# A NOTE ON THE GROTHENDIECK GROUP OF A FINITE ABELIAN GROUP

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## Introduction

For any ring $A$, by $G(A)$ we denote the Grothendieck group of left $A$-modules which are finitely generated. Let $R$ be the ring of integers of an algebraic number field $K$, and let $R\pi$ and $K\pi$ be the group rings of a finite group $\pi$ over $R$ and $K$, respectively. If $\mathfrak{O}$ is a maximal $R$-order in $K\pi$ which contains $R\pi$, then by regarding a module over $\mathfrak{O}$ as one over $R\pi$, we get a homomorphism

$$\psi\colon G(\mathfrak{O}) \to G(R\pi)$$

of Grothendieck groups. Swan [4] proved that $\psi$ is an epimorphism, and Heller and Reiner [2] described the structure of ker $\psi$ by using a map which depends on an ideal theory of the center of $\mathfrak{O}$ and the modular representations of $\pi$. The following theorem is an immediate consequence from the description.

**Theorem 1.** *Let $R_i$ be maximal orders in the center of the simple constituents $A_i$ ($i=1, \cdots, s$) of $K\pi$. If any prime ideal of $R_i$ which divides the order of $\pi$ is contained in the ray $J(R_i)$ modulo the real archimedian primes ramified in $A_i$, then $\psi$ is an isomorphism.*

The purpose of this note is to show that under certain assumptions the converse of this theorem is also true.

**Theorem 2.** *Let $\pi$ be a finite abelian group of order $n$ and $K$ be a cyclotomic field. Then $\psi$ is an isomorphism if and only if any prime ideal in $\mathfrak{O}$ which divides $n$ is principal.*

In this case $J(R_i)$ is the group of all principal ideals of $R_i$ and $\mathfrak{O}$ is the direct sum of the $R_i$. Hence the if part of this theorem is a special case of Theorem 1. Our proof of this theorem is based on a method using the conducter from $\mathfrak{O}$ to $R\pi$, which owes to Swan ([4]).

Throughout this note, modules are assumed to be left modules which are finitely generated and by $K$, $R$ and $[M]$ we denote an algebraic number field,

the ring of integers of $K$ and an element of a Grothendieck group which is represented by a module $M$, respectively.

**1.** Let $A$ be a central simple algebra over $K$ and let $\mathfrak{O}$ be a maximal $R$-order in $A$. By $I(R)$ and $J(R)$ we denote the multiplicative group of $R$-ideals in $K$ and the subgroup of $I(R)$ consisting of elements $xR$ ($x \in K$), respectively, where $x$ are positive at each real archimedian prime of $K$ which ramifies in $A$.

If $S$ is a simple $\mathfrak{O}$-module, there exists a unique prime ideal $\mathfrak{p}$ of $R$ such that $\mathfrak{p}S = 0$. Hence, when $M$ is a torsion $\mathfrak{O}$-module and $S_1, \cdots, S_k$ are the $\mathfrak{O}$-composition factors of $M$, we define the reduced order ideal $Or(M)$ of $M$ as $Or(M) = \mathfrak{p}_1 \cdots \mathfrak{p}_k$, where each $\mathfrak{p}_i$ is a unique prime ideal scuh that $\mathfrak{p}_i S_i = 0$.

We note that $G(\mathfrak{O})$ is isomorphic to the Grothendieck group of torsion-free $\mathfrak{O}$-modules. Now let $L$ be a torsion-free $\mathfrak{O}$-module such that $K \underset{R}{\otimes} L$ is a simple $A$-module and fix it. Let $M$ be a torsion-free $\mathfrak{O}$-module. Then $K \underset{R}{\otimes} M$ is isomorphic to a direct sum $(K \underset{R}{\otimes} L)^r$ of $r$-copies of $K \underset{R}{\otimes} L$ and there exists a submodule $N$ of $M$ such that $N \cong L^r$. Hence $[M] = r[L] + [M/N]$, and $M/N$ is torsion. Set $\mathfrak{a} = Or[M/N]$. By setting $\eta([M]) = (r, \bar{\mathfrak{a}})$, a map

$$\eta\colon G(\mathfrak{O}) \to \mathbf{Z} \oplus I(R)/J(R)$$

is an isomorphism, where $\bar{\mathfrak{a}}$ is an element of $I(R)/J(R)$ represented by $\mathfrak{a}$ and $\mathbf{Z}$ is the ring of rational integers, (see Swan [5] or Heller and Reiner [2]).

For any (non-zero) ideal $\mathfrak{A}$ of $\mathfrak{O}$, by $D(\mathfrak{A})$ we denote the set of prime ideals of $R$ which divide $Or(\mathfrak{O}/\mathfrak{A})$. Then by the definition of $\eta$, we have an immediate consequence.

**Lemma 3.** *Let $\mathfrak{A}$ be an ideal of $\mathfrak{O}$ and let $M$ be an $\mathfrak{O}$-module which annihilated by $\mathfrak{A}$. If any element of $D(\mathfrak{A})$ is contained in $J(R)$, then $[M] = 0$ in $G(\mathfrak{O})$.*

**2.** Let $\pi$ be a finite group and let $\mathfrak{O}$ be a maximal $R$-order in $K\pi$ which contains $R\pi$. Now we consider the epimorphism

$$\psi\colon G(\mathfrak{O}) \to G(R\pi).$$

Let $K\pi = A_1 \oplus \cdots \oplus A_s$ be the decomposition of $K\pi$ into the simple constituents. We denote by $K_i$ the center of $A_i$ and by $R_i$ the ring of integers of $K_i$. Since $\mathfrak{O}$ is a maximal $R$-order in $K\pi$, there is a decomposition $\mathfrak{O} = \mathfrak{O}_1 \oplus \cdots \oplus \mathfrak{O}_s$ of $\mathfrak{O}$, where each $\mathfrak{O}_i$ is a maximal $R_i$-order in $A_i$. By $\mathfrak{C}$ we denote the conductor from $\mathfrak{O}$ to $R\pi$ (the largest $\mathfrak{O}$-ideal contained in $R\pi$), and we write $\mathfrak{C} = \mathfrak{C}_1 \oplus \cdots \oplus \mathfrak{C}_s$, where each $\mathfrak{C}_i$ is an ideal of $\mathfrak{O}_i$. It is known that $\mathfrak{C}$ divides $n$ the order of $\pi$.

**Proposition 4.** *Suppose any element of $D(\mathfrak{C}_i)$ is contained in $J(R_i)$ for*

*each $i$. Then the map $\psi: G(\mathcal{D}) \to G(R\pi)$ is an isomorphism.*

Proof. We shall give a left inverse map $\phi$ of $\psi$. At first, we define a homomorphism

$$\phi_i: G(R\pi) \to G(\mathcal{D}_i).$$

Any element of $G(R\pi)$ is represented by $[M]-[N]$, where $M$ and $N$ are torsion-free $R\pi$-modules. It therefore suffices to consider torsion-free $R\pi$-modules. Now let $M$ be a torsion-free $R\pi$-module. Since $\mathfrak{C}_i$ is contained in $R\pi$, we can consider an $R\pi$-module $\mathfrak{C}_i M$. We define an operation of $\mathcal{D}_i$ for $\mathfrak{C}_i M$ as $x(\Sigma_t c_t m_t) = \Sigma_t (x c_t) m_t$ $(x \in \mathcal{D}_i, \Sigma_t c_t m_t \in \mathfrak{C}_i M)$. To see that this is well defined, let $\Sigma_t c_t m_t = 0$. Since $nx$ is contained in $\mathfrak{C}_i$, $n\Sigma_t (x c_t) m_t = (nx)\Sigma_t c_t m_t = 0$. However $\Sigma_t (x c_t) m_t \in M$, and $M$ is torsion-free, so we have $\Sigma_t (x c_t) m_t = 0$. Hence we can regard $\mathfrak{C}_i M$ as an $\mathcal{D}_i$-module.

We now define a homomorphism

$$\phi_i: G(R\pi) \to G(\mathcal{D}_i),$$

by $\phi_i([M]) = [\mathfrak{C}_i M]$. It is well defined as follows. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of torsion-free $R\pi$-modules. Then it induces a sequence

$$0 \to \mathfrak{C}_i M' \xrightarrow{\alpha} \mathfrak{C}_i M \xrightarrow{\beta} \mathfrak{C}_i M'' \to 0$$

of $\mathcal{D}_i$-modules, which is exact up to middle. We easily see that $\ker\beta/\operatorname{im}\alpha$ is annihilated by $\mathfrak{C}_i$, but by the assumption, any element of $D(\mathfrak{C}_i)$ is contained in $J(R_i)$. So by Lemma 3, $[\ker\beta/\operatorname{im}\alpha]$ is zero in $G(\mathcal{D}_i)$. This implies that $[\mathfrak{C}_i M] = [\mathfrak{C}_i M'] + [\mathfrak{C}_i M'']$, which showes that $\phi_i$ is well defined.

Since $G(\mathcal{D}) = \Sigma \oplus G(\mathcal{D}_i)$, we define a homomorphism $\phi$ with all $\phi_i$,

$$\phi = \Sigma \phi_i: G(R\pi) \to G(\mathcal{D}).$$

Any $\mathcal{D}_i$-module $M$ is regarded as an $\mathcal{D}$-module by setting $\mathcal{D}_j M = 0$ for $j \neq i$. Then $\phi_i \psi([M]) = [\mathfrak{C}_i M]$ and $\phi_j \psi([M]) = [\mathfrak{C}_j M] = 0$. However $[M] = [\mathfrak{C}_i M]$ in $G(\mathcal{D}_i)$ by Lemma 3. Therefore $\phi\psi = 1$, and we complete the proof.

3. Hereafter let $\pi$ be abelian. Then for each $i$, $A_i$ and $\mathcal{D}_i$ coincide with $K_i$ and $R_i$, respectively. Moreover $J(\mathcal{D}_i)$ is the group of principal ideals of $\mathcal{D}_i$, and $D(\mathfrak{C}_i)$ is the set of prime ideals of $\mathcal{D}_i$ dividing $\mathfrak{C}_i$.

Now we assume that $K$ is a cyclotomic field or any prime rational integer dividing $n$ the order of $\pi$ is unramified in $R$. Let $\rho_i$ is a map $R\pi \to \mathcal{D}_i$ induced by the projection from $K\pi$ onto each constituent $A_i$. Then $\rho_i$ is an epimorphism (see Swan [5]). Set $\mathfrak{N}_i = \operatorname{Ker}\rho_i$. Since $\rho_i$ is an epimorphism, $\rho_i(\mathfrak{N}_j)$ is an

ideal of $\mathfrak{O}_i$ for each $i$ and $j$. Any $\mathfrak{O}_i$-module $M$ is regarded as an $\mathfrak{O}$-module by setting $\mathfrak{O}_j M=0$ for $j \neq i$, and moreover this is also regarded as an $R\pi$-module by restriction of the operation. But such an operation of $R\pi$ for $M$ coincides with one induced by $\rho_i$.

**Theorem 2′.** *Let $\pi$ be an abelian group of order $n$. Assume that either $K$ is a cyclotomic field or $R$ has a property that any prime rational integer dividing $n$ is unramified in $R$. Then the map $\psi$ is an isomorphism if and only if any prime ideal dividing $\mathfrak{C}$ is principal in $\mathfrak{O}$.*

Proof. Assume that any prime ideal dividing $\mathfrak{C}$ is principal in $\mathfrak{O}$ which is equivalent to saying that for each $i$ any prime ideal dividing $\mathfrak{C}_i$ is principal in $\mathfrak{O}_i$, i.e. $D(\mathfrak{C}_i) \subset I(\mathfrak{O}_i)$. Then $\psi$ is an isomorphism by Proposition 4.

Conversely, let $\psi$ be an isomorphism. Suppose that there exists a non-principal prime ideal $\mathfrak{P}$ of $\mathfrak{O}_i$ dividing $\mathfrak{C}_i$ for some $i$. Set $M=\mathfrak{O}_i/\mathfrak{P}$. Since $\mathfrak{P}$ is a non-principal prime ideal, $[M]$ is not zero in $G(\mathfrak{O}_i)$.

On the other hand $\prod_{j \neq i} \mathfrak{R}_j \subset \bigcap_{j \neq i} \mathfrak{R}_j = \mathfrak{C}_i$ (see Bass [1]). Hence, for some $j \neq i$, $\rho_i(\mathfrak{R}_j)$ is contained in $\mathfrak{P}$. If the $\mathfrak{O}_i$-module $M$ is regarded as an $R\pi$-module (by the way mentioned above), $\mathfrak{R}_j M=\rho_i(\mathfrak{R}_j)M \subset \mathfrak{P}M=0$. Consequently, $M$ is annihilated by $\mathfrak{R}_j$, so that from the isomorphism $R\pi/\mathfrak{R}_j \cong \mathfrak{O}_j$, $M$ is also regarded as an $\mathfrak{O}_j$-module. Then the new $R\pi$-module $M$ obtained above coincides with the given $\mathfrak{O}_i$-module. It implies that in $G(R\pi)$, $[M]$ is contained in the image of $G(\mathfrak{O}_i)$ as well as of $G(\mathfrak{O}_j)$, which contradicts injectivity of $\psi$. This proves Theorem 2′.

Now, in particular let $K$ be an arbitrary cyclotomic field. Then a prime ideal $\mathfrak{P}$ of $\mathfrak{O}_i$ divides $\mathfrak{C}_i$ if and only if $\mathfrak{P}$ divides $n$ (see Bass[1]). Thus Theorem 2 is an immediate consequence of Theorem 2′.

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## References

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