Title	Emergence of Bulk Geometries from Conformal Field Theory
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Citation	大阪大学, 2019, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/72642
rights	
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Ph.D. thesis

Emergence of Bulk Geometries
from
Conformal Field Theory

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February 4, 2019

Abstract

This thesis is devoted to a possible generalization of the entanglement entropy and its implication to the holography, especially AdS/CFT correspondence. The AdS/CFT correspondence is a way to define the quantum gravity in $d + 1$-dimensional asymptotically anti-de Sitter (AdS) spacetime by using the conformal field theory (CFT) in d-dimension. In this context, it is still unclear how we can realize the $d + 1$-dimensional full bulk geometries from the d-dimensional quantum field theories. The main object of this thesis is the minimal surface not necessarily anchored on the boundary, which has not been derived from the field theory analysis. First, we review how we can obtain minimal surfaces anchored on the boundary from the entanglement entropy. Then, we introduce a generalization of the entanglement entropy for mixed states and discuss its basic properties. Finally, we argue that we can extract the aforementioned minimal surfaces ending on the bulk from the generalized version of the entanglement entropy. Our explicit derivations include the geometries in three-dimensional AdS and BTZ black holes.

Acknowledgement

First and foremost, I would like to express my deepest gratitude to Norihiro Iizuka, my supervisor. I could not reach here without his patient guidance and encouragement. I also thank him for sharing his ideas and for stimulating discussion during the collaboration. Second, I am very grateful to my collaborators, Hayato Hirai, Mitsuhiro Nishida, Sinya Aoki, and Tsuyoshi Yokoya. The discussion with them have enriched not only the contents of this thesis but also myself. Third, I would like to thank all members of the Particle Physics Theory Group at Osaka University for giving me a good environment during my days as a graduate student.

In the course of my Ph.D. studies, I have benefited from discussions with various peoples. I would like to thank especially Akinori Tanaka, Hideki Kyono, Jin-Beom Bae, Kanato Goto, Kazuya Higashide, Koji Hashimoto, Koji Umemoto, Masahiro Nozaki, Ning Bao, Pawel Caputa, Satoshi Yamaguchi, Sotaro Sugishita, Tadashi Takayanagi, Takahiro Nishinaka, Tatsuma Nishioka, Tokiro Numasawa, Yasuaki Hikida, Yoichi Kazama and Yuya Kusuki for useful discussion and interesting conversation. I also would like to thank Hideaki Sakai, Koji Hashimoto, Norihiro Iizuka, Satoshi Yamaguchi, and Tetsuya Onogi for reading this thesis carefully and giving me useful comments.

Last but not least, I would like to thank my parents for their continuous support and encouragement.

Contents

1 Introduction and Summary

1.1 Introduction: towards origin of gravity and spacetime

This thesis is devoted to a better understanding of an interesting *duality* between a family of *quantum field theories (QFTs)* and *quantum gravity*. The dualities are phenomena that the seemingly different theories turn out to be just the same one. Hence, deepening our understanding of a given duality will show us a new perspective of the original theories. In particular, our aim is to understand how and why gravity and spacetime can emerge at the low energy.

The duality which we shall discuss, *AdS/CFT correspondence* [1] is an explicit realization of *holography* or holographic principle [2, 3]. The ultimate motivation of this principle is to describe the non-perturbative nature of quantum gravity. Firstly, we explain the historical background of the holographic principle very briefly. Due to the Einstein equation, strongly self-gravitating objects eventually become the black holes. If we maintain the second law of thermodynamics even in the above situation, the black holes must have entropy. The black holes turn out to possess the entropy which is proportional to the area of the event horizon [4, 5],

$$S_{BH} = \frac{1}{4G_N}(\text{Area of event horizon}), \tag{1}$$

where G_N is the Newton constant. Through out this paper, we will use the natural unit. Surprisingly, the black hole entropy is not extensive in the usual sense. This fact suggests that the entropy (namely, degrees of freedom) in a given gravitating system should be bounded above by the area of the boundary of the system [6]. This observation leads the holographic principle [2, 3], a quantum gravity on M can be described by the (non-gravitating) quantum many-body systems on ∂M. The natural questions which we need to answer are as follows: "*How* (*Why*) can we reproduce the classical gravity via the holography?" and "What kinds of quantum many-body systems can be used?". To answer these questions, discussing concrete examples is beneficial. That is the reason why we want to study the AdS/CFT.

The statement of the AdS/CFT correspondence is that $d + 1$-dimensional quantum gravity in asymptotically anti-de Sitter (AdS) spacetime is equivalent to certain *conformal field theory* in d-dimension (CFT$_d$) which is QFT with scale invariance (and so forth) [1, 7–9]. The simplest consistency check is the agreement of the symmetry acting on the Hilbert space. Indeed, the AdS side (CFT side) enjoys the $SO(d, 2)$ symmetry as the asymptotic symmetry (conformal symmetry). We expect that certain states in CFT$_d$ can play a role of the background geometry in AdS$_{d+1}$ at classical gravity limit. Two examples, which will be also argued in this thesis, are

(i) empty AdS_{d+1} and vacuum in CFT_d and (ii) Euclidian black holes in AdS_{d+1} and thermal states in CFT_d.

Even within the framework of the AdS/CFT, there still remain fundamental questions why and how some given states ensure the existence of background geometry itself. To be more concrete, how and why we can realize Einstein gravity and local effective field theories on a fixed background in $d+1$-dimension from d-dimensional CFT. In particular, we would like to understand the fundamental mechanism behind the AdS/CFT correspondence which will hopefully teach us general lessons about holography and quantum gravity.

Over the last decade, one fundamental relation has been studied extensively, that is, the relation between geometry in AdS and *entanglement* in CFT [10–14]. A concrete realization of this relation is holographic entanglement entropy (HEE) formula [11, 12][1],

$$S(\rho_A) = \frac{1}{4G_N} \min_{\substack{\gamma_A \sim A \\ \partial\gamma_A = \partial A}} (\mathrm{Area}(\gamma_A)). \tag{2}$$

In what follows, we spell out this formula. The left-hand side is the *entanglement entropy* (EE) for boundary CFT_d with given subregion A. The EE characterizes the quantum entanglement, especially for pure states. It is defined by the von Neumann entropy of a reduced density matrix ρ_A on a subsystem A,

$$S(\rho_A) = -\mathrm{Tr}_{\mathcal{H}_A} \rho_A \log \rho_A. \tag{3}$$

We will discuss more the EE in the next section. The right-hand side is the area of the *minimal surfaces* anchored on the boundary of asymptotically AdS_{d+1} spacetime. We normalize it with the unit of Newtonian constant G_N. Here the γ_A is a co-dimension 2 surface on the bulk and supposed to be (i) homologous to the boundary subregion A and (ii) $\partial\gamma_A = \partial A$. We shall pick up the one which minimizes the corresponding area. One can see an example in the bulk side in the left panel of figure 1. Note that the right-hand side is a purely geometrical object which is determined by metric in the bulk theory. The prescription in the bulk will be modified when we consider the time-dependent background; however, the resulting quantity is again purely geometrical[2]. This relation has ignited the aforementioned expectation—geometries may emerge from the entanglement. There are also proofs of this conjecture if one accepts to assume the AdS/CFT itself [15, 16].

[1] The (2) is the case for static geometry. For the time dependent background, we should find extremal one rather than minimal one [12]. Throughout this paper, for simplicity, we will only consider the static geometry.

[2] As far as the limit of classical Einstein gravity.

One can regard the HEE formula as a generalization of the black hole entropy (1). Hence, one can imagine that the minimal surfaces play a role of "the horizon" for a given bulk subregion. In other words, the reduced density matrix in the (2) seems to describe all information outside of "the horizon". Nowadays we expect that the subregion/subregion duality — the ρ_A can reproduce all information in the *entanglement wedge* [17–19]: when we compute the minimal surfaces in (2), we have the region M_A which is surrounded by the boundary subregion A and the corresponding minimal surfaces γ_A. Then, we call M_A and its domain of dependence as entanglement wedge. Since we will only consider the time slice of the bulk in this thesis, we will loosely say time slice of entanglement wedge as entanglement wedge.

The geometrical objects obtained from the HEE formula are always constrained to the boundary of asymptotically AdS spacetime. In general, the minimal surfaces cannot probe the entire bulk region. For example, if we start from the mixed state ρ_A at the beginning, the minimal surfaces cannot probe the entire region of the entanglement wedge due to the phase transition of the minimal surfaces. The regions, the minimal surfaces (ending on the boundary) cannot probe, are referred to "entanglement shadow" [17, 20–22]. Similarly, on the other hand, the EE also cannot characterize the whole structure of entanglement for a given state. Therefore, we should find a more general version of the (2) or much newer notions.

In the light of the bulk, the geometrical objects are not necessarily to be ending on its boundary. For example, the bulk geodesics can end on the arbitrary bulk points. On the other hand, it is non-trivial to find which kind of quantities in the boundary QFT can describe it. Since the surfaces/geodesics anchored on the boundary can be regarded as a special limit of more general ones, one may imagine it can be extracted from certain generalizations of the EE.

From the viewpoint of the boundary, we do not have the unique entanglement measure for mixed states, as opposed to the EE for pure states. Naively, some measures of entanglement for mixed states would give us a nice generalization of the HEE formula. One technical challenge is that almost all quantities in the literature contain some optimization problem for its evaluation. There is an interesting conjecture that the entanglement of purification [23] which measures total correlation (namely, both classical and quantum correlation) is equivalent to the aforementioned general minimal surfaces, entanglement wedge cross section [24, 25] (as an example, see the right panel of figure 1). This conjecture is supported by agreement of non-trivial inequalities in both sides and the tensor network picture of the AdS/CFT correspondence [26]. Unfortunately, this quantity includes the optimization procedure, which makes analytical evaluation difficult even for two-qubit systems.

In this thesis, we give a possible generalization of the entanglement entropy and its holo-

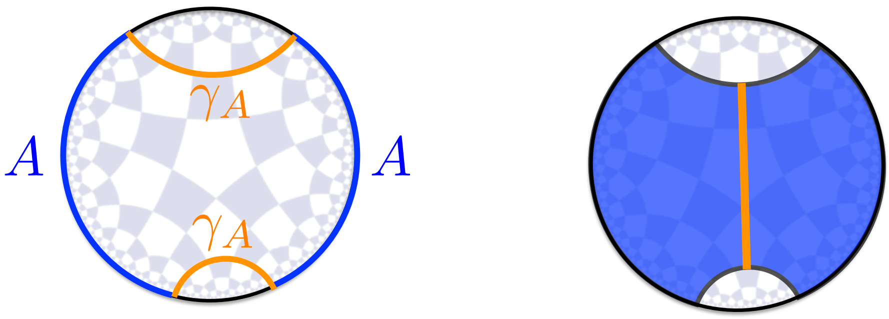


Figure 1: An example of minimal surfaces and corresponding entanglement wedge and its cross section E_W. Inside of the circles corresponds to the interior of AdS time slice. In the left panel, blue curves represent (disconnected) boundary subregions A. Orange curves are the minimal surfaces in this setup. In the right panel, we display (time slice of) the corresponding entanglement wedge as blue shaded region. An orange curve in the right panel is just the minimal cross section of the entanglement wedge E_W.

graphic interpretation. Do we have any quantities which *can be evaluated directly in QFT* and *can give nice bulk interpretation beyond the original HEE formula?* Before guessing this question, let us go back to the case of the EE. Actually, we have a nice way to evaluate the EE in QFT, which is called the replica trick [27]. In addition, we also have (logarithmic) negativity [28] which is a calculable entanglement measure[3] for mixed states. The negativity can also be calculated by using the replica trick; however, the corresponding bulk objects turn out to be no longer ones in the original spacetime [29]. This is because the bulk objects inevitably cause the serious back reaction. The issue is related to the somewhat strange analytic continuation in the replica trick. We will also see this fact briefly in the upcoming section. Therefore, it would be nice to find a cousin of negativity which can evade the aforementioned back reaction.

1.2 Summary: what the author has done

In this thesis, we introduce a novel von-Neumann-like entropy into which we assign not a reduced density matrix but a partially transposed density matrix. We will argue

- It is a generalization of the EE for mixed states,
- It enables us to extract more general minimal surfaces anchored on the bulk.

[3]Strictly speaking, the logarithmic negativity is the entanglement monotone; namely, its value is monotonically decreasing under the local operation and classical communication (LOCC). The measure of entanglement should be defined such that it cannot increase under the LOCC.

In particular, we compute this quantity explicitly for the two-dimensional CFT with the bulk dual by using the replica trick. The resulting quantity can be identified with the entanglement wedge cross section E_W, originally proposed in the context of a holographic dual of the entanglement of purification. Note that this does not conflict with the original proposal. We guess that many information theoretical quantities can have the same value at the semi-classical gravity limit. Byproducts are some non-trivial constraints on states dual to the classical spacetime. The entanglement wedge cross section E_W is defined as the minimal cross section of the entanglement wedge (see the right panel of figure 1). This cross section allows us to probe the region where the minimal surfaces anchored on the boundary cannot reach, especially in the context of subregion/subregion duality.

Organization of this thesis is as follows. In section 2, we briefly review the entanglement entropy, partial transposition, and related topics. In section 3, we introduce a generalization of entanglement entropy which we will call odd entanglement entropy (OEE). We will show some generic properties of OEE with proofs. Up to here, our argument is purely quantum mechanical. Then, in section 4, We move to the CFT in two-dimension. We compute OEE in QFTs by using the replica trick. In particular, for the holographic CFT_2 at the leading order of large-c limit, almost all calculation reduces to the calculation of the conformal blocks. We also discuss how the conformal block can tell us the geodesics in dual gravity theory. In section 5, we conclude with a few remarks and discuss future directions. We put off many technical but important parts until appendices. In appendix A, we leave some proof of useful facts which are applied in section 2 and 3. In appendix B, we review the replica trick, especially in CFT_2. In appendix C, we note the embedding formalism in AdS/CFT correspondence. In appendix D, we verify the geodesic approximation of the conformal blocks from Zamolodchikov's recursion relation [30, 31]. This thesis is mainly an extended version of the author's work,

[32] K. Tamaoka,
"Entanglement Wedge Cross Section from the Dual Density Matrix,"
arXiv:1809.09109 [hep-th],

where we have introduced the OEE and considered its holographic counterpart. Some arguments in section 4.4 were originally emphasized in

[33] H. Hirai, K. Tamaoka and T. Yokoya,
"Towards Entanglement of Purification for Conformal Field Theories,"
PTEP **2018**, no. 6, 063B03 (2018), arXiv:1803.10539 [hep-th].

2 Preliminaries: quantum entanglement

Before discussing the entanglement in quantum field theories (QFTs), we shall start our argument from the quantum mechanics that is enough to discuss some basic properties of entanglement. We review the entanglement entropy (in section 2.1) and the partial transposition (in section 2.2) very briefly.

2.1 Entanglement entropy

Let us consider the EPR state living on bi-partite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$[4], which is the spin-0 singlet consists of two spin-$1/2$ particles,

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B\rangle - |1_A 0_B\rangle), \tag{4}$$

where each Hilbert space $\mathcal{H}_A$, $\mathcal{H}_B$ spans eigenvectors for Pauli matrix $\sigma_z \left|0\right> = \left|0\right>,\ \sigma_z \left|1\right> = -\left|1\right>$. Suppose A(lice) and B(ob) are far from each other, but they share the EPR state. Although Alice cannot manage Bob's spin, she can measure her spin for z-direction, and predict Bob's spin from her outcome. This is because there are perfect (non-local) correlation between Alice's and Bob's spin. This kind of correlation is a typical feature of the quantum world and we call it *entanglement*. This is not the case for the state as like

$$|\text{product}\rangle = |0_A 1_B\rangle\,, \tag{5}$$

which has no correlation between Alice and Bob.

The *entanglement entropy (EE)* quantifies bi-partite quantum correlation for pure states with the unit of the EPR state. It is defined by the von-Neumann entropy with respect to the reduced density matrix for a subregion; namely, Alice's spin or Bob's one. Let ρ_{AB} be a state (density matrix) acting on bi-partite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, the EE is defined as

$$S(\rho_A) = -\text{Tr}_{\mathcal{H}_A} \rho_A \log \rho_A, \tag{6}$$

[4] We should keep in mind that existence of bi-partition is just an assumption. This is not the case especially for the gauge theories (and gravity) even after the lattice regularization. Due to the gauge constraints as like Gauss' law in $U(1)$ Maxwell theory, the physical Hilbert space in these theories cannot have such bi-partite structure (in other words, the bi-partite structure does break gauge-invariance). We can avoid this issue by simply starting from the extended Hilbert space which also includes gauge-variant states. This prescription is, as of now, consistent with the replica trick; hence, the AdS/CFT correspondence. For further detail, see [34–40], for example.

where

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho_{AB}. \tag{7}$$

We will abuse the similar notation in what follows. If we have a state σ_{XY} acting on $\mathcal{H}_X \otimes \mathcal{H}_Y$, σ_X is supposed to be $\sigma_X = \mathrm{Tr}_{\mathcal{H}_Y}\sigma_{XY}$ and vice versa.

In the previous example, we simply have

$$S(\rho_A) = S(\rho_B) = \log 2 \quad (\text{for } \rho_{AB} = |\text{EPR}\rangle\langle\text{EPR}|), \tag{8}$$

$$S(\rho_A) = S(\rho_B) = 0 \quad (\text{for } \rho_{AB} = |\text{product}\rangle\langle\text{product}|). \tag{9}$$

Note that the EE for pure state is mutual, $S(\rho_A) = S(\rho_B)$. Moreover, we have $S(\rho_A) = 0$ for pure states if and only if a given state is not entangled (product state). In this way, the EE nicely quantifies how a given pure state is entangled.

Unfortunately, this is not the case for mixed states. For example,

$$\rho_{AB} = \frac{1}{2}\left|0_A 0_B\right\rangle\!\left\langle 0_A 0_B\right| + \frac{1}{2}\left|1_A 1_B\right\rangle\!\left\langle 1_A 1_B\right|, \tag{10}$$

is just a classical mixture of product states $|0_A 0_B\rangle$ and $|1_A 1_B\rangle$; thus, there should be no quantum entanglement. On the other hand, its reduced density matrix gives rise to the same one for the (4). Therefore, the EE is not so good measure for the mixed states — in this case, it also counts the classical correlation.

2.2 Partial transposition

So far we did not explicitly define *entangled states*. To this end, we first define not-entangled states, *separable states*. Let ρ_{AB} is a state acting on bi-partite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We say the given state is separable if it can be written as a classical mixture of the product states,

$$\rho_{AB} = \sum_i p_i \sigma_{Ai} \otimes \tau_{Bi}, \tag{11}$$

where $0 \le p_i \le 1, \sum_i p_i = 1$. Then, we also define the entangled states as the non-separable states.

It is difficult in general to see whether a given state is entangled or not. For example, let us consider interpolation of an entangled state and a separable state,

$$\rho_{AB} = q\,|\text{EPR}\rangle\langle\text{EPR}| + (1-q)\frac{I_A}{2} \otimes \frac{I_B}{2}, \tag{12}$$

where the $I_{A(B)}$ is identity operator acting on $\mathcal{H}_{A(B)}$. In this case, we have $S(\rho_A) = \log 2$ for all values of $0 \leq q \leq 1$.

One nice way to see this is the partial transposition introduced by Peres [41]. Let ρ_{AB} be a state acting on Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $|e_i^{(A,B)}\rangle$s $(i = 1, 2, \cdots, \text{dim}\mathcal{H}_{A,B})$ be a complete set thereof. Using this basis, we can expand a given density matrix,

$$\rho_{AB} = \sum_{ik}\sum_{j\ell} \langle e_i^{(A)} e_j^{(B)}|\rho_{AB}|e_k^{(A)} e_\ell^{(B)}\rangle\, |e_i^{(A)} e_j^{(B)}\rangle\, \langle e_k^{(A)} e_\ell^{(B)}|\,. \tag{13}$$

We define the partial transposition of the ρ_{AB} with respect to $\mathcal{H}_{A,B}$ as

$$\langle e_i^{(A)} e_j^{(B)}|\rho_{AB}^{T_A}|e_k^{(A)} e_\ell^{(B)}\rangle = \langle e_k^{(A)} e_j^{(B)}|\rho_{AB}|e_i^{(A)} e_\ell^{(B)}\rangle\,, \tag{14}$$

$$\langle e_i^{(A)} e_j^{(B)}|\rho_{AB}^{T_B}|e_k^{(A)} e_\ell^{(B)}\rangle = \langle e_i^{(A)} e_\ell^{(B)}|\rho_{AB}|e_k^{(A)} e_j^{(B)}\rangle\,. \tag{15}$$

Note that the partial transposition does not change its normalization $\text{Tr}_{\mathcal{H}}\rho_{AB}^{T_A} = \text{Tr}_{\mathcal{H}}\rho_{AB}^{T_B} = \text{Tr}_{\mathcal{H}}\rho_{AB} = 1$, whereas it changes the eigenvalues. Since the partial transposition is not a completely positive map, the $\rho_{AB}^{T_B}$ can include negative eigenvalues[5]. Interestingly, there is a nice theorem which relates negative eigenvalues of partially transposed states to the entanglement of original states:

Theorem **(Peres)** Let ρ_{AB} be a physical state acting on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. If $\rho_{AB}^{T_B}$ has the negative eigenvalue, the ρ_{AB} has entanglement between subregions A and B.

For the proof, please see appendix A (very simple!). This theorem implies that the negative eigenvalues of $\rho_{AB}^{T_B}$ are just a sign of the quantum entanglement. Note that the opposite is not always true—there are some entangled states whose partial transposition have only non-negative eigenvalues. However, only for the 2-qubit or qutrit systems, the opposite turns out to be also

[5]This technical statement may be understood more intuitively as follows: the transposition for a physical state is equivalent to its complex conjugate, $\rho_{\text{phys}}^T = \rho_{\text{phys}}^*$; hence, it is closely related to the time reversal of the given physical state. Of course, the time reversal maps a given physical state to (another) physical one. On the other hand, nobody can guarantee so time reversal for *partial region* does. Eventually, the resulting "state" would be no longer a physical state — it could include the negative "probability distribution".

true [42]. Since

$$\mathrm{Spec}(\rho_{AB}^{T_B}) = \left\{ \frac{1-3q}{4}, \frac{1+q}{4}, \frac{1+q}{4}, \frac{1+q}{4} \right\}, \tag{16}$$

for the previous state (12), we can conclude that our state is entangled if $1/3 < q$, otherwise separable.

Negativity and Replica trick Motivated by this observation, there is a quantification of entanglement for the mixed states, called logarithmic negativity [28],

$$\mathcal{E} \equiv \log \left(1 + 2 \sum_{\lambda_i < 0} |\lambda_i| \right), \tag{17}$$

where λ_is are the eigenvalues of the $\rho_{AB}^{T_B}$. Although the logarithmic negativity is not the main scope of this thesis, let us play a bit with this quantity.

We introduce another way to express the right-hand side of the above equation (17), which is called the replica trick. This expression will be useful when we consider the QFTs. First, we note the EE can be written as

$$S(\rho_A) = - \lim_{n \to 1} \frac{1}{n-1} \log \mathrm{Tr}_{\mathcal{H}_A} (\rho_A)^n = - \frac{\partial}{\partial n} \mathrm{Tr}_{\mathcal{H}_A} (\rho_A)^n \Big|_{n \to 1}. \tag{18}$$

Although we need to take the analytic continuation of the integer n, computing $\mathrm{Tr}(\rho)^n$ is relatively easier than doing $-\mathrm{Tr}(\rho \log \rho)$. In the same way, we can write the negativity (and OEE introduced in the next section) by using n-th power of the $\rho_{AB}^{T_B}$. Since $\rho_{AB}^{T_B}$ may contain the negative eigenvalues, the n-th power of the $\rho_{AB}^{T_B}$ depends on the parity of n:

$$\mathrm{Tr}_{\mathcal{H}} (\rho_{AB}^{T_B})^n = \begin{cases} \sum_{\lambda_i > 0} |\lambda_i|^n - \sum_{\lambda_j < 0} |\lambda_j|^n & (n : \text{odd}), \\ \sum_{\lambda_i > 0} |\lambda_i|^n + \sum_{\lambda_j < 0} |\lambda_j|^n & (n : \text{even}), \end{cases} \tag{19}$$

If one starts from the even integer n_e and take analytic continuation to the real value, we have

$$\mathcal{E} = \lim_{n_e \to 1} \log \mathrm{Tr}_{\mathcal{H}} (\rho_{AB}^{T_B})^{n_e}. \tag{20}$$

This expression may look strange, but consistent with the original definition. From the bulk perspective, this analytic continuation gives rise to the back reaction into the original spacetime.

3 An odd generalization of the entanglement entropy

In this section, we define a generalization of the entanglement entropy which is the main theme of this thesis. We will tentatively call it "*odd entanglement entropy*", *OEE* in short. This quantity reduces to the usual entanglement entropy if a given state is a pure state. For the mixed states, it measures total correlation (both classical correlation and quantum entanglement). We start from the definition of OEE in section 3.1. Then, we see some general features of the OEE in section 3.2. We also discuss the behavior of OEE in a two-qubit system, as a simple example (section 3.3).

3.1 Definition

Let ρ_{AB} be a mixed state acting on bi-partite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then we define a quantity which measures the correlation between A and B;

$$\mathcal{S}_o^{(n_o)}(\rho_{AB}) \equiv \frac{1}{1-n_o}\left[\mathrm{Tr}_{\mathcal{H}}(\rho_{AB}^{T_B})^{n_o} - 1\right], \tag{21}$$

where T_B is the partial transposition [41] with respect to the subsystem B. Namely, we will consider the Tsallis entropy [43] for the partially transposed ρ_{AB}. We are especially interested in the limit $n_o \to 1$,

$$S_o(\rho_{AB}) \equiv \lim_{n_o \to 1} S_o^{(n_o)}(\rho_{AB}), \tag{22}$$

where n_o is the analytic continuation of the *odd* integer. Since the odd integer analytic continuation is crucial in the later discussion, we will call S_o as "odd entanglement entropy" or OEE in short. Loosely speaking, the OEE is the von Neumann entropy with respect to $\rho_{AB}^{T_B}$; however, $\rho_{AB}^{T_B}$ potentially contains negative eigenvalues.

The main difference in the OEE, compared with the negativity, is that we are just choosing the odd integer. Therefore, OEE can be formally written as

$$S_o(\rho_{AB}) = -\sum_{\lambda_i>0} |\lambda_i| \log|\lambda_i| + \sum_{\lambda_j<0} |\lambda_j| \log|\lambda_j|, \tag{23}$$

where λ_is are the eigenvalues of the $\rho_{AB}^{T_B}$.

3.2 Basic properties of OEE

Here we argue generic properties of the OEE. There must be more, but we leave it for interesting future work.

3.2.1 Mutuality

The above definition used the partial transposition with respect to the second Hilbert space $\mathcal{H}_B$. Equivalently, one can use the one for the first Hilbert space $\mathcal{H}_A$. Indeed, we have

$$\mathrm{Tr}_{\mathcal{H}}(\rho_{AB}^{T_B})^n = \mathrm{Tr}_{\mathcal{H}}(\{\rho_{AB}^{T_B}\}^n)^{T_{AB}} = \mathrm{Tr}_{\mathcal{H}}(\{\rho_{AB}^{T_B}\}^{T_{AB}})^n = \mathrm{Tr}_{\mathcal{H}}(\rho_{AB}^{T_A})^n. \tag{24}$$

Here we used the properties of the (usual) transposition $T \equiv T_{AB}$ within the trace.

3.2.2 Pure states

Let $|\Psi_{AB}\rangle$ be a pure state in bi-partite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Using the Schmidt decomposition, we can write the $|\Psi_{AB}\rangle$ as a simple form,

$$|\Psi_{AB}\rangle = \sum_{n=1}^{N} \sqrt{p_n}\, |n_A\rangle\, |n_B\rangle\,, \tag{25}$$

where $0 \leq p_n \leq 1$, $\sum_n p_n = 1$. We leave a proof of Schmidt decomposition in appendix A. The N can be taken as $\min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$. One can show that the corresponding density matrix $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ and its partial transposition $\rho_{AB}^{T_B}$ have the eigenvalues,

$$\mathrm{Spec}(\rho_{AB}) = \{1, 0, \cdots, 0\}, \tag{26}$$

$$\mathrm{Spec}(\rho_{AB}^{T_B}) = \{p_1, \cdots, p_N, +\sqrt{p_1 p_2}, -\sqrt{p_1 p_2}, \cdots, +\sqrt{p_{N-1} p_N}, -\sqrt{p_{N-1} p_N}\}. \tag{27}$$

Here each $\pm\sqrt{p_i p_j}$ $(i \neq j)$ in (27) appears just once respectively. Here we have eigenvectors of $\rho_{AB}^{T_B}$ as follows:

$$p_n : \quad |n_A\rangle\, |n_B\rangle\,, \tag{28}$$

$$\pm\sqrt{p_n p_m} : \quad \frac{|n_A\rangle\, |m_B\rangle \pm |m_A\rangle\, |n_B\rangle}{\sqrt{2}}. \tag{29}$$

Notice that $\rho_{AB}^{T_B}$ has the maximal rank N^2. In particular, from the definition of the (23), these contributions completely cancel out. Thus, one can conclude that

$$S_o(\rho_{AB}) = \sum_{n=1}^{N} (-p_n \log p_n) = S(\rho_A) \ \ \text{(for pure states)}. \tag{30}$$

3.2.3 Product states

Let $\rho_{A_1B_1} \otimes \sigma_{A_2B_2}$ be a product state with respect to the bi-partition $\mathcal{H}_{A_1B_1} \otimes \mathcal{H}_{B_2A_2}$. Then the S_o is *additive*,

$$S_o(\rho_{A_1B_1} \otimes \sigma_{A_2B_2}) = S_o(\rho_{A_1B_1}) + S_o(\sigma_{A_2B_2}). \tag{31}$$

Here we are taking the partial transposition with respect to B_1 and B_2. In particular, if

$$\tau_{AB} = \tau'_A \otimes \tau''_B, \tag{32}$$

we have $\mathrm{Tr}_{\mathcal{H}} \tau_{AB}^n = \mathrm{Tr}_{\mathcal{H}} (\tau_{AB}^{T_B})^n$. This fact immediately leads $S_o(\tau_{AB}) = S(\tau_{AB})$.

3.3 Example: two-qubit Werner state

Let us consider the following state for two-qubit system as a simple example,

$$\rho_{AB} = q \, |\mathrm{EPR}_{AB}\rangle\langle \mathrm{EPR}_{AB}| + \frac{(1-q)}{4} I_A \otimes I_B, \tag{33}$$

where the $I_{A(B)}$ is the identity operator acting on $\mathcal{H}_A(\mathcal{H}_B)$ and the $|\mathrm{EPR}_{AB}\rangle$ is the singlet EPR state, $2^{-\frac{1}{2}}(|0_A 1_B\rangle - |1_A 0_B\rangle)$. (This is just the same one in the previous section.) The eigenvalues of this density operator and its partial transposition are given by

$$\mathrm{Spec}(\rho_{AB}) = \left\{ \frac{1+3q}{4}, \frac{1-q}{4}, \frac{1-q}{4}, \frac{1-q}{4} \right\}, \tag{34}$$

$$\mathrm{Spec}(\rho_{AB}^{T_B}) = \left\{ \frac{1-3q}{4}, \frac{1+q}{4}, \frac{1+q}{4}, \frac{1+q}{4} \right\}. \tag{35}$$

Then one can easily compute the $S_o(\rho_{AB})$, the difference between OEE and EE,

$$\mathcal{E}_W(\rho_{AB}) \equiv S_o(\rho_{AB}) - S(\rho_{AB}), \tag{36}$$

and the mutual information,

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{37}$$

respectively. See figure 2.

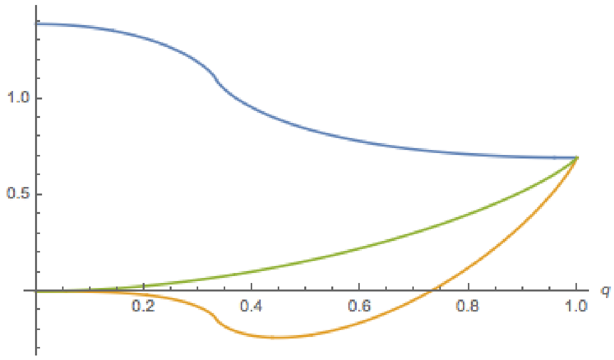


Figure 2: Plot of the $S_o(\rho_{AB})$ (blue), the $\mathcal{E}_W(\rho_{AB})$ (orange) and half of the mutual information (green) for the (33). The horizontal axis represents the value of q. For sufficiently large mixture of the entangled state and classical state, we have $\mathcal{E}_W < 0$. The inflection point appears at $q = 1/3$. The states with $q \leq 1/3$ are separable.

From this example, we can notice that both the $S_o(\rho_{AB})$ and the $\mathcal{E}_W(\rho_{AB})$ do not take the lowest value for separable states; hence, we can not say these are measures of the quantum entanglement. Please remind the Peres and Horodecki criterion [41,42] explained in the previous section. If all eigenvalues for the $\rho_{AB}^{T_B}$ is non-negative, our given state ρ_{AB} is separable. This is just the case for $0 \leq q \leq 1/3$. In other words, if $q \leq 1/3$, the ρ_{AB} can be constructed from the product state via the LOCC process.

Rather interestingly, the $S_o(\rho_{AB})$ takes the lowest value for the pure state $|\text{EPR}_{AB}\rangle$ and is bounded below by its EE $S(\rho_A)$. It implies the $S_o(\rho_{AB})$ measures total correlation between A and B.

4 AdS geometries from conformal field theory

In this section, we derive the entanglement wedge cross section from the reduced density matrix supposed to be dual to the corresponding entanglement wedge. In particular, we would like to argue

1. How the evaluation of EE and OEE in the holographic CFT_2 boils down to the conformal blocks analysis
2. How the conformal blocks can capture the geodesics in asymptotically AdS_3 spacetime

Since the classical gravity limit is quite similar to the thermodynamical limit, both EE and OEE in this limit are captured by the universal (kinematical) object, the conformal blocks.

In section 4.1, we first review the argument in the case of usual EE. From the next section, we start to compute the OEE. We consider the case of the subregion of vacuum (in section 4.2) and thermal state (in section 4.3). In both cases, we will see the agreement with the entanglement wedge cross section E_W in AdS_3 and (planar) BTZ black hole respectively,

$$\boxed{\underbrace{\mathcal{E}_W(\rho_{AB}) \equiv S_o(\rho_{AB}) - S(\rho_{AB})}_{\text{previous section}} \overset{\text{to be shown}}{=} E_W.}$$

Section 4.4 is devoted to some technical but important results on the conformal blocks which are applied extensively in this section.

4.1 EE in holographic CFT_2 (vacuum)

In this subsection, we briefly review the evaluation of the usual EE for the holographic CFT_2 (on $\mathbb{R}^{1,1}$). We take canonical time slice $\mathbb{R}$ and divide the total Hilbert space into $\mathcal{H}_A \otimes \mathcal{H}_{A^c}$[6], where the corresponding subregion A and its complement A^c are not necessarily connected.

In general, we can relate $\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n$ to correlation function of twist operators. Let us first consider the case of single interval $A = [u, v]$. In this case, we have

$$\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n \propto \langle \sigma_n(u)\bar{\sigma}_n(v)\rangle \propto \frac{1}{|u-v|^{2\Delta_n}}, \tag{38}$$

[6]We again assume that our Hilbert space is enlarged such that we can realize the tensor product structure. The way of extension gives rise to an ambiguity of the contribution from boundary of subregions A in general. Since the boundaries of our subsystem (called entangling surfaces) consist of points, the ambiguity coming from the boundary at the entangling surfaces (the boundary of our subregion) will not affect the leading term of the EE (and OEE) in two-dimensional CFT.

where $\sigma_n(u)$ ($\bar{\sigma}_n(v)$) is the (anti-)twist operator in two-dimension. For more detail, see the appendix B or [44]. Then, we can obtain

$$S(\rho_A) = -\frac{\partial}{\partial n}\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n\Big|_{n\to 1} = \frac{c}{3}\log\frac{|u-v|}{\epsilon} + \text{const.}, \tag{39}$$

where we recovered UV cutoff ϵ from dimensionality. If we use the famous relation among the central charge c in CFT_2, the radius of AdS_3 ℓ_{AdS} and the three-dimensional Newtonian constant G_N [45],

$$c = \frac{3\ell_{\mathrm{AdS}}}{2G_N}, \tag{40}$$

we can reproduce the simplest example of the Ryu-Takayanagi formula. (In the three-dimensional gravity, the area of minimal surfaces is nothing but the length of spacelike minimal geodesics in AdS_3.) Since the two-point function of local operators is universal in CFT, the (39) always agrees with the gravity calculation (superficially, of course).

This is not the case for more than one (disconnected) interval. Next, let us see the case of two disconnected intervals, $A = A_1 \cup A_2$ with $A_1 = [u_1, v_1]$ and $A_2 = [u_2, v_2]$ ($u_1 < v_1 < u_2 < v_2$). In this situation, we should compute 4-point function of twist operators,

$$\mathrm{Tr}_{\mathcal{H}_A}(\rho_{A_1A_2})^n = \langle\sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2)\rangle\,, \tag{41}$$

which depends on the dynamical information of CFT. We may expand (41) in terms of the (Virasoro) conformal blocks:

$$\begin{aligned}&\langle\sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2)\rangle\\ &\quad= \frac{1}{|u_1-v_1|^{2\Delta_n}|u_2-v_2|^{2\Delta_n}}\sum_p a_p\mathcal{F}(c, h_{\sigma_n}, h_p, x)\bar{\mathcal{F}}(c, \bar{h}_{\sigma_n}, \bar{h}_p, \bar{x}),\end{aligned} \tag{42}$$

where a_ps are the OPE coefficients. We defined the cross ratio,

$$x = \frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)}, \tag{43}$$

and impose $x = \bar{x}$ since we are interested in the time slice. Note that the above correlator is expanded in the s-channel ($x \to 0$).

To go further, we should assume the spectrum of conformal families. Since our interest is

CFT with the bulk dual, we only assume

1. Unitarity and compactness

2. Large central charge limit $c \gg 1$
   To make theory have classical bulk dual, we should take the classical limit $G_N \ll \ell_{\text{AdS}}$; hence, we will take $c \gg 1$ limit. Thus, we assume that we can take large-c limit which is consistent with the analytic continuation $n \to 1$.

3. Sparse spectrum
   Assume that we have only $\mathcal{O}(c^0)$ degeneracies for each "light" primary operators whose scaling dimension scale $\mathcal{O}(1)$. This condition is necessarily to reproduce the black hole entropy at the semi-classical regime ($c \to \infty$ and $\Delta = \mathcal{O}(c)$) [46]. Remind that the conventional Cardy formula can be applied only for the primaries with the scaling dimension $\Delta \gg c$.

Then, we can show that the dominant contribution at the large-c limit will come from a conformal family with the lowest scaling dimension in the appropriate channel [47][7]. In this correlation function, this is just the vacuum block. At the large-c limit, the Virasoro vacuum block reduces to the global vacuum block; namely, it becomes trivial. Thus, we simply have

$$\langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2)\rangle \sim \frac{1}{|u_1-v_1|^{2\Delta_n}|u_2-v_2|^{2\Delta_n}}, \tag{44}$$

When t-channel ($x \to 1$) is preferred, we instead obtain

$$\langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2)\rangle \sim \frac{1}{|u_1-v_2|^{2\Delta_n}|u_2-v_1|^{2\Delta_n}}. \tag{45}$$

Eventually, we have reproduced the minimal geodesics from the entanglement entropy,

$$S(\rho_{A_1A_2}) = \frac{c}{3}\min\left\{\log\frac{|u_1-v_1|}{\epsilon}+\log\frac{|u_2-v_2|}{\epsilon}, \log\frac{|u_1-v_2|}{\epsilon}+\log\frac{|u_2-v_1|}{\epsilon}\right\} \tag{46}$$

$$\overset{(40)}{=} \frac{1}{4G_N}\min\left\{2\ell_{\text{AdS}}\log\frac{|u_1-v_1|}{\epsilon}+2\ell_{\text{AdS}}\log\frac{|u_2-v_2|}{\epsilon},\right.$$
$$\left. 2\ell_{\text{AdS}}\log\frac{|u_1-v_2|}{\epsilon}+2\ell_{\text{AdS}}\log\frac{|u_2-v_1|}{\epsilon}\right\}. \tag{47}$$

[7] For this conclusion, we also assume that each OPE coefficients grow at most $\mathcal{O}(e^c)$.

In the similar way, we can discuss the entanglement entropy for excited states, thermal states and so forth. In every cases, we can see the agreement with the holographic calculation.

4.2 OEE in holographic CFT$_2$ (vacuum)

In this section, we compute the S_o and the $\mathcal{E}_W$ for mixed states in CFT$_2$. We again divide the total Hilbert space of CFT into $\mathcal{H}_A \otimes \mathcal{H}_{A^c}$, where the corresponding subregion A and its complement A^c are not necessarily connected. Then we can prepare a mixed state $\rho_{A_1A_2} \equiv \mathrm{Tr}_{\mathcal{H}_{A^c}} |0\rangle\langle 0|$, where $|0\rangle$ is the vacuum state in CFT. Here we further divided the remaining subspace $\mathcal{H}_A$ into two pieces, $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$. We focus only on the holographic CFT$_2$.

4.2.1 Two disjoint intervals

First, we consider disjoint interval $A_1 = [u_1, v_1], A_2 = [u_2, v_2]$ on a time slice $\tau = 0$. In order to compute the S_o and the $\mathcal{E}_W$, we can apply the replica trick. See appendix B.2 or [48, 49]. In particular, one can write n-th power of the density matrix (with partial transposition) in terms of the correlation functions of twist operators,

$$\mathrm{Tr}_{\mathcal{H}_A}(\rho_{A_1A_2}^{T_{A_2}})^n = \langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\bar{\sigma}_n(u_2)\sigma_n(v_2)\rangle\,, \tag{48}$$

where $\sigma_n(\bar{\sigma}_n)$ is the (anti-)twist operator with scaling dimension $h_{\sigma_n} = \bar{h}_{\sigma_n} = \frac{c}{24}(n - \frac{1}{n})$. Notice that the order of σ_n and $\bar{\sigma}$ are flipped in contrast to the previous section. Let us expand (48) into the conformal blocks in t-channel,

$$\begin{aligned}&\langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\bar{\sigma}_n(u_2)\sigma_n(v_2)\rangle / (|u_1 - v_2||v_1 - u_2|)^{-\frac{c}{6}(n-\frac{1}{n})}\\ &= \sum_p b_p \mathcal{F}(c, h_{\sigma_n}, h_p, 1-x)\bar{\mathcal{F}}(c, \bar{h}_{\sigma_n}, \bar{h}_p, 1-\bar{x}),\end{aligned} \tag{49}$$

where $\mathcal{F}(c, h_{\sigma_n}, h_p, x)$ and $\bar{\mathcal{F}}(c, \bar{h}_{\sigma_n}, \bar{h}_p, \bar{x})$ are the Virasoro conformal blocks and b_ps are the OPE coefficients.

The dominant contribution at the large-c limit will come from a conformal family with the lowest scaling dimension in the channel [47]. This approximation should be valid only for some specific region $x_c < x < 1$. We can not specify the lower bound x_c, but just expect $x_c \sim \frac{1}{2}$ as like previous subsection. In this channel, the dominant one is universally σ_n^2 (and $\bar{\sigma}_n^2$) due to the twist number conservation,

$$\langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\bar{\sigma}_n(u_2)\sigma_n(v_2)\rangle / (|u_1 - v_2||v_1 - u_2|)^{-\frac{c}{6}(n-\frac{1}{n})}$$
$$\sim b_{\sigma_n^2}\mathcal{F}(c, h_{\sigma_n}, h_{\sigma_n^2}, 1-x)\bar{\mathcal{F}}(c, \bar{h}_{\sigma_n}, \bar{h}_{\sigma_n^2}, 1-\bar{x}). \tag{50}$$

Next we would like to specify the analytic form of the above conformal blocks. First, the scaling dimension of the σ_n^2 depends on the parity of n [48, 49] (see also appendix B.2),

$$h_{\sigma_n^2} = \bar{h}_{\sigma_n^2} = \begin{cases} \frac{c}{24}\left(n - \frac{1}{n}\right) & (n : \text{odd}), \\ \frac{c}{12}\left(\frac{n}{2} - \frac{2}{n}\right) & (n : \text{even}). \end{cases} \tag{51}$$

Since we are interested in the odd integer case, this coincides with h_{σ_n}. Second, this contribution of the conformal block consists only of light operators in the heavy-light limit [50]. In this limit, these analytic forms are known in the literatures [33, 50–53]. In our situation, the block for σ_n^2 has a simple form,

$$\log \mathcal{F}(c, h_{\sigma_{n_o}}, h_{\sigma_{n_o}^2}, 1-x) = -h_{\sigma_{n_o}} \log\left[\frac{1+\sqrt{x}}{1-\sqrt{x}}\right], \tag{52}$$

where we assumed analytic continuation of odd integer $n \equiv n_o$ and the light limit $c \gg 1$ with fixed $h_i/c, h_p/c \ll 1$. Here we took the normalization in [51]. Since all internal/external operators are light, the resulting Virasoro conformal block reduces to the global one. To respect course of argument, we put off derivation of the (52) until section 4.4. In a nut-shell, we can use the saddle point approximation of the geodesic Witten diagrams [51] because our Virasoro conformal block can be approximated by the global one. Attentive readers may worry about the sub-leading contribution of the approximation since eventually we will take $h_i/c, h_p/c \to 0$ limit. One can independently check this is indeed consistent with the Zamolodchikov's recursion relation. See appendix D. Therefore, we have obtained

$$S_o(\rho_{A_1A_2}) = S(\rho_{A_1A_2}) + \frac{c}{6}\log\left[\frac{1+\sqrt{x}}{1-\sqrt{x}}\right] + \text{const.}\,, \tag{53}$$

where

$$S(\rho_{A_1A_2}) = \frac{c}{3}\log\frac{|u_1 - v_2|}{\epsilon} + \frac{c}{3}\log\frac{|v_1 - u_2|}{\epsilon}\,. \tag{54}$$

Here we introduced UV cutoff ϵ. The constant terms do not depend on the position. For a while,

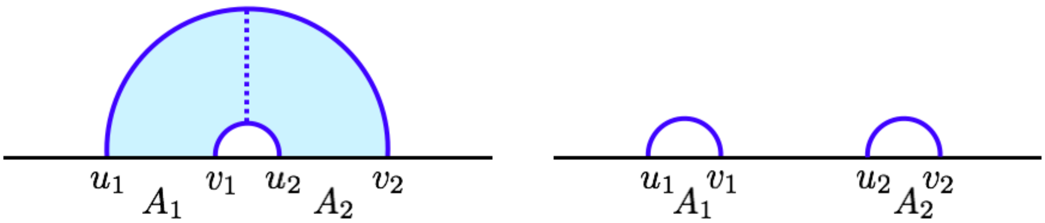


Figure 3: Left: Calculation of (53) in the bulk side. The vertical direction corresponds to the radial direction of AdS$_3$. The t-channel corresponds to the length of minimal geodesics anchored on the boundary points (blue semi-circles) plus the minimal cross section (blue dotted line) of the connected entanglement wedge (shaded region). Right: One of (56). The s-channel corresponds to the usual holographic EE calculation; hence, no connected entanglement wedge and its cross section.

we just assume the contribution from $b_{\sigma_n^2}$ is negligible at the large-c limit. This assumption will be justified when we consider the pure state limit discussed in the next subsection.

Let us briefly comment on the case of logarithmic negativity. In this case, we cannot use the light operator ($h_{\sigma_n^2}/c \ll 1$) approximation of the t-channel conformal blocks — the $\bar{h}_{\sigma_n^2}$ with even integer n is indeed heavy enough to cause the back reaction.

In the same way, we can compute the s-channel limit $x \to 0$. In this case, the dominant contribution will be the vacuum block as like the EE. Hence, we obtain

$$\begin{aligned} S_o(\rho_{A_1A_2}) &= \frac{c}{3}\log\frac{|u_1-u_2|}{\epsilon} + \frac{c}{3}\log\frac{|v_1-v_2|}{\epsilon} &(55)\\ &= S(\rho_{A_1A_2}). &(56)\end{aligned}$$

Therefore, we have confirmed

$$\mathcal{E}_W(\rho_{A_1A_2}) = \begin{cases} \dfrac{1}{4G_N}\log\left[\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right] & (\text{t-channel}, x\sim 1),\\ 0 & (\text{s-channel}, x\sim 0). \end{cases} \quad (57)$$

in the two disjoint interval case. Here we again used the relation between the central charge and the three-dimensional Newtonian constant $c = \frac{3}{2G_N}$ [45]. The (57) precisely matches the minimal entanglement wedge cross section for AdS$_3$ (see figure 3).

4.2.2 Pure state limit

Let us consider the single interval limit $u_2 \to v_1$ and $v_2 \to u_1$, that is, $A^c \to \emptyset$ limit. This corresponds to the pure state limit for the initial mixed state. In this case, our calculation reduces to two-point function of the twist operators,

$$\mathrm{Tr}_{\mathcal{H}_A}(\rho_{A_1A_2}^{T_{A_2}})^n = \langle \sigma_n^2(u_1)\bar{\sigma}_n^2(v_1)\rangle \,, \tag{58}$$

Hence, we can get the usual EE with single interval $A = [u_1, v_1]$ in this limit. Notice that this is consistent with a basic property of OEE discussed in section 3. The above is the generic statement for any CFT_2, of course, but let us see this behavior from (53). If one takes the distance $|u_2 - v_1|$ and $|v_2 - u_1|$ to the cutoff scale ϵ, the second term of right-hand side of (53) reduces to the length of the geodesics anchored on the boundary points u_1 and v_1. Moreover, this argument guarantees the constant terms from $b_{\sigma_p^2}$ is irrelevant at the large-c limit because of the position independence.

4.2.3 Multiple disjoint intervals

We briefly illustrate the case of more than two disjoint intervals. In general, the previous A_1 and A_2 themselves also have disconnected pieces within their subregions. To compute $\mathrm{Tr}_{\mathcal{H}_A}\rho_{A_1A_2}^n$, we assign twist operators σ_n and $\bar{\sigma}_n$ for each boundary of subregions. This was considered in [47] and reproduces the usual Ryu-Takayanagi formula. We can also compute the $\mathrm{Tr}_{\mathcal{H}_A}(\rho_{A_1A_2}^{T_{A_2}})^n$ from the twist operators' correlation function by just flipping the order of σ_n and $\bar{\sigma}_n$ belonging to the A_2. Let us consider

$$\langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\bar{\sigma}_n(z_3)\sigma_n(z_4)\sigma_n(z_5)\bar{\sigma}_n(z_6)\bar{\sigma}_n(z_7)\sigma_n(z_8)\rangle \,, \tag{59}$$

which corresponds to $\mathrm{Tr}_{\mathcal{H}_A}(\rho_{A_1A_2}^{T_{A_2}})^n$ for left panel of figure 4, for concreteness. If two-points belonging to A_1 and A_2 respectively are sufficiently close, the OPE channel in which twist operators σ_ns (or $\bar{\sigma}_n$s) on these points fuse into σ_n^2 ($\bar{\sigma}_n^2$s) will be favored. Here we just assume the lightest conformal block approximation could still work. Eventually, we can pick the OPE channel in right panel of figure 4.

The corresponding eight point conformal blocks factorizes into four-point ones due to the identity exchange with no external/internal heavy operators. Hence, we can reproduce the corresponding entanglement wedge cross section from the $\mathcal{E}_W$.

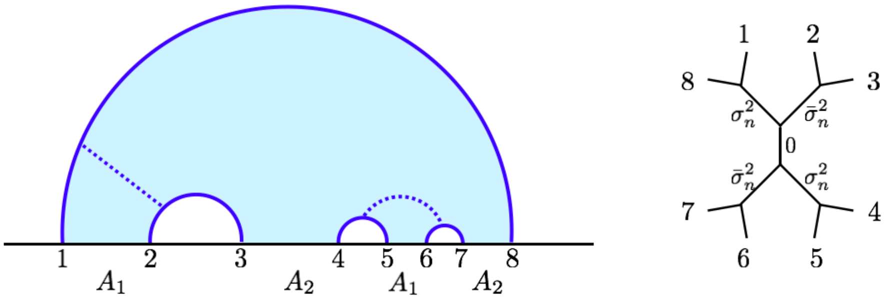


Figure 4: Left: Calculation of (59) in the bulk side. Each line and region represent the same one as figure 3. At the boundary, we have disconnected subregion for both A_1 and A_2. In the boundary calculation, we flip the order of twist operators on A_2 in contrast to the entanglement entropy. Right: the OPE channel we took in the calculation of the left panel. The $1, \cdots, 8$ corresponds to operators sitting on $z_1, \cdots, z_8$. In particular, this conformal block factorizes into two pieces due to the identity exchange.

4.3 OEE in holographic CFT$_2$ (thermal state)

In this section, we consider the thermal state in CFT$_2$ with the bulk dual, which is genuinely mixed state and is dual to the static (planar) BTZ black hole [54]. Namely, we will consider the CFT$_2$ on cylinder $S^1_\beta \times \mathbb{R}$, with single interval on the time slice, $A = [-\ell/2, \ell/2]$. The A^c denotes its complement on the slice.

In order to compute $\mathrm{Tr}_{\mathcal{H}}(\rho_{AA^c}^{T_{A^c}})^{n_o}$ by using the replica trick, one needs to take care about the location of branch cut, which cannot be realized as the naive conformal map from the plane z (previous results in section 4.2) to the cylinder $w = \sigma + i\tau$. The correct prescription [55] is given by

$$\mathrm{Tr}_{\mathcal{H}_{AA^c}}(\rho_{AA^c}^{T_{A^c}})^{n_o} = \langle \sigma_{n_o}(-L/2)\bar{\sigma}^2_{n_o}(-\ell/2)\sigma^2_{n_o}(\ell/2)\bar{\sigma}_{n_o}(L/2)\rangle_\beta \tag{60}$$

where we introduced a finite but large cutoff L so that the conformal map can work. Thus, our "complement" A^c is now $[-L/2, -\ell/2] \cup [\ell/2, L/2]$, although the true time slice is the infinite line. After taking the limit $n_o \to 1$, we let $L \to \infty$ [55]. Here the suffix of correlation function β denotes the inverse temperature. Then the corresponding $\mathrm{Tr}(\rho_{AA^c})^{n_o}$ should be

$$\mathrm{Tr}_{\mathcal{H}_{AA^c}}(\rho_{AA^c})^{n_o} = \langle \sigma_{n_o}(-L/2)\bar{\sigma}_{n_o}(L/2)\rangle_\beta \tag{61}$$

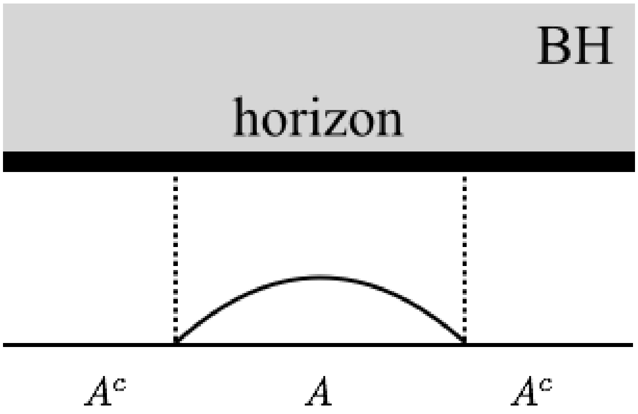


Figure 5: Calculation of E_W for the static planar BTZ black hole. The inverse temperature β is determined by the radius of the horizon. If the subsystem A is sufficiently small $\ell \ll \beta$, the E_W computes the geodesics anchored on the boundary of A (black curve) which agrees with the (64). For $\ell \gg \beta$, the E_W does the disconnected surfaces (dotted vertical lines) which is consistent with the (65).

By using the conformal map $z = e^{2\pi w/\beta}$, one can write the above correlation function as

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_{AA^c}}(\rho_{AA^c}^{T_{A^c}})^{n_o} \\
&= \left(\frac{2\pi}{\beta}\right)^{8h_{\sigma_{n_o}}} \langle \sigma_{n_o}(e^{-\frac{\pi L}{\beta}})\bar{\sigma}^2_{n_o}(e^{-\frac{\pi \ell}{\beta}})\sigma^2_{n_o}(e^{\frac{\pi \ell}{\beta}})\bar{\sigma}_{n_o}(e^{\frac{\pi L}{\beta}})\rangle \,, \qquad (62) \\
&\mathrm{Tr}_{\mathcal{H}_{AA^c}}(\rho_{AA^c})^{n_o} = \left(\frac{2\pi}{\beta}\right)^{4h_{\sigma_{n_o}}} \langle \sigma_{n_o}(e^{-\frac{\pi L}{\beta}})\bar{\sigma}_{n_o}(e^{\frac{\pi L}{\beta}})\rangle \,. \qquad (63)
\end{aligned}$$

Then one can expand the (62) by using the conformal blocks. The dominant contribution can be again approximated by the single conformal block contribution which depends on the value of the cross ratio. Here the cross ratio is $x = e^{-\frac{2\pi}{\beta}\ell}$ for sufficiently large L.

First, we consider the t-channel ($x \to 1$) limit, $\ell \ll \beta$. Then the dominant contribution from the channel is the vacuum block; hence, the (62) reduces to the product of two-point functions. After simple calculation, we obtain

$$\mathcal{E}_W = \frac{c}{3}\log\frac{\beta}{\pi\epsilon}\left(\sinh\frac{\pi\ell}{\beta}\right) + \text{const.} \quad (x \sim 1), \qquad (64)$$

where we introduced the UV cutoff ϵ form the dimensional analysis. The constant term comes from the normalization of two-point functions. This precisely matches the E_W for the planar BTZ black hole (see figure 5).

Next, we consider the s-channel ($x \to 0$) limit, $\ell \gg \beta$. The dominant contribution in the channel is now the twist operator $\sigma_n(\bar{\sigma}_n)$ because of the twist number conservation. Then we have obtained

$$\mathcal{E}_W = \frac{c}{3}\log\frac{\beta}{\pi\epsilon} + \text{const.} \quad (x \sim 0), \tag{65}$$

where the constant terms come from the normalization of two-point functions and the OPE coefficients. This again agrees with the E_W; however, it is important to note that this result is exact at the leading order of small x expansion. There is the position dependent deviation of order $\mathcal{O}(x^1)$.

4.4 Technical side: geodesics from conformal blocks

In this subsection, we will see the detailed reason why the conformal blocks for local operators in two-dimension can tell us the geodesics in AdS_3. The best way to see this fact is using the geodesic Witten diagram (GWD) [51] which is an integral representation of the (global) conformal partial waves (and blocks) for local operators in arbitrary spacetime dimension. We will use an useful notation called the embedding space formalism (or ambient space formalism in some literature). We leave the detail of the formalism extensively in appendix C, but it may not be necessary for readers who only want to scratch the surface.

It is well known that the global conformal symmetry determines the two-point and three-point correlation functions up to constant. In addition, the operator product expansion (OPE) that relates higher point functions to lower ones converges. Therefore, we have the universal basis of more than three-point functions in CFT. These are called the conformal partial waves (CPWs). For example, the four-point function of scalar operators $\mathcal{O}_{\Delta_i}(P_i)$ with scaling dimension Δ_i can be expanded as follows:

$$\begin{aligned}
&\langle \mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}_{\Delta_3}(P_3)\mathcal{O}_{\Delta_4}(P_4)\rangle \\
&\quad = \sum_{p,\ell} C_{12p}C_{34p}W_{\Delta_i,\Delta_p,\ell}(P_i) && (66)\\
&\quad = \left(\frac{P_{14}}{P_{23}}\right)^{\frac{\Delta_{34}}{2}}\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}} \frac{1}{(-2P_{12})^{\Delta_1+\Delta_2}(-2P_{34})^{\Delta_3+\Delta_4}} \sum_{\Delta_p,\ell} C_{12p}C_{34p}G_{\Delta_i,\Delta_p,\ell}(u,v), && (67)
\end{aligned}$$

where we defined the cross ratio

$$u = \frac{P_{12}P_{34}}{P_{13}P_{24}}, \quad v = \frac{P_{23}P_{14}}{P_{13}P_{24}}. \tag{68}$$

Here the C_{ijk}s represent the OPE coefficients. Each Δ_p, ℓ in the summation labels the irreducible representation of conformal algebra. We also used $P_{ij} \equiv P_i \cdot P_j$ and $\Delta_{ij} \equiv \Delta_i - \Delta_j$ for brevity. The relation between conformal partial waves and conformal blocks is given by

$$W_{\Delta_i,\Delta_p,\ell}(P_i) = \left(\frac{P_{14}}{P_{23}}\right)^{\frac{\Delta_{34}}{2}} \left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}} \frac{1}{(-2P_{12})^{\Delta_1+\Delta_2}(-2P_{34})^{\Delta_3+\Delta_4}} G_{\Delta_i,\Delta_p,\ell}(u,v), \tag{69}$$

so these are essentially the same object.

Let us consider amplitude of an exchange diagram in AdS$_{d+1}$, where we are integrating cubic interactions only on the geodesics anchored on the boundary points,

$$W_{\Delta_i,\Delta_p,0}(P_i) = \int_{\gamma_{34}} d\lambda' F_{\Delta_1,\Delta_2,\Delta_p}(P_1,P_2,X'(\lambda'))G_{b\partial}^{\Delta_3}(X'(\lambda'),P_3)G_{b\partial}^{\Delta_4}(X'(\lambda'),P_4), \tag{70}$$

where

$$F_{\Delta_1,\Delta_2,\Delta_p}(P_1,P_2,X'(\lambda')) = \int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1}(X(\lambda),P_1)G_{b\partial}^{\Delta_2}(X(\lambda),P_2)G_{bb}^{\Delta_p}(X(\lambda),X'(\lambda')). \tag{71}$$

Here the scalar bulk-boundary propagator in AdS$_{d+1}$ is given by

$$G_{b\partial}^{\Delta_i}(X,P_i) = \frac{1}{(-2X\cdot P_i)^{\Delta_i}}, \tag{72}$$

and the scalar bulk-bulk propagator satisfies

$$\left[\nabla_X^2 - \Delta_p(\Delta_p - d)\right] G_{b\partial}^{\Delta_p}(X,X') = \delta(X,X'). \tag{73}$$

For later use, we also note the explicit form in three dimension ($d=2$),

$$G_{b\partial}^{\Delta_p}(X,X')|_{\mathrm{AdS}_3} = \frac{e^{-\Delta_p\sigma(X,X')}}{1-e^{-2\sigma(X,X')}}, \tag{74}$$

where $\sigma(X,X')$ is the geodesic distance between X and X';

$$\sigma(X,X') = \log\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right),\ \xi^{-1} = -X\cdot X'. \tag{75}$$

Geodesics γ_{ij} anchored on the boundary points P_i and P_j are given by

$$X^A(\lambda) = \frac{P_i^A e^{\lambda} + P_j^A e^{-\lambda}}{\sqrt{-2P_i\cdot P_j}}. \tag{76}$$

The derivation is noted in appendix . The most important property of GWDs is that it satisfies the conformal Casimir equation,

$$-\frac{1}{2}(L_{P_1}+L_{P_2})^2 W_{\Delta_i,\Delta_p,0}(P_i) = C_{\Delta_p,0} W_{\Delta_i,\Delta_p,0}(P_i), \tag{77}$$

where $C_{\Delta_p,0} = \Delta_p(d-\Delta_p)$ is the quadratic conformal Casimir for the scalar primary field in d-dimension. The sketch of the proof is as follows:

1. The $F_{\Delta_1,\Delta_2,\Delta_p}(P_1,P_2,X')$ is invariant under simultaneous rotation about P_1, P_2, and X'. Hence, we have

$$(L_{P_1}+L_{P_2}+L_{X'})_{AB} F_{\Delta_1,\Delta_2,\Delta_p}(P_1,P_2,X') = 0. \tag{78}$$

2. Using the above relation twice, we can obtain

$$-\frac{1}{2}(L_{P_1}+L_{P_2})^2 F_{\Delta_1,\Delta_2,\Delta_p}(P_1,P_2,X') = -\frac{1}{2}L_{X'}^2 F_{\Delta_1,\Delta_2,\Delta_p}(P_1,P_2,X') \tag{79}$$

$$= \nabla_{X'}^2 F_{\Delta_1,\Delta_2,\Delta_p}(P_1,P_2,X'). \tag{80}$$

Here we use the relation between quadratic Casimir and Laplacian in AdS, $-\frac{1}{2}L_X^2 = \nabla_X^2$. This is just the same logic in the QM where we related the angular momentum L^2 to the Laplacian on S^2.

3. The (73) relates $\nabla_{X'}^2$ to $\Delta_p(\Delta_p - d)$. The point is that the delta function in the (73) does not contribute within the geodesic integral (70). Finally, we obtain the (77).

The (77) is a defining property of the conformal partial waves[8] for external/internal scalar pri-

[8] When we say conformal blocks rather than conformal partial waves, we focus on the part depend only on the cross ratio u and v.

maries. Moreover, at the OPE limit ($P_4 \to P_1$), the asymptotic behavior does match with one of the CPWs. The same local scaling properties under $P_i \to \lambda(P_i)P_i$ will obey from the properties of bulk-boundary propagators (72).

Thus, one can conclude that "GWD is CPW". Notice that we did not assume the AdS/CFT correspondence. Therefore, one can use GWD expression of CPW for any CFT even without the bulk dual. This is because CPWs are merely kinematical objects. Indeed, "which Δ_p do enter the spectrum?" is a physical question (dynamics), whereas "what is the form of the CPW with fixed Δ_p?" is nothing to do with the dynamics of CFT.

Let us consider t-channel GWD in AdS$_3$ with identical external operators $\Delta_i \equiv \Delta$. Then, the (70) reduces to

$$W_{\Delta,\Delta_p,0}(P_i) = \frac{1}{(-2P_1 \cdot P_4)^{\Delta}(-2P_2 \cdot P_3)^{\Delta}} \int_{\gamma_{14}} d\lambda \int_{\gamma_{23}} d\lambda' \, G_{b\partial}^{\Delta_p}(X(\lambda), X'(\lambda')), \tag{81}$$

where

$$X^A(\lambda) = \frac{P_1^A e^{\lambda} + P_4^A e^{-\lambda}}{\sqrt{-2P_1 \cdot P_4}}, \tag{82}$$

$$X'^A(\lambda') = \frac{P_2^A e^{\lambda'} + P_3^A e^{-\lambda'}}{\sqrt{-2P_2 \cdot P_3}}. \tag{83}$$

It turns out that the integrand only depends on the conformal cross ratio; hence, we can identify the integral part as the conformal block:

$$G_{\Delta,\Delta_p,0}(v,u) \equiv \int_{\gamma_{14}} d\lambda \int_{\gamma_{23}} d\lambda' \, G_{b\partial}^{\Delta_p}(X(\lambda), X'(\lambda')), \tag{84}$$

Compared with the (67), the order of u, v in the argument of conformal block is interchanged. This is because we are considering the t-channel expansion (just flipping all 2 and 4 in the (84)).

Let assume both Δ and Δ_p are sufficiently heavy such that $1 \ll \Delta, \Delta_p \ll c$, where c is the Virasoro central charge. In this regime, one can show that the Virasoro conformal blocks reduce to the global ones at the leading order of $1/c$ expansion. We can further use saddle point approximation for the integrand of the (81), since we have

$$G_{b\partial}^{\Delta_p}(X, X')|_{\mathrm{AdS}_3} \sim e^{-\Delta_p \sigma(X,X')}. \tag{85}$$

Then, the dominant contribution will come from the minimal distance between two geodesics

γ_{14} and γ_{23}. Since

$$\frac{\partial\sigma}{\partial\lambda} = \frac{1}{\xi^{-1}+\sqrt{(\xi^{-1})^2-1}}\left(1+\frac{\xi^{-1}}{\sqrt{(\xi^{-1})^2-1}}\right)\frac{\partial\xi^{-1}}{\partial\lambda}, \tag{86}$$

as same as $\frac{\partial\sigma}{\partial\lambda'}$, $\frac{\partial\xi^{-1}}{\partial\lambda} = 0 = \frac{\partial\xi^{-1}}{\partial\lambda'}$ are necessarily for the minimization. It determines the value of λ (λ') and then ξ^{-1};

$$\lambda = \frac{1}{4}\log\frac{P_{12}P_{13}}{P_{24}P_{34}}, \quad \lambda' = \frac{1}{4}\log\frac{P_{12}P_{24}}{P_{13}P_{34}}, \quad \xi^{-1} = \frac{1}{\sqrt{v}}\left(1+\sqrt{u}\right). \tag{87}$$

Finally, we have obtained

$$\log G_{\Delta_i,\Delta_p,0}(v,u) \sim -\Delta_p\sigma_{min} = -\Delta_p\log\left(\frac{1+\sqrt{u}+\sqrt{(1+\sqrt{u})^2-v}}{\sqrt{v}}\right). \tag{88}$$

One can also obtain the s-channel result by $u \leftrightarrow v$.

Let us rewrite it for more usual convention in the two-dimensional CFT [56],

$$\log\mathcal{F}(h,h_p,1-x) \sim -h_p\log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right), \tag{89a}$$

$$\log\mathcal{F}(\bar{h},\bar{h}_p,1-\bar{x}) \sim -\bar{h}_p\log\left(\frac{1+\sqrt{\bar{x}}}{1-\sqrt{\bar{x}}}\right), \tag{89b}$$

where we introduced a different convention,

$$x = \frac{(z_2-z_1)(z_4-z_3)}{(z_4-z_2)(z_3-z_1)}, \quad \bar{x} = \frac{(\bar{z}_2-\bar{z}_1)(\bar{z}_4-\bar{z}_3)}{(\bar{z}_4-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}, \tag{90}$$

where $z = \sigma + i\tau \in \mathbb{C}$. The dictionary is just $x = \sqrt{u}, 1-x = \sqrt{v}, h+\bar{h} = \Delta, |h-\bar{h}| = \ell$ and $G_{\Delta_i,\Delta_p,0}(u,v) = \mathcal{F}(h,h_p,x)\mathcal{F}(\bar{h},\bar{h}_p,\bar{x})$. If we assume $\tau = 0$, we have $x = \bar{x}$ in the above expression (89). Note that one can check the (89) from Zamolodchikov's recursion relation of the Virasoro conformal blocks. We leave this argument in appendix D.

5 Concluding remarks

In this section, we conclude with some discussion about our results. We also note some possible future direction including ongoing works.

Comments on $\mathcal{E}_W$

We firstly make comments on the possible connection of $\mathcal{E}_W$ to the entanglement of purification (EoP). In [33], the calculation of the EoP has been identified with the conformal blocks with internal twist operators with the aid of the holographic code model [57]. In this case, the corresponding correlation function consists of twist operators with the twist number $\pm\frac{n+1}{2}$ where n is an odd integer. From the path integral perspective, these operators and $\sigma_n(\bar{\sigma}_n)$s would play the same role effectively. The $\mathcal{E}_W$ is not the EoP in general. In particular, the $\mathcal{E}_W$ can be *negative*; thus, the $\mathcal{E}_W$ is farther from the entanglement measure than the EoP.

The $\mathcal{E}_W(\rho_{AB})$ is rather similar to the coherent information [58, 59]

$$I(A\rangle B) \equiv S(\rho_B) - S(\rho_{AB}), \tag{91}$$

or equivalently, the conditional entropy with the minus sign $S(A|B) \equiv -I(A\rangle B)$. Remarkably, these quantities can have either positive and negative values. The conditional entropy has already been discussed in the context of the differential entropy from which one can draw the bulk (closed) convex surfaces [60, 61]. In particular, these were defined together with its orientation (with $\pm$ sign) [62, 63]. For the differential entropy, one needs infinite series of density matrices associated with each infinitesimal subregion. On the other hand, our present result has been derived from a single density matrix ρ_{AB} dual to the entanglement wedge. This is a crucial difference compared with the differential entropy. It is very interesting to study operational interpretation of S_o as like the differential entropy [64].

We have seen that the holographic states should satisfy

$$\mathcal{E}_W(\rho_{AB}) \geq \frac{1}{2}I(A:B) = \frac{1}{2}(S(\rho_A) + S(\rho_B) - S(\rho_{AB})), \tag{92}$$

where ρ_{AB} is a state dual to the bulk classical geometry. On the other hand, we also discussed the 2-qubit state, which satisfied the opposite inequality. Therefore, the above relation is a special feature of the "holographic state" as like monogamy of mutual information [65]. Understanding when such constraints can be satisfied is a very interesting future direction. It might be understood as the specific nature of the "holographic states" such as the absolutely maximally entangled (AME) states [57].

Comments on OEE and future direction

It is also natural to study the general properties of OEE itself. In particular, these studies might give us series of new constraints on the holographic CFT. The proof for general properties may be technically challenging since the OEE is not convex function as opposed to the usual EE. One possible way is to study various setup such as spin systems, free theories, gauge theories on the lattice and so forth. Extensive studies would suggest some generic features of the OEE. Besides such indirect studies of holography, it is very interesting to see the OEE in the holographic code model [57] and path integral optimization [66–68].

This thesis argues only the AdS_3/CFT_2 setup, so higher dimensional cases are still conjecture. In the CFT_d with $d > 2$, the twist operators become non-local operator (co-dimension 2). Hence, we cannot use the geodesic Witten diagram argument (for local operators) at all. The geodesic Witten diagrams (GWD) [51] has been extensively discussed in the case of local operators, including spinning fields [69–74], fermions [75], thermal background [76, 77] and boundary/defect [78–80]. However, we do not know the explicit form for ones with non-local operators—it may be referred as "surface Witten diagrams" [81]. It might give us some insight in the case of higher dimension. Of course, it is not clear that we can follow the similar argument in $d = 2$. It is nice to use gravitational path integral [15, 82, 83]. It is also intriguing to consult with "quantum correction" of the OEE and the entanglement wedge cross section.

Towards the reconstruction of entire spacetime

To prove the entire spacetime region, especially the interior of the horizon in black hole geometries, the entanglement wedge cross section is not enough [84, 85]. From the viewpoint of the co-dimension 2 surfaces, we need extremal (but non-minimal) surfaces. To extract such non-minimal ones, one may need to use the internal (gauged) degrees of freedom in the string theory [22]. Besides the black hole geometries, it is still unclear how to obtain the information of time-like component of metric g_{tt} from a given state. How can we extract the time-like curves from a given state; namely, how does the holographic state realize the gravitational force? We conclude this thesis with a hope that we will be able to address these challenging but interesting issues in the future.

A Some useful facts in Quantum Mechanics

Here we note simple proofs for the facts used in section 2 (Peres' theorem) and section 3 (Schmidt decomposition).

Peres' theorem Suppose our state acting on bi-partite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable,

$$\rho_{AB} = \sum_i p_i \sigma_{iA} \otimes \tau_{iB}, \tag{93}$$

its partial transposition with respect to B is given by

$$\rho_{AB}^{T_B} = \sum_i p_i \sigma_{iA} \otimes \tau_{iB}^*. \tag{94}$$

Here we used the fact that $\tau^T = (\tau^\dagger)^T = \tau^*$ for any physical states τ. Note that the complex conjugate does not change eigenvalues of Hermitian matrices. Therefore, each $\sigma_{iA} \otimes \tau_{iB}^*$ still has non-negative eigenvalues. We can conclude that if a given state has the form (93) (a separable state), $\langle\psi| \rho_{AB}^{T_B} |\psi\rangle \geq 0$ for all $|\psi\rangle$. In other words, taking contraposition, if a given $\rho_{AB}^{T_B}$ has at least one negative eigenvalue, the state ρ_{AB} is entangled.

This simple but powerful argument has been presented by Peres [41]. After that, Horodecki-Horodecki-Horodecki [42] have displayed, explicit examples which are entangled but have no negative eigenvalues for its partial transposition.

Schmidt decomposition Any pure state on bi-partite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle = \sum_{n\alpha} \psi_{n\alpha} |n_A\rangle |\alpha_B\rangle \equiv \sum_n |n_A\rangle |\psi_B^{(n)}\rangle, \tag{95}$$

where

$$|\psi_B^{(n)}\rangle \equiv \sum_\alpha \psi_{n\alpha} |\alpha_B\rangle. \tag{96}$$

Without loss of generality, we can take the above $|n\rangle$s as the eigenvectors for the reduced density matrix of A;

$$\rho_A = \operatorname*{Tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi| = \sum_n p_n |n_A\rangle\langle n_A|, \tag{97}$$

where $0 \leq p_n \leq 1$ and $\sum_n p_n = 1$. Then, from the definition of partial trace, we should conclude

$$\sum_m \langle m_B | \psi_B^{(i)} \rangle \langle \psi_B^{(j)} | m_B \rangle = \langle \psi_B^{(j)} | \psi_B^{(i)} \rangle = p_i \delta_{ij}. \tag{98}$$

Therefore, our $|\psi^{(j)}\rangle$s must be orthogonal to each others. Hence we can identify $|n_B\rangle \equiv |\psi_B^{(n)}\rangle / \sqrt{p_n}$ which leads the Schmidt decomposition,

$$|\psi\rangle = \sum_n \sqrt{p_n} \, |n_B\rangle \, |n_B\rangle \, . \tag{99}$$

B Replica Path Integral and Partial Transposition

In this appendix, we review the replica trick which is a methodology to compute n-th power of reduced density matrices [27, 44] (and its partial transposition [48, 49, 55]). For technical reason, we are especially interested in the two-dimensional CFT. In section B.1, we explain the path integral representation of $\mathrm{Tr}\rho_A^n$ and specific simplification for CFT_2. In section B.2, we describe how we can realize the partial transposition in terms of the replica trick. In particular, we argue that the scaling dimension of some twist operators depends on the parity of n.

B.1 Path integral representation of reduced density matrices

Firstly, we write the vacuum wave functional $\Psi_0[\phi]$ by using the Euclidean path integral,

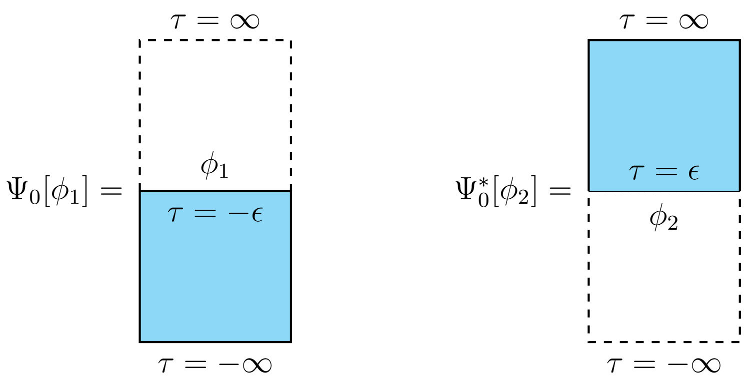
$$\Psi_0[\phi] = \langle\phi| \lim_{\Delta\tau\to\infty} e^{-\Delta\tau H} |\Psi\rangle = \langle\phi|0\rangle \,, \tag{100}$$

where $|0\rangle$ is the ground state such that $H|0\rangle = 0$. We obtain

$$\Psi_0[\phi_1] = \mathcal{N}^{-\frac{1}{2}} \int \mathcal{D}\phi \left(\prod_{\vec{x}\in A} \delta(\phi(-\epsilon,\vec{x}) - \phi_1(\vec{x}))\right) \exp\left[-\int_{-\infty}^{-\epsilon} d\tau\, L[\phi(\tau,\vec{x})]\right], \tag{101}$$

$$\Psi_0^*[\phi_2] = (\mathcal{N}^*)^{-\frac{1}{2}} \int \mathcal{D}\phi \left(\prod_{\vec{x}\in A} \delta(\phi(+\epsilon,\vec{x}) - \phi_2(\vec{x}))\right) \exp\left[-\int_{+\epsilon}^{\infty} d\tau\, L[\phi(\tau,\vec{x})]\right], \tag{102}$$

which may be pictorially written as



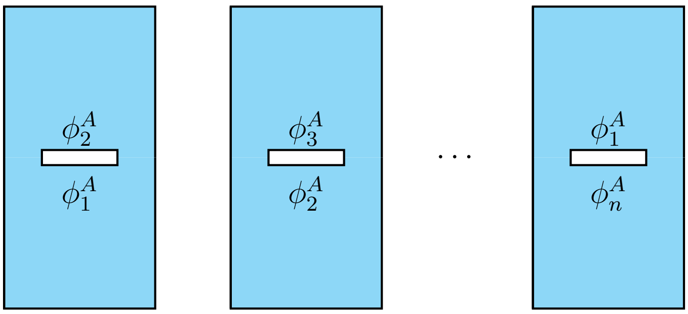
(103)

Then, we can make its reduced density matrix ρ_A just by gluing the boundary conditions imposed on A^c,

$$
\begin{aligned}
&\rho_A[\phi_1^A, \phi_2^A] \\
&= \langle \phi_1^A | \Big(\mathrm{Tr}_{\mathcal{H}_{A^c}} |0\rangle\langle 0| \Big) |\phi_2^A\rangle \\
&= |\mathcal{N}|^{-1} \int \mathcal{D}\phi \left(\prod_{\vec{x}\in A} \delta(\phi(+\epsilon, \vec{x}) - \phi_2(\vec{x}))\delta(\phi(-\epsilon, \vec{x}) - \phi_1(\vec{x})) \right) e^{-S[\phi]}.
\end{aligned} \tag{104}
$$

Demanding $\mathrm{Tr}_{\mathcal{H}_A}\rho_A = 1$, we can identify $\mathcal{N} = \int \mathcal{D}\phi e^{-S[\phi]} \equiv Z_1$. By using the above expression (104), one can express the trace of ρ_A^n in the similar way:

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n = |\mathcal{N}|^{-n} \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_n \Bigg(&\prod_{\vec{x}\in A} \delta(\phi_1(-\epsilon, \vec{x}_1) - \phi_2(\epsilon, \vec{x}_2)) \\
&\delta(\phi_2(-\epsilon, \vec{x}_2) - \phi_3(\epsilon, \vec{x}_3)) \cdots \delta(\phi_n(-\epsilon, \vec{x}_n) - \phi_1(\epsilon, \vec{x}_1)) \Bigg) \exp\left[-\sum_{i=1}^{n} S[\phi_i] \right]
\end{aligned} \tag{105}
$$

$=$ 

(106)

The last line is a schematic picture of the $\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n$ where we are gluing each boundary condition ϕ_is on a sheet into another sheet with the same ϕ_is. In this way, we can regard the $\mathrm{Tr}\rho_A^n$ as a partition function on the n-sheeted geometry (Riemann sheet),

$$
\mathrm{Tr}_{\mathcal{H}_A}\rho_A^n = Z_n/Z_1^n, \tag{107}
$$

where each Z_1 comes from the normalization factor $\mathcal{N}$.

To be concrete, we shall consider the time slice of two-dimensional conformal field theories. We can treat this Z_n in two different ways. First, by using the conformal transformation, we can relate Z_n to the theory on a single plane with a deficit angle. Second, we remind the original motivation to introduce the Riemann surface. That was to avoid the multi-valued function on a single sheet. This can be regarded as the orbifold theory $\mathrm{CFT}^n/\mathbb{Z}_n$ with insertions of twist operators which play a role of branch points. In the second viewpoint, we simply have

$$Z_n \propto \langle \sigma_n(u)\bar{\sigma}_n(v)\rangle_{\mathrm{CFT}^n/\mathbb{Z}_n}\,, \tag{108}$$

where $\sigma_n(u)$ and $\bar{\sigma}_n(v)$ are the (branch point) twist and anti-twist operators. Notice that the right-hand side is just the two-point function of local operator. We can fix it by conformal symmetry once we determine the scaling dimension of twist operators. To this end, Calabrese and Cardy computed the one point function of the stress tensor in the aforementioned two ways and equate with them. We skip the detail of calculation but finally they obtained

$$h_{\sigma_n} = \bar{h}_{\sigma_n} = \frac{c}{24}\left(n - \frac{1}{n}\right). \tag{109}$$

We can straightforwardly write the above ones in the case of multiple N-interval cases ($N > 1$). In this case, we finally obtain Z_n as partition function on the Riemann surface with genus $g = (n-1)(N-1)$ or $2N$-point correlation function for $\mathrm{CFT}^n/\mathbb{Z}_n$,

$$Z_n \propto \langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2)\cdots\sigma_n(u_N)\bar{\sigma}_n(v_N)\rangle_{\mathrm{CFT}^n/\mathbb{Z}_n}\,. \tag{110}$$

The latter can be evaluated in the holographic CFT_2 since it reduces to the vacuum conformal blocks (at the leading order of large-c limit).

B.2 Replica trick including the partial transposition

The partial transposition (of the reduced density matrix) can be also expressed in terms of the replica partition function and correlators of twist operators. For simplicity, let us consider the two disjoint interval case:

$$\langle \phi_1^A\phi_1^B|\rho_{AB}|\phi_2^A\phi_2^B\rangle = |\mathcal{N}|^{-1}\int \mathcal{D}\phi\left(\prod_{\vec{x}\in A,B}\delta(\phi(+\epsilon,\vec{x})-\phi_2(\vec{x}))\delta(\phi(-\epsilon,\vec{x})-\phi_1(\vec{x}))\right)e^{-S[\phi]}. \tag{111}$$

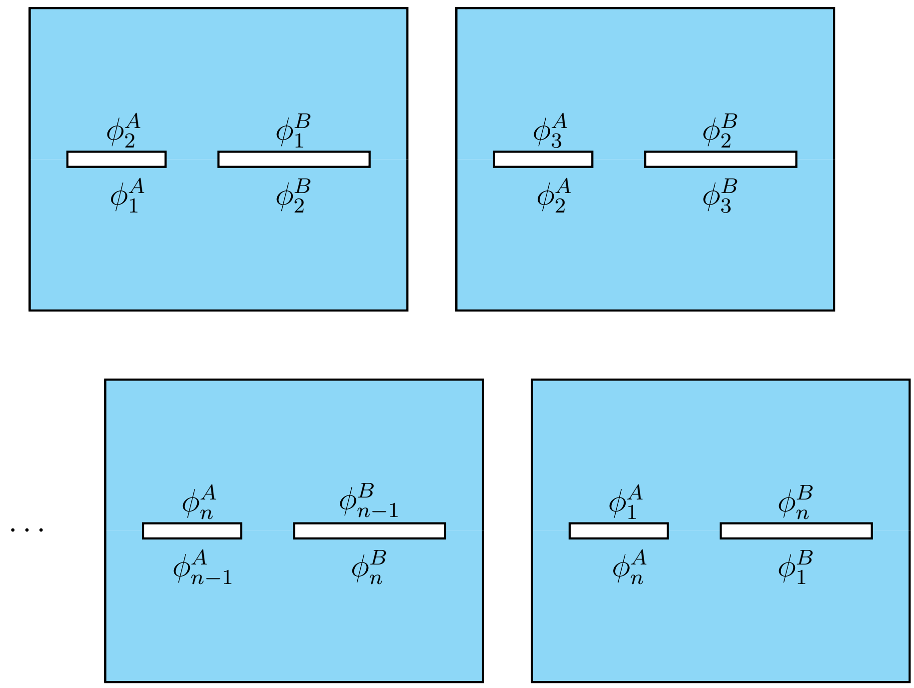
Then, we take the partial transposition for the second disconnected peace B:

$$
\begin{aligned}
\langle \phi_1^A \phi_1^B | \rho_{AB}^{T_B} | \phi_2^A \phi_2^B \rangle &= \langle \phi_1^A \phi_2^B | \rho_{AB} | \phi_2^A \phi_1^B \rangle \\
&= |\mathcal{N}|^{-1} \int \mathcal{D}\phi \left(\prod_{\vec{x} \in A} \delta(\phi(+\epsilon, \vec{x}) - \phi_2(\vec{x})) \delta(\phi(-\epsilon, \vec{x}) - \phi_1(\vec{x})) \right) \\
&\qquad \left(\prod_{\vec{x} \in B} \delta(\phi(+\epsilon, \vec{x}) - \phi_1(\vec{x})) \delta(\phi(-\epsilon, \vec{x}) - \phi_2(\vec{x})) \right) e^{-S[\phi]}.
\end{aligned}
\tag{112}
$$

Therefore, $\mathrm{Tr}_{\mathcal{H}} (\rho_{AB}^{T_B})^n$ can be written schematically as follows;

$\mathrm{Tr}_{\mathcal{H}} (\rho_{AB}^{T_B})^n$

$=$



, (113)

where the same $\phi_i^{A,B}$ are glued each others. Since the row and column in the second subregion B are interchanged, the replicated geometry is also changed compared with the original $\mathrm{Tr}_{\mathcal{H}} \rho_{AB}^n$.

We can easily write its counterpart in the orbifold CFT,

$$
\mathrm{Tr}_{\mathcal{H}} (\rho_{AB}^{T_B})^n = \langle \sigma_n(u_1) \bar{\sigma}_n(v_1) \bar{\sigma}_n(u_2) \sigma_n(v_2) \rangle_{\mathrm{CFT}^n/\mathbb{Z}_n} . \tag{114}
$$

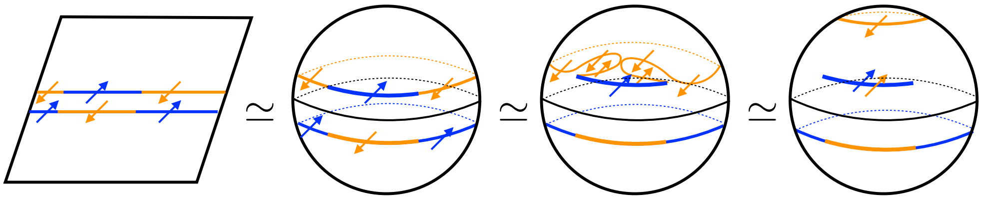


Figure 6: These figures show a pictorial explanation of the pure state limit. Let us concentrate on a single replica sheet (very left panel). We shall view each complex plane as a 2-sphere as usual (middle left panel). Each line and arrows (blue/orange) represent the branch cut connected with the upper or downer sheet. One can deform each cut in topologically equivalent manners (middle right panel; for brevity, let us do it only for upper hemisphere). Finally, we obtain a single branch cut but this cut skips the next upper sheet and connects to the next-next upper sheet (very right panel).

Notice that the order of twist and anti-twist operators on the subregion B is flipped. Let us consider the pure state limit; namely, the limit with $(A \cup B)^c \to \emptyset$. For the replicated geometry, this may be understood as like figure 6. In the terminology of (114), it corresponds to the limit with $u_2 \to v_1, v_2 \to v_1$,

$$\mathrm{Tr}_{\mathcal{H}}(\rho_{AB}^{T_B})^n \to \langle \sigma_n^2(u_1)\bar{\sigma}_n^2(v_1)\rangle_{\mathrm{CFT}^n/\mathbb{Z}_n}\,, \tag{115}$$

where $\sigma_n^2(\bar{\sigma}_n^2)$ are the (ant-)twist operators with twist number ± 2. An important point is that the n-sheeted geometry is dramatically different between even n and odd n. In particular, for even n, our replicated geometry decouples to two independent $n/2$-sheeted geometry each of which is equivalent to the path integral of $\mathrm{Tr}_{\mathcal{H}_A}\rho_A^{n/2}{}^2$. On the other hand, for odd n, it reduces to the $\mathrm{Tr}\rho_A^n$ after changing the label of each sheets. Therefore, the scaling dimension of $\sigma_n^2(\bar{\sigma}_n^2)$ can be determined from the one of $\sigma_n(\bar{\sigma}_n)$:

$$h_{\sigma_n^2} = \bar{h}_{\sigma_n^2} = \begin{cases} \dfrac{c}{24}\left(n - \dfrac{1}{n}\right) & (n:\text{odd}), \\ \dfrac{c}{12}\left(\dfrac{n}{2} - \dfrac{2}{n}\right) & (n:\text{even}). \end{cases} \tag{116}$$

In the main text, we will omit the suffix of the correlation function, $\mathrm{CFT}^n/\mathbb{Z}_n$, for brevity.

C Embedding formalism in AdS/CFT

We use the embedding definition of (Euclidian) AdS_{d+1} and its conformal boundary $\mathbb{R}^d$. This idea was first initiated by Dirac [86] and recently revived by Weinberg [87]. We also recommend to see [88–90]. Let us consider $\mathbb{R}^{d+1,1}$ and its sub-manifolds

$$\mathrm{AdS}_{d+1}: \; X^2 = -\ell_{\mathrm{AdS}}^2 \equiv -1, \; X^0 > 0, \tag{117a}$$

$$\partial\mathrm{AdS}_{d+1}: \; P^2 = 0, \; P^A \sim \lambda P^A \quad (\lambda \in \mathbb{R}). \tag{117b}$$

Here we used the light cone metric

$$A \cdot B = \eta_{AB} A^A B^B = -\frac{1}{2}\left(A^+ B^- + A^- B^+\right) + \delta_{ab} A^a B^b \quad \text{(for all vector } A, B \text{ in } \mathbb{R}^{d+1,1}\text{)}. \tag{118}$$

These sub-manifolds indeed represent the AdS_{d+1} and its conformal boundary. Especially,

$$X^A = (X^+, X^-, X^a) = \frac{1}{z}\left(1, z^2 + x^2, x^a\right), \tag{119}$$

and

$$P^A = (P^+, P^-, P^a) = \left(1, x^2, x^a\right), \tag{120}$$

describe the Poincaré patch and its boundary $\mathbb{R}^d$ (see figure 7).

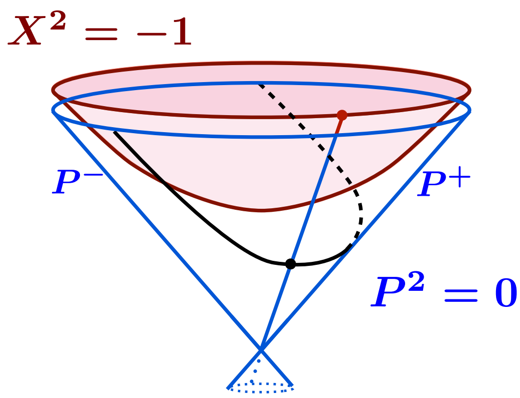


Figure 7: Euclidian AdS (red hyperboloid) and its conformal boundary (blue light cone) in the embedding space. The blue light ray shows the identification of the boundary points $P^A \sim \lambda P^A$. The black hyperbolic curve displays one choice of the flat section for CFT (the Poincaré section).

C.1 Embedded fields in AdS/CFT

We would like to embed the fields in AdS$_{d+1}$/CFT$_d$ into a flat space $\mathbb{R}^{d+1,1}$. To this end, we must impose constraints in order to adjust the total d.o.f. to the original one. We can employ "transverse condition" for tensor fields

$$P_{A_1} T_\partial^{A_1 A_2 \cdots A_\ell}(P) = 0, \quad X_{A_1} T_b^{A_1 A_2 \cdots A_\ell}(X) = 0, \tag{121}$$

where T_∂ is a tensor field in the CFT and T_b is one in the AdS. Note that these conditions are imposed only on the physical space (117). We further impose the condition to the primary field $T_\partial^{A_1 A_2 \cdots A_l}(X)$ as

$$T_\partial(\lambda P) = \lambda^{-\Delta} T_\partial(P), \tag{122}$$

which can be realized as

$$T_\partial(\lambda P) = (P^+)^{-\Delta} t_\partial(P^a/P^+), \tag{123}$$

where we omitted the tensor indices for brevity. Having the (123) with the Poincaré patch (120), one can show that t_∂ satisfies conformal algebra for primary fields with scaling dimension Δ and spin ℓ[9]. It is worth noting that the usual scaling limit of the AdS/CFT fields,

$$T_\partial(P) = \lim_{X^+ \to \infty} (X^+)^\Delta T_b(X = X^+ P + \cdots), \tag{124}$$

also reproduce the (122).

One can also embed Dirac fermions for even d [75, 87],

$$P_A \Gamma^A \Psi_\partial(P) = 0, \quad X_A \Gamma^A \Psi_b(X) = \Gamma_{\text{chiral}} \Psi_b(X), \tag{125}$$

where Γ^A (Γ_{chiral}) are the (chiral) gamma matrix in the embedding space.

The covariant derivative is given by

$$\nabla_A = G_{AB} \partial_X^B + X^B \Sigma_{AB}, \tag{126}$$

where we defined the induced metric

$$G_{AB} = \eta_{AB} + X_A X_B, \tag{127}$$

[9]Here we should use $SO(d+1,1)$ (Lorentz) generator $L_{AB}(P) = P_A \partial_B - P_B \partial_A + \Sigma_{AB}$, where Σ_{AB} is generators associated with the rotational representation.

and the generator of rotation for non-trivial spin representation Σ_{AB}. It is worth noting that $X^A \nabla_A = 0 = X^A G_{AB}$ on the AdS sub manifold $X^2 = -1$.

By using the embedding space, one can algebraically determine the two and three-point correlation functions in CFT without using the conformal Ward-Takahashi identity. Furthermore, one can also fix the bulk-boundary propagator in AdS. (Of course, we cannot do the overall constants though.) We listed the result only the case for scalars since we use it explicitly in this thesis:

$$G^{\Delta,0}_{\partial\partial}(P_1, P_2) = \frac{1}{(-2P_1 \cdot P_2)^{\Delta}}, \tag{128}$$

$$G^{\Delta,0}_{b\partial}(X, P) = \frac{1}{(-2X \cdot P)^{\Delta}}, \tag{129}$$

where the bulk scalar field has the mass $m^2 = \Delta(\Delta - d)$. Notice that these are the unique choice up to constant. This fact obeys from the scaling property of primaries, $P_i \to \lambda(P_i) P_i$. For non-trivial representation, please see [74, 75, 90] and references therein.

C.2 Geodesics in AdS

In this section, we use Lorentzian signature. We note the case of AdS$_3$ ($\mathbb{R}^{2,2}$) hereafter, but the generalization is obvious. We take our metric on $\mathbb{R}^{2,2}$ as

$$ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 - dX_3^2. \tag{130}$$

Then, AdS$_3$ is defined as a timelike surface in $\mathbb{R}^{2,2}$;

$$X^2 \equiv -X_0^2 + X_1^2 + X_2^2 - X_3^2 = -\ell^2_{\text{AdS}} \tag{131}$$

First of all, let us consider the geodesics on AdS$_3$. Geodesic equation can be derived from free particle Lagrangian,

$$L = \frac{1}{2}\dot{X}^2 + \lambda(X^2 + \ell^2_{\text{AdS}}), \tag{132}$$

where $\dot{X} \equiv \frac{dX}{d\tau}$ and λ is the Lagrange multiplier. The 2nd term is necessarily for a particle to stay on AdS$_3$. Then, the equation of motion is,

$$\ddot{X}^A = 2\lambda X^A. \tag{133}$$

Since our Lagrangian is invariant under the $SO(2,2)$ rotation, we have the conserved currents;

$$\begin{aligned}
&J_{AB} = X_A \dot{X}_B - \dot{X}_A X_B, \\
&\frac{dJ_{AB}}{d\tau} \overset{(133)}{=} 0.
\end{aligned} \tag{134}$$

We also have

$$J_{AB} X^B = \ell_{\text{AdS}}^2 \dot{X}_A, \tag{135}$$
$$J_{AB} \dot{X}^B = \dot{X}^2 X_A, \tag{136}$$
$$J_{AB} J^{AB} = -2\ell_{\text{AdS}}^2 (\dot{X})^2. \tag{137}$$

Therefore,

$$\frac{d}{d\tau}\dot{X}_A \overset{(135)}{=} \frac{d}{d\tau}\left(\frac{1}{\ell_{\text{AdS}}^2} J_{AB} X^B\right) \overset{(134)}{=} \frac{1}{\ell_{\text{AdS}}^2} J_{AB} \dot{X}^B \overset{(136)}{=} \frac{\dot{X}^2}{\ell_{\text{AdS}}^2} X_A. \tag{138}$$

Thus,

$$\lambda = \frac{\dot{X}^2}{2\ell_{\text{AdS}}^2}, \quad \dot{\lambda} \overset{(137)}{=} 0. \tag{139}$$

Let us consider spacelike geodesics. We can easily obtain the generic solution,

$$X^A(\tau) = m^A e^{\sqrt{\frac{\dot{X}^2}{\ell_{\text{AdS}}^2}}\tau} + n^A e^{-\sqrt{\frac{\dot{X}^2}{\ell_{\text{AdS}}^2}}\tau} \tag{140}$$

with

$$m^2 = n^2 = 0, \quad 2(m \cdot n) = -\ell_{\text{AdS}}^2. \tag{141}$$

Note that $\sqrt{\dot{X}^2}\tau$ is related to the geodesic distance. Therefore, the spacelike geodesics anchored on the boundary points should have the form (76). Let us define a parameter ξ as

$$\xi^{-1} = -X(\tau_1) \cdot X(\tau_2) = \ell_{\text{AdS}}^2 \cosh\left[\sqrt{\dot{X}^2}(\tau_1 - \tau_2)\right]. \tag{142}$$

Since $\sqrt{\dot{X}^2}(\tau_1 - \tau_2)$ is nothing but the geodesic distance σ, we have obtained

$$\sigma(X(\tau_1), X(\tau_2)) = \cosh^{-1}\left(\frac{1}{\xi}\right) = \log\left(\frac{1 + \sqrt{1 - \xi^2}}{\xi}\right), \tag{143}$$

where we took $\ell_{\text{AdS}} = 1$.

D Virasoro conformal blocks from recursion relation

In this appendix, we introduce another way to obtain the Virasoro conformal blocks, introduced by Zamolodchikov. The goal of this section is to reproduce the (52) by using Zamolodchikov's recursion relation.

In the two-dimensional CFT, we have the infinite-dimensional conformal symmetry. The generators obey the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{144a}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}(m^3 - m)\delta_{m+n,0}, \tag{144b}$$

where L_n and $\bar{L}_n$ corresponds to the generators for left and right movers (z and $\bar{z}$). The $c, \bar{c}$ are the central charges and we will consider the case $c = \bar{c}$. The sub-algebra associated with the global conformal symmetry is the part $n, m = 0, \pm 1$. One can construct the highest weight representation (conformal family) of each algebra labelled by the eigenvalues of $(L_0, \bar{L}_0) = (h, \bar{h})$. The highest weight state is called primary states. Due to the factorization of left/right modes in CFT_2, we can often discuss the holomorphic or anti-holomorphic part individually. For further detail, see [56].

Note that the asymptotic symmetry in the three-dimensional AdS spacetime also turns out to be the same one (144) [45]. Hence, by using the Virasoro conformal blocks, we can take into account the "quantum gravitational effect" naturally for each conformal blocks because the Virasoro confromal blocks include stress tensor contributions in the global ones.

In what follows, we consider a four-point function of the identical operator ϕ. We may expand CFT_2 correlators in terms of Virasoro conformal blocks not merely the global ones,

$$\langle \phi(0)\phi(x,\bar{x})\phi(1)\phi(\infty)\rangle = \sum_{h_{\mathcal{O}},\bar{h}_{\mathcal{O}}} a_{\mathcal{O}} \mathcal{F}(c, h_\phi, h_{\mathcal{O}}; x)\bar{\mathcal{F}}(c, \bar{h}_\phi, \bar{h}_{\mathcal{O}}; \bar{x}) \tag{145}$$

D.1 Zamolodchikov's recursion relation

The holomorphic part of Virasoro conformal blocks for the (145) can be obtained from the following recursion relation [30, 31]

$$\mathcal{F}(c,h_\phi,h_\mathcal{O};x)=[x(1-x)]^{\frac{c-1}{24}-2h_\phi}[16\mathfrak{q}(x)]^{h_\mathcal{O}-\frac{c-1}{24}}\vartheta_3(\mathfrak{q}(x))^{\frac{c-1}{2}-16h_\phi}H(c,h_\phi,h_\mathcal{O};\mathfrak{q}(z)), \tag{146}$$

$$H(c,h_\phi,h_\mathcal{O};\mathfrak{q}(x))=1+\sum_{m,n\geq 1}\frac{(16)^{mn}R_{m,n}(c,h_\phi)}{h_\mathcal{O}-h_{m,n}(c)}H(c,h_\phi,h_{m,n}(c)+mn;\mathfrak{q}(x)), \tag{147}$$

where,

$$\mathfrak{q}(z)=\exp\left[-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2},1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2},1;x)}\right]=e^{-\pi\frac{K(1-x)}{K(x)}}, \tag{148}$$

$$\vartheta_3(\mathfrak{q}(x))=\sum_{n\in\mathbb{Z}}\mathfrak{q}^{n^2}(x), \tag{149}$$

$$h_{m,n}(c)=\frac{c-1}{24}+\frac{(\alpha_+m+\alpha_-n)^2}{4},\quad \alpha_\pm=\sqrt{\frac{1-c}{24}}\pm\sqrt{\frac{25-c}{24}}, \tag{150}$$

$$R_{m,n}(c,h_\phi)=-\frac{1}{2}\prod_{r,s}\left(2\ell_\phi-\frac{\ell_{r,s}}{2}\right)^2\left(\frac{\ell_{r,s}}{2}\right)^2\prod_{a,b}\left(\frac{1}{\ell_{a,b}}\right). \tag{151}$$

Here the labels inside of $R_{m,n}$ is given by

$$r=-m+1,-m+3,\cdots,m-3,m-1 \tag{152}$$

$$s=-n+1,-n+3,\cdots,n-3,n-1 \tag{153}$$

$$a=-m+1,-m+2,\cdots,m \tag{154}$$

$$b=-n+1,-n+2,\cdots,n \tag{155}$$

except for $(a,b)=(0,0),(m,n)$.

To solve the recursion relation, first we expand $H(q)$ as

$$H(\mathfrak{q}(x))=1+\sum_{k=1}^{\infty}c_k(h_\mathcal{O})\mathfrak{q}^k(x),\quad c_0(h_\mathcal{O})=1, \tag{156}$$

and assign it into (147). Then, we get

$$\sum_{k=1}^{\infty} c_k(h_{\mathcal{O}})\mathfrak{q}^k(x) = \sum_{m,n\geq 1}\sum_{\ell=0}^{\infty} \frac{(16)^{mn}R_{m,n}}{h_{\mathcal{O}} - h_{m,n}(c)} c_\ell(h_{m,n}+mn)\mathfrak{q}^{\ell+mn}(x). \tag{157}$$

Though we have $c_\ell(h_{m,n}+mn)$ in the right-hand side rather than $c_k(h_{\mathcal{O}})$, we can obtain one for $c_\ell(h_{m,n}+mn)$ by simply replacing $h_{\mathcal{O}} = h_{m,n}+mn$.

We can solve it order by order. Firstly, if mn is an odd number, we always have $R_{m,n} = 0$. Iterating this fact, we get $c_{2k+1}(h_{\mathcal{O}}) = 0$. Then, we obtain

$$c_{2k}(h_{\mathcal{O}}) = \sum_{i=1}^{k}\sum_{mn=2i} \frac{R_{m,n}}{h_{\mathcal{O}} - h_{m,n}(c)} c_{2k-2i}(h_{m,n}+2p). \tag{158}$$

For the open source code for Mathematica, which implements the above recursion relation, see [91].

D.2 Conformal blocks with internal twist operator

Let us compare the results (52) (and (89)) with the one obtained from the above recursion relation. To obtain the analytic expression, we need to truncate the power of $\mathfrak{q}(x)$ at finite order. Therefore, let us consider the OPE limit $x \to 0$. (Here we are discussing about the s-channel conformal blocks. In terms of the t-channel blocks, flip $x \to 1-x$ and take $x \to 1$.) In this limit, the $\mathfrak{q}(x)$ and $\vartheta_3(\mathfrak{q}(x))$ behave as

$$\mathfrak{q}(x) = \frac{x}{16} + \frac{x^2}{32} + \frac{21x^3}{1024} + \frac{31x^4}{2048} + \frac{6257x^5}{524288} + \frac{10293x^6}{1048576} + \mathcal{O}(x^7), \tag{159}$$

$$\vartheta_3(\mathfrak{q}(x)) = 1 + \frac{x}{8} + \frac{x^2}{16} + \frac{21x^3}{512} + \frac{993x^4}{32768} + \frac{6273x^5}{262144} + \frac{5169x^6}{262144} + \mathcal{O}(x^7). \tag{160}$$

Then, we can expand $\log\mathcal{F}(c, c\delta, c\delta; x)$ around $x = 0$:

$$\log\mathcal{F}(c, c\delta, c\delta; x) = -c\delta\log x + c\delta\Bigg(\frac{x}{2} + \frac{3}{16}x^2 + \frac{5}{48}x^3 + \frac{35}{512}x^4 + \frac{63}{1280}x^5 + \mathcal{O}(x^6) + \mathcal{O}(\delta)\Bigg) + \mathcal{O}(c^0). \tag{161}$$

Here we first set $h_\phi = h_\mathcal{O} = c\delta$ and expanded $\mathcal{F}(c, c\delta, c\delta; x)$ around $x = 0$. Here $\delta \equiv \frac{n^2-1}{24n}$ is supposed to be very small. Then, we took the large-c limit and after that we further expanded the conformal block around $\delta = 0$. Notice that there are no singular terms like $\mathcal{O}(\delta^{-1})$. One can compare it with the saddle point approximation of the geodesic Witten diagrams. These two expression agree with order by order except the position-independent term $-c\delta \log 4$. This term appears from the small x-expansion of the geodesic Witten diagrams. However, this is just the matter of the normalization of conformal blocks. Besides choice of the normalization, we are free to choose the UV cut off ϵ for the calculation of the EE or the OEE.

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