H}. \tag{4.15}
$$

From now on, we consider the second derivative term of the Lagrangian as an example because the contributions from the Skyrme term and Wess-Zumino action are very complicated. Substituting above ansatz for the axial current Eq. (4.13), we obtain the leading contribution,

$$
\begin{aligned}
J_{\mu=0}^{5,a=0,(2)} = & \; i\frac{F_\pi^2}{16}\left(\frac{2\sqrt{2}}{F_\pi}\right)^2 \\
& \times\mathrm{tr}\left[\tau^3\left\{\frac{1}{2}\left[\xi K\left(\partial_0 K^\dagger\right)\xi^\dagger - \xi^\dagger K\left(\partial_0 K^\dagger\right)\xi\right]\right.\right. \\
& \left.\left. -\frac{1}{2}\left[\xi\left(\partial_0 K\right)K^\dagger\xi^\dagger - \xi^\dagger\left(\partial_0 K\right)K^\dagger\xi\right]\right\}\right],
\end{aligned}
\tag{4.16}
$$

where the superscript (2) shows that the contribution from the second derivative term. Here, we consider $\Lambda(1405) \to \pi^0\Sigma^0$ process, for simplicity. In Eq. (4.16), what is important is that one $K$ and one $K^\dagger$ operators appear in the current. We

Table 4.1: Identification of creation and annihilation operators for the kaon and $s$-quark.

| | Kaon | Anti-kaon | $s$-quark | $\bar{s}$-quark |
|---|---|---|---|---|
| $K$ | Annihilation | Creation | Creation | Annihilation |
| $K^\dagger$ | Creation | Annihilation | Annihilation | Creation |

identify $K$ and $K^\dagger$ with kaon annihilation and creation operators, respectively, and then we obtain several relations summarized in Tab. 4.1.

According to Tab. 4.1, we identify $K$ with the $s$-quark creation operator acting on the final state and $K^\dagger$ with the $\bar{K}$ annihilation one acting on the initial state. Therefore, $K$ is regarded as a $s$-quark in the CK appraoch while $K^\dagger$ remains a kaon in our approach after quantization,

$$\begin{cases} K \to AK_{CK} \\ K^\dagger \to K^\dagger_{EH}, \end{cases} \tag{4.17}$$

and the Hedgehog soliton is quantized by isospin rotation,

$$\xi \to A\left(t\right)\xi A^\dagger\left(t\right), \tag{4.18}$$

where $A\left(t\right)$ is an SU(2) isospin rotation matrix.

Here, we show several techniques for explicit calculations. We first consider the properties of the hedgehog soliton. As we have mentioned in Sec. 2.2, the hedgehog soliton is invariant for the simultaneous rotation in the isospin and spatial spaces, Therefore, we obtain the following relation,

$$A\left(t\right)e^{i\tau\cdot\hat{r}F(r)}A^\dagger\left(t\right) = e^{i\tau_a R_{ab}(A)\hat{r}_b F(r)}, \tag{4.19}$$

where $R_{ab}\left(A\right)$ is a spatial rotation matrix. Here, let us recall what we would like to calculate. Our purpose is to evaluate the following matrix element,

$$\begin{aligned} \int d^3x \ \langle\pi^0\Sigma^0|\,\mathcal{L}\,|\bar{K}N\rangle \ &\sim \ \int d^3x \ \langle\pi\Sigma|\,\partial^0\pi^{a=3}J_0^{5,a=3}\,|\bar{K}N\rangle \\ &= \int d^3x \ \langle\pi|\,\partial^0\pi^3\,|0\rangle\,\langle\Sigma|\,J_0^{5,3}\,|\bar{K}N\rangle\,. \end{aligned} \tag{4.20}$$

Therefore, if we change the angle variables by the spatial rotation matrix $R\left(A\right)$,

$$\hat{r} \to \hat{r}' = R\left(A\right)\hat{r}, \tag{4.21}$$

then the integral measure is also changed,

$$d\hat{r} \to d\hat{r}'. \tag{4.22}$$

However, as long as $\hat{r}$ runs over $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, the angular integral measure is invariant for the above variable transformation,

$$\int d\hat{r} = \int d\hat{r}' \left( = \int d\left(R\left(A\right)\hat{r}\right) \right). \tag{4.23}$$

Therefore, if we redefine $\hat{r}'$ as $\hat{r}$ after isospin rotation, we obtain,

$$\begin{aligned} A\left(t\right) e^{i\tau\cdot\hat{r}F(r)} A^{\dagger}\left(t\right) &= e^{i\tau_a R_{ab}(A)\hat{r}_b F(r)} \\ &= e^{i\tau_a \hat{r}'_a F(r)} \\ &\equiv e^{i\tau_a \hat{r}_a F(r)} = U_H = \xi^2, \end{aligned} \tag{4.24}$$

and the integral measure apparently does not change.

According to the redefinition, the matrix element of the pion current and the kaon operator in the CK approach change. For the former, the matrix element is given by the partially conserved axial current (PCAC) relation (Ref.),

$$\langle 0 | \partial_\mu \pi^a | \pi^b \rangle = i p_\mu e^{ipx} \delta^{ab}, \tag{4.25}$$

where $p_\mu$ is the pion momentum. Next, we expand the pion plane wave part with the Rayleigh formula,

$$\begin{aligned} e^{-ipx} &= e^{-ip_0 x_0} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = e^{-ip_0 x_0} e^{ipr\cos\theta} \\ &= e^{-ip_0 x_0} \sum_{l=0}^{\infty} \left(2l+1\right) i^l j_l\left(pr\right) P_l\left(\cos\theta\right), \end{aligned} \tag{4.26}$$

where $j_l\left(px\right)$ and $P_l\left(\cos\theta\right)$ are the spherical Bessel function and the Legendre polynomial, respectively. We show their explicit forms, for example, $l = 0, 1, 2$,

$$j_0\left(z = pr\right) = \frac{\sin z}{z} \tag{4.27}$$

$$j_1\left(z = pr\right) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \tag{4.28}$$

$$j_2\left(z = pr\right) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3\cos z}{z^2} \tag{4.29}$$

and

$$P_0\left(\cos\theta\right) = 1 \tag{4.30}$$

$$P_1\left(\cos\theta\right) = \cos\theta \tag{4.31}$$

$$P_2\left(\cos\theta\right) = \frac{1}{2}\left(3\cos^2\theta - 1\right). \tag{4.32}$$

Due to the spin-parity conservation, the s- and d-waves pion is allowed in the $\bar{K}N \to \pi\Sigma$. However, in the low-energy region, the s-wave contribution is more dominant than the d-wave one. Therefore, we consider the s-wave pion,

$$e^{-ipx} \to e^{-ip_0 x_0} \frac{\sin\left(pr\right)}{pr}, \tag{4.33}$$

which is independent of the angular variables.

For the latter, as we mentioned in Sec. 3.2.2, the Σ-hyperon is generated as a bound state of the p-wave kaon and Hedgehog soliton. Because the kaon is the p-wave, it depends on the angular variables. To make a further discussion, we need to introduce the kaon wave function in the CK approach. Therefore, we will consider again in Sec. 4.2.3

Let us go back to the derivation of the axial current. Using the quantization rule Eq. (4.17), we obtain from Eqs. (4.16),

$$\begin{aligned} J^{5,a=0,(2)}_{\mu=0} &= i\frac{F_\pi^2}{16}\left(\frac{2\sqrt{2}}{F_\pi}\right)^2 \\ &\quad\times \mathrm{tr}\left[\tau^3\left\{\frac{1}{2}\left[\xi A K_{CK}\left(\partial_0 K^\dagger_{EH}\right)\xi^\dagger - \xi^\dagger A K_{CK}\left(\partial_0 K^\dagger_{EH}\right)\xi\right]\right.\right. \\ &\qquad\left.\left. -\frac{1}{2}\left[\xi A\left(\partial_0 K_{CK}\right)K^\dagger_{EH}\xi^\dagger - \xi^\dagger A\left(\partial_0 K_{CK}\right)K^\dagger_{EH}\xi\right]\right\}\right] \\ &\quad+\mathcal{O}\left(1/N_c\right). \end{aligned} \tag{4.34}$$

Here, the $1/N_c$ contributions comes from the time-derivative of the isospin rotation matrix and we neglect them because it is higher-order correction.

### 4.2.2 Initial and final states

Here, we construct the initial $\bar{K}N$ and final Σ states. First of all, we show our phase convention for the s-quark two-component spinor and kaon isospinor.

- s-quark spinors

  If we naively introduce,

  $$s = \begin{pmatrix} s_\uparrow \\ s_\downarrow \end{pmatrix}, \tag{4.35}$$

  which means

  $$s_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_{\mu=+1/2}, \qquad s_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi_{\mu=-1/2}. \tag{4.36}$$

- Kaon isospinors

  If we introduce the kaon iso-spinor as,

  $$K \equiv \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \tag{4.37}$$

which corresponds to

$$K^{+}=\begin{pmatrix}1\\0\end{pmatrix},\qquad K^{0}=\begin{pmatrix}0\\1\end{pmatrix}. \tag{4.38}$$

. Then, we obtain

$$K^{\dagger}=\begin{pmatrix}K^{-}, & \bar{K}^{0}\end{pmatrix}, \tag{4.39}$$

which meams

$$K^{-}=\begin{pmatrix}1, & 0\end{pmatrix},\qquad \bar{K}^{0}=\begin{pmatrix}0, & -1\end{pmatrix}, \tag{4.40}$$

due to the charge conjugation.

Now, we construct the $\Sigma$ state by combining the spin 1 di-quark and spin 1/2 s-quark to form spin 1/2 by using the Clebsch-Gordan (C.G.) coefficients,

$$\begin{aligned}|\Sigma\,(J_3=1/2)\rangle &= |d\,(J=1)\,s\,(J=1/2)\rangle\\ &= \sqrt{\frac{2}{3}}\,|d\,(J_3=1)\,s_{\downarrow}\rangle-\sqrt{\frac{1}{3}}\,|d\,(J_3=0)\,s_{\uparrow}\rangle\,. \end{aligned}\tag{4.41}$$

On the other hand, for the $\bar{K}N$ state, it is given by

$$\begin{aligned}|\bar{K}N\rangle &= \sqrt{\frac{1}{2}}\,|pK^{-}\rangle-\left(-\sqrt{\frac{1}{2}}\,|n\bar{K}^{0}\rangle\right)\\ &= \sqrt{\frac{1}{2}}\,|pK^{-}\rangle+\sqrt{\frac{1}{2}}\,|n\bar{K}^{0}\rangle\,. \end{aligned}\tag{4.42}$$

### 4.2.3 Wave functions for the particles

Finally, we consider the wave functions for the s-quark, anti-kaon, di-quark, and nucleon in terms of collective coordinates. First, we introduce the nucleon and di-quark wave functions, which are defined in the SU(2) isospin space. For the nucleon wave functios, we have already shown in Sec. 2.2 and we give them again here,

$$\begin{cases}|p\uparrow\rangle=\dfrac{1}{\pi}\,(a_1+ia_2)\\ |p\downarrow\rangle=-\dfrac{i}{\pi}\,(a_0-ia_3)\\ |n\uparrow\rangle=\dfrac{i}{\pi}\,(a_0+ia_3)\\ |n\downarrow\rangle=-\dfrac{1}{\pi}\,(a_1-ia_2)\,.\end{cases}\tag{4.43}$$

For the di-quark, it is possible to construct the wave function by regarding the proton (neutron) as u-quark (d-quark),

$$\begin{cases} |p\uparrow\rangle \sim |u\uparrow\rangle \\ |p\downarrow\rangle \sim |u\downarrow\rangle \\ |n\uparrow\rangle \sim |d\uparrow\rangle \\ |n\downarrow\rangle \sim |d\downarrow\rangle . \end{cases} \tag{4.44}$$

For example, di-quark with $J_3 = 1, I_3 = 0$ state is given by

$$\sqrt{\frac{1}{2}}\left[(u\uparrow)(d\uparrow) + (d\uparrow)(u\uparrow)\right]. \tag{4.45}$$

Therefore, the wave function is written by,

$$\begin{aligned} & \sqrt{\frac{1}{2}}\left[(u\uparrow)(d\uparrow) + (d\uparrow)(u\uparrow)\right] \\ = & \sqrt{\frac{1}{2}}\left[\left(\frac{1}{\pi}(a_1 + ia_2)\right)\left(\frac{i}{\pi}(a_0 + ia_3)\right) + \left(\frac{i}{\pi}(a_0 + ia_3)\right)\left(\frac{1}{\pi}(a_1 + ia_2)\right)\right] \\ \to & \frac{\sqrt{3}}{\pi}(a_0a_1 + ia_1a_3 + ia_0a_2 - a_2a_3) \quad \text{with normalization} \\ = & \frac{\sqrt{3}}{\pi}(a_1 + ia_2)(a_0 + ia_3). \end{aligned} \tag{4.46}$$

Its detailed derivation is sown in Appendix G

Next, we consider the kaon and s-quark wave functions which are defined in the configuration space. The radial part of the kaon wave function is obtained by solving the equation of motion Eq. (3.54) and the angular part is given by the spherical harmonics $Y_0^0 = 1/\sqrt{4\pi}$ Therefore, we obtain the spatial kaon wave function in the following form,

$$K(\boldsymbol{r}) = \frac{1}{\sqrt{4\pi}}k(r), \tag{4.47}$$

where $k(r)$ is obtained by solving Eq. (3.54).

For the s-quark, the radial part is obtained by solving the equation of motion Eq. (3.23), which is the same as the case of the kaon wave function. However the angular part is different from the kaon case. As we mentioned in Sec. 3.2.4, the s-quark spin $J$ is equal to the kaon grand-spin $T = I + L$ in the CK approach. The kaon isospin is 1/2 and the lowest bound kaon is a p-wave in the CK approach. Therefore, the angular part of the s-quark is a combination of the spin 1 and spin 1/2,

$$[Y_1^m, 1/2]^{1/2}_{\mu=\pm 1/2} \quad = \quad -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{r}\chi_\mu, \tag{4.48}$$

where $\chi_\mu$ is a two-component spinor. As a result, the s-quark wave function is written as,

$$s(\boldsymbol{r}) = -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{r}\chi_\mu s(r), \tag{4.49}$$

where $s(r)$ is obtained by solving Eq. (3.23).

As we mentioned in Sec. 4.2.1, the s-quark for $\Sigma$ is in p-wave and its wave function depends on angles. Therefore, it is modified after an isospin rotation as follows,

$$\begin{aligned} AK_{CK}(\boldsymbol{r}) &= -\sqrt{\frac{1}{4\pi}}A\boldsymbol{\tau}\cdot\hat{r}\left(A^\dagger A\right)\chi_\mu s(r)e^{-iE_s t} \\ &= -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{r}'A\chi_\mu s(r)e^{-iE_s t} \\ &\equiv -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{r}A\chi_\mu s(r)e^{-iE_s t}. \end{aligned} \tag{4.50}$$

Here, we define the two kaon operators $\hat{K}_{CK}$ and $\hat{K}_{EH}$ with their wave functions. To do that, we first expand $\hat{K}_{CK}$ and $\hat{K}_{EH}$ operators as follows,

$$\begin{cases} \hat{K}_{CK} = b^\dagger_{s_\uparrow}\psi_{s_\uparrow}(r) + b^\dagger_{s_\downarrow}\psi_{s_\downarrow}(r) + \cdots \\ \hat{K}^\dagger_{CK} = b_{s_\uparrow}\psi^\dagger_{s_\uparrow}(r) + b_{s_\downarrow}\psi^\dagger_{s_\downarrow}(r) + \cdots \\ \hat{K}_{EH} = a^\dagger_{\bar{K}^0}\psi_{\bar{K}^0}(r) + a^\dagger_{K^-}\psi_{K^-}(r) + \cdots \\ \hat{K}^\dagger_{EH} = a_{\bar{K}^0}\psi^\dagger_{\bar{K}^0}(r) + a_{K^-}\psi^\dagger_{K^-}(r) + \cdots, \end{cases} \tag{4.51}$$

where dots parts $(\cdots)$ in the equations are the higher excited states, which are irrelevant for the present study. Here, we would like to explain the notations in the above equations. First, $b_i^\dagger$ and $b_i$ $(i = s_\uparrow, s_\downarrow)$ stand for s-quark creation and annihilation operators with spin $\pm 1/2$ states, respectively. In the below two equations, $a_x^\dagger$, and $a_x$ $\left(x = \bar{K}^0, K^-\right)$ represent the creation and annihilation operators, respectively. The wave functions for the particle is denoted by $\psi_j^\dagger(r)$ $\left(j = s_\uparrow, s_\downarrow, \bar{K}^0, K^-\right)$.

Using the creation operators in Eq. (4.51), we define the following one-particle quark and meson states,

$$\begin{cases} |s_\uparrow\rangle \equiv b^\dagger_{s_\uparrow}|0\rangle \\ |s_\downarrow\rangle \equiv b^\dagger_{s_\downarrow}|0\rangle \\ |\bar{K}^0\rangle \equiv a^\dagger_{\bar{K}^0}|0\rangle \\ |K^-\rangle \equiv a^\dagger_{K^-}|0\rangle. \end{cases} \tag{4.52}$$

The wave functions of the $s$-quark and anti-kaon have already been obtained,

$$\begin{cases} \psi_{s_\uparrow}(\boldsymbol{r},t) = -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{r}\chi_{s_\uparrow}s(r)\,e^{-iE_st} \\ \psi_{s_\downarrow}(\boldsymbol{r},t) = -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{r}\chi_{s_\downarrow}s(r)\,e^{-iE_st} \\ \psi_{\bar{K}^0}(\boldsymbol{r},t) = \sqrt{\frac{1}{4\pi}}\Phi_{\bar{K}^0}k(r)\,e^{-iE_Kt} \\ \psi_{K^-}(\boldsymbol{r},t) = \sqrt{\frac{1}{4\pi}}\Phi_{K^-}k(r)\,e^{-iE_Kt}, \end{cases} \tag{4.53}$$

where $\chi$ and $\Phi$ are the $s$-quark two component spinor and $\bar{K}$ isospinor, respectively.

To calculate the matrix element Eq. (4.9), we consider the correspondence between the two component spinor (isospinor) in Eq. (4.53) and the $s$-quark (anti-kaon) spin (isospin) states. To do that, we first consider the anti-kaon case because it is simpler than the $s$-quark one.

If we naively define the kaon operator $\hat{K}_{EH}$ as,

$$\hat{K}_{EH} \equiv \begin{pmatrix} \hat{K}^+ \\ \hat{K}^0 \end{pmatrix}, \tag{4.54}$$

we obtain the following operator,

$$\hat{K}^\dagger_{EH} = \begin{pmatrix} \hat{K}^-, & \hat{\bar{K}}^0 \end{pmatrix}, \tag{4.55}$$

and $K^\dagger_{EH}$ is an annihilation operator for the anti-kaon.

From Eq. (4.42), we can write the $\Lambda(1405)$ state as,

$$\begin{aligned} |\Lambda(1405)\rangle &= |\bar{K}N(I=0)\rangle \\ &= \sqrt{\frac{1}{2}}|pK^-\rangle + \sqrt{\frac{1}{2}}|n\bar{K}^0\rangle. \end{aligned} \tag{4.56}$$

Therefore, acting $\hat{K}^\dagger_{EH}$ from the left side of the ket-vetcor, we obtain,

$$\begin{aligned} &\hat{K}^\dagger_{EH}|\bar{K}N(I=0)\rangle \\ =\ & \begin{pmatrix} \hat{K}^-, & \hat{\bar{K}}^0 \end{pmatrix}\left[\sqrt{\frac{1}{2}}|pK^-\rangle + \sqrt{\frac{1}{2}}|n\bar{K}^0\rangle\right] \\ \propto\ & \begin{pmatrix} \hat{a}_{K^-}\psi_{K^-}, & \hat{a}_{\bar{K}^0}\psi_{\bar{K}^-} \end{pmatrix}\left[\sqrt{\frac{1}{2}}|pK^-\rangle + \sqrt{\frac{1}{2}}|n\bar{K}^0\rangle\right] \\ =\ & \sqrt{\frac{1}{2}}\begin{pmatrix} 1, & 0 \end{pmatrix}\psi^\dagger_{K^-}(r)|p\rangle + \sqrt{\frac{1}{2}}\begin{pmatrix} 0, & 1 \end{pmatrix}\psi^\dagger_{\bar{K}^0}(r)|n\rangle, \end{aligned} \tag{4.57}$$

where $a_x$ $\left(x = K^-, \bar{K}^0\right)$ is the annihilation operators defined in Eq. (4.51).

Next, we consider the $\hat{K}_{CK}$ operator. We start with the kaon two-component isospinor,

$$\hat{K} \equiv \begin{pmatrix} \hat{K}^+ \\ \hat{K}^0 \end{pmatrix}, \tag{4.58}$$

As we mentioned in the previous chapter, the kaon is quantized as an s-quark in the CK approach. Therefore, $\hat{K}^+$ corresponds to a spin-up state and $\hat{K}^0$ does to a spin-down one. Furthermore taking into account that the kaon does not contain the s-quark but $\bar{s}$-quark, we obtain the following relation due to the charge conjugation,

$$\hat{K}_{CK} = \begin{pmatrix} \hat{\bar{s}}_\downarrow \\ -\hat{\bar{s}}_\uparrow \end{pmatrix} \tag{4.59}$$

and $\hat{K}_{CK}$ is the annihilation operator of a $\bar{s}$-quark, equivalently the creation operator for an s-quark, from Tab. 4.1

The $\Sigma$ state has been given in Eq. (4.41)

$$\begin{aligned} |\Sigma\,(J_3 = 1/2)\rangle &= |d\,(J = 1)\,s\,(J = 1/2)\rangle \\ &= \sqrt{\frac{2}{3}}\,|d\,(J_3 = 1)\,s_\downarrow\rangle - \sqrt{\frac{1}{3}}\,|d\,(J_3 = 0)\,s_\uparrow\rangle\,. \end{aligned} \tag{4.60}$$

Therefore, $\langle\Sigma|$ is written as,

$$\langle\Sigma\,(J_3 = 1/2)| = \sqrt{\frac{2}{3}}\,\langle d\,(J_3 = 1)\,s_\downarrow| - \sqrt{\frac{1}{3}}\,\langle d\,(J_3 = 0)\,s_\uparrow|\,. \tag{4.61}$$

Operating $\hat{K}_{CK}$ from the right side, we obtain

$$\begin{aligned} &\langle\Sigma\,(J_3 = 1/2)|\,\hat{K}_{CK} \\ = &\sqrt{\frac{2}{3}}\,\langle d\,(J_3 = 1)\,s_\downarrow|\,\hat{K}_{CK} - \sqrt{\frac{1}{3}}\,\langle d\,(J_3 = 0)\,s_\uparrow|\,\hat{K}_{CK} \\ = &\sqrt{\frac{2}{3}}\,\langle d\,(J_3 = 1)\,s_\downarrow| \begin{pmatrix} \hat{\bar{s}}_\downarrow \\ -\hat{\bar{s}}_\uparrow \end{pmatrix} - \sqrt{\frac{1}{3}}\,\langle d\,(J_3 = 0)\,s_\uparrow| \begin{pmatrix} \hat{\bar{s}}_\downarrow \\ -\hat{\bar{s}}_\uparrow \end{pmatrix} \\ = &\sqrt{\frac{2}{3}}\,\langle d\,(J_3 = 1)\,s_\downarrow| \begin{pmatrix} b^\dagger_{s_\downarrow}\psi_{s_\downarrow}(r) \\ -b^\dagger_{s_\uparrow}\psi_{s_\uparrow}(r) \end{pmatrix} - \sqrt{\frac{1}{3}}\,\langle d\,(J_3 = 0)\,s_\uparrow| \begin{pmatrix} b^\dagger_{s_\downarrow}\psi_{s_\downarrow}(r) \\ -b^\dagger_{s_\uparrow}\psi_{s_\uparrow}(r) \end{pmatrix} \\ \propto &\sqrt{\frac{2}{3}}\,\langle d\,(J_3 = 1)|\,\psi_{s_\downarrow}(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sqrt{\frac{1}{3}}\,\langle d\,(J_3 = 0)|\,\psi_{s_\uparrow}(r) \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \end{aligned} \tag{4.62}$$

where $b^\dagger_i$ $(i = s_\uparrow, s_\downarrow)$ is the s-quark creation operators defined in Eq. (4.51).

From Eqs. (4.41), (4.42), (4.57), and (4.62), it is possible to evaluate the following matrix element,

$$
\begin{aligned}
& \langle\Sigma^0| \hat{K}_{CK}\hat{K}_{EH}^{\dagger} |\bar{K}N\rangle \\
= & \left\{\sqrt{\frac{2}{3}}\langle d\left(J_3=1\right)s_{\downarrow}| - \sqrt{\frac{1}{3}}\langle d\left(J_3=0\right)s_{\uparrow}|\right\}\hat{K}_{CK} \\
& \times\hat{K}_{EH}^{\dagger}\left\{\sqrt{\frac{1}{2}}|pK^-\rangle + \sqrt{\frac{1}{2}}|n\bar{K}^0\rangle\right\} \\
= & \sqrt{\frac{2}{3}}\langle d\left(J_3=1\right)s_{\downarrow}|\hat{K}_{CK}\hat{K}_{EH}^{\dagger}\sqrt{\frac{1}{2}}|pK^-\rangle \\
& +\sqrt{\frac{2}{3}}\langle d\left(J_3=1\right)s_{\downarrow}|\hat{K}_{CK}\hat{K}_{EH}^{\dagger}\sqrt{\frac{1}{2}}|n\bar{K}^0\rangle \\
& -\sqrt{\frac{1}{3}}\langle d\left(J_3=0\right)s_{\uparrow}|\hat{K}_{CK}\hat{K}_{EH}^{\dagger}\sqrt{\frac{1}{2}}|pK^-\rangle \\
& -\sqrt{\frac{1}{3}}\langle d\left(J_3=0\right)s_{\uparrow}|\hat{K}_{CK}\hat{K}_{EH}^{\dagger}\sqrt{\frac{1}{2}}|n\bar{K}^0\rangle \\
\propto & \sqrt{\frac{1}{3}}\langle d\left(J_3=1\right)|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1, & 0\end{pmatrix}|p\rangle + \sqrt{\frac{1}{3}}\langle d\left(J_3=1\right)|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0, & 1\end{pmatrix}|n\rangle \\
& -\sqrt{\frac{1}{6}}\langle d\left(J_3=0\right)|\begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}1, & 0\end{pmatrix}|p\rangle - \sqrt{\frac{1}{6}}\langle d\left(J_3=0\right)|\begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}0, & 1\end{pmatrix}|n\rangle .
\end{aligned}
\tag{4.63}
$$

In the end of this subsection, we introduce normalization conditions for the kaon and s-quark wave functions which is consistent with the solutions of the Klein-Gordon equation.

First, in the CK approach, the normalization is given by [20, 21],

$$
\begin{cases}
4\pi\int drr^2 k_n^*\left(\boldsymbol{r}\right)k_m\left(\boldsymbol{r}\right)\left[f\left(r\right)\left(\omega_n+\omega_m\right)+2\lambda\left(r\right)\right]=\delta_{nm} \\
4\pi\int drr^2 \tilde{k}_n^*\left(\boldsymbol{r}\right)\tilde{k}_m\left(\boldsymbol{r}\right)\left[f\left(r\right)\left(\tilde{\omega}_n+\tilde{\omega}_m\right)-2\lambda\left(r\right)\right]=\delta_{nm} \\
4\pi\int drr^2 k_n^*\left(\boldsymbol{r}\right)\tilde{k}_m\left(\boldsymbol{r}\right)\left[f\left(r\right)\left(\omega_n-\tilde{\omega}_m\right)+2\lambda\left(r\right)\right]=0,
\end{cases}
\tag{4.64}
$$

where $\omega_m$ an $\tilde{\omega}_m$ are the energies of the s-quark and $\bar{s}$-quark in the $m$-mode, respectively and $k_m\left(\boldsymbol{r}\right)$ and $\tilde{k}_m\left(\boldsymbol{r}\right)$ are the wave functions for the s- and $\bar{s}$-quarks, respectively. The radial dependent functions are given by,

$$
f\left(r\right) = 1+\frac{1}{\left(eF_\pi\right)^2}\left[2\frac{\sin^2 F}{r^2}+F'^2\right] \tag{4.65}
$$

$$
\lambda\left(r\right) = -\frac{N_c E}{2\pi^2 {F_\pi}^2}\frac{\sin^2 F}{r^2}F' \tag{4.66}
$$

$$
h\left(r\right) = 1+\frac{1}{\left(eF_\pi\right)^2}\left[2\frac{\sin^2 F}{r^2}\right]. \tag{4.67}
$$

These normalization conditions are derived from the canonical commutation relation. Using them, the dimension of the wave function turns out to be MeV.

On the other hand, in our approach, the normalization conditions are given by,

$$\begin{cases} 4\pi \int drr^2 k_n^* (\boldsymbol{r})\, k_m (\boldsymbol{r}) \left[ f(\omega_n + \omega_m) + 2\{\rho_1 + \lambda_1\} - \frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_2\right) \right] = \delta_{nm} \\ 4\pi \int drr^2 \tilde{k}_n^* (\boldsymbol{r})\, \tilde{k}_m (\boldsymbol{r}) \left[ f(\tilde{\omega}_n + \tilde{\omega}_m) - 2\{\rho_1 + \lambda_1\} + \frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_2\right) \right] = \delta_{nm} \\ 4\pi \int drr^2 k_n^* (\boldsymbol{r})\, \tilde{k}_m (\boldsymbol{r}) \left[ f(\omega_n - \tilde{\omega}_m) + 2\{\rho_1 + \lambda_1\} - \frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_2\right) \right] = 0, \end{cases} \tag{4.68}$$

where

$$\rho_1(r) = -\frac{4\sin^2(F/2)}{3\Lambda}\boldsymbol{I}_K\cdot\boldsymbol{I}_N\left[1+\frac{1}{(eF_\pi)^2}\left(\frac{4}{r^2}\sin^2 F + F'^2\right)\right] - \frac{\sin^2(F/2)}{\Lambda}\left[1+\frac{1}{(eF_\pi)^2}\left(\frac{5}{r^2}\sin^2 F + F'^2\right)\right] \tag{4.69}$$

$$\rho_2(r) = \frac{1}{(eF_\pi)^2}\left[\frac{\sin F}{\Lambda}F'\left(4\boldsymbol{I}_K\cdot\boldsymbol{I}_N + 3\right)\right] \tag{4.70}$$

$$\begin{aligned} \lambda_1(r) &= -\frac{N_c}{2\pi^2 {F_\pi}^2}\frac{\sin^2 F}{r^2}F' \\ &= \frac{N_c}{{F_\pi}^2}B^0, \quad B^0 = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F' \end{aligned} \tag{4.71}$$

where $\omega_m$ and $k_m(\boldsymbol{r})$ are the anti-kaon energy and the wave function, respectively, and the tilded variables stand for the kaon.

## 4.3 Result and discussion

Using the guidelines shown in the previous section, we obtain the following matrix element,

$$\begin{aligned} &\langle\Sigma^0|\, J^{5,a=3}_{\mu=0}\, |\bar{K}N\rangle \\ =\ &\langle\Sigma^0|\, J^{5,a=3,(2)}_{\mu=0}\, |\bar{K}N\rangle + \langle\Sigma^0|\, J^{5,a=3,(4)}_{\mu=0}\, |\bar{K}N\rangle + \langle\Sigma^0|\, J^{5,a=3,(WZ)}_{\mu=0}\, |\bar{K}N\rangle\,, \end{aligned} \tag{4.72}$$

where the first term is the contributions from the second derivative term, the second from the Skyrme term, and the last from the Wess-Zumino action.

In this section, we consider the contribution form the second derivative term $\langle\Sigma^0|\, J^{5,a=3,(2)}_{\mu=0}\, |\bar{K}N\rangle$ as an example. We would like to show the axial current

$J^{5,a=0,(2)}_{\mu=0}$ again,

$$\begin{aligned}
J^{5,a=0,(2)}_{\mu=0} &\simeq i\frac{F_\pi^2}{16}\left(\frac{2\sqrt{2}}{F_\pi}\right)^2 \\
&\quad\times\mathrm{tr}\left[\tau^3\left\{\frac{1}{2}\left[\xi A K_{CK}\left(\partial_0 K^\dagger_{EH}\right)\xi^\dagger-\xi^\dagger A K_{CK}\left(\partial_0 K^\dagger_{EH}\right)\xi\right]\right.\right. \\
&\qquad\left.\left.-\frac{1}{2}\left[\xi A\left(\partial_0 K_{CK}\right)K^\dagger_{EH}\xi^\dagger-\xi^\dagger A\left(\partial_0 K_{CK}\right)K^\dagger_{EH}\xi\right]\right\}\right].
\end{aligned} \tag{4.73}$$

First, we take the matrix element of $J^{5,a=0,(2)}_{\mu=0}$ with the initial $\bar{K}N$ and final $\Sigma$ states,

$$\begin{aligned}
&\langle\Sigma^0|\,J^{5,a=3,(2)}_{\mu=0}\,|\bar{K}N\rangle \\
=\ &\langle\Sigma^0|\,\frac{i}{2} \\
&\times\mathrm{tr}\left[\tau^3\left\{\frac{1}{2}\left[\xi A K_{CK}\left(\partial_0 K^\dagger_{EH}\right)\xi^\dagger-\xi^\dagger A K_{CK}\left(\partial_0 K^\dagger_{EH}\right)\xi\right]\right.\right. \\
&\qquad\left.\left.-\frac{1}{2}\left[\xi A\left(\partial_0 K_{CK}\right)K^\dagger_{EH}\xi^\dagger-\xi^\dagger A\left(\partial_0 K_{CK}\right)K^\dagger_{EH}\xi\right]\right\}\right]|\bar{K}N\rangle.
\end{aligned} \tag{4.74}$$

Here, we can replace time-derivative operators with $\pm iE$ because they act on the exponential part of the $\bar{K}$ and s-quark wave functions,

$$\begin{cases}\partial_0 K_{CK}\to -iE_s K_{CK}\\ \partial_0 K^\dagger_{EH}\to iE_{\bar{K}}K_{EH}.\end{cases} \tag{4.75}$$

Therefore, the above equation is rewritten by,

$$\begin{aligned}
&\langle\Sigma^0|\,J^{5,a=3,(2)}_{\mu=0}\,|\bar{K}N\rangle \\
=\ &\langle\Sigma^0|\,\frac{i}{2} \\
&\times\mathrm{tr}\left[\tau^3\left\{\frac{iE_{\bar{K}}}{2}\left[\xi A K_{CK}K^\dagger_{EH}\xi^\dagger-\xi^\dagger A K_{CK}K^\dagger_{EH}\xi\right]\right.\right. \\
&\qquad\left.\left.+\frac{iE_s}{2}\left[\xi A K_{CK}K^\dagger_{EH}\xi^\dagger-\xi^\dagger A K_{CK}K^\dagger_{EH}\xi\right]\right\}\right]|\bar{K}N\rangle \\
=\ &\langle\Sigma^0|-\frac{(E_{\bar{K}}+E_s)}{4} \\
&\times\mathrm{tr}\left[\tau^3\left\{\xi A K_{CK}K^\dagger_{EH}\xi^\dagger-\xi^\dagger A K_{CK}K^\dagger_{EH}\xi\right\}\right]|\bar{K}N\rangle.
\end{aligned} \tag{4.76}$$

Expanding the Hedgehog soliton as follows,

$$\xi=\cos\frac{F}{2}+i\left(\boldsymbol{\tau}\cdot\hat{r}\right)\sin\frac{F}{2}\equiv c+i\left(\boldsymbol{\tau}\cdot\hat{r}\right)s, \tag{4.77}$$

we obtain,

$$
\begin{aligned}
&\langle\Sigma^0|\, J^{5,a=3,(2)}_{\mu=0}\, |\bar{K}N\rangle \\
= \; & -\frac{i\left(E_{\bar{K}}+E_s\right)\sin F}{4} \\
& \times \langle\Sigma^0|\, \mathrm{tr}\left[\tau^3 \left\{(\boldsymbol{\tau}\cdot\hat{r})\, AK_{CK}K^{\dagger}_{EH} - AK_{CK}K^{\dagger}_{EH}\,(\boldsymbol{\tau}\cdot\hat{r})\right\}\right] |\bar{K}N\rangle \,.
\end{aligned} \tag{4.78}
$$

Next, we replace two kaon operators, $K_{CK}$ and $K_{EH}$, with their wave functions given by Eqs. (4.50) and (4.53) as we have done in Eq. (4.63),

$$
\begin{aligned}
&\langle\Sigma^0|\, J^{5,a=3,(2)}_{\mu=0}\, |\bar{K}N\rangle \\
= \; & -\frac{i\left(E_{\bar{K}}+E_s\right)\sin F}{4}\left(-\sqrt{\frac{1}{4\pi}}\right) s\left(r\right) e^{-iE_s t}\sqrt{\frac{1}{4\pi}} k^{*}\left(r\right) e^{iE_K t} \\
& \times\mathrm{tr}\left[\sqrt{\frac{1}{3}}\,\langle d\left(J_3=1\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right)\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1, & 0\end{pmatrix}|p\rangle\right. \\
& \quad +\sqrt{\frac{1}{3}}\,\langle d\left(J_3=1\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right)\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0, & 1\end{pmatrix}|n\rangle \\
& \quad -\sqrt{\frac{1}{6}}\,\langle d\left(J_3=0\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right)\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}1, & 0\end{pmatrix}|p\rangle \\
& \quad -\sqrt{\frac{1}{6}}\,\langle d\left(J_3=0\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right)\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}0, & 1\end{pmatrix}|n\rangle \\
& \quad -\sqrt{\frac{1}{3}}\,\langle d\left(J_3=1\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1, & 0\end{pmatrix}\left(\boldsymbol{\tau}\cdot\hat{r}\right)|p\rangle \\
& \quad -\sqrt{\frac{1}{3}}\,\langle d\left(J_3=1\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0, & 1\end{pmatrix}\left(\boldsymbol{\tau}\cdot\hat{r}\right)|n\rangle \\
& \quad +\sqrt{\frac{1}{6}}\,\langle d\left(J_3=0\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}1, & 0\end{pmatrix}\left(\boldsymbol{\tau}\cdot\hat{r}\right)|p\rangle \\
& \quad \left.+\sqrt{\frac{1}{6}}\,\langle d\left(J_3=0\right)|\, \tau^3\left(\boldsymbol{\tau}\cdot\hat{r}\right) A \begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}0, & 1\end{pmatrix}\left(\boldsymbol{\tau}\cdot\hat{r}\right)|n\rangle\right] \\
\xrightarrow{\int d\Omega} \; & \frac{i\left(E_{\bar{K}}+E_s\right)\sin F}{4} s\left(r\right) k^{*}\left(r\right) e^{-i\left(E_s-E_K\right)t} \\
& \times\left[\frac{4}{3}\sqrt{\frac{1}{3}}\left\{\langle d\left(J_3=1\right)|\, A_{11}\,|p\rangle - \langle d\left(J_3=1\right)|\, A_{21}\,|n\rangle\right\}\right. \\
& \quad \left.+\frac{4}{3}\sqrt{\frac{1}{6}}\left\{\langle d\left(J_3=0\right)|\, A_{12}\,|p\rangle - \langle d\left(J_3=0\right)|\, A_{22}\,|n\rangle\right\}\right],
\end{aligned} \tag{4.79}
$$

where $A$ is the SU(2) isospin rotation matrix and it related to the nucleon wave functions,

$$\begin{aligned} A &= a_0 + i\tau_i a_i, \quad i = 1, 2, 3 \\ &= \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{aligned} \tag{4.80}$$

and we use several techniques in calculations. The first one is a property of the Pauli's matrices,

$$\begin{cases} (\boldsymbol{\tau} \cdot \hat{r}) (\boldsymbol{\tau} \cdot \hat{r}) = \mathbf{1}_{2\times 2} \\ \tau_a \tau_b \tau_a = -\tau_b, \end{cases} \tag{4.81}$$

where $\mathbf{1}_{2\times 2}$ is the $2 \times 2$ unit matrix. Second, we perform the angular integral because the contribution from the pion part is independent of the angles as we have shown in Eq. (4.33). Performing tha angular integral, we obtain,

$$\int d\Omega\, (\boldsymbol{\tau} \cdot \hat{r})\, \tau_3\, (\boldsymbol{\tau} \cdot \hat{r}) = \frac{4\pi}{3} \tau_a \tau_3 \tau_a. \tag{4.82}$$

Using the wave function of the nucleon and di-quark summarized in Appendix G, we obtain,

$$\begin{aligned} & \langle \Sigma^0 |\, J^{5,a=3,(2)}_{\mu=0}\, | \bar{K} N \rangle \\ \xrightarrow{\int d\Omega} \; & \frac{i\,(E_{\bar{K}} + E_s) \sin F}{4} \frac{4}{3} s(r)\, k^*(r)\, e^{-i(E_s - E_K)t} \\ & \times \left[ \sqrt{\frac{1}{3}} \int d\mu(A)\, \psi^*_{d_{I_3=0,J_3=1}} \frac{2}{\sqrt{3}} \psi_{d_{I_3=0,J_3=1}} \right. \\ & \qquad \left. + \sqrt{\frac{1}{6}} \int d\mu(A)\, \psi^*_{d_{I_3=0,J_3=0}} \sqrt{\frac{2}{3}} \left( -\psi_{d_{I_3=0,J_3=0}} \right) \right] \\ = \; & \frac{i\,(E_{\bar{K}} + E_s) \sin F}{4} \frac{4}{3} s(r)\, k^*(r)\, e^{-i(E_s - E_K)t} \times \frac{1}{3} \\ = \; & \frac{i\,(E_{\bar{K}} + E_s) \sin F}{9} s(r)\, k^*(r)\, e^{-i(E_s - E_K)t}, \end{aligned} \tag{4.83}$$

where the integral in the isospin space is defined by,

$$\int d\mu(A) = \int_0^{\pi} d\theta_1 d\theta_2 \int_0^{2\pi} d\theta_3 \sin^2\theta_1 \sin\theta_2, \tag{4.84}$$

and the relation between the three angles and the SU(2) rotation matrix is given by,

$$\begin{cases} a_0 = \cos\theta_1 \\ a_1 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ a_2 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \\ a_3 = \sin\theta_1 \cos\theta_2. \end{cases} \tag{4.85}$$

Finally, we obtain the matrix element of the second derivative term on the effective Lagrangian Eq. (4.7),

$$
\begin{aligned}
& \int d^3x \ \langle \pi^0 \Sigma^0 | \, J^{5,a=3,(2)}_{\mu=0} \, | \bar{K} N \rangle \\
= \ & \frac{2}{F_\pi} \int d^3x \ \langle \pi | \, \partial^0 \pi^3 \, | 0 \rangle \, \langle \Sigma | \, J^{5,3}_0 \, | \bar{K} N \rangle \\
\to \ & \frac{2}{F_\pi} \int_0^\infty dr \ r^2 \left( -i E_\pi \frac{\sin (pr)}{pr} e^{-iE_\pi t} \right) \\
& \qquad \times \left\{ \frac{i \left( E_{\bar{K}} + E_s \right) \sin F}{9} s\left(r\right) k^* \left(r\right) e^{-i(E_s - E_K)t} \right\} \\
= \ & \frac{2}{F_\pi} \frac{E_\pi \left( E_{\bar{K}} + E_s \right)}{9} e^{-i(E_\pi + E_s - E_K)t} \\
& \qquad \times \int_0^\infty dr \ r^2 \left( \frac{\sin (pr)}{pr} \sin F s\left(r\right) k^* \left(r\right) \right), \qquad (4.86)
\end{aligned}
$$

where $E_{\bar{K}}$ and $E_s$ are the energies of the bound state anti-kaon and s-quark, respectively, and $p = p_\mu = (E_{pi}, \boldsymbol{p})$ is the pion four-momentum in the final state and we use Eqs. (4.25), (4.33), and (4.83). The wave functions for the anti-kaon and s-quark are denoted by $k\left(r\right)$ and $s\left(r\right)$, respectively,

It is possible to derive the other contributions with the same way as we mentioned above after painful calculations,

$$
\int d^3x \ \langle \pi^0 \Sigma^0 | \, J^{5,a=3,(4)}_{\mu=0} \, | \bar{K} N \rangle \to \frac{E_\pi}{e^2 {F_\pi}^3} e^{-i(E_\pi + E_s - E_{\bar{K}})t} \int dr \ r^2 \frac{\sin (pr)}{pr} \mathcal{I} \qquad (4.87)
$$

where

$$
\begin{aligned}
\mathcal{I} \ \equiv \ & -4\pi i \left( 2 e^2 {F_K}^2 \right) \langle \Sigma^0 | \, J_5^{a=3,\mu=0} \, | \bar{K} N \rangle \\
= \ & -\frac{\left( E_s + E_{\bar{K}} \right)}{3} s\left(r\right) k^* \left(r\right) \left[ \frac{2}{3} \sin F \left\{ \left( F' \right)^2 + \frac{\sin^2 F}{r^2} \right\} \right] \\
& + \frac{E_{\bar{K}}}{3} s\left(r\right) k^* \left(r\right) \left[ \frac{4}{3} \frac{c^2 \sin F}{r^2} \left( 5c^2 - s^2 \right) \right] + \frac{E_s}{3} s\left(r\right) k^* \left(r\right) \left[ \frac{4}{3} \frac{s^2 \sin F}{r^2} \left( -c^2 + 5s^2 \right) \right] \\
& - \frac{E_{\bar{K}}}{3} s'\left(r\right) k^* \left(r\right) \left[ 2F' \left( -\frac{5c^2}{3} + \frac{7s^2}{3} \right) \right] + \frac{E_s}{3} s\left(r\right) k^{*\prime} \left(r\right) \left[ 2F' \left( \frac{7c^2}{3} - \frac{5s^2}{3} \right) \right], \\
& \qquad (4.88)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int d^3x \ \langle \pi^0 \Sigma^0 | \, J^{5,a=3,(WZ)}_{\mu=0} \, | \bar{K} N \rangle \\
\to \ & -\frac{N_c E_\pi}{18 \pi^2 {F_\pi}^3} e^{-i(E_\pi + E_s - E_{\bar{K}})t} \\
& \times \int dr \ r^2 \frac{\sin (pr)}{pr} \left( s\left(r\right) k^* \left(r\right) \frac{4 \sin F F'}{r^2} + s'\left(r\right) k^* \left(r\right) \frac{2 \sin^2 F}{r^2} \right. \\
& \qquad \left. - s\left(r\right) k^{*\prime} \left(r\right) \frac{2 \sin^2 F}{r^2} \right). \qquad (4.89)
\end{aligned}
$$

In the above equations, $s = \sin(F'/2)$, $c = \cos(F'/2)$. The prime symbol stands for the $r$-derivative, for example,

$$F'(r) = \frac{dF(r)}{dr}. \tag{4.90}$$

Now, we have analytically derived the coupling constant for the $\Lambda(1405)$-$\pi^0\Sigma^0$ vertex as the sum of Eqs. (4.86), (4.87), and (4.89). We would like to show the numerical result for the decay width of the $\bar{K}N$ Feshbach resonance. To do that, we first show parameters in numerical calculations below.

- Pion decay constant $F_\pi$ and the Skyrme parameter $e$

  They are the basic parameters in our study and we use the same three parameter sets A, B ,and C as we have introduced in Sec.3.5. If we determine the parameter set, the profile function $F(r)$ is obtained and the binding energies of the anti-kaon and s-quark, $E_{\bar{K}}$ and $E_s$, are derived.

- Masses and energies of particles, and momentum transfer

  They are requested when we evaluate the decay width from Eq. (4.4). The particles related to our purpose are the $\Sigma$, $\pi$, s-quark, $\bar{K}$ and $\Lambda(1405)$.

  We first consider the $\Sigma$, $\pi$, and $\Lambda(1405)$. To determine the energies of $\Sigma$ and $\pi$, we need to introduce their masses because their energies are determined by,

$$E_\Sigma = \sqrt{|\boldsymbol{p}|^2 + m_\Sigma^2} \tag{4.91}$$

  and

$$E_\pi = \sqrt{|\boldsymbol{p}|^2 + m_\pi^2}, \tag{4.92}$$

  where $\boldsymbol{p}$ is obtained from the energy-momentum conservation,

$$\begin{cases} E_\Sigma + E_\pi = E_{\Lambda(1405)} = m_{\Lambda(1405)} \\ \boldsymbol{p}_\Sigma + \boldsymbol{p}_\pi = \boldsymbol{0}, \quad (|\boldsymbol{p}| = |\boldsymbol{p}_\Sigma| = |\boldsymbol{p}_\pi|) \end{cases} \tag{4.93}$$

  To decide the Energies, $E_\Sigma$ and $E_\pi$, we introduce the masse of $\Sigma$ and $\pi$. In this study, we take their masses at their experimental values,

$$m_\Sigma = 1193 \text{ MeV}, \quad m_\pi = 138 \text{ MeV}, \tag{4.94}$$

  where we take an isospin average of masses of $\pi$ and $\Sigma$

For the mass of the $\Lambda\,(1405)$, we take it at 1420 MeV which corresponds to the mass of the higher pole in the chiral unitary approach, which originates in the $\bar{K}N$ bound state [55],

$$m_{\Lambda(1405)} = 1420 \text{ MeV}. \tag{4.95}$$

Therefore, we obtain the energies of $\Sigma$ and $\pi$ from Eq. (4.93),

$$E_\Sigma = 1204 \text{ MeV}, \quad E_\pi = 216 \text{ MeV}, \quad \boldsymbol{p} = 166 \text{ MeV}. \tag{4.96}$$

As we have shown, we choose the experimental values for the masses of $\pi$, $\Sigma$, and $\Lambda\,(1405)$ because we consider that it is important to use the physical values for the realistic situations.

Next, we consider the energies of the s-quark and $\bar{K}$. They are related to their binding energies.

$$\begin{cases} E_s = m_K - E^s_{B.E.} \\ E_{\bar{K}} = m_K - E^{\bar{K}}_{B.E.}, \end{cases} \tag{4.97}$$

where $E^s_{B.E.}$ and $E^{\bar{K}}_{B.E.}$ are the binding energies of the kaon-Hedgehog and kaon-nucleon systems, respectively, and we take the mass of the kaon, $m_K$ at 495 MeV.

To summarize the numerical parameters, we would like to classify them into to group: one is independent of $F_\pi$ and $e$ and the other is dependent on $F_\pi$ and $e$. The former are the masses and energies of the hadrons ($m_\pi$, $E_\pi$, $m_K$, $m_\Sigma$, $E_\Sigma$, and $m_{\Lambda(1405)}$) and the momentum transfer $\boldsymbol{p}$ is also independent of $F_\pi$ and $e$. We summarize them in Tab. 4.2.

Table 4.2: Parameters independent of $F_\pi$ and $e$

| $m_\pi$ | $E_\pi$ | $m_K$ | $m_\Sigma$ | $E_\Sigma$ | $\Lambda\,(1405)$ | $\lvert\boldsymbol{p}\rvert$ |
|---|---|---|---|---|---|---|
| 138 MeV | 216 MeV | 495 MeV | 1193 MeV | 1204 MeV | 1420 MeV | 166 MeV |

The latter are the energies of the s-quark and anti-kaon. We summarize them for each parameter set in Tab. 4.3.

Table 4.3: Parameters for the decay width.

| | $F_\pi$ [MeV] | $e$ | $E_s$ [MeV] | $E_{\bar{K}}$ [MeV] |
|---|---|---|---|---|
| Set A | 205 | 4.67 | 360.8 | 475.0 |
| Set B | 186 | 4.82 | 345.3 | 463.7 |
| Set C | 129 | 5.45 | 276.7 | 415.5 |

We numerically calculate the coupling constant for the $\Lambda(1405)$-$\pi^0\Sigma^0$ vertex and the result is shown in Tab. 4.4. The $\Lambda(1405)$ can decay three $\pi\Sigma$ states, $\pi^0\Sigma^0$, $\pi^+\Sigma^-$, and $\pi^-\Sigma^+$. Therefore, the total coupling constant $g_{\Lambda^*\pi\Sigma}$ is given by,

Table 4.4: Coupling constant for the $\Lambda(1405)$-$\pi^0\Sigma^0$ vertex.

| | $F_\pi$ [MeV] | $e$ | $g_{\Lambda^*\pi^0\Sigma^0}$ |
|---|---|---|---|
| Set A | 205 | 4.67 | 0.187 |
| Set B | 186 | 4.82 | 0.223 |
| Set C | 129 | 5.45 | 0.333 |

$$g_{\Lambda^*\pi\Sigma} = \sqrt{3} g_{\Lambda^*\pi^0\Sigma^0}. \tag{4.98}$$

In Tab. 4.5, we show the total coupling constant and the decay width obtained from Eq. (4.4).

Table 4.5: To total coupling constant $g_{\Lambda(1405)\pi\Sigma}$ and the decay width of the $\bar{K}N$ Feshbach resonance.

| | $F_\pi$ [MeV] | $e$ | $g_{\Lambda(1405)\pi\Sigma}$ | $\Gamma_{\Lambda^*\pi\Sigma}$ [MeV] |
|---|---|---|---|---|
| Set A | 205 | 4.67 | 0.324 | 2.3 |
| Set B | 186 | 4.82 | 0.386 | 3.3 |
| Set C | 129 | 5.45 | 0.576 | 7.4 |

From this table, we find that the decay width is few MeV for the sets A and B. For the set C, the width is slightly larger than those of the other sets but it is about 7 MeV. As a result, the $\Lambda(1405)$ as the $\bar{K}N$ Feshbach resonance turns out to be a narrow resonance in our approach.

We can see that the width becomes larger as the coupling constant gets large. This is because the factor without the coupling constant in the formula for the decay width Eq. (4.4) is common among the three parameter sets in our study,

$$\Gamma_{\Lambda^*\to\pi\Sigma} = g^2_{\Lambda(1405)\pi\Sigma} \frac{|\boldsymbol{p}|}{\pi} \frac{E_\Sigma + m_\Sigma}{4(E_\Sigma + E_\pi)} \simeq g^2_{\Lambda(1405)\pi\Sigma} \times 22.3 \text{ MeV}. \tag{4.99}$$

Therefore, the reason for small width is the same as that for the coupling constant.

We consider why the coupling constant turns out to be narrow. It is related to the wave functions of the s-quark and anti-kaon. As we have shown in Eqs. (4.86), (4.87), and (4.89), the coupling constant depends on the overlap of the wave functions, $s(r)\,k^*(r)$. The s-quark wave function is obtained by the CK approach and it behaves as an s-wave shown in Fig. 3.1. On the other hand, the anti-kaon wave function is derived from our approach and it behaves a p-wave shown in Fig. 3.3. Therefore, the coupling constant is suppressed because the overlap of the two wave functions, $s\,(r)\,k^{*}\,(r)$, is that of the s-wave and p-wave.

# Chapter 5

# Summary

In this study, we have investigated the $\Lambda(1405)$ as a $\bar{K}N$ Feshbach resonance in the Skyrme model. To do that, we have first constructed a new approach to describe the kaon-nucleon systems and investigated the kaon-nucleon interactions. Our approach is based on the bound state approach in the Skyrme model proposed by Callan and Klebanov [20, 21]. In our approach, we first quantize the Hedgehog soliton as a nucleon and then introduce the kaon as fluctuations around the physical nucleon, which corresponds *variation after projection*. This is the different point from the Callan-Klebanov's (CK) approach [20, 21]. In their approach, we first introduce the kaon fluctuation around the Hedgehog soliton and then the kaon-Hedgehog system is quantized as a hyperon. The CK approach corresponds to *projection after variation*. Changing the order of variation and projection, our approach does not obey the $1/N_c$ expansion but we consider that our approach is more suitable to physical kaon-nucleon systems.

For the kaon-nucleon interaction, it contains central force with and without isospin dependence and the similar spin-orbit ones, which completes a general structure between isoscalar-pseudoscalar kaon and isospinor-spinor nucleon. A nontrivial finding in our study is that there exists a repulsive component proportional to $1/r^2$ for small $r$. This repulsion is considered to be a centrifugal-like force by the isospin rotating Hedgehog soliton. For the s-wave kaon-nucleon interaction, the resulting potential turns out to contain the repulsive core at short distances and the attractive pocket at the middle range. The attractive pocket comes from the Wess-Zumino action, which physically corresponds to the $\omega$-meson exchange. As a result, the $\bar{K}N$ bound state is generated as a weekly binding object. Furthermore, the existence of the repulsive core should influence the properties of the high density nuclear matter with anti-kaon.

For the kaon-nucleon scattering states, we have investigated the phase shifts for the s-wave $\bar{K}N$ and $KN$ scatterings. The obtained phase shifts indicate that the $\bar{K}N$ interaction is attractive and $KN$ one is repulsive, whose difference is simply explained by the Wess-Zumino action. Then, we have calculated the scattering length for the $\bar{K}N$ scattering state but the resulting length turns out to be larger than the experimental results and other theoretical calculations [52, 53, 54].

Second, we have constructed a new method to investigate the decay width of $\Lambda\,(1405)$. In the present study, we have regarded the $\Lambda\,(1405)$ as a $\bar{K}N$ Feshbach resonance as a first attempt and have evaluated its width. To do that, we have introduced an effective Lagrangian which is given by the pion axial current and the baryon axial current. Taking a matrix element of the effective Lagrangian with the initial $\Lambda\,(1405)$ and final $\pi\Sigma$ states, we have derived the coupling constant for the $\Lambda\,(1405)$-$\pi\Sigma$ vertex. To compute the matrix element, we have combined the CK and our approach: the $\Sigma$ hyperon is constructed in the CK approach and the $\Lambda\,(1405)$ appears as the $\bar{K}N$ bound state in our approach. The obtained coupling constant and width is few MeV and the $\bar{K}N$ Feshbach resonance is realized as a narrow resonance in this study.

Finally, we show our future plan. In the present work, we have considered the $\Lambda\,(1405)$ resonance as the $\bar{K}N$ Feshbach resonance. However, in the real situation, the $\Lambda\,(1405)$ is a resonance of the $\bar{K}N$ and $\pi\Sigma$ channels. We are planning to investigate the coupled channel of the $\bar{K}N$ and $\pi\Sigma$.

# Appendix A

# Spin and isospin operators

Here, we derive the spin and isospin operators in the Skyrme model. We show the Skyrme Lagrangian with the rotating Hedgehog ansatz for later convenience,

$$\begin{aligned} L &\equiv \int d^3x \mathcal{L}_{Skyrme}\left(U_H = AU_H A^\dagger\right) \\ &= -M_{sol} + \Lambda \mathrm{tr}\left[\dot{A}\dot{A}^\dagger\right] \quad \left(\dot{A} \equiv \frac{dA(t)}{dt}\right), \end{aligned} \tag{A.1}$$

where $M_{sol}$ is the classical soliton mass, $A = A(t)$ is an SU(2) isospin rotation matrix, and $\Lambda$ is the moment of inertia. The second term in Eq. (A.1) is the contribution of the rotation energy.

To derive the spin and isospin operator, we first consider the spatial and isospin rotations.

- Spatial rotation

$$AU_H A^\dagger = Ae^{i\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)}A^\dagger \to Ae^{i\tau_a R_{ab}\hat{r}_b F(r)}A^\dagger = ARe^{i\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)}R^\dagger A^\dagger, \tag{A.2}$$

where $R$ is a spatial rotation matrix with a generator $\frac{\boldsymbol{\sigma}}{2}$ and a rotation angle $\boldsymbol{\theta}$,

$$R = \exp\left[i\boldsymbol{J}\cdot\boldsymbol{\theta}\right], \qquad \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}. \tag{A.3}$$

- Isospin rotation

$$AU_H A^\dagger = Ae^{i\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)}A^\dagger \to DAe^{i\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)}A^\dagger D^\dagger, \tag{A.4}$$

where $D$ is a spatial rotation matrix with a generator $\frac{\boldsymbol{\tau}}{2}$ and a rotation angle $\boldsymbol{\phi}$,

$$D = \exp\left[i\boldsymbol{I}\cdot\boldsymbol{\phi}\right], \qquad \boldsymbol{I} = \frac{\boldsymbol{\tau}}{2}. \tag{A.5}$$

As a result, we find that the spatial and isospin rotations correspond to the following transformations,

$$\begin{cases} A \to AR & \text{for spatial rotation} \\ A \to DA & \text{for isospin rotation.} \end{cases} \tag{A.6}$$

Therefore, the rotation energy part in Eq. (A.1),

$$L_{rot} \equiv \Lambda \mathrm{tr}\left[\dot{A}\dot{A}^\dagger\right] \tag{A.7}$$

is relevant under these rotations. Thanks to the trace properties, we can easily check that $L_{rot}$ is invariant under the two transformations Eq. (A.6).

Now, we derive the spin and isospin operators as the Noether's charge. We first consider the spin operator. From Eq. (A.6), the infinitesimal spatial rotation is given by,

$$A \to AR \simeq A\left(1 + \frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\theta}\right), \tag{A.8}$$

and the Noether's current [40] related to the infinitesimal transformation is written as,

$$j^\mu = \sum_{\phi=A,A^\dagger} \frac{\partial L_{rot}}{\partial(\partial_\mu\phi)}\delta\phi, \tag{A.9}$$

Therefore Noether's charge, that is classical spin operator $J_k$, is given by,

$$\begin{aligned} J_k = j^0 &= \Lambda \mathrm{tr}\left[\dot{A}^\dagger \frac{i}{2} A\tau_k\right] + \Lambda \mathrm{tr}\left[\left(-\frac{i}{2}\tau_k A^\dagger\right)\dot{A}\right] \\ &= i\Lambda \mathrm{tr}\left[\dot{A}^\dagger A\tau_k\right] \quad \left(\because \dot{A}^\dagger A = -A^\dagger\dot{A}\right), \end{aligned} \tag{A.10}$$

where $k$ is the three-dimensional space indices, $k = 1, 2, 3$.

Next, we express the obtained spin operator with the collective coordinates, $a_\mu\,(\mu = 0, 1, 2, 3)$. To do that, we introduce $a_\mu$ as follows,

$$A = a_0 + ia_i\tau_i \ \in SU(2), \qquad (i = 1, 2, 3) \tag{A.11}$$

then $L_{rot}$ and the spin operator $J_k$ are, respectively, given by,

$$L_{rot} = 2\Lambda\left(\dot{a}_0^2 + \dot{a}_i^2\right) \tag{A.12}$$

and

$$\begin{aligned} & J_k = -2\Lambda\left(\dot{a}_0 a_k - \dot{a}_k a_0 + \epsilon_{ijk}\dot{a}_i a_j\right) \\ \Leftrightarrow \ & J_i = -2\Lambda\left(\dot{a}_0 a_i - \dot{a}_i a_0 + \epsilon_{ijk}\dot{a}_j a_k\right). \end{aligned} \tag{A.13}$$

Furthermore, the conjugate momenta is defined by,

$$\pi_\mu \equiv \frac{\partial L}{\partial a_\mu} = 4\Lambda \dot{a}_\mu. \tag{A.14}$$

Using $\pi_\mu$, the spin operator, $J_i$ is written as,

$$\begin{aligned}
& J_i = -\frac{1}{2}\left(\pi_0 a_i - \pi_i a_0 + \epsilon_{ijk}\pi_j a_k\right) \\
\rightarrow\quad & \hat{J}_i = \frac{i}{2}\left(\frac{\partial}{\partial a_0}a_i - \frac{\partial}{\partial a_i}a_0 + \epsilon_{ijk}\frac{\partial}{\partial a_j}a_k\right) \\
\Leftrightarrow\quad & \hat{J}_i = \frac{i}{2}\left(a_i\frac{\partial}{\partial a_0} - a_0\frac{\partial}{\partial a_i} - \epsilon_{ijk}a_j\frac{\partial}{\partial a_k}\right), \quad (i = 1, 2, 3),
\end{aligned} \tag{A.15}$$

where we perform the canonical quantization, $\pi_\mu \rightarrow -i\partial/\partial a_\mu$ in the second line.

The isospin operator is obtained with the same way as the case of the spin one,

$$I_a = i\Lambda \mathrm{tr}\left[\dot{A}A^\dagger \tau_a\right], \tag{A.16}$$

and

$$\hat{I}_i = \frac{i}{2}\left(a_0\frac{\partial}{\partial a_i} - a_i\frac{\partial}{\partial a_0} - \epsilon_{ijk}a_j\frac{\partial}{\partial a_k}\right), \quad (i = 1, 2, 3). \tag{A.17}$$

From Eqs. (A.10) and (A.16), we find that it is possible to obtain $J_a$ from $I_a$ by exchanging $A$ with $A^\dagger$, which reflect the strong correlation of spin and isospin in the Hedgehog ansatz.

# Appendix B

# Baryon number current

In this appendix, we derive the baryon number current from the Wess-Zumino action [22, 23, 24] as the Noether's current [40] associated with the U(1) vector transformation, The Wess-Zumino action is given by,

$$\begin{aligned}\Gamma_{WZ} &= \frac{iN_c}{240\pi^2}\int d^5x\ \epsilon^{\mu\nu\alpha\beta\gamma}\mathrm{tr}\left[L_\mu L_\nu L_\alpha L_\beta L_\gamma\right] \\ &= \frac{iN_c}{240\pi^2}\int \mathrm{tr}\left[\alpha^5\right], \end{aligned} \tag{B.1}$$

where $L_\mu = U^\dagger \partial_\mu U$ and we introduce an 1-form $\alpha$ for our convenience,

$$\alpha = U^\dagger \partial_\mu U dx^\mu = U^\dagger dU, \tag{B.2}$$

which satisfies,

$$\begin{aligned} d\alpha + \alpha^2 &= \partial_\mu\left(U^\dagger\partial_\nu U\right)dx^\mu\wedge dx^\nu + \left(U^\dagger\partial_\mu U dx^\mu\right)\wedge\left(U^\dagger\partial_\nu U dx^\nu\right) \\ &= \partial_\mu U^\dagger \partial_\nu U dx^\mu \wedge dx^\nu - \partial_\mu U^\dagger \partial_\nu U dx^\mu \wedge dx^\nu \\ &= 0 \end{aligned} \tag{B.3}$$

First of all, we derive the general expression for the vector current from the Wess-Zumino action. To do that, we consider the following vector transformation,

$$U \to gUg^\dagger, \quad g \equiv e^{i\boldsymbol{\theta}\cdot\boldsymbol{T}/2} = e^{i\phi}, \tag{B.4}$$

where $\theta$ and $T$ are a rotation angle and generator, respectively.

Under this transformation, we derive the vector current as the Noether's current. First, we consider how the variable, $\alpha$ transforms.

$$\begin{aligned} \alpha &= U^\dagger dU \\ &\to \left(gU^\dagger g^\dagger\right)d\left(gUg^\dagger\right), \quad g = e^{i\phi} \\ &\simeq (1+i\phi)\,U^\dagger\,(1-i\phi)\,d\left[(1+i\phi)\,U\,(1-i\phi)\right] \\ &= \alpha + iU^\dagger d\phi U - id\phi - i\alpha\phi + i\phi\alpha \\ &\equiv \alpha + \delta\alpha. \end{aligned} \tag{B.5}$$

Here we consider the infinitesimal transformation and we define,

$$\delta\alpha \equiv iU^\dagger d\phi U - id\phi - i\alpha\phi + i\phi\alpha, \quad d\phi = d\theta^a \frac{T^a}{2}. \tag{B.6}$$

Then we obtain,,

$$\begin{aligned}
\delta\Gamma_{WZ} &\equiv \Gamma_{WZ}\,|_{U=gUg^\dagger} - \Gamma_{WZ}\,|_{U=U} \\
&= \int \frac{iN_c}{240\pi^2}\left(\mathrm{tr}\left[(\alpha+\delta\alpha)^5\right] - \mathrm{tr}\left[\alpha^5\right]\right) \\
&\simeq \int \frac{iN_c}{240\pi^2}\left(5\mathrm{tr}\left[\delta\alpha\alpha^4\right]\right) \\
&= \int \frac{iN_c}{48\pi^2}\mathrm{tr}\left[\left(iU^\dagger d\phi U - id\phi - i\alpha\phi + i\phi\alpha\right)\alpha^4\right] \\
&= -\int \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi U\alpha^4 U^\dagger - d\phi\alpha^4\right] \\
&= \int \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi\alpha^4 - d\phi\beta^4\right],
\end{aligned} \tag{B.7}$$

where $\alpha = U^\dagger dU, \quad \beta \equiv -U\alpha U^\dagger = UdU^\dagger = R_\mu dx^\mu$, and a 1-form $\beta$ satisfies,

$$\beta^2 + d\beta = 0. \tag{B.8}$$

As we shown in Eq. (B.1), the Wess-Zumino action is defined in the five dimensional space, $D_5$. Therefore we next change the five-dimensional integral into the four-dimensional one by using the Stoke's theorem. To do that, we use Eqs. (B.3) and (B.8) for the above result,

$$\begin{aligned}
\int_{D_5} \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi\alpha^4 - d\phi\beta^4\right] &= -\int_{D_5} \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi d\alpha\alpha^2 - d\phi d\beta\beta^2\right] \\
&= \int_{D_5} \frac{N_c}{48\pi^2} d\left(\mathrm{tr}\left[d\phi\alpha\alpha^2 - d\phi\beta\beta^2\right]\right) \\
&= \int_{\partial D_5} \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi\alpha\alpha^2 - d\phi\beta\beta^2\right] \\
&\qquad (\because \text{Stoke's theorem}),
\end{aligned} \tag{B.9}$$

where $\partial D_5$ is the boundary of the five-dimensional space which corresponds to the four-dimensional space-time. We will discuss it later.

As a result, we obtain the vector current, $J^{\mu,a}$,

$$\begin{aligned}
J^{\mu,a} &= \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{T^a}{2}\left(L_\nu L_\alpha L_\beta - R_\nu R_\alpha R_\beta\right)\right] \\
&= \frac{N_c}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{T^a}{2}L_\nu L_\alpha L_\beta\right],
\end{aligned} \tag{B.10}$$

where,

$$L_\mu = U^\dagger\partial_\mu U \tag{B.11}$$

$$R_\mu = U\partial_\mu U^\dagger = -L_\mu. \tag{B.12}$$

For the case of the U(1) vector transformation, the baryon number current is given by,

$$J^\mu = \frac{N_c}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[L_\nu L_\alpha L_\beta\right] \tag{B.13}$$

Finally, we have a comment for the space where the Wess-Zumino action is defined. It is defined in the 5-dimensional space, $D_5$, given by,

$$D_5 = \mathbb{R}^3 \times \mathbb{R}^2, \tag{B.14}$$

where $\mathbb{R}^3$ is the 3-dimensional space and $\mathbb{R}^2$ stands for the time and fifth directions. Then, we compactify the these spaces as follows,

$$\begin{cases} \mathbb{R}^3 \simeq S^3 \\ \mathbb{R}^2 \simeq S^2, \end{cases} \tag{B.15}$$

where $S^n$ is the $n$-sphere which is a $n$-dimensional sphere defined in the $(n+1)$-dimensional space. Furthermore $S^2$ is generally written by

$$S^2 \simeq S^1 \times [-1, 1] \tag{B.16}$$

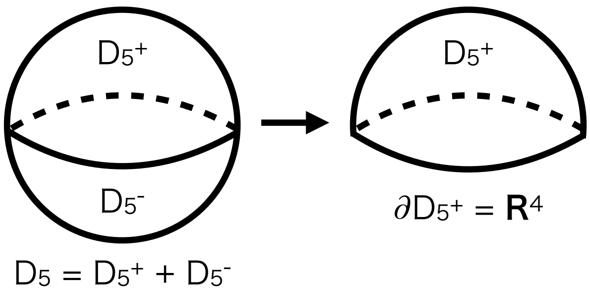


Figure B.1: Schematic description of the five dimensional space.

Next, we decompose $D_5$ into two hemisphere, $D_5^+$ and $D_5^-$,

$$\begin{cases} D_5^+ = S^3 \times S_t^1 \times [0, 1] \\ D_5^- = S^3 \times S_t^1 \times [-1, 0], \end{cases} \tag{B.17}$$

where $S_t^1$ stands for the time direction and we decompose $S^2$ into the two hemisphere with Eq. (B.16). The boundary of the 5-dimensional space, $\partial D_5$, is given by the Euclidean space, $\boldsymbol{R}^4 = S^3 \times S_t^1$, in the above equation (see Fig. B.1).

# Appendix C

# Explicit form of the interaction

In this appendix, we show the explicit form of the interaction term Eq. (3.57) in Sec. 3.4. As we shown, the interaction term are written as follows,

$$V\left(r\right)=V_{0}^{c}\left(r\right)+V_{\tau}^{c}\left(r\right)I_{KN}+V_{0}^{LS}\left(r\right)J_{KN}+V_{\tau}^{LS}\left(r\right)J_{KN}I_{KN}, \tag{C.1}$$

where the first and second terms are the isospin independent and dependent central ones, respectively, and the third and last terms are the isospin independent and dependent spin-orbit ones, respectively. Their explicit form are given by,

$$\begin{aligned}
V_{0}^{c}\left(r\right) &= -\frac{1}{4}\left(2\frac{\sin^{2}F}{r^{2}}+\left(F'\right)^{2}\right)+2\frac{s^{4}}{r^{2}}+\left[1+\frac{1}{\left(eF_{\pi}\right)^{2}}\left(F'^{2}+\frac{\sin^{2}F}{r^{2}}\right)\right]\frac{l\left(l+1\right)}{r^{2}}\\
&\quad -\frac{1}{\left(eF_{\pi}\right)^{2}}\left[2\frac{\sin^{2}F}{r^{2}}\left(\frac{\sin^{2}F}{r^{2}}+2\left(F'\right)^{2}\right)-2\frac{s^{4}}{r^{2}}\left(\left(F'\right)^{2}+\frac{\sin^{2}F}{r^{2}}\right)\right]\\
&\quad +\frac{1}{\left(eF_{\pi}\right)^{2}}\frac{6}{r^{2}}\left[\frac{s^{4}\sin^{2}F}{r^{2}}+\frac{d}{dr}\left\{s^{2}\sin FF'\right\}\right]\\
&\quad +\frac{2E}{\Lambda}s^{2}\left[1+\frac{1}{\left(eF_{\pi}\right)^{2}}\left(\left(F'\right)^{2}+\frac{5}{r^{2}}\sin^{2}F\right)\right]\\
&\quad +\frac{3}{\left(eF_{\pi}\right)^{2}}\frac{1}{r^{2}}\frac{d}{dr}\left[r^{2}\left(\frac{EF'\sin F}{\Lambda}\right)\right]\pm\frac{3}{\pi^{2}F_{\pi}^{2}}\frac{\sin^{2}F}{r^{2}}F'\left(E-\frac{s^{2}}{\Lambda}\right),
\end{aligned} \tag{C.2}$$

$$V_{\tau}^{c}\left(r\right)=\frac{8E}{3\Lambda}s^{2}\left[1+\frac{1}{\left(eF_{\pi}\right)^{2}}\left(\left(F'\right)^{2}+\frac{4}{r^{2}}\sin^{2}F\right)\right]+\frac{4}{\left(eF_{\pi}\right)^{2}}\frac{1}{r^{2}}\frac{d}{dr}\left[r^{2}\left(\frac{EF'\sin F}{\Lambda}\right)\right], \tag{C.3}$$

$$V_{0}^{LS}\left(r\right)=\frac{1}{\left(eF_{\pi}\right)^{2}}\frac{2E\sin^{2}F}{\Lambda r^{2}}\pm\frac{3}{F_{\pi}^{2}\pi^{2}}\frac{\sin^{2}F}{\Lambda r^{2}}F', \tag{C.4}$$

and

$$V_{\tau}^{LS}\left(r\right)=-\left[1+\frac{1}{\left(eF_{\pi}\right)^{2}}\left(\left(F'\right)^{2}+4\frac{\sin^{2}F}{r^{2}}\right)\right]\frac{16s^{2}}{3r^{2}}-\frac{1}{\left(eF_{\pi}\right)^{2}}\frac{8}{r^{2}}\left[\frac{d}{dr}\left(\sin FF'\right)\right]. \tag{C.5}$$

In the above equations,

$$s = \sin\left(F\left(r\right)/2\right), \tag{C.6}$$

$$F' = \frac{dF\left(r\right)}{dr}, \tag{C.7}$$

and

$$I_{KN} = \boldsymbol{I}^K \cdot \boldsymbol{I}^N, \qquad J_{KN} = \boldsymbol{L}^K \cdot \boldsymbol{J}^N, \tag{C.8}$$

where $\boldsymbol{I}^K$ and $\boldsymbol{I}^N$ stand for the isospin of the kaon and nucleon, respectively. The kaon angular momentum and nucleon spin are $\boldsymbol{L}^K$ and $\boldsymbol{J}^N$, respectively. The moment of inertia $\Lambda$ is given by

$$\Lambda = \frac{2\pi}{3} F_\pi^2 \int drr^2 \sin^2 F \left[1 + \frac{4}{\left(eF_\pi\right)^2} \left(\left(F'\right)^2 + \frac{\sin^2 F}{r^2}\right)\right]. \tag{C.9}$$

The last terms of Eq. (C.2) and Eq. (C.4) are derived from the Wess-Zumino term, which is attractive for the $\bar{K}N$ potential and repulsive for the $KN$ potential. These equations are general for any partial waves of the kaon. For instance, the $s$-wave potential is obtained by setting $l = 0$ and removing the terms including $J_{KN}$ in Eq. (3.57).

# Appendix D

# Decay width in the field theory

## D.1 Notations

We first show our notations, which is based on those of M. E. Peskin and D. V. Schroeder [56].

### D.1.1 $\gamma$-matrices

$\gamma$-matrices are defined by,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{D.1}$$

where

$$\sigma^\mu = (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}), \tag{D.2}$$

and

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{D.3}$$

### D.1.2 Free boson field

- Mode expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left[ a(\boldsymbol{p}) e^{-ipx} + b^\dagger(\boldsymbol{p}) e^{ipx} \right]. \tag{D.4}$$

  If $\phi$ is real, the $a = b$.

- Commutation relation

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \tag{D.5}$$

  and

$$\left[ a(\boldsymbol{p}), a^\dagger(\boldsymbol{q}) \right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}). \tag{D.6}$$

- One particle state

$$|\boldsymbol{p}\rangle \equiv \sqrt{2E_{\boldsymbol{p}}}a^{\dagger}(\boldsymbol{p})|0\rangle . \tag{D.7}$$

- Wave function

$$\begin{aligned}\langle 0|\phi(x)|\boldsymbol{p}\rangle &= \langle 0|\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p'}}}}\left[a(\boldsymbol{p'})e^{-ip'x}+b^{\dagger}(\boldsymbol{p'})e^{ip'x}\right]\cdot\sqrt{2E_{\boldsymbol{p}}}a^{\dagger}(\boldsymbol{p})|0\rangle \\ &= e^{-ipx}.\end{aligned} \tag{D.8}$$

### D.1.3 Free Fermion field

- Mode expansion

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_r\left[a_r(\boldsymbol{p})u_r(\boldsymbol{p})e^{-ipx}+b_r^{\dagger}(\boldsymbol{p})v_r(\boldsymbol{p})e^{ipx}\right], \tag{D.9}$$

and

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_r\left[b_r(\boldsymbol{p})\bar{v}_r(\boldsymbol{p})e^{-ipx}+a_r^{\dagger}(\boldsymbol{p})\bar{u}_r(\boldsymbol{p})e^{ipx}\right]. \tag{D.10}$$

- Anti-commutation relation

$$\left\{\psi_a(\boldsymbol{x}),\psi_b^{\dagger}(\boldsymbol{y})\right\} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{ab}, \tag{D.11}$$

and

$$\{a_r(\boldsymbol{p}),a_s(\boldsymbol{q})\} = \{b_r(\boldsymbol{p}),b_s(\boldsymbol{q})\} = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta_{rs},\ \text{etc.} \tag{D.12}$$

- Normalization and explicit form of the spinor

$$\begin{cases}\bar{u}_r(\boldsymbol{p})u_s(\boldsymbol{p}) = -\bar{v}_r(\boldsymbol{p})v_s(\boldsymbol{p}) = 2m\delta_{rs}\\ u_r^{\dagger}(\boldsymbol{p})u_s(\boldsymbol{p}) = v_r^{\dagger}(\boldsymbol{p})v_s(\boldsymbol{p}) = 2E_{\boldsymbol{p}}\delta_{rs}\\ u_r^{\dagger}(\boldsymbol{p})v_s(-\boldsymbol{p}) = 0,\end{cases} \tag{D.13}$$

and

$$\sum_s u_s(\boldsymbol{p})\bar{u}_s(\boldsymbol{p}) = \not{p}+m, \quad \sum_s v_s(\boldsymbol{p})\bar{v}_s(\boldsymbol{p}) = \not{p}-m. \tag{D.14}$$

For $r = 1, 2$

$$u_s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \chi_r \tag{D.15}$$

$$v_s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma} \\ -\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \chi_r \tag{D.16}$$

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{D.17}$$

where

$$\sigma^\mu = (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}). \tag{D.18}$$

- One particle state

$$|\boldsymbol{p}, s\rangle \equiv \sqrt{2E_{\boldsymbol{p}}}\, a_s^\dagger(\boldsymbol{p})\, |0\rangle. \tag{D.19}$$

- Wave function

$$\begin{aligned} & \langle 0|\, \psi(x)\, |\boldsymbol{p}, s\rangle \\ = & \langle 0| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p'}}}} \sum_r \left[ a_r(\boldsymbol{p'})\, u_r(\boldsymbol{p'})\, e^{-ip'x} + b_r^\dagger(\boldsymbol{p'})\, v_r(\boldsymbol{p'})\, e^{ip'x} \right] \\ & \times \sqrt{2E_{\boldsymbol{p}}}\, a_s^\dagger(\boldsymbol{p})\, |0\rangle \\ = & u_r(\boldsymbol{p})\, e^{-ipx}. \end{aligned} \tag{D.20}$$

## D.2 Decay width

We derive the decay width for the $\Lambda(1405) \to \pi\Sigma$ process from a effective Lagrangian. To do that, let us start with the following Lagrangian,

$$\mathcal{L}_{eff} = g\bar{\psi}_\Sigma^a \pi^a \psi_{\Lambda^*}, \tag{D.21}$$

where $g$ is a dimensionless coupling constant for the $\Lambda(1405)$-$\pi\Sigma$ vertex and $a$ is the isospin indices, $a = 1, 2, 3$. The field operators for $\Sigma$, $\pi$, and $\Lambda(1405)$ stand for $\bar{\psi}_\Sigma$, $\pi$, and $\psi_{\Lambda^*}$, respectively. For our convenience, we abbreviate $\Lambda(1405)$ to $\Lambda^*$ in the Lagrangian.

To derive the decay width for the $\Lambda(1405) \to \pi\Sigma$ process, we first consider the following matrix element of the effective Lagrangian,

$$\langle \pi\Sigma|\, \mathcal{L}_{eff}\, |\Lambda(1405)\rangle. \tag{D.22}$$

Using the wave functions discussed in the previous section, we obtain,

$$\langle \pi\Sigma | \mathcal{L}_{eff} | \Lambda(1405) \rangle = g\bar{u}_s(\boldsymbol{p}_\Sigma) u_r(\boldsymbol{p}_{\Lambda^*}) e^{-i(p_{\Lambda^*}-p_\pi-p_\Sigma)x}. \tag{D.23}$$

Performing space-time Integral, we obtain the invariant amplitude, which is a probability amplitude for this decay process

$$\begin{aligned} \mathcal{M}(\Lambda(1405) \to \pi\Sigma) &= \int d^4x \langle \pi\Sigma | \mathcal{L}_{eff} | \Lambda(1405) \rangle \\ &= g\bar{u}_s(\boldsymbol{p}_\Sigma) u_r(\boldsymbol{p}_{\Lambda^*}) (2\pi)^4 \delta^{(4)}(p_{\Lambda^*} - p_\pi - p_\Sigma). \end{aligned} \tag{D.24}$$

Here, we perform the space-time integral in order to estimate the total probability in the whole universe.

Therefore, we obtain the probability as follows,

$$|\mathcal{M}(\Lambda(1405) \to \pi\Sigma)|^2 = g^2 |\bar{u}_s(\boldsymbol{p}_\Sigma) u_r(\boldsymbol{p}_{\Lambda^*})|^2 \left\{(2\pi)^4 \delta^{(4)}(p_{\Lambda^*} - p_\pi - p_\Sigma)\right\}^2. \tag{D.25}$$

Now, we consider the meanings of the square of the delta-function. The delta-function is defined as,

$$(2\pi)^4 \delta^{(4)}(p_\Lambda - p_\pi - p_\Sigma) = \int d^4x e^{-i(p_\Lambda - p_\pi - p_\Sigma)x}, \tag{D.26}$$

which corresponds to the space-time volume, $VT$. For the decay width (and the cross-section), it is important to consider the unit space-time volume. Therefore, we can remove the one delta-function for our purpose.

Next, let us move on to the integral measure, which is given by

$$\frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}. \tag{D.27}$$

We need the first factor $d^3p_f/(2\pi)^3$ to sum up the final states with momentum interval $[\boldsymbol{p}_f, \boldsymbol{p}_f + d\boldsymbol{p}_f]$. The rest one, $1/(2E_f)$, reflects that there $2E_f$ particles in the unit space-time volume. To verify this, we consider the time-component of the current, $j^0$, which corresponds to the charge density. For example, we consider the fermion current. It is given by,

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{D.28}$$

therefore, we obtain

$$j^0 = \psi^\dagger\psi = 2E, \tag{D.29}$$

which shows that there is $2E$ fermions in the unit space-time volume.

Let us go back to the derivation. The decay width is given by,

$$\begin{aligned}\Gamma &= \frac{1}{2}\sum_r\sum_s\int\frac{1}{2m_{\Lambda^*}}\left(\prod_{f=\pi\Sigma}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)g^2|\bar{u}_s(\boldsymbol{p}_\Sigma)\,u_r(\boldsymbol{p}_{\Lambda^*})\,|^2\\ &\quad\times(2\pi)^4\,\delta^{(4)}(p_{\Lambda^*}-p_\pi-p_\Sigma)\,. \end{aligned} \tag{D.30}$$

In this equation, $m_{\Lambda^*}$ is the mass of the $\Lambda(1405)$, $m_{\Lambda^*}=1405$ MeV. Here the spinor part is given by,

$$\begin{aligned}&\sum_r\sum_s|\bar{u}_s(\boldsymbol{p}_\Sigma)\,u_r(\boldsymbol{p}_{\Lambda^*})\,|^2\\ &= \sum_r\sum_s\bar{u}_{s,\alpha}(\boldsymbol{p}_\Sigma)\,u_{r,\alpha}(\boldsymbol{p}_{\Lambda^*})\,u_{s,\beta}(\boldsymbol{p}_\Sigma)\,\bar{u}_{r,\beta}(\boldsymbol{p}_{\Lambda^*})\\ &= \sum_r\sum_s[u_r(\boldsymbol{p}_\Lambda)\,\bar{u}_r(\boldsymbol{p}_\Lambda)]_{\alpha\beta}\,[u_r(\boldsymbol{p}_\Lambda)\,\bar{u}_r(\boldsymbol{p}_\Lambda)]_{\beta\alpha}\\ &= \mathrm{tr}\,[(\not{p}_\Lambda+m_\Lambda)\,(\not{p}_\Sigma+m_\Sigma)]\\ &= 4\,(p_\Lambda\cdot p_\Sigma+m_\Lambda m_\Sigma)\,. \end{aligned} \tag{D.31}$$

In the decay process, the decaying particle is at rest. Therefore, due to the energy-momentum conservation, we obtain the following relations,

$$\begin{aligned}&p_{\Lambda,\mu}=(m_\Lambda,\boldsymbol{0})\,,\quad p_{\pi,\mu}=(E_\pi,\boldsymbol{p}_\pi)\,,\quad p_{\Sigma,\mu}=(E_\Sigma,\boldsymbol{p}_\Sigma)\\ &m_\Lambda=E_\Sigma+E_\pi,\quad \boldsymbol{0}=\boldsymbol{p}_\pi+\boldsymbol{p}_\Sigma, \end{aligned} \tag{D.32}$$

with $E^2=m^2+\boldsymbol{p}^2$ for each patrticle. As a result, we obtain,

$$\begin{aligned}\sum_r\sum_s|\bar{u}_s(\boldsymbol{p}_\Sigma)\,u_r(\boldsymbol{p}_\Lambda)\,|^2 &= 4\,(p_\Lambda\cdot p_\Sigma+m_\Lambda m_\Sigma)\\ &= 4\,(m_\Lambda E_\Sigma+m_\Lambda m_\Sigma)\\ &= 4m_\Lambda\,(E_\Sigma+m_\Sigma)\,. \end{aligned} \tag{D.33}$$

Using the above result, the decay width reduces to

$$\begin{aligned}\Gamma &= \frac{1}{2}\int\frac{1}{2m_\Lambda}\left(\prod_{f=\pi\Sigma}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)g^2\,[4m_\Lambda\,(E_\Sigma+m_\Sigma)]\,(2\pi)^4\,\delta^{(4)}(p_\Lambda-p_\pi-p_\Sigma)\\ &= g^2\frac{1}{(2\pi)^2}\int\frac{d^3p_\pi d^3p_\Sigma}{4E_\pi E_\Sigma}\,[(E_\Sigma+m_\Sigma)]\,\delta^{(4)}(p_\Lambda-p_\pi-p_\Sigma)\\ &= g^2\frac{1}{(2\pi)^2}\int\frac{d^3p_\pi}{4E_\pi E_\Sigma}\,[(E_\Sigma+m_\Sigma)]\,\delta\,(m_\Lambda-E_\pi-E_\Sigma)\,|_{\boldsymbol{p}_\pi=-\boldsymbol{p}_\Sigma\equiv\boldsymbol{p}}\,. \end{aligned} \tag{D.34}$$

In the last step of Eq. (D.34), we use the momentum conservation, $\boldsymbol{0}=\boldsymbol{p}_\pi+\boldsymbol{p}_\Sigma$. Defining $\boldsymbol{p}\equiv\boldsymbol{p}_\pi=-\boldsymbol{p}_\Sigma$, the energies of the $\pi$ and $\Sigma$ are written by,

$$\begin{cases}E_\pi=\sqrt{{m_\pi}^2+\boldsymbol{p}^2}\\ E_\Sigma=\sqrt{{m_\Sigma}^2+\boldsymbol{p}^2}=\sqrt{{m_\pi}^2+{E_\Sigma}^2-{m_\Sigma}^2}.\end{cases} \tag{D.35}$$

As a result, we obtain,

$$\begin{aligned}
\Gamma &= g^2 \frac{1}{(2\pi)^2} \int \frac{d^3p}{4E_\pi E_\Sigma} \left[ (E_\Sigma + m_\Sigma) \right] \delta \left( m_\Lambda - E_\pi - E_\Sigma \right) \\
&= g^2 \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^\infty |\boldsymbol{p}|^2 d|\boldsymbol{p}| \frac{E_\Sigma + m_\Sigma}{4E_\pi E_\Sigma} \delta \left( m_\Lambda - E_\pi - E_\Sigma \right) \\
&= g^2 \frac{1}{\pi} \int_0^\infty |\boldsymbol{p}|^2 d|\boldsymbol{p}| \frac{E_\Sigma + m_\Sigma}{4E_\pi E_\Sigma} \delta \left( m_\Lambda - E_\pi - E_\Sigma \right) \\
&= g^2 \frac{1}{\pi} \int_0^\infty \sqrt{{E_\Sigma}^2 - {m_\Sigma}^2} dE_\Sigma \frac{E_\Sigma + m_\Sigma}{4E_\pi} \delta \left( m_\Lambda - \sqrt{{m_\pi}^2 + {E_\Sigma}^2 - {m_\Sigma}^2} - E_\Sigma \right) \\
&= g^2 \frac{1}{\pi} \int_0^\infty \sqrt{{E_\Sigma}^2 - {m_\Sigma}^2} dE_\Sigma \frac{E_\Sigma + m_\Sigma}{4E_\pi} \frac{1}{\left| - \frac{E_\Sigma}{\sqrt{{m_\pi}^2 + {E_\Sigma}^2 - {m_\Sigma}^2}} - 1 \right|} \delta \left( E_\Sigma \right) \\
&= g^2 \frac{1}{\pi} \int_0^\infty |\boldsymbol{p}| dE_\Sigma \frac{E_\Sigma + m_\Sigma}{4E_\pi} \frac{E_\pi}{E_\Sigma + E_\pi} \delta \left( E_\Sigma \right) \\
&= g^2 \frac{1}{\pi} |\boldsymbol{p}| \frac{E_\Sigma + m_\Sigma}{4 \left( E_\Sigma + E_\pi \right)}. \qquad \text{(D.36)}
\end{aligned}$$

In this calculation, we use a formula for the delta-function,

$$\delta \left( f \left( r \right) \right) = \sum_i \frac{1}{\left| f' \left( r_i \right) \right|} \delta \left( r - r_i \right), \qquad \text{(D.37)}$$

where $f'(r) = \frac{df(r)}{dr}$ and $f(r = r_i) = 0$.

Finally, taking into account that the $\Lambda(1405)$ is allowed to decay in to the $\pi^0\Sigma^0$, $\pi^+\Sigma^-$, and $\pi^-\Sigma^+$ stats, we obtain,

$$\Gamma = g^2 \frac{3}{\pi} |\boldsymbol{p}| \frac{E_\Sigma + m_\Sigma}{4 \left( E_\Sigma + E_\pi \right)}. \qquad \text{(D.38)}$$

# Appendix E

# Axial current in the Skyrme model

In this appendix, we derive the axial current from the SU(3) Skyrme Lagrangian via the Noether's theorem [40]. To do that, let us start with the SU(3) Skyrme action,

$$\Gamma = \int d^4x \left\{ \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2} \mathrm{tr}\left[\left(\partial_\mu U\right) U^\dagger, \left(\partial_\nu U\right) U^\dagger\right]^2 + L_{SB} \right\} + \Gamma_{WZ}. \tag{E.1}$$

The SU(3) axial transformation for the variable $U$ is given by,

$$U \to g_A U g_A, \tag{E.2}$$

where $g_A$ is an axial transformation matrix,

$$g_A = e^{i\boldsymbol{\theta}\cdot\boldsymbol{T}/2} \equiv e^{i\phi} \simeq 1 + i\phi, \tag{E.3}$$

and $T^a$ $(a = 1, 2, \cdots 8)$ are the eight Gell-Mann matrices. In the following sections we derives the axial current from each term in the SU(3) action.

## E.1 Second derivative term

First, we consider the second derivative term $L^{(2)}$,

$$L^{(2)} = \frac{1}{16} F_\pi^2 \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right). \tag{E.4}$$

Under the infinitesimal axial transformation Eq. (E.3), a variation of $L^{(2)}$ is given by,

$$\begin{aligned} \delta L^{(2)} &\equiv L\left(U = g_A U g_A\right) - L\left(U\right) \\ &\simeq \frac{F_\pi^2}{16} \mathrm{tr}\left[\partial_\mu\left(i\phi\right) U \partial^\mu U^\dagger + \partial_\mu U \partial^\mu\left(-i\phi\right) U^\dagger + \partial_\mu U U^\dagger \partial^\mu\left(-i\phi\right) + U \partial_\mu\left(i\phi\right) \partial^\mu U^\dagger\right] \\ &= \frac{F_\pi^2}{16} 2i \mathrm{tr}\left[\partial_\mu \phi\left(U \partial^\mu U^\dagger - U^\dagger \partial^\mu U\right)\right], \qquad \phi = \boldsymbol{\theta}\cdot\boldsymbol{T}/2, \end{aligned} \tag{E.5}$$

where we use the following relation between $U$ and $U^\dagger$

$$U\partial_\mu U^\dagger = -\partial_\mu U U^\dagger. \tag{E.6}$$

As a result, we obtain the Axial current from the second derivative term,

$$J^{5,\mu,a,(2)} = \frac{F_\pi^2}{16} i\mathrm{tr}\left[T^a\left(U\partial_\mu U^\dagger - U^\dagger\partial_\mu U\right)\right]. \tag{E.7}$$

## E.2 Skyrme term

Next, we consider the Skyrme term,

$$L^{(4)} = \frac{1}{32e^2}\mathrm{tr}\left[\left(\partial_\mu U\right)U^\dagger, \left(\partial_\nu U\right)U^\dagger\right]^2 \tag{E.8}$$

For our later convenience, we consider the following values,

$$\begin{cases} R_\mu \equiv U\partial_\mu U^\dagger \\ L_\mu \equiv U^\dagger\partial_\mu U. \end{cases} \tag{E.9}$$

By using them, we rewrite the Skyrme term as,

$$\begin{aligned} L^{(4)} &= \frac{1}{32e^2}\mathrm{tr}\left[\left(\partial_\mu U\right)U^\dagger, \left(\partial_\nu U\right)U^\dagger\right]^2 \\ &= \frac{1}{32e^2}\mathrm{tr}\left[R_\mu, R_\nu\right]^2. \end{aligned} \tag{E.10}$$

For the trace part, we obtain,

$$\begin{aligned} & \mathrm{tr}\left[R_\mu, R_\nu\right]^2 \\ = \; & 2\mathrm{tr}\left[R_\mu R_\nu R^\mu R^\nu - R_\mu R^\mu R_\nu R^\nu\right]. \end{aligned} \tag{E.11}$$

Because only the trace part of the Skyrme term changes under the axial transformation, we concentrate on it for a while. Under the axial transformation Eq. (E.3), $R_\mu$ transforms,

$$\begin{aligned} R_\mu \equiv U\partial_\mu U^\dagger \;\; &\rightarrow \;\; gUg\partial_\mu\left(g^\dagger U^\dagger g^\dagger\right) \\ &\simeq \;\; -iU\left(\partial_\mu\phi\right)U^\dagger + R_\mu + i\left(\phi R_\mu - R_\mu\phi\right) - i\partial_\mu\phi \\ &= \;\; R_\mu - iU\left(\partial_\mu\phi\right)U^\dagger + i\left(\phi R_\mu - R_\mu\phi\right) - i\partial_\mu\phi \\ &= \;\; R_\mu^{(0)} + R_\mu^{(1)} \quad \left(= \left(R + R^{(1)}\right)_\mu\right) \equiv R'_\mu, \end{aligned} \tag{E.12}$$

where the superscripts, (0) and (1), stand for the order of $\phi$,

$$R_\mu^{(0)} = R_\mu, \tag{E.13}$$

and

$$R_\mu^{(1)} = -iU\left(\partial_\mu \phi\right)U^\dagger + i\left(\phi R_\mu - R_\mu \phi\right) - i\partial_\mu \phi. \tag{E.14}$$

Therefore, we obtain,

$$\begin{aligned}
\delta L^{(4)} &\equiv L^{(4)}\left(R'_\mu\right) - L^{(4)}\left(R_\mu\right) \\
&\simeq \frac{1}{32e^2}\cdot 4\mathrm{tr}\left[R_\mu^{(1)}R_\nu R^\mu R^\nu + R_\mu^{(1)}R_\nu R^\mu R^\nu - \left(R_\mu^{(1)}R^\mu R_\nu R^\nu + R_\mu^{(1)}R_\nu R^\nu R^\mu\right)\right] \\
&\equiv \frac{1}{32e^2}\cdot 4\mathrm{tr}\left[2R_\mu^{(1)}X^\mu - \left(R_\mu^{(1)}Y^\mu + R_\mu^{(1)}Z^\mu\right)\right],
\end{aligned} \tag{E.15}$$

where we define,

$$\begin{cases} X^\mu = R_\nu R^\mu R^\nu \\ Y^\mu = R^\mu R_\nu R^\nu \\ Z^\mu = R_\nu R^\nu R^\mu. \end{cases} \tag{E.16}$$

Substituting the explicit form of $R_\mu^{(1)}$, Eq. (E.14), we obtain,

$$\begin{aligned}
\delta L^{(4)} &\simeq \frac{1}{32e^2}\cdot 4\mathrm{tr}\left[2\left\{-iU\left(\partial_\mu\phi\right)U^\dagger - i\partial_\mu\phi\right\}X^\mu\right. \\
&\qquad\qquad \left. - \left\{-iU\left(\partial_\mu\phi\right)U^\dagger - i\partial_\mu\phi\right\}\left(Y^\mu + Z^\mu\right)\right] \\
&= \frac{1}{32e^2}\cdot 4i\mathrm{tr}\left[\partial_\mu\phi\left\{\left(Y^\mu + Z^\mu - 2X^\mu\right) + U^\dagger\left(Y^\mu + Z^\mu - 2X^\mu\right)U\right\}\right] \\
&= \frac{1}{32e^2}\cdot 4i\mathrm{tr}\left[\partial_\mu\phi\left\{-\left[R^\nu,\left[R^\mu,R_\nu\right]\right] + \left[L^\nu,\left[L^\mu,L_\nu\right]\right]\right\}\right],
\end{aligned} \tag{E.17}$$

where $\phi = \boldsymbol{\theta}\cdot\boldsymbol{T}/2$. As a result, we obtain the axial current from the Skyrme term,

$$\begin{aligned}
J^{5,\mu,a,(4)} &= \frac{i}{16e^2}\mathrm{tr}\left[T^a\left\{-\left[R^\nu,\left[R^\mu,R_\nu\right]\right] + \left[L^\nu,\left[L^\mu,L_\nu\right]\right]\right\}\right] \\
&= \frac{i}{16e^2}\mathrm{tr}\left[T^a\left\{\left[R^\nu,\left[R_\nu,R^\mu\right]\right] - \left[L^\nu,\left[L_\nu,L^\mu\right]\right]\right\}\right]
\end{aligned} \tag{E.18}$$

with $R_\mu = U\partial_\mu U^\dagger$ and $L_\mu = U^\dagger\partial_\mu U$.

## E.3 Wess-Zumino term

Finally, we consider the Wess-Zumino term,

$$\begin{aligned}
\Gamma_{WZ} &= \frac{iN_c}{240\pi^2}\int d^5x\ \epsilon^{\mu\nu\alpha\beta\gamma}\mathrm{tr}\left[L_\mu L_\nu L_\alpha L_\beta L_\gamma\right] \\
&= \frac{iN_c}{240\pi^2}\int \mathrm{tr}\left[\alpha^5\right],
\end{aligned} \tag{E.19}$$

where $\alpha = U^\dagger dU$ is an 1-form introduced in Appendix B and it satisfies,

$$d\alpha + \alpha^2 = 0. \tag{E.20}$$

After the axial transformation Eq. (E.3), we obtain,

$$\begin{aligned}\alpha &\simeq \alpha + iU^\dagger d\phi U + i\left(\alpha\phi - \phi\alpha\right) + id\phi \\ &\equiv \alpha + \delta\alpha,\end{aligned} \tag{E.21}$$

where we define as

$$\delta\alpha \equiv iU^\dagger d\phi U + i\left(\alpha\phi - \phi\alpha\right) + id\phi, \quad d\phi = d\theta^a \frac{T^a}{2}. \tag{E.22}$$

Using the above result, The Wess-Zumino action changes as follows,

$$\begin{aligned}\delta\Gamma_{WZ} &\equiv \Gamma_{WZ}\,|_{U=gUg} - \Gamma_{WZ}\,|_{U=U} \\ &\simeq \int \frac{iN_c}{48\pi^2}\mathrm{tr}\left[\delta\alpha\alpha^4\right] \\ &= -\int \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi\alpha^4 + d\phi\beta^4\right],\end{aligned} \tag{E.23}$$

where

$$\alpha = U^\dagger dU, \tag{E.24}$$

$$\beta \equiv U dU^\dagger, \tag{E.25}$$

and

$$\beta^2 + d\beta = 0. \tag{E.26}$$

Using the Stoke's theorem as we have done in Appendix B, we obtain,

$$\int_{D_5} \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi\alpha^4 + d\phi\beta^4\right] = -\int_{\partial D_5} \frac{N_c}{48\pi^2}\mathrm{tr}\left[d\phi\alpha\alpha^2 + d\phi\beta\beta^2\right] \tag{E.27}$$

As a result, we obtain the axial current, $J^{5,\mu,a}$, from the Wess-Zumino term

$$J^{5,\mu,a,(WZ)} = -\frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{T^a}{2}\left(L_\nu L_\alpha L_\beta + R_\nu R_\alpha R_\beta\right)\right], \tag{E.28}$$

where,

$$L_\mu = U^\dagger \partial_\mu U \tag{E.29}$$

$$R_\mu = U \partial_\mu U^\dagger. \tag{E.30}$$

# Appendix F

# Dominant contribution of the axial current

In this Appendix, we consider the dominant contribution of the axial current in the non-relativistic limit. To do that, we first explain the general properties of the axial current. After that, we consider the $\bar{K}N$ Feshbach resonance. In the non-relativistic limit, the axial current for the $\beta$-decay is given by

$$J_\mu^5 \simeq \bar{u}\left(p_f\right) \gamma_\mu \gamma_5 u\left(p_i\right), \tag{F.1}$$

where $\bar{u}\left(p_f\right)$ and $u\left(p_i\right)$ are the two-component spinor wave function with final and initial momentum $p_f$ and $r_i$, respectively, and, in the non-relativistic limit, $u\left(p\right)$ is given by

$$u\left(p\right) \simeq \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{2M} \end{pmatrix} \chi, \qquad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{F.2}$$

In the above equation, $M$ is the mass of the particle, $\boldsymbol{\sigma}$ is the Pauli matrix, and $\chi$ is the two-component spinor.

Using it, we obtain the axial current for $\mu = 0$ and $\mu = k$ components,

$$\begin{aligned} J_{\mu=0}^5 &\simeq \chi_f^\dagger \begin{pmatrix} 1 & \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_f}{2M} \end{pmatrix} \gamma_0 \gamma_\mu \gamma_5 \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_i}{2M} \end{pmatrix} \chi_i \\ &= \chi_f^\dagger \frac{\boldsymbol{\sigma}}{2M} \cdot \left(\boldsymbol{p}_i + \boldsymbol{p}_f\right) \chi_i \end{aligned} \tag{F.3}$$

$$\begin{aligned} J_{\mu=k}^5 &\simeq \chi_f^\dagger \begin{pmatrix} 1 & \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_f}{2M} \end{pmatrix} \gamma_0 \gamma_k \gamma_5 \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_i}{2M} \end{pmatrix} \chi_i \\ &= \chi_f^\dagger \left\{ \sigma_k + \frac{1}{\left(2M\right)^2} \left(\boldsymbol{\sigma} \cdot \boldsymbol{p}_f\right) \sigma_k \left(\boldsymbol{\sigma} \cdot \boldsymbol{p}_i\right) \right\} \chi_i. \end{aligned} \tag{F.4}$$

Furthermore, in the non-relativistic limit, the velocity of the particle, $\boldsymbol{v} \sim \boldsymbol{p}/M$, is much smaller then 1,

$$\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2M}\right)^2 \ll \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2M} \ll 1. \tag{F.5}$$

Therefore, the spatial component of the axial current is dominant,

$$J^5_{\mu=0} \ll J^5_{\mu=k} \simeq \chi^\dagger_f \sigma_k \chi_i. \tag{F.6}$$

Now, let us consider the $\bar{K}N$ Feshbach resonance. To do that, we consider the following matrix element,

$$\langle\pi\Sigma|\,\partial^\mu\pi^a J^{5,a}_\mu\,|\bar{K}N\,(I=0)\rangle = \langle\pi|\,\partial^\mu\pi^a\,|0\rangle\,\langle\Sigma|\,J^{5,a}_\mu\,|\bar{K}N\,(I=0)\rangle\,. \tag{F.7}$$

What is important here is that the parity is different in the initial and final states for the matrix element of the baryon axial current, $\langle\Sigma|\,J^{5,a}_\mu\,|\bar{K}N\,(I=0)\rangle$: $\Sigma\left(J^P=1/2^+\right)$ and $\bar{K}N\left(J^P=1/2^-\right)$. Therefore, we need to employ the axial current with parity minus, in order to satisfy the parity invariance.

According to this, for the $\bar{K}N$ Feshbach resonance, we use the following current,

$$J^5_\mu \simeq \bar{u}\left(p_f\right)\gamma_\mu u\left(p_i\right). \tag{F.8}$$

The spatial and time components of the axial current Eq. (F.8) is given by,

$$\begin{aligned}
J^5_{\mu=0} &\simeq \chi^\dagger_f \left(1 \quad \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_f}{2M}\right)\gamma_0\gamma_\mu \begin{pmatrix}1\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_i}{2M}\end{pmatrix}\chi_i \\
&= \chi^\dagger_f\left\{1+\frac{1}{(2M)^2}\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}_f\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}_i\right)\right\}\chi_i
\end{aligned} \tag{F.9}$$

$$\begin{aligned}
J^5_{\mu=k} &\simeq \chi^\dagger_f \left(1 \quad \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_f}{2M}\right)\gamma_0\gamma_k \begin{pmatrix}1\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_i}{2M}\end{pmatrix}\chi_i \\
&= \chi^\dagger_f\left(\sigma_k\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_i}{2M}+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_f}{2M}\sigma_k\right)\chi_i.
\end{aligned} \tag{F.10}$$

Therefore, in the non-relativistic limit, the time component is dominant,

$$J^5_{\mu=k} \ll J^5_{\mu=0} \sim \chi^\dagger_f\chi_i. \tag{F.11}$$

# Appendix G

# Di-quark wave function in the isospin space

In this appendix, we construct the di-quark functions in the isospin space. To do that, we first show the normalized nucleon wave functions which are given in Ref. [38],

$$\begin{cases} |p\uparrow\rangle = \dfrac{1}{\pi}\left(a_1 + ia_2\right) \\ |p\downarrow\rangle = -\dfrac{i}{\pi}\left(a_0 - ia_3\right) \\ |n\uparrow\rangle = \dfrac{i}{\pi}\left(a_0 + ia_3\right) \\ |n\downarrow\rangle = -\dfrac{1}{\pi}\left(a_1 - ia_2\right), \end{cases} \tag{G.1}$$

where $\uparrow$ ($\downarrow$) shows the spin up (down) state and $a_\mu\,(a = 0, 1, 2, 3)$ are the SU(2) collective coordinates defined by,

$$A\left(t\right) = a_0 + ia_i\tau_i \in SU(2), \qquad \sum_{\mu=0}^{3} {a_\mu}^2 = 1. \tag{G.2}$$

To construct di-quark wave functions, we identify the proton (neutron) with the an up quark (down quark). Therefore, we obtain the following relations,

$$\begin{cases} |p\uparrow\rangle = \dfrac{1}{\pi}\left(a_1 + ia_2\right) \sim |u\uparrow\rangle \\ |p\downarrow\rangle = -\dfrac{i}{\pi}\left(a_0 - ia_3\right) \sim |u\downarrow\rangle \\ |n\uparrow\rangle = \dfrac{i}{\pi}\left(a_0 + ia_3\right) \sim |d\uparrow\rangle \\ |n\downarrow\rangle = -\dfrac{1}{\pi}\left(a_1 - ia_2\right) \sim |d\downarrow\rangle, \end{cases} \tag{G.3}$$

where $u$ and $d$ stand for the up and down quarks, respectively, and the meanings of the up- and down-arrows are the same as those of the nucleon wave functions.

Using Eq. (G.3), we construct di-quark wave functions. For example, the di-quark with $J = I = 1$ wave function is constructed by the sum of two $p\uparrow$ states,

$$D\left(J_3 = 1, I_3 = 1\right) = \left[\left(u\uparrow\right)\left(u\uparrow\right)\right], \tag{G.4}$$

where, $D$ stand for the d-quark in this Appendix, $J$ and $J_3$ are the spin and its third component, respectively, and $I$ and $I_3$ are the isospin and its third component, respectively. Therefore, we obtain the wave function without normalization,

$$\begin{aligned} D\left(J_3 = 1, I_3 = 1\right) = \left(u\uparrow\right)\left(u\uparrow\right) &\propto \left[\left(a_1 + ia_2\right)\right]^2 \\ &= \left(a_1^2 - a_2^2 + 2ia_1a_2\right). \end{aligned} \tag{G.5}$$

Actually, we can easily check that it has $J = I = 1$ by acting the spin and isospin operators on the wave function, and the operators are given by [38],

$$\begin{cases} I_k = \dfrac{1}{2}i\left(a_0\dfrac{\partial}{\partial a_k} - a_k\dfrac{\partial}{\partial a_0} - \epsilon_{klm}a_l\dfrac{\partial}{\partial a_m}\right) \\ J_k = \dfrac{1}{2}i\left(a_k\dfrac{\partial}{\partial a_0} - a_0\dfrac{\partial}{\partial a_k} - \epsilon_{klm}a_l\dfrac{\partial}{\partial a_m}\right) \end{cases} \tag{G.6}$$

Finally, we show the di-quark wave functions for $(J_3, I_3) = (1, 0)$ and $(J_3, I_3) = (0, 0)$ states. For the other states, we can derive the wave functions in the same way.

- $D\left(J_3 = 1, I_3 = 0\right)$

$$\begin{aligned} D\left(J_3 = 1, I_3 = 0\right) &= \sqrt{\frac{1}{2}}\left[\left(u\uparrow\right)\left(d\uparrow\right) + \left(d\uparrow\right)\left(u\uparrow\right)\right] \\ &= \frac{\sqrt{3}}{\pi}\left(a_1 + ia_2\right)\left(a_0 + ia_3\right). \end{aligned} \tag{G.7}$$

- $D\left(I_3 = 0, J_3 = 0\right)$

$$\begin{aligned} &D\left(I_3 = 0, J_3 = 0\right) \\ &= \sqrt{\frac{1}{2}}\left[\sqrt{\frac{1}{2}}\left\{\left(u\uparrow\right)\left(d\downarrow\right) + \left(u\downarrow\right)\left(d\uparrow\right)\right\} + \sqrt{\frac{1}{2}}\left\{\left(d\uparrow\right)\left(u\downarrow\right) + \left(d\downarrow\right)\left(u\uparrow\right)\right\}\right] \\ &= \sqrt{\frac{3}{2}}\frac{i}{\pi}\left({a_0}^2 - {a_1}^2 - {a_2}^2 + {a_3}^2\right). \end{aligned} \tag{G.8}$$

Using Eq. (G.6), we verify that the wave functions have their proper quantum numbers.

# Appendix H

# Normalization of the wave functions in the CK and EH approaches

In this appendix, we show the normalization conditions for an s-quark in the CK approach and for a kaon in our (EH) one. Under the normalization condition, their wave functions carries a dimension of MeV, which is consistent with the solution of the Klein-Gordon equation.

### The CK approach

First, we review the normalization condition in the CK approach [21]. In their approach, the Lagrangian and the equation of motion for the kaon are, respectively, written by,

$$\begin{aligned}\mathcal{L} = f\left(r\right)\dot{k}^{\dagger}\dot{k} + i\lambda\left(r\right)\left(k^{\dagger}\dot{k} - \dot{k}^{\dagger}k\right) \\ -h\left(r\right)\frac{dk^{\dagger}}{dr}\frac{dk}{dr} - k^{\dagger}k\left[m_k^2 + V_{eff}\left(r;T=I+L,L\right)\right]\end{aligned} \tag{H.1}$$

and

$$-f\left(r\right)\ddot{k} + 2i\lambda\left(r\right)\dot{k} + Ok = 0, \tag{H.2}$$

where

$$Ok \equiv \frac{1}{r^2}\frac{d}{dr}\left[\left(r^2 h\left(r\right)\right)\frac{dk}{dr}\right] - m_K^2 k - V_{eff}\left(r;T=I+L,L\right)k \tag{H.3}$$

is an hermitian operator. The explicit expressions for $f(r)$ and the other terms are given in Ref. [20, 21],

$$\begin{aligned}
f(r) &= 1+\frac{1}{(eF_\pi)^2}\left[2\frac{\sin^2 F}{r^2}+F'^2\right] && \text{(H.4)}\\
\lambda(r) &= -\frac{N_c E}{2\pi^2 {F_\pi}^2}\frac{\sin^2 F}{r^2}F' && \text{(H.5)}\\
h(r) &= 1+\frac{1}{(eF_\pi)^2}\left[2\frac{\sin^2 F}{r^2}\right]. && \text{(H.6)}
\end{aligned}$$

and $k=k_{TL}(r,t)$ is the radial part of the kaon field, $K(\boldsymbol{r},t)=\sum Y_{TLT_z}(\hat{r})\,k_{TL}(r,t)$, and the second term in the above Lagrangian comes from the Wess-Zumino term. Expanding the kaon as follows,

$$k(r,t)=\sum_{n>0}\left[\tilde{k}_n(\boldsymbol{r})\,e^{i\tilde{\omega}_n t}b_n^\dagger+k_n(\boldsymbol{r})\,e^{-i\omega_n t}a_n\right], \tag{H.7}$$

we obtain the following equations for each eigen mode,

$$\left[f(r)\,{\omega_n}^2+2\lambda(r)\,\omega_n+O\right]k_n=0, \qquad \left[f(r)\,\tilde{\omega}_n^2-2\lambda(r)\,\tilde{\omega}_n+O\right]\tilde{k}_n=0, \tag{H.8}$$

where $\tilde{\omega}_n$ and $\omega_n$ are the energies including the rest mass of the kaon.

Using the hermiticity of $O$, we derive the following orthonormal relations,

$$\begin{cases}
4\pi\int drr^2 k_n^*(\boldsymbol{r})\,k_m(\boldsymbol{r})\,[f(r)\,(\omega_n+\omega_m)+2\lambda(r)]=\delta_{nm}\\
4\pi\int drr^2 \tilde{k}_n^*(\boldsymbol{r})\,\tilde{k}_m(\boldsymbol{r})\,[f(r)\,(\tilde{\omega}_n+\tilde{\omega}_m)-2\lambda(r)]=\delta_{nm}\\
4\pi\int drr^2 k_n^*(\boldsymbol{r})\,\tilde{k}_m(\boldsymbol{r})\,[f(r)\,(\omega_n-\tilde{\omega}_m)+2\lambda(r)]=0.
\end{cases} \tag{H.9}$$

To verify them, we recall the orthogonal condition and the definition of the Hermicity in the elementary quantum mechanics. For the orthogonality, we suppose that the eigen energies for $n$ and $m$ modes are different, $E_n \neq E_m$ $(E \in \boldsymbol{R})$. Then, the energy eigen equation for an operator, $H$, is defined by,

$$H\psi_n=E_n\psi_n, \tag{H.10}$$

where $\psi_n$ is the eigen state for $n$-mode. Now, we consider the following values,

$$\begin{cases}
(H\psi_n)^\dagger\,\psi_m=(E_n\psi_n)^\dagger\,\psi_m\\
(H\psi_m)\,\psi_n^\dagger=(E_m\psi_m)\,\psi_n^\dagger
\end{cases}
\Leftrightarrow
\begin{cases}
\left(\psi_n^* H^\dagger\right)\psi_m=E_n\psi_n^*\psi_m\\
\psi_n^*\,(H\psi_m)=E_m\psi_n^*\psi_m.
\end{cases} \tag{H.11}$$

If $H$ is Hermitian, then

$$\begin{cases}
\psi_n^*\,(H\psi_m)=E_n\psi_n^*\psi_m\\
\psi_n^*\,(H\psi_m)=E_m\psi_n^*\psi_m.
\end{cases} \tag{H.12}$$

Therefore, we obtain,

$$0 = \int d^3x \ \left(E_n - E_m\right) \psi_n^* \psi_m. \tag{H.13}$$

Because $E_n \neq E_m$, we obtain the orthogonal relation by spatial integral,

$$\int d^3r \psi_n^* \psi_m \propto \delta_{nm}. \tag{H.14}$$

Next let us recall the definition of the Hermicity. For an operator, $\hat{H}$, defined in the Hilbert space, $H$, if the following relation satisfies,

$$\langle \hat{H}\xi | \eta \rangle = \langle \xi | \hat{H}\eta \rangle \quad \text{for any } \xi, \eta \in H, \tag{H.15}$$

the operator, $\hat{H}$, is the Hermitian operator. In other words,

$$\int d^3x \ \left[\left(\hat{H}\xi\right)^\dagger \eta\right] = \int d^3x \ \left[\xi^\dagger \left(\hat{H}\eta\right)\right]. \tag{H.16}$$

Now, let us go back to the verification for the orthonormal conditions. The equation of motion Eq. (H.8) is rewritten as follows

$$\left[f\omega_n^2 + 2\lambda\omega_n + O\right] k_n \equiv E_n^{CK} k_n = 0, \quad \left[f\tilde{\omega_n}^2 - 2\lambda\tilde{\omega}_n + O\right] \tilde{k}_n \equiv \tilde{E}_n^{CK} \tilde{k}_n = 0. \tag{H.17}$$

Therefore, we obtain as we have done in Eq. (H.11),

$$\begin{cases} \left(E_n^{CK} k_n\right)^\dagger k_m = 0 \\ k_n^\dagger \left(E_m^{CK} k_m\right) = 0. \end{cases} \tag{H.18}$$

Therefore, from Eq. (H.18), we obtain

$$\begin{cases} \int d^3x \left[\left(E_n^{CK} k_n\right)^\dagger k_m\right] = 0 \\ \int d^3x \left[k_n^\dagger \left(E_m^{CK} k_m\right)\right] = 0 \end{cases} \tag{H.19}$$

$$\Leftrightarrow \quad \begin{cases} \int d^3x \left[\omega_n^2 f\left(r\right) k_n^* k_m + 2\omega_n \lambda\left(r\right) k_n^* k_m + k_n^* O k_m\right] = 0 \\ \int d^3x \left[\omega_m^2 f\left(r\right) k_n^* k_m + 2\omega_m \lambda\left(r\right) k_n^* k_m + k_n^* O k_m\right] = 0, \end{cases} \tag{H.20}$$

where $O$ is Hermitian. If $n \neq m \left(\omega_n \neq \omega_m\right)$, we can derive from the above equations,

$$\begin{aligned} &\int d^3x \left[\left(\omega_n^2 - \omega_m^2\right) f\left(r\right) k_n^* k_m + 2\left(\omega_n - \omega_m\right) \lambda\left(r\right) k_n^* k_m\right] = 0 \\ \Leftrightarrow \quad &\int d^3x \left[\left(\omega_n + \omega_m\right) f\left(r\right) k_n^* k_m + 2\lambda\left(r\right) k_n^* k_m\right] = 0, \end{aligned} \tag{H.21}$$

which is nothing but the orthogonal condition for $k_n$ in Eq. (4.64). For $\tilde{k}_n$, the situation is the same as the above case. However we keep in mind that the sign of

$\lambda(r)$ changes because it comes from the Wess-Zumino term. The last relation is easily derived from the equations,

$$\begin{cases} \left(E_n^{CK} k_n\right)^\dagger \tilde{k}_m = 0 \\ k_n^\dagger \left(\tilde{E}_m^{CK} \tilde{k}_m\right) = 0. \end{cases} \tag{H.22}$$

From now on, we identify $a_n$ and $b_n$ with the annihilation operators for the $s$- and $\bar{s}$-quarks, respectively. In other words, $k_n$ corresponds to the $s$-quark wave function and $\tilde{k}_n$ to the $\bar{s}$-quark one. Next, we derive the momentum conjugate to $k$,

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{k}} = f(r)\,\dot{k}^\dagger + i\lambda(r)\,k^\dagger. \tag{H.23}$$

Then canonical commutation relation is given by,

$$\left[k_n(\boldsymbol{r},t), \pi_m(\boldsymbol{q},t)\right] = i\delta_{nm}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{q}). \tag{H.24}$$

By substituting the explicit forms of $K$ and $\pi$ for the commutation relation, we verify the normalization condition. To do that, we impose the operators $a_n$ and $b_n$ on the following commutation relation,

- For the bound states,

$$\left[a_n, a_m^\dagger\right] = \left[b_n, b_m^\dagger\right] = \delta_{nm}. \tag{H.25}$$

- For the continuum states,

$$\left[a(\boldsymbol{k}), a^\dagger(\boldsymbol{p})\right] = \text{Const.} \times \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}). \tag{H.26}$$

Therefore,

$$\begin{aligned} \left[k_n(\boldsymbol{r},t), \pi_m(\boldsymbol{r},t)\right] &= \left[k_n, f\dot{k}_m^\dagger + i\lambda k_m^\dagger\right] \\ &= f\left[k_n, \dot{k}_m^\dagger\right] + i\lambda\left[k_n, k_m^\dagger\right]. \end{aligned} \tag{H.27}$$

Using Eq. (**??**),

$$\begin{aligned} \left[k_n(\boldsymbol{r},t), \pi_m(\boldsymbol{r},t)\right] &= f(-i\tilde{\omega}_m)\left\{\tilde{k}_n\tilde{k}_m^* e^{i(\tilde{\omega}_n-\tilde{\omega}_m)t}\right\}\left[b_n^\dagger, b_m\right] \\ &\quad f(i\omega_m)\left\{k_n k_m^* e^{-i(\omega_n-\omega_m)t}\right\}\left[a_n, a_m^\dagger\right] \\ &\quad +i\lambda\left\{\tilde{k}_n\tilde{k}_m^* e^{i(\tilde{\omega}_n-\tilde{\omega}_m)t}\right\}\left[b_n^\dagger, b_m\right] \\ &\quad +i\lambda\left\{k_n k_m^* e^{-i(\omega_n-\omega_m)t}\right\}\left[a_n, a_m^\dagger\right] \\ &= i(f\tilde{\omega}_n - \lambda)\tilde{k}_n\tilde{k}_n^* + i(f\omega_n + \lambda)k_n k_n^* \\ &= i\delta^{(3)}(\boldsymbol{0}) \end{aligned} \tag{H.28}$$

After operating the spatial integral, $\int d^3r$, we obtain the canonical commutation relation,

$$[k, \pi] = i. \tag{H.29}$$

**The EH approach**

From now on, we consider the orthonormal condition for the anti-kaon field, $k_n$ in the EH approach, with the same way as we have done for the CK one. In our approach, the Lagrangian is given as follows,

$$\begin{aligned}\mathcal{L} = & f(r)\dot{k}^\dagger \dot{k} + i\rho_1(r)\left(k^\dagger \dot{k} - \dot{k}^\dagger k\right) + i\rho_2(r)\left(\frac{dk^\dagger}{dr}\dot{k} - \dot{k}^\dagger \frac{dk}{dr}\right) \\ & + i\lambda_1(r)\left(k^\dagger \dot{k} - \dot{k}^\dagger k\right) - h(r)\frac{dk^\dagger}{dr}\frac{dk}{dr} + J(r)\left(\frac{dk^\dagger}{dr}k + k^\dagger \frac{dk}{dr}\right) \\ & - k^\dagger k\left[m_k^2 + V_{eff}\left(r; I^K, L^K, I^N, J^N\right) + \lambda_2(r)\right], \end{aligned} \tag{H.30}$$

where

$$f(r) = 1 + \frac{1}{(eF_\pi)^2}\left[2\frac{\sin^2 F}{r^2} + F'^2\right] \tag{H.31}$$

$$\begin{aligned}\rho_1(r) = & -\frac{4\sin^2(F/2)}{3\Lambda}\boldsymbol{I}_K\cdot\boldsymbol{I}_N - \frac{\sin^2(F/2)}{\Lambda} \\ & -\frac{1}{(eF_\pi)^2}\left[\frac{4\sin^2(F/2)}{3\Lambda}\left(\frac{4}{r^2}\sin^2 F + F'^2\right)\boldsymbol{I}_K\cdot\boldsymbol{I}_N\right. \\ & \left. - \frac{\sin^2(F/2)}{\Lambda}\left(\frac{5}{r^2}\sin^2 F + F'^2\right)\right] \\ = & -\frac{4\sin^2(F/2)}{3\Lambda}\boldsymbol{I}_K\cdot\boldsymbol{I}_N\left[1 + \frac{1}{(eF_\pi)^2}\left(\frac{4}{r^2}\sin^2 F + F'^2\right)\right] \\ & - \frac{\sin^2(F/2)}{\Lambda}\left[1 + \frac{1}{(eF_\pi)^2}\left(\frac{5}{r^2}\sin^2 F + F'^2\right)\right] \end{aligned} \tag{H.32}$$

$$\rho_2(r) = \frac{1}{(eF_\pi)^2}\left[\frac{\sin F}{\Lambda}F'\left(4\boldsymbol{I}_K\cdot\boldsymbol{I}_N + 3\right)\right] \tag{H.33}$$

$$\begin{aligned}\lambda_1(r) = & -\frac{N_c}{2\pi^2 {F_\pi}^2}\frac{\sin^2 F}{r^2}F' \\ = & \frac{N_c}{{F_\pi}^2}B^0, \quad B^0 = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F' \end{aligned} \tag{H.34}$$

$$\lambda_2(r) = \frac{N_c}{\pi^2 {F_\pi}^2}\frac{\sin^2(F/2)}{\Lambda}\frac{\sin^2 F}{r^2}F' \tag{H.35}$$

$$h(r) = 1 + \frac{1}{(eF_\pi)^2}\left[2\frac{\sin^2 F}{r^2}\right] \tag{H.36}$$

$$J(r) = \frac{1}{(eF_\pi)^2}\left[6\frac{\sin^2(F/2)\sin F}{r^2}F'\right]. \tag{H.37}$$

In the above equations, $\lambda_1(r)$ and $\lambda_2(r)$ come from the Wess-Zumino term and the other functions come from the normal SU(3) Skyrme Lagrangian. The equation of motion is derived from the Lagrangian Eq. (H.30) by taking an variation for $k^\dagger$, that is,

$$\int d^3x\mathcal{L}\left(k,k^\dagger+\delta k^\dagger\right)-\int d^3x\mathcal{L}\left(k,k^\dagger\right)=0. \tag{H.38}$$

As a result, we obtain,

$$\begin{aligned}
& -f\left(r\right)\ddot{k}+2i\rho_1\left(r\right)\dot{k}+2i\lambda_1\left(r\right)\dot{k}-i\frac{1}{r^2}\frac{d}{dr}\left[r^2\rho_2\left(r\right)\right]\dot{k}+Ok=0 \\
\Leftrightarrow\quad & -f\left(r\right)\ddot{k}+2iX_1\left(r\right)\dot{k}-iY_2\left(r\right)\dot{k}+Ok=0, && \text{(H.39)}
\end{aligned}$$

where we define the real functions, $X_1$, $Y_1$, and $O$ as follows,

$$\begin{aligned}
X_1\left(r\right) &\equiv \rho_1\left(r\right)+\lambda_1\left(r\right) && \text{(H.40)}\\
Y_2\left(r\right) &\equiv \frac{1}{r^2}\frac{d}{dr}\left[r^2\rho_2\left(r\right)\right] \\
&= \frac{2}{r}\rho_2\left(r\right)+\frac{d\rho_2\left(r\right)}{dr} && \text{(H.41)}\\
Ok &\equiv \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{d}{dr}\right)-\frac{1}{r^2}\frac{d}{dr}\left(r^2J\right)\right. \\
&\qquad \left.-\left(m_k^2+V_{eff}\left(r;I^K,L^K,I^N,J^N\right)+\lambda_2\right)\right]k. && \text{(H.42)}
\end{aligned}$$

Here, the problem is whether $O$ is hermitian or not. To verify this, we calculate the following value with eigen function $k_1, k_2 \in \boldsymbol{R}$,

$$\begin{aligned}
& \int d^3x\left(Ok_1\right)^\dagger k_2 \\
= & \int d^3x\left(\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{d}{dr}\right)-\frac{1}{r^2}\frac{d}{dr}\left(r^2J\right)\right.\right. \\
& \qquad \left.\left.-\left(m_k^2+V_{eff}\left(r;I^K,L^K,I^N,J^N\right)+\lambda_2\right)\right]k_1\right)^\dagger k_2 \\
\equiv & \int d^3x\left(\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{d}{dr}\right)-V\right]k_1\right)^\dagger k_2. && \text{(H.43)}
\end{aligned}$$

for the first term,

$$\begin{aligned}
\int d^3x\left(Ok_1\right)^\dagger k_2 &= \int d^3x\left(\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{d}{dr}\right)\right]k_1\right)^\dagger k_2 \\
&= \int d^3x\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{dk_1^*}{dr}\right)\right)k_2 \\
&= 4\pi\int dr\left[-\left(r^2h\frac{dk_1^*}{dr}\frac{dk_2}{dr}\right)\right] \\
&= 4\pi\int dr\left[k_1^*\frac{d}{dr}\left(r^2h\frac{dk_2}{dr}\right)\right] \\
&= \int d^3x\left[k_1^*\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{dk_2}{dr}\right)\right]. && \text{(H.44)}
\end{aligned}$$

and for the second term,

$$\begin{aligned}
\int d^3x\,(Ok_1)^\dagger\,k_2 &= -\int d^3x\,(Vk_1)^\dagger\,k_2 \\
= \quad -\int d^3xVk_1^*k_2 &= \int d^3xk_1^*\,(-Vk_2)\,,
\end{aligned} \tag{H.45}$$

Therefore,

$$\begin{aligned}
&\int d^3x\,(Ok_1)^\dagger\,k_2 \\
\equiv\quad &\int d^3x\left(\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{d}{dr}\right)-V\right]k_1\right)^\dagger k_2 \\
=\quad &\int d^3x\left(k_1^\dagger\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2h\frac{d}{dr}\right)k_2\right]\right) \\
=\quad &\int d^3xk_1^\dagger\,(Ok_2)
\end{aligned} \tag{H.46}$$

Therefore, we conclude that $O$ is hermitian. Next, we expand kaon field and derive the eigen equations,

$$k\,(r,t)=\sum_{n>0}\left[\tilde{k}_n\,(r)\,e^{i\tilde{\omega}_nt}b_n^\dagger+k_n\,(r)\,e^{-i\omega_nt}a_n\right], \tag{H.47}$$

and we obtain,

$$\begin{cases}
\left[f\omega_n{}^2+2X_1\omega_n-Y_2\omega_n+O\right]k_n\equiv E_n^{EH}k_n=0\\
\left[f\tilde{\omega}_n^2-2X_1\tilde{\omega}_n+Y_2\tilde{\omega}_n+O\right]\tilde{k}_n\equiv\tilde{E}_n^{EH}\tilde{k}_n=0.
\end{cases} \tag{H.48}$$

Now, we consider the orthonormal condition. For $k_n$, we obtain,

$$\begin{cases}
\left(E_n^{EH}k_n\right)^\dagger k_m=0\\
k_n^\dagger\left(E_m^{EH}k_m\right)=0.
\end{cases} \tag{H.49}$$

Therefore, from Eq. (H.49), we obtain

$$\begin{cases}
\int d^3x\left[\left(E_n^{EH}k_n\right)^\dagger k_m\right]=0\\
\int d^3x\left[k_n^\dagger\left(E_m^{EH}k_m\right)\right]=0
\end{cases} \tag{H.50}$$

$$\Leftrightarrow\quad\begin{cases}
\int d^3x\left[f\omega_n^2k_n^*k_m+2X_1\omega_nk_n^*k_m-Y_2\omega_nk_n^*k_m+k_n^*Ok_m\right]=0\\
\int d^3x\left[f\omega_m^2k_n^*k_m+2X_1\omega_mk_n^*k_m-Y_2\omega_mk_n^*k_m+k_n^*Ok_m\right]=0,
\end{cases} \tag{H.51}$$

where we use the fact that $O$ is Hermitian as we have proven above. If $n\neq m$ $(\omega_n\neq\omega_m)$, we can derive from the above equations,

$$\begin{aligned}
&\int d^3x\left[f\left(\omega_n^2-\omega_m^2\right)k_n^*k_m+(2X_1-Y_2)\,(\omega_n-\omega_m)\,k_n^*k_m\right]=0\\
\Leftrightarrow\quad&\int d^3x\left[f\,(\omega_n+\omega_m)\,k_n^*k_m+(2X_1-Y_2)\,k_n^*k_m\right]=0.
\end{aligned} \tag{H.52}$$

For $\tilde{k}_n$,

$$\begin{cases} \int d^3x \left[ \left( \tilde{E}_n^{EH} \tilde{k}_n \right)^\dagger \tilde{k}_m \right] = 0 \\ \int d^3x \left[ \tilde{k}_n^\dagger \left( \tilde{E}_m^{EH} \tilde{k}_m \right) \right] = 0 \end{cases} \tag{H.53}$$

$$\Leftrightarrow \quad \begin{cases} \int d^3x \left[ f\tilde{\omega}_n^2 \tilde{k}_n^* \tilde{k}_m - 2X_1 \tilde{\omega}_n \tilde{k}_n^* \tilde{k}_m + Y_2 \tilde{\omega}_n \tilde{k}_n^* \tilde{k}_m + \tilde{k}_n^* O \tilde{k}_m \right] = 0 \\ \int d^3x \left[ f\tilde{\omega}_m^2 \tilde{k}_n^* \tilde{k}_m - 2X_1 \tilde{\omega}_m \tilde{k}_n^* \tilde{k}_m + Y_2 \tilde{\omega}_m \tilde{k}_n^* \tilde{k}_m + \tilde{k}_n^* O \tilde{k}_m \right] = 0, \end{cases} \tag{H.54}$$

If $n \neq m \ (\omega_n \neq \omega_m)$, we can derive from the above equations,

$$\begin{aligned} &\int d^3x \left[ f \left( \tilde{\omega}_n^2 - \tilde{\omega}_m^2 \right) \tilde{k}_n^* \tilde{k}_m - (2X_1 - Y_2) \left( \tilde{\omega}_n - \tilde{\omega}_m \right) \tilde{k}_n^* \tilde{k}_m \right] = 0 \\ \Leftrightarrow \quad &\int d^3x \left[ f \left( \tilde{\omega}_n + \tilde{\omega}_m \right) \tilde{k}_n^* \tilde{k}_m - (2X_1 - Y_2) \tilde{k}_n^* \tilde{k}_m \right] = 0. \end{aligned} \tag{H.55}$$

Finally,

$$\begin{cases} \int d^3x \left[ \left( E_n^{EH} k_n \right)^\dagger \tilde{k}_m \right] = 0 \\ \int d^3x \left[ k_n^\dagger \left( \tilde{E}_m^{EH} \tilde{k}_m \right) \right] = 0. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \int d^3x \left[ f\omega_n^2 k_n^* \tilde{k}_m + 2X_1 \omega_n k_n^* \tilde{k}_m - Y_2 \omega_n k_n^* \tilde{k}_m + k_n^* O \tilde{k}_m \right] = 0 \\ \int d^3x \left[ f\tilde{\omega}_m^2 k_n^* \tilde{k}_m - 2X_1 \tilde{\omega}_m k_n^* \tilde{k}_m + Y_2 \tilde{\omega}_m k_n^* \tilde{k}_m + k_n^* O \tilde{k}_m \right] = 0. \end{cases} \tag{H.56}$$

As a result, we obtain,

$$\begin{aligned} &\int d^3x \left[ f \left( \omega_n^2 - \tilde{\omega}_m^2 \right) k_n^* \tilde{k}_m + (2X_1 - Y_2) \left( \omega_n + \tilde{\omega}_m \right) k_n^* \tilde{k}_m \right] = 0 \\ \Leftrightarrow \quad &\int d^3x \left[ f \left( \omega_n - \tilde{\omega}_m \right) k_n^* \tilde{k}_m + (2X_1 - Y_2) k_n^* \tilde{k}_m \right] = 0. \end{aligned} \tag{H.57}$$

Let us summarize the above results for the normalization,

$$\begin{cases} 4\pi \int drr^2 k_n^* (\boldsymbol{r}) \, k_m (\boldsymbol{r}) \left[ f \left( \omega_n + \omega_m \right) + 2 \left\{ \rho_1 + \lambda_1 \right\} - \dfrac{1}{r^2} \dfrac{d}{dr} \left( r^2 \rho_2 \right) \right] = \delta_{nm} \\ 4\pi \int drr^2 \tilde{k}_n^* (\boldsymbol{r}) \, \tilde{k}_m (\boldsymbol{r}) \left[ f \left( \tilde{\omega}_n + \tilde{\omega}_m \right) - 2 \left\{ \rho_1 + \lambda_1 \right\} + \dfrac{1}{r^2} \dfrac{d}{dr} \left( r^2 \rho_2 \right) \right] = \delta_{nm} \\ 4\pi \int drr^2 k_n^* (\boldsymbol{r}) \, \tilde{k}_m (\boldsymbol{r}) \left[ f \left( \omega_n - \tilde{\omega}_m \right) + 2 \left\{ \rho_1 + \lambda_1 \right\} - \dfrac{1}{r^2} \dfrac{d}{dr} \left( r^2 \rho_2 \right) \right] = 0 \end{cases} \tag{H.58}$$

To verify them, we introduce the momentum conjugate to $k$ given by,

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{k}} = f(r) \, \dot{k}^\dagger + i\rho_1(r) \, k^\dagger + i\rho_2(r) \, k'^\dagger + i\lambda_1(r) \, k^\dagger, \tag{H.59}$$

where $O$ is hermitian.

Now, we perform the canonical-quantization. To do that, let us start with,

$$\begin{cases} \left[k_n\left(\boldsymbol{q},t\right),\pi_m\left(\boldsymbol{r},t\right)\right]=i\delta_{nm}\delta^{(3)}\left(\boldsymbol{q}-\boldsymbol{r}\right) \\ \pi\equiv\dfrac{\partial\mathcal{L}}{\partial\dot{k}}=f\dot{k}^{\dagger}+i\left(\rho_1+\lambda_1\right)k^{\dagger}+i\rho_2 k'^{\dagger} \end{cases} \tag{H.60}$$

and require the following commutation relation,

$$\left[a_n,a_m^{\dagger}\right]=\left[b_n,b_m^{\dagger}\right]=\delta_{nm}. \tag{H.61}$$

Therefore, we obtain,

$$\begin{aligned}\left[k_n\left(\boldsymbol{r},t\right),\pi_m\left(\boldsymbol{r},t\right)\right] &= f\left[k_n,\dot{k}_m^{\dagger}\right]+i\left(\rho_1+\lambda_1\right)\left[k_n,k_m^{\dagger}\right]+i\rho_2\left[k_n,k_m'^{\dagger}\right] \\ &= if\left(\tilde{\omega}_n\tilde{k}_n\tilde{k}_n^*+\omega_n k_n k_n^*\right) \\ &\quad +i\left(\rho_1+\lambda_1\right)\left(-\tilde{k}_n\tilde{k}_n^*+k_n k_n^*\right) \\ &\quad +i\rho_2\left(-\tilde{k}_n\tilde{k}_n'^*+k_n k_n'^*\right) \\ &= i\delta^{(3)}\left(\mathbf{0}\right). \end{aligned} \tag{H.62}$$

Here, what is the problem is the third term is the left-hand-side. Therefore, we consider the spatial integral of this part, especially for $k_n$ as an example,

$$\begin{aligned}\int d^3x\ \rho_2 k_n k_n'^* &= 4\pi\int dr\ r^2\rho_2 k_n k_n'^* \\ &= 4\pi\int dr\ \frac{d}{dr}\left[r^2\rho_2 k_n k_n^*\right]-4\pi\int dr\ \frac{d}{dr}\left[r^2\rho_2 k_n\right]k_n^* \\ &= -4\pi\int dr\ \frac{d}{dr}\left[r^2\rho_2\right]k_n k_n^*-\int d^3x\ \rho_2 k_n' k_n^*. \end{aligned} \tag{H.63}$$

Because the wave function is $k_n$ is real, $k_n=k_n^*$, we obtain

$$\begin{aligned}2\int d^3x\ \rho_2 k_n k_n'^* &= -4\pi\int dr\ \frac{d}{dr}\left[r^2\rho_2\right]k_n k_n^* \\ &= -4\pi\int dr\ r^2\frac{1}{r^2}\frac{d}{dr}\left[r^2\rho_2\right]k_n k_n^*. \end{aligned} \tag{H.64}$$

As a result, after spatial integral we obtain,

$$\begin{aligned} & 4\pi\int drr^2 k_n^* k_m\left[f\omega_n+\left\{\rho_1+\lambda_1\right\}-\frac{1}{2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_2\right)\right] \\ & \quad +4\pi\int drr^2\tilde{k}_n^*\tilde{k}_m\left[f\tilde{\omega}_n-\left\{\rho_1+\lambda_1\right\}+\frac{1}{2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_2\right)\right] \\ = & \ 1. \end{aligned} \tag{H.65}$$

This equation is completed due to the orthonormal conditions Eq. (H.58).

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