Tsushima, Y.
Osaka J. Math.
27 (1990), 947-962

ON CERTAIN PROJECTIVE MODULES FOR FINITE GROUPS OF LIE TYPE

Dedicated to Professor Tosiro Tsuzuku on his 60th birthday

YUKIO TSUSHIMA

(Received October 31, 1989)

1. Introduction

Let q be a power of a prime number p, F_q a finite field with q elements and K an algebraic closure of F_q. Let G_0 be a classical linear group written in $GL(n, q)$; we are particularly interested in $SL(l+1, q)$, $Sp(2l, q)$, $\Omega(2l+1, q)$, $\Omega_{\pm 1}(2l, q)$ and $SU(l+1, q)$. Let $V=K^n$, the vector space of column vectors of size n over K, and let St be the Steinberg module for G_0. In [8] Lusztig showed that $St\otimes V$ is a principal indecomposable module for $G_0=GL(n, q)$, provided $q>2$. In this paper we shall prove this fact in all the classical linear groups, with the treatment of the case of $q=2$. Our methods rely heavily on Steinberg's tensor product theorem on the representation of semisimple algebraic groups over K. So we shall begin our arguments with a review of some standard facts about (universal) Chevalley groups over K.

For modules M, N over a ring A, we write $N\langle\oplus M$ if N is isomorphic to a direct summand of M, and $N\ll M$ if N is isomorphic to an irreducible constituen of M. We abbreviate $\otimes_K$ to $\otimes$ and denote by e_j the unit vector of K^n with 1 at the j-th entry. We refer to Borel [1], Carter [3] [4], Steinberg [10] [11] and Suzuki [12] for the general theories of Chevalley groups and their modular representations.

We mention here that our results in the cases of $SL(l+1, q)$ and $Sp(2l, q)$ were already obtained by Okuyama [9] by different methods.

2. Background materials

Let $\mathfrak{g}$ be a simple Lie algebra over the complex field $\boldsymbol{C}$ of type A_l, B_l, C_l or D_l, so that $\mathfrak{g}\subset\mathfrak{gl}(n, \boldsymbol{C})$ and $n=l+1$, $2l+1$, $2l$'s according to the order of the occurrence of the above types. Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$, Φ the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ a simple root systme of Φ, Φ^+ the set of positive roots of Φ with respect to Π, and W_Π the Weyl group of Φ. More generally, for $J\subset\Pi$, we let Φ_J be the root system with basis J and

W_J be the Weyl group of Φ_J. There is a unique $w_0 \in W_\Pi$ such that $w_0\Pi = -\Pi$. Let h_α be the coroot of $\alpha \in \Phi$ and $\{e_\alpha, h_\beta;\ \alpha \in \Phi, \beta \in \Pi\}$ be a Chevalley basis of $\mathfrak{g}$. For simplicity we write h_i for h_{α_i}.

Define $\lambda_i \in \mathfrak{h}^* = \mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{h}, \boldsymbol{C})$ by

$$\lambda_i(\mathrm{diag}(t_1, \cdots, t_n)) = t_i \qquad (1 \leq i \leq l),$$

where we write $\mathrm{diag}(t_1, \cdots, t_{2l})$ for $\mathrm{diag}(0, t_1, \cdots, t_{2l})$ in case $\mathfrak{g}$ is of type B_l. Since each λ_i is a weight of $\mathfrak{h}$ in the $\mathfrak{g}$-module $\boldsymbol{C}^n$, it takes integral values on all h_α. Let ω_i be the fundamental dominant weight corresponding to α_i, i.e., $\omega_i(h_j) = \delta_{ij}$ $(1 \leq i, j \leq l)$, and let $X = \sum_{i=1}^{l} \boldsymbol{Z}\omega_i$. In X we set $X^+ = \{\sum_i m_i\omega_i;\ m_i \geq 0\}$ and $X_q = \{\sum_i m_i\omega_i;\ 0 \leq m_i \leq q-1\}$.

Recall that $SL(l+1, q)$, $\Omega(2l+1, q)$, $Sp(2l, q)$ and $\Omega_{+1}(2l, q)$ are the Chevalley groups over F_q associated to the embedding $\mathfrak{g} \rightarrow \mathfrak{gl}(n, \boldsymbol{C})$. In order to give a unified treatment of them with the Steinberg groups $SU(l+1, q)$ and $\Omega_{-1}(2l, q)$ of types ${}^2A_l$ and ${}^2D_l$ respectively, let us consider a universal Chevalley group over K:

$$\tilde{G} = \langle x_\alpha(t);\ \alpha \in \Phi, t \in K \rangle .$$

We know that $\tilde{G}$ is a simply connected, semisimple algebraic group defined over F_p, which has $\tilde{H} = \langle h_\alpha(t);\ \alpha \in \Phi, t \in K^\times = K/\{0\} \rangle = \langle h_i(t);\ t \in K^\times, 1 \leq i \leq l \rangle$ and $B = \langle \tilde{H}, x_\alpha(t);\ \alpha \in \Phi, t \in K \rangle$ as a maximal torus and a Borel subgroup respectively, where $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ and $h_\alpha(t) = w_\alpha(t)w_\alpha(-1)$. Also, we have that $N_{\tilde{G}}(\tilde{H}) = \langle w_\alpha(t);\ \alpha \in \Phi, t \in K^\times \rangle$ with factor group modulo $\tilde{H}$ isomorphic to W_Π via $w_\alpha(1) \mapsto w_\alpha$, where w_α is the reflection in the hyperplane orthogonal to α.

Let $X(\tilde{H})$ be the group of rational characters of $\tilde{H}$. For $\lambda \in X$, we define $\tilde{\lambda} \in X(\tilde{H})$ by $\tilde{\lambda}(h_\alpha(t)) = t^{\lambda(h_\alpha)}$. Then there is an isomorphism $X \simeq X(\tilde{H})$ sending λ onto $\tilde{\lambda}$, which is compatible with the actions of the Weyl group W_Π on both sides. The set of weights of $\tilde{H}$ in $V = K^n$ is given by

$\{\tilde{\lambda}_i;\ 1 \leq i \leq l+1\}$ if $\mathfrak{g}$ is of type A_l;
$\{\tilde{0}, \pm\tilde{\lambda}_i;\ 1 \leq i \leq l\}$ if $\mathfrak{g}$ is of type B_l;
$\{\pm\tilde{\lambda}_i;\ 1 \leq i \leq l\}$ if $\mathfrak{g}$ is of type C_l or D_l.

The weight λ_1 coincides with the first fundamental dominant weight ω_1 in each case.

Throughout we fix a Frobenius endomorphism σ of $\tilde{G}$ such that

$$\sigma(x_\alpha(t)) = x_{\tau(\alpha)}(\varepsilon_\alpha t^q) ,$$

where either τ is the identity and all $\varepsilon_\alpha = 1$, or else τ is the symmetry of order 2 on the Dynkin diagram of type A_l or D_l and $\varepsilon_\alpha = \pm 1$. Let $G=\tilde{G}^\sigma$, the finite subgroup of τ-stable points of $\tilde{G}$, and also $H=\tilde{H}^\sigma$, $B=\tilde{B}^\sigma$. Let G_0 be one of the classlcal linear groups $SL(l+1, q)$, $\Omega(2l+1, q)$, $Sp(2l, q)$, $\Omega_{\pm 1}(2l, q)$ and $SU(l+1, q)$. There is a natural epimorphism $\psi: G \to G_0$, whose kernel is a central subgroup of G (provided, of course, that the underlying Lie algebras of them are the same). In this sense we often regard a G_0-module as a G-module.

For each $\lambda \in X^+$, there is a simple $\tilde{G}$-module $L(\lambda)$ with highest weight $\tilde{\lambda}$, which means that $\tilde{\lambda}$ is a weight of $\tilde{H}$ in $L(\lambda)|_{\tilde{H}}$ (the restriction to $\tilde{H}$) and that all other weights are of the form $\tilde{\lambda} - \sum_i m_i \tilde{\alpha}_i$ with non-negative integers m_i. The set $\{L(\lambda); \lambda \in X^+\}$ provides a complete set of representatives of the underlying $\tilde{G}$-modules for the non-equivalent irreducible rational representations of $\tilde{G}$ over K. Furthermore the set $\{L(\lambda)' = L(\lambda)|_G; \lambda \in X_q\}$ gives a complete set of representatives of non-isomorphic simple G-modules. The canonical module K^n for G_0 is, when considered as a G-module, isomorphic to $L(\omega_1)'$ and the Steinberg module to $L((q-1)\rho)'$, where $\rho = \sum_{i=1}^{l} \omega_i$.

REMARK. In case that $\mathfrak{g}$ is of type B_l and $p=2$, we have

$$G_0 = \Omega(2l+1, q) = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & Sp(2l, q) \end{array}\right] \simeq Sp(2l, q)$$

and $V = K^{2l+1}$ decomposes into $V = K \oplus K^{2l}$ in a natural manner. Hence the canonical module for $\Omega(2l+1, q)$ in this case has been and will be understood to be the one K^{2l} for $Sp(2l, q)$.

For $\lambda, \mu \in X$ we write $\lambda \leq \mu$, if $\mu - \lambda$ is a non-negative integral linear combination of the simple roots α_i. Also, following Jantzen, we write $\lambda \leq_Q \mu$, if $\mu - \lambda$ is a non-negative rational linear combination of the simple roots α_i. We remark that given $\mu \in X^+$, there are only a finite number of $\lambda \in X^+$ such that $\lambda \leq_Q \mu$. In particular, the induction over $\leq_Q$ may be carried out. The following well-known fact will be used throughout this paper.

Lemma 1. *Let* $\lambda, \mu, \gamma \in X^+$.

(1) *The K-dual $L(\lambda)^*$ of $L(\lambda)$ is isomorphic to $L(-w_0\lambda)$.*

(2) *If $L(\gamma) \ll L(\lambda) \otimes L(\mu)$, then $\gamma \leq \lambda + \mu$.*

(3) *$L(\lambda+\mu)$ appears as a constituent of $L(\lambda) \otimes L(\mu)$ with multiplicity one. If $\lambda + \mu \in X$, then the same is true as G-modules.*

For $\lambda \in X_q$, let $\lambda^0 = (q-1)\rho + w_0\lambda \in X_q$ and let $U(\lambda)$ be a projective cover of the simple G-module $L(\lambda)'$.

The next lemma is noted by Jantzen [6].

Lemma 2. *Suppose that G is a universal Chavalley group over F_q. For*

$\lambda \in X_q$, *we have*

$$St \otimes L(\lambda)' \simeq U(\lambda^0) \oplus \bigoplus_{\mu} m(\lambda, \mu) U(\mu),$$

where the sum is taken over those $\mu \in X_q$ *such that* $\lambda^0 <_Q \mu$, *and* $m(\lambda, \mu)$ *denotes the multiplicity of* $U(\mu)$, *so that*

$$\begin{aligned} m(\lambda, \mu) &= \dim \mathrm{Hom}_{KG}(L(\mu)', St \otimes L(\lambda)') \\ &= \dim \mathrm{Hom}_{KG}(L(\mu)' \otimes L(-w_0 \lambda)', St). \end{aligned}$$

This result is valid for the universal Steinberg group $\Omega_{-1}(2l, q)$ ($l \geq 4$), too. In fact, a slight modification of Jantzen's argument covers the proof of this case. To see this, it is sufficient, by Lemma 1, to show the following lemma.

Lemma 3. *Let* G *be a universal Chevalley group over* F_q *or a universal Steinberg group over* F_{q^2} *of type* ${}^2D_l$ ($l \geq 4$). *Let* $\gamma \in X_q$ *and* $\lambda, \mu \in X^+$. *Then, if* $L(\gamma)' \ll L(\lambda)' \otimes L(\mu)'$, *we have* $\gamma \leq_Q \lambda + \mu$.

Proof. We argue by induction over $\leq_Q$. There is $\nu \in X^+$ such that $L(\nu) \ll L(\lambda) \otimes L(\mu)$ and that $L(\gamma)' \ll L(\nu)'$. If $\nu \in X_q$, then $\gamma = \nu \leq \lambda + \mu$. Suppose that $\nu \notin X_q$, and write $\nu = \nu_0 + q\nu_1$ with $\nu_0 \in X_q$, $\nu_1 \in X^+$. Since $L(q\nu_1) \simeq L(\tau\nu_1) \circ \sigma$, we get by Steinberg's tensor product theorem (cf. Steinberg [11] Theorem 13.1)

$$L(\nu) \simeq L(\nu_0) \otimes L(\tau\nu_1) \circ \sigma$$

and since σ is trivial on $G = \tilde{G}^\sigma$, we have

$$L(\nu)' \simeq L(\nu_0)' \otimes L(\tau\nu_1)' \gg L(\gamma)'.$$

We claim that $\nu_0 + \tau\nu_1 <_Q \nu$, which is trivial if τ is the identity. Suppose that τ is the symmetry of order 2 on the Dynkin diagram of type D_l ($l \geq 4$), so $\tau(i) = i$ ($1 \leq i \leq l-2$), $\tau(l-1) = l$ and $\tau(l) = l-1$. Write $\nu_1 = \sum_i b_i \omega_i$. Then

$$\nu - (\nu_0 + \tau\nu_1) = q\nu_1 - \tau\nu_1 = \sum_{i=1}^{l-2} (q-1) b_i \omega_i + (q b_{l-1} - b_l)\omega_{l-1} + (q b_l - b_{l-1})\omega_l.$$

Expressing ω_{l-1} and ω_l as linear combinations of $\alpha_1, \cdots, \alpha_l$ (cf. Bourbaki [2]), we find easily that

$$(q b_{l-1} - b_l)\omega_{l-1} + (q b_l - b_{l-1})\omega_l \ {}_Q\!\geq 0,$$

provided $l \geq 4$. This proves the claim and we have that $\nu_0 + \tau\nu_1 <_Q \lambda + \mu$. Then by the inductive hypothesis we get that $\gamma \leq_Q \nu_0 + \tau\nu <_Q \lambda + \mu$, completing the proof of the lemma.

The above lemma (hence Lemma 2) still holds for the universal Steinberg groups of type ${}^3D_4$ and ${}^2E_6$, but is false for $SU(l+1, q)$. For instance, we have

$L((q-1)\omega_1)\otimes L(\omega_1)\gg L(q\omega_1)=L(\omega_l)\circ\sigma$ and hence $L((q-1)\omega_1)'\otimes L(\omega_1)'\gg L(\omega_l)'$, But it is not generally true that $\omega_l\leq_Q q\omega_1$. In this case, however, we have an alternative version, which is weaker than the ordering $\leq_Q$, but sufficient for our purpose. Namely we have

Lemma 4. *Suppose $G_0=SU(l+1, q)$. For $\lambda=\sum a_i\omega_i\in X$, let $|\lambda|=\sum_i a_i$.*

(1) *If λ, $\mu\in X$ and $\lambda\leq\mu$, then $|\lambda|\leq|\mu|$.*

(2) *Let $\gamma\in X_q$ and λ, $\mu\in X^+$.*

(a) *If $L(\gamma)'\ll L(\lambda)'\otimes L(\mu)'$, then $|\gamma|\leq|\lambda+\mu|$.*

(b) *If $L(\gamma)'\ll L(\lambda)'$ and $|\gamma|=|\lambda|$, then $\gamma=\lambda$.*

Proof. (1) It suffices to show that if $\lambda\geq 0$, then $|\lambda|\geq 0$ (this is not necessarily true for other types of Lie algebras). We write $\lambda=\sum_i a_i\omega_i$ with $a_i\in\boldsymbol{Z}$. The coefficients of α_1 and α_l in λ are gievn by

$$1/l+1(la_1+(l-1)a_2+\cdots+a_l)$$

and

$$1/l+1(a_1+2a_2+\cdots+la_l)$$

respectively. Both are non-negative integers by assumption, so that by adding them, we get $|\lambda|=\sum_i a_i\geq 0$.

Part (a) of (2) can be proved similarly as Lemma 3 via induction on $|\lambda+\mu|$, using (1). For the proof of (b), write $\lambda=\lambda_0+q\lambda_1$ with $\lambda_0\in X_q$ and $\lambda_1\in X^+$. Then $L(\lambda)'\simeq L(\lambda_0)'\otimes L(\tau\lambda_1)'\gg L(\gamma)'$, and so $|\gamma|\leq|\lambda_0+\tau\lambda_1|\leq|\lambda_0+q\lambda_1|=|\lambda|$. Hence $|\lambda_0+\tau\lambda_1|=|\lambda_0+q\lambda_1|$, and thus $\lambda_1=0$. Therefore $\lambda=\lambda_0\in X_q$, whence $\lambda=\gamma$.

To apply Lemma 2 to $St\otimes V$, we need the following fact.

Lemma 5. *Let $\mathfrak{g}$ be as above.*

(1) *$\delta=(q-1)\rho$ is the only weight in X_q such that $\omega_1^0<_Q\delta$, except for type B_2, in which case ω_2^0 also satisfies that $\omega_1^0<_Q\omega_2^0$.*

(2) *If $\mathfrak{g}$ is of type A_l, then (1) is true for all ω_k in place of ω_1 ($1\leq k\leq l$).*

Proof. Although we have to distinguish the cases, the proof is easy. Suppose that $\mu=\sum_i c_i\omega_i\in X_q$ satisfies $\omega_1^0<_Q\mu$. If $\mathfrak{g}$ is of type other than A_l, then $w_0\omega_1=-\omega_1$, so that

$$\mu-\omega_1^0 = (c_1-(q-2))\omega_1+\sum_{i\geq 2}(c_i-(q-1))\omega_i\,{}_Q>0\,.$$

Since $0\leq c_i\leq q-1$, we find readily $c_1=q-1$. Expressing each ω_i as a (non-negative) rational linear combinations of the simple roots and looking at the coefficients of α_{l-1} and α_l, we find easily $c_i=q-1$ for all i, except for the case of type B_2. In that case there is one exception that $\omega_1^0 <_Q \omega_2^0$.

Now, let $\mathfrak{g}$ is of type A_l. Then $w_0\omega_k = -\omega_{l+1-k}$ for all $k \leq l$ and so

$$\mu - \omega_k^0 = (c_{l+1-k} - (q-2))\omega_{l+1-k} + \sum_{l \neq l+1-k} (c_i - (q-1))\omega_i \;{}_Q\!> 0 ,$$

whence we have $c_{l+1-k} = q-1$. Suppose that $c_i - (q-1) < 0$ for some i. If $k > l-i+1$, then $i/l+1 > l+1-k/l+1$. Since $i/l+1$ is the coefficient of α_l in ω_i, this implies that the coefficient of α_l, in $\mu - \omega_k^0$ is negative, contradicting the assumption. If, on the other hand, $k < l-i+1$, then we find that the coefficient of α_1 in $\mu - \omega_k^0$ is negative again, contradicting the assumption. Therefore we have $c_i = q-1$ for all i. This completes the proof of the lemma.

The last preliminary lemma is the following.

Lemma 6. *Let G_0 be $SL(l+1, q)$, $\Omega(2l+1, q)$, $Sp(2l, q)$, $\Omega_{\pm 1}(2l, q)$ or $SU(l+1, q)$. Then we have*

(1) $St \otimes V \simeq U(\omega_1^0) \oplus m_1 St \quad (m_1 \geq 0)$.

(2) *If $G_0 = SL(l+1, q)$ or $SU(l+1, q)$, then for all $k \leq l$*
$St \otimes L(\omega_k)' \simeq U(\omega_k^0) \oplus m_k St \quad (m_k \geq 0)$.

Proof. (1) By Lemmas 2 and 5, we need only prove the assertion in the cae case of $G_0 = \Omega(5, q)$ with odd prime power q. We want to show that $L(\omega_2^0)'$ is not a constituent of $St \otimes V$. Suppose the contrary. Then there exists $\lambda = a_1\omega_1 + a_2\omega_2 \in X^+$ such that $L(\lambda) \ll St \otimes L(\omega_1)$ and that $L(\omega_2^0)' \ll L(\lambda)'$. In particular we have $\lambda \leq (q-1)\rho + \omega_1$. Since $\omega_1 = \alpha_1 + \alpha_2$ and $\omega_2 = 1/2\alpha_1 + \alpha_2$, we find from the above that $(q-a_1) + 1/2(q-1-a_2)$ is a non-negative integer and that

$$2a_1 + a_2 \leq 3q-1, \quad a_1 + a_2 \leq 2q-1 .$$

If $a_1, a_2 \leq q-1$, then $\lambda = \omega_2^0 \leq (q-1)\rho + \omega_1$, so that $\omega_1 + \omega_2 \geq 0$, which is impossible. If $a_1 \geq q$, then $a_2 \leq q-1$. Write $a_1 = q+b$ with $0 \leq b \leq q-1$. Then

$$L(\lambda)' = L(b\omega_1 + a_2\omega_2 + q\omega_1)' \simeq L(b\omega_1 + a_2\omega_2)' \otimes L(\omega_1)' \gg L(\omega_2^0)'$$

whence $(b+1)\omega_1 + a_2\omega_2 \;{}_Q\!\geq \omega_2^0$ and we have

$$(b+2-q) + 1/2(a_2-q+2) \geq 0 ,$$
$$(b+2-q) + (a_2-q+2) \geq 0 .$$

From the first inequality we have $2a_1 + a_2 \geq 5q-6$, so that $3q-1 \geq 5q-6$, i.e., $q \leq 2$, contradicting the assumption. If $a_2 \geq q$, then $a_1 \leq q-1$. Write $a_2 = q+c$ with $0 \leq c \leq q1$. Then

$$L(\lambda)' \simeq L(a_1\omega_2 + c\omega_2)' \otimes L(\omega_2)' \gg L(\omega_2^0)'$$

whence $a_1\omega_1 + (c+1)\omega_2 \;{}_Q\!\geq \omega_2^0$ and we have

$$a_1-(q-1)+1/2(c-q+3)\geq 0\,,$$
$$a_1-(q-1)+(c-q+3)\geq 0\,.$$

From the second inequality we have $a_1+a_2\geq 3q-4$, so that $2q-1\geq 3q-4$, i.e., $q\leq 3$. But the case that $q=3$ occurs if and only if $a_1=q-1=2$ and $a_2=q=3$. Then $q-a_1+1/2(q-1-a_2)=1/2$ is not an integer. As noted above, this is a contradiction.

For the proof of (2), we may assume $G_0=SU(l+1, q)$. Take $\mu=\sum_i a_i\omega_i\in X_q$. We want to show that if $St=L((q-1)\rho)'\ll L(\mu)'\otimes L(-w_0\omega_k)'$, then $\mu=\omega_k^0$ or $(q-1)\rho$. There is $\gamma\in X^+$ such that $L(\gamma)\ll L(\mu)\otimes L(-w_0\omega_k)$ and that $St\ll L(\gamma)'$. Since $\gamma\leq\mu+(-w_0\omega_k)=\mu+\omega_{\tau(k)}$, we have by Lemma 4

$$(q-1)l\leq|\gamma|\sum_i a_i+1\leq(q-1)l+1\,.$$

If $a_i=q-1$ for all i, we have $\mu=(q-1)\rho$; otherwise we have $(q-1)l=|\gamma|=\sum_i a_i+1$. This implies that $\mu=(q-1)\rho-\omega_j$ for some $j\leq l$ and we have $\gamma=(q-1)\rho$ by Lemma 4. Since $\gamma\leq\mu+\omega_{\tau(k)}$ we have $\omega_{\tau(k)}\geq\omega_j$ from the above, whence $j=\tau(k)$. Therefore $\mu=(q-1)\rho-\omega_{\tau(k)}=\omega_k^0$ as desired.

For convenience of later arguments, we list here the standard unipotent elements $x_i(t)$ of each Chevalley group G_0 corresponding to the simple root α_i (cf. Carter [3]). I is the identity matrix and e_{ij} the matrix unit. We remark that the element $x_{-i}(t)$ corresponding to $-\alpha_i$ is given by ${}^tx_i(t)$, except for $x_{-l}(t)\in\Omega(2l+1, q)$.

[A_l] $G_0=SL(l+1, q)$ $(=G)$.

$$\Pi=\{\alpha_1=\lambda_1-\lambda_2,\cdots,\alpha_l=\lambda_l-\lambda_{l+1}\}\,,$$
$$x_i(t)=I+te_{i,i+1}\qquad(1\leq i\leq l)\,.$$

[B_l] $G_0=\Omega(2l+1, q)$

$$\Pi=\{\alpha_1=\lambda_1-\lambda_2,\cdots,\alpha_{l-1}=\lambda_{l-1}-\lambda_l,\ \alpha_l=\lambda_l\}\,,$$
$$x_i(t)=I+t(e_{i,i+1}-e_{-(i+1),-i})\qquad(1\leq i\leq l-1)\,,$$
$$x_l(t)=I+t(2e_{l,0}-e_{0,-l})-t^2e_{l,-l}\,.$$

(Rows and columns are numbered $0, 1, \cdots, l, -1, \cdots, -l$.)

[C_l] $G_0=Sp(2l, q)$ $(=G)$

$$\Pi=\{\alpha_1=\lambda_1-\lambda_2,\cdots,\alpha_{l-1}=\lambda_{l-1}-\lambda_l,\ \alpha_l=2\lambda_l\}\,,$$
$$x_i(t)=I+t(e_{i,i+1}-e_{-(i+1),-i})\qquad(1\leq i\leq l-1)\,,$$
$$x_l(t)=I+te_{l,-l}\,.$$

[D_l] $G_0=\Omega_{+1}(2l, q)$

$$\Pi = \{\alpha_1 = \lambda_1-\lambda_2, \cdots, \alpha_{l-1} = \lambda_{l-1}-\lambda_l, \alpha_l=\lambda_{l-1}+\lambda_l\},$$
$$x_i(t) = I+t(e_{i,i+1}-e_{-(i+1),-i}) \qquad (1\leq i\leq l-1),$$
$$x_l(t) = I+t(e_{l-1,-l}-e_{l,-(l-1)}).$$

For $J\subset\Pi$, let $G_J=\langle x_\alpha(t);\ \alpha\in\Phi_J,\ t\in F_q\rangle\subset G_0$. This occupies parts of the main diagonal blocks of G_0. If I and J are mutually orthogonal subsets of Π, then $G_{I\cup J}=G_I\times G_J$.

The action of $h_j(t)$ on the unit vectors $e_{\pm i}(1\leq i\leq l)$ of V is written as

$$h_j(t)e_{\pm i} = t^{\pm\lambda_i(h_j)}e_{\pm i};$$
$$h_j(t)e_0 = e_0 \quad (\text{only for } \Omega(2l+1, q)),$$

where in the case of $SL(l+1, q)$ no e_{-i} appears, but e_{l+1} is possible instead.

The standard diagonal subgroups H_1 and H_2 of the universal Steinberg groups of type ${}^2A_l$ and ${}^2D_l$ are as follows respectively:

$$H_1 = \langle h_i(t)h_{l+1-i}(t^q);\ t\in F_{q^2}^\times,\quad 1\leq i\leq l\rangle,$$
$$H_2 = \langle h_i(u),\ h_{l-1}(t)h_l(t^q);\ u\in F_q^\times,\ t\in F_{q^2}^\times,\quad 1\leq i\leq l-2\rangle.$$

3. Reduction to Levi subgroups

Let G be as before. We consider G as a group with a split (B, N)-pair (with $B=\tilde{H}^\sigma$, $N=N_{\tilde{G}}(\tilde{H})^\sigma$); see § 1.18 of Carter [4], which will be referred to for the general theory of groups with a (B, N)-pair. Our notations are mostly the same as in the book.

For a τ-invariant subset J of Π, let P_J, L_J, and St_{L_J} be the standard parabolic subgroup $(\tilde{B}W_J\tilde{B})^\sigma$, the Levi subgroup $\langle\tilde{H},\ x_\alpha(t);\ \alpha\in\Phi_J,\ t\in K\rangle^\sigma$ of P_J, and the Steinberg character of L_J respectively. As a complex character of G, St is defined by

$$St = \sum_J (-1)^{|J/\tau|}(1_{P_J})^G$$

where J runs over the τ-invariant subsets of Π and $|J/\tau|$ denotes the number of the τ-orbits on J. We know that $St|_{P_J}=(St_{L_J})^{P_J}$ and $(St, (1_B)^G)=1$. In particular, it follows that if $J=\phi$, then $L_\phi=H=\tilde{H}^\sigma$ and $St_H=1_H$. Also we have $St|_B\simeq(K_H)^B$ as KB-modules, which give a principal indecomposable KB-module corresponding to the trivial module, since H is a p-complement of B. Let be φ the Brauer character defined by $V=K^n$. Since St is projective, we see, with the notation of Lemma 6, that $m_1=\dim \mathrm{Hom}_{KG}(St, St\otimes V)$ is just the inner product $(St, St\,\varphi)$ of the Brauer characters. Thus

$$m_1 = (St, \sum_J (-1)^{|J/\tau|}(\varphi|_{P_J})^G) = \sum_J (-1)^{|J/\tau|}(St|_{P_J}, \varphi|_{P_J})$$
$$= \sum_J (-1)^{|J/\tau|}(St_{L_J}, \varphi|_{L_J}).$$

We put $m_J=(St_{L_J}, \varphi|_{L_J})=\dim \mathrm{Hom}_{KL_J}(St_{L_J}, V|_{L_J})$. We now prove

Theorem 1. *Suppose $q\geq 3$. Then we have*

$$St\otimes V\simeq\begin{cases} U(\omega_1^0) \text{ for } SL(l+1, q),\ Sp(2l, q),\ \Omega_{\pm 1}(2l, q) \text{ and } SU(l+1, q);\\ U(\omega_1^0)\oplus St \text{ for } \Omega(2l+1, q).\end{cases}$$

Proof. We want to show $m_J=0$ for any τ-invariant subset J of Π. Suppose to the contrary that $m_J\neq 0$ for some J. Since St_{L_J} is injective, it follows that $St_{L_J}\langle\oplus V|_{L_J}$ and hence V contains a nonzero element fixed under H. But this is clearly impossible in the groups $SL(l+1, q)$, $Sp(2l, q)$, $\Omega_{\pm 1}(2l, q)$ and $SU(l+1, q)$, provided $q\geq 3$. So let us assume that $G_0=\Omega(2l+1, q)$ with $p>2$. Then the first unit vector e_0 is a unique element, up to scalar multiples, fixed under H. If $J\not\ni\alpha_l$, then $L_J=\langle H, x_\alpha(t);\, \alpha\in\Phi_J,\, t\in F_q\rangle$ is mapped under ψ: $G\to G_0$ into the set of the elements of the form $\left[\begin{array}{c|c} 1 & 0\\ \hline 0 & *\end{array}\right]$. Hence $V=Ke_0\oplus W$ ($W=K^{2l}$) is a direct sum as a KL_J-module. If $J=\phi$, then $L_J=H$ and $St_H=1_H$, hence $m_\phi=1$. If, on the other hand, $J\neq\phi$ and $St_{L_J}\langle\oplus V|_{L_J}$, then $St_{L_J}\langle\oplus W$. This is impossible because $Ke_0\cap W=0$ and thus $m_J=0$. If $J\ni\alpha_l$, then $x_l(t)$ does not fix e_0, so that no nonzero element of V is stable under the subgroup $B_J=\langle H, x_\alpha(t);\, \alpha\in\Phi_J^+,\, t\in F_q\rangle$ of L_J, and we have again $m_J=0$. (Remember that L_J has a split (B_J, N_J)-pair (Carter [4] Proposition 2.6.3).)

Now, we concentrate on $G_0=SL(l+1, q)$ or $SU(l+1, q)$. For $k\leq l$, we know that $L(\omega_k)'\simeq\overset{k}{\wedge} V$, the module of skew-symmetric tensors of degree k (cf. Wong [13]). Using Lemma 6(2), we prove

Theorem 2. *Let $G_0=SL(l+1, q)$ or $SU(l+1, q)$ with $q\geq 3$. Then we have*

$$St\otimes\overset{k}{\wedge} V\simeq U(\omega_k^0)\qquad \text{for all}\quad k\leq l.$$

Proof. The weight of the standard diagonal subgroup H of G_0 in $\overset{k}{\wedge} V$ are of the form $\tilde\delta$ for some $\delta=\lambda_{p_1}+\cdots+\lambda_{p_k}\in X$ with $1\leq p_1\leq\cdots<p_k\leq l+1$. We show that $\tilde\delta$ is not trivial on H. We may assume that $p_k\leq l$, because $\lambda_{l+1}=-(\lambda_1+\cdots+\lambda_l)$. If $G_0=SL(l+1, q)$, $H=\langle h_i(t);\, t\in F_q^\times,\, 1\leq i\leq l\rangle$ and $\delta(h_{p_k})=2(\delta, \alpha_{p_k})/(\alpha_{p_k}, \alpha_{p_k})=(\delta, \lambda_{p_k}-\lambda_{p_k+1})=1$. If $G_0=SU(l+1, q)$, then, by a similar computation, we have

$$\begin{aligned}&\delta(h_{p_k}+qh_{l+1-p_k})=1\,, &&\text{if}\quad p_k<l+1/2;\\ &\delta(h_{p_k}+qh_{l+1-p_k})\in\{1,\, 1\pm q\}\,, &&\text{if}\quad p_k\geq l+1/2\,.\end{aligned}$$

Therefore, with the notation at the end of the section 2, $\tilde\delta$ is not trivial on H_1, provided $q\geq 3$, i.e., H_1 has no fixed point on $\overset{k}{\wedge} V$ other than zero. Since the same formula as m_1 written above Theorem 1 holds for m_k, with V replaced by $\overset{k}{\wedge} V$, Theorem 2 is now immediate.

4. Case of $q=2$

In this section we shall discuss the case of $q=2$ and determine the multiplicity m_1 of St in $S \otimes V$. This will be done for $G_0=SU(l+1, 2)$ in the next section. In the remaining linear groups, it is clear that $m_1 \geq 1$; for $St \otimes V = L(\rho)' \otimes L(\omega_1)' \gg L(\rho+\omega_1)' = (L(\rho-\omega_1) \otimes L(\omega_1))' \gg L(\rho)' = St$. Actually we have $m_1=1$ as will be shown below.

We first assume that $G_0=SL(l+1, 2)$, $Sp(2l, 2)$ or $\Omega_{+1}(2l, 2)$, and compute $m_J = \dim \mathrm{Hom}_{KL_J}(St_{L_J}, V|_{L_J})$ for a non-empty subset J of Π. Let $J=\bigcup_{i=1}^{r} J_i$ be the partition into the connected components J_i of J. Here, for certain technical reason, we suppose in the case of $\Omega_{+1}(2l, 2)$ that α_{l-1} and α_l are connected, whenever J contains both. Since $H=1$, $G_J=L_J$ for all $J \subset \Pi$ and so $L_J = L_{J_1} \times \cdots \times L_{J_r}$. We write L_i for L_{J_i} for simplicity. Corresponding to this direct product, we have

$$V = V_1 \oplus \cdots \oplus V_r \oplus U,$$

in which each L_i acts on V_i in a natural manner, but trivially on other V_j and U. For example, if $J=\{\alpha_1\}$ and $G_0=Sp(2l, 2)$, $V=V_1 \oplus U$ with $V_1=Ke_1 \oplus Ke_2 \oplus Ke_{-1} \oplus Ke_{-2}$ and $U=\bigoplus_j Ke_j (j \neq \pm 1, \pm 2)$ (note that if $G_0=\Omega_{+1}(2l, 2)$, then $L_{\{\alpha_{l-1}\}}$ and $L_{\{\alpha_l\}}$ act non-trivially on the same subspace $Ke_{l-1} \oplus Ke_l \oplus Ke_{-(l-1)} \oplus Ke_{-l}$, for this reason α_{l-1} and α_l are supposed to be connected). We see that $\dim V_i$ is either $|J_i|+1$, $2|J_i|$ or $2(|J_i|+1)$. If $St_{L_J} \langle \oplus V|_{L_J}$, then $St_{L_J} \langle \oplus \bigoplus_{i=1}^{r} V_i$ and hence $St_{L_J} \langle \oplus V_j$ for a unique $j \leq r$. But since $St_{L_J} = \bigotimes_{i=1}^{r} St_{L_i}$, this forces $r=1$. Recall that $\dim St_{L_J} = 2^a$, where $a=|\Phi_J^+|$. Hence

$$(*) \qquad 2^a \leq \dim V_1 \leq 2(|J|+1).$$

Suppose for the time being that $J \neq \{\alpha_{l-1}, \alpha_l\}$ in case $G_0=\Omega_{+1}(2l, 2)$. If $|J| \geq 2$, then $a \geq |J|+1$, which contradicts $(*)$. Therefore we have $|J|=1$. Summarizing the above, we have $|J|=1$, whenever $m_J \neq 0$ for a nonempty subset J of Π. Write $J=\{\alpha_i\}$ nad $V=V_1 \oplus U$. Since $L_J \simeq SL(2, 2)$, the canonical module K^2 gives the Steinberg module for L_J.

If $G_0=SL(l+1, 2)$, then $V_1 \simeq K^2 = St_{L_J}$ and so $m_J=1$. Since $m_\phi = \dim V = l+1$, we have $m_1 = \sum_J (-1)^{|J|} m_J = (l+1)-l=1$.

Let $G_0=Sp(2l, 2)$. If $i \leq l-1$, then $V_1=V^{(1)} \oplus V^{(2)}$ with $V^{(1)}=Ke_i \oplus Ke_{i+1}$ and $V^{(2)}=Ke_{-i} \oplus Ke_{-(i+1)}$. Since $V^{(1)} \simeq V^{(2)} \simeq K^2$, we have $m_J=2$. On the other hand, if $J=\{\alpha_l\}$, then $V_1=Ke_l \oplus Ke_{-l} \simeq K^2$, whence we have $m_J=1$.

Therefore $m_1=2l-2(l-1)-1=1$.

Let $G_0=\Omega_{+1}(2l, 2)$. If $i\leq l-1$, we are in the same situation as $Sp(2l, 2)$, hence $m_J=2$. If $J=\{\alpha_l\}$, $V_1=V^{(1)}\oplus V^{(2)}$ with $V^{(1)}=Ke_{l-1}\oplus Ke_{-l}$, $V^{(2)}=K(e_{l-1}+e_l)\oplus K(e_{-(l-1)}+e_{-l})$. Since $V^{(1)}\simeq K^2\simeq V^{(2)}$, we have $m_J=2$ again. Now we assume $J=\{\alpha_{l-1}, \alpha_l\}$. Then J has two connected components $J_1=\{\alpha_{l-1}\}$ and $J_2=\{\alpha_l\}$. As noted above, L_1 and L_2 act on the same subspace $V_1=Ke_{l-1}\oplus Ke_l\oplus Ke_{-(l-1)}\oplus Ke_{-l}$. It is easy to see that V_1 is irreducible as an $L_J=L_1\times L_2$-module, which necessarily gives the Steinberg module for it. Hence we have $m_J=1$. Combining the aboves, we get $m_1=2l-2l+1=1$.

We next consider the group $\Omega_{-1}(2l, 2)$. This coincides with the universal Steinberg group $\Omega_+(2l, K)^\sigma$ (since $p=2$) and the standard diagonal subgroup is written as

$$H = \langle h_{l-1}(t)h_l(t^2);\ t\in F_4^\times\rangle$$

$$= \left\{\left(\begin{array}{cc|cc} I & & 0 & \\ & t & & \\ \hline & & I & \\ 0 & & & t^{-1} \end{array}\right);\ t\in F_4^\times\right\},$$

where I denotes the identity matrix of degree $l-1$. For a τ-stable subset J of Π, let $J=\bigcup_{i=1}^{r} J_i$ be the partition into the connected components J_i of J, where we assume α_{l-1} and α_l are connected, as before, if J contains both. If J contains none of α_{l-1} and α_l, we have

$$L_J = G_1\times\cdots\times G_r\times H$$

with $G_i=\langle x_\alpha(1);\ \alpha\in\Phi_{J_i}\rangle$. Hence the corresponding decomposition of V is written as

$$V = V_1\oplus\cdots\oplus V_r\oplus U\,,$$

in which each G_i acts on V_i in a natural manner, but trivially on other V_j and U. In particular H acts trivially on each V_i. Hence the same argument applies as in $\Omega_{+1}(2l, 2)$, yielding $m_J=2$.

If some J_i, say J_r, contains one of α_{l-1} and α_l, then it contains the other by our assumption. We have

$$L_J = G_1\times\cdots\times G_{r-1}\times L_r\,.$$

By the same argument as in $\Omega_{+1}(2l, 2)$ we get $|J|=|J_r|=2$, i.e., $L_J=L_r=\Omega_{-1}(4, 2)(\simeq SL(2, 4))$, provided $m_J\neq 0$. A direct computation shows that $V_s=Ke_{l-1}\oplus Ke_l\oplus Ke_{-(l-1)}\oplus Ke_{-l}$ is irreducible, so that $V_r=St_{L_J}$ and thus $m_J=1$. Therefore $m_1=\sum_J(-1)^{|J/\tau|}m_J=2(l-1)-2(l-2)-1=1$.

Summarizing the aboves, we get

Theorem 3. *For* $SL(l+1, 2)$, $Sp(2l, 2)\simeq\Omega(2l+1, 2)$ *and* $\Omega_{\pm 1}(2l, 2)$, *we have*

$$St\otimes V\simeq U(\omega_1^0)\oplus St\,.$$

5. More on modules of skew-symmetric tensors

We begin with the following combinatorial facts. For the first two assertions, see Lovász [7], Problems 1.31 and 1.42 (g).

Lemma 7. *Let n, k be natural numbers.*

(1) *The number of the subsets of $\{1, 2, \cdots. n\}$ with cardinality r which contains no successive pair of integers is equal to the binomial coefficient* $\binom{n+1-r}{r}$.

(2) $\sum_{r=0}^{[n/2]}(-1)^r\binom{n-r}{r}(1/4)^r=n+1/2^n$.

(3) $\sum_{r=0}^{k}(-1)^r\binom{k}{r}\binom{n-r}{k}=1$.

Proof. (3) From $(1-x^{-1})^k=\sum_{r=0}^{k}(-1)^r\binom{k}{r}x^{-r}$ we have

$$x^n(1-x^{-1})^k = \sum_{r=0}^{k} -1)^r\binom{k}{r}x^{n-r}\,.$$

Evaluating the value of the k-th derivatives at $x=1$ on both sides we get the assertion.

Theorem 4. *For* $SL(l+1, 2)$ *we have*

$$St\otimes \overset{k}{\wedge} V\simeq U(\omega_k^0)\oplus St \qquad (1\leq k\leq l)\,.$$

Proof. Let us fix $k\leq l$ and $J\subset\Pi$, and compute the integer $m(J, k)=\dim \mathrm{Hom}_{KL_J}(St_{L_J}, \overset{k}{\wedge} V)$. Using the same notation as in the proof of the preceding theorem, we have

$$L_J = L_1\times\cdots\times L_r$$

and

$$V = V_1\oplus\cdots\oplus V_r\oplus U, \text{ with } \dim V_i = |J_i|+1\,.$$

As is well-known, we have (cf. Curtis and Reiner [5], § 12)

$$\overset{k}{\wedge} V = \oplus\overset{s_1}{\wedge} V_1\otimes\cdots\otimes\overset{s_r}{\wedge} V_r\oplus\overset{s}{\wedge} U\,,$$

where the direct sum is taken over the sequences $(s_1, \cdots, s_r, s)$ of $r+1$ integers such that $k=s_1+\cdots+s_r+s$, $0\leq s_i\leq|J_i|+1$. Since $L_i\simeq SL(|J_i|+1, 2)$, each $\overset{s_i}{\wedge} V_i$ is irreducible as an L_i-module and we have $St_{L_J}=\bigotimes_{i=1}^{r} St_{J_i}$. Therefore, if

$m(J, k)\neq 0$, i.e., $St_{L_J} < \oplus \overset{k}{\wedge} V$, then there exists a $(s_1, \cdots, s_r, s)$ such that $St_{J_i} \simeq \overset{s_i}{\wedge} V_i$ for all $i \leq r$. Then, considering the dimension of St_{J_i}, we get $|J_i|=1$ and hence $s_i=1$, $\dim V_i=2$, for all i. If this is the case, then $m(J, k)=\dim \overset{s}{\wedge} U = \binom{\dim V-2r}{k-r} = \binom{l+1-2r}{k-r}$. Since no pair of elements of J is connected and $|J|=r$, the number of choices of such J is $\binom{l+1-r}{r}$ by Lemma 7(1). Therefore we have

$$m_k = \sum_J (-1)^{|J/\tau|} m_J = \sum_J (-1)^{|J/\tau|} \binom{l+1-2r}{k-r}\binom{l+1-r}{r}$$
$$= \sum_{r=0}^{k} (-1)^r \binom{k}{r}\binom{l+1-r}{k},$$

which is equal to 1 by Lemma 7 (3). This completes the proof of the theorem.

Finally we show the following result.

Theorem 5. *For $SU(l+1,2)$ and $k\leq l$, we have*

$$St\otimes \overset{k}{\wedge} V = \begin{cases} U(\omega_k^0)\oplus St, & \text{if } l \text{ is odd and } k=l+1/2; \\ U(\omega_k^0), & \text{otherwise}. \end{cases}$$

Proof. Since $L(\omega_k)^* \simeq L(-w_0\omega_k) = L(\omega_{l+1-k})$, we may assume $k\leq l+1/2$. The matrix form of the standard diagonal subgroup H_1 of $SU(l+1, q)$ is in general described as

$$H_1 = \{\mathrm{diag}(t_1, \cdots, t_{l+1});\ \prod_{i=1}^{l+1} t_i = 1,\ t_i^q t_{l+2-i} = 1,\ t_i \in F_{q^2}^{\times}\rangle$$

and so in our case

$$H_1 = \{\mathrm{diag}(t_1, \cdots, t_{l+1});\ \prod_{l=1}^{l+1} t_i = 1,\ t_i = t_{l+2-i},\ t_i \in F_4^{\times}\rangle .$$

In particular, for $\mathrm{diag}(t, \cdots, t_{l+1}) \in H_1$, we have

$$(**) \qquad \begin{cases} (t_1 \cdots t_s)^2 t_{s+1} = 1, & \text{if } l=2s; \\ t_1 \cdots t_{s+1} = 1, & \text{if } l=2s+1. \end{cases}$$

Using this, we first show that H_1 has a non-zero fixed point in $\overset{k}{\wedge} V$ only if l is odd and $k=l+1/2$, which will establish the second statement of the theorem. The set $\{e_{p_1}\wedge \cdots \wedge e_{p_k};\ 1\leq p_1 < \cdots < p_k \leq l+1\}$ forms a basis of $\overset{k}{\wedge} V$ and we have

$$\mathrm{diag}(t_1, \cdots, t_{l+1}) e_{p_1}\wedge \cdots \wedge e_{p_k} = t_{p_1} \cdots t_{p_k} e_{p_1}\wedge \cdots \wedge e_{p_k} .$$

So, if $e_{p_1}\wedge \cdots \wedge e_{p_k}$ is H_1-stable, $t_{p_1} \cdots t_{p_k}=1$ for all $\mathrm{diag}(t_1, \cdots, t_{l+1}) \in H_1$. Replacing t_{p_i} with t_{l+2-p_i} if $p_i \geq s+2$, we see easily from (**) that this occurs only if l is odd and $k=l+1/2$. And when this is the case, the H_1-stable element $e_{p_1}\wedge\cdots\wedge e_{p_k}$ is obtained from $e_1\wedge\cdots\wedge e_{s+1}$ by replacing some of $e_1, \cdots, e_{s+1}$, say $e_i, \cdots, e_j$, with $e_{l+2-i}, \cdots, e_{l+2-j}$ respectively.

We now assume that $l=2s+1$, $k=s+1$, and prove the first statement of the theorem. From the above, the subspace of the H_1-stable points of $\overset{s+1}{\wedge}V$ has dimension $\sum_{j=0}^{s+1}\binom{s+1}{j}=2^{s+1}$. For a τ-stable subset $J\neq\phi$ of Π, let $\tilde{G}_J=\langle x_\alpha(t);\ \alpha\in\Phi_J,\ t\in K\rangle$ and let $\tilde{L}_J=\langle\tilde{H}, \tilde{G}_J\rangle$, the Levi subgroup (as before). Since $\tilde{G}_J$ is a connected normal subgroup of $\tilde{L}_J$, it follows from the Lang-Steinberg theorem that $L_J=\tilde{L}_J^\sigma=\langle H_1, G_J\rangle$ with $G_J=\tilde{G}_J$. We say that J is τ-connected if either J is connected and contains α_{s+1}, or else J is of the form $J=I\cup\tau(I)$ for some connected subset I not containing α_{s+1}. In the former case we have that $G_J\simeq SU(|J|+1, 2)$, while in the latter case, $\tilde{G}_J=\tilde{G}_I\times\tilde{G}_{\tau(I)}\simeq SL(|I|+1, K)\times SL(|I|+1, K)$ is a universal Chevalley group over K. Hence $G_J=\langle U, U'\rangle$ where $U=\langle x_\alpha(t);\ \alpha\in\Phi_J^+,\ t\in K\rangle^\sigma$ and $U'=\langle x_{-\alpha}(t);\ \alpha\in\Phi_J^+,\ t\in K\rangle^\sigma$.

Now, let $J=\bigcup_{i=1}^r J_i$ be the partition into the τ-connected components J_i of J. Then $G_J=G_{J_1}\times\cdots\times G_{J_r}$. Write $G_i=G_{J_i}$ and $n_i=|J_i|$. We have

$$V=V_1\oplus\cdots\oplus V_r\oplus U\,,$$

in which G_i acts naturally on V_i, but trivially on other V_j and U. We want to show that $m_{s+1}=\sum_J(-1)^{|J/\tau|}m_J$ is 1, where J runs over the τ-stable subsets of Π and $m_J=\dim\operatorname{Hom}_{KL_J}(St_{L_J}, \overset{s+1}{\wedge}V)$. Since L_J and G_J have the same Sylow 2-subgroups, $(St_{L_J})|_{G_J}$ must be irreducible, which therefore gives the Steinberg module for G_J.

If $J\ni\alpha_{t+1}$, we arrange the indices so that $J_r\ni\alpha_{s+1}$. Hence, if $J\not\ni\alpha_{s+1}$, we shall ignore in the following the terms that involve r or $s+1$ as subscripts. As noted above, $G_r\simeq SU(n_r+1, 2)$.

If $i\neq r$, then n_i is even and we have

$$G_i=\left\{\left[\begin{array}{c|c}x&0\\\hline 0&\sigma x\end{array}\right];\ x\in SL(n_i/2+1, 4)\right\}\simeq SL(n_i/2+1, 4)\,,$$

so that we have a KG_i-decomposition $V_i=V_i^{(1)}\oplus V_i^{(2)}$ with $\dim V_i^{(1)}=\dim V_i^{(2)}=n_i/2+1$.

The Steinberg module for G_J may be written as $\bigotimes_{i=1}^{r-1}M_i\otimes M_r$, where M_i is the Steinberg module for $SL(n_i/2+1, 4)$ and $M_r=St_{G_r}$. Suppose $m_J\neq 0$. Then by the same argument as in the proof of Theorem 4, we get $|J_i/\tau|=1$ for all $i\leq r$ and $St_{G_J}\langle\oplus\bigotimes_{i=1}^{r-1}(V_i^{(1)}\otimes V_i^{(2)})\otimes V_r$; for instance, that $St_{G_i}\langle\oplus\overset{s_1}{\wedge}V_i^{(1)}\otimes\overset{s_2}{\wedge}V_i^{(2)}$

implies that, putting $m=n_i/2+1$, $4^{m(m-1)/2}\leq\binom{m}{s_1}\binom{m}{s_2}<2^{2m}$, whence $n_i=2$. We then have $G_r\simeq SL(2, 2)$, and hence $V_r\simeq K^2$ is the Steinberg module for it. If $i\neq r$, then $V_i^{(1)}\simeq L(\omega)'$ and $V_i^{(2)}\simeq L(2\omega)'$, where ω is the first (and unique) fundamental dominant weight in the canonical module K^2 for $SL(2, 4)$. Thus as an $SL(2, 4)$-module we have

$$V_i^{(1)}\otimes V_i^{(2)} = L(\omega)'\otimes L(2\omega)'\gg L(3\omega)' .$$

Since $L(3\omega)'$ is the Steinberg module for $SL(2, 4)$, $\dim L(3\omega)'=4$ and so $V_i^{(1)}\otimes V_i^{(2)}\simeq L(3\omega)'$. Thus, we conclude that $St_{G_J}\simeq\bigotimes_{i=1}^{n-1}(V_i^{(1)}\otimes V_i^{(2)})\otimes V_r$ (provided $m_J\neq 0$). Write $J=\{\alpha_{p_i}, \alpha_{\tau(p_i)}, \alpha_{s+1};\ 1\leq i\leq f,\ 1\leq p_i\leq s\}$. Remember that $|p_i-p_j|\geq 2$ whenever $i\neq j$. Since a highest weight vector of St_{L_J} in $\overset{s+1}{\wedge} V$ is stable under the subgroup $\langle H_1,\ x_{p_i}(t)x_{\tau(p_i)}(t^2),\ x_{s+1}(1);\ 1\leq i\leq f,\ t\in F_4^\times\rangle$, it takes the form

$$e(J, u) = \bigotimes_{i=1}^{f}(e_{p_i}\otimes e_{\tau(p_i)})\otimes e_{s+1}\otimes u \qquad \text{for some } u\in \overset{t}{\wedge} U \quad (t = s-2f)\,.$$

Now we devide the cases.

Case 1. $J\ni\alpha_{s+1}$.

Take a subset R of $\Pi\backslash J$ with cardinality t and let $R'=\{j;\ \alpha_j\in R\}$. We have, using that $t_{\tau(p_i)}=t_{l+1-p_i}=t_{p_i+1}$,

$$\mathrm{diag}\,(t_1, \cdots, t_{l+1})e(J, \textstyle\bigwedge_{j\in R'} e_j) = t_{p_1}t_{p_1+1}\cdots t_{p_f}t_{p_f+1}t_{s+1}\prod_{j\in R'}t_j e(J, \textstyle\bigwedge_{j\in R'}e_j)\,.$$

Noting that p_i, p_i+1 and $s+1$ are all distinct ($1\leq i\leq f$), we find easily that the coefficient of $e(J, \bigwedge_{j\in R'}e_j)$ on the right-hand side of the aboe above is 1 if and only if $\prod_{j\in R'}t_j=\prod_i t_i$, where i runs over $\{1, \cdots, s\}\backslash\{p_i, p_i+1;\ 1\leq i\leq f\}$. Since $t_i=t_{l+2-i}$, there are exactly 2^{s-2f} choices of such $\wedge e_j$'s and this gives the multiplicity m_J of St_{L_J} in $\overset{s+1}{\wedge} V$. Once $|J|=2f+1$ is fixed, the number of the subsets J under consideratoin is, from the aboves, equal to the number the subsets of $\{1, \cdots, s-1\}$ with cardinality f that contain no successive pair integers, which is $\binom{s-f}{f}$ by Lemma 7(1). Since $|J/\tau|=f+1$, the terms in m_k involving m_J with $J\ni\alpha_{s+1}$ are given by

$$\sum_{f=0}^{[s/2]}(-1)^{f+1}\binom{s-f}{f}2^{s-2f}$$

and this equals $-(s+1)$ by virtue of Lemma 7(2).

Case 2. $J\not\ni\alpha_{s+1}$.

By the same argument as above, we find that the terms in m_k involving m_J with $J \not\ni \alpha_{s+1}$ are given by

$$\sum_{f=1}^{[s+1/2]}(-1)^f\binom{s+1-f}{f}2^{s+1-2f}$$

which equals $(s+2)-2^{s+1}$.

Now, summarizing the aboves, we get $m_k=2^{s+1}-(s+1)+(s+2)-2^{s+1}=1$, as desired.

Professor Jantzen informed the author that the results in this paper can be extended to $V=L(\lambda)$ with highest weight λ being minuscule or the unique dominant short root using some general results on the representations of algebraic groups due to himself in part.

References

[1] A. Borel: *Properties and linear representations of Chevalley groups,* Lect. notes in math. 131 (1970), Springer, 1–55.

[2] N. Bourbaki: Groupes et algèbras de Lie, Chap. 4–6, Hermann, Paris, 1968.

[3] R.W. Carter: Simple groups of Lie type, John Wiley & Sons, London, 1972.

[4] ———: Finite groups of Lie type, John Wiley & Sons, Chichester, 1985.

[5] C.W. Curtis and I. Reiner: Methods of representation theory I, Wiley, New-York, 1981.

[6] J.C. Jantzen: *Representations of Chevalley groups in their own characteristic,* Proc. Symposia in pure math., Amer. Math. Soc. 47, part I (1987), 127–146.

[7] L. Lovász: Combinatorial problems and exercises, Akadémiai Kiadó, Budapest, 1979.

[8] G. Lusztig: The discrete series of GL_n over a finite field, Ann. of Math. Studies 81, Princeton Univ. Press, 1974.

[9] T. Okuyama: *On p-block theory of finite groups with a (B, N)-pair,* Proc. Symposium on groups and their representations (1984), 176–185 (in Japanese).

[10] R. Steinberg: Lectures on Chevalley groups, Yale Univ. 1967.

[11] ———: Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80 (1968).

[12] M. Suzuki: Finite simple groups (in Japanese), Kinokuniya, Tokyo, 1987.

[13] W.J. Wong: *Representations of Chevalley groups in characteristic p,* Nagoya Math. J. **45** (1987), 39–78.

Department of Mathematics
Osaka City University
558 Osaka Japan