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LIMIT THEOREMS FOR SHIFT SELFSIMILAR ADDITIVE RANDOM SEQUENCES

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1. Introduction

We introduce shift selfsimilar random sequences, as a discrete time analogue of semi-selfsimilar processes. They are also extensions of stationary random sequences. We study limit theorems for those sequences having independent increments. Our results will be a potential resource for studying Galton-Watson branching trees and diffusions on fractals. Let $\mathbf{R}^d$ be the d-dimensional Euclidean space and let $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$ and $\mathbf{N} = \{1, 2, \ldots\}$. We consider $\mathbf{R}^d$ as the totality of d-dimensional column vectors and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^d$. In this paper, we use the words "increase" and "decrease" in the wide sense allowing flatness.

Definition 1.1. An $\mathbf{R}^d$-valued random sequence $\{X(n), n \in \mathbf{Z}\}$ is said to be *shift a-selfsimilar* if there exists a non-zero real number a such that

$$\{X(n+1), n \in \mathbf{Z}\} \stackrel{\mathrm{d}}{=} \{aX(n), n \in \mathbf{Z}\}, \tag{1.1}$$

where $\stackrel{\mathrm{d}}{=}$ denotes the equality in finite-dimensional distributions.

Let $\{X(n), n \in \mathbf{Z}\}$ be a shift a-selfsimilar random sequence. Then we see that, for positive integer m, the distribution of $X(mn)$ is the same as that of $a^{n(m-1)}X(n)$. Thus the shift selfsimilar random sequence is not selfsimilar in the usual sense.

An $\mathbf{R}^d$-valued stochastic process $\{Y(t), t \geq 0\}$ is said to be *semi-selfsimilar* if there exist $c \in (0,1) \cup (1,\infty)$ and $a > 0$ such that

$$\{Y(ct), t \geq 0\} \stackrel{\mathrm{d}}{=} \{aY(t), t \geq 0\}. \tag{1.2}$$

Strictly semi-stable Lévy processes in $\mathbf{R}^d$ and a Brownian motion on the unbounded Sierpinski gasket are important examples of semi-selfsimilar stochastic processes. If $\{Y(t), t \geq 0\}$ is semi-selfsimilar, then the random sequence $\{X(n), n \in \mathbf{Z}\}$ defined by $X(n) = Y(c^n t_0)$ is shift a-selfsimilar for every $t_0 > 0$. We extend the property (1.1) to an operator version as follows.

DEFINITION 1.2. Let A be a real invertible $d \times d$ matrix. An $\mathbf{R}^d$-valued random sequence $\{X(n), n \in \mathbf{Z}\}$ is called *shift A-selfsimilar* if

$$\{X(n+1), n \in \mathbf{Z}\} \stackrel{\mathrm{d}}{=} \{AX(n), n \in \mathbf{Z}\}. \tag{1.3}$$

DEFINITION 1.3. An $\mathbf{R}^d$-valued random sequence $\{X(n), n \in \mathbf{Z}\}$ is said to have independent increments if, for every $n \in \mathbf{Z}$, $\{X(k), k \leq n\}$ and $X(n+1) - X(n)$ are independent. It is equivalent to the condition that, for every $n \in \mathbf{Z}$, $X(n)$, $X(n+1) - X(n)$, $X(n+2) - X(n+1), \ldots$ are independent. It is called to have independent increments in the weak sense if $X(n+1) - X(n)$, $n \in \mathbf{Z}$, are independent. A random sequence with independent increments is also called an *additive* random sequence.

After this, we investigate for shift A-selfsimilar additive random sequences several problems which have already been studied for selfsimilar (or semi-selfsimilar) processes with independent increments. As to the latter processes, general results are written in [26], problems of recurrence and transience are discussed in [27], [36] and [37] although any criterion to classify recurrence and transience is not yet known, and problems on the rate of growth in increasing case are studied in [25] and [31] in comparison with the results for subordinators in [6] and [7].

The contents of this paper are the following. In Section 2 we give a characterization for non-degenerate shift A-selfsimilar additive random sequences. See Theorem 2.2. In Section 3 we prove that non-degenerate shift A-selfsimilar additive random sequences are transient if and only if A has an eigenvalue whose absolute value is greater than 1. See Corollary 3.2. Next we discuss in detail the rate of growth of shift a-selfsimilar additive random sequences in the "liminf" case for increasing sequences in Section 4, and in the "limsup" case for general sequences in Section 5 as follows. Let $a > 1$, $\mathcal{G}_0 = \{g(x) : g(x)$ is positive and decreasing on $[0, \infty)\}$ and $\mathcal{G}_1 = \{g(x) : g(x)$ is positive and increasing on $[0, \infty)\}$. Suppose that $\{X(n), n \in \mathbf{Z}\}$ is an increasing shift a-selfsimilar additive random sequence in (1.4) and that it is an $\mathbf{R}^d$-valued non-zero shift a-selfsimilar additive random sequence in (1.5) below. We first prove that, for every $g_0 \in \mathcal{G}_0$ and $g_1 \in \mathcal{G}_1$, there exist c_0 and $c_1 \in [0, \infty]$ such that

$$\liminf_{n \to \pm\infty} \frac{X(n)}{a^n g_0(|n|)} = c_0 \quad \text{a.s.} \tag{1.4}$$

and

$$\limsup_{n \to \pm\infty} \frac{|X(n)|}{a^n g_1(|n|)} = c_1 \quad \text{a.s.} \tag{1.5}$$

Here the abbreviation "a.s." means "almost surely". Then we obtain a necessary and sufficient condition for the existance of $g_0 \in \mathcal{G}_0$ such that (1.4) holds for $c_0 = 1$. In the case where there does not exist $g_0 \in \mathcal{G}_0$ such that (1.4) holds for $c_0 = 1$, we give a criterion which classifies $g_0 \in \mathcal{G}_0$ with $c_0 = 0$ or $g_0 \in \mathcal{G}_0$ with $c_0 = \infty$ in (1.4). Further, changing the roles of $g_0 \in \mathcal{G}_0$ and $\{X(n), n \in \mathbf{Z}\}$, we fix $a > 1$ and $g_0 \in \mathcal{G}_0$ then consider the family of the sequences $\{X(n), n \in \mathbf{Z}\}$ which satisfy (1.4). We obtain a necessary and sufficient condition for the existance of $\{X(n), n \in \mathbf{Z}\}$ such that (1.4) holds for $c_0 = 1$. In the case where there does not exist $\{X(n), n \in \mathbf{Z}\}$ such that (1.4) holds for $c_0 = 1$, we give a criterion which classifies $\{X(n), n \in \mathbf{Z}\}$ with $c_0 = 0$ or $\{X(n), n \in \mathbf{Z}\}$ with $c_0 = \infty$ in (1.4). Moreover we get all of the above results replacing $g_0 \in \mathcal{G}_0$, c_0 and (1.4) by $g_1 \in \mathcal{G}_1$, c_1 and (1.5), respectively. Finally we give in Section 6 some examples of the results in Sections 4 and 5. The main results are as follows. The distribution of $X(0) - X(-1)$ is denoted by ρ_1. The Laplace transform of a probability distribution μ on $[0, \infty)$ is denoted by $L_\mu(t)$ for $t \geq 0$, that is, $L_\mu(t) = \int_{[0,\infty)} e^{-tx}\mu(dx)$.

Theorem 4.2. *There exists $g(x) \in \mathcal{G}_0$ satisfying (1.4) with $c_0 = 1$ if and only if $\rho_1(\{0\}) = 0$.*

Corollary 4.2. *Let $g(x) \in \mathcal{G}_0$. Suppose that $\lambda := \rho_1(\{0\}) > 0$. If*

$$\int_0^\infty K_\lambda\left(\frac{1}{g(x)}\right) dx = \infty \quad (\text{resp. } < \infty),$$

then

$$\liminf_{n\to\pm\infty} \frac{X(n)}{a^n g(|n|)} = 0 \quad (\text{resp. } = \infty) \qquad \text{a.s.}$$

where $K_\lambda(x)$ is regularly varying with index $\log\lambda/\log a$ and defined on $(0, \infty)$ as

$$K_\lambda(x) = x^{\log\lambda/\log a} \exp\left(\int_1^x \frac{\log L_{\rho_1}(u) - \log\lambda}{u \log a} du\right).$$

Theorem 4.3. *Let $g(x) \in \mathcal{G}_0$. There exists $\{X(n), n \in \mathbf{Z}\}$ satisfying (1.4) with $c_0 = 1$ if and only if*

$$\liminf_{x\to\infty} \frac{-\log g(x)}{\log x} = 0.$$

Corollary 4.3. *Let $g(x) \in \mathcal{G}_0$. Suppose that $\rho_1(\{0\}) = 0$ and*

$$\liminf_{x\to\infty} \frac{-\log g(x)}{\log x} > 0.$$

Then we have

$$\liminf_{n\to\pm\infty} \frac{X(n)}{a^n g(|n|)} = \infty \quad \text{a.s.}$$

Define a function $\rho_1^*(x)$ on $[0,\infty)$ as $\rho_1^*(x) = P(|X(0) - X(-1)| > x)$. A positive measurable function $f(t)$ on $(0,\infty)$ is said to belong to the class OR if, for every $\delta > 1$, $\limsup_{t\to\infty} f(\delta t)/f(t) < \infty$ and $\liminf_{t\to\infty} f(\delta t)/f(t) > 0$. Define the inverse function $g^{-1}(x)$ on $[0,\infty)$ of $g(x) \in \mathcal{G}_1$ as

$$g^{-1}(x) = \sup\{y \geq 0 : g(y) < x\}$$

with understanding that $\sup\emptyset = 0$.

Theorem 5.2. *There exists $g(x) \in \mathcal{G}_1$ satisfying* (1.5) *with $c_1 = 1$ if and only if $\rho_1^*(x) \notin OR$.*

Corollary 5.1. *Let $g(x) \in \mathcal{G}_1$. Suppose that $\rho_1^*(x) \in OR$. If*

$$\int_0^\infty \rho_1^*(g(x))\,dx < \infty \quad (\text{resp. } = \infty),$$

then

$$\limsup_{n\to\pm\infty} \frac{|X(n)|}{a^n g(|n|)} = 0 \quad (\text{resp. } = \infty) \quad \text{a.s.}$$

Theorem 5.3. *Let $g(x) \in \mathcal{G}_1$. There exists $\{X(n), n \in \mathbf{Z}\}$ satisfying* (1.5) *with $c_1 = 1$ if and only if $g^{-1}(x) + \log(1+x) \notin OR$.*

Corollary 5.2. *Let $g(x) \in \mathcal{G}_1$. Suppose that $g^{-1}(x) + \log(1+x) \in OR$. If*

$$\int_{\mathbf{R}^d} g^{-1}(|x|)\rho_1(dx) < \infty \quad (\text{resp. } = \infty),$$

then

$$\limsup_{n\to\pm\infty} \frac{|X(n)|}{a^n g(|n|)} = 0 \quad (\text{resp. } = \infty) \quad \text{a.s.}$$

REMARK 1.1. Let $g(x) \in \mathcal{G}_1$. Then $g^{-1}(x) + \log(1+x) \notin OR$ provided that

$$\liminf_{x\to\infty} \frac{\log g(x)}{\log x} = 0.$$

For every shift A-selfsimilar random sequence $\{X(n), n \in \mathbf{Z}\}$, the sequence $\{S(n), n \in \mathbf{Z}\}$ defined by $S(n) = A^{-n}X(n)$ is a stationary random sequence. Obviously the converse relation is also true. While shift A-selfsimilar random sequences have independent increments in some cases, stationary random sequences cannot have independent increments except in the trivial case. Thus the sequence $\{S(n), n \in \mathbf{Z}\}$ cannot inherit the independence of increments from the sequence $\{X(n), n \in \mathbf{Z}\}$ through the above correspondence.

For every semi-selfsimilar process $\{Y(t), t \geq 0\}$ with independent increments satisfying (1.2) with $c > 1$, the sequence $\{X(n), n \in \mathbf{Z}\}$ defined by $X(n) = Y(c^n)$ is a shift a-selfsimilar additive random sequence. This example shows the existence of a rich class of shift a-selfsimilar additive random sequences, but it is formal. A non-formal interesting example of a shift selfsimilar additive random sequence is found in the hitting time sequence for the Brownian motion $\{B(t), t \geq 0\}$ starting at the origin on the unbounded Sierpinski gasket $\widehat{G}$ in $\mathbf{R}^2$ as below. Define the sets F_n in $\widehat{G}$ as $F_n = \{x \in \widehat{G} : |x| = 2^n\}$ for $n \in \mathbf{Z}$ and let T_n be the first hitting time of the set F_n for the process $\{B(t), t \geq 0\}$, that is, $T_n = \inf\{t > 0 : B(t) \in F_n\}$. Then $\{T_n, n \in \mathbf{Z}\}$ is an increasing shift 5-selfsimilar additive random sequence. The sequence $\{T_n, n \in \mathbf{Z}\}$ plays a key role in the theory of the Brownian motion $\{B(t), t \geq 0\}$ on $\widehat{G}$. See [1]. Our results will be applied in a forthcoming paper [34] to this example and its extensions which are associated with supercritical branching processes. In particular, we shall give an estimate of the unknown constants in two types of laws of the iterated logarithm for the process $\{B(t), t \geq 0\}$ on $\widehat{G}$. Moreover those studies will be the first step to consider the exact Hausdorff and packing measures for the boundary of a Galton-Watson branching tree, which are discussed in [9], [14] and [15].

Selfsimilar processes were introduced in [13] under the name of semi-stable processes. Some extensions in operator versions are found in [12] and then [10]. The meaning of selfsimilarity in the theory of stochastic processes is stronger than that in the theory of selfsimilar sets and measures which were introduced in [11]. Thus Maejima and Sato introduced in [18] the notion of semi-selfsimilarity in stochastic processes. They proved that the marginal distributions of stochastically continuous semi-selfsimilar processes $\{Y(t), t \geq 0\}$ with independent increments are semi-selfdecomposable in the sense introduced in [17] and conversely any semi-selfdecomposable distribution can be the distribution of $Y(1)$ for some (not necessarily unique in law) $\{Y(t), t \geq 0\}$. While the marginal distributions of stochastically continuous semi-selfsimilar processes are infinitely divisible, those of shift A-selfsimilar additive random sequences are not necessarily infinitely divisible. We show, in Theorem 2.1, that in the case where A is a real invertible $d \times d$ matrix all of whose eigenvalues have absolute values greater than 1, the marginal distributions of shift A-selfsimilar additive random sequences $\{X(n), n \in \mathbf{Z}\}$ are A^{-1}-decomposable in the sense of [16] and [35] and conversely any A^{-1}-decomposable distribution can be the distribution of $X(0)$ for some (not necessarily unique in law) shift A-selfsimilar additive random sequence $\{X(n), n \in \mathbf{Z}\}$. In this way we can have random sequences of this kind on selfsimilar sets such as the Cantor sets and the Sierpinski gasket. They cannot be expressed as $\{Y(c^n), n \in \mathbf{Z}\}$ for any stochastically continuous semi-selfsimilar processes $\{Y(t), t \geq 0\}$ with independent increments satisfying (1.2) with $c > 1$. See Example 6.1.

We finish this section by mentioning that, since shift a-selfsimilar additive random sequences with $a > 1$ are considered as sums of independent and shift a-selfsimilarly distributed random variables, it is of interest to compare our results with those on random walks. See, for recurrence and transience, [5], [20], [29] and [30], and, for the rate of growth, [22], [23] and [24].

2. Characterization

Denote by $\langle z, x\rangle$ and $|x|$ the Euclidean inner product of z and x and the Euclidean norm of x in $\mathbf{R}^d$, respectively. Denote by A' the transpose of a real matrix A and by $\|Q\|$ the operator norm of a real $d \times d$ matrix Q on $\mathbf{R}^d$, that is, $\|Q\| = \sup_{|x|=1} |Qx|$. The symbol $\delta_a(dx)$ stands for the probability distribution on $\mathbf{R}^d$ concentrated at $a \in \mathbf{R}^d$. Let $\widehat{\mu}(z)$ and S_μ be the characteristic function and the support of a probability distributuion μ on $\mathbf{R}^d$, respectively. We denote by $\bar{\mu}$ the reflection of μ, that is, $\bar{\mu}(E) = \mu(-E)$ for Borel sets E in $\mathbf{R}^d$. Denote by $\mu * \rho$ the convolution of probability distributions μ and ρ on $\mathbf{R}^d$. Let B be a real invertible $d \times d$ matrix all of whose eigenvalues have absolute values less than 1. A probability distribution μ on $\mathbf{R}^d$ is said to be *B-decomposable* if there exists a probability distribution ρ on $\mathbf{R}^d$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(B'z)\widehat{\rho}(z). \tag{2.1}$$

Note that B-decomposable distributions are not necessarily infinitely divisible. A probability distribution μ on $\mathbf{R}^d$ is B-decomposable if and only if there exists a probability distribution ρ on $\mathbf{R}^d$ such that

$$\int_{\mathbf{R}^d} \log(1 + |x|)\rho(dx) < \infty \tag{2.2}$$

and

$$\widehat{\mu}(z) = \prod_{n=0}^{\infty} \widehat{\rho}\big((B')^n z\big). \tag{2.3}$$

Any distribution ρ in (2.1) can be used as the distribution ρ in (2.2) and (2.3). Thus a B-decomposable distribution μ on $\mathbf{R}^d$ is determined by ρ in (2.1) but ρ is not necessarily determined by μ. In the case where $S_\mu \subset [0,\infty)^d$, ρ is uniquely determined by μ. The class of all B-decomposable distributions is rather broad and contains many important limit distributions such as operator semi-stable distributions and selfsimilar measures. See [4], [19] and [32]. Since B^2 is expressed as e^Q with Q being a real $d \times d$ matrix all of whose eigenvalues have negative real parts, B-decomposable distributions are always e^Q-decomposable. A probability distribution μ on $\mathbf{R}^d$ is said to be full if S_μ is not contained in any proper hyperplane in $\mathbf{R}^d$. Wolfe showed in [35] that every full e^Q-decomposable distribution is either continuous singular or absolutely continuous. Continuity properties of e^Q-decomposable distributions are studied in [32] and [33]. An $\mathbf{R}^d$-valued random sequence $\{Y(n), n \in \mathbf{Z}\}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a zero sequence if $P(Y(n) = 0 \text{ for } n \in \mathbf{Z}) = 1$. It is called deterministic if there exists a non-random sequence $a_n \in \mathbf{R}^d$ such that $P(Y(n) = a_n \text{ for } n \in \mathbf{Z}) = 1$. It is called non-degenerate if the distributions of $Y(n)$ are full for all $n \in \mathbf{Z}$.

From now on, let $\{X(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued shift A-selfsimilar random sequence with independent increments in the weak sense on a probability space $(\Omega, \mathcal{F}, P)$. Denote by μ_n and ρ_l the distribution of $X(n)$ for $n \in \mathbf{Z}$ and that of $X(0) - X(-l)$ for $l \in \mathbf{N}$, respectively.

Theorem 2.1. *Let A be a real invertible $d \times d$ matrix all of whose eigenvalues have absolute values greater than 1 and let $B = A^{-1}$.*
(i) *Let $\{X(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued shift A-selfsimilar random sequence with independent increments in the weak sense. The probability distribution ρ_1 satisfies that*

$$\int_{\mathbf{R}^d} \log(1 + |x|)\rho_1(dx) < \infty. \tag{2.4}$$

The distributions μ_n are B-decomposable and their characteristic functions are represented as

$$\widehat{\mu}_n(z) = \prod_{k=0}^{\infty} \widehat{\rho}_1\big((B')^{k-n}z\big). \tag{2.5}$$

Moreover, $\{X(n), n \in \mathbf{Z}\}$ has independent increments and

$$\lim_{n \to -\infty} X(n) = 0 \quad \text{a.s.}$$

(ii) *Conversely, if a probability distribution ρ satisfying* (2.2) *is given, then there is a unique* (*in law*) *shift A-selfsimilar additive random sequence $\{X(n), n \in \mathbf{Z}\}$ satisfying $\rho_1 = \rho$. That is, for every B-decomposable distribution μ on $\mathbf{R}^d$, there is a* (*not necessarily unique in law*) *shift A-selfsimilar additive random sequence $\{X(n), n \in \mathbf{Z}\}$ satisfying $\mu_0 = \mu$.*

Proof. We see from the shift A-selfsimilarity that

$$\widehat{\mu}_n(z) = \widehat{\mu}_0\big((B')^{-n}z\big).$$

Since $(B')^n \to O$ as $n \to \infty$, we have $\lim_{n \to -\infty} \widehat{\mu}_n(z) = 1$ for any $z \in \mathbf{R}^d$. Thus μ_n converges weakly to $\delta_0(dx)$ as $n \to -\infty$. Hence $X(-n)$ converges in probability to 0 as $n \to \infty$. Therefore, there are $n_k \uparrow \infty$ such that

$$\lim_{k \to \infty} X(-n_k) = 0 \quad \text{a.s.} \tag{2.6}$$

Thus we see from the independence of increments in the weak sense and the shift A-selfsimilarity that

$$\widehat{\mu}_0(z) = \lim_{n\to\infty} E\exp\bigl(i\langle z, X(0) - X(-n)\rangle\bigr) = \lim_{n\to\infty} \prod_{j=0}^{n-1} \widehat{\rho}_1\bigl((B')^j z\bigr).$$

We have in like manner

$$\widehat{\mu}_{-1}(z) = \lim_{n\to\infty} \prod_{j=1}^{n-1} \widehat{\rho}_1\bigl((B')^j z\bigr).$$

Hence we obtain that

$$\widehat{\mu}_0(z) = \widehat{\mu}_{-1}(z)\widehat{\rho}_1(z) = \widehat{\mu}_0(B'z)\widehat{\rho}_1(z),$$

that is, μ_0 is B-decomposable and $\int_{\mathbf{R}^d} \log(1 + |x|)\rho_1(dx) < \infty$. By the same way, we get that

$$\widehat{\mu}_n(z) = \widehat{\mu}_n(B'z)\widehat{\rho}_1\bigl((B')^{-n} z\bigr) = \prod_{k=0}^{\infty} \widehat{\rho}_1\bigl((B')^{k-n} z\bigr).$$

Therefore μ_n are B-decomposable for $n \in \mathbf{Z}$. Thus we have proved the first assertion of (i). Taking a sufficiently large positive integer k satisfying $\|B^k\| < 1$, (see Lemma 2.6 of [25]) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} P\bigl(|X(-n) - X(-n-1)| \geq \|B^k\|^{n/(2k)}\bigr) \\
&\quad \leq \sum_{n=0}^{\infty} P\bigl(|X(0) - X(-1)| \geq c_1\|B^k\|^{-n/(2k)}\bigr) \\
&\quad \leq c_2 \int_{\mathbf{R}^d} \log(2 + |x|)\rho_1(dx) < \infty,
\end{aligned}$$

where c_1 and c_2 are positive constants. Hence we see from the Borel-Cantelli lemma that $\{X(-n)\}$ is a Cauchy sequence in $\mathbf{R}^d$ as $n \to \infty$ almost surely. Since we already showed in (2.6) that $\lim_{k\to\infty} X(-n_k) = 0$ almost surely, we get $\lim_{n\to-\infty} X(n) = 0$ almost surely. Hence it is evident that

$$X(n) = \sum_{k=0}^{\infty} \bigl(X(n-k) - X(n-k-1)\bigr) \quad \text{a.s.}$$

This shows that $\{X(n), n \in \mathbf{Z}\}$ has independent increments. Next we prove the assertion (ii). Let $Y(n)$, $n \in \mathbf{Z}$, be independent identically distributed $\mathbf{R}^d$-valued random variables with the distribution ρ. The sum $\sum_{n=0}^{\infty} B^n Y(-n)$ is convergent almost surely if and only if (2.2) holds. See [4] and [19]. Then the sequence $\{X(n), n \in \mathbf{Z}\}$ defined by $X(n) = \sum_{j=-\infty}^{n} A^j Y(j)$ is a shift A-selfsimilar additive random sequence with $\rho_1 = \rho$. Uniqueness is obviously true because all finite dimensional distributions of $\{X(n), n \in \mathbf{Z}\}$ are determined by ρ_1. □

REMARK 2.1. Let A be a real invertible $d \times d$ matrix all of whose eigenvalues have absolute values greater than 1 and let $B = A^{-1}$. Let $\{X(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued shift A-selfsimilar additive random sequence. We see from Theorem 2.1 that the distribution of $\{X(n), n \in \mathbf{Z}\}$ is determined by ρ_1. Thus properties of $\{X(n), n \in \mathbf{Z}\}$ should be characterized in terms of ρ_1. If ρ_1 is a discrete probability distribution, then μ_n are selfsimilar for $n \in \mathbf{Z}$ in the following sense. Let N^* be a positive integer or $N^* = \infty$. Define a mapping T_a on $\mathbf{R}^d$ as $T_a x = Bx + a$ for $a \in \mathbf{R}^d$. Define a probability distribution $T_a\mu$ on $\mathbf{R}^d$ for a probability distribution μ on $\mathbf{R}^d$ as $T_a\mu(E) = \mu(T_a^{-1}E)$ for Borel sets E in $\mathbf{R}^d$. If $\rho_1(dx) = \sum_{j=1}^{N^*} p_j \delta_{a_j}(dx)$ for $a_j \in \mathbf{R}^d$ and for $p_j \geq 0$ with $\sum_{j=1}^{N^*} p_j = 1$, then $\mu_n = \sum_{j=1}^{N^*} p_j T_{B^{-n}a_j}\mu_n$ for $n \in \mathbf{Z}$. This selfsimilarity of probability measures is slightly different from the original one introduced in [11]. A relationship between the upper Hausdorff dimension of μ_0 and the entropy of ρ_1 is discussed in [33].

The following lemma is well known. See Lemma 13.9 of [26].

Lemma 2.1. *Let μ be a probability distribution on $\mathbf{R}^d$. If $|\widehat{\mu}(z)| = 1$ on a neighborhood of $z = 0$, then $\mu(dx) = \delta_a(dx)$ for some $a \in \mathbf{R}^d$.*

Let G be a real invertible $d \times d$ matrix and let $m(x)$ be its minimal polynomial. Assume that $m(x) = \prod_{j=1}^{k} \{m_j(x)\}^{r_j}$, where $m_j(x)$ are distinct irreducible monic polynomials over $\mathbf{R}^1$ and r_j are positive integers. Each polynomial $m_j(x)$ has a unique real zero α_j or has two non-real zeros α_j and $\overline{\alpha}_j$, where $\overline{\alpha}_j$ is the complex conjugate of α_j. Let $W_j = \ker(\{m_j(G)\}^{r_j})$ and $W_j' = \ker(\{m_j(G')\}^{r_j})$ for $1 \leq j \leq k$. Then W_j are G-invariant and W_j' are G'-invariant. We have direct sum decompositions

$$\mathbf{R}^d = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

and

$$\mathbf{R}^d = W_1' \oplus W_2' \oplus \cdots \oplus W_k'.$$

Lemma 2.2. *Let G be a real invertible $d \times d$ matrix. Let ζ be a full probability distribution on $\mathbf{R}^d$. Suppose that there exists a probability distribution η on $\mathbf{R}^d$ such that*

$$\widehat{\zeta}(z) = \widehat{\zeta}(G'z)\widehat{\eta}(z). \tag{2.7}$$

Then the following statements are true.
(i) *All eigenvalues of G have absolute values less than or equal to* 1. *Those with absolute value* 1 *are simple zeros of the minimal polynomial of G.*
(ii) *Let V_1 be the direct sum of all W_j satisfying $|\alpha_j| < 1$ and let V_2 be the direct sum of all W_j satisfying $|\alpha_j| = 1$. Let T_l be the projector to V_l for $l = 1, 2$ in the direct sum decomposition $\mathbf{R}^d = V_1 \oplus V_2$. Define probability distributions ζ_l and η_l on $\mathbf{R}^d$ for $l = 1, 2$ as $\zeta_l(E) = \zeta(T_l^{-1}(E \cap V_l))$ and $\eta_l(E) = \eta(T_l^{-1}(E \cap V_l))$ for Borel sets E in $\mathbf{R}^d$. Then we have*

$$\widehat{\zeta}(z) = \widehat{\zeta}_1(z)\widehat{\zeta}_2(z) \quad \textit{and} \quad \widehat{\zeta}_l(z) = \widehat{\zeta}_l(G'z)\widehat{\eta}_l(z) \quad \textit{for} \quad l = 1,\ 2 \tag{2.8}$$

with $\int_{V_1} \log(1 + |x|)\eta_1(dx) < \infty$ and $\eta_2(dx) = \delta_a(dx)$ for some $a \in V_2$.

Proof. Let $\mathrm{Leb}(dx)$ be the Lebesgue measure on $\mathbf{R}^d$. By considering the Jordan canonical form of the matrix G, we have

$$\left\{x \in \mathbf{R}^d : \liminf_{n\to\infty} |G^n x| < \infty\right\} = \left\{x \in \mathbf{R}^d : \limsup_{n\to\infty} |G^n x| < \infty\right\}.$$

Denote the above set by H. Note that H is a subspace in $\mathbf{R}^d$. Denote the closed ball with radius r and the center 0 by B_r. For any $\delta > 0$, we can take sufficiently small $r > 0$ such that $|\widehat{\zeta}(z)|^2 \geq 1 - \delta$ for $z \in B_r$. We see from the Riemann-Lebesgue theorem that

$$\lim_{n\to\infty} \int_{B_r} \cos\langle z, G^n x\rangle\, dz = 0 \qquad \text{for } x \in H^c.$$

Note from (2.7) that

$$|\widehat{\zeta}(z)| \leq |\widehat{\zeta}((G')^n z)| \qquad \text{for } n \in \mathbf{Z}_+.$$

Hence we obtain that

$$\begin{aligned}(1-\delta)\,\mathrm{Leb}(B_r) &\leq \int_{B_r} |\widehat{\zeta}(z)|^2\, dz \\ &\leq \limsup_{n\to\infty} \int_{\mathbf{R}^d} \zeta * \overline{\zeta}(dx) \int_{B_r} \cos\langle z, G^n x\rangle\, dz \\ &\leq \zeta * \overline{\zeta}(H)\,\mathrm{Leb}(B_r).\end{aligned}$$

It follows that $\zeta * \overline{\zeta}(H) = 1$, that is, $H = \mathbf{R}^d$ by the fullness of ζ. Hence the assertion (i) is true. Next we prove the assertion (ii). We define V_l' for $l = 1, 2$ by replacing W_j with W_j' in the definition of V_l. We see from (2.7) that

$$\widehat{\zeta}_l(z) = \widehat{\zeta}_l(G'z)\widehat{\eta}_l(z) \qquad \text{for } l = 1,\ 2.$$

Let G_l be the restriction of G to V_l and let I_l be the identity operator on V_l. We see that ζ_1 is B-decomposable on V_1 with $B = G_1$ and hence $\int_{V_1} \log(1 + |x|)\eta_1(dx) < \infty$. Since $(G_2')^n$, $n \in \mathbf{Z}_+$, are relatively compact in operator norm on V_2', there is a sequence m_k of integers such that $m_{k+1} - m_k \uparrow \infty$ and $(G_2')^{m_k}$ converges to some operator G_0' in the operator norm on V_2' as $k \to \infty$. Note that the absolute values of all eigenvalues of G_0' are 1 and hence G_0' is invertible on V_2'. Thus there is a sequence $n_k := m_{k+1} - m_k \uparrow \infty$ of integers such that $(G_2')^{n_k} \to I_2'$ in operator norm on V_2' as $k \to \infty$. Hence we obtain that

$$|\widehat{\zeta}_2(z)| = \lim_{k\to\infty} |\widehat{\zeta}_2\big((G_2')^{n_k} z\big)| \prod_{j=0}^{n_k-1} |\widehat{\eta}_2((G_2')^j z)| \le |\widehat{\zeta}_2(z)||\widehat{\eta}_2(z)| \qquad \text{for } z \in V_2'.$$

Noting that $|\widehat{\zeta}_2(z)| > 0$ on $V_2' \cap B_r$ for sufficiently small $r > 0$, we find that $|\widehat{\eta}_2(z)| = 1$ on $V_2' \cap B_r$ and hence, by Lemma 2.1, $\eta_2(dx) = \delta_a(dx)$ for some $a \in V_2$. Define probability distributions σ_k on $\mathbf{R}^d$ as $\widehat{\sigma}_k(z) = \prod_{j=0}^{n_k-1} \widehat{\eta}((G')^j z)$. We obtain from (2.7) that

$$\widehat{\zeta}(z) = \widehat{\zeta}\big((G')^{n_k} z\big)\widehat{\sigma}_k(z).$$

Since $(G')^{n_k} z \to T_2' z$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \widehat{\zeta}\big((G')^{n_k} z\big) = \widehat{\zeta}_2(z).$$

Taking a subsequence, if necessary, we see from Lemma of [16] or Theorem 2.1 of [21] that σ_k converges weakly to a probability measure ν on $\mathbf{R}^d$ as $k \to \infty$. It follows that

$$\widehat{\zeta}(z) = \lim_{k\to\infty} \widehat{\zeta}\big((G')^{n_k} z\big)\widehat{\sigma}_k(z) = \widehat{\zeta}_2(z)\widehat{\nu}(z). \tag{2.9}$$

We have $\widehat{\zeta}_2(T_1' z) = \widehat{\zeta}_2(0) = 1$ for $z \in \mathbf{R}^d$ and hence by (2.9)

$$\widehat{\nu}(T_1' z) = \frac{\widehat{\zeta}(T_1' z)}{\widehat{\zeta}_2(T_1' z)} = \widehat{\zeta}_1(z) \qquad \text{for } z \in \mathbf{R}^d.$$

On the other hand, we get by (2.9) that, for some small $r > 0$,

$$\widehat{\nu}(T_2' z) = \frac{\widehat{\zeta}(T_2' z)}{\widehat{\zeta}_2(z)} = 1 \qquad \text{for } z \in B_r,$$

and hence $S_\nu \subset V_1$. It follows that $\widehat{\nu}(z) = \widehat{\nu}(T_1' z) = \widehat{\zeta}_1(z)$, that is, $\nu = \zeta_1$. Thus we have by (2.9)

$$\widehat{\zeta}(z) = \widehat{\zeta}_1(z)\widehat{\zeta}_2(z).$$

The proof of the lemma is complete. □

Next we intend to characterize shift A-selfsimilar additive random sequences. However, since it is difficult to treat the general case, we discuss only the non-degenerate case.

Theorem 2.2. *Let A be a real invertible $d \times d$ matrix. Let $\{X(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued non-degenerate shift A-selfsimilar additive random sequence. Then the following statements are true.*
(i) *All eigenvalues of A have absolute values greater than or equal to* 1. *Those with absolute value* 1 *are simple zeros of the minimal polynomial of A.*
(ii) *There is a direct sum decomposition $\mathbf{R}^d = V_1 \oplus V_2$ such that V_1 and V_2 are A-invariant, all eigenvalues of A on V_1 have absolute values greater than* 1, *and those on V_2 have absolute value* 1. *Let A_l be the restriction of A to V_l for $l = 1, 2$. Then $\{X(n), n \in \mathbf{Z}\}$ is decomposed as the sum of two independent random sequences $\{X_l(n), n \in \mathbf{Z}\}$, $l = 1, 2$, such that, for each l, $\{X_l(n), n \in \mathbf{Z}\}$ is a shift A_l-selfsimilar additive random sequence on V_l, and almost surely $\{X_2(n), n \in \mathbf{Z}\}$ has deterministic increments and almost surely $\sup_{n\in\mathbf{Z}} |X_2(n)| < \infty$.*

Proof. We see from the shift A-selfsimilarity and the independence of increments that

$$\widehat{\mu}_0(z) = \widehat{\mu}_0((A^{-1})'z)\widehat{\rho}_1(z).$$

By setting $G = A^{-1}$, $\zeta = \mu_0$, and $\eta = \rho_1$, we can use Lemma 2.2 and hence the assertion (i) is true. Define $\{X_l(n), n \in \mathbf{Z}\}$ by $X_l(n) = T_l X(n)$ for $l = 1, 2$. Then obviously $X(n) = X_1(n) + X_2(n)$, and $\{X_l(n), n \in \mathbf{Z}\}$ are shift A_l-selfsimilar additive random sequences on V_l for $l = 1, 2$. We see from (ii) of Lemma 2.2 that the distribution of $X_2(0) - X_2(-1)$ is a delta distribution δ_a with $a \in V_2$ and hence $\{X_2(n), n \in \mathbf{Z}\}$ has deterministic increments almost surely. The fact that $\sup_{n\in\mathbf{Z}} |X_2(n)| < \infty$ a.s. is clear from the fact that $\sup_{n\in\mathbf{Z}_+} |\sum_{j=0}^{n} A^j a| < \infty$ and $\sup_{n\in\mathbf{Z}_+} |\sum_{j=-n}^{0} A^j a| < \infty$ because all eigenvalues of A on V_2 have absolute value 1 and are simple zeros of the minimal polynomial of A. Finally we prove the independence of two random sequences $\{X_l(n), n \in \mathbf{Z}\}$ for $l = 1, 2$ by using (ii) of Lemma 2.2 and the independence of increments of $\{X(n), n \in \mathbf{Z}\}$. Let N be an arbitrary positive integer, $z_l^{(1)}$, $z_l^{(2)}$ be arbitrary in $\mathbf{R}^d$ and define $w_l = T_1' z_l^{(1)} + T_2' z_l^{(2)}$ for $1 \leq l \leq N$. Note that $\langle z_l^{(1)}, X_1(n_l)\rangle = \langle w_l, X_1(n_l)\rangle$ and $\langle z_l^{(2)}, X_2(n_l)\rangle) = \langle w_l, X_2(n_l)\rangle$ for any $n_l \in \mathbf{Z}$ with $1 \leq l \leq N$. We obtain from (2.8) and the properties of increments of $\{X(n), n \in \mathbf{Z}\}$, $\{X_1(n), n \in \mathbf{Z}\}$, and $\{X_2(n), n \in \mathbf{Z}\}$ that, for any strictly increasing sequence $n_l \in \mathbf{Z}$

with $0 \le l \le N$,

$$
\begin{aligned}
&E \exp\left(i \sum_{l=1}^{N} \left(\langle z_l^{(1)}, X_1(n_l) \rangle + \langle z_l^{(2)}, X_2(n_l) \rangle \right) \right) \\
&= E \exp\left(i \sum_{l=1}^{N} \langle w_l, X(n_l) \rangle \right) \\
&= \left[\prod_{j=1}^{N} E \exp\left(i \sum_{l=j}^{N} \langle w_l, X(n_j) - X(n_{j-1}) \rangle \right) \right] E \exp\left(i \sum_{l=1}^{N} \langle w_l, X(n_0) \rangle \right) \\
&= \prod_{k=1}^{2} \left[\prod_{j=1}^{N} E \exp\left(i \sum_{l=j}^{N} \langle w_l, X_k(n_j) - X_k(n_{j-1}) \rangle \right) \right] E \exp\left(i \sum_{l=1}^{N} \langle w_l, X_k(n_0) \rangle \right) \\
&= E \exp\left(i \sum_{l=1}^{N} \langle w_l, X_1(n_l) \rangle \right) E \exp\left(i \sum_{l=1}^{N} \langle w_l, X_2(n_l) \rangle \right) \\
&= E \exp\left(i \sum_{l=1}^{N} \langle z_l^{(1)}, X_1(n_l) \rangle \right) E \exp\left(i \sum_{l=1}^{N} \langle z_l^{(2)}, X_2(n_l) \rangle \right)
\end{aligned}
$$

Thus we have established the independence of $\{X_1(n), n \in \mathbf{Z}\}$ and $\{X_2(n), n \in \mathbf{Z}\}$. □

3. Transience

An $\mathbf{R}^d$-valued random sequence $\{Y(n), n \in \mathbf{Z}\}$ is said to be transient if $P(\lim_{n\to\infty} |Y(n)| = \infty) = 1$. In this section we prove the following theorem.

Theorem 3.1. *Let A be a real invertible $d \times d$ matrix all of whose eigenvalues have absolute values greater than* 1. *Then all non-zero $\mathbf{R}^d$-valued shift A-selfsimilar additive random sequences $\{X(n), n \in \mathbf{Z}\}$ are transient.*

Corollary 3.1. *Let $\{Y(t), t \ge 0\}$ be an $\mathbf{R}^d$-valued stochastically continuous semi-selfsimilar process with independent increments satisfying* (1.2) *with $c > 1$. If $P(Y(t_1) \ne 0) > 0$ for some $t_1 > 0$, then the random sequence $\{Y(c^n t_0), n \in \mathbf{Z}\}$ is transient for every $t_0 > 0$.*

REMARK 3.1. In case $\{Y(t), t \ge 0\}$ is a strictly stable Lévy process in $\mathbf{R}^d$, the corollary above is already shown in [3]. The assertion of the corollary remains true in the case where $\{Y(t), t \ge 0\}$ is a strictly operator semi-stable Lévy process in $\mathbf{R}^d$.

We obtain the following corollary, combining Theorems 2.2 and 3.1.

Corollary 3.2. *Let A be a real invertible $d \times d$ matrix. Non-degenerate $\mathbf{R}^d$-valued shift A-selfsimilar additive random sequences $\{X(n), n \in \mathbf{Z}\}$ are transient if and only if A has an eigenvalue whose absolute value is greater than* 1.

Lemma 3.1. *Let $\{Y(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued random sequence. If*

$$\sum_{n=0}^{\infty} P(|Y(n)| \le a) < \infty \quad \text{for } \forall a > 0,$$

then $\{Y(n), n \in \mathbf{Z}\}$ is transient.

Proof. Proof is clear from the Borel-Cantelli lemma. □

REMARK 3.2. There exists a non-zero, non-transient shift A-selfsimilar random sequence. It is shown as follows. Let A be a real invertible $d \times d$ matrix. Let $\{Y(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued shift A-selfsimilar random sequence such that $Y(n)$, $n \in \mathbf{Z}$, are independent. Then it follows from the Borel-Cantelli lemma that $\{Y(n), n \in \mathbf{Z}\}$ is transient if and only if

$$\sum_{n=0}^{\infty} P\bigl(|Y(n)| \le a\bigr) < \infty \quad \text{for } \forall a > 0.$$

In the case where A is a real invertible $d \times d$ matrix all of whose eigenvalues have absolute values greater than 1, it is equivalent from the shift A-selfsimilarity to $E(-\log(|Y(0)| \wedge 1)) < \infty$. Thus the first assertion is obviously true.

Lemma 3.2. *Let $\{Y(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued random sequence and let η_n be the distribution of $Y(n)$ for $n \in \mathbf{Z}$. If there exists $a_0 > 0$ such that*

$$\sum_{n=0}^{\infty} \int_{|z| \le a_0} |\widehat{\eta}_n(z)|\, dz < \infty, \tag{3.1}$$

then $\{Y(n), n \in \mathbf{Z}\}$ is transient.

Proof. Let $x = (x_j)_{j=1}^d \in \mathbf{R}^d$. Define a function $f_a(x)$ on $\mathbf{R}^d$ for $a > 0$ as

$$f_a(x) = \prod_{j=1}^{d} \left(\frac{2 \sin(a x_j/2)}{a x_j} \right)^2$$

with understanding that $(\sin 0)/0 = 1$. Then the Fourier transform $\widehat{f}_a(z)$ of $f_a(x)$ is given by

$$\begin{aligned}\widehat{f}_a(z) &= \int_{\mathbf{R}^d} \exp\bigl(i\langle z, x\rangle\bigr) f_a(x)\, dx \\ &= (2\pi a^{-1})^d \prod_{j=1}^{d} \bigl(1 - a^{-1}|z_j|\bigr) 1_{[-a,a]}(z_j),\end{aligned}$$

where $1_{[-a,a]}(x)$ is the indicator function of the interval $[-a, a]$. We see from the Perseval's equality that

$$\begin{aligned}\sum_{n=0}^{\infty} E\bigl(f_a(Y(n))\bigr) &= (2\pi)^{-d} \sum_{n=0}^{\infty} \int_{\mathbf{R}^d} \widehat{\eta}_n(z) \widehat{f}_a(-z)\, dz \\ &\le a^{-d} \sum_{n=0}^{\infty} \int_{|z| \le \sqrt{d}\, a} |\widehat{\eta}_n(z)|\, dz.\end{aligned}$$

Hence we find that (3.1) implies that $\sum_{n=0}^{\infty} E(f_a(Y(n))) < \infty$ for any $a \in (0, a_0/\sqrt{d}\,)$. Thus the lemma follows from Lemma 3.1. □

Let $b \in (0, 1)$. Define $I(c)$ for $c > 0$ as

$$I(c) = \sum_{n=0}^{\infty} \int_0^{2\pi} \exp\left(c \sum_{k=0}^{n} (\cos(b^{-k} u) - 1)\right) du.$$

Let $D = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ with $\beta \ne 0$ and $\alpha^2 + \beta^2 < 1$. Let E be a square in $\mathbf{R}^2$ having the area $4\pi^2$ with a vertex at 0. Define $J(c, y, E)$ for $c > 0$ and $y \in \mathbf{R}^2$ satisfying $|y| = 1$ as

$$J(c, y, E) = \sum_{n=0}^{\infty} \int_E \exp\left(c \sum_{k=0}^{n} \bigl(\cos\langle (D')^{-k} z, y\rangle - 1\bigr)\right) dz.$$

Lemma 3.3. (i) *$I(c) < \infty$ for all $c > 0$.*
(ii) *$\sup_{|y|=1} J(c, y, E) < \infty$ for all $c > 0$.*

Proof. We first prove the assertion (i). Let $N \in \mathbf{Z}_+$ be sufficiently large and let δ be an arbitrary real number. Define sequences $\phi_l(\delta)$ for $l \in \mathbf{Z}_+$ and a_m for $m \in \mathbf{Z}_+$ as

$$\phi_l(\delta) = \sum_{n=0}^{l} \int_0^{2\pi} \exp\left(c \sum_{k=0}^{n} \bigl(\cos(b^{-kN}(v + \delta)) - 1\bigr)\right) dv,$$

and

$$a_m = \inf_{0 \le x \le 2\pi} \bigl(1 - \cos(b^N(x + 2m\pi) + \delta)\bigr).$$

Denote $M = [b^{-N}] + 1$ and $r = b^N(2M/3 + e^{-c}M/3)$, where $[x]$ stands for the largest integer not exceeding a real number x. We have

(3.2)
$$\begin{aligned}\phi_l(\delta) &= \phi_0(\delta) + b^N \sum_{n=1}^{l} \int_0^{b^{-N}2\pi} \exp\left(c\sum_{k=0}^{n}\bigl(\cos(b^{-(k-1)N}(u+b^{-N}\delta)) - 1\bigr)\right) du \\ &\le \phi_0(\delta) + b^N \sum_{n=1}^{l}\sum_{m=0}^{M-1} \int_0^{2\pi} \exp\left(c\sum_{k=0}^{n}\bigl(\cos(b^{-(k-1)N}(v+2m\pi+b^{-N}\delta)) - 1\bigr)\right) dv \\ &\le \phi_0(\delta) + b^N \sum_{m=0}^{M-1} \phi_{l-1}(2m\pi + b^{-N}\delta)\exp(-ca_m).\end{aligned}$$

Since N is sufficiently large, so is M and we can assume that the number of m satisfying $a_m \ge 1$ for $0 \le m \le M-1$ is more than $M/3$. Hence we see that $0 < r < 1$ and

$$b^N \sum_{m=0}^{M-1} \exp(-ca_m) \le r.$$

Noting that $\phi_0(\delta) \le 2\pi$, we obtain from (3.2) that

$$\phi_l(\delta) \le \sum_{j=0}^{l} 2\pi r^j \le \frac{2\pi}{1-r}.$$

It follows that

$$\begin{aligned}(3.3)\qquad I(c) &= \lim_{l\to\infty} \sum_{n=0}^{(l+1)N-1} \int_0^{2\pi} \exp\left(c\sum_{k=0}^{n}\bigl(\cos(b^{-k}u) - 1\bigr)\right) du \\ &\le \lim_{l\to\infty} \sum_{n=0}^{l}\sum_{j=0}^{N-1} \int_0^{2\pi} \exp\left(c\sum_{k=0}^{n}\bigl(\cos(b^{-(kN+j)}u) - 1\bigr)\right) du \\ &= \lim_{l\to\infty} \sum_{n=0}^{l}\sum_{j=0}^{N-1} b^j \int_0^{2\pi b^{-j}} \exp\left(c\sum_{k=0}^{n}\bigl(\cos(b^{-kN}v) - 1\bigr)\right) dv \\ &\le \lim_{l\to\infty} \sum_{n=0}^{l}\sum_{j=0}^{N-1}\sum_{m=0}^{[b^{-j}]} b^j \int_0^{2\pi} \exp\left(c\sum_{k=0}^{n}\bigl(\cos\bigl(b^{-kN}(u+2m\pi)\bigr) - 1\bigr)\right) du \\ &= \lim_{l\to\infty} \sum_{j=0}^{N-1}\sum_{m=0}^{[b^{-j}]} b^j \phi_l(2m\pi) \le \frac{4N\pi}{1-r} < \infty.\end{aligned}$$

Next we prove the assertion (ii). Let $D = b\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ with $b = \sqrt{\alpha^2+\beta^2}$ and $0 \le \theta < 2\pi$. We continue to use M and N as above. Let $\xi \in \mathbf{R}^2$ be arbitrary. Define a sequence $\Phi_l(\xi, y, E)$ for $l \in \mathbf{Z}_+$ as

$$\Phi_l(\xi, y, E) = \sum_{n=0}^{l} \int_E \exp\left(c \sum_{k=0}^{n} \left(\cos\langle (D')^{-kN}(z+\xi), y\rangle - 1\right)\right) dz.$$

Denote $E_1 = \begin{pmatrix} \cos N\theta & -\sin N\theta \\ \sin N\theta & \cos N\theta \end{pmatrix} E$ and denote the vertices of E_1 by $\{0, e_1, e_2, e_1+e_2\}$. Define $a(m_1, m_2)$ for $m_1, m_2 \in \mathbf{Z}_+$ as

$$a(m_1, m_2) = \inf_{x \in E_1} \left(1 - \cos\left\langle (D')^N \left(x + \sum_{j=1}^{2} m_j e_j \right) + \xi, y \right\rangle \right).$$

We have, as in (3.2),

$$\begin{aligned}
&\Phi_l(\xi, y, E) - \Phi_0(\xi, y, E) \\
&= b^{2N} \sum_{n=1}^{l} \int_{b^{-N}E_1} \exp\left(c \sum_{k=0}^{n} \left(\cos\left\langle (D')^{-(k-1)N}(z + (D')^{-N}\xi), y\right\rangle - 1\right)\right) dz \\
&\le b^{2N} \sum_{n=1}^{l} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} \int_{E_1} \exp(c \sum_{k=0}^{n} (\cos\langle (D')^{-(k-1)N}(z + \sum_{j=1}^{2} m_j e_j + (D')^{-N}\xi), y\rangle - 1))\, dz \\
&\le b^{2N} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} \Phi_{l-1}\left(\sum_{j=1}^{2} m_j e_j + (D')^{-N}\xi, y, E_1 \right) \exp\left(-c a(m_1, m_2)\right).
\end{aligned}$$

Since M is sufficiently large, we can assume that there is a positive absolute constant $\delta \in (0,1)$ such that the number of (m_1, m_2) for $0 \le m_1, m_2 \le M-1$ satisfying $a(m_1, m_2) \ge 1$ is more than δM^2. Denote $s = b^{2N}((1-\delta)M^2 + \delta M^2 e^{-c})$. Then we see that $0 < s < 1$ and

$$b^{2N} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} \exp\left(-c a(m_1, m_2)\right) \le s.$$

Noting that

$$\Phi_0(\xi, y, E) \le 4\pi^2,$$

we have

$$\Phi_l(\xi, y, E) \le \sum_{j=0}^{l} 4\pi^2 s^j \le \frac{4\pi^2}{1-s}.$$

Hence we obtain by the same manner in (3.3) that

$$\sup_{|y|=1} J(c, y, E) \le \frac{8\pi^2 N}{1-s} \quad \text{for } \forall c > 0.$$

Thus the proof is complete. □

Proof of Theorem 3.1. We can assume without loss of generality that there is no proper subspace W such that $P(X(n) \in W \text{ for } n \in \mathbf{Z}) = 1$. Let $S = RAR^{-1}$ be the real Jordan canonical form of the matrix A with a real invertible $d \times d$ matrix R. Since we have

$$\{RX(n+1), n \in \mathbf{Z}\} \stackrel{\mathrm{d}}{=} \{SRX(n), n \in \mathbf{Z}\},$$

the random sequence $\{RX(n),\ n \in \mathbf{Z}\}$ is shift S-selfsimilar and additive. If $\{RX(n), n \in \mathbf{Z}\}$ is transient, then $\{X(n), n \in \mathbf{Z}\}$ is also transient. Thus we can assume that $A = S$. There are two possible cases.

CASE 1. An eigenvalue of S is real.

CASE 2. No eigenvalue of S is real.

Let $H_l = \{(x_1, \dots, x_d)' \in \mathbf{R}^d : x_j = 0 \text{ for } 1 \leq j \leq d - l\}$ and let P_l be the orthogonal projector to H_l for $l = 1,\ 2$. Define the random sequence $\{Y_l(n), n \in \mathbf{Z}\}$ by $Y_l(n) = P_l X(n)$ for $l = 1,\ 2$. In Case 1, there is a Jordan block with a real eigenvalue $b^{-1} \in (-\infty, -1) \cup (1, \infty)$ in S. We can assume that this Jordan block lies in the lowest position in S. Thus $\{Y_1(n), n \in \mathbf{Z}\}$ is a non-zero shift b^{-1}-selfsimilar additive random sequence on H_1. In Case 2, $\{Y_2(n), n \in \mathbf{Z}\}$ is a non-zero shift $P_2 S$-selfsimilar additive random sequence on H_2. Thus it is enough to prove the transience in the case of $d = 1$ and in the case where $d = 2$ and $A^{-1} = D$ with $\beta \neq 0$ and $\alpha^2 + \beta^2 < 1$. We treat only the latter case. The proof of the first case is similar by virtue of (i) of Lemma 3.3 and is omitted. By using the inequalities $|x| \leq e^{|x|-1}$ and $|\cos x| - 1 \leq 4^{-1}(\cos 2x - 1)$, we obtain from (2.5) that

$$
\begin{aligned}
(3.4) \qquad |\widehat{\mu}_n(z)|^2 &= \prod_{k=0}^{\infty} \left| \int_{\mathbf{R}^2} \cos\langle (D')^{k-n} z, x\rangle \rho_1 * \overline{\rho_1}(dx) \right| \\
&\leq \exp\left(\sum_{k=0}^{\infty} \int_{\mathbf{R}^2} \bigl(|\cos\langle (D')^{k-n} z, x\rangle| - 1 \bigr) \rho_1 * \overline{\rho_1}(dx) \right) \\
&\leq \exp\left(4^{-1} \sum_{k=0}^{\infty} \int_{\mathbf{R}^2} \bigl(\cos\langle 2(D')^{k-n} z, x\rangle - 1 \bigr) \rho_1 * \overline{\rho_1}(dx) \right).
\end{aligned}
$$

If $\rho_1 * \overline{\rho_1}(dx) = \delta_0(dx)$, then $\rho_1(dx) = \delta_a(dx)$ for some $a \in \mathbf{R}^d$ and hence $\{X(n), n \in \mathbf{Z}\}$ has deterministic increments and transient. Note that $a \neq 0$ because $\{X(n), n \in \mathbf{Z}\}$ is not a zero sequence. Thus we can choose a compact set K in $\mathbf{R}^2$ not containing 0 and a positive number a_0 such that $C := \rho_1 * \overline{\rho_1}(K) > 0$ and $a_0 \sup_{y \in K} |y| \leq \pi$. Let $E_1 = [0, 2\pi] \times [0, 2\pi]$, $E_2 = [-2\pi, 0] \times [0, 2\pi]$, $E_3 = [-2\pi, 0] \times [-2\pi, 0]$ and $E_4 = [0, 2\pi] \times [-2\pi, 0]$. Then by using Jensen's inequality and letting $w = 2|x|z$, we conclude from (3.4) and (ii) of Lemma 3.3 that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\int_{|z|\le a_0}|\widehat{\mu}_n(z)|\,dz\\
&\le \sum_{n=0}^{\infty}\int_{|z|\le a_0}\exp\left(8^{-1}\sum_{k=0}^{\infty}\int_K\big(\cos\langle 2(D')^{k-n}z,x\rangle-1\big)\rho_1*\overline{\rho_1}(dx)\right)dz\\
&\le \sum_{n=0}^{\infty}\int_K\frac{\rho_1*\overline{\rho_1}(dx)}{4C|x|^2}\sum_{j=1}^{4}\int_{E_j}\exp\left(8^{-1}C\sum_{k=0}^{\infty}\left(\cos\left\langle (D')^{k-n}w,\frac{x}{|x|}\right\rangle-1\right)\right)dw\\
&\le \sum_{j=1}^{4}\int_K J\left(\frac{C}{8},\frac{x}{|x|},E_j\right)\frac{\rho_1*\overline{\rho_1}(dx)}{4C|x|^2}<\infty.
\end{aligned}$$

It follows from Lemma 3.2 that $\{X(n), n\in\mathbf{Z}\}$ is transient. □

4. Rate of growth I

In this section, let $\{X(n), n\in\mathbf{Z}\}$ be an increasing shift a-selfsimilar additive random sequence with non-deterministic increments, that is, $a>1$, $S_{\rho_1}\subset[0,\infty)$ and $\rho_1(dx)\ne\delta_c(dx)$ for any $c\ge 0$. Note that all distributions μ_n are continuous thanks to Wolfe's theorem in [35]. We investigate the rate of growth of $\{X(n), n\in\mathbf{Z}\}$ in the "liminf" case. We state the results only as $n\to\infty$ except in Theorem 4.1. The results and their proofs as $n\to-\infty$ are similar and omitted. Define

$$\mathcal{G}_0=\{g(x): g(x)\text{ is positive and decreasing on }[0,\infty)\}.$$

The abbreviation "i.o." means "infinitely often". First we study some preliminary results.

Lemma 4.1. *Let B be a real invertible $d\times d$ matrix all of whose eigenvalues have absolute values less than* 1. *Let ζ_n, $n\ge 1$, be B-decomposable distributions on $\mathbf{R}^d$ such that*

$$\widehat{\zeta}_n(z)=\widehat{\zeta}_n(B'z)\widehat{\eta}_n(z), \tag{4.1}$$

where η_n are probability distributions on $\mathbf{R}^d$ satisfying $\int_{\mathbf{R}^d}\log(1+|x|)\eta_n(dx)<\infty$. Suppose that η_n converges weakly to a probability distribution η_∞ on $\mathbf{R}^d$ as $n\to\infty$ and

$$\lim_{N\to\infty}\sup_{n\ge 1}\int_{|x|\ge N}\log(1+|x|)\eta_n(dx)=0. \tag{4.2}$$

Then ζ_n converges weakly to some B-decomposable distribution ζ_∞ on $\mathbf{R}^d$ as $n\to\infty$, *which is defined by*

$$\widehat{\zeta}_\infty(z) = \widehat{\zeta}_\infty(B'z)\widehat{\eta}_\infty(z).$$

Proof. Fix $z \in \mathbf{R}^d$ and let $M \in \mathbf{N}$. We obtain from (4.1) that

$$\widehat{\zeta}_n(z) = \prod_{l=0}^{M-1} \widehat{\eta}_n\big((B')^l z\big) \prod_{l=M}^{\infty} \left(1 + \int_{\mathbf{R}^d} \big(e^{i\langle z, B^l x\rangle} - 1\big)\eta_n(dx)\right). \tag{4.3}$$

Choose $k \in \mathbf{N}$ satisfying $\|B^k\| < 1$. Then we have

$$\begin{aligned}\sum_{l=M}^{\infty} \left| \int_{|x| \le \|B^k\|^{-l/(2k)}} \big(e^{i\langle z, B^l x\rangle} - 1\big)\eta_n(dx)\right| &\le \sum_{l=M}^{\infty} \int_{|x| \le \|B^k\|^{-l/(2k)}} |\langle z, B^l x\rangle| \eta_n(dx) \\ &\le c_1 \sum_{l=M}^{\infty} |z| \|B^k\|^{l/(2k)} \le c_2 |z| \|B^k\|^{M/(2k)}\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\sum_{l=M}^{\infty} \left| \int_{|x| \ge \|B^k\|^{-l/(2k)}} (e^{i\langle z, B^l x\rangle} - 1)\eta_n(dx)\right| &\le 2 \sum_{l=M}^{\infty} \int_{|x| \ge \|B^k\|^{-l/(2k)}} \eta_n(dx) \\ &\le c_3 \int_{|x| \ge \|B^k\|^{-M/(2k)}} \log\big(2 + |x|\big)\eta_n(dx),\end{aligned} \tag{4.5}$$

where c_j, $j = 1, 2, 3$, are positive constants. We see from (4.2), (4.4) and (4.5) that

$$\begin{aligned}&\limsup_{M\to\infty} \sup_{n \ge 1} \sum_{l=M}^{\infty} \left| \int_{\mathbf{R}^d} \big(e^{i\langle z, B^l x\rangle} - 1\big)\eta_n(dx)\right| \\ &\le \limsup_{M\to\infty} \left(c_2 |z| \|B^k\|^{M/(2k)} + c_3 \sup_{n \ge 1} \int_{|x| \ge \|B^k\|^{-M/(2k)}} \log\big(2 + |x|\big)\eta_n(dx)\right) = 0.\end{aligned} \tag{4.6}$$

Note that

$$\lim_{n\to\infty} \prod_{l=0}^{M-1} \widehat{\eta}_n\big((B')^l z\big) = \prod_{l=0}^{M-1} \widehat{\eta}_\infty\big((B')^l z\big).$$

It follows from (4.3) and (4.6) that

$$\lim_{n\to\infty} \widehat{\zeta}_n(z) = \lim_{n\to\infty} \prod_{l=0}^{\infty} \widehat{\eta}_n\big((B')^l z\big) = \prod_{l=0}^{\infty} \widehat{\eta}_\infty\big((B')^l z\big) = \widehat{\zeta}_\infty(z).$$

Thus ζ_n converges weakly to ζ_∞ as $n \to \infty$. □

For two positive functions $f(t)$ and $g(t)$ on $[1,\infty)$, we define a relation $f(t) \asymp g(t)$ as $\limsup_{t\to\infty} g(t)/f(t) < \infty$ and $\liminf_{t\to\infty} g(t)/f(t) > 0$, and a relation $f(t) \sim g(t)$ as $\lim_{t\to\infty} g(t)/f(t) = 1$. As mentioned in Section 1, a positive measurable function $f(t)$ on $(0,\infty)$ is said to belong to the class OR if, for every $\delta > 1$, $\limsup_{t\to\infty} f(\delta t)/f(t) < \infty$ and $\liminf_{t\to\infty} f(\delta t)/f(t) > 0$. The Laplace transform of a probability distribution μ on $[0,\infty)$ is denoted by $L_\mu(t)$ for $t \ge 0$, that is, $L_\mu(t) = \int_{[0,\infty)} e^{-tx}\mu(dx)$. The following lemma is a version of Theorem 1 of [8]. The proof is similar and omitted.

Lemma 4.2. *Let μ be a probability distribution on $[0,\infty)$. Then the following are equivalent.*
(i) $\mu([0,1/t]) \in OR$.
(ii) $L_\mu(t) \in OR$.
(iii) $\mu([0,1/t]) \asymp L_\mu(t)$ *as* $t \to \infty$.

Define a regularly varying function $K_\lambda(x)$ on $(0,\infty)$ with the index $-\log\lambda/\log b$ for a probability distribution ρ on $[0,\infty)$ with $\lambda := \rho(\{0\}) > 0$ as

$$K_\lambda(x) = x^{-\log\lambda/\log b} \exp\left(\int_1^x \frac{\log\lambda - \log L_\rho(u)}{u\log b}\,du\right). \tag{4.7}$$

The following proposition is an extension of Theorem 1.6 of [28] concerning one-sided selfdecomposable distributions.

Proposition 4.1. *Let $B = b \in (0,1)$ and let μ be a B-decomposable distribution on $[0,\infty)$ with ρ in (2.1). If $\lambda := \rho(\{0\}) > 0$, then*

$$\mu\left(\left[0,\frac{1}{t}\right]\right) \asymp K_\lambda(t) \quad \text{as } t\to\infty. \tag{4.8}$$

Proof. We have by (2.1)

$$L_\mu(t) = \prod_{n=0}^{\infty} L_\rho(b^n t).$$

Hence we get that

$$\lim_{t\to\infty}\frac{L_\mu(b^{-1}t)}{L_\mu(t)} = \lim_{t\to\infty} L_\rho(b^{-1}t) = \lambda.$$

Since $L_\mu(t)$ is decreasing, it follows that $L_\mu(t) \in OR$. Define $N(t) = [-\log t/\log b]$. Since $tb^{N(t)} \asymp 1$ and

$$L_\mu(t) = \prod_{n=0}^{N(t)} L_\rho(b^n t)\prod_{n=0}^{\infty} L_\rho(b^{N(t)+n+1}t),$$

we see that

$$\begin{aligned} L_\mu(t) &\asymp \prod_{n=0}^{N(t)} L_\rho(b^n t) = \exp\left(\sum_{n=0}^{N(t)} \log L_\rho(b^n t)\right) \\ &\asymp \exp\left(-\int_1^t \frac{\log L_\rho(u)}{u\log b}\,du\right) = K_\lambda(t). \end{aligned}$$

Therefore, we obtain (4.8) from Lemma 4.2. □

Proposition 4.2. *Let $B = b \in (0,1)$ and let μ be a B-decomposable distribution on $[0,\infty)$ with ρ in (2.1). Then there are positive constants c_1 and c_2 such that, for $0 < t < b$ and $0 < \varepsilon < 1 - b$,*

$$\mu([0,t]) \le c_1 \exp\left(-\int_t^1 \frac{\log\rho([0,u])}{u\log b}\,du\right) \tag{4.9}$$

and

$$\mu([0,t]) \ge c_2 \exp\left(\int_{b\varepsilon t}^1 \frac{\log\rho([0,u])}{u\log(b^{-1}(1-\varepsilon))}\,du\right). \tag{4.10}$$

Proof. We use c_j, $j \ge 1$, as positive constants. We see from (2.1) that

$$\mu([0,t]) = \int_{[0,t]} \mu\big([0, b^{-1}(t-x)]\big)\rho(dx). \tag{4.11}$$

Define $\widetilde{N}(t) = [\log t/\log b]$ and $M(t) = [-\log t/\log(b^{-1}(1-\varepsilon))]$. We have by (4.11)

$$\begin{aligned} \mu([0,t]) &\le \mu\big([0,b^{-1}t]\big)\rho\big([0,t]\big) \\ &\le \prod_{n=0}^{\widetilde{N}(t)} \rho\big([0,b^{-n}t]\big)\mu\big([0,b^{-\widetilde{N}(t)-1}t]\big) \end{aligned} \tag{4.12}$$

$$\le c_3 \exp\left(\sum_{n=0}^{\widetilde{N}(t)} \log \rho([0, b^{-n}t])\right)$$
$$\le c_1 \exp\left(-\int_t^1 \frac{\log \rho([0,u])}{u \log b}\, du\right).$$

On the other hand, we see from (4.11) that, for any $\varepsilon \in (0, 1-b)$,

$$\begin{aligned}\mu([0,t]) &\ge \mu\big([0,(b^{-1}(1-\varepsilon))t]\big)\rho\big([0,\varepsilon t]\big)\\ &\ge \prod_{n=0}^{M(t)} \rho\big([0,\varepsilon(b^{-1}(1-\varepsilon))^n t]\big)\mu\big([0,(b^{-1}(1-\varepsilon))^{M(t)+1}t]\big)\\ &\ge c_4 \exp\left(\sum_{n=0}^{M(t)} \log \rho\big([0,\varepsilon(b^{-1}(1-\varepsilon))^n t]\big)\right)\\ &\ge c_2 \exp\left(\int_{b\varepsilon t}^1 \frac{\log \rho([0,u])}{u \log(b^{-1}(1-\varepsilon))}\, du\right).\end{aligned}$$

Thus the proof of the proposition is complete. □

REMARK 4.1. Under the same assumption as in Proposition 4.2, we see from (4.9) that, if $\rho(\{0\}) \ne 1$, then

$$\int_0^1 \mu\big([0,t]\big)t^{-1}dt < \infty.$$

Now we present a key theorem in this section.

Theorem 4.1. *Let $g(x) \in \mathcal{G}_0$. If*

$$\int_0^\infty P\big(X(0) \le g(x)\big)\, dx < \infty \quad \text{(resp. } = \infty), \tag{4.13}$$

then

$$P\big(X(n) \le a^n g(n) \quad \text{i.o. as} \quad n \to \infty\big) = 0 \quad \big(\text{resp.} \quad = 1\big) \tag{4.14}$$

and

$$P\big(X(n) \le a^n g(-n) \quad \text{i.o. as } n \to -\infty\big) = 0 \quad \big(\text{resp.} \quad = 1\big). \tag{4.15}$$

Proof. Suppose that

$$\int_0^\infty P\big(X(0) \le g(x)\big)\, dx < \infty,$$

that is,

$$\sum_{n=0}^{\infty} P\big(X(0) \le g(n)\big) < \infty.$$

Then we have by the shift a-selfsimilarity

$$\sum_{n=0}^{\infty} P\big(X(n) \le a^n g(n)\big) < \infty.$$

Hence we see from the Borel-Cantelli lemma that almost surely $X(n) > a^n g(n)$ for all large n, that is,

$$P(X(n) \le a^n g(n) \quad \text{i.o. as } n \to \infty) = 0.$$

Conversely, suppose that

$$\int_0^{\infty} P\big(X(0) \le g(x)\big)\, dx = \infty,$$

that is,

$$\sum_{n=0}^{\infty} P\big(X(0) \le g(n)\big) = \infty.$$

Then we get by the shift a-selfsimilarity

$$\sum_{n=0}^{\infty} P\big(X(n) \le a^n g(n)\big) = \infty. \tag{4.16}$$

Define the events A_n and a sequence s_n with $l \in \mathbf{N}$ as

$$A_n = \{\omega : X(n) \le a^n g(n)\}$$

and

$$s_n = P\big(X(jl) - X(nl) > a^{jl} g(jl) \quad \text{for } \forall j \ge n+1\big).$$

We find from (4.16) that there is j with $0 \le j \le l-1$ such that $\sum_{n=0}^{\infty} P(A_{nl+j}) = \infty$. We assume that $j = 0$. Discussion in the case $j \ne 0$ is similar. We have by the independence of increments

$$1 \ge P\left(\bigcup_{n=0}^{\infty} A_{nl}\right) \ge \sum_{n=0}^{\infty} P\left(\left(\bigcup_{j=n+1}^{\infty} A_{jl}\right)^c \cap A_{nl}\right) \ge \sum_{n=0}^{\infty} P(A_{nl}) s_n.$$

Hence we see that there is a subsequence $S(k) := s_{n_k}$ such that $S(k) \to 0$ as $k \to \infty$. Define a sequence $T(k)$ and a sequence $p_k(x)$ of functions on $[0, \infty)$ as

$$T(k) = P(X(nl) > a^{nl} g(nl) \quad \text{for } \forall n \geq n_k + 1)$$

and

$$p_k(x) = P\big(X(nl) - X((n_k + 1)l) > a^{nl} g(nl) - a^{(n_k+1)l} x \quad \text{for } \forall n \geq n_k + 2\big).$$

Note that $p_k(x)$ is increasing and bounded in x. We have

$$S(k) = \int_{(g((n_k+1)l), \infty)} p_k(x) \rho_l(dx)$$

and

$$T(k) = \int_{(g((n_k+1)l), \infty)} p_k(x) \mu_0(dx).$$

We show that $T(k) \to 0$ as $k \to \infty$ by considering two possible cases. Case (i). $M := \sup S_{\mu_0} < \infty$; Case (ii). $M = \infty$. In Case (i) we can choose, for any $\varepsilon > 0$, sufficiently large l such that $M - \varepsilon < \sup S_{\rho_l}$. If $\lim_{x\to\infty} g(x) \geq M$, then trivially $T(k) = 0$ for all $k \geq 0$. Thus we can and do assume that $\lim_{x\to\infty} g(x) < M$. Hence we obtain that, for sufficiently small $\varepsilon > 0$,

$$0 = \lim_{k\to\infty} S(k) \geq \lim_{k\to\infty} p_k(M - \varepsilon) \rho_l\big([M - \varepsilon, M]\big),$$

that is, $\lim_{k\to\infty} p_k(M - \varepsilon) = 0$. We have

$$T(k) \leq \int_{[0, M-\varepsilon)} p_k(M - \varepsilon) \mu_0(dx) + \mu_0\big([M - \varepsilon, M]\big).$$

Letting $k \to \infty$ and then $\varepsilon \downarrow 0$, we see from the continuity of μ_0 that $\lim_{k\to\infty} T(k) = 0$. In Case (ii), we can prove by the same way that $\lim_{k\to\infty} T(k) = 0$. Denote the events B_k as

$$B_k = \{\omega : X(n) \leq a^n g(n) \text{ for some } n \geq n_k l\}.$$

Then B_k is decreasing and $P(B_k) \geq 1 - T(k)$. It follows that

$$P(B_k) = P\left(\bigcap_{k=1}^{\infty} B_k\right) = 1,$$

that is,

$$P\big(X(n) \leq a^n g(n) \quad \text{i.o. as } n \to \infty\big) = 1.$$

The proof of (4.15) is similar and omitted. Thus we have proved the theorem. □

Corollary 4.1. *Let* $g(x) \in \mathcal{G}_0$ *and* $c \in [0, \infty]$. *Then*

$$\liminf_{n\to\infty} \frac{X(n)}{a^n g(n)} = c \quad \text{a.s.} \tag{4.17}$$

if and only if

$$\int_0^\infty P\big(X(0) \le \delta g(x)\big)\, dx \begin{cases} < \infty & \text{for } 0 < \delta < c \\ = \infty & \text{for } \delta > c. \end{cases} \tag{4.18}$$

Thus, for any $g(x) \in \mathcal{G}_0$, *there exists* $c \in [0, \infty]$ *such that* (4.17) *holds.*

Proof. The corollary is clear from Theorem 4.1. □

REMARK 4.2. We see from Remark 4.1 and Corollary 4.1 that, for any $\varepsilon \in (0, a)$,

$$\lim_{n\to\infty} \frac{X(n)}{(a-\varepsilon)^n} = \infty \quad \text{a.s.}$$

Theorem 4.2. *There exists* $g(x) \in \mathcal{G}_0$ *satisfying*

$$\liminf_{n\to\infty} \frac{X(n)}{a^n g(n)} = 1 \quad \text{a.s.} \tag{4.19}$$

if and only if $\rho_1(\{0\}) = 0$.

Proof. Suppose that $\rho_1(\{0\}) > 0$. Since μ_0 is a^{-1}-decomposable, we have as in (4.11)

$$\begin{aligned} \mu_0([0,x]) &= \int_{[0,x]} \mu_0\big([0, a(x-u)]\big)\rho_1(du) \\ &\ge \mu_0\big([0, ax]\big)\rho_1(\{0\}). \end{aligned} \tag{4.20}$$

If there is $g(x) \in \mathcal{G}_0$ satisfying (4.19), then we get by Corollary 4.1 that

$$\int_0^\infty \mu_0\big([0, \sqrt{a}\, g(x)]\big)\, dx = \infty \text{ and } \int_0^\infty \mu_0\left(\left[0, \frac{g(x)}{\sqrt{a}}\right]\right) dx < \infty.$$

But they contradict (4.20). Hence if $\rho_1(\{0\}) > 0$, then there is no $g(x) \in \mathcal{G}_0$ satisfying (4.19). Conversely, suppose that $\rho_1(\{0\}) = 0$. If $M := \inf S_{\rho_1} > 0$, then $\inf S_{\mu_0} = M(1-a^{-1})^{-1} > 0$. Define $g(x) = M(1-a^{-1})^{-1}$ on $[0,\infty)$. Then we have (4.18) with $c = 1$ and hence (4.19) by Corollary 4.1. Thus it is enough to construct $g(x) \in \mathcal{G}_0$ satisfying (4.19) under the assumption that $0 \in S_{\rho_1}$ and $\rho_1(\{0\}) = 0$.

We see from the assumption that there is a positive and decreasing sequence a_n such that

$$\sum_{n=0}^{\infty} \rho_1\big([0, a_n]\big) < \infty. \tag{4.21}$$

We see as in (4.12) that

$$\mu_0\big([0, x]\big) \le \mu_0\big([0, ax]\big)\rho_1\big([0, x]\big). \tag{4.22}$$

Hence we obtain that

$$\mu_0\big([0, a^{-1}a_n]\big) \le \mu_0\big([0, a_n]\big)\rho_1\big([0, a_n]\big) \tag{4.23}$$

and

$$\mu_0\big([0, aa_n]\big) \ge \frac{\mu_0([0, a_n])}{\rho_1([0, a_n])}. \tag{4.24}$$

We define an increasing sequence b_n by induction as follows. Set $b_0 = 0$. Assume that b_n are defined for $0 \le n \le k$. Then we define b_{k+1} considering two cases. If $\mu_0([0, a_k]) \le \rho_1([0, a_k])$, then choose b_{k+1} satisfying

$$\rho_1\big([0, a_k]\big) \le \mu_0\big([0, a_k]\big)(b_{k+1} - b_k) \le 1.$$

If $\mu_0([0, a_k]) > \rho_1([0, a_k])$, then set $b_{k+1} = b_k + 1$. Note that $b_{n+1} \ge b_n + 1$ and hence $\lim_{n\to\infty} b_n = \infty$. Define $g(x) \in \mathcal{G}_0$ as $g(x) = a_n$ on $[b_n, b_{n+1})$ for $n \in \mathbf{Z}_+$. Then we have by (4.21) and (4.23)

$$\begin{aligned}\int_0^{\infty} \mu_0\big([0, a^{-1}g(x)]\big)\, dx &\le \sum_{n=0}^{\infty} \mu_0\big([0, a_n]\big)\rho_1\big([0, a_n]\big)(b_{n+1} - b_n)\\ &\le \sum_{n=0}^{\infty} \rho_1\big([0, a_n]\big) < \infty.\end{aligned}$$

On the other hand we get by (4.24)

$$\int_0^{\infty} \mu_0\big([0, ag(x)]\big)\, dx \ge \sum_{n=1}^{\infty} \frac{\mu_0([0, a_n])}{\rho_1([0, a_n])}(b_{n+1} - b_n) = \infty.$$

It follows from Corollary 4.1 that there is $c \in [a^{-1}, a]$ satisfying (4.18). Thus we get (4.19) using $cg(n)$ in place of $g(n)$. □

Corollary 4.2. *Let $g(x) \in \mathcal{G}_0$. Let $K_\lambda(x)$ be the function defined in* (4.7) *with $\rho = \rho_1$ and $b = a^{-1}$. Suppose that $\lambda := \rho_1(\{0\}) > 0$. If*

$$\int_0^\infty K_\lambda\left(\frac{1}{g(x)}\right) dx = \infty \quad (\text{resp. } < \infty),$$

then

$$\liminf_{n\to\infty} \frac{X(n)}{a^n g(n)} = 0 \quad (\text{resp. } = \infty) \qquad \text{a.s.}$$

Proof. We see from Proposition 4.1 that

$$\int_0^\infty P\big(X(0) \le \delta g(x)\big)\, dx = \infty \quad (\text{resp. } < \infty) \quad \text{for } \forall\delta > 0$$

if and only if

$$\int_0^\infty K_\lambda\left(\frac{1}{g(x)}\right) dx = \infty \quad (\text{resp. } < \infty).$$

Therefore the corollary follows from Corollary 4.1. □

In the following theorem, we fix $a > 1$ and consider the family of all increasing shift a-selfsimilar additive random sequences $\{X(n), n \in \mathbf{Z}\}$.

Theorem 4.3. *Let $g(x) \in \mathcal{G}_0$. There exists $\{X(n), n \in \mathbf{Z}\}$ satisfying* (4.19) *if and only if*

$$\liminf_{x\to\infty} \frac{-\log g(x)}{\log x} = 0. \tag{4.25}$$

Proof. We use c_j as positive constants. Without loss of generality, we can assume that $g(1) < 1$. Suppose that (4.25) is not true and that there is $\{X(n), n \in \mathbf{Z}\}$ satisfying (4.19). Then we see that $\rho_1(\{0\}) = 0$ by Theorem 4.2 and there is $L > 0$ such that

$$g(x) \le x^{-L} \qquad \text{on } [1, \infty).$$

Noting that $\rho_1([0, u]) \downarrow 0$ as $u \downarrow 0$, we have by (4.9), for any $\delta > 0$,

$$\begin{aligned}\int_0^\infty P\big(X(0) \le \delta g(x)\big)\, dx &\le 1 + \int_1^\infty P\big(X(0) \le \delta x^{-L}\big)\, dx \\ &\le 1 + c_1 \int_1^\infty \exp\left(\int_{\delta x^{-L}}^1 \frac{\log \rho_1([0, u])}{u \log a}\, du\right) dx < \infty.\end{aligned}$$

It follows from Corollary 4.1 that

$$\liminf_{n\to\infty} \frac{X(n)}{a^n g(n)} = \infty \quad \text{a.s.}$$

This is a contradiction. Thus if there is $\{X(n),\ n \in \mathbf{Z}\}$ satisfying (4.19), then (4.25) is true. Conversely, suppose that the condition (4.25) is true. In case $N := \lim_{x\to\infty} g(x) > 0$, define ρ_1 as $\rho_1(\{(1-a^{-1})N\}) = \rho_1(\{2(1-a^{-1})N\}) = 2^{-1}$. Then (4.19) is true by Corollary 4.1. Thus we can assume that $N = 0$. We show the existence of $\{X(n), n \in \mathbf{Z}\}$ satisfying (4.19) by constructing the measure ρ_1. The condition (4.25) says that there are sequences $x_n \uparrow \infty$ and $\delta_n \downarrow 0$ for $n \in \mathbf{Z}_+$ such that $x_0 = 1$, $x_{n+1} > 2x_n$,

$$\varepsilon^{-1} a g(2^{-1}x_{n+1}) \le e^{-1} g(x_n) \quad \text{with } \varepsilon = 1 - a^{-1/2} \tag{4.26}$$

and

$$g(x_n) \ge x_n^{-\delta_n}. \tag{4.27}$$

We construct $\rho_1([0,u])$ together with $n_k \uparrow \infty$ and $M_k \uparrow \infty$ for $k \in \mathbf{Z}_+$ by induction in such a way that with $b_k := x_{n_k}$

$$M_{k+1} \ge M_k + 1 \text{ and } \rho_1([0,u]) = e^{-M_k \log a} \text{ on } \big[g(b_k), g(b_{k-1})\big) \text{ for } k \in \mathbf{Z}_+$$

and

$$2^{-1} \le \int_{2^{-1}b_k}^{b_k} \mu_0\big([0, \varepsilon^{-1} a g(x)]\big)\,dx + \int_{b_{k-1}}^{2^{-1}b_k} \mu_0\big([0, e^{-1} g(x)]\big)\,dx \le 2 \text{ for } k \ge 1. \tag{4.28}$$

First set $n_{-1} = n_0 = 0$ and $M_0 = 0$. Let $l \ge 0$. Let $\{X^{(l)}(n), n \in \mathbf{Z}\}$ be an increasing shift a-selfsimilar additive random sequence with

$$\rho_1^{(l)}([0,u]) = \begin{cases} e^{-M_k \log a}, & \text{on } \big[g(b_k), g(b_{k-1})\big) \text{ for } 0 \le k \le l \\ 0, & \text{on } \big[0, g(b_l)\big). \end{cases}$$

Denote the distribution of $X^{(l)}(0)$ by $\mu_0^{(l)}$. Define, for $1 \le k \le l$,

$$J^{(l)}(k) = \int_{2^{-1}b_k}^{b_k} \mu_0^{(l)}\big([0, \varepsilon^{-1} a g(x)]\big)\,dx + \int_{b_{k-1}}^{2^{-1}b_k} \mu_0^{(l)}\big([0, e^{-1} g(x)]\big)\,dx.$$

Assume that $M_k \ge M_{k-1} + 1$ and $2^{-1} + 2^{-l} \le J^{(l)}(k) \le 2 - 2^{-l}$ for $1 \le k \le l$. Temporarily set, for some $n \ge n_l + 1$ and $M \ge M_l + 1$,

$$\rho_1([0,u]) = \begin{cases} e^{-M_k \log a}, & \text{on } \big[g(b_k), g(b_{k-1})\big) \text{ for } 0 \le k \le l \\ e^{-M \log a}, & \text{on } \big[g(x_n), g(b_l)\big) \\ 0, & \text{on } \big[0, g(x_n)\big) \end{cases}$$

and define

$$I(n, M) = \int_{2^{-1}x_n}^{x_n} \mu_0\big([0, \varepsilon^{-1}ag(x)]\big)\,dx + \int_{b_l}^{2^{-1}x_n} \mu_0\big([0, e^{-1}g(x)]\big)\,dx.$$

Fix $M \geq M_l + 1$, then we obtain from (4.10) and (4.27) that

$$\begin{aligned}
\liminf_{n\to\infty} I(n, M) &\geq \liminf_{n\to\infty} \int_{2^{-1}x_n}^{x_n} \mu_0\big([0, \varepsilon^{-1}ag(x)]\big)\,dx \\
&\geq \liminf_{n\to\infty} \mu_0\big([0, \varepsilon^{-1}ag(x_n)]\big)2^{-1}x_n \\
&\geq c_2 \liminf_{n\to\infty} \exp\left(-\int_{g(x_n)}^{1} \frac{M\log a}{u\log(a(1-\varepsilon))}\,du\right)x_n \\
&= c_2 \liminf_{n\to\infty} g(x_n)^{2M}x_n \geq c_2 \lim_{n\to\infty} x_n^{(1-2M\delta_n)} = \infty.
\end{aligned}$$

On the other hand, fix $n \geq n_l + 1$, then we get by (4.9) and (4.26) that

$$\begin{aligned}
\limsup_{M\to\infty} I(n, M) &\leq 2^{-1} \limsup_{M\to\infty} \mu_0\big([0, e^{-1}g(b_l)]\big)x_n + \limsup_{M\to\infty} \int_{b_l}^{2^{-1}x_n} \mu_0\big([0, e^{-1}g(b_l)]\big)\,dx \\
&\leq \limsup_{M\to\infty} \mu_0\big([0, e^{-1}g(b_l)]\big)x_n \\
&\leq c_3 \limsup_{M\to\infty} \exp\left(-\int_{e^{-1}g(b_l)}^{g(b_l)} \frac{M}{u}\,du\right)x_n \\
&= c_3 \lim_{M\to\infty} \exp(-M)x_n = 0.
\end{aligned}$$

Hence we have $I(n, M_l + 1) \geq 1$ for sufficiently large n satisfying $n \geq n_l + 1$. Since $I(n, M)$ is continuous in M on account of Lemma 4.1, we can take $M = M(n) \geq M_l + 1$ such that $I(n, M(n)) = 1$ for sufficiently large n. Since $M(n) \to \infty$ as $n \to \infty$, ρ_1 and μ_0 are convergent weakly to $\rho_1^{(l)}$ and $\mu_0^{(l)}$, respectively as $n \to \infty$ by virtue of Lemma 4.1. Hence we can choose sufficiently large $n = n_{l+1}$ and define $\rho_1^{(l+1)}$ such that $M_{l+1} = M(n_{l+1})$ and $2^{-1} + 2^{-l-1} \leq J^{(l+1)}(k) \leq 2 - 2^{-l-1}$ for $1 \leq k \leq l + 1$. Finally we define ρ_1 as the weak limit of $\rho_1^{(l)}$ as $l \to \infty$. Let $\{X(n), n \in \mathbf{Z}\}$ be the corresponding increasing shift a-selfsimilar additive random sequence. Then (4.28) is satisfied clearly by virtue of Lemma 4.1. Hence we see that

$$\int_0^\infty \mu_0\big([0, \varepsilon^{-1}ag(x)]\big)\,dx = \infty$$

and that

$$\int_0^\infty \mu_0\big([0, (ae)^{-1}g(x)]\big)\,dx \leq 1 + \sum_{k=1}^{\infty} \int_{b_{k-1}}^{b_k} \mu_0\big([0, e^{-1}g(x)]\big)\rho_1\big([0, g(x)]\big)\,dx$$

$$\leq 1 + 2 \sum_{k=1}^{\infty} e^{-M_k \log a} < \infty,$$

using (4.22) and noting $M_{k+1} \geq M_k + 1$. It follows from Corollary 4.1 that (4.18) holds for some $c \in [(ae)^{-1}, \varepsilon^{-1} a]$ and hence (4.19) is true by replacing $\{X(n), n \in \mathbf{Z}\}$ with $\{c^{-1} X(n), n \in \mathbf{Z}\}$. □

In the proof of the theorem above, we have proved the following corollary.

Corollary 4.3. *Let $g(x) \in \mathcal{G}_0$. Suppose that $\rho_1(\{0\}) = 0$ and*

$$\liminf_{x \to \infty} \frac{-\log g(x)}{\log x} > 0.$$

Then we have

$$\liminf_{n \to \infty} \frac{X(n)}{a^n g(n)} = \infty \quad \text{a.s.}$$

5. Rate of growth II

In this section, let $\{X(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued non-zero shift a-selfsimilar additive random sequence for some $a > 1$. We study the rate of growth of $\{X(n), n \in \mathbf{Z}\}$ in the "limsup" case. We state the results only as $n \to \infty$ except in Remark 5.1. Define

$$\mathcal{G}_1 = \{g(x) : g(x) \text{ is positive and increasing on } [0, \infty)\}.$$

Define the inverse function $g^{-1}(x)$ on $[0, \infty)$ of $g(x) \in \mathcal{G}_1$ as

$$g^{-1}(x) = \sup\{y \geq 0 : g(y) < x\}$$

with understanding that $\sup \emptyset = 0$. Define a function $\rho_1^*(x)$ on $[0, \infty)$ as

$$\rho_1^*(x) = P\big(|X(0) - X(-1)| > x\big).$$

A positive measurable function $h(x)$ on $[0, \infty)$ is said to be *submultiplicative* if there is a positive constant C_1 such that

$$h(x + y) \leq C_1 h(x) h(y) \qquad \text{for } \forall x, \ \ y \geq 0.$$

Lemma 5.1. *Let $g(x) \in \mathcal{G}_1$ and let $l \in \mathbf{N}$. If*

$$\int_0^{\infty} P\big(|X(0) - X(-l)| > g(x)\big)\, dx = \infty,$$

then, for all $k \in \mathbf{N}$ *and all* $\varepsilon \in (0,1)$,

$$\int_0^\infty P\big(|X(0) - X(-kl)| > (1-\varepsilon)g(x)\big)\,dx = \infty.$$

Proof. Note that, for any $j \in \mathbf{N}$,

$$\int_0^\infty P\big(|X(0) - X(-j)| > g(x)\big)\,dx = \int_{\mathbf{R}^d} g^{-1}(|x|)\rho_j(dx). \tag{5.1}$$

There are two cases. Case 1. $M := \lim_{x\to\infty} g(x) < \infty$; Case 2. $M = \infty$. In Case 1, $\rho_{kl}(\{x : |x| > (1-\varepsilon)M\}) > 0$ for all $k \in \mathbf{N}$ and all $\varepsilon \in (0,1)$ whenever $\rho_l(\{x : |x| > (1-\varepsilon)M\}) > 0$ for all $\varepsilon \in (0,1)$. Hence we see from (5.1) that, if

$$\int_0^\infty P\big(|X(0) - X(-l)| > g(x)\big)\,dx = \infty,$$

then, for all $k \in \mathbf{N}$ and all $\varepsilon \in (0,1)$,

$$\int_0^\infty P\big(|X(0) - X(-kl)| > (1-\varepsilon)g(x)\big)\,dx = \infty.$$

In Case 2, we find from (5.1) that, if

$$\int_0^\infty P\big(|X(0) - X(-l)| > g(x)\big)\,dx = \infty,$$

then

$$\int_{\mathbf{R}^d} g^{-1}(|x|)\rho_l(dx) = \infty.$$

Choose $N > 0$ such that $\rho_l(\{x : |x| \le N\}) > 0$. We get, for all $k \in \mathbf{N}$ and all $\delta > 0$, that

$$\begin{aligned}
&\int_{\mathbf{R}^d} g^{-1}\big((1+\delta)|x|\big)\rho_{kl}(dx)\\
&\quad = \int_{(\mathbf{R}^d)^k} g^{-1}\left((1+\delta)\left|\sum_{n=0}^{k-1} a^{-nl}x_n\right|\right)\prod_{n=0}^{k-1}\rho_l(dx_n)\\
&\quad \ge \{\rho_l(\{x : |x| \le N\})\}^{k-1}\int_{\mathbf{R}^d} g^{-1}\big((1+\delta)|x| - N(1+\delta)(1-a^{-l})^{-1}\big)\rho_l(dx)\\
&\quad \ge c_1 \int_{\mathbf{R}^d} g^{-1}(|x|)\rho_l(dx),
\end{aligned}$$

where c_1 is a positive constant. Note that we used the condition $M = \infty$ in the last inequality. Thus the lemma is true from (5.1). □

Lemma 5.2. *Let* $g(x) \in \mathcal{G}_1$ *and let* $l \in \mathbf{N}$.
(i) *If*

$$\int_0^\infty P\big(|X(0) - X(-l)| > g(x)\big)\, dx = \infty, \tag{5.2}$$

then

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} \geq 1 \quad \text{a.s.} \tag{5.3}$$

(ii) *If*

$$\int_0^\infty P\big(|X(0) - X(-l)| > g(x)\big)\, dx < \infty, \tag{5.4}$$

then

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} \leq (1 - a^{-l})^{-1} \quad \text{a.s.} \tag{5.5}$$

Proof. Suppose the condition (5.2) holds. That is,

$$\sum_{n=0}^{\infty} P\big(|X(0) - X(-l)| > g(n)\big) = \infty.$$

Then we see from the shift a-selfsimilarity that

$$\sum_{n=0}^{\infty} P\big(|X(n) - X(n-l)| > a^n g(n)\big) = \infty.$$

Hence there is j $(0 \leq j \leq l-1)$ such that

$$\sum_{n=0}^{\infty} P\big(|X(nl+j) - X((n-1)l+j)| > a^{nl+j} g(nl+j)\big) = \infty.$$

Define $b_n = nl + j$ for $n \in \mathbf{Z}$. It follows from the Borel-Cantelli lemma that

$$P\big(|X(b_n) - X(b_{n-1})| > a^{b_n} g(b_n) \text{ i.o. as } n \to \infty\big) = 1.$$

Owing to Kolmogorov's 0-1 law, we see that

$$P\left(\limsup_{n\to\infty} \frac{|X(b_n)|}{a^{b_n} g(b_n)} \geq 1\right) = 0 \text{ or } 1.$$

If the probability above is 1, then (5.3) is true. Thus we can assume that the probability above is 0 and thereby we get

$$\limsup_{n\to\infty} \frac{|X(b_n)|}{a^{b_n} g(b_n)} < 1 \quad \text{a.s.}$$

and hence $|X(b_{n-1})| < a^{b_{n-1}} g(b_{n-1})$ for all large n almost surely. Hence we obtain that

$$\begin{aligned} &P\big(|X(b_n)| > (1-a^{-l})a^{b_n} g(b_n) \text{ i.o. as } n\to\infty\big)\\ &\geq P\big(|X(b_n) - X(b_{n-1})| > (1-a^{-l})a^{b_n} g(b_n) + |X(b_{n-1})| \text{ i.o. as } n\to\infty\big)\\ &\geq P\big(|X(b_n) - X(b_{n-1})| > a^{b_n} g(b_n) \text{ i.o. as } n\to\infty\big) = 1. \end{aligned}$$

Thus we have

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} \geq 1 - a^{-l} \quad \text{a.s.} \tag{5.6}$$

Hence we get (5.3) by using Lemma 5.1.

Suppose the condition (5.4) holds. Namely,

$$\sum_{n=0}^{\infty} P\big(|X(0) - X(-l)| > g(n)\big) < \infty.$$

Then we find from the shift a-selfsimilarity that

$$\sum_{n=0}^{\infty} P\big(|X(n) - X(n-l)| > a^n g(n)\big) < \infty.$$

It follows from the Borel-Cantelli lemma that there is $N(\omega) \in \mathbf{Z}_+$ such that

$$P\big(|X(n) - X(n-l)| \leq a^n g(n) \text{ for } \forall n \geq N(\omega)\big) = 1. \tag{5.7}$$

Let $n \geq N(\omega)$ and let $m = m(\omega)$ be the largest integer satisfying $n \geq ml + N(\omega)$. We obtain from (5.7) that, for $0 \leq j \leq m$,

$$|X(n-jl)| - |X(n-(j+1)l)| \leq a^{n-jl} g(n-jl) \quad \text{a.s.} \tag{5.8}$$

Summing up (5.8) in j, we have

$$|X(n)| - |X(n-(m+1)l)| \leq \sum_{j=0}^{m} a^{n-jl} g(n-jl) \leq \frac{a^n g(n)}{1-a^{-l}} \quad \text{a.s.}$$

Hence we see that

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} \le \frac{1}{1-a^{-l}} \quad \text{a.s.}$$

Thus we establish the inequality (5.5). □

REMARK 5.1. The assertions of the lemma above remain valid as $n \to -\infty$. However, we must replace a^{-l} by a_1^{-l} for some $a_1 \in (1, a)$ in (5.5) and (5.6) as $n \to -\infty$. We need an analogue of Lemma 4.4 of [31], which is proved by virtue of (2.4).

Theorem 5.1. *Let $g(x) \in \mathcal{G}_1$ and $C \in [0, \infty]$. Then*

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} = C \quad \text{a.s.} \tag{5.9}$$

if and only if

$$\int_0^\infty P\big(|X(0)-X(-l)| > \delta g(x)\big)\,dx \begin{cases} = \infty & \text{for } 0 < \forall\delta < C \text{ and } \exists l(\delta) \in \mathbf{N} \\ < \infty & \text{for } \forall\delta > C \text{ and } \forall l \in \mathbf{N}. \end{cases} \tag{5.10}$$

Thus, for every $g(x) \in \mathcal{G}_1$, there exists $C \in [0, \infty]$ such that (5.9) *holds.*

Proof. The proof is clear from Lemma 5.2. □

Theorem 5.2. *There exists $g(x) \in \mathcal{G}_1$ satisfying*

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} = 1 \quad \text{a.s.} \tag{5.11}$$

if and only if $\rho_1^*(x) \notin OR$.

Proof. Suppose that $\rho_1^*(x) \in OR$ and there is $g(x) \in \mathcal{G}_1$ satisfying (5.11). Then we see from Lemma 5.2 that

$$\int_0^\infty P\big(|X(0) - X(-1)| > 2^{-1}(1 - a^{-1})g(x)\big)\,dx = \infty$$

and

$$\int_0^\infty P\big(|X(0) - X(-1)| > 2g(x)\big)\,dx < \infty.$$

But they contradict the condition $\rho_1^*(x) \in OR$. Hence if $\rho_1^*(x) \in OR$, then there is no $g(x) \in \mathcal{G}_1$ satisfying (5.11). Conversely, suppose that $\rho_1^*(x) \notin OR$. Then there is a positive sequence $y_n \uparrow \infty$ for $n \in \mathbf{Z}_+$ such that $2^{-n}\rho_1^*(y_n) \ge \rho_1^*(2y_n)$ for $n \in \mathbf{Z}_+$. In case $M := \sup_{x\in S_{\rho_1}} |x| < \infty$, we define $g(x) \in \mathcal{G}_1$ as $g(x) = M(1-a^{-1})^{-1}$ on $[0,\infty)$. Then it is evident that

$$\sup_{x\in S_{\rho_l}} |x| = \frac{(1-a^{-l})M}{1-a^{-1}},$$
$$\int_0^\infty P\big(|X(0)-X(-l)| > g(x)\big)\,dx = 0 \quad \text{for } \forall l \geq 1$$

and

$$\int_0^\infty P\big(|X(0)-X(-l)| > (1-a^{-l+1})g(x)\big)\,dx = \infty \quad \text{for } \forall l \geq 1.$$

Hence we see from Lemma 5.2 that (5.11) is true. In case $M = \infty$, we define $g(x) \in \mathcal{G}_1$ together with $x_n \uparrow \infty$ as $x_0 = 0$ and $g(x) = y_n$ on $[x_n, x_{n+1})$ satisfying

$$1 \leq \rho_1^*(y_n)(x_{n+1}-x_n) \leq 2 \quad \text{for } n \in \mathbf{Z}_+.$$

Then we obtain that

$$\int_0^\infty P\big(|X(0)-X(-1)| > g(x)\big)\,dx = \sum_{n=0}^\infty \rho_1^*(y_n)(x_{n+1}-x_n) = \infty$$

and

$$\int_0^\infty P\big(|X(0)-X(-1)| > 2g(x)\big)\,dx \leq \sum_{n=0}^\infty 2^{-n}\rho_1^*(y_n)(x_{n+1}-x_n) < \infty.$$

It follows from Lemma 5.2 that there is $C \in [1, 2(1-a^{-1})^{-1}]$ satisfying (5.9). Thus we have (5.11) by replacing $g(n)$ with $Cg(n)$. □

Corollary 5.1. *Let $g(x) \in \mathcal{G}_1$. Suppose that $\rho_1^*(x) \in OR$. If*

$$\int_0^\infty \rho_1^*\big(g(x)\big)\,dx < \infty \quad (\text{resp. } = \infty),$$

then

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} = 0 \quad (\text{resp. } = \infty) \quad \text{a.s.}$$

Proof. Proof is clear from Lemma 5.2 and Theorem 5.2. □

As in Theorem 4.3, we fix $a > 1$ and consider the family of all $\mathbf{R}^d$-valued shift a-selfsimilar additive random sequences $\{X(n), n \in \mathbf{Z}\}$ in the following theorem.

Theorem 5.3. *Let $g(x) \in \mathcal{G}_1$. There exists $\{X(n), n \in \mathbf{Z}\}$ satisfying* (5.11) *if and only if $g^{-1}(x) + \log(1+x) \notin OR$.*

Proof. We obtain from (5.1) that

$$\int_0^\infty \rho_1^*\big(g(x)\big)\,dx = \int_{\mathbf{R}^d} g^{-1}(|x|)\rho_1(dx). \tag{5.12}$$

Suppose that $g^{-1}(x) + \log(1+x) \in OR$ and there is $\{X(n), n \in \mathbf{Z}\}$ satisfying (5.11). By the same way as in the proof of the preceding theorem, we see from (5.12) that absurdity occurs. Thus if $g^{-1}(x) + \log(1+x) \in OR$, then there is no $\{X(n), n \in \mathbf{Z}\}$ satisfying (5.11). Conversely, suppose that $g^{-1}(x) + \log(1+x) \notin OR$. In case $g^{-1}(x)$ is not finite, $M := \lim_{x\to\infty} g(x) < \infty$. Let $z = ((1-a^{-1})M, 0, \dots, 0)' \in \mathbf{R}^d$. Define ρ_1 as $\rho_1(\{0\}) = 1/2$ and $\rho_1(\{z\}) = 1/2$. Then we see as in the proof of Theorem 5.2 that (5.11) is true. Thus we can assume that $g^{-1}(x)$ is finite on $[0,\infty)$. So there is $x_n \uparrow \infty$ for $n \in \mathbf{Z}_+$ such that $x_0 = 1$ and $2^{-n}(g^{-1}(x_n) + \log(1+x_n)) \ge g^{-1}(2^{-1}x_n) + \log(1+2^{-1}x_n)$ for $n \in \mathbf{Z}_+$ and $C_2 := \sum_{n=0}^\infty 1/(g^{-1}(x_n) + \log(1+x_n)) < \infty$. Choose $y_n \in \mathbf{R}^d$ for $n \in \mathbf{Z}_+$ satisfying $|y_n| = x_n$. Define ρ_1 as

$$\rho_1(\{y_n\}) = \frac{1}{C_2(g^{-1}(x_n) + \log(1+x_n))} \quad \text{and} \quad \rho_1\left(\left(\bigcup_{n=0}^\infty \{y_n\}\right)^c\right) = 0.$$

Then we obtain that

$$\int_{\mathbf{R}^d} \big(g^{-1}(|x|) + \log(1+|x|)\big)\rho_1(dx) = \sum_{n=0}^\infty C_2^{-1} = \infty$$

and

$$\int_{\mathbf{R}^d} \big(g^{-1}(2^{-1}|x|) + \log(1+2^{-1}|x|)\big)\rho_1(dx)$$
$$= \sum_{n=0}^\infty \frac{g^{-1}(2^{-1}x_n) + \log(1+2^{-1}x_n)}{C_2(g^{-1}(x_n) + \log(1+x_n))} \le C_2^{-1}\sum_{n=0}^\infty 2^{-n} < \infty.$$

It follows from Theorem 5.1 and (5.12) that (5.10) holds for some $C \in [1, 2(1-a^{-1})^{-1}]$ and hence (5.11) is true by replacing $\{X(n),\ n \in \mathbf{Z}\}$ with $\{C^{-1}X(n), n \in \mathbf{Z}\}$. ∎

Corollary 5.2. *Let $g(x) \in \mathcal{G}_1$. Suppose that $g^{-1}(x) + \log(1+x) \in OR$. If*

$$\int_{\mathbf{R}^d} g^{-1}(|x|)\rho_1(dx) < \infty \quad (\text{resp. } = \infty),$$

then

$$\limsup_{n\to\infty} \frac{|X(n)|}{a^n g(n)} = 0 \quad (\text{resp.} \ = \infty) \quad \text{a.s.}$$

Proof. Proof is evident from Lemma 5.2, Theorem 5.3 and (5.12). □

REMARK 5.2. We see from (2.4) and Corollary 5.2 that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|X(n)|}{(a+\varepsilon)^n} = 0 \quad \text{a.s.}$$

It follows from Remark 4.2 that, if $\{X(n), n \in \mathbf{Z}\}$ is increasing and not zero, then

$$\lim_{n\to\infty} \frac{\log X(n)}{n} = \log a \quad \text{a.s.}$$

Corollary 5.3. *Let $g(x) \in \mathcal{G}_1$ and $C \in [0,\infty]$. Suppose that $g^{-1}(x)$ is finite and submultiplicative on $[0,\infty)$. Then (5.9) is true if and only if*

$$\int_{\mathbf{R}^d} g^{-1}(\delta|x|)\rho_1(dx) \begin{cases} < \infty & \text{for } 0 < \delta < C^{-1} \\ = \infty & \text{for } \delta > C^{-1}. \end{cases}$$

Proof. We prove that, for $0 < \delta$,

$$\int_{\mathbf{R}^d} g^{-1}(\delta|x|)\rho_1(dx) < \infty \text{ implies } \int_{\mathbf{R}^d} g^{-1}(\delta|x|)\rho_l(dx) < \infty \text{ for } \forall l \geq 2. \tag{5.13}$$

We have, for $l \geq 2$,

$$\begin{aligned} \int_{\mathbf{R}^d} g^{-1}(\delta|x|)\rho_l(dx) &= \int_{(\mathbf{R}^d)^l} g^{-1}\left(\delta\left|\sum_{n=0}^{l-1} a^{-n}x_n\right|\right)\prod_{n=0}^{l-1}\rho_1(dx_n) \\ &\leq c_1 \prod_{n=0}^{l-1}\int_{\mathbf{R}^d} g^{-1}(\delta a^{-n}|x_n|)\rho_1(dx_n) \\ &\leq c_1\left\{\int_{\mathbf{R}^d} g^{-1}(\delta|x|)\rho_1(dx)\right\}^l, \end{aligned}$$

where c_1 is a positive constant. Thus (5.13) is true. Therefore the corollary follows from (5.1) and Theorem 5.1. □

REMARK 5.3. In the case where ρ_1 is an infinitely divisible distribution on $\mathbf{R}^d$, we can replace ρ_1 and $\mathbf{R}^d$ by the Lévy measure of ρ_1 and $\{x : |x| > 1\}$, respectively in the integral of Corollary 5.3. See Theorem 25.3 of [26].

6. Examples

In this section, we give some examples for the results in Sections 4 and 5. Let $\{X(n), n \in \mathbf{Z}\}$ be an $\mathbf{R}^d$-valued non-zero shift a-selfsimilar additive random sequence for some $a > 1$. In Examples 6.1 and 6.2, we assume that $d = 1$ and $\{X(n), n \in \mathbf{Z}\}$ is increasing. More interesting examples will be found in [34].

EXAMPLE 6.1. Suppose that $\rho_1(dx) = p\delta_0(dx)+(1-p)\delta_1(dx)$ with $0 < p < 1$. Denote $\gamma = -\log p/\log a$, $d_1 = (-p\log p-(1-p)\log(1-p))/\log a$ and $d_2 = \log 2/\log a$. Then μ_n are called infinite Bernoulli convolutions with upper Hausdorff dimension d_1 for $a > p^{-p}(1-p)^{-(1-p)}$ and S_{μ_n} are the Cantor sets with Hausdorff dimension d_2 for $a > 2$.
(i) Let $g(x) \in \mathcal{G}_0$. If

$$\int_0^\infty \{g(x)\}^\gamma \, dx < \infty \quad (\text{resp. } = \infty),$$

then

$$\liminf_{n\to\pm\infty} \frac{X(n)}{a^n g(|n|)} = \infty \quad (\text{resp. } = 0) \qquad \text{a.s.}$$

(ii) We have

$$\limsup_{n\to\pm\infty} \frac{X(n)}{a^n(1-a^{-1})^{-1}} = 1 \qquad \text{a.s.}$$

Proof. Since we have

$$K_\lambda(x) = x^{-\gamma} \exp\left(-\int_1^x \frac{\log p - \log(p+(1-p)e^{-u})}{u\log a}\,du\right) \asymp x^{-\gamma},$$

the assertion (i) follows from Corollary 4.2. The assertion (ii) is essentially proved in the proof of Theorem 5.3. □

EXAMPLE 6.2. Let $0 < \alpha < 1$ and let $\xi(x)$ be a measurable function on $[0,\infty)$ such that $\lambda_1 \le \xi(x) \le \lambda_2$ on $[0,\infty)$ and $\lim_{x\downarrow 0}\xi(x) = \lambda_0$ for some positive constants λ_0, λ_1, and λ_2. Denote the constant C_α as

$$C_\alpha = \left(\frac{1-\alpha}{\alpha}\right)^{(1-\alpha)/\alpha} \left(\frac{\Gamma(1-\alpha)\lambda_0}{1-a^{-\alpha}}\right)^{1/\alpha}.$$

Suppose that ρ_1 is an infinitely divisible distribution on $[0,\infty)$ given by

$$L_{\rho_1}(t) = \exp\left(\int_0^\infty (e^{-tx}-1)\frac{\xi(x)}{x^{\alpha+1}}\,dx\right).$$

(i) We have

$$\liminf_{n\to\pm\infty} \frac{X(n)}{a^n(\log|n|)^{(\alpha-1)/\alpha}} = C_\alpha \quad \text{a.s.}$$

(ii) Let $g(x) \in \mathcal{G}_1$. If

$$\int_0^\infty \{g(x)\}^{-\alpha}\, dx < \infty \quad (\text{resp.} \ = \infty),$$

then

$$\limsup_{n\to\pm\infty} \frac{X(n)}{a^n g(|n|)} = 0 \quad (\text{resp.} \ = \infty) \quad \text{a.s.}$$

Proof. First we prove the assertion (i). We have

$$L_{\mu_0}(t) = \prod_{n=0}^{\infty} L_{\rho_1}(a^{-n}t) = \exp\left(\sum_{n=0}^{\infty}\int_0^\infty (e^{-tx}-1)\frac{\xi(a^n x)a^{-n\alpha}}{x^{\alpha+1}}\,dx\right).$$

Note that

$$\lim_{x\downarrow 0}\sum_{n=0}^{\infty}\xi(a^n x)a^{-n\alpha} = \frac{\lambda_0}{1-a^{-\alpha}}.$$

Hence we see from Theorem 8.2.2 of [2] that

$$-\log\mu_0\left(\left[0,\frac{1}{t}\right]\right) \sim \frac{1-\alpha}{\alpha}\left(\frac{\Gamma(1-\alpha)\lambda_0}{1-a^{-\alpha}}\right)^{1/(1-\alpha)} t^{\alpha/(1-\alpha)} \quad \text{as } t\to\infty. \tag{6.1}$$

Define $g(x) = (\log(x \vee e))^{-(1-\alpha)/\alpha}$. Then $g(x) \in \mathcal{G}_0$ and we obtain (4.18) from (6.1) with $c = C_\alpha$. It follows from Corollary 4.1 that the assertion (i) is true. Next we prove the assertion (ii). We see from Proposition 4.1 of [31] that $\rho_1((x,\infty)) \in OR$ and $\rho_1((x,\infty)) \asymp x^{-\alpha}$ as $x\to\infty$. Hence the assertion (ii) follows from Corollary 5.1. □

EXAMPLE 6.3. Let $\alpha > 0$. If

$$\int_{\mathbf{R}^d} |x|^\alpha \rho_1(dx) < \infty \quad (\text{resp.} \ = \infty),$$

then

$$\limsup_{n\to\pm\infty} \frac{|X(n)|}{a^n|n|^{1/\alpha}} = 0 \quad (\text{resp.} \ = \infty) \quad \text{a.s.}$$

Proof. Proof is clear from Corollary 5.2. □

EXAMPLE 6.4. Let $\beta > 0$ and $C \in [0, \infty]$. Then we have

$$\limsup_{n\to\pm\infty} \frac{|X(n)|}{a^n(\log|n|)^{1/\beta}} = C \quad \text{a.s.} \tag{6.2}$$

if and only if

$$\int_{\mathbf{R}^d} \exp(\delta|x|^\beta)\rho_l(dx) \begin{cases} < \infty & \text{for } 0 < \forall\delta < C^{-\beta} \text{ and } \forall l \in \mathbf{N} \\ = \infty & \text{for } \forall\delta > C^{-\beta} \text{ and } \exists l(\delta) \in \mathbf{N}. \end{cases}$$

In the case where $0 < \beta \leq 1$, (6.2) holds if and only if

$$\int_{\mathbf{R}^d} \exp(\delta|x|^\beta)\rho_1(dx) \begin{cases} < \infty & \text{for } 0 < \delta < C^{-\beta} \\ = \infty & \text{for } \delta > C^{-\beta}. \end{cases}$$

Proof. Let $g(x) = (\log(x \vee e))^{1/\beta}$ on $[0, \infty)$. Obviously, $g(x) \in \mathcal{G}_1$ and $g^{-1}(x) = \exp(x^\beta)$ on $[1, \infty)$. The first assertion is due to (5.1) and Theorem 5.1. Since $g^{-1}(x)$ is submultiplicative on $[0, \infty)$ for $0 < \beta \leq 1$, the second assertion follows from Corollary 5.3. □

REMARK 6.1. Let $\{W(t), t \geq 0\}$ be a Brownian motion in $\mathbf{R}^d$ and set $X(n) = W(a^{2n})$ for $n \in \mathbf{Z}$. Then the equation (6.2) with $\beta = 2$ and $C = \sqrt{2}$ is a discrete analogue of the classical law of the iterated logarithm for the Brownian motion in $\mathbf{R}^d$.

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