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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 15(1) P.161–P.182</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1978</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/7291">https://doi.org/10.18910/7291</a></td>
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<td>DOI</td>
<td>10.18910/7291</td>
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ON THE ALEXANDER POLYNOMIALS OF SLICE LINKS

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(Received November 26, 1976)
(Revised September 1, 1977)

The purpose of this note is to generalize the theorem that the Alexander polynomial of a slice knot is of the form $f(t) f(t^{-1})$ for an integral polynomial $f(t)$ with $|f(1)|=1$ (see [3]). We will show the following:

**Theorem.** Let $L$ be a slice link with $\mu$ components in the strong sense, then there exists an integral polynomial $F(t_1, \ldots, t_\mu)$ with $|F(1, \ldots, 1)|=1$ and the Alexander polynomial $A(t_1, \ldots, t_\mu)$ of $L$ is of the form

$$A(t_1, \ldots, t_\mu) \equiv F(t_1, \ldots, t_\mu) \cdot F(t_1^{-1}, \ldots, t_\mu^{-1})^{(*)}.$$  

Conversely for a given integral polynomial $F(t_1, \ldots, t_\mu)$ with $|F(1, \ldots, 1)|=1$, there exists a slice link with $\mu$ components in the strong sense whose Alexander polynomial is $F(t_1, \ldots, t_\mu) \cdot F(t_1^{-1}, \ldots, t_\mu^{-1})$.

To prove the above Theorem, we will consider two theorems. In §2 the necessary condition of the Alexander polynomials will be considered for not only slice links in the strong sense, but also cobordant links. We will prove the following:

**Theorem 1.** For cobordant links $L_i$, $i=1, 2$, with $\mu$ components, there exist two integral polynomials $F_i(t_1, \ldots, t_\mu)$, $i=1, 2$, with $|F_1(1, \ldots, 1)|=1$ such that

$$A(t_1, \ldots, t_\mu) \cdot F_1(t_1, \ldots, t_\mu) \cdot F_1(t_1^{-1}, \ldots, t_\mu^{-1})$$

$$\equiv A_2(t_1, \ldots, t_\mu) \cdot F_2(t_1, \ldots, t_\mu) \cdot F_2(t_1^{-1}, \ldots, t_\mu^{-1}),$$

where $A_i$ is the Alexander polynomial of the link $L_i$.

Since a slice link $L$ with $\mu$ components in the strong sense is cobordant to the trivial link with $\mu$ components, the following corollary will be obtained.

**Corollary.** The Alexander polynomial $A(t_1, \ldots, t_\mu)$ of a slice link $L$ with $\mu$ components in the strong sense necessarily satisfies $A(t_1, \ldots, t_\mu) \equiv F(t_1, \ldots, t_\mu)$

* The notation "$\equiv$" means equal up to $\pm t_1^{m_1} t_2^{m_2} \cdots t_\mu^{m_\mu}$ for suitable integers $m_1, \ldots, m_\mu$.  


\[ \times F(t_1^{-1}, \cdots, t_\mu^{-1}) \] for an integral polynomial \( F(t_1, \cdots, t_\mu) \) with \( |F(1, \cdots, 1)| = 1 \).

In §3, it will be shown that the condition in the Cor. to Theorem 1 is sufficient; i.e., the following theorem will be proved:

**Theorem 2.** For a given integral polynomial \( F(t_1, \cdots, t_\mu) \) with \( |F(1, \cdots, 1)| = 1 \), there exists a slice link \( L \) with \( \mu \) components in the strong sense whose Alexander polynomial is \( F(t_1, \cdots, t_\mu) \cdot F(t_1^{-1}, \cdots, t_\mu^{-1}) \).

In §4, some examples will be considered.

A. Kawauchi [5] has obtained some of the results of this paper. Our work is independent of his; on the other hand, it was useful to us in that it showed the re-definition of the Alexander polynomials and the numerical invariant \( \beta \). By Fox’s definition [1], slice links in the strong sense have 0-Alexander polynomials for \( \mu \geq 2 \).

Throughout the paper, spaces are considered in the piecewise-linear category, and the Alexander polynomials are non-zero.

1. Preliminaries and definitions

A link is the disjoint union of piecewise-linearly embedded, oriented 1-spheres in the oriented 3-sphere \( S^3 \). Two links \( L_1 \) and \( L_2 \) with \( \mu \) components are cobordant, if there exist mutually disjoint, locally flat, piecewise-linearly embedded proper annuli \( F_1, \cdots, F_\mu \) in \( S^3 \times [0, 1] \) spanning \( S^3 \times 0 \) and \( S^3 \times 1 \) such that \( (F_1 \cup \cdots \cup F_\mu) \cap (S^3 \times 0) = L_1 \times 0 \) and \( (F_1 \cup \cdots \cup F_\mu) \cap (S^3 \times 1) = (-L_2) \times 1 \), where \( -L_2 \) is \( L_2 \) with orientation reversed. A link that is cobordant to the trivial link is called a slice link in the strong sense ([1]). For cobordant links \( L_i \), \( i = 1, 2 \), with \( \mu \) components the Alexander polynomials \( A_i(t_1, \cdots, t_\mu) \) of \( L_i \) should be chosen to be the Alexander polynomials associated with the meridian bases of \( H_1(S^3 - L_i; \mathbb{Z}) \) consistent through the cobordism annuli \( F_1, \cdots, F_\mu \).

Let \( L \subset S^3 \) be a link with \( \mu \) components and \( B_1, \cdots, B_\nu \) be mutually disjoint 2-cells in \( S^3 \) such that for each \( j \), \( B_j \cap L = \partial B_j \cap L \) consists of two arcs. The resulting link \( L' = (L - \bigcup_{j=1}^\nu \partial B_j \cap L) \cup \bigcup_{j=1}^\nu \text{cl} (\partial B_j - L) \) with the induced orientation from \( L - \bigcup_{j=1}^\nu \partial B_j \cap L \) is called the (oriented) link obtained from \( L \) by the hyperbolic transformations along the bands \( B_1, \cdots, B_\nu \). If the number of the components of \( L' \) is \( \mu - \nu \), then the link \( L' \) is said to be obtained from \( L \) by the fusion* along \( B_1, \cdots, B_\nu \).

Let a link \( L \) consist of sublinks \( L_1 \) and \( L_2 \) that are separated by a 2-sphere in \( S^3 \). Then the link \( L \) is denoted by \( L_1 \cup L_2 \). Let \( O^\nu = \underbrace{O \cdots O}_\nu \) be the trivial link with \( \nu \) components.

* This terminology is the same as in [6], but more general than that of F. Hosokawa [4].
2. Proof of Theorem 1

**Theorem 1.** For cobordant links \( L_i, i=1, 2, \) with \( \mu \) components, there exist two integral polynomials \( F_i(t_1, \ldots, t_\mu), i=1, 2, \) with \(|F_i(1, \ldots, 1)| = 1\) such that

\[
A_i(t_1, \ldots, t_\mu) = F_i(t_1, \ldots, t_\mu) \cdot F_i(t_1^{-1}, \ldots, t_\mu^{-1})
\]

\[
= A_2(t_1, \ldots, t_\mu) \cdot F_2(t_1, \ldots, t_\mu) \cdot F_2(t_1^{-1}, \ldots, t_\mu^{-1}),
\]

where \( A_i \) is the Alexander polynomial of the link \( L_i \).

To prove Theorem 1, it is enough to consider the following lemmas.

**Lemma 1.** Let \( L_1 \) and \( L_2 \) be cobordant links with \( \mu \) components. Then there exist integers \( v_1, v_2 \geq 0 \) and a link \( \tilde{L} \) with \( \mu \) components such that for each \( i, i=1, 2, \) \( \tilde{L} \) is obtained from the \((\mu+v_i)\)-component link \( L_1 \circ O^{v_i} \) by the fusion along certain bands \( B_1^{(i)}, \ldots, B_\mu^{(i)} \) joining each component of \( O^{v_i} \) with the link \( L_i \).

This lemma is generally known. (See [2], [4] and [6].)

**Lemma 2.** If a \( \mu \)-component link \( \tilde{L} \) is obtained from the \((\mu+\nu)\)-component link \( L \circ O^\nu \) by the fusion along bands \( B_1, \ldots, B_\mu \) joining each component of \( O^\nu \) with \( L \), then there exists a polynomial \( F(t_1, \ldots, t_\mu) \) such that \( \tilde{A}(t_1, \ldots, t_\mu) \equiv (t_1, \ldots, t_\mu) \times F(t_1, \ldots, t_\mu) \cdot F(t_1^{-1}, \ldots, t_\mu^{-1}), |F(1, \ldots, 1)| = 1 \), where \( A \) and \( \tilde{A} \) are the Alexander polynomials of \( L \) and \( \tilde{L} \), respectively.

Proof of Theorem 1. It is straightforward from Lemmas 1 and 2.

Proof of Lemma 2. We will consider a case in which \( \mu=2, \nu=3 \) to avoid unnecessary complexity, but as we will see later, the calculation method will not depend on the numbers \( \mu \) and \( \nu \).

Consider the plane projection of \( L \) as in Fig. 1. The link group \( G(L) \) can be then presented as follows:

- **generators:** \( x_1, \ldots, x_n \), \( y_1, \ldots, y_n \),
- **relators:**
  - \( r_1^{(x)} = x_ia^p_b x_ia^{-1}_i x_ia^{-p} \) \( (i = 1, \ldots, n-1) \)
  - \( r_1^{(y)} = y_ia^p_b y_ia^{-1}_i y_ia^{-p} \) \( (i = 1, \ldots, n-1) \)
  - \( r_1^{(p)} = y_ia^p_b y_ia^{-1}_i y_ia^{-p} \),

where \( x_* \) is an element in the set \( \{x_i, y_j; i = 1, \ldots, n, j = 1, \ldots, n\} \), and \( \epsilon_p = +1 \) or \(-1\).
Let $\alpha$ be the Alexander matrix of $L$, then $\alpha$ is equivalent to the following matrix with entries in $\mathbb{Z}[x, y]$, where $\{x, y\}$ is the meridian base of $G(L)/G(L')$.

$$
\begin{pmatrix}
1 & -w_x & \cdots & 1 & -w_x & \cdots & 1 & \cdots & 1 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{pmatrix} = \alpha.
$$
Let us use this presentation of $G(L)$ to consider a presentation of $G(\tilde{L})/G(\tilde{L})''$. Let $x', y', z', a_i, b_j$ and $c_k$ be the generators corresponding to the trivial link and the attaching bands as in Fig. 1.

We will study how the upper paths of $L$ are divided by the attaching bands in the projection of $\tilde{L}$;

Fig. 2

Fig. 3
The upper path $x_i$ is divided into $x_{i1}, \ldots, x_{it}$ by the attaching bands (see Fig. 2). The relators obtained from these parts are as follows:

(I) \[ \begin{align*}
x_{i1} &= \alpha_{i1}^{e_1} x_{i1-1} \alpha_{i1}^{-e_1} \\
x_{i2} &= \alpha_{i2}^{e_2} x_{i2} \alpha_{i2}^{-e_2}.
\end{align*} \]

Here, $e_1$ is $+1$ or $-1$, and $\alpha_{it}$ is one of $a_\ast, b_\ast, c_\ast$. Thus, we get $t_i$ generators instead of one generator of $G(L)$ and $t_i - 1$ defining relators (I).

Assume that the attaching bands attach at the upper paths $x_{i1}, x_{i2}$ and $y_j$ of $L$ (see, for example, Fig. 3), so that the resulting upper paths of $L$ are denoted by $x_{i1}'$ and $x_{i2}'$, and $y_j'$ and $y_j''$.

More generators and relators related to $O_i \cup O_2 \cup O_3$ and the attaching bands have to be considered (see, for example, Fig. 4).

As a result, one presentation of $G(L)/G(L)''$ is as follows*:

generators; $x_{it}, y_{jm}, (i = 1, \ldots, i_1, i_2, \ldots, i_{t1}, i_{t2}, \ldots, n, j = 1, \ldots, j_1, j_2, \ldots, n_j)$

* In addition to the relators stated below, the generators of $G(L)''$ should be added as the relators of $G(L)/G(L)''$. But these relators become 0 by Fox's free calculus on each generator of $G(L)/G(L)''$. Hence we need not think of these relators for our purpose and omit.
relators; \( r'_i = w_{k_i} w_{l_i} w_{r_i}^{-1} w_{s_i}^{-1} \) or \( w_{a_i} w_{k_i} w_{l_i} w_{r_i}^{-1} w_{s_i}^{-1} \), caused from the presentation of \( G(L) \), where \( \{x_{il} y_{jm}\} \) and \( n = n_x \) or \( n_y \), \( i = 1, 2, \ldots, n_x + n_y \).

From (I), \( x_{il} = Ax_{il} A^{-1} \) \( (i = 1, \ldots, n_x; j = 1, \ldots, n_y) \)

\( y_{jm} = By_{jm} B^{-1} \),

where \( A \) and \( B \) are some words of \( \{a_{\pm 1}^*, b_{\pm 1}^*, c_{\pm 1}^*\} \).

From (III), \( S_1 = s_1 \ldots s_{n_x} s_{n_x}^{-1} \ldots s_{i}^{-1} a_{n_x} a_{n_x}^{-1} \)

\( S_2 = s'_1 \ldots s'_{n_x} s'_{n_x}^{-1} \ldots s'_{i}^{-1} c_{n_x} c_{n_x}^{-1} \)

\( S_3 = s''_1 \ldots s''_x s''_y s''_{n_x} s''_{n_x} \ldots s''_{i}^{-1} b_{n_y} b_{n_y}^{-1} \),

where \( s_i, s'_i, s''_i \) are some of \( a_{\pm 1}^*, b_{\pm 1}^* \) and \( c_{\pm 1}^* \).

From (IV), \( R_1 = w_1 \ldots w_{n_x} w_{n_x}^{-1} \ldots w_1^{-1} x_1 \)

\( R'_1 = w_1 \ldots w_1 x_1^{-1} \ldots w_{n_x}^{-1} x_1 \)

\( R_2 = w_1 \ldots w_{n_x} w_{n_x}^{-1} \ldots w_1^{-1} x_2 \)

\( R'_2 = w_1 \ldots w_1 x_2^{-1} \ldots w_{n_x}^{-1} x_2 \)

\( R_3 = w_1 \ldots w_{n_x} w_{n_x}^{-1} \ldots w_1^{-1} y_1 \)

\( R'_3 = w_1 \ldots w_1 y_1^{-1} \ldots w_{n_x}^{-1} y_1 \)

where \( w_i, w'_i \) and \( w''_i \) are some of \( \{x_{il}^*, y_{jm}^*, a_{il} \} \).

Since \( a_* \), \( b_* \), and \( c_* \) are the elements of \( G(L)' \), these generators are commutative mutually, so that their indices are changed only after the attaching bands crossing under the upper paths of \( O_1 \circ O_2 \circ O_3 \circ L \);

\[
\begin{align*}
(a_2 &= a_{a_1} a_{a_1}^{-1} \\
&\vdots \\
a_{n_x} &= a_{n_x-1} \ldots a_{a_1} a_{a_1}^{-1} \ldots a_{a_1}^{-1} \\
(2 &= \gamma_{e_1} \gamma_{e_1}^{-1} \\
&\vdots \\
\gamma_{n_x} &= \gamma_{n_x-1} \ldots \gamma_{e_1} \gamma_{e_1}^{-1} \ldots \gamma_{e_1}^{-1} \\
b_2 &= b_{b_1} b_{b_1}^{-1} \\
&\vdots \\
b_{n_x} &= b_{n_x-1} \ldots b_{b_1} b_{b_1}^{-1} \ldots b_{b_1}^{-1}
\end{align*}
\]

where \( a_* \), \( b_* \), and \( \gamma_* \) are some of \( x_{il} \) and \( y_{jm} \), since \( x_{il} \) and \( y_{jm} \) have the form in (I).

For the same reason, \( S_1 \), \( S_2 \) and \( S_3 \) are equivalent to the following:

\[
S_1 = a_{n_x}^{-1} (a_{e_1} a_{e_1}^{-1} \ldots a_{e_1}) (c_{e_1} c_{e_1}^{-1} \ldots c_{e_1}) (b_{e_k} b_{e_k}^{-1} \ldots b_{e_k}) x_1^{-1} \times (b_{e_k}^{-1} \ldots b_{e_k}^{-1}) (c_{e_1}^{-1} c_{e_1} \ldots c_{e_1}^{-1}) (a_{e_1}^{-1} a_{e_1} \ldots a_{e_1}) x_1
\]
where \( n_s > i_1 > \cdots > i_n > 1, \ n_s > j_1 > \cdots > j_m > 1, \ n_s > k_1 > \cdots > k_n > 1, \) and so on.

Since the sets (I) and (V) are the defining relations, \( x_i, (l+1), y_j, (n+1), \ a_i, (i+1), b_j, (j+1) \) and \( c_k, (k+1) \) vanish. Let us use \( x_i, y_j, a, b \) and \( c \) instead of \( x^{i_1}, y^{j_1}, a_1, b_1 \) and \( c_1 \), respectively.

After \( a_i, b_j, \) and \( c_k \) vanishing, let us use these notations as words having the following forms:

\[
\begin{align*}
a_i &= \alpha_{i-1}^a \cdots \alpha_1^a x_1^{-1} \cdots x_1^{-1} \\
b_j &= \beta_{j-1}^b \cdots \beta_1^b y_1^{-1} \cdots y_1^{-1} \\
c_k &= \gamma_{k-1}^c \cdots \gamma_1^c z_1^{-1} \cdots z_1^{-1}.
\end{align*}
\]

Then, the presentation of \( G(L)/G(L)' \) is the following:

**Generators:** \( x_i, \cdots, x_{i_1}, y_{j_1}, \cdots, x_{i_2}, y_{j_2}, \cdots, x_{i_n}, \)
\( y_1, \cdots, y_{j_1}, y_{j_2}, \cdots, y_{j_m}, \)
\( x', y', z', \)
\( a, b, c, \)

**Relators:**

\[
\begin{align*}
r_i &= A_i w_i A_i^{-1} \cdot W_i w_i^1 W_i^{-1} \cdot A_i^* w_i^1 A_i^{-1} \cdot W_i^{-1} w_i^{-1} W_i \\
(\ell = 1, \cdots, n_s + n_j),
\end{align*}
\]

where \( A_* \) and \( W_* \) are some words of \( \{a_i^{\pm 1}, b_j^{\pm 1}, c_k^{\pm 1}\} \), and \( w_* \) is some of \( \{x_i^{\pm 1}, y_j^{\pm 1}\} \), and \((\ell', \ell') = (k, k+1) \) or \((n_s, 1) \) or \((n_j, 1) \).

\[
\begin{align*}
R_1 &= W_1(x_i, y_j, a_*, b_*, c_*, x', y', z') x_i^{-1} W_1^{-1}(x_i, \cdots, z') x_i^{-1} \\
R_1' &= W_1(x_i, y_j, a_*, b_*, c_*, x', y', z') x_i^{-1} W_1^{-1}(x_i, \cdots, z') x_i^{-1} \\
R_2 &= W_2(x_i, y_j, a_*, b_*, c_*, x', y', z') z_i^{-1} W_2^{-1}(x_i, \cdots, z') x_i^{-1} \\
R_2' &= W_2(x_i, \cdots, z') z_i^{-1} W_2^{-1}(x_i, \cdots, z') x_i^{-1} \\
R_3 &= W_3(x_i, \cdots, z') y' b_n^{-1} W_3^{-1}(x_i, \cdots, z') y_1^{-1} \\
R_3' &= W_2(x_i, \cdots, z') y' W_3^{-1}(x_i, \cdots, z') y_1^{-1},
\end{align*}
\]

where \( W_1, W_2 \) and \( W_3 \) are the words of \( \{x_i, y_j, a_*, b_*, c_*, x', y', z'\} \).

\[
\begin{align*}
S_1 &= a_n^{-1}(c_{i_1}^{\pm 1} \cdots a_{i_1}^{\pm 1}(c_j^{\pm 1} \cdots c_{j_m}^{\pm 1}(b_{k_1}^{\pm 1} \cdots b_{k_n}^{\pm 1} x')^{-1} \\
& \times (b_{k_n}^{\pm 1} \cdots b_{k_1}^{\pm 1}) (c_{j_m}^{\pm 1} \cdots c_j^{\pm 1}(a_i^{\pm 1} \cdots a_i^{\pm 1} x').
\end{align*}
\]
Before considering the Alexander matrix of $G(\widetilde{L})/G(\widetilde{L})^\vee$, we will introduce several properties of the free calculus.

**Proposition 1.**

\[
\frac{\partial r^*_i}{\partial w} = 0 \quad (w = x', y', z', i = 1, \ldots, n_x + n_y).
\]

Proof. If $w$ appears in $r_i^*$, then $w$ is contained in the words $A_*$ or $W_*$ that have the special forms; for example, let us consider the form of $A_*$,

\[
A_* = a_i \cdots a_i, b_j, \cdots b_{j_m}, c_k, \cdots c_k
\]

\[
= (\alpha_{i_1} \cdots \alpha_{i_s} a^{s_1} \alpha_{i_1}^{\chi_1} \cdots \alpha_{i_s}^{\chi_s}) \times
\]

\[
\cdots \times (\gamma_{j_1}^{s_1} \cdots \gamma_{j_s}^{s_1})
\]

Since $a$, $b$, and $c$ are mapped to 1 by the abelianized map, $a_i$, $b_j$, and $c_k$ are also mapped to 1. Let us consider the case that $\alpha_j = w$, which appears in $a_i$, then

\[
\frac{\partial a_i}{\partial w} = \alpha_{i-1} \cdots \alpha_{i-j+1}(1 + w\alpha_{i-j-1} \cdots \alpha a^{\chi_1} \cdots \alpha_{j-1}(-w^{-1}))
\]

\[= 0.\]

In the case that $w$ appears in $a_i$ in more than one place, it is easy to get the same result by using a similar calculation as above.

So, it is not difficult to get $\frac{\partial A_*}{\partial w} = 0$, since $A_*$ consists of only $\{a_i^{\chi_1}\}$, $\{b_j^{\chi_1}\}$ and $\{c_k^{\chi_1}\}$.

**Proposition 2.**

\[
\frac{\partial r^*_i}{\partial w} = \frac{\partial r_i}{\partial w} \quad (w = x_i, y_j, i = i_1, i_2, j, i_1 - 1, i_2 - 1, j - 1).
\]

Proof. In the case that $w$ appears in some of $A_*$ and $W_*$, there is no change in this part, by the same reasoning introduced in the previous proposition, since the words $A_*$ and $W_*$ in $r_i^*$ are mapped to 1 by abelianization;
\[ = A_i \frac{\partial w_i}{\partial w} + A_i w_i A_i^{-1} W^i \frac{\partial w}{\partial w} + \ldots \]
\[ = \frac{\partial w}{\partial w} + \frac{\partial w}{\partial w} + w_i \frac{\partial w}{\partial w} + W^i \frac{\partial w}{\partial w} + \frac{\partial w}{\partial w} \]
\[ = \frac{\partial (w_i \frac{\partial w}{\partial w} + W^i \frac{\partial w}{\partial w})}{\partial w} = \frac{\partial r_i}{\partial w} . \]

The following is similarly obtained:

**Proposition 3.**
\[ \frac{\partial r'_i}{\partial w} = \frac{\partial r_i}{\partial w} \quad (w \neq x_{i1}, x_{i2}, x_{i12}, x_{i22}, y_{j1}, y_{j2}) , \]
\[ \frac{\partial r'_{i1}}{\partial x_{i1}} = \frac{\partial r_{i1}}{\partial x_{i1}} \quad \frac{\partial r'_{i1-1}}{\partial x_{i1}} = \frac{\partial r_{i1-1}}{\partial x_{i1}} , \]
\[ \frac{\partial r'_{i2}}{\partial x_{i2}} = \frac{\partial r_{i2}}{\partial x_{i2}} \quad \frac{\partial r'_{i2-1}}{\partial x_{i2}} = \frac{\partial r_{i2-1}}{\partial x_{i2}} , \]
\[ \frac{\partial r'_j}{\partial y_{j2}} = \frac{\partial r_j}{\partial y_{j2}} \quad \frac{\partial r'_{j-1}}{\partial y_{j2}} = \frac{\partial r_{j-1}}{\partial y_{j2}} . \]

**Proposition 4.**
\[ \frac{\partial R_1}{\partial w} = \frac{\partial R'_1}{\partial w} \quad (w = x_i (i \neq i_1, i_12), y_j, x', y', z') , \]
\[ \frac{\partial R_2}{\partial w} = \frac{\partial R'_2}{\partial w} \quad (w = w_i (i \neq i_1, i_12) y_j, x', y', z') , \]
\[ \frac{\partial R_3}{\partial w} = \frac{\partial R'_3}{\partial w} \quad (w = x_{i1}, y_{j1} (j \neq j_1, j_2), x', y', z') , \]
\[ \frac{\partial R_1}{\partial x_{i1}} = \frac{\partial R'_1}{\partial x_{i1}} - 1 , \quad \frac{\partial R'_1}{\partial x_{i1}} = \frac{\partial R_1}{\partial x_{i1}} - 1 , \]
\[ \frac{\partial R_2}{\partial x_{i2}} = \frac{\partial R'_2}{\partial x_{i2}} - 1 , \quad \frac{\partial R'_2}{\partial x_{i2}} = \frac{\partial R_2}{\partial x_{i2}} - 1 , \]
\[ \frac{\partial R_3}{\partial y_{j2}} = \frac{\partial R'_3}{\partial y_{j2}} - 1 , \quad \frac{\partial R'_3}{\partial y_{j2}} = \frac{\partial R_3}{\partial y_{j2}} - 1 . \]

**Proof.** The differences between \( R_1 \) and \( R'_1 \) are in the last letters and the center parts. Since \( \frac{\partial a_{s-1}}{\partial w} = 0 \) and \( a_{s-1} \) is mapped to 1 by abelianization, we have

\[ \frac{\partial R_1}{\partial x_{i1}} = \frac{\partial (W_i x' x W_{i1}^{-1})}{\partial x_{i1}} + W_i x' a_{s-1} W_{i1}^{-1} (1-x_{i1}) \]
\[ = \frac{\partial (W_i x' x W_{i1}^{-1})}{\partial x_{i1}} - 1 . \]
By using a similar calculation, the other equations are also obtained.

**Proposition 5.**

\[
\frac{\partial S_i}{\partial w} = 0 \quad (i = 1, 2, 3, w = x_i, y_j, x', y', z')
\]

**Proof.** This is easily derived considering the forms of \(a_i, b_j\) and \(c_k\).

Now consider the Alexander matrix of \(L\). By Propositions 1, 4 and 5, this matrix is equivalent to the following matrix:

\[
\begin{array}{ccccccccc}
& x_1 & \cdots & x_n & y_1 & \cdots & y_s & x_{i1} & x_{i2} & x_{i3} & y_{j1} & y_{j2} & x' & y' & a & c & b \\
R_1 & p_1 & \cdots & \cdots & p_n & P_1 - 1 & P'_1 & P_{i1} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_1' & p_1 & \cdots & \cdots & p_n & P_1 - 1 & P_{i1} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_2 & q_1 & \cdots & \cdots & \cdots & q_{i2} & P_2 - 1 & P'_2 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_2' & q_1 & \cdots & \cdots & \cdots & q_{i2} & P_2 - 1 & P'_2 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_3 & r_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_3' & r_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
S_1 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_1 & & & & & & & & & & & & & & & & \\
S_3 & & & & & & & & & & & & & & & & \\
\end{array}
\]

where \(P_i = \frac{\partial R'_i}{\partial x_{i1}}, P'_1 = \frac{\partial R'_1}{\partial x_{i2}}, P_2 = \frac{\partial R'_2}{\partial x_{i1}}, P'_2 = \frac{\partial R'_2}{\partial x_{i2}}, P_3 = \frac{\partial R'_3}{\partial y_{j1}}, P'_3 = \frac{\partial R'_3}{\partial y_{j2}}\).

By Proposition 3, each entry of \((x_{i1}-\text{th row} + x_{i2}-\text{th row})\) is equal to the \(x_i-\text{th row}\), and \((x_{i1}-\text{th row} + x_{i2}-\text{th row})\) and \((y_{j1}-\text{th row} + y_{j2}-\text{th row})\) are equal to the \(x_{i1}-\text{th row}\) and the \(y_{j1}-\text{th row}\) of the Alexander matrix of \(L\), so that this matrix is equivalent to the following:
By Proposition 4, this matrix is equivalent to the following:
(substitute $R'_i - R_i$ to $R'_i$ for $i=1, 2, 3$)

$$
\begin{pmatrix}
\ldots \\
1 \\
\ldots \\
0 \\
\ldots \\
0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_{i1} \\
x_{i2} \\
y_1 \\
y_{i1} \\
y_{i2}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
x'_1 \\
x'_{i1} \\
x'_{i2} \\
y'_1 \\
y'_{i1} \\
y'_{i2}
\end{pmatrix}
= 
\begin{pmatrix}
a \\
0 \\
0 \\
b
\end{pmatrix}
$$

where $M_1 = \left(\frac{\partial S_i}{\partial w}\right)$ ($i=1, 2, 3, w=a, b, c$) and $M_2 = \left(\frac{\partial R_i}{\partial w}\right)$ ($i=1, 2, 3, w=x', y', z'$).
To complete the proof of the special case, it suffices to show that if \( \det M_1 = F(x, y) \), then \( \det M_2 = F(x^{-1}, y^{-1}) \) and \( |F(1,1)| = 1 \), since the first non-zero polynomial of the above matrix is a product of the first non-zero polynomial of \((a)\), \( \det M_1 \) and \( \det M_2 \). Therefore, consider \( \frac{\partial R_i}{\partial w} \) \((i=1, 2, 3, w=x', y', z')\) and \( \frac{\partial S_i}{\partial w} \) \((i=1, 2, 3, w=a, b, c)\).

Since the words of \( a_i \)'s, \( b_j \)'s and \( c_k \)'s are the conjugates of \( a, b \) and \( c \), we obtain the following:

**Proposition 6.**

\[
\begin{align*}
\frac{\partial R_1}{\partial w} &= \frac{\partial (W_1(x_i, y_j, 1, 1, 1, x', y', z') \cdot x' \cdot W_1^{-1}(x_i, \ldots, z') x_{11}^{-1})}{\partial w} \\
\frac{\partial R_2}{\partial w} &= \frac{\partial (W_2(x_i, y_j, 1, 1, 1, x', y', z') \cdot z' \cdot W_2^{-1}(x_i, \ldots, z') x_{12}^{-1})}{\partial w} \\
\frac{\partial R_3}{\partial w} &= \frac{\partial (W_3(x_i, y_j, 1, 1, 1, x', y', z') \cdot y' \cdot W_3^{-1}(x_i, \ldots, z') y_{j1}^{-1})}{\partial w},
\end{align*}
\]

where \( w=x', y' \) or \( z' \).

Let us consider the words \( \tilde{W}_i = W_i(x_i, y_j, 1, 1, 1, x', y', z') \) \((i=1, 2, 3)\). Since the relators \( R_i \) \((i=1, 2, 3)\) are obtained from the edges of the attaching bands, the length of \( \tilde{W}_i \) are related to the indices \( n_a, n_b \) and \( n_c \). The indices of \( a_*, b_* \) and \( c_* \) are changed when the attaching bands pass under the edges of \( O_1 \cup O_2 \cup O_3 \cup L \), at the same time the length of \( \tilde{W}_i \) increases by just one letter.

Assume that

\[
\begin{align*}
\tilde{W}_1 &= w_1 w_2 \cdots w_n \\
\tilde{W}_2 &= v_1 v_2 \cdots v_m \quad (w_*, v_*, u_* = x_i^t, y_j^t, x_i'^t, y_j'^t, z_j'^t, \epsilon = \pm1) \\
\tilde{W}_3 &= u_1 u_2 \cdots u_l,
\end{align*}
\]

where \( n=n_a-1, m=n_b-1, l=n_c-1 \).

Since \( a_n, b_n \) and \( c_n \) are obtained by the paths of the attaching bands, it follows that

\[
\begin{align*}
a_n &= w_n^{-1} \cdots w_1^{-1} a w_1 \cdots w_n \\
b_n &= u_1^{-1} \cdots u_l^{-1} b u_l \cdots u_1 \\
c_n &= v_m^{-1} \cdots v_1^{-1} c v_1 \cdots v_m.
\end{align*}
\]

Similarly,

\[
\begin{align*}
a_i &= w_i^{-1} \cdots w_{i-1}^{-1} a w_{i-1} \cdots w_i \\
b_j &= u_j^{-1} \cdots u_{j-1}^{-1} b u_{j-1} \cdots u_j \\
c_k &= v_k^{-1} \cdots v_{k-1}^{-1} c v_{k-1} \cdots v_k.
\end{align*}
\]
Proposition 7.

\[ \frac{\partial S_1}{\partial a} = -w_{i-1} \cdots w_1 \cdot (1-x^{-1}) \left( \varepsilon_i \cdot w_{i-1} \cdots w_1 \cdot \cdots + \varepsilon_j \cdot w_{j-1} \cdots w_1 \right) \]

\[ \frac{\partial S_1}{\partial b} = (1-x^{-1}) \left( \varepsilon_{i_k} \cdot u_{i-k-1} \cdots u_1 \cdot \cdots + \varepsilon_{i_j} \cdot u_{j-1} \cdots u_1 \right) \]

\[ \frac{\partial S_1}{\partial c} = \left( \frac{1}{x^{-1}} \right) \left( \varepsilon_{i_l} \cdot v_{l-1} \cdots v_1 \cdot \cdots + \varepsilon_{j_m} \cdot v_{m-1} \cdots v_1 \right) \]

\[ \frac{\partial S_2}{\partial a} = (1-x^{-1}) \left( \varepsilon_{i_k} \cdot u_{i-k-1} \cdots w_{i-1} \cdot \cdots + \varepsilon_{i_j} \cdot w_{j-1} \cdots w_1 \right) \]

\[ \frac{\partial S_2}{\partial b} = (1-x^{-1}) \left( \varepsilon_{i_k} \cdot w_{i-k-1} \cdots u_{i-1} \cdot \cdots + \varepsilon_{i_j} \cdot u_{j-1} \cdots u_1 \right) \]

\[ \frac{\partial S_2}{\partial c} = \left( \frac{1}{y^{-1}} \right) \left( \varepsilon_{i_k} \cdot w_{i-k-1} \cdots v_{i-1} \cdot \cdots + \varepsilon_{i_j} \cdot v_{j-1} \cdots v_1 \right) \]

\[ \frac{\partial S_3}{\partial a} = \left( \frac{1}{y^{-1}} \right) \left( \varepsilon_{i_k} \cdot v_{i-k-1} \cdots w_{i-1} \cdot \cdots + \varepsilon_{i_j} \cdot w_{j-1} \cdots w_1 \right) \]

\[ \frac{\partial S_3}{\partial b} = \left( \frac{1}{y^{-1}} \right) \left( \varepsilon_{i_k} \cdot v_{i-k-1} \cdots u_{i-1} \cdot \cdots + \varepsilon_{i_j} \cdot u_{j-1} \cdots u_1 \right) \]

\[ \frac{\partial S_3}{\partial c} = \left( \frac{1}{y^{-1}} \right) \left( \varepsilon_{i_k} \cdot v_{i-k-1} \cdots v_{i-1} \cdot \cdots + \varepsilon_{i_j} \cdot v_{j-1} \cdots v_1 \right) \]

Proof. These are deduced from the forms of \( S_i \).

To calculate \( \frac{\partial R_1}{\partial w} \) \((i=1, 2, 3, w=x', y', z')\), we check where \( x', y' \) and \( z' \) appear. When the attaching bands cross under \( O_1 \cup O_2 \cup O_3 \), then \( x', y' \) or \( z' \) appears in \( \widetilde{W}' \).

Let us consider \( a_i'^{\ast} \) in \( S_i \). There are two cases (see, Fig. 5).

Case (I). If \( a_i \) crosses over \( O_i \) from left to right, then \( \varepsilon_i = 1 \) and \( a_i = x'a_{i-1}^{-1}x' \).

So there exists \( w_{i-1} \) in \( R_1 \) \((1 \leq i = 1 \leq n, \) such that \( w_{i-1} = x'^{-1} \).

Case (II). If \( a_i \) crosses over \( O_1 \) from right to left, then \( \varepsilon_i = -1 \) and \( a_i = -1 \).
Alexander Polynomials of Slice Links

\[ x'^{-1}a_ix'. \] So there exists \( w_{i_k} \) in \( R_i \) \((1 \leq i_k \leq n)\), such that \( w_{i_k} = x'. \)

Then, for \( a_{i_k} \) in \( S_l \), there exists index \( i_k - 1 \) or \( i_k \), and a letter \( w_{i_k-1} \) or \( w_{i_k} \) in \( R_i \), such that \( w_{i_k-1} = x'^{-1} \) or \( w_{i_k} = x'. \) Corresponding to this letter,

\[
\frac{\partial R_i}{\partial x'} = \cdots + w_1 \cdots w_{i_k-1}(x'^{-1}) + \cdots \quad (\varepsilon_{i_k} = 1)
\]

\[
= \cdots + (\varepsilon_{i_k}w_1 \cdots w_{i_k-1}) + \cdots .
\]

By the same reasoning, corresponding to the letters \( b_{i_k} \) and \( c_{j_k} \) in \( S_l \), we obtain the equations

\[
\frac{\partial R_j}{\partial x'} = \cdots + (\varepsilon_{k}u_1 \cdots u_{k-1}) + \cdots,
\]

respectively.

Using these equations, we can prove Proposition 8.

**Proposition 8.**

\[
\frac{\partial R_1}{\partial x'} = w_1 \cdots w_n -(1-x) (\varepsilon_{i_1}w_1 \cdots w_{i_1-1} + \cdots + \varepsilon_{i_n}w_1 \cdots w_{i_n-1})
\]

\[
\frac{\partial R_1}{\partial y'} = -(1-x) (\varepsilon_{i_1}w_1 \cdots w_{i_1-1} + \cdots + \varepsilon_{i_n}w_1 \cdots w_{i_n-1})
\]

\[
\frac{\partial R_1}{\partial z'} = -(1-x) (\varepsilon_{i_1}w_1 \cdots w_{i_1-1} + \cdots + \varepsilon_{i_n}w_1 \cdots w_{i_n-1})
\]

\[
\frac{\partial R_1}{\partial x'} = -(1-x) (\varepsilon_{i_1}v_1 \cdots v_{j_1-1} + \cdots + \varepsilon_{j_n}v_1 \cdots v_{j_n-1})
\]

\[
\frac{\partial R_1}{\partial y'} = -(1-x) (\varepsilon_{i_1}v_1 \cdots v_{j_1-1} + \cdots + \varepsilon_{j_n}v_1 \cdots v_{j_n-1})
\]

\[
\frac{\partial R_1}{\partial z'} = v_1 \cdots v_m -(1-x) (\varepsilon_{i_1}v_1 \cdots v_{j_1-1} + \cdots + \varepsilon_{j_n}v_1 \cdots v_{j_n-1})
\]

\[
\frac{\partial R_3}{\partial x'} = u_1 \cdots u_l -(1-y) (\varepsilon_{k_1}u_1 \cdots u_{k_1-1} + \cdots + \varepsilon_{k_n}u_1 \cdots u_{k_n-1})
\]

\[
\frac{\partial R_3}{\partial y'} = u_1 \cdots u_l -(1-y) (\varepsilon_{k_1}u_1 \cdots u_{k_1-1} + \cdots + \varepsilon_{k_n}u_1 \cdots u_{k_n-1})
\]

\[
\frac{\partial R_3}{\partial z'} = -(1-y) (\varepsilon_{k_1}u_1 \cdots u_{k_1-1} + \cdots + \varepsilon_{k_n}u_1 \cdots u_{k_n-1})
\]

**Proof.** For example, consider the form of \( R_1 \). Except for the letter \( x' \) in the
center of $R_1$, all letters $x'$ appear in $\tilde{W}_1$ corresponding to the parts of the attaching bands crossing over $O_1$. Then, it is not difficult to get the desired equation of $\frac{\partial R_1}{\partial x'}$. And all $y'$ (or $z'$) appear in $\tilde{W}_1$ corresponding to the parts of the attaching bands crossing over $O_2$.

Using Propositions 7 and 8, let $\frac{\partial S_1}{\partial a} = f_1(x, y), \frac{\partial S_1}{\partial b} = (1 - x^{-1})f_2(x, y), \frac{\partial S_1}{\partial c} = -(1 - x^{-1})f_3(x, y)$,

\[ (1 - x^{-1})g_1(x, y), \frac{\partial S_2}{\partial a} = (1 - x^{-1})g_2(x, y), \frac{\partial S_2}{\partial b} = g_3(x, y), \frac{\partial S_2}{\partial c} = g_4(x, y), \]

\[ (1 - y^{-1})h_1(x, y), \frac{\partial S_3}{\partial a} = h_2(x, y), \frac{\partial S_3}{\partial b} = h_3(x, y), \frac{\partial S_3}{\partial c} = (1 - y^{-1})h_4(x, y). \]

Then, \[ \frac{\partial R_1}{\partial x'} = -f_1, \quad \frac{\partial R_2}{\partial x'} = -(1 - x)f_2, \quad \frac{\partial R_3}{\partial x'} = -(1 - y)f_3, \]

\[ \frac{\partial R_1}{\partial x'} = -(1 - x)g_1, \quad \frac{\partial R_2}{\partial x'} = -g_2, \quad \frac{\partial R_3}{\partial x'} = -(1 - y)g_3, \]

\[ \frac{\partial R_1}{\partial y'} = -(1 - x)h_1, \quad \frac{\partial R_2}{\partial y'} = -(1 - y)h_2, \quad \frac{\partial R_3}{\partial y'} = -h_3, \]

where $f_i$ means $f_i(x^{-1}, y^{-1})$ and so on.

We have

\[
M_1 \sim S_2 \begin{pmatrix} a & f_1 \ (1 - x^{-1})f_3 \ (1 - x^{-1})f_2 \\ c & (1 - x^{-1})g_1 & g_3 & (1 - x^{-1})g_2 \\ b & (1 - y^{-1})h_1 & (1 - y^{-1})h_3 & h_2 \end{pmatrix}
\]

and

\[
M_2 \sim R_2 \begin{pmatrix} x' & \ -f_1 \ -(1 - x)g_1 \ -h_1 \\ z' & -(1 - x)f_2 & -g_3 & -(1 - y)h_2 \\ y' & -(1 - y)f_3 & -h_3 & -h_2 \end{pmatrix}
\]

Thus, $F(x, y) = \det M_1 = - (1 - x^{-1}) (1 - y^{-1}) f_1 g_2 h_3 - (1 - x^{-1}) (1 - y^{-1}) f_1 g_2 h_1$

\[ - (1 - x^{-1})^2 f_2 g_1 h_2 + (1 - x^{-1})(1 - y^{-1}) f_2 g_3 h_1 \]

\[ + f_1 g_3 h_2 + (1 - x^{-1})(1 - y^{-1}) f_2 g_3 h_3, \]

and $\det M_2 = (1 - x) (1 - y) f_1 g_2 h_3 + (1 - x) (1 - y) f_1 g_2 h_1 + (1 - x) f_3 g_3 h_1$

\[ - (1 - x)^2 f_2 g_2 h_1 - f_1 g_3 h_2 - (1 - x)(1 - y) f_2 g_3 h_3 \]

\[ = - F(x^{-1}, y^{-1}). \]

It is immediate that

\[ |F(1, 1)| = |f_1(1, 1, 1) \cdot g_3(1, 1, 1) \cdot h_2(1, 1, 1)| = 1. \]
For general cases of \( \mu \) and \( \nu \), it is sufficient only to check the matrices \( M_1 \) and \( M_2 \) as in the previous step. These matrices are related to the trivial link \( O_1 \cup \cdots \cup O_6 \) and the attaching bands.

Instead of \( a, b, c, x', y', z', R, S, (i=1, 2, 3) \), we need generators \( a_i, x_i' (i=1, 2, \cdots, \nu) \) and relators \( R_i, S_i (i=1, 2, \cdots, \nu) \). Since the situation is just the same as in the previous case,

\[
M_1 \sim \begin{pmatrix}
S_1 & \frac{\partial S_1}{\partial a_i} \\
S_2 & \frac{\partial S_2}{\partial a_i}
\end{pmatrix}
\]

and

\[
M_2 \sim \begin{pmatrix}
R_1 & \frac{\partial R_1}{\partial x_i'} \\
R_2 & \frac{\partial R_2}{\partial x_i'}
\end{pmatrix}
\]

Let \( \frac{\partial S_i}{\partial a_i} = f_{i\cdot}(x_1, \cdots, x_n) \) \((i=1, \cdots, \nu)\)

\( \frac{\partial S_i}{\partial a_i} = (1-x_i'^{-1})f_{i\cdot}(x_1, \cdots, x_n) \) \((i=1, \cdots, \nu, \rho \neq i) \), (*)

then

\( \frac{\partial R_i}{\partial x_i'} = -f_{i\cdot} \) \((i=1, \cdots, \nu)\)

\( \frac{\partial R_i}{\partial x_i'} = -(1-x_i')f_{i\cdot} \) \((i=1, \cdots, \nu, \rho \neq i)\).

So, \( \det M_1 = | f_{11} \ldots (1-x_i'^{-1})f_{1\cdot} \ldots (1-x_i'^{-1})f_{1v} | \)

\( (1-x_i'^{-1})f_{21} \ldots f_{2\cdot} \ldots f_{2v} | \)

\( (1-x_i'^{-1})f_{31} \ldots \ldots \ldots f_{3\cdot} \ldots f_{3v} | \)

\( (1-x_i'^{-1})f_{41} \ldots \ldots \ldots \ldots \ldots \ldots f_{4\cdot} \ldots f_{4v} | \)

\( (1-x_i'^{-1})f_{51} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots f_{5\cdot} \ldots f_{5v} | \)

\( (1-x_i'^{-1})f_{61} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots f_{6\cdot} \ldots f_{6v} | \)

\( \det M_2 = | -f_{11} \ldots -(1-x_i')f_{12} \ldots -(1-x_i')f_{1v} | \)

\( -(1-x_i')f_{21} \ldots f_{22} \ldots f_{2v} | \)

\( -(1-x_i')f_{31} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots f_{32} \ldots f_{3v} | \)

\( -(1-x_i')f_{41} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots f_{42} \ldots f_{4v} | \)

\( -(1-x_i')f_{51} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots f_{52} \ldots f_{5v} | \)

\( -(1-x_i')f_{61} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots f_{62} \ldots f_{6v} | \)

* Here, \( x_i' \) denotes a suitable letter in \( \{x_1, \cdots, x_n\} \).
\[ Y. \text{ Nakagawa} \]

Thus, there exists a polynomial \( \sum_{1}^{n} \) such that

\[ \det M_1 = (-1) \det M. \]

This completes the proof of Lemma 2.

REMARK. In the proof of Lemma 2, we can also find that the integer \( \beta(L) \) is the invariant of PL cobordant links [5]. To see this, let \( L_i, i=1, 2, \) be PL cobordant links. \( L_1 \) is cobordant to a link \( L'_1, \) where each component of \( L'_2 \) is obtained from a component of \( L_2 \) by tying a knot in a small 3-cell. We have \( \beta(L_1)=\beta(L_2), \) since \( \det M_1 \neq 0 \) and \( \det M_2 \neq 0 \) in the proof of Lemma 2 imply that \( \beta(L) \) is the cobordism invariant. \( \beta(L'_2)=\beta(L_2) \) easily follows from a direct use of Fox's free calculus. Hence \( \beta(L_1)=\beta(L_2). \)

3. Proof of Theorem 2

Theorem 2. For a given polynomial \( F(t_1, \ldots, t_\mu) \) with \( |F(1, \ldots, 1)| = 1, \) there exists a slice link \( L \) with \( \mu \) components in the strong sense whose Alexander polynomial is \( F(t_1, \ldots, t_\mu) \cdot F(t_1^{-1}, \ldots, t_\mu^{-1}). \)

To avoid unnecessary complexity, let us consider the case that \( \mu=3, \) but the construction of a slice link \( L \) with \( \mu \) components in the strong sense are completely done by the same way.

Theorem 2'. For a given polynomial \( F(x, y, z) \) with \( |F(1, 1, 1)| = 1, \) there exists a slice link \( L \) with 3 components in the strong sense whose Alexander polynomial is \( F(x, y, z) \cdot F(x^{-1}, y^{-1}, z^{-1}). \)

Proof. Since \( |F(1, 1, 1)| = 1, \) we can assume that \( F(x, y, z) \) will be splitted into the form

\[ F(x, y, z) = 1 - (1-x)f_1(x, y, z) - (1-y)f_2(x, y, z) - (1-z)f_3(x, y, z). \]

In order to construct a slice link \( L, \) it's enough to get the informations of attaching bands. So we need relators \( R_i \) and \( S_i (i=1, 2, 3). \) Since the relators \( S_i \) can be automatically obtained from \( R_i, \) let us consider \( R_i. \) Therefore, we
have to consider a part of the Alexander matrix $M_2 = \left( \frac{\partial R_i}{\partial w} \right)$, $(w=x', y', z')$.

To consider the matrix $M_2$, let us deform the polynomial $F(x, y, z)$ as follows:

$$F(x, y, z) = \left\{ 1 - (1-x) f_1(x, y, z) \right\} \left\{ 1 - (1-y) f_2(x, y, z) \right\} \left\{ 1 - (1-z) f_3(x, y, z) \right\}$$

$$- (1-x) (1-y) f_1(x, y, z) f_2(x, y, z) - (1-y) (1-z) f_2(x, y, z) f_3(x, y, z)$$

$$- (1-z) (1-x) f_3(x, y, z) f_1(x, y, z)$$

$$+ (1-x) (1-y) (1-z) f_1(x, y, z) f_2(x, y, z) f_3(x, y, z).$$

It's easy to check that this form is the determinant of the following matrix $M$:

$$M \sim \begin{bmatrix} 1 - (1-x) f_1 & - (1-x) f_1 & - (1-x) f_1 \\ - (1-y) f_2 & 1 - (1-y) f_2 & - (1-y) f_2 \\ - (1-z) f_3 & - (1-z) f_3 & 1 - (1-z) f_3 \end{bmatrix}.$$

Let us take the matrix $M$ as $M_2$; i.e.,

$$\frac{\partial R_1}{\partial x} = 1 - (1-x) f_1, \quad \frac{\partial R_1}{\partial y} = -(1-x) f_1, \quad \frac{\partial R_1}{\partial z} = -(1-x) f_1,$$

$$\frac{\partial R_2}{\partial x} = -(1-y) f_2, \quad \frac{\partial R_2}{\partial y} = 1 - (1-y) f_2, \quad \frac{\partial R_2}{\partial z} = -(1-y) f_2,$$

$$\frac{\partial R_3}{\partial x} = -(1-z) f_3, \quad \frac{\partial R_3}{\partial y} = - (1-z) f_3, \quad \frac{\partial R_3}{\partial z} = 1 - (1-z) f_3.$$

Instead of the relator $R_i$ is a word of $x, x', y, y', z, z', a, b,$ and $c$, it's enough to construct $R_i$ as a word of $x, x', y, y', z, z', a, b,$ and $c$. Since

$$R_1 = w_1 w_2 \cdots w_n x' a_1^{-1} w_1^{-1} \cdots w_1^{-1} x_*,$$

$$R_2 = w_1' \cdots w_m' y' b_1^{-1} w_1'^{-1} \cdots w_1'^{-1} y_*'$$

$$R_3 = w_1'' \cdots w_{m'}'' z' c_1^{-1} w_1''^{-1} \cdots w_1''^{-1} z_*'$$

we will make the words $w_1 \cdots w_n, w_1' \cdots w_m'$ and $w_1'' \cdots w_{m'}''$.

For example let us assume that

$$f_1(x, y, z) = \varepsilon_1 x^{a_1} y^{b_1} z^{c_1} + \varepsilon_2 x^{a_2} y^{b_2} z^{c_2} + \cdots + \varepsilon_r x^{a_r} y^{b_r} z^{c_r}.$$

Then, $w_1 \cdots w_n$ contains $x', y', z'$ in $r$ places, respectively.

Let us put these $3r$ letters as the following manner;

$$z'^{-t_1} \cdots z'^{-t_r} \cdots z'^{-t_r} \cdots y'^{-t_1} \cdots y'^{-t_r} \cdots x'^{-t_1} \cdots x'^{-t_r}.$$

Decide $w_1 \cdots w_n$ to satisfy the equation

$$\frac{\partial (w_1 \cdots w_n)}{\partial x'} = \varepsilon_1 x^{a_1} y^{b_1} z^{c_1}.$$
Depending on $\alpha_1 > 0$ or $\alpha_1 < 0$, letters among $w_1 \cdots w_\#$ are $x$ or $x^{-1}$, and $\beta_1$ letters among $w_1' \cdots w_\#'$ are $y$ or $y^{-1}$. If $\epsilon_1 > 0$ and $\gamma_1 > 0$, there must be $\gamma_1 + 1$ letters $z$ among $w_1 \cdots w_\#$. Then,

$$w_1 \cdots w_\# z^{-1} = x^{\epsilon_1 \pm 1} \cdots x^{\epsilon_\# \pm 1} y^{\gamma_1 \pm 1} \cdots y^{\gamma_\# \pm 1} z \cdots z_{\#}^{-1}.$$  

Let us repeat this step $3r - 1$ times more. After these steps, the word $w_1 \cdots w_m$ can be obtained. By the same way, $w_1' \cdots w_m'$ and $w_1'' \cdots w_m''$ can also be obtained.

By using these words, the following relators will be read;

$$a_n = w_n^{-1} \cdots w_1^{-1} \cdot a_1 \cdot w_1 \cdots w_n$$

$$b_n = w_n'^{-1} \cdots w_1'^{-1} \cdot b_1 \cdot w_1' \cdots w_n'$$

$$c_n = w_n''^{-1} \cdots w_1''^{-1} \cdot c_1 \cdot w_1'' \cdots w_n''$$

Since the indices of $a$, $b$ and $c$ are changed when the attaching bands cross under the overpasses of the link, the relators above give us the informations about the attaching bands;

$$a_2 = w_1^{-1} \cdot a_1 \cdot w_1$$

$$b_2 = w_1'^{-1} \cdot b_1 \cdot w_1'$$

$$c_2 = w_1''^{-1} \cdot c_1 \cdot w_1''$$

$$a_3 = w_2^{-1} \cdot a_2 \cdot w_2$$

$$b_3 = w_2'^{-1} \cdot b_2 \cdot w_2'$$

$$c_3 = w_2''^{-1} \cdot c_2 \cdot w_2''$$

$$\vdots$$

$$a_n = w_n^{-1} \cdot a_n \cdot w_n$$

$$b_n = w_n'^{-1} \cdot b_n \cdot w_n'$$

$$c_n = w_n''^{-1} \cdot c_n \cdot w_n''$$

To construct a link $L$, put the trivial link with 6 components, representing the generators $x$, $x'$, $y$, $y'$, $z$ and $z'$, and contact two components representing $x$ and $x'$ by a attaching band which goes over according as relators $a_i$, and so on. Since the attaching bands can freely go over the other parts of attaching bands, it is possible to combine two components to represent the relators $a_i$, $b_i$ and $c_i$.

This completes the proof.

Note that there may be many different links satisfying conditions for the given polynomial.

4. Some examples

Example 1. Let $F(x, y) = x - xy + y$. Then, this polynomial can be deformed into

$$FF(x, y) = 1 - y + x^{-1} y = 1 - (1 - x) (-x^{-1} y).$$

This is the determinant of a matrix $M_2$;

$$M_2 \sim \begin{pmatrix}
1 - (1 - x)(-x^{-1} y) & 0 \\
0 & 1
\end{pmatrix}$$
Then \( \frac{\partial R_1}{\partial x'} = 1 - (1-x)(-x^{-1}y) \), \( \frac{\partial R_1}{\partial y'} = 0 \),
\( \frac{\partial R_2}{\partial x'} = 0 \), \( \frac{\partial R_2}{\partial y'} = 1 \).

So there are three \( x' \) but no \( y' \) among letters of \( R_i \). Since \( \frac{\partial (w_1 \cdots w_n)}{\partial x'} = -x^{-1}y \),
\( \varepsilon_i = -1 \), \( w_1 = x^{-1} \), \( w_2 = y \), \( w_3 = x' \), and since \( \frac{\partial (w_4 \cdots w_n)}{\partial x'} = 1 \), \( n = 4 \) and \( w_4 = y^{-1} \).

Then, we get
\[
 R_1 = x^{-1}y x' y^{-1} x_x x_x^{-1} y x' x^{-1} x, 
\]
and
\[
 a_2 = w_1^{-1} a_2 w_1 = x a_1 x^{-1}, 
\]
\[
 a_3 = w_2^{-1} a_3 w_2 = y^{-1} a_3 y, 
\]
\[
 a_4 = w_3^{-1} a_4 w_3 = x' a_4 x', 
\]
\[
 a_5 = w_4^{-1} a_5 w_4 = y a_4 y^{-1}. 
\]

Similarly, \( R_2 = y' b_n y_j^{-1} \).

By using these relators, it's possible to construct a link \( L \) with 2 components.

**Example 2**. Let \( F(x, y, z) = -x + yz + xyz \). Then

\[
 F(x, y, z) = 1 + x - x y z - (1-x) (1-y) (1-z) y z 
\]
\[
 = (1-(1-x) (1-y) (1-z)) y (1-y) (1-z) y z 
\]
\[
 - (1-x) (1-y) (1-z) y z 
\]
\[
 - (1-z) (1-x) y z (1-y) (1-x) (1-y) (1-z) y z. 
\]

* In example 2, \( w \) means \( w^{-1} \).
Then,

\[
M_z \sim \begin{bmatrix}
1-(1-x)(1-yz) & -(1-x)(1-yz) & -(1-x)(1-yz) \\
-(1-y)\bar{y} & 1-(1-y)\bar{y} & -(1-y)\bar{y} \\
-(1-z)\bar{y}z & -(1-z)\bar{y}z & 1-(1-z)\bar{y}z
\end{bmatrix}
\]

Let us construct \( R_1 \);

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \bar{y}z & \bar{y} & \bar{y}' & \bar{x}' & \bar{x} \\
1 & \bar{y}z & \bar{y} & \bar{y}' & \bar{x}' & \bar{x} \\
\end{array}
\]

Since \( \frac{\partial (w_1 \cdots z')}{\partial z'} = 1 \), \( w_1 = z \) and \( w_2 = z' \) and \( z \bar{z} \frac{\partial (w_3 \cdots z')}{\partial z'} = -\bar{y}z, w_3 = \bar{y}, w_4 = z \) and \( w_5 = z' \). Since \( z \bar{z} \bar{y}z' \frac{\partial (w_6 \cdots z')}{\partial z'} = 1 \), \( w_6 = y, w_7 = y \) and \( w_8 = y' \). By the similar way, we can get

\[
R_1 = z \bar{z} | \bar{y} \bar{z} \bar{z}' | y y' \bar{y}' | \bar{y} z \bar{z}' | \bar{z} \bar{z} \bar{z}' | a \bar{z}_{n_z}^{-1}
\times z \bar{z} \bar{y} xx' \bar{x} \bar{y} \bar{z} y y' \bar{y} \bar{y} z \bar{y} z \bar{z}' z \bar{z} x^{-1},
\]

and \( n_z = 21 \)

\[
\begin{align*}
a_2 &= 2a_1z \\
a_3 &= z'a_2z' \\
a_4 &= ya_3\bar{y} \\
a_5 &= za_4z \\
a_6 &= z'a_5z' \\
a_7 &= ya_6\bar{y} \\
a_8 &= \bar{y}a_7y \\
a_9 &= y'a_6\bar{y}' \\
a_{10} &= ya_9\bar{y} \\
a_{11} &= za_{10}z \\
a_{12} &= \bar{y}'z_{11}y' \\
a_{13} &= za_{12}z \\
a_{14} &= xa_{13}\bar{z} \\
a_{15} &= a_{14}\bar{z} \\
a_{16} &= ya_{15}\bar{y} \\
a_{17} &= za_{16}z \\
a_{18} &= xa_{17}\bar{z} \\
a_{19} &= ya_{18}\bar{y} \\
a_{20} &= \bar{y}a_{19}y \\
a_{21} &= za_{20}z \\
a_{22} &= x'a_{14}\bar{z}'
\end{align*}
\]

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References