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Ph.D. Thesis

Analysis on entanglement entropy for two-dimensional lattice gauge theories with matter fields

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Abstract

We study entanglement entropy for 2-dimensional lattice gauge theories with matter fields. For gauge theories, even in lattice space we fail to define entanglement entropy as usual way due to non-local excitations characteristic of theories with constraints. To solve this problem, we apply extended Hilbert space formalism and define entanglement entropy in the generalized way. At first we review the formulation and outcomes by using it. As a result, in addition to ordinary quantum correlation (Bell pair part), two new types of contribution we call Shannon part and color part, will appear. After outlining this effect explicitly, we analyze entanglement entropy of the ground state for 2-dimensional $SU(N)$ gauge theory with fundamental scalar matter field. As tools to perform that, we use transfer matrix method and hopping parameter expansion(HPE), roughly the hopping parameter corresponds to inverse square of the mass for the matter. The evaluation is carried out in the perturbation from infinity mass limit. In the analysis, we observe that all of the three types of contribution emerge, while Bell pair part does in higher order than Shannon and color part. We discuss the implication by the result.

Contents

1 Introduction

Quantum correlation, or entanglement is a special feature of quantum mechanics and it is one of the key concepts in theoretical physics in various areas. Especially, as a quantity to measure such correlation, entanglement entropy has been the primal subject of study.

In condensed matter physics, entanglement entropy can be the index of quantum phase transition[1, 2]. For example, in topologically ordered systems in 2+1-dimension, topological entanglement entropy[3] can be used to distinguish orders.

On the other hand, in information theory, entanglement is considered as "resources" for tasks which classical systems never accomplish. One such example is quantum teleportation[4], where given quantum states can be transmitted from a sub-system to another one without quantum communication. This can be achieved by using entanglement shared in the two sub-systems prepared in advance. Entanglement can also be used to protect information from quantum errors. The protocols are known as quantum error correcting code[5]. In quantum computation theory, many algorithms to solve various problems have been devised, and by using them we can achieve higher performance than using classical ones. A famous example is Shor's algorithm[6], which solves the problem of finding prime numbers for given integer. The problem is considered hard to solve by classical computer, and Shor's algorithm has offered more efficient computation. In quantum communication and quantum cryptography theory, entanglement is used to the anonymity of information being transferred[7].

Generically, the quantum correlation in quantum states can be considered as follows. Take some quantum state whose density operator is represented as ρ, and pick up spatial sub-region V. Let us suppose that the total Hilbert space $\mathcal{H}$ has the *tensor product structure*(TPS),

$$\mathcal{H} = \mathcal{H}_{\mathrm{V}} \otimes \mathcal{H}_{\bar{\mathrm{V}}}, \tag{1}$$

where $\mathcal{H}_{\mathrm{V}}$ is Hilbert space composed by d.o.f. supported by sub-region V and so $\mathcal{H}_{\bar{\mathrm{V}}}$ by $\bar{\mathrm{V}}$.[1] If ρ has the structure,

$$\rho = \sum_i p_i \rho_i = \sum_i p_i \rho_{i\mathrm{V}} \otimes \rho_{i\bar{\mathrm{V}}}, \tag{2}$$

[1] In general, the TPS needs not to be based on spatial structure, and *any* TPSs allow us to define the entanglement entropy corresponding to that[8]. In this paper we consider spatial structure case only.

where p_i is probability distribution and density operators ρ_{V} and $\rho_{\bar{\mathrm{V}}}$ act on $\mathcal{H}_{\mathrm{V}}$ and $\mathcal{H}_{\bar{\mathrm{V}}}$ respectively. Then we say that the state is *separable.* If not, we say that the state is *non-separable* or *entangled*[2]. One of the concrete quantification of the entanglement is entanglement entropy.

Naively, *Entanglement entropy* for the state ρ associated to sub-region V is defined as von Neumann entropy,

$$S_{\mathrm{EE}}(V,\rho) = -\mathrm{Tr}_{\mathcal{H}_{\mathrm{V}}}\rho_{\mathrm{V}}\log\rho_{\mathrm{V}}. \tag{3}$$

where

$$\rho_{\mathrm{V}} = \mathrm{Tr}_{\mathcal{H}_{\bar{\mathrm{V}}}}\rho. \tag{4}$$

It is obvious that the definition depends on the TPS. With after extension in mind, we call this quantity as *ordinary entanglement entropy.*

At least for pure states, entanglement entropy quantifies quantum correlation between two sub-systems[9]. Roughly speaking, it counts the number of Bell pairs spreading over sub-region V and its complement to which the entanglement of the state corresponds. Here for the term Bell pairs we mean 2 qubit states like

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle\,|\uparrow\rangle + |\downarrow\rangle\,|\downarrow\rangle\right). \tag{5}$$

More detailed explanation will be given at §2.

Although entanglement can be considered for any states in principle, how the ground state(vacuum) in the theory is entangled is particularly interesting question, because it gives much of information about the theory. In the seminal work [10], the ground state in massless free field is analyzed in the context of black hole entropy. In [11], entanglement entropy of the ground state in spin chain model is studied to know the phase structure of the theory.

Gauge theories, on the other hand, are playing a central role in elementary particle physics. In the standard model, among four types of fundamental forces, three of them except gravitational one are described as quantum gauge fields. Establishing quantum theory of gravitation (quantum gravity) is one of the biggest problem in the present particle physics.

One of the important key to understand quantum gravity is given by gauge/gravity duality[12, 13]. In the framework, it is argued that quantum

[2]We simply use the term "entanglement" as equivalent to quantum correlation.

gravity in some special setup(*e.g.* the geometry is asymptotic anti-de Sitter space) is equivalent to certain gauge theory non-perturbatively. Although enormous works have studied that, the essential part of the mechanism is still mysterious.

As one of the most remarkable progress, Ryu and Takayanagi proposed a formula[14] which has shed a light to entanglement entropy, offering that it is related deeply to geometrical quantity in gravity side. Ryu-Takayanagi formula states that,

$$S_{\mathrm{EE}} = \frac{A_{\mathrm{ext}}}{4G_N}, \tag{6}$$

up to leading order in $1/N$ expansion, where r.h.s. refers the area of extremal surface on bulk-side. This formula suggests that quantum entanglement, especially for gauge theories, is a important clue to unravel quantum gravity.

It happens, however, that the TPS (1) breaks so that we cannot exploit the definition (3) directly. Without TPS, we would fail to have the reduced density matrix(4), so to entanglement entropy. As one of the reason, it is known that in continuum quantum field theory we don't have any spatial TPS due to UV structure[15]. To get spatial TPS, we need some UV regularization. In this paper we focus on the analysis with lattice regularized theories to avoid that situation.

Even in lattice theories, however, TPS still breaks in gauge theories. It is because gauge theories inevitably include non-local fundamental d.o.f., such as Wilson loops. These objects spreading over sub-region V and $\bar{\mathrm{V}}$ *cannot* be described by d.o.f. supported by only V or $\bar{\mathrm{V}}$. In order to have well-defined entanglement entropy, we need to modify the definition.

There has been many works trying to cure this problem[16, 17, 18, 19, 20, 21, 22, 23, 24, 25] by generalizing the definition of entanglement entropy[3]. Roughly we can classify them in two approaches. One is to extend the Hilbert space by allowing gauge *variant* states(nonphysical states) to join to the Hilbert space[22, 23, 25]. As a result the Hilbert space becomes factorized and we can define reduced density matrix as usual way, and also entanglement entropy. In this paper we call this approach the *extended Hilbert space formalism.* The other approach[17, 18, 19, 21, 24] is to apply new definition for the reduced density matrix using operator algebra. We call this approach the *algebraic formalism.*

In this paper we exploit extended Hilbert space approach. The reason to

[3]We call entanglement entropy by original definition *ordinary entanglement entropy.*

use it is that for lattice theories, the realization of extended Hilbert space[4] becomes very natural and the calculation of entanglement entropy can be done straightforward way. The comparison with algebraic approach is given in appendix C.

The remarkable outcome by extending the definition of entanglement entropy is that they generically produce two *new* types of contributions to entanglement entropy, adding to original contribution we saw above (we refer this original contribution as *Bell pair part*). Concretely, in the §3, we will see that entanglement entropy for extended Hilbert space formalism have the form,

$$S_{\rm EE} = -\sum_{\mathbf{k}} p_{\mathbf{k}} \log p_{\mathbf{k}} + \sum_{\mathbf{k}} p_{\mathbf{k}} \left(\sum_{j=1}^{n_b} \log d_{\mathbf{k}_j} \right) + \sum_{\mathbf{k}} p_{\mathbf{k}} S_{\text{Bell},\mathbf{k}}, \tag{7}$$

where $\mathbf{k}$ labels reducible representations of gauge field penetrating each boundary and p_i are probability associated to it. $d_{\mathbf{k}_j}$ is the number of dimension of the representation. One of the new contributions is that we call *Shannon part*(first term). This is roughly the correlation of the information about which representation of gauge group each boundary belongs. This is just due to gauge constraints. The constraints make "superselection rule" for partial systems, and we cannot regard them as quantum correlation simply and we have to treat this more carefully to make it distinguishable from Bell pair part. The other contribution is what we call *color part*(second term). This occurs when the number of dimension of the gauge group representation is more than one and have color d.o.f.. Non-trivial representations in non-abelian theories are the case.

Although we have general form of (7), explicit evaluation of how each term contribute, especially for non-abelian theories is hard and such works have been restricted. One important reason is that non-abelian theories inevitably have self-interaction and the explicit calculation of entanglement entropy itself is difficult even in free theories without some techniques we can exploit for some special situation.

In this paper we analyze entanglement entropy of the ground state for lattice gauge theories. To understand the behavior of entanglement in continuum theory, we have to understand how the result in the ground state in lattice theory is connected to that in continuum theory.

[4]In contrast to the extended Hilbert space, throughout this paper, we call original Hilbert space before the extension *physical Hilbert space.* and represent them as $\mathcal{H}_{ext}$ and $\mathcal{H}_{phys}$. Without gauge constraints, $\mathcal{H}_{ext} = \mathcal{H}_{phys}$

Generally, the Hamiltonian for pure lattice gauge theories in general dimension is given by [26]

$$H = \frac{g_{YM}^2}{2a}\sum_l \hat{J}_l^2 + \frac{1}{g_{YM}^2 a}\sum(\text{plaquette terms}), \tag{8}$$

where a is the lattice spacing, g_{YM} is the bare gauge coupling on the lattice, and $\hat{J}_l$ is the generator of the gauge transformation at the link $l = (i, j)$, which satisfies $\hat{J}^2_{(i,j)} = \hat{J}^2_{(j,i)}$. The second plaquette terms are given by the trace of each plaquette combination of gauge "position" operators[5]. To get explicit form of the ground state, we have to diagonalize the Hamiltonian, but this is hard task generically.

In the strong coupling limit that $g_{YM} \to \infty$, the first terms of (8) become dominant and we get the ground state, which we call the *strong coupling ground state* $|0\rangle_{\text{strong}}$. Obviously, this is given as,

$$|0\rangle_{\text{strong}} = \bigotimes_l |0\rangle_l \ , \tag{9}$$

where $|0\rangle_l$ satisfies $\hat{J}_l^2 |0\rangle_l = 0$. This state is separable for any division of the system so gives *no* entanglement entropy. In the weak coupling limit, in contrast, in (8) the second plaquette terms become dominant.

In 2-dimensional[6] theory, however, the situation will change because the plaquette terms are absent for 1-spatial dimension. Thus for arbitrary value of g_{YM}, the ground state is $|0\rangle_{\text{strong}}$ and there is *no* entanglement[27]. On the other hand, the vacuum in continuum gauge theories, which we call the *continuum ground state*, is known to be entangled. Due to the asymptotic freedom of gauge theories, the continuum gauge theory with non-zero renormalized coupling (the IR theory) is obtained from the lattice gauge theory in the limit of zero bare gauge coupling (the UV theory).

Therefore, it is important to understand how the strong coupling ground state approaches the entangled continuum ground state in the process of the continuum limit. Although seeing this approach is difficult for general dimension, for 2-dimensional theories the task become easier drastically. Indeed, for pure gauge theories we can calculate entanglement entropy for any states at an arbitrary coupling constant [27], so that we can take the continuum limit analytically. Unfortunately, quantum entanglement, i.e., the Bell pair

[5]This measures which element of gauge group the field belongs. The explicit definition will be given at §2.

[6]In this paper we mean euclidean 2-dimensional theory.

part, vanishes in 2-dimensional pure gauge theories even in the continuum limit [27] as is expected. This is because we cannot have any local excitation in the situation.

Once we add matter fields to pure gauge theories in 2-dimensions, entanglement emerges due to the existence of local degrees of freedom.We thus take these gauge plus matter theories as toy models of pure gauge theories in higher dimensions, since gauge plus adjoint matters in 2-dimensions, for example, are expected to have analogous behaviors as higher dimensional pure Yang-Mills theories with compactified extra $(d-2)$ dimensions. While pure gauge theories plus matters can not be solved analytically even in 2-dimensions,[7] we can include effects of matter fields order by order in the hopping parameter expansion (HPE) for the small hopping parameter $K \equiv 1/(2+(ma)^2)$, where m is the bare mass of matter field and ma must be large for the HPE to work.[8]

In this paper, using the HPE but at an arbitrary gauge coupling, we demonstrate how the three types of entanglement entropy emerges for the ground state of gauge plus matter fields in 2-dimensions. We mainly consider matter fields in the fundamental representation, but an essential idea works similarly for adjoint matters and other representations. Adding adjoint matters is an interesting set-up, since it resembles the large N D1-brane gauge theory, which is dual to the string theory in the curved space-time [29].

In the HPE perturbation, the strong coupling ground state is corrected by "meson-like" objects, where "meson-like" pairs with some length are always connected by gauge field flux. What comes to mind as correlation from these objects is that of how the flux penetrates boundaries[9], or correlation of color d.o.f. the flux itself has. These correlation is one by gauge constraints. Indeed, they will turn to be Shannon part and color part we introduced in (7) respectively. It tempts us to conclude that the ground state consists of correlation by gauge constraints (Shannon part and color part) only. We will show this intuition is *incorrect*. At a higher order in HPE expansion[10], "decomposition" to reducible representations in the gauge group emerges and it makes quantum correlation (Bell pair part) . This mechanism is distinctive of gauge theories and provides a basis for discussion about entanglement in

[7]Unless we take large N limit [28].

[8]The massless theory or the continuum limit with the finite mass corresponds to $K = 1/2$, its maximum value.

[9]For example, when observer which can access to some sub-space only observed a flux is penetrating some boundary point, he/she will realize that an observer in another subregion across the boundary will also observe a flux penetrating the boundary.

[10]To be exact, K^6 order.

gauge theories.

The organization of this paper is as follows. In §2, we review the ordinary definition of entanglement entropy and related quantities and extended Hilbert space formalism we use to define the entanglement entropy for gauge theories. In §3, we see how new types of contribution to entanglement entropy occur by using explicit examples. In §4, we introduce two techniques need to the analysis, transfer matrix and hopping parameter expansion. §5 is the main part of this work. Here, we analyze the entanglement entropy of 1+1 dimensional gauge theories with matter, mainly focusing on how three types of contribution appear. In §6, we summarize the result and make discussion.

This paper is mainly based on the work [30].

2 Entanglement entropy and Extended Hilbert space formalism

In this section we review and introduce extended Hilbert space formalism[22, 23, 25] and generalize the definition of entanglement entropy using it. Before doing that, firstly we look back to the ordinal definition and properties of Shannon entropy, von Neumann entropy, and ordinary entanglement entropy, emphasizing informational point of view.

2.1 Ordinary entanglement entropy

2.1.1 Shannon entropy

Let us start with Shannon entropy[31]. Suppose that we are given some probability distribution $\{p_i\}, i = 1, 2, \ldots N$, where each element is related to some probability event. They satisfy[11] $p_i \geq 0$ for all i and $\sum_i p_i = 1$. Then the *Shannon entropy* for this distribution is defined as

$$H(\{p_i\}) = -\sum_i p_i \log p_i. \tag{10}$$

Here we defined as $0 \log 0 = 0$.

[11] Here we are assuming that the number of elements is at most countably infinite. For continuum case some property(e.g. positivity) will break.

From the informational point of view, we can interpret $H(\{p_i\})$ as a quantification of how the probability distribution is uncertain. Indeed, for degenerate distribution $p_i = \delta_{ij}$ (j is fixed) case (this is most certain case), obviously $H(\{p_i\}) = 0$ and for uniform distribution $p_i = 1/N$ (for all i) case (most uncertain case), it gives maximal value $H(\{p_i\}) = \log N$.

2.1.2 von Neumann entropy

Suppose that we are given some Hilbert space $\mathcal{H}$ and a bounded self-adjoint operator ρ acting on $\mathcal{H}$, which is semi-positive and satisfies $\mathrm{Tr}_{\mathcal{H}}\rho = 1$. Then *von Neumann entropy*[32] for ρ is defined as

$$S_{\mathrm{vN}}(\rho) = -\mathrm{Tr}_{\mathcal{H}}\rho \log \rho. \tag{11}$$

Throughout this work we interpret ρ as density operator (the conditions we stated above are necessary and sufficient for ρ to be a density operator) in quantum theory. Generically, by using some complete orthonormal system $|\psi_i\rangle$ $(i = 1, 2, \dots)$, the density operator can be represented as

$$\rho = \sum_i p_i \hat{P}_i, \tag{12}$$

where $\{p_i\}$ is some probability distribution and $\hat{P}_i$ is the projection operator to subspace generated by $|\psi_i\rangle$ (that is, $\hat{P}_i = |\psi_i\rangle \langle\psi_i|$). When the probability distribution is discrete (i.e. for some i, $p_i = 1$), we call the state as *pure state*, and if not, *mixed state*. $S_{\mathrm{vN}}(\rho)$ vanishes iff the state is pure. We can interpret $S_{\mathrm{vN}}(\rho)$ as the quantification of how the state is mixed. If we consider this mixture as the uncertainly of which pure state $|\psi_i\rangle$ the state belongs, we can regard the von Neumann entropy as a generalization of the Shannon entropy.

2.1.3 Ordinary entanglement entropy

As mentioned in introduction, with tensor product structure $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{\bar{V}}$, the ordinary entanglement entropy can be defined as

$$S_{EE}(\rho, \mathrm{V}) = S_{\mathrm{vN}}(\rho_V), \tag{13}$$

where $\rho_V = \mathrm{Tr}_{\mathcal{H}_{\bar{V}}}\rho$.

The reason that (at least for pure states) we can regard this as the quantum correlation between sub-region V and $\bar{\mathrm{V}}$ is as follows. Generally, even if ρ is pure (and its von Neumann entropy vanishes), partially traced operator ρ_V may become mixed and gives non-zero von Neumann entropy. This means that totally fixed(pure) state becomes cloudy(in informational sense) by losing the information supported by sub-region $\bar{\mathrm{V}}$. This cannot happen in local classical theory, because classically we can have pure state by only fixing all of local d.o.f. for total region and losing information in the sub-region $\bar{\mathrm{V}}$ does not make any effect to the information supported region V. So we can say that the variation of the von Neumann entropy by partially tracing out is totally attributable to the quantum effect, non-locality. If the firstly prepared ρ is mixed, hiding information of sub-region causes the change of probability distribution and we cannot regard entanglement entropy as totally quantum correlation. On the other hand, even for pure states, if we have non-local objects due to constraints, we also have another contributions to entanglement entropy. Gauge theories are just the case.

2.1.4 Entanglement entropy as entanglement measure

As stated above, entanglement entropy captures quantitative property of entanglement, that is non-locality. Generally quantities which quantify entanglement are called *entanglement measure.* Indeed it is known that[33] *for pure states*, entanglement entropy is unique entanglement measure which satisfy properties we assume that entanglement measure should have.

However, even for separable, mixed states can give non-zero entanglement entropy. This suggests that the value of entanglement entropy itself does not mean entanglement directory, that is it is not entanglement measure generically.

For mixed states, several quantities have been suggested for entanglement measure for mixed states.In general, for such quantity $E(\rho, \mathrm{V})$, we require the quantities agree with entanglement entropy for pure states[12],

$$E(\rho, \mathrm{V}) = S_{EE}(\rho, \mathrm{V}). \text{ (for pure states)} \tag{14}$$

In this meaning, we can interpret these quantities as generalizations of entanglement entropy, putting physical meaning on it. This will be useful for physical interpretation in extended Hilbert space formalism we will explain in the next subsection.

Among entanglement measure we pick up *entanglement of distillation*[35, 36] $E_D(\rho, \mathrm{V})$ as an example. Roughly, this quantity measures how many Bell

[12] Occasionally we relax this condition. One of such quantity is logarithmic negativity[34]

pairs the state can be converted to the given state by some special family of action so called *local operation and classical communication*(LOCC). By local operation we mean operation (unitary transformations and measurements) supported by only the sub-region we consider or its complement, and by classical communication we mean the exchange of information between the sub-region and its complement, by using only classical way. If we represent general LOCC transformation as $\Psi(\rho)$ and n copies of Bell state as $\Phi(2^n)$, formally we can define the quantity as

$$E_D(\rho, \mathrm{V}) = \sup\left\{r \ : \ \lim_{n\to\infty}\left[\inf_{\Psi}\mathrm{Tr}|\Psi(\rho^{\otimes n}) - \Phi(2^{rn})|\right] = 0\right\}. \tag{15}$$

Here we use LOCC to n copies of given state ρ, making it close to Bell pairs state (we are measuring the "distance" by the trace). The entanglement of distillation is maximal rate of the number of given state and Bell state under "optimal" LOCC. The reason we take number limit is that LOCC includes measurements where outcomes are probabilistic and we have to see asymptotic behavior. Entanglement of distillation satisfies that

$$E_D(\rho, \mathrm{V}) = 0, \ \ \text{(for separable states)} \tag{16}$$

one of the necessary condition to be entanglement measure.

2.2 Extended Hilbert space formalism

Here we introduce so called extended Hilbert space formalism and extended definition of entanglement entropy based on it in lattice theories.

In this work we consider 2-dimensional (Euclidean) lattice gauge theories. In lattice theory gauge transformation is defined naturally and we do not need to chose a gauge in principle (even without gauge fixing, it does not give infinite quantity). However, by gauge fixing before quantization the calculation becomes simple[37]. In the work we exploit temporal gauge where link variables along temporal direction are set to be unit element in the gauge group. In the temporal gauge, even for continuum theory there is no negative norm state, but the physical states are constrained by "Gauss law" condition[13].

Consider spatial lattice with fixed timeThe lattice is composed by vertices and links connecting them. We describe oriented links as $l = (a, b)$, where a

[13]The difficulty of canonical quantization in gauge theories is caused by the fact that there is no momentum conjugate of temporal gauge field A_0. Temporal gauge fixing eliminate A_0 and then one of the e.o.m. becomes static and "Gauss law" condition.

and b refer neighboring starting vertex and ending vertex of the link respectively. We represent the associated reversed links as $\bar{l} = (b, a)$.

Suppose that a gauge field lives on links and a matter field on vertices[14]. The gauge field variable are assigned on links and each one is an element in the some gauge group G equipped to it,

$$U_l = g \in G, \tag{17}$$

where l refers the link the variable lives on. For the link $\bar{l}$, we have

$$U_{\bar{l}} = g^{-1}. \tag{18}$$

The matter field variable are assigned on vertices and we assume that it is coupling to the gauge field as the some representation R in the gauge group G.

Firstly, let us consider pure gauge theory case. For each link l we introduce "position" operator $(\hat{U}_l^{\mathbf{R}})^i{}_j$ which measures the gauge field variable component in representation $\mathbf{R}$ and its eigenstate $|U_l\rangle$,

$$(\hat{U}_l^{\mathbf{R}})^i{}_j \, |U_l\rangle = (U_l^{\mathbf{R}})^i{}_j \, |U_l\rangle \, . \tag{19}$$

We can use these eigenstates as basis of quantum state, with orthogonal condition,

$$\langle U_l | U_l' \rangle = \delta(U_l, U_l'), \tag{20}$$

where δ should be understood as Dirac's delta function or Kronecker's delta respecting the gauge group is continuum or not. Then we can have *link Hilbert space* $\mathcal{H}_l$,

$$\mathcal{H}_l = \mathrm{span}(|g\rangle) \;\; g \in G, \tag{21}$$

here span means linear span. For the whole system we can define *extended Hilbert space*,

$$\mathcal{H}_{\mathrm{ext}} = \bigotimes_l \mathcal{H}_l. \tag{22}$$

The space have tensor product structure for any spatial division. If we assign a set of links to "sub-region" V, we have the structure,

[14]For simplicity we consider single gauge field and matter field case. Multiple case also can be considered straightforwardly.

$$\mathcal{H}_{\text{ext}} = \mathcal{H}_{\text{V}} \otimes \mathcal{H}_{\bar{\text{V}}} = \bigotimes_{l \in \text{V}} \mathcal{H}_l \otimes \bigotimes_{l' \in \bar{\text{V}}} \mathcal{H}_{l'}, \tag{23}$$

and the entanglement entropy associated to it can be defined as,

$$S_{EE}(\rho, \text{V}) = S_{\text{vN}}(\rho_V), \tag{24}$$

where $\rho_V = \text{Tr}_{\mathcal{H}_{\bar{V}}} \rho$.

Next we introduce (left) link operator $\hat{L}_{l,g}$, which induces the change of gauge group element, as

$$\hat{L}_{l,g} \left| U_l \right\rangle = \left| g U_l \right\rangle . \tag{25}$$

For non-abelian gauge group case, adding to them, we have to consider also right link operator $\hat{\mathcal{L}}_{l,g}$,

$$\hat{\mathcal{L}}_{l,g} \left| U_l \right\rangle = \left| U_l g \right\rangle . \tag{26}$$

Even with matter field, the framework of the formalism is almost same. The different point is that adding to links we have gauge d.o.f. on vertices also. We should add *vertex Hilbert space*[15],

$$\mathcal{H}_v = \text{span}(|g, s\rangle) \;\; g \in G \tag{27}$$

the extended Hilbert space in this case becomes,

$$\mathcal{H}_{\text{ext}} = \bigotimes_{l} \mathcal{H}_l \otimes \bigotimes_{v} \mathcal{H}_v \tag{28}$$

Next we introduce left (right) vertex operator $\hat{V}_{i,g}$ ($\hat{\mathcal{V}}_{i,g}$) corresponding to the change of gauge group element on the vertex i, then the gauge transformation at i, $\hat{\mathcal{G}}_{i,g}$ is given as,

$$\hat{\mathcal{G}}_{i,g} = \hat{V}_{i,g} \prod_{a} \hat{L}_{(i,a),g}. \tag{29}$$

For physical (gauge-invariant) states, we have

$$\hat{\mathcal{G}}_{i,g} = 1 \;\; \text{(physical states)}, \tag{30}$$

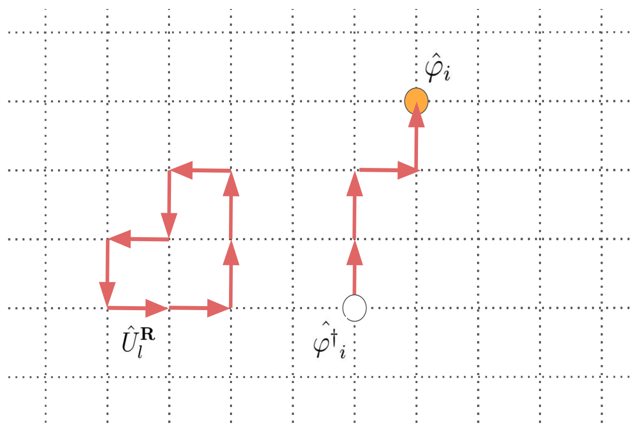

Figure 1: Two types of gauge invariant excitation in 2 spatial dimensional theory, loop excitation(left) and meson excitation(right).

for arbitrary i. This is gauge constraint equation.

Now let us consider general gauge-invariant states. Most simple one is *strong coupling ground state*,

$$|0\rangle_{\text{strong}} = \bigotimes_{l,v} |0\rangle_{l,v} = \bigotimes_{l,v} \sum_g |g\rangle_{l,v} \,, \tag{31}$$

where summing up by group elements should change to be integration with Haar measure for continuum group case.

The general gauge-invariant states can be generated by acting two types of excitation operator on $|0\rangle_{\text{strong}}$(figure 1). one is *loop excitation* operators

$$\text{Tr} \prod_{l \in \text{loop}} \hat{U}_l^{\mathbf{R}}. \tag{32}$$

the second one is *meson excitation* operators, like

$$\hat{\varphi}^\dagger{}_i \hat{U}^{\mathbf{F}}_{i,i+1} \hat{\varphi}_{i+1}, \tag{33}$$

[15]Here s refers physical d.o.f. the matter field can have regardless to gauge d.o.f. For example, in $U(1)$ theory, complex scalar field can have phase factor as gauge d.o.f. and magnitude as physical d.o.f. We will omit s, as it does not play a important role in this paper.

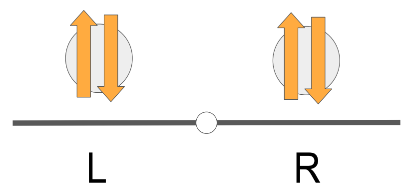

Figure 2: Toy model with a vertex and two links (left and right). 2 spins are set on links.

for with fundamental scalar field, for instance. The explicit example of them will be given in the next section.

When spatial dimension is 1, loop excitation can occur only as global loop with periodic boundary condition. For considering the ground state, we may focus on meson excitation operators only.

3 Three types of contribution to entanglement entropy

In this section we review how gauge constraints make new types of contribution in EE[22, 23, 25]. In the first subsection we consider two spin model with a constraint as a toy model. In the next subsection we take up Z_2 lattice gauge model. For generic states, we see one of the two contributions emerge by gauge constraints (Shannon part). In the next subsection we apply the formalism for $U(1)$ gauge theory and $SU(N)$ gauge theory case, For latter case, the other contribution (colour part) will appear due to the non-commutativity of the gauge group. In the last subsection we pick up examples of states and calculate entanglement entropy explicitly.

3.1 Two spin model with a Z_2 constraint

Before tackling to gauge field theories, it is helpful to see more simple toy model, two spin with a constraint.

Let us consider two links sharing one vertex (See figure (2)) and two q-bit valued spins living on links. We use Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{34}$$

and the basis

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{or} \quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{35}$$

where former basis are eigenstates of Z and latter are of X. The Hilbert space $\mathcal{H}_{\text{ext}}$ is generated by the basis,

$$\mathcal{H}_{\text{ext}} = \text{span}(|\uparrow\rangle_{\text{L}} |\uparrow\rangle_{\text{R}}, |\uparrow\rangle_{\text{L}} |\downarrow\rangle_{\text{R}}, |\downarrow\rangle_{\text{L}} |\uparrow\rangle_{\text{R}}, |\downarrow\rangle_{\text{L}} |\downarrow\rangle_{\text{R}}) \tag{36}$$

$$= \text{span}(|+\rangle_{\text{L}} |+\rangle_{\text{R}}, |+\rangle_{\text{L}} |-\rangle_{\text{R}}, |-\rangle_{\text{L}} |+\rangle_{\text{R}}, |-\rangle_{\text{L}} |-\rangle_{\text{R}}). \tag{37}$$

Then we introduce the "gauge" transformation $\mathcal{G} = X_{\text{L}} X_{\text{R}}$. That is, we require that physical states $|\text{phys}\rangle$ are invariant under this transformation,

$$\mathcal{G} |\text{phys}\rangle = |\text{phys}\rangle . \tag{38}$$

In other words, in the physical Hilbert space $\mathcal{H}_{\text{phys}}$, the gauge transformation acts as identity,

$$\mathcal{G} = X_{\text{L}} X_{\text{R}} = 1 \text{ for } \mathcal{H}_{\text{phys}}. \tag{39}$$

In this model we can easily specify the $\mathcal{H}_{\text{phys}}$ as

$$\mathcal{H}_{\text{phys}} = \text{span}(|+\rangle_{\text{L}} |+\rangle_{\text{R}}, |-\rangle_{\text{L}} |-\rangle_{\text{R}}). \tag{40}$$

With the extended Hilbert space formalism we introduced in previous subsection, we can define entanglement entropy for this model and general states gives non-zero entanglement entropy. For instance, let us consider a "Bell" state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_{\text{L}} |+\rangle_{\text{R}} + |-\rangle_{\text{L}} |-\rangle_{\text{R}}) \tag{41}$$

$$= \frac{1}{\sqrt{2}}(|\uparrow\rangle_{\text{L}} |\uparrow\rangle_{\text{R}} + |\downarrow\rangle_{\text{L}} |\downarrow\rangle_{\text{R}}). \tag{42}$$

This state obviously have the same form as usual Bell state, and by using extended definition, it gives entanglement entropy $S_{EE} = \log 2$.

However, we have to take more careful consideration about physical interpreting of this. The correlation here should be interpreted in totally different way because this is caused by the gauge constraint. This correlation is fatal property where *any* physical states should have and there is no way to break it (that would be unphysical state). In other words, *any* local operation (except identity) to the state will break the constraint and there is no room to distillate any entanglement from the state. Thus we should consider the correlation as *classical* one, rather than quantum one, so we refer this as Shannon contribution.

Let us see this fact in detail. Adding to the state $\rho_1 = |\psi\rangle\langle\psi|$ in (41), consider a state

$$\rho_2 = \frac{1}{2}\Big(|+\rangle_{\mathrm{L}}|+\rangle_{\mathrm{R}}\langle+|_{\mathrm{L}}\langle+|_{\mathrm{R}} + |-\rangle_{\mathrm{L}}|-\rangle_{\mathrm{R}}\langle-|_{\mathrm{L}}\langle-|_{\mathrm{R}}\Big). \tag{43}$$

If we introduce projection operators P_+ and P_-, projecting to "+ sector" and "- sector", and using the fact that any local operation $\mathcal{O}$ cannot change the sector, that is $[\mathcal{O}, P_+] = [\mathcal{O}, P_-] = 0$, expectation value of O becomes as

$$\begin{aligned}\mathrm{Tr}(\mathcal{O}\rho_2) &= \mathrm{Tr}\left[\mathcal{O}\left(P_+\rho_1 P_+ + P_-\rho_1 P_-\right)\right] \\ &= \mathrm{Tr}(\mathcal{O}\rho_1),\end{aligned} \tag{44}$$

where we used cyclic property of the trace. This means that in the level of LOCC, we cannot distinguish pure state ρ_1 and ρ_2. Thus we have $E_D(\rho_1, \mathrm{V}) = E_D(\rho_2, \mathrm{V})$. Since ρ_2 is separable state, we conclude that $E_D(\rho_1, \mathrm{V}) = 0$, implicating that $S_{\mathrm{EE}}(\rho_1, \mathrm{V}) = \log 2$ is *not* quantum correlation, becoming classical new contribution. Therefore, even for pure states, entanglement entropy can attribute to classical correlation. This is just a result from the constraint.

3.2 Z_2 lattice gauge theory

Next we consider lattice gauge theory for most simple nontrivial gauge group case, Z_2 group.

Actually, the structure of the theory is almost same as the toy model considered in last subsection. We can consider qubit on links as Z_2 gauge field and "position" operator and link operator becomes like

$$\hat{U}_l = Z_l, \quad \hat{L}_l = X_l. \tag{45}$$

Remark that Z_2 group has only single non-trivial representation and abelian. We don't need to consider the orientation of links.

Further, we can add a scalar matter field living on vertices[38]. Then we have vertex operator,

$$\hat{V}_i = Z_i. \tag{46}$$

We have gauge transformation on the vertex i,

$$\mathcal{G}_i = X_{(i-1,i)} X_i X_{(i,i+1)}, \tag{47}$$

and constraint equation,

$$X_{(i-1,i)} X_i X_{(i,i+1)} = 1 \ \text{ for } \mathcal{H}_{\text{phys}}. \tag{48}$$

The strong coupling ground state is $\bigotimes_{l,v} |+\rangle_{l,v}$. (Global) loop excited state is

$$|\text{loop}\rangle = \prod_{\text{loop}} Z_l |0\rangle_{\text{strong}} = \bigotimes_l |-\rangle_l \otimes \bigotimes_v |+\rangle_v . \tag{49}$$

Meson like states are given like

$$\begin{aligned} |i,j\rangle &= Z_i Z_{(i,i+1)} Z_{(i+1,i+2)} \cdots Z_{(j-1,j)} Z_j |0\rangle_{\text{strong}} \\ &= |-\rangle_i \otimes \bigotimes_{l=(a,b)|i\leq a\leq j-1, a<b} |-\rangle_l \otimes |-\rangle_j \otimes \bigotimes_{l,v\in\text{others}} |+\rangle_{l,v} . \end{aligned} \tag{50}$$

This state have meson like excitation between vertex i and j.

As seen in the toy model in previous subsection, we have Shannon part in the entanglement entropy for general states.

3.3 Lattice gauge theory for abelian and non-abelian group

Next we consider $U(1)$ gauge theory. In this case we can use phase parameter $\theta \in [0, 2\pi)$[16] as gauge d.o.f.

The "position" operators are given as,

[16]For simplicity we consider compact group here.

$$\hat{U}_l^n = e^{in\hat{\theta}_l}, \;\; \hat{\theta}_l \left|\theta\right\rangle_l = \theta \left|\theta\right\rangle_l . \tag{51}$$

Where integer n corresponds to its representation. As $U(1)$ group is continuous, we can introduce the generator of link operator $\hat{J}_l$ as,

$$\hat{L}_{l,\varepsilon} = e^{i\varepsilon \hat{J}_l}, \;\; e^{i\varepsilon \hat{J}_l} \left|\theta\right\rangle_l = \left|\theta + \varepsilon\right\rangle_l . \tag{52}$$

Let us introduce matter field, fundamental complex scalar φ as simplest case. Then "position" operator of the field and vertex operators become

$$\begin{aligned} \hat{\varphi}_v \left|\varphi\right\rangle_v &= \varphi \left|\varphi\right\rangle_v , \\ \hat{V}_{v,\varepsilon} = e^{i\varepsilon\hat{\rho}_v}, \;\; e^{i\varepsilon\hat{\rho}_v} \left|\varphi\right\rangle_v &= \left|e^{i\varepsilon}\varphi\right\rangle_v . \end{aligned} \tag{53}$$

The gauge transformation for the vertex i is

$$\mathcal{G}_{v,\varepsilon} = \hat{L}_{(i-1,i),-\varepsilon} \hat{V}_{i,\varepsilon} \hat{L}_{(i,i+1),\varepsilon}. \tag{54}$$

For infinitesimal transformation, we get gauge constraint equation,

$$\hat{J}_{(i-1,i)} - \hat{J}_{(i,i+1)} = \hat{\rho}_i. \tag{55}$$

This is just Gauss's law.

The formation of gauge invariant states are given in straightforward way as Z_2 case. As Z_2 group case, we have Shannon part in the entanglement entropy for general states. We can classify the state by fixing the group representation penetrating each boundary, making each superselection sector.

Now let us consider non-abelian gauge theory case. For simplicity we consider $SU(2)$ gauge theory as an example. For each links we have three (left) generators,

$$\hat{J}_l^a \quad a = 1, 2, 3, \tag{56}$$

$$[\hat{J}_l^a, \hat{J}_l^b] = i\epsilon^{abc} \hat{J}_l^c, \tag{57}$$

The most significant difference with abelian case is that the gauge transformation become *non-gauge invariant* itself. It is due to color d.o.f. by the redundant d.o.f. In the case, the representation is decided by Casimir operators (in $SU(2)$ case $\sum_a (\hat{J}^a)^2$).

Even after fixing the superselection sector (then we don't have Shannon part, because it is from the superposition among different sectors), it can still have the correlation which *cannot* be regarded as entanglement. As an example, by using usually diagonalized state $|j, m\rangle$, where $\sum_a (\hat{J}^a)^2 |j, m\rangle = j(j+1)|j, m\rangle$ and $\hat{J}^3 |j, m\rangle = m|j, m\rangle$, let us consider the state in single vertex space as in the figure 2,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\frac{1}{2}, -\frac{1}{2}\rangle_{\rm L} |\frac{1}{2}, +\frac{1}{2}\rangle_{\rm R} + |\frac{1}{2}, +\frac{1}{2}\rangle_{\rm L} |\frac{1}{2}, -\frac{1}{2}\rangle_{\rm R} \right). \tag{58}$$

We can see this state has same form as Bell state, giving entanglement entropy $\log 2$. However this correlation is that of color d.o.f. and by using the fact that any *physical* local operation is gauge-invariant, again it cannot be interpreted as entanglement. We have to regard this as a new type of contribution in entanglement entropy. We call these third contribution as *color part* in entanglement entropy.

Summary of three types of contributions We can classify three types of contributions for general states $|\Psi\rangle$ as follows. First, we consider representations penetrating each boundary vertex labeled by $i = 1, 2, \ldots n_b$. Then we regards the state as "superposition" of representations $\mathbf{k} = (\mathrm{k}_1, \mathrm{k}_2, \ldots \mathrm{k}_{n_b})$ where each j is label of each irreducible representation,

$$|\Psi\rangle = \sum_{\mathbf{k}} \sqrt{p_{\mathbf{k}}} \, |\Psi\rangle_{\mathbf{k}} \, . \tag{59}$$

Here $|\Psi\rangle_{\mathbf{k}}$ are states in each sector with properly normalization. Then we have entanglement entropy

$$S_{\rm EE} = -\sum_{\mathbf{k}} p_{\mathbf{k}} \log p_{\mathbf{k}} + \sum_{\mathbf{k}} p_{\mathbf{k}} \left(\sum_{j=1}^{n_b} \log d_{\mathbf{k}_j} \right) + \sum_{\mathbf{k}} p_{\mathbf{k}} S_{\mathrm{Bell},\mathbf{k}}, \tag{60}$$

where each term is Shannon part, color part, Bell pair part respectively. Here d_j is the dimension of representation j.

3.4 Examples of gauge invariant states

In this subsection, we show explicit calculation of entanglement entropy for some excited states. We will consider the 7 vertex spatial lattice given in Fig. 3 as a simple example, which is good enough to see the essential points, and one can easily generalize the results in this section to more general cases. Here we continue the analysis of entanglement entropy for non-abelian gauge theory by considering specific excitations as example.

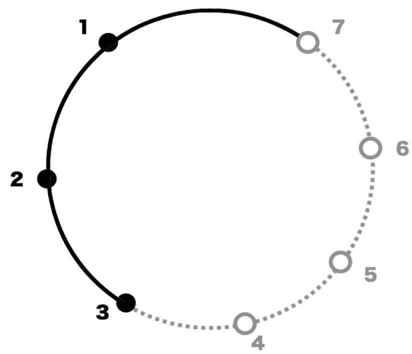

Figure 3: Toy seven vertex lattice setup. Black vertices and solid lines belongs to “inside" and white vertices and dotted lines to “outside".

3.4.1 Loop excited state

Consider following wave function

$$\mathbf{R}(U) \equiv \chi_{\mathbf{F}}(U) = \mathop{\mathrm{Tr}}_{\mathbf{F}}(U) \quad (U \equiv U_{12}U_{23}U_{34}U_{45}U_{56}U_{67}U_{71}), \tag{61}$$

where $U_{ij} \in SU(N)$ is the spatial gauge link variable between the vertices i and j, which satisfies $U_{ji} \equiv U_{ij}^{\dagger}$, and $\chi_{\mathbf{F}}(U)$ is the character for the ‘fundamental representation’ $\mathbf{F}$.[17]

Straightforward calculation shows that the reduced density matrix becomes

$$\begin{aligned}
&\langle U_{12}, U_{23}, U_{71} |\, \rho \,| V_{12}, V_{23}, V_{71} \rangle \\
&\quad = \int dW_{34}dW_{45}dW_{56}dW_{67}\chi_{\mathbf{F}}(U_{71}U_{12}U_{23}W_{34}W_{45}W_{56}W_{67}) \\
&\qquad\quad \times \chi_{\mathbf{F}}(W_{67}^{\dagger}W_{56}^{\dagger}W_{45}^{\dagger}W_{34}^{\dagger}V_{23}^{\dagger}V_{12}^{\dagger}V_{71}^{\dagger}) \\
&\quad = \frac{1}{N}\chi_{\mathbf{F}}(U_{71}U_{12}U_{23}V_{23}^{\dagger}V_{12}^{\dagger}V_{71}^{\dagger}),
\end{aligned} \tag{62}$$

where we used (212) and integrated out “outside”-link variables $W_{34}, W_{45}, W_{56}, W_{67}$. Therefore the square of the reduced density matrix is

[17] We take the temporal gauge $A_0 = 0$ throughout this paper. This is a loop excited state. As will be seen later, this $\mathbf{R}(U)$ is the eigenfunction of the transfer matrix [27].

$$
\begin{aligned}
\langle U_{12}, U_{23}, U_{71}| \rho^2 |V_{12}, V_{23}, V_{71}\rangle &= \frac{1}{N^2} \int dW_{12} dW_{23} dW_{71} \chi_{\mathbf{F}}(U_{71}U_{12}U_{23}W_{23}^\dagger W_{12}^\dagger W_{71}^\dagger) \\
&\quad \times \chi_{\mathbf{F}}(W_{71}W_{12}W_{23}V_{23}^\dagger V_{12}^\dagger V_{71}^\dagger) \\
&= \frac{1}{N^3}\chi_{\mathbf{F}}(U_{71}U_{12}U_{23}V_{23}^\dagger V_{12}^\dagger V_{71}^\dagger) \\
&= \frac{1}{N^2} \langle U_{12}, U_{23}, U_{71}| \rho |V_{12}, V_{23}, V_{71}\rangle \,, \qquad (63)
\end{aligned}
$$

where again we used (212). This implies

$$
\mathrm{Tr}\rho^n = \frac{1}{N^{2(n-1)}} \,. \qquad (64)
$$

As a result, we obtain an entanglement entropy S_{EE} as

$$
S_{EE} \equiv -\mathrm{Tr}\rho \log \rho = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho^n = 2\log N = n_b \log N \,. \qquad (65)
$$

This is consistent with the "area-law" of the entanglement entropy [10], where the boundary is consists of two sites, *i.e.,* site 3 and 7, so the "boundary site number" $n_b = 2$. To see this further, as an example of $n_b = 4$, we consider a different separation of in and out regions in such a way that link 2-3 and 5-6 are outside and others are inside. Then using (212) and (213), it is straightforward to check the reduced density matrix and its square become

$$
\begin{aligned}
\langle U_{in}| \rho |V_{in}\rangle &= \int dW_{23} dW_{56} \chi_{\mathbf{F}}(U_{71}U_{12}W_{23}U_{34}U_{45}W_{56}U_{67}) \\
&\quad \times \chi_{\mathbf{F}}(V_{67}^\dagger W_{56}^\dagger V_{45}^\dagger V_{34}^\dagger W_{23}^\dagger V_{12}^\dagger V_{71}^\dagger) \\
&= \frac{1}{N^2}\chi_{\mathbf{F}}(U_{67}U_{71}U_{12}V_{12}^\dagger V_{71}^\dagger V_{67}^\dagger)\chi_{\mathbf{F}}(U_{34}U_{45}V_{45}^\dagger V_{34}^\dagger) \,, \qquad (66) \\
\langle U_{in}| \rho^2 |V_{in}\rangle &= \frac{1}{N^4} \int dW_{12} dW_{71} dW_{67} \chi_{\mathbf{F}}(U_{67}U_{71}U_{12}W_{12}^\dagger W_{71}^\dagger W_{67}^\dagger) \\
&\quad \times \chi_{\mathbf{F}}(W_{67}W_{71}W_{12}V_{12}^\dagger V_{71}^\dagger V_{67}^\dagger) \\
&\quad \times \int dW_{34} dW_{45} \chi_{\mathbf{F}}(U_{34}U_{45}W_{45}^\dagger W_{34}^\dagger)\chi_{\mathbf{F}}(W_{34}W_{45}V_{45}^\dagger V_{34}^\dagger) \\
&= \frac{1}{N^4} \langle U_{in}| \rho |V_{in}\rangle \,, \qquad (67)
\end{aligned}
$$

so that we obtain

$$
S_{EE} = 4\log N = n_b \log N \,, \qquad (68)
$$

for $n_b = 4$. It is easy to see in general that

$$
S_{EE} = n_b \log d_{\mathbf{R}} \,, \qquad (69)
$$

where $d_{\mathbf{R}}$ is the dimension of the irreducible representation $\mathbf{R}$. This is the essential results of [22, 27]. Before we end this section, we have several comments.

Since there is no physical degrees of freedoms in the 2-dimensional pure gauge theory, the result (69) cannot represent the Bell pair part entanglement in the sprint of the information theory, which is equivalent to the number of Bell pairs obtained in the entanglement distillation. See §4 of [25], for example.

All calculations in the above are done in the extended Hilbert space definition [23, 22, 25]. The Hilbert space in the gauge theory cannot be written as a tensor product of "inside" Hilbert space and "outside" Hilbert space. In above calculations, however, we trace over all of the out states without worrying about the gauge constraint. This is possible *only* in the extended Hilbert space.

In the extended Hilbert space, we can define the entanglement entropy, which consists of three contributions as is given (7). Different superselection sectors are distinguished by the electric flux for the Abelian gauge theory and by the quadratic Casimir for the non-Abelian gauge theory at each boundary, and the different Casimir corresponds to the different "spin", or representation. Due to the Gauss's law in 1+1 dimension, we have only one sector, $p_{\mathbf{F}} = 1$, in our wave function (61), restricted in the fundamental representation. Therefore (69) gives only the second term in (7), as the first and the third term in (7) vanish.

Clearly this entanglement entropy (69) is associated with the fact that in and out link variables connected with each other at the boundary vertex cannot take values freely due to the gauge invariance constraint, and this gauge invariance correlates the two link variables. As a result, this correlation produces the entanglement obtained in (69), which is the "color entanglement".

3.4.2 Meson states

Now we consider the 2-dimensional gauge theory with the fundamental scalar field. Again we consider the Fig. 3 lattice setup. For each vertex n, there is a scalar field φ_n, in addition to the link variable $U_{ij} \equiv U_{ji}^{\dagger}$ on each link (ij).

Let us consider the following wave function,

$$\Psi(\varphi_i, U_{ij}) \equiv \frac{1}{\mathcal{N}} \left[\varphi_1^\dagger U_{12}U_{23}U_{34}U_{45}\varphi_5\right] \prod_{m=1}^{7} e^{-\frac{\gamma}{2}\varphi_n^\dagger\varphi_n}\,, \tag{70}$$

$$|\mathcal{N}|^2 = \frac{N}{\gamma^2}\left(\frac{\pi}{\gamma}\right)^{7N}\,, \tag{71}$$

where $\mathcal{N}$ is the normalization constant. This is a single "meson" state composed by a scalar "quark" (at site $n = 1$) and "anti-quark" (at site $n = 5$) pair. For the wave function of the scalar field to be normalizable, we have introduced the Gaussian suppression factor $\propto e^{-\frac{\gamma}{2}\varphi^\dagger\varphi}$ with the Gaussian parameter γ. The normalization constant $\mathcal{N}$ is obtained from the condition

$$1 = \int [d\varphi_1 d\varphi_2 \cdots d\varphi_7] \int [dU_{12}dU_{23}\cdots dU_{71}]\Psi^*(\varphi_i, U_{ij})\Psi(\varphi_i, U_{ij})\,,$$

where we use (199) and (208). Similarly, using (199), (200) and (71), the reduced density matrix $\rho(\varphi_{in}, U_{in}; \phi_{in}, V_{in})$ becomes

$$\begin{aligned}
\rho(\varphi_{in}, U_{in}; \phi_{in}, V_{in}) &= \int [d\tilde{\varphi}_4\cdots d\tilde{\varphi}_7]\int [dW_{34}\cdots dW_{67}]\Phi(\varphi_{in}, \tilde{\varphi}_{out}; U_{in}, W_{out}) \\
&\qquad \times \Phi^*(\phi_{in}, \tilde{\varphi}_{out}; V_{in}, W_{out}) \\
&= \frac{\gamma}{N}\left(\frac{\pi}{\gamma}\right)^{-3N}\left[(\varphi_1^\dagger U_{12}V_{12}^\dagger\phi_1)\prod_{n=1}^{3} e^{-\frac{\gamma}{2}\varphi_n^\dagger\varphi_n - \frac{\gamma}{2}\phi_n^\dagger\phi_n}\right]\,,
\end{aligned} \tag{72}$$

and a square of the reduced density matrix thus is given by

$$\begin{aligned}
\rho^2(\varphi, U; \phi, V) &= \int [d\tilde{\varphi}dW]\ \rho(\varphi, U; \tilde{\varphi}, W)\rho(\tilde{\varphi}, W; \phi, V) \\
&= \left[\frac{\gamma}{N}\left(\frac{\pi}{\gamma}\right)^{-3N}\right]^2 \prod_{n=1}^{3} e^{-\frac{\gamma}{2}\varphi_n^\dagger\varphi_n - \frac{\gamma}{2}\phi_n^\dagger\phi_n} \\
&\quad \times \int [d\tilde{\varphi}dW]\left(\varphi_1^\dagger U_{12}W_{12}^\dagger\tilde{\varphi}_1\right)\left(\tilde{\varphi}_1^\dagger W_{12}V_{12}^\dagger\phi_1\right)\ e^{-\gamma(\tilde{\varphi}_1^\dagger\tilde{\varphi}_1 + \tilde{\varphi}_2^\dagger\tilde{\varphi}_2 + \tilde{\varphi}_3^\dagger\tilde{\varphi}_3)} \\
&= \left[\frac{\gamma}{N}\left(\frac{\pi}{\gamma}\right)^{-3N}\right]^2 \prod_{n=1}^{3} e^{-\frac{\gamma}{2}\varphi_n^\dagger\varphi_n - \frac{\gamma}{2}\phi_n^\dagger\phi_n}\left(\varphi_1^\dagger U_{12}\right)_c\left(V_{12}^\dagger\phi_1\right)^b \\
&\quad \times \int [d\tilde{\varphi}_2 d\tilde{\varphi}_3]\ e^{-\gamma(\tilde{\varphi}_2^\dagger\tilde{\varphi}_2 + \tilde{\varphi}_3^\dagger\tilde{\varphi}_3)}\frac{1}{N}\delta^a{}_d\delta^c{}_b\int [d\tilde{\varphi}_1]\ \tilde{\varphi}_1^d\tilde{\varphi}_{1a}^\dagger\, e^{-\gamma(\tilde{\varphi}_1^\dagger\tilde{\varphi}_1)} \\
&= \frac{1}{N}\rho(\varphi_{in}, U_{in}; \phi_{in}, V_{in})\,,
\end{aligned} \tag{73}$$

where we have performed the W_{12} integral using the formula (208) in the third equality, and then the φ integral using (200) in the fourth equality.

From eq. (73), the entanglement entropy is obtained as

$$S_{EE}^{\text{Fund.}} = -\text{Tr}\rho \log \rho = \log N\,. \tag{74}$$

Here $\log N$ simply represents the color charge entanglement between scalar quark and anti-quark in the fundamental representation.

For completeness, we show the result with the adjoint matter field Φ. We take

$$\Psi(\Phi_i, U_{ij}) = \frac{1}{\mathcal{N}}\left[\chi(\Phi_1 U_{12} U_{23} U_{34} U_{45} \Phi_5 U_{45}^\dagger U_{34}^\dagger U_{23}^\dagger U_{12}^\dagger)\right] \prod_{i=1}^{7} e^{-\beta \text{Tr}\Phi_i^2} \tag{75}$$

for the wave function with the adjoint scalar field Φ at the vertex 1 and 5, where β is the Gaussian suppression factor. The lattice setup is same as Fig. 3.

Applying (205) and (209) to the condition

$$\begin{aligned} 1 =& \frac{1}{|\mathcal{N}|^2}\int [d\Phi][dU]\, \chi(\Phi_1 U_{12} U_{23} U_{34} U_{45} \Phi_5 U_{45}^\dagger U_{34}^\dagger U_{23}^\dagger U_{12}^\dagger) \\ & \times \chi(\Phi_1 U_{12} U_{23} U_{34} U_{45} \Phi_5 U_{45}^\dagger U_{34}^\dagger U_{23}^\dagger U_{12}^\dagger) \prod_{i=1}^{7} e^{-2\beta \text{Tr}\Phi_i^2}, \end{aligned} \tag{76}$$

the normalization constant is determined as

$$\frac{1}{|\mathcal{N}|^2} = \frac{16\beta^2}{N^2-1}\left(\sqrt{\frac{2\beta}{\pi}}\right)^{7(N^2-1)}\,. \tag{77}$$

Then, the reduced density matrix is given by

$$\begin{aligned} &\langle \tilde{\Phi}_{in}, V_{in}|\,\rho\,|\Phi_{in}, U_{in}\rangle \\ =& \frac{1}{|\mathcal{N}|^2}\prod_{i=1}^{3} e^{-\beta\text{Tr}\Phi_i^2 - \beta\text{Tr}\tilde{\Phi}_i^2}\int [dX_{4,6,7}] \prod_{i=4,6,7} e^{-2\beta\text{Tr}X_i^2}\int [dW][dX_5] \\ &\qquad \times \chi(\Phi_1 U_{12} U_{23} W_{34} W_{45} X_5 W_{45}^\dagger W_{34}^\dagger U_{23}^\dagger U_{12}^\dagger) \\ &\qquad\quad \times \chi(\tilde{\Phi}_1 V_{12} V_{23} W_{34} W_{45} X_5 W_{45}^\dagger W_{34}^\dagger V_{23}^\dagger V_{12}^\dagger) e^{-2\beta\text{Tr}X_5^2} \\ =& \underbrace{\frac{4\beta}{N^2-1}\left(\sqrt{\frac{2\beta}{\pi}}\right)^{3(N^2-1)}}_{:=A} \chi(U_{23}^\dagger U_{12}^\dagger \Phi_1 U_{12} U_{23} V_{23}^\dagger V_{12}^\dagger \tilde{\Phi}_1 V_{12} V_{23}) \prod_{i=1}^{3} e^{-\beta\text{Tr}\Phi_i^2 - \beta\text{Tr}\tilde{\Phi}_i^2}, \end{aligned} \tag{78}$$

and its square becomes

$$
\begin{aligned}
\langle\tilde{\Phi}_{in}, V_{in}|\,\rho^2\,|\Phi_{in}, U_{in}\rangle =\ & A^2\int[dX_2][dX_3]\prod_{i=2,3}e^{-2\beta\mathrm{Tr}X_i^2}\\
&\times\int[dX_1][dW]\chi(U_{23}^\dagger U_{12}^\dagger\Phi_1U_{12}U_{23}W_{23}^\dagger W_{12}^\dagger X_1W_{12}W_{23})\\
&\times\chi(W_{23}^\dagger W_{12}^\dagger X_1W_{12}W_{23}V_{23}^\dagger V_{12}^\dagger\tilde{\Phi}_1V_{12}V_{23})e^{-2\beta\mathrm{Tr}X_1^2}\prod_{i=1}^{3}e^{-\beta\mathrm{Tr}\Phi_i^2-\beta\mathrm{Tr}\tilde{\Phi}_i^2}\\
= A\,\frac{4\beta}{N^2-1}&\left(\sqrt{\frac{2\beta}{\pi}}\right)^{3(N^2-1)}\frac{1}{4\beta}\left(\sqrt{\frac{\pi}{2\beta}}\right)^{3(N^2-1)}\\
&\times\chi(U_{23}^\dagger U_{12}^\dagger\Phi_1U_{12}U_{23}V_{23}^\dagger V_{12}^\dagger\tilde{\Phi}_1V_{12}V_{23})\prod_{i=1}^{3}e^{-\beta\mathrm{Tr}\Phi_i^2-\beta\mathrm{Tr}\tilde{\Phi}_i^2}\\
= \frac{1}{N^2-1}&\,\langle\tilde{\Phi}_{in}, V_{in}|\,\rho\,|\Phi_{in}, U_{in}\rangle\,,
\end{aligned}
\tag{79}
$$

Therefore, the entanglement entropy is obtained as

$$
S_{EE} = \log(N^2-1)\,, \tag{80}
$$

which confirms that the argument of log counts a dimension of the representation for the flux at the boundary vertex.

4 Transfer matrix and hopping parameter expansion

In the previous sections, we consider the entanglement entropy for various states, which are chosen by hand, in order to demonstrate how the two new contribution, Shannon and color part, appear in the 1+1 lattice gauge theories with scalar fields.

Our next task is to calculate the entanglement entropy for the grand state of the 1+1 dimensional $SU(N)$ gauge theories with the fundamental scalar field on the lattice. In this section, we give several definitions and formula useful for this purpose. The calculation of the entanglement entropy will be given in the next section.

4.1 Lattice action and Transfer matrix

We start from reviewing Transfer Matrix [18].

We set a as spacial lattice size and a_t lattice size for time direction. We use integer t, x and label the point as (t, x) . The lattice action of 1+1 lattice gauge theory with fundamental scalar field is given as

$$S = S_G + S_M, \tag{81}$$

$$S_G = \beta \sum_t \sum_x \chi_{\mathrm{F}} \left(U_{\mathrm{P},t,x} + U^\dagger_{\mathrm{P},t,x} - 2 \right), \beta = \frac{1}{g^2_{\mathrm{YM}} a_t a}, \tag{82}$$

$$S_M = a_t a \sum_t \sum_x \varphi^\dagger_{t,x} (\nabla^2 - m^2) \varphi_{t,x}, \tag{83}$$

Where $U_{\mathrm{P},t,x} = U_{(t,x)\to(t,x+1)} U_{(t,x+1)\to(t+1,x+1)} U_{(t+1,x+1)\to(t+1,x)} U_{(t+1,x)\to(t,x)}$.

Here covariant derivatives are given as

$$a_t^2 \nabla_t^2 \varphi_{t,x} = U_{(t,x)\to(t+1,x)} \varphi_{t+1,x} + U_{(t,x)\to(t-1,x)} \varphi_{t-1,x} - 2\varphi_{t,x}, \tag{84}$$

$$a^2 \nabla_x^2 \varphi_{t,x} = U_{(t,x)\to(t,x+1)} \varphi_{t,x+1} + U_{(t,x)\to(t,x-1)} \varphi_{t,x-1} - 2\varphi_{t,x}. \tag{85}$$

From now we use temporal gauge $U_{(t,x)\to(t+1,x)} = 1$ and use integer i, j to label spatial position. For each link and vertex, We use the basis

$$\hat{\varphi} \left|\varphi\right\rangle = \varphi \left|\varphi\right\rangle, \tag{86}$$

$$(\hat{U}^{\mathbf{R}})^i{}_j \left|U\right\rangle = (U^{\mathbf{R}})^i{}_j \left|U\right\rangle, \tag{87}$$

$$\langle\phi|\varphi\rangle = \delta(\phi - \varphi), \tag{88}$$

$$\langle V|U\rangle = \delta(V, U), \tag{89}$$

$$\int [d\varphi] \left|\varphi\right\rangle \left\langle\varphi\right| = 1, \tag{90}$$

$$\int [dU] \left|U\right\rangle \left\langle U\right| = 1. \tag{91}$$

The entanglement entropy for the ground state of the theory is often calculated in the path integral formalism using the replica method. In this paper,

[18]We summarized transfer matrix for harmonic oscillator, in appendix.

however, in order to distinguish all three contributions, we employ the operator formalism, as in the previous case for the pure gauge theories [27], where the transfer matrix and its eigenstates (instead of the Hamiltonian) were used to calculate the entanglement entropy. The transfer matrix $\hat{T}$ is defined to generate the time translation by one (temporal) lattice unit [37, 39] and thus is symbolically denoted as

$$\hat{T}(a_t, a) \equiv e^{-a_t H_L(a_t,a)} . \tag{92}$$

where $H_L(a_t, a)$ is the lattice "Hamiltonian" for the discrete time. In the $a_t \to 0$ limit while keeping the spatial lattice spacing a non-zero, we recover the lattice Hamiltonian (8) for the continuous time as

$$H = \lim_{a_t \to 0} H_L(a_t, a) = - \lim_{a_t \to 0} \frac{1}{a_t} \log \hat{T}(a_t, a) . \tag{93}$$

Although eigenvalues and eigenstates are different between H and $\hat{T}$ at non-zero a_t, they agree in the continuum limit that $(a_t, a) \to (0, 0)$. In particular, the eigenstate for the largest eigenvalue of $\hat{T}$ corresponds to the ground state of the theory at $a_t = a \neq 0$ in one to one, and it approaches to the ground state of the continuum theory as $a \to 0$.

To derive the transfer matrix from the path integral with the given action (81), we define $\hat{T}$ as

$$\langle \Psi_{\text{out}} | (\hat{T})^{N_t} | \Psi_{\text{in}} \rangle = \int_{\Psi_0=\Psi_{\text{in}}}^{\Psi_{N_t}=\Psi_{\text{out}}} \prod_{n_0=1}^{N_t-1} \mathcal{D}\Psi_{n_0} \, e^{S_G+S_M} , \tag{94}$$

where $\Psi_{n_0} = \{U_{n_0}, \varphi_{n_0}\}$ represents the gauge field $U_{n_0} = \{U_{\vec{n},1}\}$ and the scalar fields $\varphi_{n_0} = \{\varphi_{\vec{n}}\}$ at a give time slice n_0, and we fix them to Ψ_{in} at $n_0 = 0$ and Ψ_{out} at $n_0 = N_t$.

We next rewrite the left-hand side of (94) as

$$\int_{\Psi_0=\Psi_{\text{in}}}^{\Psi_{N_t}=\Psi_{\text{out}}} \prod_{n_0=1}^{N_t-1} \mathcal{D}\Psi_{n_0} \prod_{n_0=0}^{N_t-1} \langle \Psi_{n_0+1} | \hat{T} | \Psi_{n_0} \rangle , \tag{95}$$

which must be equal to the right-hand side.

The transfer matrix is given as

$$\hat{T} = \hat{T}_G \hat{T}_M \hat{T}_0^2 \tag{96}$$

$$\langle V, \phi | \, \hat{T}_0 \, | U, \varphi \rangle = \prod_i \exp\left\{ \frac{a_t}{a} \left(\varphi_i^\dagger U_{i,i+1} \varphi_{i+1} \right.\right. \\ \left.\left. + \varphi_{i+1}^\dagger U_{i,i+1}^\dagger \varphi_i - (2 + a^2 m^2) \varphi_i^\dagger \varphi_i \right) \right\} \delta(\phi_i - \varphi_i) \delta(V, U) \tag{97}$$

$$\langle \phi | \, \hat{T}_M \, | \varphi \rangle = c_M \prod_i \exp\left\{ \frac{a}{a_t} \left(\phi_i^\dagger \varphi_i + \varphi_i^\dagger \phi_i - 2 \varphi_i^\dagger \varphi_i \right) \right\} \tag{98}$$

$$\langle V | \, \hat{T}_G \, | U \rangle = c_G \prod_i \exp\left\{ \beta \chi_{\mathrm{F}} \left(U_{i,i+1} V_{i,i+1}^\dagger + V_{i,i+1} U_{i,i+1}^\dagger - 2 \right) \right\}. \tag{99}$$

Here we allowed transfer matrix to be asymmetric.

With $a_t \to 0$, by demanding that $\hat{T}_M = \hat{T}_G = 1$, we get

$$c_M = \left(\sqrt{\frac{a}{\pi a_t}} \right)^{N_s} \quad (a_t \to 0), \tag{100}$$

$$c_G = \left(\sqrt{\frac{\beta}{\pi}} \right)^{N_L} \quad (a_t \to 0). \tag{101}$$

We can fix c_G more by demanding $\hat{T}_G = 1$ in the trivial representation for arbitrary parameter. In Ref. [27], the character expansion is applied to the pure gauge part of the transfer matrix T_G as

$$\exp\left\{ \beta \chi_{\mathbf{F}} \left(U V^\dagger + V U^\dagger - 2 \right) \right\} = \sum_{\mathrm{R}} d_{\mathbf{R}} \lambda_{\mathbf{R}}(\beta) \chi_{\mathbf{R}}(U V^\dagger), \tag{102}$$

where $\chi_{\mathbf{R}}(U) = \mathrm{tr}\, U(\mathbf{R})$ is a character for the irreducible representation $\mathbf{R}$ with its dimension $d_{\mathbf{R}} = \chi_{\mathbf{R}}(\mathbf{1})$, and $\mathbf{R} = \mathbf{1}$ denotes the trivial representation, and The expansion coefficient is given by

$$\lambda_{\mathbf{R}}(\beta) = \frac{1}{d_{\mathbf{R}}} \int dU \chi_{\mathbf{R}}(U) \exp\left\{ \beta \chi_{\mathbf{F}} (U + U^\dagger - 2) \right\}. \tag{103}$$

The coefficient c_G is decided as $c_G = \left\{ \frac{1}{\lambda_1(\beta)} \right\}^{N_L}$.

If we take $a_t = a$, then the transfer matrix becomes

$$\hat{T} = \hat{T}_G \hat{T}_M \hat{T}_0^2 \tag{104}$$

$$\langle V, \phi | \hat{T}_0 | U, \varphi \rangle = \prod_i \exp\Bigg(\varphi_i^\dagger U_{i,i+1} \varphi_{i+1} + \varphi_{i+1}^\dagger U_{i,i+1}^\dagger \varphi_i - (2 + a^2 m^2) \varphi_i^\dagger \varphi_i \Bigg) \delta(\phi_i - \varphi_i) \delta(V, U) \tag{105}$$

$$\langle \phi | \hat{T}_M | \varphi \rangle = \left(\sqrt{\frac{1}{\pi}} \right)^{N_s} \prod_i \exp\left(\phi_i^\dagger \varphi_i + \varphi_i^\dagger \phi_i - 2 \varphi_i^\dagger \varphi_i \right) \tag{106}$$

$$\langle V | \hat{T}_G | U \rangle = \left(\sqrt{\frac{\beta}{\pi}} \right)^{N_L} \prod_i \exp\left\{ \beta \chi_{\mathrm{F}} \left(U_{i,i+1} V_{i,i+1}^\dagger + V_{i,i+1} U_{i,i+1}^\dagger - 2 \right) \right\}. \tag{107}$$

Here $\beta = \dfrac{1}{g_{\mathrm{YM}}^2 a^2}$.

4.2 Hopping parameter expansion (HPE)

We rescale $\hat{T}$ so that coefficients do not appear any more. We also rescale scalar fields as $\varphi_n \to \sqrt{K} \varphi_n$ and $\phi_n \to \sqrt{K} \phi_n$ with the hopping parameter $K = 1/(m^2 a^2 + 2)$, so that T_0^2 and T_M becomes

$$T_0(\Psi) = \prod_{n=0}^{N_l - 1} \exp\left[-\varphi_n^\dagger \varphi_n + K \left\{ \varphi_n^\dagger U_n \varphi_{n+1} + \varphi_{n+1}^\dagger U_n^\dagger \varphi_n \right\} \right], \tag{108}$$

$$T_M(\varphi, \phi) = \prod_{n=0}^{N_l - 1} \exp\left[K \left(\varphi_n \phi_n^\dagger + \varphi_n^\dagger \phi_n \right) \right]. \tag{109}$$

Assuming that K is small, we can expand the transfer matrix around $K = 0$, which is called the hopping parameter expansion (HPE) [40, 41]. In this case, the Feynman rule for the scalar field is given by

$$\langle (\varphi_n^\dagger)_a \varphi_m^b \rangle = \delta_{nm} \delta^b{}_a, \qquad \langle \phi_n^a \phi_m^b \rangle = \langle (\phi_n^\dagger)_a (\phi_m^\dagger)_b \rangle = 0\,, \tag{110}$$

$$\langle (\varphi_{n_a}^\dagger)_a \varphi_{n_b}^b (\varphi_{n_c}^\dagger)_c \varphi_{n_d}^d \rangle = \delta^b{}_a \delta^d{}_c \delta_{n_a,n_b} \delta_{n_c,n_d} + \delta^d{}_a \delta^b{}_c \delta_{n_a,n_d} \delta_{n_c,n_b}. \tag{111}$$

We define states as

$$\langle \Phi^B | n, m \rangle = \phi_n^\dagger V_{n \to m} \phi_m, \qquad V_{n \to m} \equiv V_n V_{n+1} \cdots V_{m-1} \tag{112}$$

$$\langle \Phi^B | 0 \rangle = 1. \tag{113}$$

We then calculate $\hat{T}|0\rangle$ up to the order K^4 and $\hat{T}|n,m\rangle$ up to the order K^3, which are given below.

$$\begin{aligned} T|0\rangle &= \left(1 + K^2 N N_l + \frac{3}{2} K^4 N N_l + \frac{1}{2} K^4 N^2 N_l^2\right) |0\rangle \\ &+ \sum_n (K^2 + 2K^4 + K^4 N N_l)|n,n\rangle \\ &+ \frac{1}{2} K^4 \sum_n |n,n\rangle\,|n,n\rangle + K^4 \sum_{n\neq m} |n,n\rangle\,|m,m\rangle \\ &+ \sum_n K^3 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right) \{|n,n+1\rangle + |n,n-1\rangle\} \\ &+ \sum_n K^4 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 \{|n,n+2\rangle + |n,n-2\rangle\}\ , \end{aligned} \tag{114}$$

$$\begin{aligned} T|n,n\rangle &= N\{1 + 2K^2(N+1) + K^2 N (N_l - 2)\}|0\rangle + K^2\,|n,n\rangle \\ &+ K^2 N \sum_m |m,m\rangle \\ &+ K^3 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right) \Big(|n,n+1\rangle + |n+1,n\rangle \\ &+ |n,n-1\rangle + |n-1,n\rangle \Big) \\ &+ K^3 N \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right) \sum_m \Big(|m,m+1\rangle + |m+1,m\rangle \Big), \end{aligned} \tag{115}$$

$$\begin{aligned} T|n,n+1\rangle &= NK\left\{1 + 4(N+1)K^2 + N(N_l - 3)K^2\right\}|0\rangle \\ &+ K^2 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right) |n,n+1\rangle \\ &+ K^3 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 \{|n,n+2\rangle + |n-1,n+1\rangle\} \\ &+ K^3 \{|n,n\rangle + |n+1,n+1\rangle\} \\ &+ K^3 N \sum_m |m,m\rangle\,, \end{aligned} \tag{116}$$

$$
\begin{aligned}
T|n,n-1\rangle &= NK\left\{1+4(N+1)K^2+N(N_l-3)K^2\right\}|0\rangle \\
&+K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)|n,n-1\rangle \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2\{|n,n-2\rangle+|n+1,n-1\rangle\} \\
&+K^3\{|n,n\rangle+|n-1,n-1\rangle\} \\
&+K^3N\sum_m|m,m\rangle\,, \qquad (117)
\end{aligned}
$$

$$
\begin{aligned}
T|n,n+2\rangle &= NK^2|0\rangle+K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2|n,n+2\rangle \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^3\{|n,n+3\rangle+|n-1,n+2\rangle\} \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)\{|n,n+1\rangle+|n+1,n+2\rangle\}\ , \qquad (118)
\end{aligned}
$$

$$
\begin{aligned}
T|n,n-2\rangle &= NK^2|0\rangle+K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2|n,n-2\rangle \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^3\{|n,n-3\rangle+|n+1,n-2\rangle\} \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)\{|n,n-1\rangle+|n-1,n-2\rangle\}\ , \qquad (119)
\end{aligned}
$$

$$
\begin{aligned}
T|n,n+3\rangle &= NK^3|0\rangle+K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^3|n,n+3\rangle \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^4\{|n,n+4\rangle+|n-1,n+3\rangle\} \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2\{|n,n+2\rangle+|n+1,n+3\rangle\}\ , \qquad (120)
\end{aligned}
$$

$$
\begin{aligned}
T|n,n-3\rangle &= NK^3|0\rangle+K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^3|n,n-3\rangle \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^4\{|n,n-4\rangle+|n+1,n-3\rangle\} \\
&+K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2\{|n,n-2\rangle+|n-1,n-3\rangle\}\ , \qquad (121)
\end{aligned}
$$

$$
\begin{aligned}
T|n,n+l\rangle &= K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^l|n,n+l\rangle \\
&\quad +K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^{l+1}\{|n,n+l+1\rangle+|n-1,n+l\rangle\} \\
&\quad +K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^{l-1}\{|n,n+l-1\rangle+|n+1,n+l\rangle\}, \\
&\qquad\qquad (\text{for } l>3) \qquad (122)
\end{aligned}
$$

$$
\begin{aligned}
T|n,n-l\rangle &= K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^l|n,n-l\rangle \\
&\quad +K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^{l+1}\{|n,n-l-1\rangle+|n+1,n-l\rangle\} \\
&\quad +K^3\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^{l-1}\{|n,n-l+1\rangle+|n-1,n-l\rangle\}, \\
&\qquad\qquad (\text{for } l>3) \qquad (123)
\end{aligned}
$$

There are mixings among states, therefore we have to diagonalize them. Up to the K^2 order, the states $|n,n+l\rangle$ and $|n,n-l\rangle$ for $l\geq 3$ are the eigenstates for the transfer matrix, since

$$
T|n,n\pm l\rangle = K^2\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^l|n,n\pm l\rangle, \quad (\text{for } l\geq 3). \qquad (124)
$$

Thus at this order, all we have to do is to diagonalize the mixing among $|0\rangle, |n,n\rangle, |n,n\pm 1\rangle$, and $|n,n\pm 2\rangle$ states.

5 Entanglement entropy for the ground state by the HPE

5.1 Taking into higher order corrections in K

In §3.4 and 3.4.2, we have seen that a single Wilson loop or a single meson state holds nonzero entanglement entropy due to the second term of (7), which is associated with the color entanglement. In §3, we discussed multiple meson states, whose fluxes connect quarks-antiquarks through the boundary. In this case, by decomposing the wave function into irreducible representations, we obtain multiple superselection sectors, and as a result, nonzero entanglement entropy associated with the first term (the classical Shannon entropy for the probability distribution of each irreducible representation) as well as the second term (the color entanglement part) of (7) appear. We have shown these explicit examples, in order to illustrate how we obtain these non-Bell terms in the entanglement entropy in the extended Hilbert space definition.

One might wonder whether the Bell pair part of the entanglement, third term of (7), never appears in 2-dimensional gauge theory. In the pure gauge theory, we cannot have any Bell pairs due to the absence of local degrees of freedom [27]. In gauge theories with matter fields, of course, we can always prepare an appropriate linear combination of meson states by hand, which produces the Bell pair part in (7). Our main interest/concern here, however, is how the ground state of the gauge theory (the strong coupling ground state) acquires entanglements including Bell pairs from matter fields, and how entanglements for the ground state of the continuum gauge theory can be understood in terms of the lattice ground state.

In the 2-dimensional gauge theory without matter fields, which corresponds to the leading order of the HPE ($K = 0$), the ground state can be calculated exactly at an arbitrary coupling without strong coupling expansion,[19] and it is written by the tensor product of a trivial state on each link satisfying $\hat{J}_{ij}^2 \left|0\right\rangle_l = 0$ as

$$\left|0\right\rangle_{\text{strong}} = \bigotimes_l \left|0\right\rangle_l \ . \tag{125}$$

Thus the entanglement entropy of the strong coupling ground state $\left|0\right\rangle_{\text{strong}}$ vanishes at $K = 0$.[20]

Therefore, in this section, we study how the higher order in K of the HPE makes the strong coupling ground state entangled, and which part of (7) appears. We will show the following properties.

- The strong coupling ground state has no entanglement up to order K^2 in HPE (§5.2).
- The first term (the Shannon part for the superselection sector distribution) and the second term (the color entanglement part) first appear at the order K^3 for the ground state (§5.3).

[19] In 2-dimensions, there is no plaquette term (*i.e.,* magnetic field), therefore its Hamiltonian has a similar structure to the strong coupling limit of higher dimensional ones.

[20] This state corresponds to the wave function $\chi_{\mathbf{1}}(U)$, while the wave function $\chi_{\mathbf{R}}(U)$ with $\mathbf{R} \neq \mathbf{1}$ describes an excited state, which yields nonzero entanglement entropy as (69).

- The third term (the Bell pair part) first appears at the order K^6 for the ground state (§5.5).

Since all these contributions are positive definite order by order in the HPE, they never cancel each other. Therefore, the above observations imply that the 2-dimensional Yang-Mills theory with matter fields keeps all three types of entanglements in (7) in the continuum limit.

From now on, we simply denote the strong coupling ground state $|0\rangle_{\text{strong}}$ as $|0\rangle$.

5.2 Eigenstates and eigenvalues of $\hat{T}$ up to $\mathcal{O}(K^2)$

We first consider contributions at $\mathcal{O}(K^2)$, and diagonalize the transfer matrix $\hat{T}$. At this order, the generic state $|\Psi\rangle_K$ which mixes with the strong coupling ground state $|0\rangle$ can be expressed as

$$\begin{aligned} |\Psi\rangle_K \quad \equiv \quad & f_0 |0\rangle + \sum_n a_n |n,n\rangle + \sum_n b_n |n,n+1\rangle + \sum_n c_n |n,n-1\rangle \\ & + \sum_n d_n |n,n+2\rangle + \sum_n e_n |n,n-2\rangle \, . \end{aligned} \tag{126}$$

We thus determine the K dependent coefficients a_n, b_n, c_n, d_n, e_n, and f_0 in such a way that

$$\hat{T} |\Psi\rangle_K \propto |\Psi\rangle_K \tag{127}$$

is satisfied. As long as the HPE converges, the ground state in the HPE must contain $|0\rangle$, so that we will consider the state with $f_0 \neq 0$. We can set $f_0 \equiv 1$ without loss of generality, and we denote it as

$$\begin{aligned} |G^+\rangle_K \equiv |0\rangle + & \sum_n a_n |n,n\rangle + \sum_n b_n |n,n+1\rangle + \sum_n c_n |n,n-1\rangle \\ & + \sum_n d_n |n,n+2\rangle + \sum_n e_n |n,n-2\rangle \, . \end{aligned} \tag{128}$$

At the $O(K^2)$, using the transfer matrix $\hat{T}$ given in §4.2, the ground state is given by

$$|G^+\rangle_K = |0\rangle + \sum_n a_n^+ |n,n\rangle\,, \quad \text{where} \quad a_n^+ = \frac{K^2}{G_K^+ - (1+NN_\ell)K^2}\,, \tag{129}$$

$$\begin{aligned} G_K^+ &= \frac{1}{2}\{1+K^2(1+2NN_\ell)\} \\ &\quad + \frac{1}{2}\sqrt{1-2(1-2NN_\ell)K^2 + \{1+4N(NN_\ell+2)N_\ell\}K^4}. \end{aligned} \tag{130}$$

The complete list of all other eigenstates and eigenvalues at this order are given in the appendix E.

In the $K \to 0$ limit, this state $|G^+\rangle_K$ has a maximum eigenvalue of the transfer matrix, $G_K^+ = 1$, which corresponds to "zero energy", since the transfer matrix is related to the "Hamiltonian" as $T \approx e^{-aH}$. We therefore identify this state as the ground state at $O(K^2)$, which is composed of the *strong coupling* ground state $|0\rangle$ and *lattice* point-like exited meson states $|n,n\rangle$. It is thus clear that this state does not have any entanglement. More precisely, we can write this ground state as a product state as

$$|G^+\rangle_K = \left(|0\rangle_{\text{in}} + K^2 \sum_{\text{in}} |n,n\rangle\right)\left(|0\rangle_{\text{out}} + K^2 \sum_{\text{out}} |n,n\rangle\right) + \mathcal{O}(K^3). \tag{131}$$

This means that there is no correlation between inside and outside and thus no entanglement at this order.

On the other hand, the vacuum state $|0\rangle_{\text{cont.}}$ in the continuum gauge theory is expected to have non-zero entanglement. So there still remains a qualitative difference (whether it is entangled or not) between the ground state $|G^+\rangle_K$ at $O(K^2)$ and the continuum ground state $|0\rangle_{\text{cont.}}$. This indicates we need higher order of the HPE than K^2. Indeed, since the vacuum state in the continuum theory is realized in the continuum limit as

$$\lim_{\substack{K\to 1/2,\\ \beta\to\infty}} |G^+\rangle_K \to |0\rangle_{\text{cont.}}\,, \tag{132}$$

where $K = \left(2+(ma)^2\right)^{-1} \to 1/2$ and $\beta = \left(g_{\text{YM}}^2 a^2\right)^{-1} \to \infty$ as $a \to 0$ for finite mass m and coupling g_{YM}, the higher order terms in the HPE become more and more important as we approach the continuum limit. Note that our calculations include all order of the gauge coupling constant at each order of the HPE. What we will see next is that once we take into account higher order corrections, $|G^+\rangle_K$ contains various contributions of the entanglement in (7).

5.3 Entanglement appear at $\mathcal{O}(K^3)$ corrections

As a next step, we check how K^3 order effects modify the properties of $|G^+\rangle_K$. At the order K^3, $|G^+\rangle_K$ becomes

$$|G^+\rangle_K = |0\rangle + K^2 \sum_n |n,n\rangle + K^3 \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \sum_n (|n,n+1\rangle + |n,n-1\rangle) + \mathcal{O}(K^4). \tag{133}$$

We therefore see that the $\mathcal{O}(K^3)$ contributions (quark-antiquark pairs separated with unit length) give the entanglement, once we divide the system into inside and outside.

Before we will see that the first and the second terms of (7) for the entanglement entropy becomes nonzero at this order, let us first explain how we obtain the above result. The eigenvalue equation is given by

$$\hat{T}\,|G^+\rangle_K = G^+_K\,|G^+\rangle_K\,, \tag{134}$$

which must be solved order by order. Expanding $\hat{T}$, $|G^+\rangle_K$, and G^+_K in power series of K, and using the resuts at $O(K^2)$ in (129) and (130), we have

$$\hat{T} = \hat{T}_0 + K^1\hat{T}_1 + K^2\hat{T}_2 + K^3\hat{T}_3 + \mathcal{O}(K^4)\,, \tag{135}$$

$$\begin{aligned} |G^+\rangle_K &= |G^+_0\rangle + K^1\,|G^+_1\rangle + K^2\,|G^+_2\rangle + K^3\,|G^+_3\rangle + \mathcal{O}(K^4) \\ &= |0\rangle + 0 + K^2 \sum_n |n,n\rangle + K^3\,|G^+_3\rangle + \mathcal{O}(K^4)\,, \end{aligned} \tag{136}$$

$$\begin{aligned} G^+_K &= G^+_0 + K^1 G^+_1 + K^2 G^+_2 + K^3 G^+_3 + \mathcal{O}(K^4) \\ &= 1 + 0 + K^2 2NN_\ell + K^3 G^+_3 + \mathcal{O}(K^4)\,, \end{aligned} \tag{137}$$

and solve the equations at each order in K.

Since (129) and (130) satisfy eigenvalue equation (134) up to $\mathcal{O}(K^2)$, it is enough to consider only $\mathcal{O}(K^3)$ terms. Left hand side of (134) becomes

$$K^3(\hat{T}_3\,|G^+_0\rangle + \hat{T}_2\,|G^+_1\rangle + \hat{T}_1\,|G^+_2\rangle + \hat{T}_0\,|G^+_3\rangle)\,. \tag{138}$$

while the right hand side of (134) is

$$K^3(G_3\,|G^+_0\rangle + G_2\,|G^+_1\rangle + G_1\,|G^+_2\rangle + G_0\,|G^+_3\rangle)\,. \tag{139}$$

We therefore obtain

$$\hat{T}_3\,|0\rangle + \hat{T}_0\,|G^+_3\rangle = G^+_3\,|0\rangle + |G^+_3\rangle\,, \tag{140}$$

where we used $|G_1^+\rangle = 0$ and $G_1^+ = 0$, which are seen from (129) and (130), and $\hat{T}_1 = 0$ for $|n,n\rangle$ from (115). Since $\hat{T}_3\,|0\rangle = \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_n(|n,n+1\rangle + |n,n-1\rangle)$ from (114), the above equation is equivalent to

$$\hat{T}_0\,|G_3^+\rangle + \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_n(|n,n+1\rangle + |n,n-1\rangle) = G_3^+\,|0\rangle + |G_3^+\rangle\,. \tag{141}$$

By substituting the ansatz that

$$|G_3^+\rangle = \omega\,|0\rangle + \sum_n \alpha_n\,|n,n\rangle + \sum_n \beta_n\,|n,n+1\rangle + \sum_n \gamma_n\,|n,n-1\rangle\,, \tag{142}$$

into (141), together with the relation

$$\hat{T}_0\,|0\rangle = |0\rangle\ ,\quad \hat{T}_0\,|n,n\rangle = N\,|n,n\rangle\ ,\quad \text{(and the rest is zero)} \tag{143}$$

from (114) - (123), we have

$$\begin{aligned}\omega\,|0\rangle &+ N\sum_n \alpha_n\,|0\rangle + \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_n(|n,n+1\rangle + |n,n-1\rangle)\\ &= (G_3^+ + \omega)\,|0\rangle + \sum_n \alpha_n\,|n,n\rangle + \sum_n \beta_n\,|n,n+1\rangle + \sum_n \gamma_n\,|n,n-1\rangle\,.\end{aligned} \tag{144}$$

Comparing l.h.s. and r.h.s., we finally obtain,

$$\beta_n = \gamma_n = \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\,,\quad G_3^+ = \alpha_n = 0\,. \tag{145}$$

while ω is an arbitrary constant.

In conclusion, we have obtained the eigenstate at the order of K^3 as

$$\begin{aligned}|G^+\rangle_K &= (1+\omega K^3)\,|0\rangle + K^2\sum_n |n,n\rangle + K^3\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_n(|n,n+1\rangle + |n,n-1\rangle) + \mathcal{O}(K^4)\\ &= (1+\omega K^3)\Big[\,|0\rangle + \frac{1}{(1+\omega K^3)}\Big\{K^2\sum_n |n,n\rangle\\ &\qquad + K^3\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_n(|n,n+1\rangle + |n,n-1\rangle) + \mathcal{O}(K^4)\Big\}\Big]\\ &\propto |0\rangle + K^2\sum_n |n,n\rangle + K^3\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_n(|n,n+1\rangle + |n,n-1\rangle) + \mathcal{O}(K^4),\end{aligned} \tag{146}$$

$$G^+ = 1 + 2NN_\ell K^2 + \mathcal{O}(K^4)\,. \tag{147}$$

This exactly gives eq. (133).

At this order, the ground state includes terms such as $|i, i+1\rangle$ and $|i+1, i\rangle$, where i-th vertex is located in the inside and $(i+1)$-th vertex is located in the outside. Thus there appears the non-trivial electric flux penetrating the boundary, so that we have a nontrivial superselection sector distribution. Namely, the term $|i, i+1\rangle$ ($|i+1, i\rangle$) belongs to a (anti-)fundamental sector, wheres the other terms to a singlet sector. Then the state makes the non-zero entanglement entropy corresponding to the first and second terms in (7).

We can confirm that there is *no* Bell pairs at this order by investigating each superselection sector. For simplicity, we here assume that there is only one boundary between i-th inner vertex and $(i+1)$-th outer vertex with the *outer* link variable $U_{i,i+1}$.

The singlet sector for the ground state still shows the tensor product structure,

$$\begin{aligned} |G^+\rangle_K\big|_{\text{singlet}} = &\left(|0\rangle_{\text{in}} + K^2 \sum_{\text{in}} |n,n\rangle + K^3 \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \sum_{\text{in}} (|n,n+1\rangle + |n,n-1\rangle) \right) \\ &\otimes \left(|0\rangle_{\text{out}} + K^2 \sum_{\text{out}} |n,n\rangle + K^3 \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \sum_{\text{out}} (|n,n+1\rangle + |n,n-1\rangle) \right) \\ &+ \mathcal{O}(K^4). \end{aligned} \tag{148}$$

Thus the singlet sector is *not* entangled at all.

Next let us focus on the fundamental sector (the discussion for the anti-fundamental sector is almost same). In this sector the state is simply $|i, i+1\rangle$ up to its normalization. If we explicitly denote the color degree of freedom $a(= 1, 2, \ldots N)$, the state can be represented as

$$\begin{aligned} |G^+\rangle_K\big|_{\text{fundamental}} &\propto K^3 \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} |i, i+1\rangle + \mathcal{O}(K^4) \\ &= K^3 \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \sum_a \left(|i, \text{bdy}\rangle_{a\,\text{in}} \otimes |\text{bdy}, i+1\rangle^a_{\text{out}} \right) + \mathcal{O}(K^4), \end{aligned} \tag{149}$$

where $|i, \text{bdy}\rangle_a$ corresponds to a quark at i-th vertex with flux going to outside area, and $|\text{bdy}, i+1\rangle^a$ to the similar object. (As the wave function, these objects are represented as $(\varphi_i^\dagger)_a$ and $(U_{i,i+1}\varphi_{i+1})^a$, respectively.) Clearly the state gives the entanglement entropy $\log N$ originating entirely from the color degree of freedom. For each color, the state shows the tensor product structure, indicating the absence of Bell pairs.

Before closing this subsection, we calculate the entanglement entropy for this ground state, which is given by

$$|G^+\rangle_K = |G^+\rangle_K\big|_{\text{singlet}} + |G^+\rangle_K\big|_{\text{fundamental}} + |G^+\rangle_K\big|_{\text{anti-fundamental}} \,, \tag{150}$$

up to $\mathcal{O}(K^4)$, where the state in the singlet sector $|G^+\rangle_K|_{\text{singlet}}$ is given by eq. (148) while the one in the fundamental sector $|G^+\rangle_K|_{\text{fundamental}}$ by eq. (149). The corresponding reduced density matrix $\rho_{\text{red.}}$ becomes

$$\rho_{\text{red.}} = p_{\mathbf{1}}\rho_{\mathbf{1}} + p_{\mathbf{F}}\rho_{\mathbf{F}} + p_{\bar{\mathbf{F}}}\rho_{\bar{\mathbf{F}}} \,, \tag{151}$$

where

$$p_{\mathbf{1}} = \frac{|\mathcal{N}_{\text{in}}|^2|\mathcal{N}_{\text{out}}|^2}{|\mathcal{N}|^2} \,, \qquad p_{\mathbf{F}} = p_{\bar{\mathbf{F}}} = \frac{c_F^2 N}{|\mathcal{N}|^2} \,, \tag{152}$$

$$\rho_{\mathbf{1}} = \frac{1}{|\mathcal{N}_{\text{in}}|^2}{}_{\text{in}}|\mathbf{1}\rangle\,\langle\mathbf{1}|_{\text{in}} \,, \quad \rho_{\mathbf{F}} = \frac{1}{N}{}_{\text{in}}|\mathbf{F}\rangle\,\langle\mathbf{F}|_{\text{in}} \,, \quad \rho_{\bar{\mathbf{F}}} = \frac{1}{N}{}_{\text{in}}|\bar{\mathbf{F}}\rangle\,\langle\bar{\mathbf{F}}|_{\text{in}} \,, \tag{153}$$

with

$$|\mathcal{N}_{\text{in/out}}|^2 = (1 + K^2 N N_{\text{in/out}})^2 + K^4 N N_{\text{in/out}} + 2c_F^2 N(N_{\text{in/out}} - 1) \,, \tag{154}$$

$$|\mathcal{N}|^2 = |\mathcal{N}_{\text{in}}|^2|\mathcal{N}_{\text{out}}|^2 + 2c_F^2 N \,, \qquad c_F \equiv K^3\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \,, \tag{155}$$

$$|\mathbf{1}\rangle_{\text{in}} = |0\rangle_{\text{in}} + K^2\sum_{\text{in}}|n,n\rangle + c_F\sum_{\text{in}}\left(|n,n+1\rangle + |n+1,n\rangle\right) , \tag{156}$$

$$|\mathbf{F}\rangle_{\text{in}} = \sum_a |i,\text{bdy}\rangle_{a\,\text{in}} \,, \quad |\bar{\mathbf{F}}\rangle_{\text{in}} = \sum_{\bar{a}} |\text{bdy},i\rangle_{\text{in}}^{\bar{a}} \,. \tag{157}$$

Here $N_{\text{in(out)}}$ is a number of sites in the inside (outside) region, thus $N_l = N_{\text{in}} + N_{\text{out}}$, and $|\mathcal{N}|^2$ and $|\mathcal{N}_{\text{in/out}}|^2$ are defined as $|\mathcal{N}|^2 = {}_K\langle G^+|G^+\rangle_K$, $|\mathcal{N}_{\text{in/out}}|^2 = {}_{\text{in/out}}\langle\mathbf{1}|\mathbf{1}\rangle_{\text{in/out}}$. It is easy to see

$$\rho_{\mathbf{1}}^2 = \rho_{\mathbf{1}} \,, \quad \rho_{\mathbf{F}}^2 = \frac{1}{N}\rho_{\mathbf{F}} \,, \quad \rho_{\bar{\mathbf{F}}}^2 = \frac{1}{N}\rho_{\bar{\mathbf{F}}} \,. \tag{158}$$

The total entanglement entropy S_{EE} for this state is given by

$$S_{EE} \;=\; \sum_{\mathbf{R}=\mathbf{1},\mathbf{F},\bar{\mathbf{F}}}\left\{-p_{\mathbf{R}}\log p_{\mathbf{R}} + p_{\mathbf{R}}\log d_{\mathbf{R}}\right\} , \tag{159}$$

where $d_{\mathbf{1}} = 1, d_{\mathbf{F}} = d_{\bar{\mathbf{F}}} = N$.

5.4 $\mathcal{O}(K^4)$ and $\mathcal{O}(K^5)$

By almost the same way as the previous subsection, we obtain $\mathcal{O}(K^4)$ correction to the state $|G^+\rangle_K$ and eigenvalue G^+ as

$$\begin{aligned}
|G^+\rangle_K = |0\rangle + K^2 \sum_n |n,n\rangle + K^3 \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \sum_n (|n,n+1\rangle + |n,n-1\rangle) \\
+ K^4 \Bigg(3 \sum_n |n,n\rangle + 2 \sum_n |n,n\rangle |n,n\rangle + \sum_{n\neq m} |n,n\rangle |m,m\rangle \\
+ \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 \sum_n (|n,n+2\rangle + |n,n-2\rangle) \Bigg) + \mathcal{O}(K^5) \,, \qquad (160)
\end{aligned}$$

$$\begin{aligned}
G^+ &= 1 + 2NN_\ell K^2 \\
&+ \left(7 + 2\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} + 2NN_l\right) NN_l K^4 + \mathcal{O}(K^5) \,, \qquad (161)
\end{aligned}$$

again having three sectors (singlet, fundamental, and anti-fundamental).

This is obtained from the equation (134) at order K^4 as follows. Using expansions (135), (136) and (137) at order K^4, we obtain

$$\begin{aligned}
&\hat{T}_4 |0\rangle + \hat{T}_2 \sum_n |n,n\rangle + \hat{T}_1 \frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \sum_n (|n,n+1\rangle + |n,n-1\rangle) + \hat{T}_0 |G_4^+\rangle \\
&= G_4^+ |0\rangle + 2NN_l \sum_n |n,n\rangle + |G_4^+\rangle \,. \qquad (162)
\end{aligned}$$

A comparison between the l.h.s and r.h.s. in (162), together with the formula (114) and the ansatz

$$\begin{aligned}
|G_4^+\rangle = \omega |0\rangle + \sum_n \alpha_n |n,n\rangle + \sum_n \alpha_{n,n} |n,n\rangle |n,n\rangle + \sum_{n\neq m} \alpha_{n,m} |n,n\rangle |m,m\rangle \\
+ \sum_n \beta_n |n,n+1\rangle + \sum_n \gamma_n |n,n-1\rangle \\
+ \sum_n \delta_n |n,n+2\rangle + \sum_n \varepsilon_n |n,n-2\rangle \ , \qquad (163)
\end{aligned}$$

gives

$$\delta_n = \varepsilon_n = \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 , \quad \beta_n = \gamma_n = 0 , \tag{164}$$

$$\alpha_{n,m} = 1 \ \text{(for } \ n \neq m) , \quad \alpha_{n,n} = \frac{1}{2} , \quad \alpha_n = 3 , \tag{165}$$

$$G_4^+ = \left(7 + 2\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} + 2NN_l\right) NN_l . \tag{166}$$

These lead to results (160) and (161).

Let us consider whether the ground state wave function (160) at $\mathcal{O}(K^4)$ in the HPE contains the Bell pair part of the entanglement entropy. To see this, we examine singlet sector and (anti-)fundamental sector separately. Again we assume a single boundary between the i-th inner vertex and the $(i+1)$-th outer vertex.

We first analyze the singlet sector in the following way. If we assume that the Bell pair part is absent, we immediately notice that the term $|n,n\rangle_{\text{in}}\, |m,m\rangle_{\text{out}}$, where the n-th vertex is in the inside and the m-th vertex is in the outside, must appear in the ground state as

$$|G^+\rangle_K\big|_{\text{singlet}} \supset c_4\, K^4\, |n,n\rangle_{\text{in}}\, |m,m\rangle_{\text{out}} \tag{167}$$

with the coefficient $c_4 = 1$, which is determined from the result at the lower order given in (148), since such a term must be a part of the tensor product of inside-only excited states and outside-only excited states. Inversely, if $c_4 \neq 1$, such a state can *not* be written as a tensor product state given in (148). The result (165) indeed shows $c_4 = 1$ for our wave function (160) at $\mathcal{O}(K^4)$. Therefore *no* Bell pair part appears in this sector.

In the higher orders, we can employ the similar analysis. With the assumption on the tensor product structure, we can predict coefficients of new terms at the higher order from results at lower orders. At $\mathcal{O}(K^5)$, for instance, the term $|n,n\rangle\,|m,m\rangle$ cannot exist since there is no corresponding inside-only or outside-only excited terms at lower orders. Indeed we cannot construct $|n,n\rangle\,|m,m\rangle$ states from $|0\rangle$ by the $\mathcal{O}(K^5)$ part of $\hat{T}$, since we need at least $\mathcal{O}(K^6)$ terms, which consist of two "U"-shaped contributions.[21]

The (anti-)fundamental sector at K^4 order has almost the same structure as the K^3 order case, where only difference is the distance of (anti-)quark from the boundary. As is the case of $\mathcal{O}(K^3)$, we can explicitly represent the state as

[21] Each "U"-shape is $\mathcal{O}(K^3)$, see appendix D for details.

$$
\begin{aligned}
|G^+\rangle_K\big|_{\text{fundamental}} &\propto K^3\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\,|i,i+1\rangle + K^4\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2(|i,i+2\rangle + |i-1,i+1\rangle) \\
&\quad + \mathcal{O}(K^5) \\
&= K^3\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_a (|i,\text{bdy}\rangle_{a\ \text{in}}\otimes|\text{bdy},i+1\rangle^a_{\text{out}}) \\
&\quad + K^4\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2\sum_{a'}\left(|i,\text{bdy}\rangle_{a'\ \text{in}}\otimes|\text{bdy},i+2\rangle^{a'}_{\text{out}}\right) \\
&\quad + K^4\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2\sum_{a''}\left(|i-1,\text{bdy}\rangle_{a''\ \text{in}}\otimes|\text{bdy},i\rangle^{a''}_{\text{out}}\right) + \mathcal{O}(K^5) \\
&= K^3\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\sum_a\left(|i,\text{bdy}\rangle_a + K\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\,|i-1,\text{bdy}\rangle_a\right)_{\text{in}} \\
&\quad \otimes\left(|\text{bdy},i+1\rangle^a + K\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\,|\text{bdy},i+2\rangle^a\right)_{\text{out}} + \mathcal{O}(K^5),
\end{aligned}
\tag{168}
$$

again without producing any Bell pairs.

We can apply the similar analysis to the $\mathcal{O}(K^5)$ case, and get the tensor product structure. With the fact that there appears *no* new superselection sector at $\mathcal{O}(K^5)$,[22] we thus conclude that there is no Bell pair at this order.

In the next subsection we will see that once we take into account $\mathcal{O}(K^6)$ corrections, the ground state can *not* be written as a tensor product state predicted from lower order results. As a consequence, we obtain the Bell pair part at $O(K^6)$.

5.5 Bell pair appears at $\mathcal{O}(K^6)$ corrections

To show that the Bell pair part appears in the ground state at $\mathcal{O}(K^6)$, we perform the same analysis.

Suppose again that the i-th vertex is located in the inside while the $(i+1)$-th vertex is in the outside. We focus on the singlet sector of the ground state, and we thus look at the coefficient c_6, which is associated with the term at $\mathcal{O}(K^6)$ as

$$
|G^+\rangle_K\big|_{\text{singlet}} \supset c_6\,K^6\,|i,i\rangle_{\text{in}}\,|i+1,i+1\rangle_{\text{out}}\,. \tag{169}
$$

As was discussed in the previous subsection, if there is no Bell pair, $|G^+\rangle_K|_{\text{singlet}}$

[22] At the $\mathcal{O}(K^6)$, a new adjoint sector appears.

must be the tensor product of the inside-only excited state and the outside-only excited state, and vice versa. Then, the term $|i,i\rangle_{\text{in}}\,|i+1,i+1\rangle_{\text{out}}$ must come from the product of $|i,i\rangle_{\text{in}}$ and $|i+1,i+1\rangle_{\text{out}}$ at lower order in the HPE. Eq. (160) and the absence of terms such as $|i,i\rangle_{\text{in}}$ or $|i+1,i+1\rangle_{\text{out}}$ at $\mathcal{O}(K^5)$ imply that the c_6 term at $\mathcal{O}(K^6)$ in (169) must be obtained from lower orders as

$$
\begin{aligned}
&\Big[|0\rangle_{\text{in}} + K^2\,|i,i\rangle_{\text{in}} + 3K^4\,|i,i\rangle_{\text{in}} + \mathcal{O}(K^6)\Big]_{\text{in}} \\
&\qquad \otimes \Big[|0\rangle_{\text{out}} + K^2\,|i+1,i+1\rangle_{\text{out}} + 3K^4\,|i+1,i+1\rangle_{\text{out}} + \mathcal{O}(K^6)\Big]_{\text{out}} \\
&\supset K^2\,|i,i\rangle_{\text{in}} \otimes 3K^4\,|i+1,i+1\rangle_{\text{out}} + 3K^4\,|i,i\rangle_{\text{in}} \otimes K^2\,|i+1,i+1\rangle_{\text{out}} \\
&= 6K^6\,|i,i\rangle_{\text{in}}\,|i+1,i+1\rangle_{\text{out}}\ ,
\end{aligned}
\tag{170}
$$

which gives $c_6 = 6$. Inversely if $c_6 \neq 6$, which is the case we will see, there are Bell pairs in this ground state.

To calculate c_6, we consider the corresponding terms in the eigenstate equation,

$$
\hat{T}\,|G^+\rangle_K = G^+_K\,|G^+\rangle_K\ . \tag{171}
$$

Since at least the forth order part of the transfer matrix in the HPE is needed to generate the $|i,i\rangle\,|i+1,i+1\rangle$ state in the future time, together with $|G^+_1\rangle = 0$, the relevant part of the left hand side can be calculated as

$$
\begin{aligned}
\left(\hat{T}_6\,|G^+_0\rangle + \hat{T}_4\,|G^+_2\rangle\right)\Big|_{K^6,|i,i\rangle|i+1,i+1\rangle} &= \left(\hat{T}_6\,|0\rangle + \hat{T}_4 \sum_n |n,n\rangle\right)\Bigg|_{K^6,|i,i\rangle|i+1,i+1\rangle} \\
&= 6 + 2NN_l + \frac{1}{N}\,.
\end{aligned}
\tag{172}
$$

See §D.3.2 for the explicit calculation to derive this result.

On the other hand, since $|i,i\rangle\,|i+1,i+1\rangle$ term appears only at K^n $(n \geq 4)$ order and $G^+_1 = 0$, the right hand side is evaluated as

$$
\left(G^+_0\,|G^+_6\rangle + G^+_2\,|G^+_4\rangle\right)\big|_{K^6,|i,i\rangle|i+1,i+1\rangle} = 1 \times c_6 + 2NN_l \times 1 = c_6 + 2NN_l. \tag{173}
$$

Thus eq. (171) leads to

$$
c_6 = 6 + \frac{1}{N} \Rightarrow c_6 \neq 6\,. \tag{174}
$$

We therefore conclude that there is the Bell pair part of the entanglement entropy in the singlet sector for the ground state.

Finally, we estimate the Bell pair part of the entanglement on the singlet sector at K^6 order. Since the ground state $|G^+\rangle_K$ in eq. (160) has the following structure

$$|G^+\rangle_K\big|_{\text{Non-singlet}} = O(K^3)\,, \tag{175}$$

the probability distribution $p_{\mathbf{1}}$ for the singlet sector ($\mathbf{k}=\mathbf{1}$) and $p_{\mathbf{k}\neq\mathbf{1}}$ for the non-singlet sector ($\mathbf{k}\neq\mathbf{1}$) are given by

$$p_{\mathbf{1}} = 1+\mathcal{O}(K^6)\,,\quad p_{\mathbf{k}\neq\mathbf{1}} = \mathcal{O}(K^6)\,. \tag{176}$$

Therefore, the Bell pair part, the third term of (7), is estimated in the HPE as

$$S_{EE}^{\text{Bell}} \equiv -\sum_{\mathbf{k}} p_{\mathbf{k}} \mathrm{Tr}_{\hat{\mathcal{H}}_{\text{in}}^{\mathbf{k}}} \rho_{\text{in}}^{\mathbf{k}} \log \rho_{\text{in}}^{\mathbf{k}} = -\mathrm{Tr}_{\hat{\mathcal{H}}_{\text{in}}^{\mathbf{1}}} \rho_{\text{in}}^{\mathbf{1}} \log \rho_{\text{in}}^{\mathbf{1}} + \mathcal{O}(K^6)\,, \tag{177}$$

In fact one can explicitly show that for the ground state wave function up to $\mathcal{O}(K^6)$, the Bell pair part of the entanglement appears only from the singlet sector. Therefore we here focus on the singlet sector of the ground state $|G^+\rangle_K$ and evaluate the leading contribution of the Bell pair part in the HPE.

As discussed, the singlet sector of the ground state has the following structure.

$$|G^+\rangle_K\big|_{\text{singlet}} = |\Psi\rangle_{\text{in}} \otimes |\Psi\rangle_{\text{out}} + \frac{K^6}{N}\,|i,i\rangle_{\text{in}} \otimes |i+1,i+1\rangle_{\text{out}} + \mathcal{O}(K^7)\,, \tag{178}$$

Here $|\Psi\rangle_{\text{in}} \otimes |\Psi\rangle_{\text{out}}$ corresponds to the l.h.s. of (170) if we focus only on the i-th and $i+1$-th vertices. In addition, $|\Psi\rangle_{\text{in}}$ and $|\Psi\rangle_{\text{out}}$ of course contain also purely inside only and outside only excitations, respectively. In particular, $|\Psi\rangle_{\text{in/out}}$ becomes $|0\rangle_{\text{in/out}}$ at $K=0$ as we have seen in previous section. Since the first term of (178) has a tensor product structure, the second term is crucial to generate the Bell pair part of the entanglement.

From (178), we can obtain the reduced density matrix $\rho_{\text{red.}}$ neglecting $\mathcal{O}(K^7)$ for the singlet state as

$$\begin{aligned}|\mathcal{N}_{\text{singlet}}|^2\rho_{\text{red.}} = {}& |\Psi\rangle_{\text{in out}}\langle\Psi|\Psi\rangle_{\text{out in}}\langle\Psi| + \tfrac{K^6}{N}\,|\Psi\rangle_{\text{in out}}\langle i+1,i+1|\Psi\rangle_{\text{out in}}\langle i,i| \\ &+\tfrac{K^6}{N}\,|i,i\rangle_{\text{in out}}\langle\Psi|i+1,i+1\rangle_{\text{out in}}\langle\Psi| \\ &+\left(\tfrac{K^6}{N}\right)^2 |i,i\rangle_{\text{in out}}\langle i+1,i+1|i+1,i+1\rangle_{\text{out in}}\langle i,i|\;.\end{aligned} \tag{179}$$

Here the norm $|\mathcal{N}_{\text{singlet}}|^2$ is

$$
\begin{aligned}
|\mathcal{N}_{\text{singlet}}|^2 &\equiv {}_K\langle G^+|G^+\rangle_K|_{\text{singlet}} \\
&= {}_{\text{in}}\langle\Psi|\Psi\rangle_{\text{in out}}\langle\Psi|\Psi\rangle_{\text{out}} + \tfrac{K^6}{N}{}_{\text{in}}\langle i,i|\Psi\rangle_{\text{in out}}\langle i+1,i+1|\Psi\rangle_{\text{out}} \\
&\quad +\tfrac{K^6}{N}{}_{\text{in}}\langle\Psi|i,i\rangle_{\text{in out}}\langle\Psi|i+1,i+1\rangle_{\text{out}} \\
&\quad + \left(\tfrac{K^6}{N}\right)^2 {}_{\text{in}}\langle i,i|i,i\rangle_{\text{in out}}\langle i+1,i+1|i+1,i+1\rangle_{\text{out}} \,. \qquad (180)
\end{aligned}
$$

To diagonalize the reduced density matrix (179), we would like to solve the following eigenvalue problem

$$
\rho_{\text{red.}}\,|P\rangle \;=\; p\,|P\rangle\;, \quad |P\rangle = \alpha\,|\Psi\rangle_{\text{in}} + \beta\,|i,i\rangle_{\text{in}}\;, \qquad (181)
$$

which leads to

$$
\begin{pmatrix} \rho_{11}-p & \rho_{12} \\ \rho_{21} & \rho_{22}-p \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \;=\; 0\,, \qquad (182)
$$

where

$$
|\mathcal{N}_{\text{singlet}}|^2\rho_{11} = {}_{\text{in}}\langle\Psi|\Psi\rangle_{\text{in out}}\langle\Psi|\Psi\rangle_{\text{out}} + \frac{K^6}{N}\,{}_{\text{in}}\langle i,i|\Psi\rangle_{\text{in out}}\langle i+1,i+1|\Psi\rangle_{\text{out}}\,, \qquad (183)
$$

$$
|\mathcal{N}_{\text{singlet}}|^2\rho_{12} = {}_{\text{in}}\langle\Psi|i,i\rangle_{\text{in out}}\langle\Psi|\Psi\rangle_{\text{out}} + \frac{K^6}{N}\,{}_{\text{in}}\langle i,i|i,i\rangle_{\text{in out}}\langle i+1,i+1|\Psi\rangle_{\text{out}}\,, \qquad (184)
$$

$$
\begin{aligned}
|\mathcal{N}_{\text{singlet}}|^2\rho_{21} &= \frac{K^6}{N}\,{}_{\text{in}}\langle\Psi|\Psi\rangle_{\text{in out}}\langle\Psi|i+1,i+1\rangle_{\text{out}} \\
&\quad + \left(\frac{K^6}{N}\right)^2 {}_{\text{in}}\langle i,i|\Psi\rangle_{\text{in out}}\langle i+1,i+1|i+1,i+1\rangle_{\text{out}}\,, \qquad (185)
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{N}_{\text{singlet}}|^2\rho_{22} &= \frac{K^6}{N}\,{}_{\text{in}}\langle\Psi|i,i\rangle_{\text{in out}}\langle\Psi|i+1,i+1\rangle_{\text{out}} \\
&\quad + \left(\frac{K^6}{N}\right)^2 {}_{\text{in}}\langle i,i|i,i\rangle_{\text{in out}}\langle i+1,i+1|i+1,i+1\rangle_{\text{out}}\,. \qquad (186)
\end{aligned}
$$

Thus, the eigenvalue is given by

$$
p \;=\; \frac{\rho_{11}+\rho_{22} \pm \sqrt{(\rho_{11}-\rho_{22})^2 + 4\rho_{12}\rho_{21}}}{2}\,. \qquad (187)
$$

To evaluate this, we use

$$
\begin{aligned}
{}_{\text{in/out}}\langle\Psi|\Psi\rangle_{\text{in/out}} &= 1 + \mathcal{O}(K^2)\,, &(188)\\
{}_{\text{in/out}}\langle n,n|\Psi\rangle_{\text{in/out}} &= N + \mathcal{O}(K^2)\,, &(189)\\
{}_{\text{in/out}}\langle n,n|n,n\rangle_{\text{in/out}} &= N(N+1)\,, &(190)
\end{aligned}
$$

which can be obtained by recalling ${}_{\rm in/out}\langle n, n|0\rangle_{\rm in/out} = N$ and ${}_{\rm in/out}\langle 0|0\rangle_{\rm in} = 1$, together with the fact that in the leading order in HPE, we have $|\Psi\rangle_{\rm in} = |0\rangle_{\rm in} + \mathcal{O}(K^2)$. Then the leading contribution of (187) yields

$$p \simeq 1 - K^{12}\,,\ K^{12}\,. \tag{191}$$

We therefore obtain the entanglement entropy $S_{EE}^{\rm Bell}$ for the singlet state as

$$\begin{aligned} S_{EE}^{\rm Bell} &= -(1-K^{12})\log(1-K^{12}) - K^{12}\log K^{12} + \mathcal{O}(K^{14}) \\ &= \left(1 - \log K^{12}\right) K^{12} + \mathcal{O}(K^{14})\,. \end{aligned} \tag{192}$$

Note that we obtain entangled $\mathcal{O}(K^{12}N^0)$ Bell pairs in the HPE from the $\mathcal{O}(K^6N^{-1})$ term in the wave function (178).

6 Discussion

In this thesis we evaluated the entanglement entropy for 2-dimensional $SU(N)$ gauge theories with matter field by using extended Hilbert space formalism. In the analysis, we have taken hopping parameter expansions and see the effect of mass of the matter from infinite mass limit perturbatively. As a result, the Shannon color part appears from the ground state at the $\mathcal{O}(K^3)$, while the Bell pair part appears at $\mathcal{O}(K^6)$. The result suggests that, even the ground state is pure, for gauge theories we need to consider both of classical and quantum contributions for the entanglement entropy. In other words, only quantum correlation is not sufficient to capture the all of the property of the entanglement entropy.

It is beneficial to compare this result with (massive) scalar theory, where the action is given as

$$S = a_t a \sum_t \sum_x \varphi^\dagger_{t,x}(\partial^2 - m^2)\varphi_{t,x}, \tag{193}$$

with

$$a_t^2\partial_t^2\varphi_{t,x} = \varphi_{t+1,x} + \varphi_{t-1,x} - 2\varphi_{t,x}, \tag{194}$$
$$a^2\partial_x^2\varphi_{t,x} = \varphi_{t,x+1} + \varphi_{t,x-1} - 2\varphi_{t,x}. \tag{195}$$

After introducing hopping parameter, the transfer matrix becomes

$$T_0(\varphi) = \prod_{n=0}^{N_l-1} \exp\left[-\varphi_n^\dagger\varphi_n + K\left\{\varphi_n^\dagger\varphi_{n+1} + \varphi_{n+1}^\dagger\varphi_n\right\}\right], \tag{196}$$

$$T_M(\varphi,\phi) = \prod_{n=0}^{N_l-1} \exp\left[K\left(\varphi_n\phi_n^\dagger + \varphi_n^\dagger\phi_n\right)\right]. \tag{197}$$

With comparison with (81), (108), (109), we can see that the method we used (transfer matrix and HPE expansion) can be applied straightforwardly to this theory, and it gives same value of entanglement entropy in gauge theory with matter case, by setting $\beta \to \infty$. Since in the scalar theory there is no constraint and we do not need to extend the Hilbert space, entanglement entropy is totally realized as Bell pair part. If we consider the "effect" by introducing gauge fields, they connect meson-like pairs by flux and "freeze" entanglement entropy classical. Non-trivial point is that even dominant part become frozen, some quantum correlation will survive. This is caused by the decomposition of reducible representation at boundaries, as we have seen.

The effect mentioned above is universal to a certain extent. Although we mainly focused on fundamental scalar matter case, we can expect similar result(dominant classical part and non-zero Shannon part) to remain in other representation case (e.g. adjoint matter) or fermion matter case. In higher-dimensional theories, similar effect can occur even for pure gauge theories.

To see the implication for continuum theory, we have to consider continuum limit,

$$\lim_{\substack{K\to 1/2,\\ \beta\to\infty}} |G^+\rangle_K \to |0\rangle_{\text{cont.}} \,. \tag{198}$$

In the limit, the mechanism how three types of contribution emerged is unchanged. Although the analysis in the work is restricted for perturbative region in the HPE, the positivity of entanglement entropy does not cancel the existence of contribution itself and it suggests that qualitative behavior of entanglement we have found in this work will remain. However, as stated in introduction, the continuum theories break TPS even without gauge constraints, suggesting that we need more general formulation to treat entanglement entropy in continuum theories.

In the theses, we have unraveled explicit mechanism of how each part of entanglement appears in the ground state of the theory. This is the first step to clarify the property of entanglement characteristic of gauge theories. This should becomes important when basic "unit" responsible for entanglement (in the model we analyzed this is meson-like pairs) is composed by flux. Aside from the purely theoretical consideration of gauge theories themselves, it may happen to be useful for high energy collider physics or early universe when we analyze entanglement. The analysis for more realistic model may give quantitative prediction for experimental physics. To achieve that, we need more physical protocols to distinguish three types of entanglement entropy.

Here we point out some problems as next steps. One direction is to consider more general quantum informational quantities, such as tripartite informationand analyze the structure of entanglement explicitly. It may allow us to extract more general property about correlations which gauge constraints give to the system.

It is also valuable to see the model with large N limit and corrections. It is necessary for direct comparison with Ryu-Takayanagi formula in the holographic context. For field theories with gravity dual, how three types of contribution in entanglement entropy can be understood is important.

Another direction we have to consider is to take more general states, mixed states. In that case the von Neumann entropy for the total density operator includes classical correlation in itself, and it will make the situation more complicated. For mixed states there are many generalizations of entanglement entropy proposed as entanglement measure to treat the correlation more easily. As such quantities, for example, we have entanglement cost, relative entropy, or negativity etc. In addition to them, there also many interesting quantity, not entanglement measure but capture *some* quantum character of the state. For such quantities we have mutual information, entanglement of purification, etc. The analysis of such quantities for gauge theories should be done to clearly the role of gauge constraints in correlation more deeply.

Considering higher dimensional case such as 2+1 dimensional theories is also important and interesting. In that case, generally additional "plaquett" gauge-invariant excitation join to states and it will make the situation complicated. What's more, In higher dimensional case we have much boundary vertices and we have to see which representation each boundary vertex belongs. In this case we no longer take arbitrary values of gauge coupling by using character expansion, so for explicitly calculation we need another tools such as numerical methods.

For quantum gravity, we have to apply the implication to holography framework. In AdS/CFT correspondence, quantum informational quantities are considered for theories with conformal symmetry and there are techniques to exploit the symmetry. Although the model we have analyzed in this thesis is inevitably not conformal theory, making attempt to extend the analysis to conformal theories is interesting as a future direction.

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A Useful formulas

We use a, b, c, d, $\cdots$, and i, j, $\cdots$ as color indices in fundamental representation (which run $1, \cdots, N$) of the $SU(N)$ gauge group.

A.1 Matter fields

Scalar field φ in Fundamental representation For the scalar field φ^c in the fundamental representation with $\varphi^\dagger \varphi \equiv \varphi_c^\dagger \varphi^c$, we have following useful Gaussian integral formulas:

$$\int [d\varphi]\; e^{-a\varphi^\dagger\varphi} = \left(\sqrt{\frac{\pi}{a}}\right)^{2N} , \tag{199}$$

$$\int [d\varphi]\; \varphi_c^\dagger \varphi^d e^{-a\varphi^\dagger\varphi} = \delta^d{}_c \frac{1}{a} \left(\sqrt{\frac{\pi}{a}}\right)^{2N} , \tag{200}$$

$$\int [d\varphi]\; \varphi_a^\dagger \varphi^b \varphi_c^\dagger \varphi^d e^{-a\varphi^\dagger\varphi} = \left(\delta^b{}_a \delta^d{}_c + \delta^d{}_a \delta^b{}_c\right) \frac{1}{a^2} \left(\sqrt{\frac{\pi}{a}}\right)^{2N} . \tag{201}$$

The last formula gives

$$\int [d\varphi]\; (\sum_{b=1}^{N} \varphi_b^\dagger \varphi^b)(\sum_{d=1}^{N} \varphi_d^\dagger \varphi^d) e^{-\sum_{c=1}^{N} a\varphi_c^\dagger \varphi^c} = \frac{N(N+1)}{a^2} \left(\sqrt{\frac{\pi}{a}}\right)^{2N} . \tag{202}$$

Hermitian $N \times N$ matrix scalar $X^c{}_d$ field Next we consider the Gaussian integral for the Hermitian $N\times N$ matrix field. This is an adjoint representation matter field for gauge group $U(N)$, whose Gaussian integral becomes

$$\int [dX] \exp\left(-a\mathrm{Tr}X^2\right) = \left(\sqrt{\frac{\pi}{a}}\right)^{N^2}, \tag{203}$$

$$\int [dX] X^a{}_b X^c{}_d \exp\left(-a\mathrm{Tr}X^2\right) = \delta^a{}_d \delta^c{}_b \frac{1}{2a}\left(\sqrt{\frac{\pi}{a}}\right)^{N^2}, \tag{204}$$

while the Gaussian integral for the field in the adjoint representation of the gauge group $SU(N)$ leads to

$$\int [dX] X^a{}_b X^c{}_d \exp\left(-a\mathrm{Tr}X^2\right)$$
$$= \left(\delta^a{}_d \delta^c{}_b - \frac{1}{N}\delta^a{}_b \delta^c{}_d\right)\frac{1}{2a}\left(\sqrt{\frac{\pi}{a}}\right)^{N^2-1}, \tag{205}$$

where the traceless condition is used. The above formulae are obtained by expanding

$$X = \sum_{A=0}^{N^2-1} t^A X_A\,, \qquad \sum_{A=0}^{N^2-1} (t^A)^a{}_b (t^A)^c{}_d = \delta^a{}_d \delta^c{}_b \tag{206}$$

for $U(N)$ and

$$X = \sum_{A=1}^{N^2-1} t^A X_A\,, \qquad \sum_{A=1}^{N^2-1} (t^A)^a{}_b (t^A)^c{}_d = \delta^a{}_d \delta^c{}_b - \frac{1}{N}\delta^a{}_b \delta^c{}_d \tag{207}$$

for $SU(N)$, where X_A is real and $\mathrm{tr}\,(t^A t^B) = \delta^{AB}$.

A.2 Link variables (= exponential of gauge fields)

For link variables $U^a{}_b, U^c{}_d, \cdots$ in the fundamental representation $(a, b, c, d = 1, \cdots, N)$, the integration over the group with the invariant Haar measure $[dU]$ gives

$$\int [dU]\; U^a{}_b U^{\dagger c}{}_d = \frac{1}{N}\delta^a{}_d \delta^c{}_b\,, \tag{208}$$

which can be derived from the symmetry under group transformation $U \to LUR$ [26]. Similarly, one can show [26]

$$\int [dU]\ U^a{}_b U^c{}_d U^{\dagger i}{}_j U^{\dagger k}{}_\ell$$
$$= \frac{1}{N^2-1}\Big[\delta^a{}_j\delta^i{}_b\delta^c{}_\ell\delta^k{}_d + \delta^a{}_\ell\delta^k{}_b\delta^c{}_j\delta^i{}_d - \frac{1}{N}\left(\delta^a{}_j\delta^k{}_b\delta^c{}_\ell\delta^i{}_d + \delta^a{}_\ell\delta^i{}_b\delta^c{}_j\delta^k{}_d\right)\Big]. \tag{209}$$

where not only a, b, c, d but also i, j, k, l are indices of the fundamental/anti-fundamental representation and thus run from 1 to N.

For generic representations $\mathbf{R}$ and $\mathbf{R}'$, eq. (208) is replaced with

$$\int [dU]\ U^a{}_b(\mathbf{R}) U^{\dagger c}{}_d(\mathbf{R}') = \frac{1}{d_\mathbf{R}}\delta_{\mathbf{R}\mathbf{R}'}\delta^a{}_d\delta^c{}_b\,, \tag{210}$$

where $d_\mathbf{R}$ is the dimension of the representation $\mathbf{R}$ ($d_\mathbf{R} = N$ for the fundamental and $d_\mathbf{R} = N^2-1$ for the adjoint) and $a, b, c, d = 1, \cdots, d_\mathbf{R}$ in this case. Furthermore eq. (209) becomes

$$\int [dU]\ U^a{}_b(\mathbf{R}) U^c{}_d(\mathbf{R}) U^{\dagger i}{}_j(\mathbf{R}) U^{\dagger k}{}_\ell(\mathbf{R})$$
$$= \frac{1}{d_\mathbf{R}^2-1}\Big[\delta^a{}_j\delta^i{}_b\delta^c{}_\ell\delta^k{}_d + \delta^a{}_\ell\delta^k{}_b\delta^c{}_j\delta^i{}_d - \frac{1}{d_\mathbf{R}}\left(\delta^a{}_j\delta^k{}_b\delta^c{}_\ell\delta^i{}_d + \delta^a{}_\ell\delta^i{}_b\delta^c{}_j\delta^k{}_d\right)\Big]. \tag{211}$$

A.3 Characters

From (210), we have useful formulas for characters;

$$\int [dU]\,\chi_\mathbf{R}(AU)\chi_{\mathbf{R}'}(U^\dagger B) = \frac{1}{d_\mathbf{R}}\delta_{\mathbf{R}\mathbf{R}'}\chi(AB)\,, \tag{212}$$

$$\int [dU]\,\chi_\mathbf{R}(AUBU^\dagger) = \frac{1}{d_\mathbf{R}}\chi_\mathbf{R}(A)\chi_\mathbf{R}(B)\,, \tag{213}$$

We will often use these formulas for fundamental/adjoint representation, where $d_\mathbf{R} = N, N^2-1$, respectively.

Especially from (212), setting A and B as unit matrix, we obtain

$$\int [dU]\,\chi_\mathbf{R}(U)\chi_{\mathbf{R}'}(U^\dagger) = \delta_{\mathbf{R}\mathbf{R}'}\,, \tag{214}$$

which is the orthogonality property of the characters (and more general expression than the $SU(2)$ case).

For Fundamental representation, from (209), we have

$$\int [dU]\chi_{\mathbf{F}}(AUBU^\dagger)\chi_{\mathbf{F}}(CUDU^\dagger) = \frac{1}{N^2-1}\left[\chi_{\mathbf{F}}(A)\chi_{\mathbf{F}}(B)\chi_{\mathbf{F}}(C)\chi_{\mathbf{F}}(D) + \chi_{\mathbf{F}}(AC)\chi_{\mathbf{F}}(BD) - \frac{1}{N}\left(\chi_{\mathbf{F}}(A)\chi_{\mathbf{F}}(C)\chi_{\mathbf{F}}(BD) + \chi_{\mathbf{F}}(B)\chi_{\mathbf{F}}(D)\chi_{\mathbf{F}}(AC)\right)\right], \tag{215}$$

$$\int [dU]\chi_{\mathbf{F}}(AUBU^\dagger CUDU^\dagger) = \frac{1}{N^2-1}\left[\chi_{\mathbf{F}}(B)\chi_{\mathbf{F}}(D)\chi_{\mathbf{F}}(AC) + \chi_{\mathbf{F}}(A)\chi_{\mathbf{F}}(C)\chi_{\mathbf{F}}(BD) - \frac{1}{N}\left(\chi_{\mathbf{F}}(AC)\chi_{\mathbf{F}}(BD) + \chi_{\mathbf{F}}(A)\chi_{\mathbf{F}}(B)\chi_{\mathbf{F}}(C)\chi_{\mathbf{F}}(D)\right)\right], \tag{216}$$

where all of the $\chi_{\mathbf{F}}$ are characters in Fundamental representation.

B Transfer matrix in harmonic oscillator

We start from the lattice action

$$S = -\frac{am}{2}\sum_i \left\{\left(\frac{x_{i+1}-x_i}{a}\right)^2 + \omega^2 x_i^2\right\}. \tag{217}$$

By using basis

$$\hat{x}\,|x\rangle = x\,|x\rangle\,, \tag{218}$$

$$\langle x'|x\rangle = \delta(x'-x), \tag{219}$$

$$1 = \int dx\,|x\rangle\,\langle x|\,, \tag{220}$$

we define transfer matrix $\hat{T}$ as

$$\hat{T} = \hat{T}_0\hat{T}_K\hat{T}_0, \tag{221}$$

$$\langle x'|\,\hat{T}\,|x\rangle = c\exp\left\{-\frac{am\omega^2}{4}x'^2 - \frac{m}{2a}(x'-x)^2 - \frac{am\omega^2}{4}x^2\right\}, \tag{222}$$

$$\langle x'|\,\hat{T}_0\,|x\rangle = \exp\left(-\frac{am\omega^2}{4}x^2\right)\delta(x'-x), \tag{223}$$

$$\langle x'|\,\hat{T}_K\,|x\rangle = c\exp\left\{-\frac{m}{2a}(x'-x)^2\right\}. \tag{224}$$

$\hat{T}_0$,and $\hat{T}_K$ correspond to potential part, and kinetic part. As the operator to $|x\rangle$, we have

$$\hat{T}\,|x\rangle = c\exp\left\{-\frac{am\omega^2}{4}x'^2 - \frac{m}{2a}(x'-x)^2 - \frac{am\omega^2}{4}x^2\right\}, \tag{225}$$

$$\hat{T}_0\,|x\rangle = \exp\left(-\frac{am\omega^2}{4}x^2\right)|x\rangle\,, \tag{226}$$

$$\hat{T}_K\,|x\rangle = c\int dx'\exp\left\{-\frac{m}{2a}(x'-x)^2\right\}|x'\rangle\,, \tag{227}$$

c is fixed by demanding $\lim_{a\to 0}\hat{T}_K = 1$ as $c = \sqrt{\frac{m}{2\pi a}}$.

In the path integral formalism, transfer matrix relates to Hamiltonian $\hat{H}$, as

$$\hat{T} = e^{-a\hat{H}}. \tag{228}$$

Then the energy E associated to eigenvalue of transfer matrix T is

$$E = -\lim_{a\to 0}\frac{1}{a}\log T. \tag{229}$$

This is linear part of a.

C Algebraic approach

In this paper we used extended Hilbert space formalism to analyze entanglement entropy in lattice gauge theories. Here we review another approach, algebraic approach[19, 21, 24]. In algebraic approach, entanglement entropy is defined by fixing sub-algebra, generated by a set of observable basis $\{\mathcal{O}_i\}$,

$$\rho(\{\mathcal{O}_i\}) = \sum_i \langle\mathcal{O}_i^{-1}\rangle\,\mathcal{O}_i, \tag{230}$$

where we labeled them by i. "Generation" means that the set is closed by them (this is just for consistency to observation's closure) and it should include the identity operator (it corresponds to observe nothing). If we chose all of observables in the theory, the density operator associating to that is usual density operator ρ. Unless above conditions are vi orated, we can make the sub-algebra arbitrary, but to impose spatial meaning to the entanglement entropy, we consider sub-algebras which is supported only by some sub-region.

There are two important things we should remark. Firstly, on the boundary between the sub-region and its complement, there occurs ambiguity whether each operator on the boundary is included to the sub-region or not. Secondary, for gauge theories the sub-algebra may have center operators, which commute with all of the other operators *with* gauge constraint equation.

If we fix the sub-algebra $\mathcal{A}_V$, and respectively the its center, we can have algebraic entanglement entropy as follows. At first we diagonalize the state by the operators of the center, each corresponding to superselection sector. After that we have block-diagonalized density operator

$$\rho(\mathcal{A}_V) = \begin{pmatrix} p_{\mathbf{k_1}}\rho_{\mathbf{k_1}} & & \\ & p_{\mathbf{k_2}}\rho_{\mathbf{k_2}} & \\ & & \dots \end{pmatrix}, \tag{231}$$

where each $\mathbf{k_i}$ is labeled for the superselection sector and each $\rho_{\mathbf{k_i}}$ is normalized as $\mathrm{Tr}\rho_{\mathbf{k_i}} = 1$. Then we have entanglement entropy as the von Neumann entropy of that,

$$S_{\mathrm{EE}} = -\sum_{\mathbf{k_i}} p_{\mathbf{k_i}} \log p_{\mathbf{k_i}} - \sum_{\mathbf{k_i}} p_{\mathbf{k_i}} \mathrm{Tr}\rho_{\mathbf{k_i}} \log \rho_{\mathbf{k_i}}. \tag{232}$$

The first term corresponds to classical correlation and the second term corresponds to quantum correlation, entanglement. This reminds us of Shannon part and Bell pair part, although it does not agree generally due to the ambiguity stated above. This is because observable basis $\{\mathcal{O}_i\}$ is composed by gauge invariant operators only. Even we fix the sub-algebra so that we have same sector structure as in extend Hilbert space, we lack color part for non-abelian case. This is because in algebraic approach we don't extend the Hilbert space and correlation of nonphysical d.o.f. is 'invisible'.

As stated above, the ambiguity of choosing sub-algebra causes ambiguity of center operators, and entanglement entropy. Although this happens generically, for $n = 1$ spatial dimension, the story is different. In the case, we have only single vertex (or link) boundary for each, and any spacial choice of sub-algebra does *not* result in the ambiguity of entanglement entropy. This is due to lacking of plaquette invariant operators also.

D Feynman diagrams for transfer matrix in the HPE

The hopping parameter expansions (HPE) for the transfer matrix can be evaluated efficiently using Feynman diagrams. We consider the $SU(N)$ gauge theory with fundamental scalar fields in 2-dimensional lattice space-time, where the horizontal direction corresponds to the spatial direction while the vertical direction corresponds to the Euclidean time direction, respectively.

The transfer matrix is defined in §4. As is clear from the expression, it represents a transition from a "current state" (which we denote as $\Psi^B = \{\phi, V\}$) to a "future state" (which we denote as $\Psi^A = \{\varphi, U\}$) by unit time shift. As mentioned, we take the temporal gauge, therefore all gauge link variables along the time direction are set to unity.

D.1 Diagrams

D.1.1 States

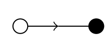
The gauge invariant "quark-antiquark" states $|n, m\rangle$ labeled by site positions (n, m) are defined as

$$\begin{aligned} \langle\Psi^A|n, n\rangle &= \varphi_n^\dagger \varphi_n \,, \\ \langle\Psi^A|n, m\rangle &= \varphi_n^\dagger U_{n\to m} \varphi_m \quad (n < m), \\ \langle\Psi^A|n, m\rangle &= \varphi_n^\dagger U_{m\to n}^\dagger \varphi_m \quad (n > m), \end{aligned} \tag{233}$$

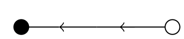
where

$$U_{n\to m} = U_{n,n+1} U_{n+1,n+2} \cdots U_{m-1,m} \,. \tag{234}$$

These states can be represented graphically as

$\langle\Psi^B|n, n\rangle =$  n , $\langle\Psi^B|n, n+1\rangle =$  n $n{+}1$ (235)

for the "current" states, and

$\langle\Psi^A|n, n\rangle =$  n , $\langle\Psi^A|n+2, n\rangle =$  n $n{+}2$ (236)

for the "future" states. Here a matter field is represented as a white or black circle for $\varphi^\dagger$ or φ respectively, while a (spatial) gauge field is a line with direction. "Current" fields are on the bottom and "future" fields are on the top such that the (Euclidean) time goes upward.

The ground state $|0\rangle$ is represented as an empty diagram.

D.1.2 Transfer matrix

The transfer matrix $\hat{T}$ is given by[23]

$$\langle \Psi^A | \hat{T} | \Psi^B \rangle \;=\; T_G(U,V) T_M(\varphi,\phi) T_0^2(\Psi^B). \tag{237}$$

Using hopping parameter K, we can represent $T_0(\Psi^B), T_M(U,V)$ as

$$\begin{aligned}
T_0^2(\Psi^B) &= \prod_{n=0}^{N_l-1} \left(\exp\left[-\phi_n^\dagger \phi_n + K \left\{ \phi_n^\dagger V_n \phi_{n+1} + \phi_{n+1}^\dagger V_n^\dagger \phi_n \right\} \right] \right) \\
&= \prod_{n=0}^{N_l-1} A_n \exp\left[K \left\{ \underset{n}{\circ}\!\!\rightarrow\!\!\underset{n+1}{\bullet} + \underset{n}{\bullet}\!\!\leftarrow\!\!\underset{n+1}{\circ} \right\} \right], \\
&= A \prod_{n=0}^{N_l-1} \sum_{h_n=0}^{\infty} \frac{K^{h_n}}{h_n!} \left(\underset{n}{\circ}\!\!\rightarrow\!\!\underset{n+1}{\bullet} + \underset{n}{\bullet}\!\!\leftarrow\!\!\underset{n+1}{\circ} \right)^{h_n},
\end{aligned} \tag{238}$$

$$\begin{aligned}
T_M(\varphi,\phi) &= \prod_{n=0}^{N_l-1} \exp\left[K \left(\varphi_n^\dagger \phi_n + \phi_n^\dagger \varphi_n \right) \right] \\
&= \prod_{n=0}^{N_l-1} \exp\left[K \left(\underset{n}{\begin{smallmatrix}\bullet\\ | \\ \circ\end{smallmatrix}} + \underset{n}{\begin{smallmatrix}\circ\\ | \\ \bullet\end{smallmatrix}} \right) \right] \\
&= \prod_{n=0}^{N_l-1} \sum_{v_n=0}^{\infty} \frac{K^{v_n}}{v_n!} \left(\underset{n}{\begin{smallmatrix}\bullet\\ | \\ \circ\end{smallmatrix}} + \underset{n}{\begin{smallmatrix}\circ\\ | \\ \bullet\end{smallmatrix}} \right)^{v_n},
\end{aligned} \tag{239}$$

In the last line of both equations, we expand them in the power series of K (HPE). Here we define $A_n = e^{-\phi_n^\dagger \phi_n}$ and $A = \prod_n A_n$, which give damping factors under the ϕ integral for normalization[24].

Ignoring the difference between a meson and its Hermitian conjugation, we have two types of diagrams, horizontal pairs and vertical pairs. Notice that vertical lines have no direction, due to the temporal gauge we take. Vertical lines are simply connecting color degrees of freedom on both ends in the (anti)fundamental representation.

[23] In this appendix, we use rescaled $\hat{T}$ which is used after §4.2. Therefore c_G does not appear here.

[24] We here ignore irrelevant constants such as powers of π's.

D.2 Evaluating the transfer matrix in HPE

In this subsection, we explicitly evaluate the action of the transfer matrix to some states. At the $\mathcal{O}(K^3)$ in the HPE, generic matrix elements are given by $\langle\Psi^A|\hat{T}|\alpha\rangle$ where $|\alpha\rangle = \{|0\rangle\,, |n,m\rangle\}$. In other words, the ground state $|0\rangle$ mix with at most a single meson state, and one can neglect multi-meson states at this order[25].

By inserting the completeness relation, we get

$$\langle\Psi^A|\hat{T}|\alpha\rangle = \int d\Psi^B \,\langle\Psi^A|\hat{T}|\Psi^B\rangle \,\langle\Psi^B|\alpha\rangle\,. \tag{240}$$

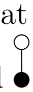
We thus get $\langle\Psi^A|\hat{T}|\alpha\rangle$ at $\mathcal{O}(K^s)$ order from the following rules,

1. Start from the diagram representing $\langle\Psi^B|\alpha\rangle$.
2. Expand T_0 and T_M in terms of K and pick up all allowed terms, *i.e.,* terms which satisfy $\sum_n(h_n + v_n) \leq s$, where h_n and v_n are numbers of horizontal and vertical pairs, respectively. Then act these terms on the above $\langle\Psi^B|\alpha\rangle$ (graphically putting corresponding diagrams), and integrate ϕ (=current matter fields) in the total diagrams.
3. Finally act T_G on the diagrams, and integrate V (=current link variables).

We have several comments for integrals of matters and link variables.

- The integration of ϕ can be done by using correlation functions for scalar fields such as

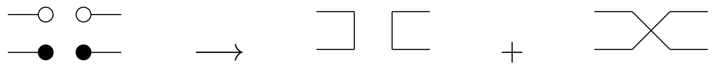
$$\begin{gathered}\langle(\phi_n^\dagger)_a\phi_m^b\rangle = \delta_{nm}\delta^b{}_a\,, \quad \langle\phi_n^a\phi_m^b\rangle = \langle(\phi_n^\dagger)_a(\phi_m^\dagger)_b\rangle = 0\,,\\ \langle(\phi_{n_a}^\dagger)_a\phi_{n_b}^b(\phi_{n_c}^\dagger)_c\phi_{n_d}^d\rangle = \delta^b{}_a\delta^d{}_c\delta_{n_a,n_b}\delta_{n_c,n_d} + \delta^d{}_a\delta^b{}_c\delta_{n_a,n_d}\delta_{n_c,n_b}\,,\end{gathered} \tag{241}$$

  where $a,b,c,d = 1,2,\dots N$ are color index[26]. Non-zero contributions can be obtained if and only if the integrand contains same number of ○ and ● at each site at the bottom ("current"). In addition we see that the number of  and  must be globally equal and the total number of vertical pairs must be even.

[25]The multi-meson states are important once we take into account higher order corrections in the HPE.

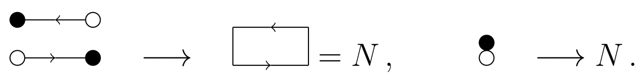
[26]We take the irrelevant multiplicative constant of T_0 to normalize the first equation

- In the diagrammatic representation, the integration by ϕ at the bottom ("current") connects a line attaching to a white circle with a line attaching to a black circle at the same site, and then remove these circles. For example,



(242)

If a closed loop or a shrunk point without links appear after the integral, a factor N must be attached as



(243)

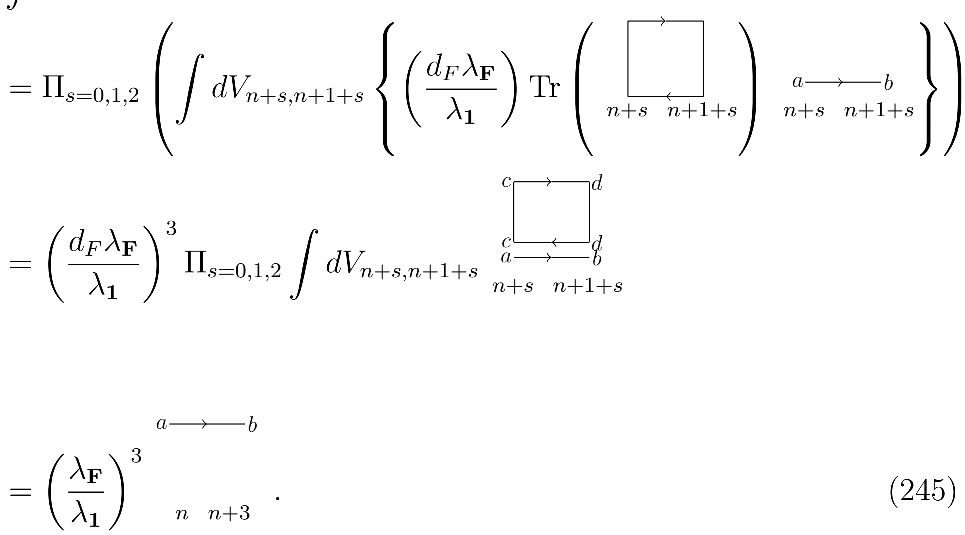
We can explicitly check the above rules using (241).

- As explained in Section §4, T_G can be expanded in terms of characters as

$$T_G(U,V) = \prod_{n=0}^{N_l-1} \sum_{\mathbf{R}} d_{\mathbf{R}} \frac{\lambda_{\mathbf{R}}(\beta)}{\lambda_{\mathbf{1}}(\beta)} \chi_{\mathbf{R}}(U_{n,1} V_{n,1}^{\dagger}) \,. \tag{244}$$

With the orthogonality condition (214), one can easily perform the gauge field integration on each link. For example, if $T_G(U,V)$ acts on gauge fields $(V_{n,n+3})^a{}_b$ and V's are integrated, we can represent this procedure graphically as

$$\int (\Pi_{n=1,\cdots,N_l} dV_{n,n+1})\, T_G(U,V)\, (V_{n,n+3})^a{}_b$$



(245)

We see in this case that T_G plays a role of uplifting gauge fields with the factor $\lambda_{\mathbf{F}}/\lambda_{\mathbf{1}}$ for each link.

- More generally, acting on links which belong to the irreducible representation $\mathbf{R}$, T_G uplifts gauge fields with the factor $\lambda_R/\lambda_{\mathbf{1}}$.

- For more complicated links which do not belong to one irreducible representation such as a product of links in some representations, we should decompose them into irreducible representations before the integration. For example, $V^a{}_b V^{\dagger d}{}_c$, which belongs to fundamental $\times$ anti-fundamental representations, can be decomposed into singlet and adjoint part as

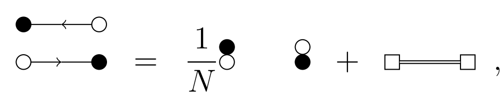
$$V^a{}_b V^{\dagger d}{}_c = \left(\frac{1}{N}\delta^a{}_c \delta^d{}_b\right) + \left(V^a{}_b V^{\dagger d}{}_c - \frac{1}{N}\delta^a{}_c \delta^d{}_b\right). \tag{246}$$

  We can visualize this as

$$\begin{matrix} c \longleftarrow d \\ a \longrightarrow b \end{matrix} \;=\; \frac{1}{N}\delta^a{}_c \quad \delta^d{}_b + (a,c) \Longleftrightarrow (b,d)\,, \tag{247}$$

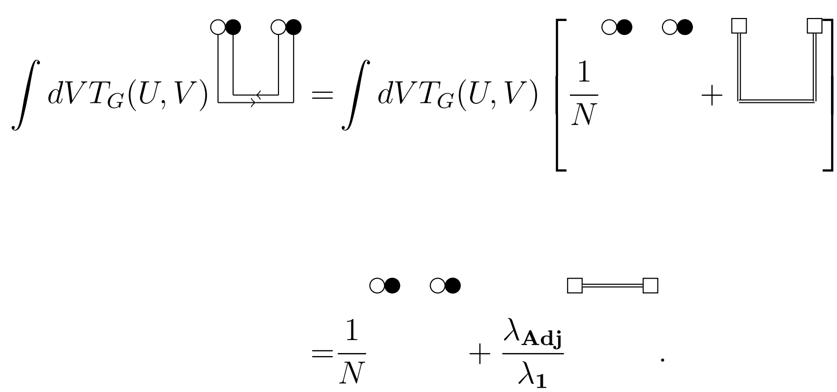
  where the doubled line without direction represents the gauge field in the adjoint representation. Each pair (a, c) or (b, d) correspond to an index of the adjoint representation of the gauge field, whose dimension is $N^2 - 1$.

- With matter fields, we can represent the decomposition of $\mathbf{F} \times \bar{\mathbf{F}}$ as:



(248)

  where squares corresponds to the adjoint parts of the matter field. This leads to the following relation we will use later.



(249)

D.3 Some examples

D.3.1 $\langle\Psi^A|\hat{T}|n,n\rangle$ at $\mathcal{O}(K^3)$

We derive the explicit form of $\langle\Psi^A|\hat{T}|n,n\rangle$ at $\mathcal{O}(K^3)$. We start from the diagram ○●.

At K^0 order we only have

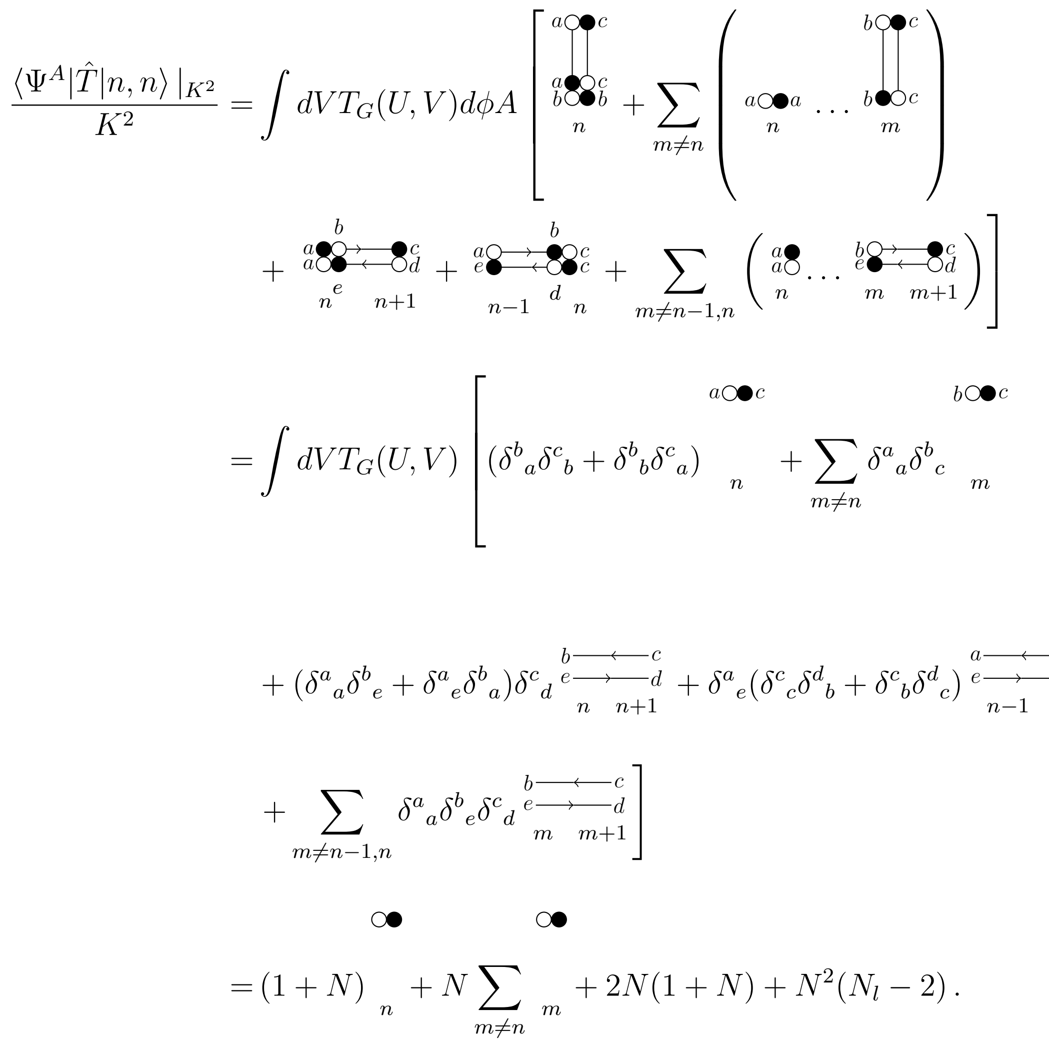
$$\langle\Psi^A|\hat{T}|n,n\rangle\,|_{K^0} = \int d\Psi^B A \underset{n}{\overset{aa}{\circ\!\bullet}} = \delta^a{}_a = N, \tag{250}$$

where we denote color indices explicitly. We thus obtain

$$\hat{T}\,|n,n\rangle\,|_{K^0} = N\,|0\rangle\,. \tag{251}$$

At K^1, from the comment we gave before, the acting pair must be horizontal. However one horizontal pair can not make even number matter fields on each site, so there are no contribution at this order.

At K^2 order, next, we can consider two vertical pairs or two horizontal pairs. In both cases two pairs must share the same link as



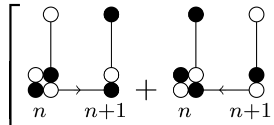
(252)

We finally obtain

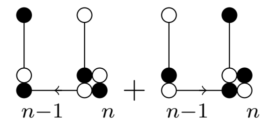
$$\hat{T}\,|n,n\rangle\,|_{K^2} = K^2 N(NN_l+2)\,|0\rangle + K^2\,|n,n\rangle + K^2 N \sum_m |m,m\rangle\,. \qquad (253)$$

At $\mathcal{O}(K^3)$ order, there are three horizontal or vertical pairs. Only the "U" shape diagram, consisting of two vertical and one horizontal pairs, are allowed, since other cases lead to an odd number of scalar fields on some site. Employing rules (242) and (243) and taking care for the direction, we have

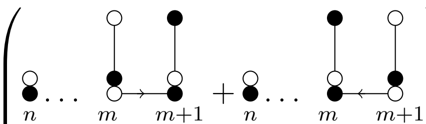
$$\frac{\langle \Psi^A | \hat{T} | n, n \rangle \,|_{K^3}}{K^3} = \int dV T_G(U,V) d\phi A \Bigg[$$



$$+$$

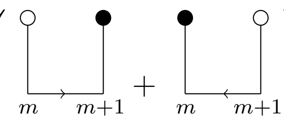


$$+ \sum_{m \neq n, n-1} \Bigg($$

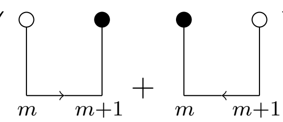


$$\Bigg)\Bigg]$$

$$= \int dV T_G(U,V) \Bigg[(N+1) \sum_{m=n-1,n} \Bigg($$

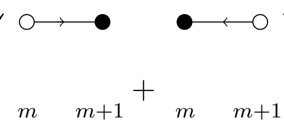


$$\Bigg) + N \sum_{m \neq n-1, n} \Bigg($$



$$\Bigg)\Bigg]$$

$$= \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right) \sum_m (N + \delta_{m,n-1} + \delta_{m,n}) \Bigg($$



$$\Bigg) . \qquad (254)$$

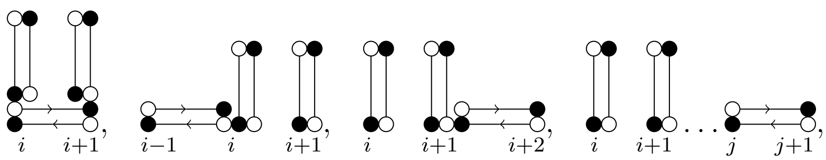
As a result, we obtain

$$\begin{aligned} \hat{T}\,|n,n\rangle\,|_{K^3} &= K^3 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right) (|n, n+1\rangle + |n+1, n\rangle + |n-1, n\rangle + |n, n-1\rangle) \\ &\quad + K^3 N \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right) \sum_m (|m, m+1\rangle + |m, m-1\rangle) \, . \qquad (255) \end{aligned}$$

D.3.2 The detail for the calculation of (172)

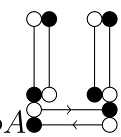
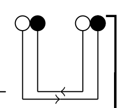
Here we show the derivation of (172), coefficient of $|i,i\rangle\,|i+1,i+1\rangle$ term at K^6 order. All we have to consider is $\hat{T}_6\,|G_0^+\rangle = \hat{T}_6\,|0\rangle$ and $\hat{T}_4\,|G_2^+\rangle = \sum_n \hat{T}_4\,|n,n\rangle$.

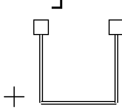
First let us consider $\hat{T}_6\,|0\rangle$. We have six meson-like pairs in $\hat{T}$, which act on $|0\rangle$. The four of them must be devoted to construct the future state $|i,i\rangle\,|i+1,i+1\rangle$ and the other two must be conjugated with each other in the horizontal direction. So we have following patterns of configurations to integrate:

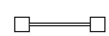


(256)

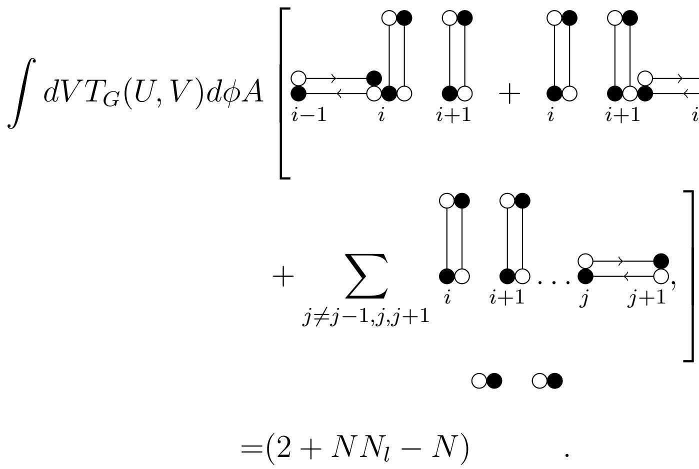
where $j \neq i-1, i, i+1$. For the first configuration, the integration can be done as

$$\int dV T_G(U,V) d\phi A$$  $$= \int dV T_G(U,V) \Bigg[(N+2)$$  $$+$$  $$\Bigg]$$

$$= \int dV T_G(U,V) \Bigg[(N+2+\frac{1}{N})$$  $$+$$  $$\Bigg]$$

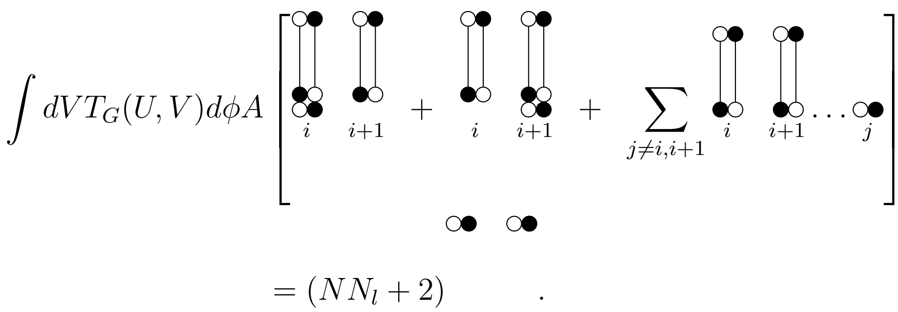
$$= (N+2+\frac{1}{N})$$  $$+ \frac{\lambda_{adj}}{\lambda_{\mathbf{1}}}$$  , (257)

where we use (249). For the other configurations, we have



(258)

For $\hat{T}_4 \sum_n |n,n\rangle$, all of pairs in $\hat{T}$ should be used to make $|i,i\rangle\,|i+1\rangle\,|i+1\rangle$. So we have



(259)

Combining all results, the coefficient of $|i,i\rangle\,|i+1,i+1\rangle$ becomes $2NN_l + 6 + \dfrac{1}{N}$.

E $\mathcal{O}(K^2)$ eigenstates and eigenvalues of $\hat{T}$

In this appendix, we derive eigenvalues and their eigenfunctions of the transfer matrix $\hat{T}$ at $O(K^2)$, where $|0\rangle$ and $|n,m\rangle$ with $|n-m| \leq 2$ mix with each other. First, we classify these eigenstates depending on the value of f_0 (zero or nonzero) as

$$
\begin{aligned}
|G\rangle_K \equiv f_0\,|0\rangle &+ \sum_n a_n\,|n,n\rangle + \sum_n b_n\,|n,n+1\rangle + \sum_n c_n\,|n,n-1\rangle \\
&+ \sum_n d_n\,|n,n+2\rangle + \sum_n e_n\,|n,n-2\rangle\ , \\
|E\rangle_K \equiv &\sum_n a_n\,|n,n\rangle + \sum_n b_n\,|n,n+1\rangle + \sum_n c_n\,|n,n-1\rangle \\
&+ \sum_n d_n\,|n,n+2\rangle + \sum_n e_n\,|n,n-2\rangle\,, \qquad (260)
\end{aligned}
$$

which correspond to $f_0 \neq 0$ case and $f_0 = 0$ case, respectively. Here $|G\rangle_K$'s should include $|0\rangle$ while $|E\rangle_K$'s denote the complement of $|G\rangle_K$'s.[27]

All relevant eigenvalues and eigenfunctions are obtained as follows.

- States $|G^{\pm}\rangle_K$ with eigenvalues $G^{\pm}_K$ are given by

$$
|G^{\pm}\rangle_K := |0\rangle + \sum_n a^{\pm}_n\,|n,n\rangle\,, \quad \text{where} \quad a^{\pm}_n = \frac{K^2}{G^{\pm}_K - (1+NN_\ell)K^2}\,, \qquad (261)
$$

$$
\begin{aligned}
G^{\pm}_K = &\ \frac{1}{2}\{1+K^2(1+2NN_\ell)\} \\
&\pm \frac{1}{2}\sqrt{1-2(1-2NN_\ell)K^2+\{1+4N(NN_\ell+2)N_\ell\}K^4}. \qquad (262)
\end{aligned}
$$

- State $|G^{bc}\rangle_K$ with the eigenvalue G^{bc}_K is given by

$$
\begin{aligned}
|G^{bc}\rangle_K := K\,|0\rangle &+ \frac{K}{\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} - (1+NN_\ell)} \sum_n |n,n\rangle \\
&+ \sum_n (b^G_n\,|n,n+1\rangle + c^G_n\,|n,n-1\rangle)\,, \qquad (263)
\end{aligned}
$$

$$
G^{bc}_K = K^2 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right), \qquad (264)
$$

where coefficients b^G_n and c^G_n must satisfy

[27] G for $|G\rangle_K$ means that it contains the strong coupling ground state $|0\rangle$, while E for $|E\rangle_K$ represents the lattice excited states. Their subscript K denotes that the state depends on K.

$$\sum_n (b_n^G + c_n^G) = -\frac{1}{N} - \frac{N_\ell}{\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} - (1 + NN_\ell)} + K^2 \left[\frac{1}{N} \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \right) - N_\ell - (NN_\ell + 2) \frac{N_\ell}{\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} - (1 + NN_\ell)} \right] . \tag{265}$$

- State $|G^{de}\rangle_K$ with the eigenvalue G_K^{de} is given by

$$|G^{de}\rangle_K := K^2 |0\rangle + \frac{K^2}{\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 - (1 + NN_\ell)} \sum_n |n, n\rangle + \sum_n (d_n^G |n, n+2\rangle + e_n^G |n, n-2\rangle) \tag{266}$$

$$G_K^{de} = K^2 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \right)^2 . \tag{267}$$

where coefficients d_n^G and e_n^G must satisfy

$$\sum_n (d_n^G + e_n^G) = -\frac{1}{N} - \frac{N_\ell}{\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 - (1 + NN_\ell)} + K^2 \left[\frac{1}{N} \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \right)^2 - N_\ell - (NN_\ell + 2) \frac{N_\ell}{\left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 - (1 + NN_\ell)} \right] . \tag{268}$$

- State $|E^a\rangle_K$ with the eigenvalue E_K^a is given by

$$|E^a\rangle_K := \sum_n a_n^E |n, n\rangle \quad \text{where} \quad \sum_n a_n^E = 0 \, , \tag{269}$$

$$E_K^a = K^2 \, , \tag{270}$$

- State $|E^{bc}\rangle_K$, which gives the eigenvalues E_K^{bc}, defined as

$$|E^{bc}\rangle_K := \sum_n (b_n^E |n, n+1\rangle + c_n^E |n, n-1\rangle) \, , \text{ where } \sum_n (b_n^E + c_n^E) = 0 \, , \tag{271}$$

$$E_K^{bc} = K^2 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}} \right) . \tag{272}$$

- State $|E^{de}\rangle_K$ with the eigenvalue E^{de}_K is given by

$$|E^{de}\rangle_K := \sum_n (d^Y_n |n, n+2\rangle + e^Y_n |n, n-2\rangle) \,, \text{ where } \sum_n (d^Y_n + e^Y_n) = 0 \,, \tag{273}$$

$$E^{de}_K = K^2 \left(\frac{\lambda_{\mathbf{F}}}{\lambda_{\mathbf{1}}}\right)^2 \,. \tag{274}$$

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