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*CORRECTIONS AND SUPPLEMENTS TO*  
**“INDEX OF THE EXPONENTIAL MAP ON A  
COMPLEX SIMPLE LIE GROUP”**

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In this note, we shall fix up some gaps in my previous papers [2] and [3], and will discuss related topics. Since [2] is a special case of [3], we may restrict our attention to [3] and will retain the notation adopted there throughout the paper.

In [3], the proofs of Lemma 1 and Proposition in Section 3 were incomplete. After some preparations, we will give a detailed proof for each of them in §3. On the way, we can see some relations between the index of a connected complex semisimple Lie group and the index of its Borel subgroup, which will be stated in the last part of this note.

I am indebted to Professor Morikuni Goto who pointed out my mistake and gave me great help to correct it.

**1. On a theorem of Kostant concerning three dimensional subalgebras**

Let  $G$  be a complex semisimple Lie algebra,  $A$  a nilpotent element ( $\neq 0$ ) in  $G$ . According to Kostant [1], we can find a semisimple  $h$  and a nilpotent  $B$  in  $G$  so that

$$(*) \quad [h, A] = A, \quad [h, B] = -B, \quad [A, B] = h.$$

Furthermore, the three dimensional subalgebra  $S = Ch + CA + CB$  is uniquely determined by  $A$  up to conjugacy, i.e. if  $A$  is conjugate with  $A'$ , then a three dimensional subalgebra  $S' = Ch' + CA' + CB'$  corresponding to  $A'$  is conjugate with  $S$ , and so is  $h'$  to  $h$ .

**Proposition B.** *If  $A$  is a regular nilpotent element in  $G$  (c.f. [4]), then the  $h$  satisfying (\*) must be a regular semisimple element.*

*Proof.* By [4], we can suppose that  $A = \sum_{j=1}^l e_{\alpha_j}$ . We shall choose  $B \in G$

such that  $h = \sum_{j=1}^k h_j$ ,  $A$  and  $B$  satisfy (\*). Notice that  $\alpha_j(h) = 1$  for  $j = 1, \dots, l$ , and  $h$  is regular.

Let  $c_{kj} = -2\langle \alpha_k, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle$  ( $k, j = 1, \dots, l$ ) be Cartan integers. The Cartan matrix  $(-c_{kj})$  is known to be non-singular and so the system of linear equations

$$\sum_{k=1}^l y_k c_{jk} = 1 \quad (j = 1, \dots, l)$$

has a unique solution  $y_1, \dots, y_l$ . We put

$$B = \sum_{k=1}^l \frac{2}{\langle \alpha_k, \alpha_k \rangle} y_k e^{-\alpha_k}.$$

Then

$$\begin{aligned} [A, B] &= \sum_{k=1}^l \frac{2}{\langle \alpha_k, \alpha_k \rangle} y_k [e_{\alpha_k}, e_{-\alpha_k}] = -\sum_{k=1}^l \frac{2}{\langle \alpha_k, \alpha_k \rangle} y_k h_{\alpha_k} \\ &= -\sum_{k=1}^l y_k h_{\alpha_k}^* = -\sum_{k=1}^l y_k \left( -\sum_{j=1}^l c_{jk} h_j \right) \\ &= \sum_{j=1}^l h_j = h, \end{aligned}$$

and it is easy to see that

$$[h, A] = A, \quad [h, B] = -B. \quad \text{Q.E.D.}$$

## 2. A key lemma

$G$  still denotes a complex semisimple Lie algebra of rank  $l$ .

**Lemma C.** *Let  $G_1$  be a semisimple subalgebra of  $G$  with the same rank  $l$ ,  $A$  a nilpotent element which is regular in  $G_1$ . Then  $z_G(A)$  (the centralizer of  $A$  in  $G = \{x \in G; [x, A] = 0\}$ ) is composed of nilpotent elements.*

*Proof.* It suffices to prove: for any nonzero semisimple element  $x_0 \in G$ ,  $A$  is not conjugate with any (nilpotent) element in  $z_G(x_0)$ .

Any semisimple element  $x_0 \in G$  is conjugate to some element  $h_0$  in the (fixed) Cartan subalgebra  $H$ . Moreover, if  $x_0 = \text{Ad}_g \cdot h_0$ , then  $z_G(x_0) = \text{Ad}_g \cdot z_G(h_0)$ . So we may assume that  $x_0 \neq 0$  lies in  $H$ .

Next, any  $x_0 \in H$  can be expressed as  $x_0 = \sum_{j=1}^l (c_j + id_j) h_j$  ( $c_j, d_j \in \mathbf{R}$ ). Denote  $x_1 = \sum_{j=1}^l c_j h_j$ ,  $x_2 = \sum_{j=1}^l d_j h_j$ , so that  $x_0 = x_1 + ix_2$  with  $x_1, x_2 \in H_0 (= \sum_{\alpha \in \Delta} \mathbf{R} h_\alpha = \sum_{j=1}^l \mathbf{R} h_j)$ . Notice that  $z_G(x_0)$  is generated by  $H$  and those  $e_\alpha$ 's with  $\alpha \in \Delta$  satisfying  $0 = \alpha(x_0) = \alpha(x_1) + i\alpha(x_2)$ , which implies that the two real numbers  $\alpha(x_1)$  and  $\alpha(x_2)$  must be zero, i.e.  $z_G(x_0) = z_G(x_1) \cap z_G(x_2)$ . By assumption,  $x_0 \neq 0$ , so  $x_1 \neq 0$  or  $x_2 \neq 0$ . Without loss of generality, we may assume that  $x_1 \neq 0$ . Since  $z_G(x_0) \subset z_G(x_1)$ , to

prove the Lemma, we may replace  $x_0$  by  $x_1$ , i.e. we assume that  $x_0 \in H_0$ .

Denote  $\bar{W}_0 = \{x \in H_0; \alpha_j(x) > 0 \quad j=1, \dots, l\}$ . Since the Weyl group  $Ad(\Delta)$  acts transitively on the set of Weyl chambers, any element  $x_0$  in  $H_0$  is conjugate with an element in  $\bar{W}_0$ . So we reduce our problem to the case that  $x_0 \in \bar{W}_0$ , i.e.  $x_0 = y_1 h_1 + \dots + y_l h_l$  with  $y_j \geq 0$  ( $j=1, \dots, l$ ). It is easy to see that  $z_G(x_0)$  is generated by  $H$  and those  $e_\alpha$ 's with  $\alpha \in \Delta$  satisfying the following condition: If  $\alpha = \sum_{j=1}^l n_j \alpha_j$ , then  $n_j = 0$  whenever  $y_j \neq 0$ ; i.e.  $z_G(x_0) = H + \sum_{\beta \in \Delta_1} C e_\beta$  where  $\Delta_1 = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i$  with  $I = \{i; 1 \leq i \leq l \text{ and } y_i = 0\}$ . The assumption  $x_0 \neq 0$  implies that  $y_j$  cannot be all zero, say  $y_k \neq 0$ . Thus we have  $z_G(x_0) \subset z_G(h_k)$ . Let us prove that a regular nilpotent element  $A$  is not conjugate with any nilpotent element in  $z_G(h_k)$ .

Any nilpotent element  $A' \in z_G(h_k)$  lies in  $G_2 = [z_G(h_k), z_G(h_k)]$ . Let  $h$  and  $B$  (resp.  $h'$  and  $B'$ ) be chosen to satisfy relation (\*) for the element  $A$  (resp.  $A'$ ) in the last section. We may choose  $h, B \in G_1$  and  $h', B' \in G_2$  (because  $G_1$  and  $G_2$  are semisimple subalgebras). It is easy to see that  $G_2$  has  $\sum_{j \neq k} \mathbb{C} h_j$  as its Cartan subalgebra, and any element  $x \in \sum_{j \neq k} \mathbb{C} h_j$  satisfies  $\alpha_k(x) = 0$ , so  $x$  is non-regular when considered as an element in  $G$ . Therefore, the semisimple element  $h' \in G_2$  is non-regular in  $G$ . On the other hand, Proposition B shows that  $h$  is regular in  $G_1$ , so  $h$  is regular in  $G$  because  $\text{rank } G = \text{rank } G_1$ . Hence  $h$  cannot be conjugate with  $h'$ . By Kostant's theorem,  $A$  cannot be conjugate with  $A'$ . This finishes our proof.

### 3. Corrections to [3]

Throughout this section, we shall follow the notation used in [3].

First, we give a correct proof for Lemma 1 [3].

Let  $h_0$  and  $\beta_1, \dots, \beta_l$  be the same as in p. 563 [3].

The argument in p. 563 [3] proves that we can find a positive integer  $d$  and some element  $h \in \Omega'$  such that  $\beta_j(dh_0 + h) = 0$  for  $j=1, \dots, r$ , i.e.  $\alpha(dh_0 + h) = 0$  for all  $\alpha \in \Delta(h_0)$ . Let  $d$  be the smallest positive integer for this to be true, then  $\text{ind}(\exp h_0 \cdot \exp N)$  is a factor of  $d$ .

Assume that  $\beta_i = \sum_{j=1}^l q_{ij} \alpha_j$ . Consider the following system of linear equations in the unknowns  $y_1, \dots, y_l$ :

$$\begin{aligned} q_{i1}y_1 + \dots + q_{il}y_l &= 2\pi i k_i & i &= 1, \dots, r; \\ q_{i1}y_1 + \dots + q_{il}y_l &= 0 & i &= r+1, \dots, l. \end{aligned}$$

Since  $(q_{i,j})$  is a nonsingular matrix (because  $\beta_1, \dots, \beta_l$  is linearly independent), this has a (unique) solution which is nontrivial because some  $k_i \neq 0$  by our assumption on  $h_0$ .

Let  $h_0' = \sum_{j=1}^l y_j h_j$ , then  $\beta_1(h_0') = \beta_1(h_0), \dots, \beta_r(h_0') = \beta_r(h_0)$  and  $\beta_1, \dots, \beta_l \in$

$\Delta(h_0')$ . Suppose that  $d'$  is the smallest positive integer for which we can find  $h' \in \Omega'$  satisfying  $\beta_j(d'h_0' + h') = 0$  for  $j=1, \dots, l$ , then  $\alpha(d'h_0' + h') = 0$  for any  $\alpha \in \Delta$  because  $\alpha$  can be written as a rational linear combination of  $\beta_1, \dots, \beta_l$  (they are linearly independent and  $l = \text{rank } G$ ), this implies that  $d'h_0' + h' = 0$ , or  $d'h_0' \in \Omega'$ , and  $d'$  is the smallest positive integer for this to hold.

On the other hand,  $\beta_j(d'h_0 + h') = \beta_j(d'h_0) + \beta_j(h') = \beta_j(d'h_0') + \beta_j(h') = \beta_j(d'h_0' + h') = 0$  for  $j=1, \dots, r$ , so that  $d'$  must be a multiple of  $d$ , and hence a multiple of  $\text{ind}(\exp h_0 \cdot \exp N)$ .

The proof of Lemma 1[3] will be complete after we prove the following lemma.

**Lemma 1 A.** *There exists a nilpotent element  $N' \in \sum_{\beta \in \Delta(h_0)'} \mathcal{C}e_\beta$  so that*

$$\text{ind}(\exp h_0' \cdot \exp N') = d'.$$

Also, we need a more detailed discussion in the proof of section 3 [3] for that element we chose to have index exactly equal to  $p_j m_j$ .

Proof of Lemma 1 A. Recall that  $h_0' \in H$  was chosen so that  $\pi(h_0')$  has cardinality  $l = \text{rank } G$ , i.e.  $G_1 = G(1, \text{Ad } \exp h_0')$  is a semisimple subalgebra of  $G$  with rank  $l$ . If  $\mathfrak{G}_1$  is the connected Lie subgroup of  $\mathfrak{G}$  with  $G_1$  as its Lie algebra, then  $\exp h_0'$  is a central element in  $\mathfrak{G}_1$ . Note that  $d'$  as we have chosen is exactly equal to the order of this central element.

Let  $N' \in \sum_{\beta \in \Delta(h_0)'} \mathcal{C}e_\beta$  be a regular nilpotent element in  $G_1$ , and  $g = \exp h_0' \cdot \exp N'$ . We claim that  $\text{ind}(g) = d'$ .

Let  $q$  be a positive integer so that  $g^q = \exp x$  for some  $x \in G$ . Consider the Jordan decomposition of  $x: x = x_0 + N_0$ , where  $x_0$  is semisimple,  $N_0$  is nilpotent and  $[x_0, N_0] = 0$ . The equality  $\exp x_0 \cdot \exp N_0 = \exp x = g^q = \exp qh_0' \cdot \exp qN'$  and the uniqueness of decomposition imply that  $\exp N_0 = \exp qN'$ . But the exponential map is one-one on the nilpotent part, so  $N_0 = qN'$ . Therefore  $x = x_0 + qN'$  with  $x_0$  semisimple and  $[x_0, N'] = 0$ . Since  $N'$  is a regular nilpotent element in the semisimple subalgebra  $G_1$ , which has rank  $l = \text{rank } G$ , we conclude from Lemma C that  $x_0 = 0$ , i.e.  $(\exp h_0')^q = \exp x_0 = 1$ . This implies that  $q$  must be a multiple of  $d'$  and proves that  $\text{ind}(g) = d'$ . Q.E.D.

A similar argument can prove the assertion we made in section 3 [3]. Let  $h_0 = 2\pi i h_j / m_j$ ,  $N = \sum_{0 \leq i \leq l, i \neq j} \epsilon_{\alpha_i}$ ,  $N$  is a regular nilpotent element in the semisimple subalgebra  $G_1 = G(1, \text{Ad } \exp h_0)$ , and  $\text{rank } G_1 = l = \text{rank } G$ .

Let  $g = \exp h_0 \cdot \exp N$ . If  $q$  is a positive integer so that  $g^q = \exp x$  for some  $x \in G$ , then  $x = x_0 + qN$  with  $x_0$  semisimple satisfying  $[x_0, N] = 0$ . Apply Lemma C again, we conclude that  $x_0 = 0$ , so  $(\exp h_0)^q = \exp x_0 = 1$ . This implies that  $qh_0 \in \Omega^*$ , the smallest  $q$  for this to hold is  $p_j m_j$ , so that  $\text{ind}(g) = p_j m_j$ .

This gives a complete proof for the main theorem in [3].

#### 4. $\text{ind}(\mathfrak{B})$ ( $\mathfrak{B}$ denotes a Borel subgroup of $\mathfrak{G}$ )

In this section, we let  $\mathfrak{G}$  be a connected complex semisimple Lie group,  $\mathfrak{B}$  a Borel subgroup of  $\mathfrak{G}$  (it is well known that  $\mathfrak{B}$  is uniquely determined up to conjugacy). We like to study the relation between  $\text{ind}_{\mathfrak{B}}(g)$  and  $\text{ind}_{\mathfrak{G}}(g)$  for  $g \in \mathfrak{B}$ , where  $\text{ind}_{\mathfrak{B}}(g)$  (resp.  $\text{ind}_{\mathfrak{G}}(g)$ ) denotes the index of  $g$  regarded as an element in  $\mathfrak{B}$  (resp. in  $\mathfrak{G}$ ). Clearly,  $\text{ind}_{\mathfrak{B}}(g) \geq \text{ind}_{\mathfrak{G}}(g)$  for any  $g \in \mathfrak{B}$ .

We may assume that the Lie subalgebra  $B$  corresponding to  $\mathfrak{B}$  is given by  $B = H + \sum_{\alpha \in \Delta^+} \mathcal{C}e_{\alpha}$ .

Any element in  $\mathfrak{B}$  can be expressed as  $\exp h_0 \cdot \exp N$  with  $\exp h_0 \cdot \exp N = \exp N \cdot \exp h_0$ . In the proof of Lemma 1[3], we may choose  $\beta_1, \dots, \beta_l$  from  $\Delta^+$ , so that the discussion can be restricted on  $B$ . We have proved that  $\text{ind}_{\mathfrak{G}}(g)$  is a factor of  $d'$ . Moreover, for the particular element  $h_0'$  we chose and the corresponding regular nilpotent element  $N'$  in  $\sum_{\beta \in \Delta(h_0')} \mathcal{C}e_{\beta} (\subset \sum_{\alpha \in \Delta^+} \mathcal{C}e_{\alpha})$ ,  $\text{ind}_{\mathfrak{G}}(\exp h_0' \cdot \exp N') = d'$ . But it is clear that

$$(\exp h_0' \cdot \exp N')^{d'} = \exp d'N' \in \mathfrak{B}$$

and  $(\exp h_0 \cdot \exp N)^d = \exp(dh_0 + h + dN) \in \mathfrak{B}$  (for some  $h \in \Omega'$ ). Therefore  $\text{ind}_{\mathfrak{G}}(\exp h_0 \cdot \exp N)$  is also a factor of  $d'$  and  $\text{ind}_{\mathfrak{B}}(\exp h_0' \cdot \exp N') = d'$ . We have proved the following:

For any  $g \in \mathfrak{B}$ , we can find  $g' \in \mathfrak{B}$ , so that  $\text{ind}_{\mathfrak{B}}(g)$ , as well as  $\text{ind}_{\mathfrak{G}}(g)$  is a factor of  $\text{ind}_{\mathfrak{B}}(g') = \text{ind}_{\mathfrak{G}}(g')$ .

If  $\text{ind}_{\mathfrak{B}}(g') = \text{ind}_{\mathfrak{G}}(g') = d = qr$ , then clearly  $\text{ind}_{\mathfrak{B}}(g'^q) = \text{ind}_{\mathfrak{G}}(g'^q) = r$ . We conclude that:

**Proposition.** *If  $\mathfrak{G}$  is a connected complex semisimple Lie group,  $\mathfrak{B}$  its Borel subgroup, then  $\{\text{ind}_{\mathfrak{B}}(g); g \in \mathfrak{B}\} = \{\text{ind}_{\mathfrak{G}}(g); g \in \mathfrak{B}\} = \{\text{ind}_{\mathfrak{G}}(g); g \in \mathfrak{G}\}$ . In particular  $\text{ind}(\mathfrak{B}) = \text{ind}(\mathfrak{G})$ .*

*This is also true for a connected complex reductive Lie group.*

REMARK. It is not clear to me whether  $\text{ind}_{\mathfrak{B}}(g) = \text{ind}_{\mathfrak{G}}(g)$  for any  $g \in \mathfrak{B}$ .

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