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A NOTE ON THE GROTHENDIECK GROUP OF A FINITE ABELIAN GROUP

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Introduction

For any ring A, by $G(A)$ we denote the Grothendieck group of left A-modules which are finitely generated. Let R be the ring of integers of an algebraic number field K, and let $R\pi$ and $K\pi$ be the group rings of a finite group π over R and K, respectively. If $\mathfrak{O}$ is a maximal R-order in $K\pi$ which contains $R\pi$, then by regarding a module over $\mathfrak{O}$ as one over $R\pi$, we get a homomorphism

$$\psi: G(\mathfrak{O}) \rightarrow G(R\pi)$$

of Grothendieck groups. Swan [4] proved that ψ is an epimorphism, and Heller and Reiner [2] described the structure of ker ψ by using a map which depends on an ideal theory of the center of $\mathfrak{O}$ and the modular representations of π. The following theorem is an immediate consequence from the description.

Theorem 1. *Let R_i be maximal orders in the center of the simple constituents A_i ($i=1, \cdots, s$) of $K\pi$. If any prime ideal of R_i which divides the order of π is contained in the ray $J(R_i)$ modulo the real archimedian primes ramified in A_i, then ψ is an isomorphism.*

The purpose of this note is to show that under certain assumptions the converse of this theorem is also true.

Theorem 2. *Let π be a finite abelian group of order n and K be a cyclotomic field. Then ψ is an isomorphism if and only if any prime ideal in $\mathfrak{O}$ which divides n is principal.*

In this case $J(R_i)$ is the group of all principal ideals of R_i and $\mathfrak{O}$ is the direct sum of the R_i. Hence the if part of this theorem is a special case of Theorem 1. Our proof of this theorem is based on a method using the conducter from $\mathfrak{O}$ to $R\pi$, which owes to Swan ([4]).

Throughout this note, modules are assumed to be left modules which are finitely generated and by K, R and $[M]$ we denote an algebraic number field, the ring of integers of K and an element of a Grothendieck group which is represented by a module M, respectively.

1. Let A be a central simple algebra over K and let $\mathfrak{O}$ be a maximal R-order in A. By $I(R)$ and $J(R)$ we denote the multiplicative group of R-ideals in K and the subgroup of $I(R)$ consisting of elements xR ($x \in K$), respectively, where x are positive at each real archimedian prime of K which ramifies in A.

If S is a simple $\mathfrak{O}$-module, there exists a unique prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p}S=0$. Hence, when M is a torsion $\mathfrak{O}$-module and $S_1, \cdots, S_k$ are the $\mathfrak{O}$-composition factors of M, we define the reduced order ideal $Or(M)$ of M as $Or(M)=\mathfrak{p}_1 \cdots \mathfrak{p}_k$, where each $\mathfrak{p}_i$ is a unique prime ideal scuh that $\mathfrak{p}_i S_i=0$.

We note that $G(\mathfrak{O})$ is isomorphic to the Grothendieck group of torsion-free $\mathfrak{O}$-modules. Now let L be a torsion-free $\mathfrak{O}$-module such that $K \underset{R}{\otimes} L$ is a simple A-module and fix it. Let M be a torsion-free $\mathfrak{O}$-module. Then $K \underset{R}{\otimes} M$ is isomorphic to a direct sum $(K \underset{R}{\otimes} L)^r$ of r-copies of $K \underset{R}{\otimes} L$ and there exists a submodule N of M such that $N \cong L^r$. Hence $[M]=r[L]+[M/N]$, and M/N is torsion. Set $\mathfrak{a}=Or[M/N]$. By setting $\eta([M])=(r, \bar{\mathfrak{a}})$, a map

$$\eta\colon\ G(\mathfrak{O}) \to \boldsymbol{Z} \oplus I(R)/J(R)$$

is an isomorphism, where $\bar{\mathfrak{a}}$ is an element of $I(R)/J(R)$ represented by $\mathfrak{a}$ and $\boldsymbol{Z}$ is the ring of rational integers, (see Swan [5] or Heller and Reiner [2]).

For any (non-zero) ideal $\mathfrak{A}$ of $\mathfrak{O}$, by $D(\mathfrak{A})$ we denote the set of prime ideals of R which divide $Or(\mathfrak{O}/\mathfrak{A})$. Then by the definition of η, we have an immediate consequence.

Lemma 3. *Let $\mathfrak{A}$ be an ideal of $\mathfrak{O}$ and let M be an $\mathfrak{O}$-module which annihilated by $\mathfrak{A}$. If any element of $D(\mathfrak{A})$ is contained in $J(R)$, then $[M]=0$ in $G(\mathfrak{O})$.*

2. Let π be a finite group and let $\mathfrak{O}$ be a maximal R-order in $K\pi$ which contains $R\pi$. Now we consider the epimorphism

$$\psi\colon\ G(\mathfrak{O}) \to G(R\pi)\,.$$

Let $K\pi=A_1 \oplus \cdots \oplus A_s$ be the decomposition of $K\pi$ into the simple constituents. We denote by K_i the center of A_i and by R_i the ring of integers of K_i. Since $\mathfrak{O}$ is a maximal R-order in $K\pi$, there is a decomposition $\mathfrak{O}=\mathfrak{O}_1 \oplus \cdots \oplus \mathfrak{O}_s$ of $\mathfrak{O}$, where each $\mathfrak{O}_i$ is a maximal R_i-order in A_i. By $\mathfrak{C}$ we denote the conductor from $\mathfrak{O}$ to $R\pi$ (the largest $\mathfrak{O}$-ideal contained in $R\pi$), and we write $\mathfrak{C}=\mathfrak{C}_1 \oplus \cdots \oplus \mathfrak{C}_s$, where each $\mathfrak{C}_i$ is an ideal of $\mathfrak{O}_i$. It is known that $\mathfrak{C}$ divides n the order of π.

Proposition 4. *Suppose any element of $D(\mathfrak{C}_i)$ is contained in $J(R_i)$ for* *each i. Then the map* ψ: $G(\mathfrak{O})\to G(R\pi)$ *is an isomorphism.*

Proof. We shall give a left inverse map ϕ of ψ. At first, we define a homomorphism

$$\phi_i\colon\ G(R\pi)\to G(\mathfrak{O}_i)\,.$$

Any element of $G(R\pi)$ is represented by $[M]-[N]$, where M and N are torsion-free $R\pi$-modules. It therefore suffices to consider torsion-free $R\pi$-modules. Now let M be a torsion-free $R\pi$-module. Since $\mathfrak{C}_i$ is contained in $R\pi$, we can consider an $R\pi$-module $\mathfrak{C}_iM$. We define an operation of $\mathfrak{O}_i$ for $\mathfrak{C}_iM$ as $x(\Sigma_t c_t m_t)=\Sigma_t(xc_t)m_t$ ($x\in\mathfrak{O}_i$, $\Sigma_t c_t m_t\in\mathfrak{C}_iM$). To see that this is well defined, let $\Sigma_t c_t m_t=0$. Since nx is contained in $\mathfrak{C}_i$, $n\Sigma_t(xc_t)m_t=(nx)\Sigma_t c_t m_t=0$. However $\Sigma_t(xc_t)m_t\in M$, and M is torsion-free, so we have $\Sigma_t(xc_t)m_t=0$. Hence we can regard $\mathfrak{C}_iM$ as an $\mathfrak{O}_i$-module.

We now define a homomorphism

$$\phi_i\colon\ G(R\pi)\to G(\mathfrak{O}_i)\,,$$

by $\phi_i([M])=[\mathfrak{C}_iM]$. It is well defined as follows. Let

$$0\to M'\to M\to M''\to 0$$

be an exact sequence of torsion-free $R\pi$-modules. Then it induces a sequence

$$0\to\mathfrak{C}_iM'\xrightarrow{\alpha}\mathfrak{C}_iM\xrightarrow{\beta}\mathfrak{C}_iM''\to 0$$

of $\mathfrak{O}_i$-modules, which is exact up to middle. We easily see that $\ker\beta/\mathrm{im}\,\alpha$ is annihilated by $\mathfrak{C}_i$, but by the assumption, any element of $D(\mathfrak{C}_i)$ is contained in $J(R_i)$. So by Lemma 3, $[\ker\beta/\mathrm{im}\,\alpha]$ is zero in $G(\mathfrak{O}_i)$. This implies that $[\mathfrak{C}_iM]=[\mathfrak{C}_iM']+[\mathfrak{C}_iM'']$, which showes that ϕ_i is well defined.

Since $G(\mathfrak{O})=\Sigma\oplus G(\mathfrak{O}_i)$, we define a homomorphism ϕ with all ϕ_i,

$$\phi=\Sigma\phi_i\colon\ G(R\pi)\to G(\mathfrak{O})\,.$$

Any $\mathfrak{O}_i$-module M is regarded as an $\mathfrak{O}$-module by setting $\mathfrak{O}_jM=0$ for $j\neq i$. Then $\phi_i\psi([M])=[\mathfrak{C}_iM]$ and $\phi_j\psi([M])=[\mathfrak{C}_jM]=0$. However $[M]=[\mathfrak{C}_iM]$ in $G(\mathfrak{O}_i)$ by Lemma 3. Therefore $\phi\psi=1$, and we complete the proof.

3. Hereafter let π be abelian. Then for each i, A_i and $\mathfrak{O}_i$ coincide with K_i and R_i, respectively. Moreover $J(\mathfrak{O}_i)$ is the group of principal ideals of $\mathfrak{O}_i$, and $D(\mathfrak{C}_i)$ is the set of prime ideals of $\mathfrak{O}_i$ dividing $\mathfrak{C}_i$.

Now we assume that K is a cyclotomic field or any prime rational integer dividing n the order of π is unramified in R. Let ρ_i is a map $R\pi\to\mathfrak{O}_i$ induced by the projection from $K\pi$ onto each constituent A_i. Then ρ_i is an epimorphism (see Swan [5]). Set $\mathfrak{N}_i=\mathrm{Ker}\,\rho_i$. Since ρ_i is an epimorphism, $\rho_i(\mathfrak{N}_j)$ is an ideal of $\mathfrak{O}_i$ for each i and j. Any $\mathfrak{O}_i$-module M is regarded as an $\mathfrak{O}$-module by setting $\mathfrak{O}_j M=0$ for $j \neq i$, and moreover this is also regarded as an $R\pi$-module by restriction of the operation. But such an operation of $R\pi$ for M coincides with one induced by ρ_i.

Theorem 2'. *Let π be an abelian group of order n. Assume that either K is a cyclotomic field or R has a property that any prime rational integer dividing n is unramified in R. Then the map ψ is an isomorphism if and only if any prime ideal dividing $\mathfrak{C}$ is principal in $\mathfrak{O}$.*

Proof. Assume that any prime ideal dividing $\mathfrak{C}$ is principal in $\mathfrak{O}$ which is equivalent to saying that for each i any prime ideal dividing $\mathfrak{C}_i$ is principal in $\mathfrak{O}_i$, i.e. $D(\mathfrak{C}_i) \subset I(\mathfrak{O}_i)$. Then ψ is an isomorphism by Proposition 4.

Conversely, let ψ be an isomorphism. Suppose that there exists a non-principal prime ideal $\mathfrak{P}$ of $\mathfrak{O}_i$ dividing $\mathfrak{C}_i$ for some i. Set $M=\mathfrak{O}_i/\mathfrak{P}$. Since $\mathfrak{P}$ is a non-principal prime ideal, $[M]$ is not zero in $G(\mathfrak{O}_i)$.

On the other hand $\prod_{j \neq i} \mathfrak{R}_j \subset \bigcap_{j \neq i} \mathfrak{R}_j = \mathfrak{C}_i$ (see Bass [1]). Hence, for some $j \neq i$, $\rho_i(\mathfrak{R}_j)$ is contained in $\mathfrak{P}$. If the $\mathfrak{O}_i$-module M is regarded as an $R\pi$-module (by the way mentioned above), $\mathfrak{R}_j M = \rho_i(\mathfrak{R}_j) M \subset \mathfrak{P} M = 0$. Consequently, M is annihilated by $\mathfrak{R}_j$, so that from the isomorphism $R\pi/\mathfrak{R}_j \simeq \mathfrak{O}_j$, M is also regarded as an $\mathfrak{O}_j$-module. Then the new $R\pi$-module M obtained above coincides with the given $\mathfrak{O}_i$-module. It implies that in $G(R\pi)$, $[M]$ is contained in the image of $G(\mathfrak{O}_i)$ as well as of $G(\mathfrak{O}_j)$, which contradicts injectivity of ψ. This proves Theorem 2'.

Now, in particular let K be an arbitrary cyclotomic field. Then a prime ideal $\mathfrak{P}$ of $\mathfrak{O}_i$ divides $\mathfrak{C}_i$ if and only if $\mathfrak{P}$ divides n (see Bass[1]). Thus Theorem 2 is an immediate consequence of Theorem 2'.

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