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CORRECTIONS AND SUPPLEMENTS TO
**“INDEX OF THE EXPONENTIAL MAP ON A
COMPLEX SIMPLE LIE GROUP”**

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In this note, we shall fix up some gaps in my previous papers [2] and [3], and will discuss related topics. Since [2] is a special case of [3], we may restrict our attention to [3] and will retain the notation adopted there throughout the paper.

In [3], the proofs of Lemma 1 and Proposition in Section 3 were incomplete. After some preparations, we will give a detailed proof for each of them in §3. On the way, we can see some relations between the index of a connected complex semisimple Lie group and the index of its Borel subgroup, which will be stated in the last part of this note.

I am indebted to Professor Morikuni Goto who pointed out my mistake and gave me great help to correct it.

1. On a theorem of Kostant concerning three dimensional subalgebras

Let G be a complex semisimple Lie algebra, A a nilpotent element ($\neq 0$) in G . According to Kostant [1], we can find a semisimple h and a nilpotent B in G so that

$$(*) \quad [h, A] = A, \quad [h, B] = -B, \quad [A, B] = h.$$

Furthermore, the three dimensional subalgebra $S = Ch + CA + CB$ is uniquely determined by A up to conjugacy, i.e. if A is conjugate with A' , then a three dimensional subalgebra $S' = Ch' + CA' + CB'$ corresponding to A' is conjugate with S , and so is h' to h .

Proposition B. *If A is a regular nilpotent element in G (c.f. [4]), then the h satisfying (*) must be a regular semisimple element.*

Proof. By [4], we can suppose that $A = \sum_{j=1}^l e_{\alpha_j}$. We shall choose $B \in G$

such that $h = \sum_{j=1}^k h_j$, A and B satisfy (*). Notice that $\alpha_j(h) = 1$ for $j = 1, \dots, l$, and h is regular.

Let $c_{kj} = -2\langle \alpha_k, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle$ ($k, j = 1, \dots, l$) be Cartan integers. The Cartan matrix $(-c_{kj})$ is known to be non-singular and so the system of linear equations

$$\sum_{k=1}^l y_k c_{jk} = 1 \quad (j = 1, \dots, l)$$

has a unique solution y_1, \dots, y_l . We put

$$B = \sum_{k=1}^l \frac{2}{\langle \alpha_k, \alpha_k \rangle} y_k e^{-\alpha_k}.$$

Then

$$\begin{aligned} [A, B] &= \sum_{k=1}^l \frac{2}{\langle \alpha_k, \alpha_k \rangle} y_k [e_{\alpha_k}, e_{-\alpha_k}] = -\sum_{k=1}^l \frac{2}{\langle \alpha_k, \alpha_k \rangle} y_k h_{\alpha_k} \\ &= -\sum_{k=1}^l y_k h_{\alpha_k}^* = -\sum_{k=1}^l y_k \left(-\sum_{j=1}^l c_{jk} h_j \right) \\ &= \sum_{j=1}^l h_j = h, \end{aligned}$$

and it is easy to see that

$$[h, A] = A, \quad [h, B] = -B. \quad \text{Q.E.D.}$$

2. A key lemma

G still denotes a complex semisimple Lie algebra of rank l .

Lemma C. *Let G_1 be a semisimple subalgebra of G with the same rank l , A a nilpotent element which is regular in G_1 . Then $z_G(A)$ (the centralizer of A in $G = \{x \in G; [x, A] = 0\}$) is composed of nilpotent elements.*

Proof. It suffices to prove: for any nonzero semisimple element $x_0 \in G$, A is not conjugate with any (nilpotent) element in $z_G(x_0)$.

Any semisimple element $x_0 \in G$ is conjugate to some element h_0 in the (fixed) Cartan subalgebra H . Moreover, if $x_0 = \text{Ad}_g \cdot h_0$, then $z_G(x_0) = \text{Ad}_g \cdot z_G(h_0)$. So we may assume that $x_0 \neq 0$ lies in H .

Next, any $x_0 \in H$ can be expressed as $x_0 = \sum_{j=1}^l (c_j + id_j) h_j$ ($c_j, d_j \in \mathbf{R}$). Denote $x_1 = \sum_{j=1}^l c_j h_j$, $x_2 = \sum_{j=1}^l d_j h_j$, so that $x_0 = x_1 + ix_2$ with $x_1, x_2 \in H_0 (= \sum_{\alpha \in \Delta} \mathbf{R} h_\alpha = \sum_{j=1}^l \mathbf{R} h_j)$. Notice that $z_G(x_0)$ is generated by H and those e_α 's with $\alpha \in \Delta$ satisfying $0 = \alpha(x_0) = \alpha(x_1) + i\alpha(x_2)$, which implies that the two real numbers $\alpha(x_1)$ and $\alpha(x_2)$ must be zero, i.e. $z_G(x_0) = z_G(x_1) \cap z_G(x_2)$. By assumption, $x_0 \neq 0$, so $x_1 \neq 0$ or $x_2 \neq 0$. Without loss of generality, we may assume that $x_1 \neq 0$. Since $z_G(x_0) \subset z_G(x_1)$, to

prove the Lemma, we may replace x_0 by x_1 , i.e. we assume that $x_0 \in H_0$.

Denote $\bar{W}_0 = \{x \in H_0; \alpha_j(x) > 0 \quad j=1, \dots, l\}$. Since the Weyl group $Ad(\Delta)$ acts transitively on the set of Weyl chambers, any element x_0 in H_0 is conjugate with an element in \bar{W}_0 . So we reduce our problem to the case that $x_0 \in \bar{W}_0$, i.e. $x_0 = y_1 h_1 + \dots + y_l h_l$ with $y_j \geq 0$ ($j=1, \dots, l$). It is easy to see that $z_G(x_0)$ is generated by H and those e_α 's with $\alpha \in \Delta$ satisfying the following condition: If $\alpha = \sum_{j=1}^l n_j \alpha_j$, then $n_j = 0$ whenever $y_j \neq 0$; i.e. $z_G(x_0) = H + \sum_{\beta \in \Delta_1} C e_\beta$ where $\Delta_1 = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i$ with $I = \{i; 1 \leq i \leq l \text{ and } y_i = 0\}$. The assumption $x_0 \neq 0$ implies that y_j cannot be all zero, say $y_k \neq 0$. Thus we have $z_G(x_0) \subset z_G(h_k)$. Let us prove that a regular nilpotent element A is not conjugate with any nilpotent element in $z_G(h_k)$.

Any nilpotent element $A' \in z_G(h_k)$ lies in $G_2 = [z_G(h_k), z_G(h_k)]$. Let h and B (resp. h' and B') be chosen to satisfy relation (*) for the element A (resp. A') in the last section. We may choose $h, B \in G_1$ and $h', B' \in G_2$ (because G_1 and G_2 are semisimple subalgebras). It is easy to see that G_2 has $\sum_{j \neq k} \mathbb{C} h_j$ as its Cartan subalgebra, and any element $x \in \sum_{j \neq k} \mathbb{C} h_j$ satisfies $\alpha_k(x) = 0$, so x is non-regular when considered as an element in G . Therefore, the semisimple element $h' \in G_2$ is non-regular in G . On the other hand, Proposition B shows that h is regular in G_1 , so h is regular in G because $\text{rank } G = \text{rank } G_1$. Hence h cannot be conjugate with h' . By Kostant's theorem, A cannot be conjugate with A' . This finishes our proof.

3. Corrections to [3]

Throughout this section, we shall follow the notation used in [3].

First, we give a correct proof for Lemma 1 [3].

Let h_0 and β_1, \dots, β_l be the same as in p. 563 [3].

The argument in p. 563 [3] proves that we can find a positive integer d and some element $h \in \Omega'$ such that $\beta_j(dh_0 + h) = 0$ for $j=1, \dots, r$, i.e. $\alpha(dh_0 + h) = 0$ for all $\alpha \in \Delta(h_0)$. Let d be the smallest positive integer for this to be true, then $\text{ind}(\exp h_0 \cdot \exp N)$ is a factor of d .

Assume that $\beta_i = \sum_{j=1}^l q_{ij} \alpha_j$. Consider the following system of linear equations in the unknowns y_1, \dots, y_l :

$$\begin{aligned} q_{i1}y_1 + \dots + q_{il}y_l &= 2\pi i k_i & i &= 1, \dots, r; \\ q_{i1}y_1 + \dots + q_{il}y_l &= 0 & i &= r+1, \dots, l. \end{aligned}$$

Since $(q_{i,j})$ is a nonsingular matrix (because β_1, \dots, β_l is linearly independent), this has a (unique) solution which is nontrivial because some $k_i \neq 0$ by our assumption on h_0 .

Let $h_0' = \sum_{j=1}^l y_j h_j$, then $\beta_1(h_0') = \beta_1(h_0), \dots, \beta_r(h_0') = \beta_r(h_0)$ and $\beta_1, \dots, \beta_l \in$

$\Delta(h_0')$. Suppose that d' is the smallest positive integer for which we can find $h' \in \Omega'$ satisfying $\beta_j(d'h_0' + h') = 0$ for $j=1, \dots, l$, then $\alpha(d'h_0' + h') = 0$ for any $\alpha \in \Delta$ because α can be written as a rational linear combination of β_1, \dots, β_l (they are linearly independent and $l = \text{rank } G$), this implies that $d'h_0' + h' = 0$, or $d'h_0' \in \Omega'$, and d' is the smallest positive integer for this to hold.

On the other hand, $\beta_j(d'h_0 + h') = \beta_j(d'h_0) + \beta_j(h') = \beta_j(d'h_0') + \beta_j(h') = \beta_j(d'h_0' + h') = 0$ for $j=1, \dots, r$, so that d' must be a multiple of d , and hence a multiple of $\text{ind}(\exp h_0 \cdot \exp N)$.

The proof of Lemma 1[3] will be complete after we prove the following lemma.

Lemma 1 A. *There exists a nilpotent element $N' \in \sum_{\beta \in \Delta(h_0)'} \mathcal{C}e_\beta$ so that*

$$\text{ind}(\exp h_0' \cdot \exp N') = d'.$$

Also, we need a more detailed discussion in the proof of section 3 [3] for that element we chose to have index exactly equal to $p_j m_j$.

Proof of Lemma 1 A. Recall that $h_0' \in H$ was chosen so that $\pi(h_0')$ has cardinality $l = \text{rank } G$, i.e. $G_1 = G(1, \text{Ad } \exp h_0')$ is a semisimple subalgebra of G with rank l . If \mathfrak{G}_1 is the connected Lie subgroup of \mathfrak{G} with G_1 as its Lie algebra, then $\exp h_0'$ is a central element in \mathfrak{G}_1 . Note that d' as we have chosen is exactly equal to the order of this central element.

Let $N' \in \sum_{\beta \in \Delta(h_0)'} \mathcal{C}e_\beta$ be a regular nilpotent element in G_1 , and $g = \exp h_0' \cdot \exp N'$. We claim that $\text{ind}(g) = d'$.

Let q be a positive integer so that $g^q = \exp x$ for some $x \in G$. Consider the Jordan decomposition of $x: x = x_0 + N_0$, where x_0 is semisimple, N_0 is nilpotent and $[x_0, N_0] = 0$. The equality $\exp x_0 \cdot \exp N_0 = \exp x = g^q = \exp qh_0' \cdot \exp qN'$ and the uniqueness of decomposition imply that $\exp N_0 = \exp qN'$. But the exponential map is one-one on the nilpotent part, so $N_0 = qN'$. Therefore $x = x_0 + qN'$ with x_0 semisimple and $[x_0, N'] = 0$. Since N' is a regular nilpotent element in the semisimple subalgebra G_1 , which has rank $l = \text{rank } G$, we conclude from Lemma C that $x_0 = 0$, i.e. $(\exp h_0')^q = \exp x_0 = 1$. This implies that q must be a multiple of d' and proves that $\text{ind}(g) = d'$. Q.E.D.

A similar argument can prove the assertion we made in section 3 [3]. Let $h_0 = 2\pi i h_j / m_j$, $N = \sum_{0 \leq i \leq l, i \neq j} \epsilon_{\alpha_i}$, N is a regular nilpotent element in the semisimple subalgebra $G_1 = G(1, \text{Ad } \exp h_0)$, and $\text{rank } G_1 = l = \text{rank } G$.

Let $g = \exp h_0 \cdot \exp N$. If q is a positive integer so that $g^q = \exp x$ for some $x \in G$, then $x = x_0 + qN$ with x_0 semisimple satisfying $[x_0, N] = 0$. Apply Lemma C again, we conclude that $x_0 = 0$, so $(\exp h_0)^q = \exp x_0 = 1$. This implies that $qh_0 \in \Omega^*$, the smallest q for this to hold is $p_j m_j$, so that $\text{ind}(g) = p_j m_j$.

This gives a complete proof for the main theorem in [3].

4. $\text{ind}(\mathfrak{B})$ (\mathfrak{B} denotes a Borel subgroup of \mathfrak{G})

In this section, we let \mathfrak{G} be a connected complex semisimple Lie group, \mathfrak{B} a Borel subgroup of \mathfrak{G} (it is well known that \mathfrak{B} is uniquely determined up to conjugacy). We like to study the relation between $\text{ind}_{\mathfrak{B}}(g)$ and $\text{ind}_{\mathfrak{G}}(g)$ for $g \in \mathfrak{B}$, where $\text{ind}_{\mathfrak{B}}(g)$ (resp. $\text{ind}_{\mathfrak{G}}(g)$) denotes the index of g regarded as an element in \mathfrak{B} (resp. in \mathfrak{G}). Clearly, $\text{ind}_{\mathfrak{B}}(g) \geq \text{ind}_{\mathfrak{G}}(g)$ for any $g \in \mathfrak{B}$.

We may assume that the Lie subalgebra B corresponding to \mathfrak{B} is given by $B = H + \sum_{\alpha \in \Delta^+} \mathcal{C}e_{\alpha}$.

Any element in \mathfrak{B} can be expressed as $\exp h_0 \cdot \exp N$ with $\exp h_0 \cdot \exp N = \exp N \cdot \exp h_0$. In the proof of Lemma 1[3], we may choose β_1, \dots, β_l from Δ^+ , so that the discussion can be restricted on B . We have proved that $\text{ind}_{\mathfrak{G}}(g)$ is a factor of d' . Moreover, for the particular element h_0' we chose and the corresponding regular nilpotent element N' in $\sum_{\beta \in \Delta(h_0')} \mathcal{C}e_{\beta} (\subset \sum_{\alpha \in \Delta^+} \mathcal{C}e_{\alpha})$, $\text{ind}_{\mathfrak{G}}(\exp h_0' \cdot \exp N') = d'$. But it is clear that

$$(\exp h_0' \cdot \exp N')^{d'} = \exp d'N' \in \mathfrak{B}$$

and $(\exp h_0 \cdot \exp N)^d = \exp(dh_0 + h + dN) \in \mathfrak{B}$ (for some $h \in \Omega'$). Therefore $\text{ind}_{\mathfrak{G}}(\exp h_0 \cdot \exp N)$ is also a factor of d' and $\text{ind}_{\mathfrak{B}}(\exp h_0' \cdot \exp N') = d'$. We have proved the following:

For any $g \in \mathfrak{B}$, we can find $g' \in \mathfrak{B}$, so that $\text{ind}_{\mathfrak{B}}(g)$, as well as $\text{ind}_{\mathfrak{G}}(g)$ is a factor of $\text{ind}_{\mathfrak{B}}(g') = \text{ind}_{\mathfrak{G}}(g')$.

If $\text{ind}_{\mathfrak{B}}(g') = \text{ind}_{\mathfrak{G}}(g') = d = qr$, then clearly $\text{ind}_{\mathfrak{B}}(g'^q) = \text{ind}_{\mathfrak{G}}(g'^q) = r$. We conclude that:

Proposition. *If \mathfrak{G} is a connected complex semisimple Lie group, \mathfrak{B} its Borel subgroup, then $\{\text{ind}_{\mathfrak{B}}(g); g \in \mathfrak{B}\} = \{\text{ind}_{\mathfrak{G}}(g); g \in \mathfrak{B}\} = \{\text{ind}_{\mathfrak{G}}(g); g \in \mathfrak{G}\}$. In particular $\text{ind}(\mathfrak{B}) = \text{ind}(\mathfrak{G})$.*

This is also true for a connected complex reductive Lie group.

REMARK. It is not clear to me whether $\text{ind}_{\mathfrak{B}}(g) = \text{ind}_{\mathfrak{G}}(g)$ for any $g \in \mathfrak{B}$.

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