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GENERALIZATIONS OF THEOREMS OF FULLER

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Let R be a right artinian ring and e a primitive idempotent of R . In [6, Corollary 3.2 and Theorem 3.4] (also see Anderson-Fuller [1, Theorem 31.3].) K. Fuller showed that the following conditions are equivalent.

- (1) eR is an injective right R -module.
- (2) There exists a primitive idempotent f of R such that
(2*) $S(eR) \cong T(fR)$ and $S(Rf) \cong T(Re)$, where $S(M)$ and $T(M)$ denote the socle and the top of M , respectively.
- (3) There exists a primitive idempotent f of R such that
(3l) $\ell_{eR}(r_{Rf}(I)) = eI$ for each left ideal I , and
(3r) $r_{Rf}(\ell_{eR}(K)) = Kf$ for each right ideal K of R , where $r_{Rf}(I) = \{a \in Rf \mid Ia = 0\}$ and $\ell_{eR}(K) = \{b \in eR \mid bK = 0\}$.

Let R be a semiprimary ring and e and f primitive idempotents of R . Then (eR, Rf) is called an i -pair in [3] if the above condition (2*) is satisfied. In [3, Theorem 1, Proposition 4 and Corollary 1], Y. Baba and K. Oshiro extended these results to semiprimary rings to show the following statements.

- (a) If R is a semiprimary ring, then the condition (1) is satisfied if and only if both (2) and (3r) are satisfied.
- (b) If R is a semiprimary ring satisfying (2) and the condition (*) below, then (1) is satisfied.
- (*) The lattice $\{r_{Rf}(X) \mid X \subseteq eR\}$ satisfies the ascending chain condition. Moreover, in [3, Theorem 2], they showed the following statement (c).
- (c) If R is a semiprimary ring and (eR, Rf) is an i -pair for primitive idempotents e and f of R , then the following are equivalent.
 - (c1) Rf is artinian as a right fRf -module.
 - (c2) eR is artinian as a left eRe -module.
 - (c3) eR is an injective right R -module and Rf is an injective left R -module.

In this note, for a right R -module M with $S(M) \cong T(fR)$ and $P = \text{End}M$, we consider a pair $({}_P M, Rf{}_f Rf)$ instead of an i -pair $(eRe eR, Rf{}_f Rf)$ and give generalizations of the results (a), (b) and (c) above (in Sections 1 and 2). In particular, in Section 1, for a module N_Q , we give some properties for the pair $({}_P M, N_Q)$, which are very similar to Theorem 1.1 in Morita-Tachikawa [11]. Moreover, in Section 3, by applying results obtained in Sections 1 and 2, we give elementary proofs of

Theorems 1 and 2 in Baba [2], which are related to some results in Fuller [6].

Throughout this note we always assume that every ring has an identity and every module is unitary. In particular, R always stands for a semiprimary ring with the Jacobson radical J . For a ring H , by M_H (${}_H M$) we stress what M is a right (left) H -module. Let M be a module. Then $L \leq M$ (resp. $L < M$) means that L is a submodule of M (resp. $L \leq M$ and $L \neq M$). By $S(M)$, $T(M)$ and $E(M)$, we denote the socle, the top and an injective hull of M , respectively, and by $|M|$ we denote the composition length of M . Assume every homomorphism always operates from opposite side of scalar. “Acc” (“dcc”) means the ascending (descending) chain condition. We denote the set of primitive idempotents of R by $\text{Pi}(R)$.

1. Colocal pairs of modules

Let P and Q be rings and ${}_P M$, N_Q and ${}_P U_Q$ be a left P -module, a right Q -module and a P - Q -bimodule, respectively. Let $\varphi : M \times N \rightarrow U$ be a P - Q -bilinear map, i. e., a map satisfying the following properties:

- (1) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)$,
- (2) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2)$,
- (3) $\varphi(px, yq) = p\varphi(x, y)q$;

for any $x, x_1, x_2 \in M, y, y_1, y_2 \in N, p \in P$ and $q \in Q$.

Then, we say that $({}_P M, N_Q)$ is a pair with respect to φ or simply a pair.

Let $({}_P M, N_Q)$ be a pair with respect to φ . Then for $x \in M, y \in N$ and for ${}_P X \leq_P M, Y_Q \leq N_Q$, by xy we denote the element $\varphi(x, y)$, and by XY we denote the P - Q -subbimodule of ${}_P U_Q$ generated by $\{xy | x \in X, y \in Y\}$. Moreover, for $A \subseteq M$ and $B \subseteq N$, we define submodules $r(A)$ ($= r_N(A)$) of N_Q and $\ell(B)$ ($= \ell_M(B)$) of ${}_P M$, as follows: $r(A) = \{y \in N | Ay = 0\}$ and $\ell(B) = \{x \in M | xB = 0\}$, and we call $r(A)$ (resp. $\ell(B)$) the right (resp. left) annihilator of A (resp. of B).

Let $({}_P M, N_Q)$ be a pair and put $U = MN$. For submodules $X' \leq X \leq {}_P M, Y' \leq Y \leq N_Q$ with $XN' = X'Y = 0$, we have a pair $({}_P X/X', Y/Y'_Q)$ by defining $(x + X')(y + Y') = xy$. This is called a pair induced from (M, N) . For an arbitrary ring H , we call an H -module V colocal if V has the (non-zero) smallest submodule. We call a pair $({}_P M, N_Q)$ colocal if the module $U (= MN)$ is colocal both as a left P -module and as a right Q -module. Note, in case $({}_P M, N_Q)$ is a colocal pair, we have $S({}_P U) = S(U_Q)$. We call a pair (M, N) left faithful (resp. right faithful) if $\ell(N) = 0$ (resp. $r(M) = 0$), and a pair (M, N) faithful if it is left and right faithful. We denote the class of right annihilator submodules in N_Q by $Ar(M, N)$; that is $Ar(M, N) = \{Y \leq N_Q | Y = r\ell(Y)\}$, and similarly $Al(M, N) = \{X \leq {}_P M | X = \ell r(X)\}$, and the lattice of submodules of ${}_P M$ (resp. N_Q) by $\text{Lat}({}_P M)$ (resp. $\text{Lat}(N_Q)$). We say that a pair $({}_P M, N_Q)$ satisfies r -ann (resp. ℓ -ann) if $Ar(M, N) = \text{Lat}(N_Q)$ (resp. $Al(M, N) = \text{Lat}({}_P M)$).

Let P be a ring, Q a subring of R , M a P - R -bimodule and I a left ideal of R which is also an R - Q -bimodule. In this case, unless otherwise stated, by the notation

$({}_P M, I_Q)$ we always mean a pair with respect to the bilinear map $\varphi : M \times I \rightarrow MI$ defined by $\varphi(m, a) = ma; m \in M, a \in I$. In case P is a subring of R and Q is a ring, for a right ideal K of R which is also a P - R -bimodule and for an R - Q -bimodule N , we consider the pair $({}_P K, N_Q)$ in the same way.

Lemma 1.1. *Let $({}_P M, N_Q)$ be a colocal pair, and $Y' < Y \leq N_Q$ with $Y' = r\ell(Y')$. If $(Y/Y')_Q$ is simple, then ${}_P(\ell(Y')/\ell(Y))$ is also simple and $Y = r\ell(Y)$.*

Proof. Put $U = MN$, $X = \ell(Y)$ and $X' = \ell(Y')$. By the assumption, there exists an element $y \in Y$ such that $Y = yQ + Y' \leq N_Q$. From $r\ell(Y') = Y' < Y \leq r\ell(Y)$, we obtain $X = \ell(Y) < \ell(Y') = X'$. For any $x \in X'$, the left multiplication map $\hat{x} : (Y/Y')_Q \rightarrow xY_Q$ by x is an epimorphism, so we have $xY_Q \leq S(U_Q)$. Hence $X'Y = S(U_Q) = S({}_P U)$, which shows that ${}_P X'Y$ is simple. On the other hand, the map $\eta : {}_P X'/X \rightarrow {}_P X'Y$ defined by $\eta(x + X) = xy$ is a monomorphism. Thus ${}_P(\ell(Y')/\ell(Y)) (= {}_P(X'/X))$ is simple. By the same argument, it follows that $(r\ell(Y)/r\ell(Y'))_Q$ is simple. Hence we have $Y = r\ell(Y)$ from $r\ell(Y') = Y' < Y \leq r\ell(Y)$. □

Lemma 1.2. *Let $({}_P M, N_Q)$ be a colocal pair, and Y and Z submodules of N_Q with $Z = r\ell(Z) \leq Y_Q$. If $|(Y/Z)_Q| < \infty$, then $Y = r\ell(Y)$.*

In particular, if $({}_P M, N_Q)$ is right faithful and $|Y_Q| < \infty$, then $Y = r\ell(Y)$.

Proof. The assertion is immediate from Lemma 1.1 by induction on the length $|(Y/Z)_Q|$. □

Lemma 1.3 (See [11, Theorem 1.1] (or [15, Theorem 1.1])). *Let $({}_P M, N_Q)$ be a colocal pair, and put $M' = \ell(N) \leq M$ and $N' = r(M) \leq N$. Then $|(N/N')_Q| < \infty$ if and only if $|{}_P(M/M')| < \infty$.*

Moreover, in case the above conditions are satisfied, we have $X = \ell r(X)$ (resp. $Y = r\ell(Y)$) for any X with $M' \leq X \leq {}_P M$ (resp. for any Y with $N' \leq Y \leq N_Q$), and $|{}_P(M/M')| = |(N/N')_Q|$.

Proof. We denote Y/N' (resp. X/M') by \bar{Y} (resp. \bar{X}). If $|(N/N')_Q| = n$ and $\bar{N}' = \bar{Y}_0 < \bar{Y}_1 < \dots < \bar{Y}_n = \bar{N}$ is a composition series of $\bar{N}_Q = (N/N')_Q$, then for $X_i = \ell(Y_i)$, $\bar{M}' = \bar{X}_n < \bar{X}_{n-1} < \dots < \bar{X}_0 = \bar{M}$ is a composition series of \bar{M} by Lemma 1.1 and in particular $|{}_P(M/M')| = |(N/N')_Q| = n$. It follows from Lemma 1.2 that $X = \ell r(X)$ and $Y = r\ell(Y)$. □

REMARK 1. Let $({}_P M, N_Q)$ be a colocal pair and put $U = {}_P M N_Q$, $M' = \ell_M(N)$ and $N' = r_N(M)$. Then the following condition (**) is satisfied.

(**) ${}_P U_Q$ - dual takes simple left P -modules and simple right Q -modules to

simples or zero.

In order to show this, let $K = xQ$ be a simple right Q -module. If $0 \neq {}_P\text{Hom}_Q(K_Q, {}_P U_Q)$, then $\alpha(x)Q = \alpha(K) = S(U_Q) = S({}_P U) = P\alpha(x)$ for any $\alpha \in {}_P\text{Hom}_Q(K_Q, {}_P U_Q)$. Hence $P\alpha(x) \geq P\beta(x)$ for any $0 \neq \alpha, \beta \in {}_P\text{Hom}_Q(K_Q, {}_P U_Q)$ and consequently $P\alpha \geq P\beta$, which implies ${}_P\text{Hom}_Q(K_Q, {}_P U_Q)$ is simple.

On the other hand, by the proof of Morita-Tachikawa [11, Theorem 1.1], in case the condition $(**)$ is satisfied, we have that $|{}_P(M/M')| < \infty$ if and only if $|(N/N')_Q| < \infty$. Thus Lemma 1.3 is obtained as a corollary to [11, Theorem 1.1].

Theorem 1.4 (See [3, Lemma 3 and Proposition 5]). *Let Q be a semiprimary ring. Assume $({}_P M, N_Q)$ is a colocal pair and put $M' = \ell(N) \leq M$ and $N' = r(M) \leq N$. Then the following conditions are equivalent :*

- (1) *$Ar(M, N)$ satisfies acc, (or equivalently $Al(M, N)$ satisfies dcc).*
- (2) *$|(N/N')_Q| < \infty$.*
- (3) *$|{}_P(M/M')| < \infty$.*

Moreover, in case the above conditions are satisfied, we have $X = \ell r(X)$ (resp. $Y = r\ell(Y)$) for any X with $M' \leq X \leq {}_P M$ (resp. for any Y with $N' \leq Y \leq N_Q$), and $|{}_P(M/M')| = |(N/N')_Q|$.

Proof. The implication (2) \Rightarrow (1) is trivial and the equivalence (2) \iff (3) follows from Lemma 1.3. Hence we only show the implication (1) \Rightarrow (2). Assume $|(N/N')_Q| = \infty$. Then we can take an infinite-chain $N' = Y_0 < Y_1 < Y_2 < \dots < N_Q$ of submodules of N_Q such that $|(Y_i/N')_Q| = i$ for any $i \geq 0$. By Lemma 1.2, $Y_i = r\ell(Y_i)$ for any $i \geq 0$. Hence, from the assumption we have $Y_n = Y_{n+1} = \dots$ for some $n \geq 0$, which is a contradiction. □

We call a pair $({}_P M, N_Q)$ right (resp. left) finite provided the lattice $Ar({}_P M, N_Q)$ (resp. $Al({}_P M, N_Q)$) satisfies acc and $({}_P M, N_Q)$ finite provided $({}_P M, N_Q)$ is left finite and right finite. As a special case of Theorem 1.4, we have the following corollary.

Corollary 1.5. *Let Q be a semiprimary ring and $({}_P M, N_Q)$ a right finite faithful colocal pair. Then it holds that $|{}_P M| = |N_Q| < \infty$ and $({}_P M, N_Q)$ satisfies r -ann and ℓ -ann.*

2. Indecomposable injective modules

As mentioned in the introduction, we assume that R always stands for a semiprimary ring with the Jacobson radical J .

Lemma 2.1 (See [6, Lemma 1.1]). *Let M be a right R -module and f a prim-*

itive idempotent of R and put $Q = fRf$. Consider the following conditions.

- (1) $S(M) \cong T(fR)$.
- (2) $\ell_M(Rf) = 0$.
- (3) $\ell_M(I) = \ell_M(If)$ for any $I_R \leq R_R$.
- (4) $S(Mf_Q) = S(M_R)f_Q$.

Then the implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) hold.

In particular, if M is injective with $S(M) \cong T(fR)$ (i.e. $M \cong E(T(fR_R))$), then $r_{Rf}(M) = 0$ and $\ell_M(Rf) = 0$.

Proof. The implication (1) \Rightarrow (2) follows from $T(fR) \cong S(M) \leq xR$ for any $(0 \neq) x \in M$. (2) \Rightarrow (3) is easily seen from $If = IRf$. We show the implication (3) \Rightarrow (4). $S(M)f \subseteq S(Mf)$ is clear. Since $S(Mf_Q)Jf = S(Mf_Q)fJf = 0$, we have $S(Mf)J = 0$ from $\ell_M(J) = \ell_M(Jf)$. Therefore we have $S(Mf) \subseteq S(M)f$ and consequently $S(Mf_Q) = S(M_R)f$.

We assume M_R is injective with $S(M) \cong T(fR)$. Then we have $\ell_M(Rf) = 0$ from the implication (1) \Rightarrow (2). If $0 \neq a \in Rf$, then we have a non-zero map $\theta : aR \rightarrow M$. Hence by the injectivity of M , $xa = \theta(a) \neq 0$ holds for some $x \in M$. Thus $r_{Rf}(M) = 0$. \square

Let L_R be a simple right R -module and $f \in \text{Pi}(R)$. Then note that Lf_{fRf} is a simple right fRf -module or zero (cf. Baba [2, Lemma 1]).

Let M be a right R -module. Then we call M quasi-injective if for any submodule L of M , any homomorphism $\theta : L \rightarrow M$ can be extended to some endomorphism of M . By [9, Theorem 1.1], M is quasi-injective if and only if $HM = M$, where $H = \text{End}E(M_R)$. Hence in case M is quasi-injective, we have a surjective ring homomorphism $H \rightarrow \text{End}(M_R)$ ($\alpha \mapsto \alpha|_M$ for any $\alpha \in H$) and we denote the map by ρ_M . As easily seen, any quasi-injective right R -module M is colocal if and only if M is end-local (i. e., $\text{End}M_R$ is a local ring.). By Harada [8], a module M is called simple-injective if for any modules L and N with $L \leq N$, any homomorphism $\theta : L \rightarrow M$ with a simple image $\theta(L)$ can be extended to some homomorphism $\varphi : N \rightarrow M$. The following lemma shows that Proposition 1 in Baba-Oshiro [3] is also verified in case M is not necessarily projective.

Lemma 2.2 (See [3, Proposition 1]). *If M is an end-local and simple-quasi-injective right R -module, then M is colocal.*

Proof. See the proof of the implication (1) \Rightarrow (2), (i) in [14, Lemma 1, 2] in which L_1 and L_2 are simple. \square

Lemma 2.3 ([3, Proposition 2]). *Let M be a colocal right R -module. If M is R -simple-injective, then M is injective.*

Lemma 2.4. *Let M be a right R -module, and put $P = \text{End}M$ and $Q = fRf$ ($\cong \text{End}_R Rf$); $f \in \text{Pi}(R)$. Then the following are equivalent.*

- (1) $({}_P M, Rf_Q)$ is a left faithful colocal pair.
- (2) ${}_P Mf$ is colocal and $S(M_R) \cong T(fR_R)$.

Moreover, in case the conditions are satisfied, any endomorphism α of $S(M_R)$ can be extended to some endomorphism of M .

Proof. (1) \Rightarrow (2). Since, by the assumption, $xRf \neq 0$ for any $0 \neq x \in S(M_R)$, we have $S(M_R) = \bigoplus_{i \in I} L_i$ with $L_i \cong T(fR_R)$ for each $i \in I$. But $S(M_R)f_Q$ ($= S(Mf_Q)$) is simple by Lemma 2.1 and $L_i f_Q$ is also simple for any i , so I is a set consisting of a single element. This shows $S(M_R) \cong T(fR_R)$.

(2) \Rightarrow (1). This is immediate from the implication (1) \Rightarrow (2), (4) in Lemma 2.1. We assume that (1) and (2) are satisfied and let $\alpha : S(M_R) \rightarrow S(M_R)$ be a map. Clearly $S(M_R) = xR$ holds for some $x = xf \in S(M_R)$. Then $\alpha(x) \in S(M_R)f_Q = xQ = Px$, which implies $\alpha(x) = \varphi(x)$ for some $\varphi \in P$. □

Lemma 2.5. *Let M be an injective (resp. quasi-injective) right R -module with $S(M_R) \cong T(fR_R)$; $f \in \text{Pi}(R)$. Then $({}_P M, Rf_Q)$ is a faithful (resp. left faithful) colocal pair, where $P = \text{End}M$ and $Q = fRf$.*

Proof. Assume that M_R is quasi-injective with $S(M_R) \cong T(fR_R)$. By Lemma 2.1, $S(Mf_Q) = S(M_R)f_Q$ is simple and the pair $({}_P M, Rf_Q)$ is left faithful. We show that ${}_P Mf$ is colocal. Let $0 \neq x = xf \in S(Mf_Q)$ and $0 \neq y = yf \in Mf_Q$. Since $(xfJ)Rf = x(fJf) = 0$, we have $xfJ = 0$ by Lemma 2.1, which shows $r_{fR}(y) \leq fJ = r_{fR}(x)$. Hence the map $\theta : yR \rightarrow M$ with $\theta(yc) = xc$ ($c \in R$) is well-defined. Therefore θ is extended to some $\varphi \in \text{End}M = P$, and in particular $x = \varphi(y)$. Thus we have $Px \leq Py$. This shows that ${}_P Mf$ is colocal. In case M_R is injective, it follows from Lemma 2.1 that $r_{Rf}(M) = 0$, so $({}_P M, Rf_Q)$ is faithful. □

REMARK 2. Let e be a primitive idempotent of R such that eR_R is quasi-injective and assume the lattice $Ar(R, R)$ satisfies acc. Then $S(eR_R) \cong T(fR)$ for some $f \in \text{Pi}(R)$, and by Lemma 2.5, $({}_{eRe} eR, Rf_f Rf)$ is a right finite left faithful colocal pair. Hence by Theorem 1.4, ${}_{eRe} eR$ is artinian. Thus in [14, Proposition 2.7], without using torsion theory we can prove that R is a left artinian ring.

As an immediate consequence of Lemma 2.5, we have Corollary 2.6 below, which was obtained by Baba-Oshiro [3] (by Fuller [6] in case R is one-sided artinian). The corollary is useful and its proof is simple. So we give a proof directly in spite of [3], [6] and Lemma 2.5. The proof is similar to that of the implication (3) \Rightarrow (2) in Kato [10, Lemma 2].

Corollary 2.6. (Baba-Oshiro [3, Proposition 4] (and Fuller [6, Theorem 3.1] for a right artinian ring R)). *Let e and f be primitive idempotents of R . If eR is an injective right R -module with $S(eR_R) = aR_R; a = eaf$, then $S({}_R Rf) = {}_R Ra \cong T({}_R Re)$. (That is: If eR is an injective right R -module with $S(eR_R) \cong T(fR_R)$, then $S({}_R Rf) \cong T({}_R Re)$.)*

Proof. It is clear that $r_{fR}(b) \leq fJ = r_{fR}(a)$ for any $0 \neq b \in Rf$. Hence the map $\theta : bR \rightarrow eR$ with $\theta(bc) = ac$ ($c \in R$) is well-defined. Therefore by the injectivity of eR_R we have $a = hb$ for some $h \in eR$. So $a \in Rb$, which implies that $S({}_R Rf) = Ra$ is simple. \square

The following theorem is a slight generalization of Baba-Oshiro [3, Theorem 1]. But for the sake of completeness, we give a proof.

Theorem 2.7 (See [3, Theorem 1]). *Let M be an indecomposable right R -module. Then the following conditions are equivalent.*

- (1) M is injective.
- (2) $({}_P M, Rf_Q)$ is a faithful colocal pair satisfying $r\text{-ann}$ for some $f \in \text{Pi}(R)$, where $P = \text{End}M_R$ and $Q = fRf$.

Proof. By Lemmas 2.4 and 2.5, we may assume that $({}_P M, Rf_Q)$ is a faithful colocal pair with $S(M_R) \cong T(fR_R); f \in \text{Pi}(R)$. Then by lemma 2.1, $\ell(I) = \ell(If)$ is satisfied for any $I \leq R_R$.

(2) \Rightarrow (1). It suffices to show that M_R is R -simple-injective by Lemma 2.3. Let $I \leq R_R$, and $\theta : I \rightarrow M$ a homomorphism with a simple image $\theta(I) = S(M_R)$ and put $K = \text{Ker}\theta$. Then θ induces an isomorphism $\bar{\theta} : I/K \rightarrow S(M_R)$. Since $Kf < If \leq Rf_Q$, it holds that $r\ell(Kf) = Kf < If = r\ell(If)$ by the assumption. Hence $\ell(K) = \ell(Kf) > \ell(If) = \ell(I)$, so there exists an element $x \in \ell(K) - \ell(I)$. Let $\hat{x} : R \rightarrow M$ be the left multiplication map by x and $\eta : I \rightarrow M$ the restriction map to I of \hat{x} . Then η induces an isomorphism $\bar{\eta} : I/K \rightarrow S(M_R)$. If $\alpha : S(M_R) \rightarrow S(M_R)$ is the automorphism with $\alpha\bar{\eta} = \bar{\theta}$ (i.e. $\alpha = \bar{\theta}\bar{\eta}^{-1}$), then α is extended to an endomorphism φ of M_R by Lemma 2.4, which shows $\varphi\eta = \theta$. Hence $\theta : I \rightarrow M$ is extended to a map $\varphi\hat{x} : R \rightarrow M$, so M_R is R -simple-injective.

(1) \Rightarrow (2). Assume that there exists a submodule Lf of Rf_Q with $Lf < r\ell(Lf)$. Then $LfR < r\ell(Lf)R$. Put $I = r\ell(Lf)R$. Since R is a semiprimary ring, we can take a maximal submodule K of I_R with $LfR \leq K < I_R$. Then $\ell(K) = \ell(I)$ holds since $\ell(LfR) = \ell(Lf) = \ell(r\ell(Lf)) = \ell(I)$. On the other hand we have $Kf < If$, because $Kf = If$ implies $I = IfR = KfR \leq K$, which is a contradiction. Hence $(I/K)f \neq 0$, so we have an isomorphism $\alpha : I/K \rightarrow S(M_R)$. Let $\theta : I \rightarrow M$ be a composition map $\theta = \mu\alpha\lambda$, where $\lambda : I \rightarrow I/K$ and $\mu : S(M_R) \rightarrow M_R$ are canonical maps. By the assumption, there exists $x \in M$ such that $\theta(a) = xa$ for any

$a \in I$. From $\theta(I) \neq 0$ and $\theta(K) = 0$, we have $x \in \ell(K) - \ell(I)$, which contradicts $\ell(K) = \ell(I)$. □

The following theorem shows that in case (M, Rf) is finite, the converse of Lemma 2.5 holds.

Theorem 2.8 (See [3, Theorem 1 and Corollary 1]). *Let M be a right R -module. If $({}_P M, Rf_Q)$ is a right finite faithful (resp. left faithful) colocal pair for some $f \in \text{Pi}(R)$, where $P = \text{End}M_R$ and $Q = fRf$, then M_R is injective (resp. quasi-injective) with $S(M_R) \cong T(fR_R)$.*

Proof. Assume $({}_P M, Rf_Q)$ is a right finite left faithful colocal pair. Then by Lemma 2.4 $S(M_R) \cong T(fR_R)$. In case that $({}_P M, Rf_Q)$ is faithful, M is injective from Corollary 1.5 and Theorem 2.7. Putting $I = r_R(M)$, then $If = r_{Rf}(M)$. Hence the pair $({}_P M, Rf_Q)$ induces a right finite faithful colocal pair $({}_P M, Rf/If_Q)$. Moreover M can be regarded as a right \bar{R} -module canonically and it holds that $P \cong \text{End}M_{\bar{R}}, \bar{R}Rf/If \cong \bar{R}\bar{R}\bar{f}$ and $Q/fIf \cong \bar{f}\bar{R}\bar{f}$ (canonically), where $\bar{R} = R/I$ and $\bar{f} = f + I \in \text{Pi}(\bar{R})$. Hence, considering the pair $({}_P M_{\bar{R}}, \bar{R}\bar{R}\bar{f}_{\bar{f}\bar{R}\bar{f}})$, then by the same argument as above, $M_{\bar{R}}$ is injective and consequently M_R is quasi-injective. □

REMARK 3. Let M_R be a right R -module with $P = \text{End}M_R$. Consider the following conditions:

- (1) $M = \ell_{E(M)}r_R(M)$,
- (2) $M_{\bar{R}}$ is injective, where $\bar{R} = R/r_R(M)$,
- (3) M_R is quasi-injective.

Then by [7, Theorem 1.2], (1) \iff (2) \implies (3) hold, and by [4, Theorem 19.14] (or [5, Corollary 5.6A]), in case ${}_P M$ is finitely generated, (3) \implies (2) holds.

But in this note (e.g., in the proof of Theorem 3.5), we consider a colocal module M_R (or a colocal module M_R with $|{}_P M| < \infty; P = \text{End}M_R$). In this case, above implications follow from Theorems 2.7, 2.8 and their proofs.

Proposition 2.9 (See [3, Theorems 1 and 2]). *Let M be an indecomposable right R -module and $({}_P M, Rf_Q)$ a faithful colocal pair, where $f \in \text{Pi}(R)$, $P = \text{End}M_R$ and $Q = fRf$. Then the following are equivalent.*

- (1) *The pair $({}_P M, Rf_Q)$ is right finite.*
- (2) *The pair $({}_P M, Rf_Q)$ satisfies r -ann and ℓ -ann.*
- (3) *M_R is injective and the pair $({}_P M, Rf_Q)$ satisfies ℓ -ann.*

Proof. The implication “(1) \implies (2)” follows from Corollary 1.5, and “(2) \iff (3)” follows from Theorem 2.7.

(2) \Rightarrow (1). By the equivalence (2) \iff (3), M_R is injective. First we shows that ${}_P M$ is linearly compact. The proof is the almost same as Mueller [12, Lemma 4]. Let $(x_i, X_i)_{i \in I}$ be a finitely solvable family of ${}_P M$. Then by the assumption, $X_i = \ell_M r_{Rf}(X_i)$, so $X_i = \ell_M r_R(X_i)$ because of $r_{Rf}(X_i) = r_R(X_i)f$ and $\ell_M(K) = \ell_M(Kf)$ for any $K \leq R_R$. Put $Y_{iR} = r_R(X_i)$ and consider a map $\theta : \sum_{i \in I} Y_i \rightarrow M$ with $\theta(\sum_{i \in F} y_i) = \sum_{i \in F} x_i y_i$, where F is a finite subset of I and $y_i \in Y_i$. By the assumption, for any finite subset F of I , there exists an element $x \in M$ such that $x_i - x \in X_i$. Then $(x_i - x)y_i \in X_i Y_i = 0$, so $\sum_{i \in F} x_i y_i = x \sum_{i \in F} y_i$, which shows θ is well-defined. Since M is injective, there exists an element $x_0 \in M$ such that $x_i y_i = x_0 y_i$ for any $y_i \in Y_i$ and any $i \in I$. Hence $(x_i - x_0)Y_i = 0$, and consequently $x_i - x_0 \in \ell_M r_R(X_i) = X_i$. Thus ${}_P M$ is linearly compact.

By the assumption, we have $\text{Lat}({}_P M) = A\ell({}_P M, Rf_Q)$ and $\text{Lat}(Rf_Q) = Ar({}_P M, Rf_Q)$, so $\text{Lat}({}_P M)$ is anti-isomorphic to $\text{Lat}(Rf_Q)$ by the correspondence $X \rightarrow Y$; where $X = \ell_M(Y)$ and $Y = r_{Rf}(X)$. Since Q is semiprimary, Rf_Q has the upper Loewy series $Rf = Y_0 > Y_1 > \dots > Y_n = 0$. Then, $0 = \ell(Y_0) < \ell(Y_1) < \dots < \ell(Y_n) = M$ is the lower Loewy series of ${}_P M$, and $\ell(Y_i)/\ell(Y_{i-1})$ is a semisimple left P -module for each $i = 1, \dots, n$. Since ${}_P M$ is linearly compact, so is ${}_P \ell(Y_i)/\ell(Y_{i-1})$ (see e.g. [13, Proposition 2.2]). Thus each module ${}_P \ell(Y_i)/\ell(Y_{i-1})$ has a finite composition length (see e.g. [13, Lemma 2.3]), and hence $|{}_P M| < \infty$. \square

Corollary 2.10 (Baba-Oshiro [3, Theorem 2]). *Let (eR, Rf) be an i -pair and $P = eRe, Q = fRf$, where $e, f \in \text{Pi}(R)$. Then the following are equivalent.*

- (1) ${}_P eR$ is artinian.
- (2) Rf_Q is artinian.
- (3) Both eR_R and $R_R f$ are injective.

3. Application of colocal pairs

In this section, we give elementary proofs of Theorems 1 and 2 in Baba [2]. “Quasi-projective” for a module is defined as the dual notion to “quasi-injective”. See [16] for the definition of a quasi-projective module and its characterization. Note that a right R -module M_R is end-local and quasi-projective if and only if $M_R \cong eR/eI$ for some primitive idempotent e of R and for some two sided ideal I of R .

Let $({}_P M, N_Q)$ be a pair and put $\overline{P} = P/\ell_P(M)$ and $\overline{Q} = Q/r_Q(N)$. Then we have a pair $(\overline{P}M, N_{\overline{Q}})$ naturally. It is clear that $({}_P M, N_Q)$ is colocal if and only if so is $(\overline{P}M, N_{\overline{Q}})$. Hence note that we may identify $({}_P M, N_Q)$ with $(\overline{P}M, N_{\overline{Q}})$ through the canonical maps $P \rightarrow \overline{P}$ and $Q \rightarrow \overline{Q}$.

Lemma 3.1 ([2, Theorem 1]). *Let $E = Rf/If \cong E(T({}_R Re))$ for some left ideal I of R , and put $P = eRe$ and $Q = fRf$ where $e, f \in \text{Pi}(R)$. If E is quasi-*

projective, then the following hold.

- (1) $r_{Rf}(eR) = If$.
- (2) $({}_P eR, Rf_Q)$ is a left faithful colocal pair.

Proof. (1) Since Rf/If is quasi-projective and injective with $S({}_R Rf/If) \cong T({}_R Re)$, we have $If(fRf) = If$ and $\ell_{eR}(Rf/If) = 0$ by Lemma 2.1. Hence $eIf(Rf/If) = 0$, so $(eR)If = eIf = 0$. Thus we have $r_{Rf}(eR) \geq If$. On the other hand, $S(Rf/If) \cong T({}_R Re)$ implies $r_{Rf/If}(eR) = 0$ by Lemma 2.1. If $eRa = 0$ for an element $a = af \in Rf$, then $eR(a+If) = 0+If$ in Rf/If . Hence $a+If = 0+If$ in Rf/If and $a \in If$, so $r_{Rf}(eR) \leq If$. Thus we have $r_{Rf}(eR) = If$.

(2) By (1), ${}_P eE_Q \cong {}_P eRf_Q$ holds, hence we can identify the pair $({}_P eR, E_Q)$ with the pair $({}_P eR, Rf/If_Q)$ induced from $({}_P eR, Rf_Q)$. Moreover we may assume $Q/fIf = \text{End}_R E$ since $E = Rf/If$ is quasi-projective. It follows from Lemma 2.5 that the pair $({}_P eR, E_Q)$ is faithful colocal, so $({}_P eR, Rf_Q)$ is left faithful colocal. □

Lemma 3.2 ([2, Theorem 1]). *Let $({}_P eR, Rf_Q)$ be a right (or left) finite left faithful colocal pair with $P = eRe$ and $Q = fRf$ where $e, f \in \text{Pi}(R)$, and put ${}_R E = {}_R E(T({}_R Re))$. Then the following hold.*

- (1) ${}_R E$ is quasi-projective with $T({}_R E) \cong T({}_R Rf)$.
- (2) eR_R is quasi-injective with $S(eR_R) \cong T(fR_R)$.

Proof. (1) Putting $I = r_R(eR)$, then $If = r_{Rf}(eR)$. Since ${}_R Rf/If$ is quasi-projective, we can regard Q/fIf as $\text{End}_R Rf/If$. Moreover $({}_P eR, Rf/If_Q)$ is a finite faithful colocal pair since ${}_P e(Rf/If)_Q \cong {}_P eRf_Q$. It follows from Theorem 2.8 that ${}_R Rf/If$ is an injective module with $S({}_R Rf/If) \cong T({}_R Re)$. Thus we have $E \cong {}_R Rf/If$, which implies (1).

(2) By Theorem 2.8. □

Theorem 3.3 (Baba [2, Theorem 1]). *Let e and f be primitive idempotents of R and put $E = E(T({}_R Re))$, $P = eRe$ and $Q = fRf$. If $\text{Ar}({}_P eR, Rf_Q)$ satisfies acc or dcc, then the following conditions are equivalent.*

- (1) ${}_R E$ is quasi-projective with $T({}_R E) \cong T({}_R Rf)$.
- (2) eR_R is quasi-injective with $S(eR_R) \cong T(fR_R)$.
- (3) $({}_P eR, Rf_Q)$ is a left faithful colocal pair.
- (4) ${}_P eRf$ is colocal and $S(eR_R) \cong T(fR_R)$.

Proof. This is an immediate consequence of Lemmas 2.4, 2.5, 3.1 and 3.2. □

Lemma 3.4. *Let $({}_P eR, Rf_Q)$ be a right (or left) finite colocal pair with*

$eS({}_R Rf) \neq 0$, where $e, f \in \text{Pi}(R)$, $P' = eRe$ and $Q = fRf$. Put $K = ReS({}_R Rf)$, $E = E(T(fR_R))$ and $P = \text{End}E_R$. Then the following hold.

- (1) ${}_R K$ is a unique simple submodule of ${}_R Rf$ satisfying $K \cong T({}_R Re)$.
- (2) There exists a local quasi-injective submodule M of E_R such that $({}_P M, Rf_Q)$ is a finite left faithful colocal pair, $T(M_R) \cong T(eR_R)$ and $MK \neq 0$.

Proof. (1) Since $S({}_{P'} eRf)$ is simple, we have $S({}_{P'} eRf) = eS({}_R Rf)$. If $S({}_R Rf) = \bigoplus_{i \in A} K_i$ with simple submodules K_i , then $eS({}_R Rf) = \bigoplus_{i \in A} eK_i$. Hence there exists only one index $i \in A$ such that $eS({}_R Rf) = eK_i$. Thus $K = ReS({}_R Rf) = K_i$ is simple.

(2) Putting $I = \ell_R(Rf)$, then we have $eI = \ell_{eR}(Rf)$ and $eIf = eIRf = 0$. Hence $({}_{P'} eR/eI, Rf_Q)$ is a finite left faithful colocal pair with $P'/\ell_{P'}(eR/eI) (= eRe/eIe) \cong \text{End}eR/eI_R$. It follows from Theorem 2.8 that eR/eI_R is quasi-injective. Since $S(eR/eI_R) \cong T(fR_R) \cong S(E_R)$, there exists a submodule M of E_R with $M \cong eR/eI$. Then M_R is quasi-injective and we have the surjective ring homomorphism $\rho_M : P \rightarrow \text{End}M_R$. Therefore $({}_P M, Rf_Q)$ is a finite left faithful colocal pair with $T(M_R) \cong T(eR_R)$. Moreover if $MReS({}_R Rf) = 0$, then $(eR/eI)eS({}_R Rf) = 0$, so $eS({}_R Rf) \leq eI$ and $eS({}_R Rf) \leq eIf = 0$, a contradiction. Hence $MK = MReS({}_R Rf) \neq 0$. Thus M satisfies the property in (2). □

Theorem 3.5 (Baba [2, Theorem 2]). *Let $E = E(T(fR_R))$ and let $({}_{P_i} e_i R, Rf_Q)$ be a right (or left) finite colocal pair for any $i = 1, \dots, n$, where $e_i, f \in \text{Pi}(R)$, $P_i = e_i Re_i$ and $Q = fRf$. Put $P = \text{End}E_R$. Then the following conditions are equivalent.*

- (1) $S({}_R Rf) \cong T({}_R Re_1) \oplus \dots \oplus T({}_R Re_n)$.
- (2) $T(E_R) \cong T(e_1 R_R) \oplus \dots \oplus T(e_n R_R)$.

Moreover in case the conditions are satisfied, $S({}_R Rf)$ (or equivalently $T(E_R)$) is square-free and the pair $({}_P E, Rf_Q)$ is finite.

Proof. Note that for any $e \in \text{Pi}(R)$, the following property (P) holds.

(P) $eS({}_R Rf) \neq 0$ implies $T(E_R)e \neq 0$.

If $eS({}_R Rf) \neq 0$, then by Lemma 2.1 $EeS({}_R Rf) \neq 0$ holds and we have $EJS({}_R Rf) = 0$ clearly, which shows that (P) holds.

(1) \Rightarrow (2). Assume (1). Then $S({}_R Rf)$ is square-free by Lemma 3.4 (1). Hence by the property (P), $T(E_R)$ has a direct summand isomorphic to $T(e_1 R_R) \oplus \dots \oplus T(e_n R_R)$. By (1) we have $S({}_R Rf) = K_1 \oplus \dots \oplus K_n$ for some $K_i \leq {}_R Rf$ with $K_i \cong T({}_R Re_i); i = 1, \dots, n$. By Lemma 3.4, for each $i = 1, \dots, n$, there exists a quasi-injective submodule M_i of E_R such that $({}_P M_i, Rf_Q)$ is a finite left faithful colocal pair with $T(M_i) \cong T(e_i R)$ and $M_i K_i \neq 0$. Putting $M = M_1 + \dots + M_n$, then M_R is a quasi-injective module with $|{}_P M| < \infty$. Since $({}_P E, Rf_Q)$ is a left faithful colocal pair, so is $({}_P M, Rf_Q)$. If $0 \neq a \in Rf$, then $Ra \geq K_i$ for some i and

consequently $Ma \geq MK_i \neq 0$. Hence $({}_P M, Rf_Q)$ is a finite faithful colocal pair. Moreover we have the surjective ring homomorphism $\rho_M : P \rightarrow \text{End}(M)$. Therefore M_R is injective by Theorem 2.8, which implies $E = M$. Thus $T(E_R)$ ($= T(M_R)$) is isomorphic to a direct summand of $T(e_1 R_R) \oplus \cdots \oplus T(e_n R_R)$ and consequently we have (2). Moreover (E, Rf) is finite because of $E = M$.

(2) \Rightarrow (1). Assume (2). Then by the property (P) and Lemma 3.4 (1), we may assume that $S({}_R Rf) \cong T({}_R R e_1) \oplus \cdots \oplus T({}_R R e_m)$ for some $m; 1 \leq m \leq n$. Therefore from the implication (1) \Rightarrow (2), $T(E_R) \cong T(e_1 R_R) \oplus \cdots \oplus T(e_m R_R)$ is obtained. Thus $m = n$ holds and consequently we have (1). \square

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