Nakagawa, Y.
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ON THE ALEXANDER POLYNOMIALS OF SLICE LINKS

Yoko NAKAGAWA

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The purpose of this note is to generalize the theorem that the Alexander polynomial of a slice knot is of the form $f(t)\cdot f(t^{-1})$ for an integral polynomial $f(t)$ with $|f(1)|=1$ (see [3]). We will show the following:

Theorem. *Let L be a slice link with μ components in the strong sense, then there exists an integral polynomial $F(t_1, \cdots, t_\mu)$ with $|F(1, \cdots, 1)|=1$ and the Alexander polynomial $A(t_1, \cdots, t_\mu)$ of L is of the form*

$$A(t_1, \cdots, t_\mu) \doteq F(t_1, \cdots, t_\mu)\cdot F(t_1^{-1}, \cdots, t_\mu^{-1})^{(*)} .$$

Conversely for a given integral polynomial $F(t_1, \cdots, t_\mu)$ with $|F(1, \cdots, 1)|=1$, there exists a slice link with μ components in the strong sense whose Alexander polynomial is $F(t_1, \cdots, t_\mu)\cdot F(t_1^{-1}, \cdots, t_\mu^{-1})$.

To prove the above Theorem, we will consider two theorems. In §2 the necessary condition of the Alexander polynomials will be considered for not only slice links in the strong sense, but also cobordant links. We will prove the following:

Theorem 1. *For cobordant links L_i, $i=1, 2$, with μ components, there exist two integral polynomials $F_i(t_1, \cdots, t_\mu)$, $i=1, 2$, with $|F_i(1, \cdots, 1)|=1$ such that*

$$\begin{aligned} &A_1(t_1, \cdots, t_\mu)\cdot F_1(t_1, \cdots, t_\mu)\cdot F_1(t_1^{-1}, \cdots, t_\mu^{-1}) \\ &\doteq A_2(t_1, \cdots, t_\mu)\cdot F_2(t_1, \cdots, t_\mu)\cdot F_2(t_1^{-1}, \cdots, t_\mu^{-1}) , \end{aligned}$$

where A_i is the Alexander polynomial of the link L_i.

Since a slice link L with μ components in the strong sense is cobordant to the trivial link with μ components, the following corollary will be obtained.

Corollary. *The Alexander polynomial $A(t_1, \cdots, t_\mu)$ of a slice link L with μ components in the strong sense necessarily satisfies $A(t_1, \cdots, t_\mu) \doteq F(t_1, \cdots, t_\mu)$*

* The notation "$\doteq$" means equal up to $\pm t_1^{n_1} t_2^{n_2} \cdots t_\mu^{n_\mu}$ for suitable integers $n_1, \cdots, n_\mu$.

$\times F(t_1^{-1}, \cdots, t_\mu^{-1})$ *for an integral polynomial* $F(t_1, \cdots, t_\mu)$ *with* $|F(1, \cdots, 1)|=1$.

In §3, it will be shown that the condition in the Cor. to Theorem 1 is sufficient; i.e., the following theorem will be proved:

Theorem 2. *For a given integral polynomial* $F(t_1, \cdots, t_\mu)$ *with* $|F(1, \cdots, 1)| =1$, *there exists a slice link* L *with* μ *components in the strong sense whose Alexander polynomial is* $F(t_1, \cdots, t_\mu)\cdot F(t_1^{-1}, \cdots, t_\mu^{-1})$.

In §4, some examples will be considered.

A. Kawauchi [5] has obtained some of the results of this paper. Our work is independent of his; on the other hand, it was useful to us in that it showed the re-definition of the Alexander polynomials and the numerical invariant β. By Fox's definition [1], slice links in the strong sense have 0-Alexander polynomials for $\mu\geq 2$.

Throughout the paper, spaces are considered in the piecewise-linear category, and the Alexander polynomials are non-zero.

1. Preliminaries and definitions

A *link* is the disjoint union of peicewise-linearly embedded, oriented 1-spheres in the oriented 3-sphere S^3. Two links L_1 and L_2 with μ components are *cobordant,* if there exist mutually disjoint, locally flat, piecewise-linearly embedded proper annuli $F_1, \cdots, F_\mu$ in $S^3\times[0, 1]$ spanning $S^3\times 0$ and $S^3\times 1$ such that $(F_1\cup\cdots\cup F_\mu)\cap(S^3\times 0)=L_1\times 0$ and $(F_1\cup\cdots\cup F_\mu)\cap(S^3\times 1)=(-L_2)\times 1$, where $-L_2$ is L_2 with orientation reversed. A link that is cobordant to the trivial link is called a *slice link in the strong sense* ([1]). For cobordant links L_i, $i=1, 2$, with μ components the Alexander polynomials $A_i(t_1, \cdots, t_\mu)$ of L_i should be chosen to be the Alexander polynomials associated with the meridian bases of $H_1(S^3-L_i; Z)$ consistent through the cobordism annuli $F_1, \cdots, F_\mu$.

Let $L\subset S^3$ be a link with μ components and $B_1, \cdots, B_\nu$ be mutually disjoint 2-cells in S^3 such that for each j, $B_j\cap L=\partial B_j\cap L$ consists of two arcs. The resulting link $L'=(L-\bigcup_{j=1}^{\nu}\partial B_j\cap L)\cup\bigcup_{j=1}^{\nu} cl(\partial B_j-L)$ with the induced orientation from $L-\bigcup_{j=1}^{\nu}\partial B_j\cap L$ is called the (*oriented*) *link obtained from* L *by the hyperbolic transformations along the bands* $B_1, \cdots, B_\nu$. If the number of the components of L' is $\mu-\nu$, then the link L' is said to *be obtained from* L *by the fusion* along* $B_1, \cdots, B_\nu$.

Let a link L consist of sublinks L_1 and L_2 that are separated by a 2-sphere in S^3. Then the link L is denoted by $L_1\circ L_2$. Let $O^\nu=\underbrace{O\circ\cdots\circ O}_{\nu}$ be the trivial link with ν components.

* This terminology is the same as in [6], but more general than that of F. Hosokawa [4].

2. Proof of Theorem 1

Theorem 1. *For cobordant links L_i, $i=1, 2$, with μ components, there exist two integral polynomials $F_i(t_1, \cdots, t_\mu)$, $i=1, 2$, with $|F_i(1, \cdots, 1)|=1$ such that*

$$A_1(t_1, \cdots, t_\mu)\cdot F_1(t_1, \cdots, t_\mu)\cdot F_1(t_1^{-1}, \cdots, t_\mu^{-1})$$
$$\doteq A_2(t_1, \cdots, t_\mu)\cdot F_2(t_1, \cdots, t_\mu)\cdot F_2(t_1^{-1}, \cdots, t_\mu^{-1}),$$

where A_i is the Alexander polynomial of the link L_i.

To prove Theorem 1, it is enough to consider the following lemmas.

Lemma 1. *Let L_1 and L_2 be cobordant links with μ components. Then there exist integers $\nu_1, \nu_2 \geq 0$ and a link $\tilde{L}$ with μ components such that for each i, $i=1, 2$, $\tilde{L}$ is obtained from the $(\mu+\nu_i)$-component link $L_i \circ O^{\nu_i}$ by the fusion along certain bands $B_1^{(i)}, \cdots, B_{\nu_i}^{(i)}$ joining each component of O^{ν_i} with the link L_i.*

This lemma is generally known. (See [2], [4] and [6].)

Lemma 2. *If a μ-component link $\tilde{L}$ is obtained from the $(\mu+\nu)$-component link $L\circ O^\nu$ by the fusion along bands $B_1, \cdots, B_\mu$ joining each component of O^ν with L, then there exists a polynomial $F(t_1, \cdots, t_\mu)$ such that $\tilde{A}(t_1, \cdots, t_\mu)\doteq(t_1, \cdots, t_\mu)\times F(t_1, \cdots, t_\mu)\cdot F(t_1^{-1}, \cdots, t_1^{-1})$, $|F(1, \cdots, 1)|=1$, where A and $\tilde{A}$ are the Alexander polynomials of L and $\tilde{L}$, respectively.*

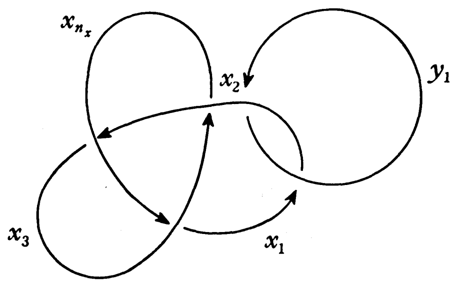
Proof of Theorem 1. It is straightforward from Lemmas 1 and 2.

Proof of Lemma 2. We will consider a case in which $\mu=2$, $\nu=3$ to avoid unnecessary complexity, but as we will see later, the calculation method will not depend on the numbers μ and ν.

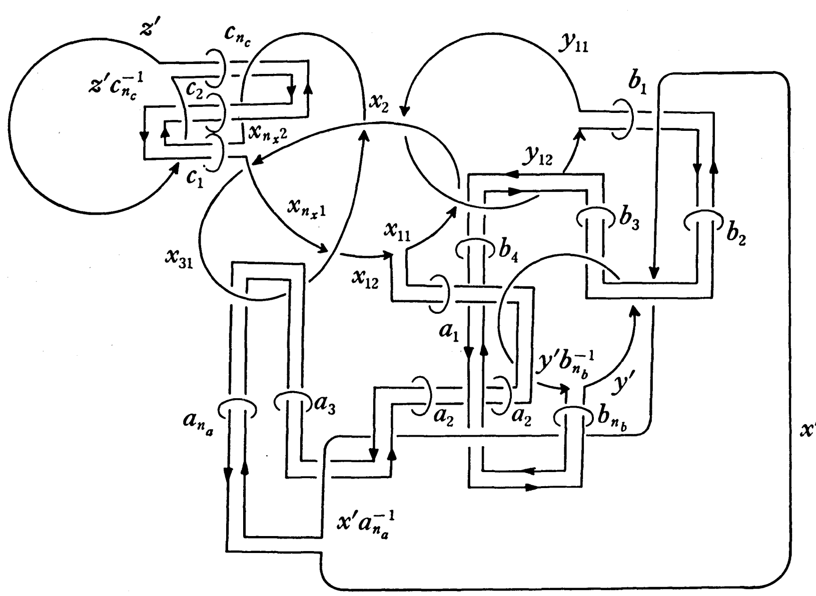
Consider the plane projection of L as in Fig. 1. The link group $G(L)$ can be then presented as follows:

$$\begin{array}{lll}
\text{generators}; & x_1, \cdots, x_{n_x} & \\
 & y_1, \cdots, y_{n_y}, & \\
\text{relators} & ;\ r_i^{(x)} = x_i w_p^{\varepsilon_p} x_{i+1}^{-1} w_p^{-\varepsilon_p} & (i=1, \cdots, n_x-1) \\
 & r_{n_x}^{(x)} = x_{n_x} w_p^{\varepsilon_p} x_1^{-1} w_p^{-\varepsilon_p} & \\
 & r_i^{(y)} = y_i w_p^{\varepsilon_p} y_{i+1}^{-1} w_p^{-\varepsilon_p} & (i=1, \cdots, n_y-1) \\
 & r_{n_y}^{(y)} = y_{n_y} w_p^{\varepsilon_p} y_1^{-1} w_p^{-\varepsilon_p}, &
\end{array}$$

where w_* is an element in the set $\{x_i, y_j;\ i=1, \cdots, n_x, j=1, \cdots, n_y\}$, and $\varepsilon_p= +1$ or -1.



L



$\tilde{L}$

Fig. 1

Let $\mathfrak{a}$ be the Alexander matrix of L, then $\mathfrak{a}$ is equivalent to the following matrix with entries in $Z[x, y]$, where $\{x, y\}$ is the meridian base of $G(L)/G(L)'$.

$$\begin{array}{c} \\ r_1^{(x)} \\ \vdots \\ r_i^{(x)} \\ \vdots \\ r_{n_x}^{(x)} \\ r_1^{(y)} \\ \vdots \\ r_{n_y}^{(y)} \end{array}
\begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & \cdots & x_i & x_{i+1} & \cdots & x_{n_x} & y_1 & \cdots & y_{n_y} \end{array} \\
\left(\begin{array}{cccccccccc}
1 & -w_* & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & 1 & -w_* & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-w_* & \vdots & & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & 1 & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & 1
\end{array}\right) \end{array} = \mathfrak{a}\,.$$

Let us use this presentation of $G(L)$ to consider a presentation of $G(\tilde{L})/G(\tilde{L})''$. Let x', y', z', a_i, b_j and c_k be the generators corresponding to the trivial link and the attaching bands as in Fig. 1.

We will study how the upper paths of L are divided by the attaching bands in the projection of $\tilde{L}$;

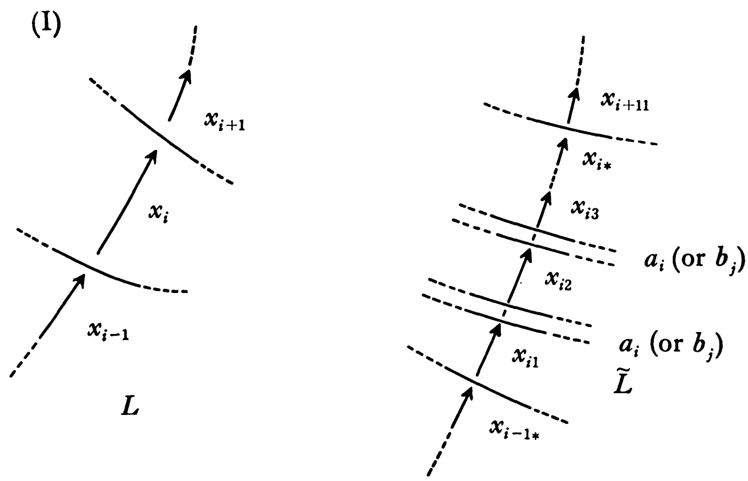


Fig. 2

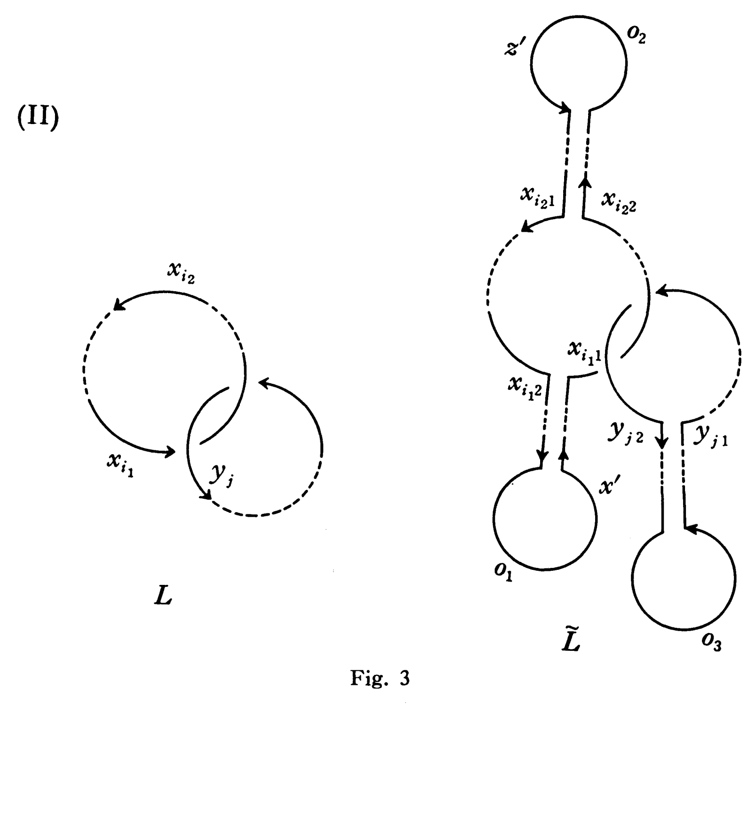


Fig. 3

The upper path x_i is divided into $x_{i1}, \cdots, x_{ii_x}$ by the attaching bands (see Fig. 2). The relators obtained from these parts are as follows:

$$\text{(I)}\quad \begin{cases} x_{ii_x} = \alpha_*^{\varepsilon_*} x_{ii_{x-1}} \alpha_*^{-\varepsilon_*} \\ \vdots \\ x_{i2} = \alpha_*^{\varepsilon_*} x_{i1} \alpha_*^{-\varepsilon_*}\,. \end{cases}$$

Here, ε_* is $+1$ or -1, and α_* is one of a_*, b_*, c_*. Thus, we get i_x generators instead of one generator of $G(L)$ and $i_x - 1$ defining relators (I).

Assume that the attaching bands attach at the upper paths x_{i_1}, x_{i_2} and y_j of L (see, for example, Fig. 3), so that the resulting upper paths of $\tilde{L}$ are denoted by $x_{i_1 1}$ and $x_{i_1 2}$, $x_{i_2 1}$ and $x_{i_2 2}$, and y_{j1} and y_{j2}.

More generators and relators related to $O_1 \cup O_2 \cup O_3$ and the attaching bands have to be considered (see, for example, Fig. 4).

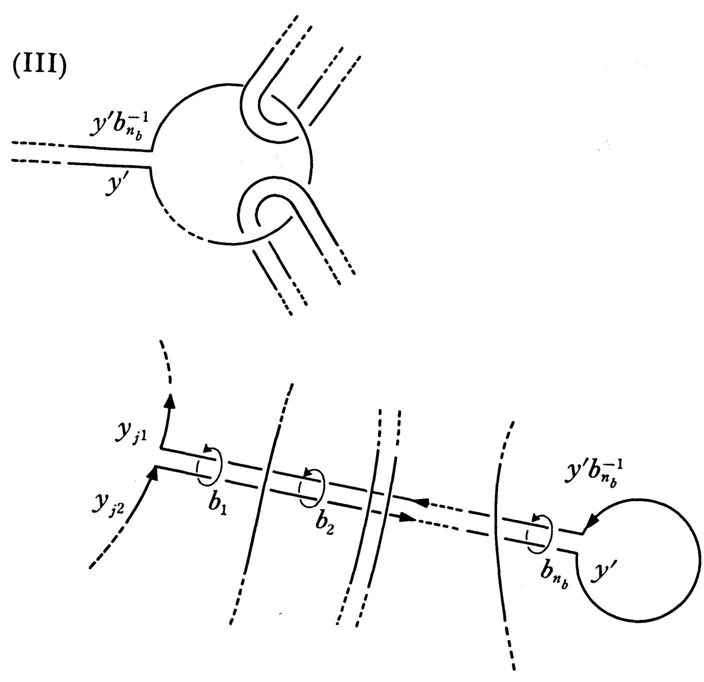


Fig. 4

As a result, one presentation of $G(\tilde{L})/G(\tilde{L})''$ is as follows*:

generators; x_{il}, y_{jm}, $(i = 1, \cdots, i_1 1, i_1 2, \cdots, i_2 1, i_2 2, \cdots, n_x,$
$j = 1, \cdots, j1, j2, \cdots n_y)$

* In addition to the relators stated below, the generators of $G(\tilde{L})''$ should be added as the relators of $G(\tilde{L})/G(\tilde{L})''$. But these relators become 0 by Fox's free calculus on each generator of $G(\tilde{L})/G(\tilde{L})''$. Hence we need not think of these relators for our purpose and omit.

relators; $r'_\iota = w_{k*}w_{l*}^{\varepsilon l}w_{k+1*}^{-1}w_{l*}^{-\varepsilon l}$ or $w_{n*}w_{l*}^{\varepsilon l}w_{1*}^{-1}w_{l*}^{-\varepsilon l}$, caused from the presentation of $G(L)$, where $w_{**}\in\{x_{il}, y_{jm}\}$ and $n=n_x$ or n_y, $\iota=1, 2, \cdots, n_x+n_y$.

From (I), $\begin{aligned} x_{il} &= Ax_{i1}A^{-1} \qquad (i=1,\cdots,n_x,\ j=1,\cdots,n_y)\\ y_{jm} &= By_{j1}B^{-1}, \end{aligned}$

where A and B are some words of $\{a_i^{\pm1}, b_k^{\pm1}, c_j^{\pm1}\}$.

From (III), $\begin{aligned} S_1 &= s_1\cdots s_{n_s}x's_{n_s}^{-1}\cdots s_1^{-1}\cdot a_{n_a}\cdot x'^{-1}\\ S_2 &= s'_1\cdots s'_{m_s}z's'^{-1}_{m_s}\cdots s'^{-1}_1c_{n_c}z'^{-1}\\ S_3 &= s''_1\cdots s''_{l_s}y's''^{-1}_{l_s}\cdots s''^{-1}_1b_{n_b}y'^{-1}, \end{aligned}$

where $s_\iota, s'_\iota, s''_\iota$ are some of $a_i^{\pm1}$, $b_k^{\pm1}$ and $c_j^{\pm1}$.

From (IV), $\begin{aligned} R_1 &= w_1\cdots w_nx'a_{n_a}^{-1}w_n^{-1}\cdots w_1^{-1}x_{i_11}^{-1}\\ R'_1 &= w_1\cdots w_nx'w_n^{-1}\cdots w_1^{-1}x_{i_12}^{-1}\\ R_2 &= w'_1\cdots w'_mz'c_{n_c}^{-1}w'^{-1}_m\cdots w'^{-1}_1x_{i_22}^{-1}\\ R'_2 &= w'_1\cdots w'_mz'w'^{-1}_m\cdots w'^{-1}_1x_{i_22}^{-1}\\ R_3 &= w''_1\cdots w''_ly'b_{n_b}^{-1}w''^{-1}_l\cdots w''^{-1}_1y_{j1}^{-1}\\ R'_3 &= w''_1\cdots w''_ly'w''^{-1}_l\cdots w''^{-1}_1y_{j2}^{-2}, \end{aligned}$

where w_ι, w'_ι and w''_ι are some of $\{x_{il}^{\pm1}, y_{jm}^{\pm1}, a_i^{\pm1}, b_k^{\pm1}, c_j^{\pm1}\}$.

Since a_*, b_*, and c_* are the elements of $G(\tilde{L})'$, these generators are commutative mutually, so that their indices are changed only after the attaching bands crossing under the upper paths of $O_1\circ O_2\circ O_3\circ L$;

$$\text{(V)}\quad\left\{\begin{aligned} a_2 &= \alpha_1a_1\alpha_1^{-1}\\ &\vdots\\ a_{n_a} &= \alpha_{n_a-1}\cdots\alpha_1a_1\alpha_1^{-1}\cdots\alpha_{n_a-1}^{-1}\\ c_2 &= \gamma_1c_1\gamma_1^{-1}\\ &\vdots\\ c_{n_c} &= \gamma_{n_c-1}\cdots\gamma_1c_1\gamma_1^{-1}\cdots\gamma_{n_c-1}^{-1}\\ b_2 &= \beta_1b_1\beta_1^{-1}\\ &\vdots\\ b_{n_b} &= \beta_{n_b-1}\cdots\beta_1b_1\beta_1^{-1}\cdots\beta_{n_b-1}^{-1} \end{aligned}\right.$$

where α_*, β_*, and γ_* are some of x_{i1} and y_{j1}, since x_{il} and y_{jm} have the form in (I).

For the same reason, S_1, S_2 and S_3 are equivalent to the following:

$$\begin{aligned} S_1 = a_{n_a}^{-1}&(a_{i_1}^{\varepsilon i_1}a_{i_2}^{\varepsilon i_2}\cdots a_{i_l}^{\varepsilon i_l})(c_{j_1}^{\varepsilon j_1}c_{j_2}^{\varepsilon j_2}\cdots c_{j_m}^{\varepsilon j_m})(b_{k_1}^{\varepsilon k_1}b_{k_2}^{\varepsilon k_2}\cdots b_{k_n}^{\varepsilon k_n})x'^{-1}\\ &\times(b_{k_n}^{-\varepsilon k_n}\cdots b_{k_1}^{-\varepsilon k_1})(c_{j_m}^{-\varepsilon j_m}\cdots c_{j_1}^{-\varepsilon j_1})(a_{i_l}^{-\varepsilon i_l}\cdots a_{i_1}^{-\varepsilon i_1})x' \end{aligned}$$

$$S_2 = c_{n_c}^{-1}(c_{j_1'}^{\varepsilon j_1'} \cdots c_{j_m'}^{\varepsilon j_m'})(a_{i_1'}^{\varepsilon i_1'} \cdots a_{i_l'}^{\varepsilon i_l'})(b_{k_1'}^{\varepsilon k_1'} \cdots b_{k_n'}^{\varepsilon k_n'})z'^{-1}$$
$$\times(b_{k_n'}^{-\varepsilon k_n'} \cdots b_{k_1'}^{-\varepsilon k_1'})(a_{i_l'}^{-\varepsilon i_l'} \cdots a_{i_1'}^{-\varepsilon i_1'})(c_{j_m'}^{-\varepsilon j_m'} \cdots c_{j_1'}^{-\varepsilon j_1'})z'$$
$$S_3 = b_{n_b}^{-1}(b_{k_1''}^{\varepsilon k_1''} \cdots b_{k_n''}^{\varepsilon k_n''})(a_{i_1''}^{\varepsilon i_1''} \cdots a_{i_l''}^{\varepsilon i_l''})(c_{j_1''}^{\varepsilon j_1''} \cdots c_{j_m''}^{\varepsilon j_m''})y'^{-1}$$
$$\times(c_{j_m''}^{-\varepsilon j_m''} \cdots c_{j_1''}^{-\varepsilon j_1''})(a_{i_l''}^{-\varepsilon i_l''} \cdots a_{i_1''}^{-\varepsilon i_1''})(b_{k_n''}^{-\varepsilon k_n''} \cdots b_{k_1''}^{-\varepsilon k_1''})y'$$

where $n_a > i_1 > \cdots > i_l > 1$, $n_c > j_1 > \cdots > j_m > 1$, $n_a > k_1 > \cdots > k_n > 1$, and so on.

Since the sets (I) and (V) are the defining relations, $x_{il}(l \neq 1)$, $y_{jn}(n \neq 1)$, $a_i(i \neq 1)$, $b_j(j \neq 1)$ and $c_k(k \neq 1)$ vanish. Let us use x_i, y_j, a, b and c instead of x_{i1}, y_{j1}, a_1, b_1 and c_1, respectively.

After a_i, b_j and c_k vanishing, let us use these notations as words having the following forms:

$$a_i = \alpha_{i-1}^{\varepsilon_{i-1}} \cdots \alpha_1^{\varepsilon_1} a \alpha_1^{-\varepsilon_1} \cdots \alpha_{i-1}^{-\varepsilon_{i-1}}$$
$$b_j = \beta_{j-1}^{\varepsilon_{j-1}} \cdots \beta_1^{\varepsilon_1} b \beta_1^{-\varepsilon_1} \cdots \beta_{j-1}^{-\varepsilon_{j-1}}$$
$$c_k = \gamma_{k-1}^{\varepsilon_{k-1}} \cdots \gamma_1^{\varepsilon_1} c \gamma_1^{-\varepsilon_1} \cdots \gamma_{k-1}^{-\varepsilon_{k-1}} .$$

Then, the presentation of $G(\tilde{L})/G(\tilde{L})''$ is the following:

generators; $x_1, \cdots, x_{i_11}, x_{i_12}, \cdots, x_{i_21}, x_{i_22}, \cdots, x_{n_x}$,
$y_1, \cdots, y_{j1}, y_{j2}, \cdots, y_{n_y}$,
x', y', z',
a, b, c,

relators; $r'_\iota = A_{\iota'} w_{\iota'} A_{\iota'}^{-1} \cdot W_\rho w_\rho^\varepsilon W_\rho^{-1} \cdot A_{\iota''} w_{\iota''}^{-1} A_{\iota''}^{-1} \cdot W_\rho^{-1} w_\rho^{-\varepsilon} W_\rho$
$(\iota = 1, \cdots, n_x + n_y)$,

where A_* and W_* are some words of $\{a_i^{\pm 1}, b_j^{\pm 1}, c_k^{\pm 1}\}$, and w_* is some of $\{x_i^{\pm 1}, y_j^{\pm 1}\}$, and $(\iota', \iota'') = (k, k+1)$ or $(n_x, 1)$ or $(n_y, 1)$.

$$R_1 = W_1(x_i, y_j, a_*, b_*, c_*, x', y', z') x' a_{n_a}^{-1} W_1^{-1}(x_i, \cdots, z') x_{i_11}^{-1}$$
$$R_1' = W_1(x_i, y_j, a_*, b_*, c_*, x', y', z') x' W_1^{-1}(x_i, \cdots, z') x_{i_12}^{-1}$$
$$R_2 = W_2(x_i, y_j, a_*, b_*, c_*, x', y', z') z' c_{n_c}^{-1} W_2^{-1}(x_i, \cdots, z') x_{i_21}^{-1}$$
$$R_2' = W_2(x_i, \cdots, z') z' W_2^{-1}(x_i, \cdots, z') x_{i_22}^{-1}$$
$$R_3 = W_3(x_i, \cdots, z') y' b_{n_b}^{-1} W_3^{-1}(x_i, \cdots, z') y_{j_1}^{-1}$$
$$R_3' = W_3(x_i, \cdots, z') y' W_3^{-1}(x_i, \cdots, z') y_{j_2}^{-1} ,$$

where W_1, W_2 and W_3 are the words of $\{x_i, y_j, a_*, b_*, c_*, x', y', z'\}$.

$$S_1 = a_{n_a}^{-1}(a_{i_1}^{\varepsilon i_1} \cdots a_{i_l}^{\varepsilon i_l})(c_{j_1}^{\varepsilon j_1} \cdots c_{j_m}^{\varepsilon j_m})(b_{k_1}^{\varepsilon k_1} \cdots b_{k_n}^{\varepsilon k_n})x'^{-1}$$
$$\times(b_{k_n}^{-\varepsilon k_n} \cdots b_{k_1}^{-\varepsilon k_1})(c_{j_m}^{-\varepsilon j_m} \cdots c_{j_1}^{-\varepsilon j_1})(a_{i_l}^{-\varepsilon i_l} \cdots a_{i_1}^{-\varepsilon i_1})x'$$

$$\begin{aligned}S_2 &= c_{n_c}^{-1}(c_{j_1'}^{\varepsilon j_1'} \cdots c_{j_m'}^{\varepsilon j_m'})(a_{i_1'}^{\varepsilon i_1'} \cdots a_{i_l'}^{\varepsilon i_l'})(b_{k_1'}^{\varepsilon k_1'} \cdots b_{k_n'}^{\varepsilon k_n'})z'^{-1} \\ &\quad \times (b_{k_n'}^{-\varepsilon k_n'} \cdots b_{k_1'}^{-\varepsilon k_1'})(a_{i_l'}^{-\varepsilon i_l'} \cdots a_{i_1'}^{-\varepsilon i_1'})(c_{j_m'}^{-\varepsilon j_m'} \cdots c_{j_1'}^{-\varepsilon j_1'})z' \\ S_3 &= b_{n_b}^{-1}(b_{k_1''}^{\varepsilon k_1''} \cdots b_{k_n''}^{\varepsilon k_n''})(a_{i_1''}^{\varepsilon i_1''} \cdots a_{i_l''}^{\varepsilon i_l''})(c_{j_1''}^{\varepsilon j_1''} \cdots c_{j_m''}^{\varepsilon j_m''})y'^{-1} \\ &\quad \times (c_{j_m''}^{-\varepsilon j_m''} \cdots c_{j_1''}^{-\varepsilon j_1''})(a_{i_l''}^{-\varepsilon i_l''} \cdots a_{i_1''}^{-\varepsilon i_1''})(b_{k_n''}^{-\varepsilon k_n''} \cdots b_{k_1''}^{-\varepsilon k_1''})y' .\end{aligned}$$

Before considering the Alexander matrix of $G(\tilde{L})/G(\tilde{L})''$, we will introduce several properties of the free calculus.

Proposition 1.

$$\frac{\partial r_\iota'}{\partial w} = 0 \qquad (w = x', y', z', \iota = 1, \cdots, n_x + n_y) .$$

Proof. If w appeares in r_ι', then w is contained in the words A_* or W_* that have the special forms; for example, let us consider the form of A_*,

$$\begin{aligned}A_* &= a_{i_1} \cdots a_{i_l} \cdot b_{j_1} \cdots b_{j_m} \cdot c_{k_1} \cdots c_{k_n} \\ &= (\alpha_{i_1 1} \cdots \alpha_{i_1 n} a^{\varepsilon_1} \alpha_{i_1 n}^{-1} \cdots \alpha_{i_1 1}^{-1})(\alpha_{i_2 1} \cdots a^{\varepsilon_2} \cdots) \times \\ &\quad \cdots \times (\gamma_{i_k 1} \cdots \gamma_{i_k *} c^{\varepsilon} \cdots \gamma_{i_k 1}^{-1}) .\end{aligned}$$

Since a, b and c are mapped to 1 by the abelianized map, a_i, b_j and c_k are also mapped to 1. Let us consider the case that $\alpha_j = w$, which appears in a_i, then

$$\begin{aligned}\frac{\partial a_i}{\partial w} &= \alpha_{i-1} \cdots \alpha_{i-j+1}(1 + w\alpha_{i-j-1} \cdots \alpha_1 a \alpha_1^{-1} \cdots \alpha_{j-j-1}^{-1}(-w^{-1})) \\ &= 0 .\end{aligned}$$

In the case that w appears in a_i in more than one place, it is easy to get the same result by using a similar calculation as above.

So, it is not difficult to get $\dfrac{\partial A_*}{\partial w} = 0$, since A_* consists of only $\{a_i^{\pm 1}\}$, $\{b_j^{\pm 1}\}$ and $\{c_k^{\pm 1}\}$.

Proposition 2.

$$\frac{\partial r_\iota'}{\partial w} = \frac{\partial r_\iota}{\partial w} \qquad \begin{aligned}&(w = x_i, y_j , \\ &\iota \neq i_1, i_2, j, i_1 - 1, i_2 - 1, j - 1) .\end{aligned}$$

Proof. In the case that w appears in some of A_* and W_*, there is no change in this part, by the same reasoning introduced in the previous proposition, since the words A_* and W_* in r_ι' are mapped to 1 by abelianization;

$$\frac{\partial r_\iota'}{\partial w} = \frac{\partial A_\iota}{\partial w} + A_\iota \frac{\partial w_\iota}{\partial w} + A_\iota w_\iota \frac{\partial A_\iota^{-1}}{\partial w} + \cdots$$

$$= A_\iota \frac{\partial w_\iota}{\partial w} + A_\iota w_\iota A_\iota^{-1} W_\rho \frac{\partial w_\rho^\varepsilon}{\partial w} + \cdots$$

$$= \frac{\partial w_\iota}{\partial w} + w_\iota \frac{\partial w_\rho^\varepsilon}{\partial w} + w_\iota w_\rho^\varepsilon \frac{\partial w_{\iota+1}^{-1}}{\partial w} + w_\iota w_\rho^\varepsilon w_{\iota+1}^{-1} \frac{\partial w_\rho^{-\varepsilon}}{\partial w}$$

$$= \frac{\partial(w_\iota w_\rho^\varepsilon w_{\iota+1} w_\rho^{-\varepsilon})}{\partial w} = \frac{\partial r_\iota}{\partial w} .$$

The following is similarly obtained:

Proposition 3.

$$\frac{\partial r_\iota'}{\partial w} = \frac{\partial r_\iota}{\partial w} \qquad (w \neq x_{i_11}, x_{i_12}, x_{i_21}, x_{i_22}, y_{j1}, y_{j2}) ,$$

$$\frac{\partial r_{i_1}'}{\partial x_{i_12}} = \frac{\partial r_{i1}}{\partial x_{i1}} , \qquad \frac{\partial r_{i_1-1}'}{\partial x_{i_11}} = \frac{\partial r_{i_1-1}}{\partial x_{i1}} ,$$

$$\frac{\partial r_{i_22}'}{\partial x_{i_21}} = \frac{\partial r_{i2}}{\partial x_{i2}} , \qquad \frac{\partial r_{i_2-1}'}{\partial x_{i_21}} = \frac{\partial r_{i_2-1}}{\partial x_{i2}} ,$$

$$\frac{\partial r_j'}{\partial y_{j2}} = \frac{\partial r_j}{\partial y_j} , \qquad \frac{\partial r_{j-1}'}{\partial y_{j1}} = \frac{\partial r_{j-1}}{\partial y_j} .$$

Proposition 4.

$$\frac{\partial R_1}{\partial w} = \frac{\partial R_1'}{\partial w} \qquad (w = x_i \ (i \neq i_11, i_12), y_j, x', y', z') ,$$

$$\frac{\partial R_2}{\partial w} = \frac{\partial R_2'}{\partial w} \qquad (w = w_i \ (\neq i_21, i_22) \ y_j, x', y', z') ,$$

$$\frac{\partial R_3}{\partial w} = \frac{\partial R_3'}{\partial w} \qquad (w = x_i, y_j \ (j \neq j_1, j_2), x', y', z') ,$$

$$\frac{\partial R_1}{\partial x_{i_11}} = \frac{\partial R_1'}{\partial x_{i_11}} - 1 , \quad \frac{\partial R_1'}{x_{i_12}} = \frac{\partial R_1}{x_{i_12}} - 1 ,$$

$$\frac{\partial R_2}{\partial x_{i_21}} = \frac{\partial R_2'}{\partial x_{i_21}} - 1 , \quad \frac{\partial R_2'}{\partial x_{i_22}} = \frac{\partial R_2}{\partial x_{i_22}} - 1 ,$$

$$\frac{\partial R_3}{\partial y_{j1}} = \frac{\partial R_3'}{\partial y_{j1}} - 1 , \quad \frac{\partial R_3'}{\partial y_{j2}} = \frac{\partial R_3}{\partial y_{j2}} - 1 .$$

Proof. The differences between R_1 and R_1' are in the last letters and the center parts. Since $\frac{\partial a_{n_a}^{-1}}{\partial w} = 0$ and $a_{n_a}^{-1}$ is mapped to 1 by abelianization, we have

$$\frac{\partial R_1}{\partial x_{i_11}} = \frac{\partial(W_1 x' W_1^{-1})}{\partial x_{i_11}} + W_1 x' a_{n_a}^{-1} W_1^{-1}(-x_{i_11})$$

$$= \frac{\partial(W_1 x' W_1^{-1})}{\partial x_{i_11}} - 1$$

$$= \frac{\partial R'_1}{\partial x_{i_1 1}} - 1 .$$

By using a similar calculation, the other equations are also obtained.

Proposition 5.

$$\frac{\partial S_\iota}{\partial w} = 0 \qquad (\iota = 1, 2, 3, w = x_i, y_j, x', y', z') .$$

Proof. This is easily derived considering the forms of a_i, b_j and c_k.

Now consider the Alexander matrix of $\tilde{L}$. By Propositions 1, 4 and 5, this matrix is equivalent to the following matrix:

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} x_1\cdots x_{n_x} & y_1 & \cdots & y_{n_y} & x_{i_1 1} & x_{i_1 2} & x_{i_2 1} & x_{i_2 2} & y_{j1} & y_{j2} & x'\ z'\ y' & a\ c\ b \end{array} \\
\begin{array}{c} r'_1 \\ \vdots \\ r'_{i_1-1} \\ r'_{i_1} \\ \vdots \\ \vdots \\ r'_{j-1} \\ r'_j \\ \vdots \\ r'_{n_x+n_y} \\ \hline R_1 \\ R'_1 \\ R_2 \\ R'_2 \\ R_3 \\ R'_3 \\ \hline S_1 \\ S_1 \\ S_3 \end{array} &
\left(\begin{array}{cccccccccc|c|c}
 & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
\cdots & \cdots & \cdots & \cdots & -w_* & 0 & \vdots & \vdots & \vdots & \vdots & & \\
 & & & & 0 & 1 & \vdots & \vdots & \vdots & \vdots & & \\
 & & & & \vdots & \vdots & -w_* & 0 & \vdots & \vdots & & \\
 & & & & \vdots & \vdots & 0 & 1 & \vdots & \vdots & 0 & * \\
 & & & & \vdots & \vdots & \vdots & \vdots & -w_* & 0 & & \\
\cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & & \\
 & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
\hline
p_1\cdots & \cdots & \cdots & p_{n_y} & P_1-1 & P'_1 & P_{i_2 1} & \vdots & \vdots & p_{j2} & & \\
p_1\cdots & \cdots & \cdots & p_{n_y} & P_1 & P'_1-1 & P_{i_2 1} & \vdots & \vdots & p_{j2} & & \\
q_1\cdots & \cdots & \cdots & \cdots & \cdots & q_{i_1 2} & P_2-1 & P'_2 & \vdots & \vdots & & \\
q_1\cdots & \cdots & \cdots & \cdots & \cdots & q_{i_1 2} & P_2 & P'_2-1 & \vdots & \vdots & * & * \\
r_1\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & r_{i_2 2} & P_3-1 & P'_3 & & \\
r_1\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & r_{i_2 2} & P_3 & P'_3-1 & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & 0 & & & & & 0 & * \\
 & & & & & & & & & & &
\end{array}\right)
\end{array} ,$$

where $P_1 = \frac{\partial R'_1}{\partial x_{i_1 1}}$, $P'_1 = \frac{\partial R_1}{\partial x_{i_1 2}}$, $P_2 = \frac{\partial R'_2}{\partial x_{i_2 1}}$, $P'_2 = \frac{\partial R_2}{\partial x_{i_2 2}}$, $P_3 = \frac{\partial R'_3}{\partial y_{j1}}$ and $P'_3 = \frac{\partial R_3}{\partial y_{j2}}$.

By Proposition 3, each entry of ($x_{i_1 1}$-th row$+x_{i_1 2}$-th row) is equal to the x_{i1}-th row, and ($x_{i_2 1}$-th row$+x_{i_2 2}$-th row) and (y_{j1}-th row$+y_{j2}$-th row) are equal to the x_{i2}-th row and the y_j-th row of the Alexander matrix of L, so that this matrix is equivalent to the following:

$$
\begin{array}{c}
\\ r'_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ r'_{n_x+n_y} \\ \hline R_1 \\ R'_1 \\ R_2 \\ R'_2 \\ R_3 \\ R'_3 \\ \hline S_1 \\ S_2 \\ S_3
\end{array}
\left(
\begin{array}{cccc|ccc|c|c}
x_1\cdots & x_{i_11} & x_{i_21} & y_{j1} & x_{i_12} & x_{i_22} & y_{j2} & x'\ z'\ y' & a\ c\ b \\
 & & & & 0 & 0 & 0 & & \\
 & & & & \vdots & \vdots & \vdots & & \\
 & & & & 0 & \vdots & \vdots & & \\
 & & & & 1 & 0 & \vdots & & \\
 & & & & 0 & 1 & \vdots & & \\
 & \alpha & & & \vdots & 0 & 0 & 0 & * \\
 & & & & \vdots & \vdots & 1 & & \\
 & & & & \vdots & \vdots & 0 & & \\
 & & & & \vdots & \vdots & \vdots & & \\
 & & & & 0 & 0 & 0 & & \\
\hline
\vdots & P_1+P'_1-1 & \vdots & \vdots & P'_1 & P_{i_22} & P_{j2} & & \\
\vdots & P_1+P'_1-1 & \vdots & \vdots & P'_1-1 & P_{i_22} & P_{j2} & & \\
\vdots\ \vdots & \vdots & P_2+P'_2-1 & \vdots & q_{i_12} & P'_2 & q_{j2} & * & * \\
\vdots\ \vdots & \vdots & P_2+P'_2-1 & \vdots & q_{i_12} & P'_2-1 & q_{j2} & & \\
\vdots\ \vdots & \vdots & \vdots & P_3+P'_3-1 & r_{i_11} & r_{i_22} & P'_3 & & \\
\vdots & \vdots & \vdots & P_3+P'_3-1 & r_{i_11} & r_{i_22} & P'_3-1 & & \\
\hline
 & & & & & & & & \\
 & & & & 0 & & & & * \\
 & & & & & & & &
\end{array}
\right)
$$

By Proposition 4, this matrix is equivalent to the following: (substitute R'_i-R_i to R'_i for $i=1, 2, 3$)

$$
\begin{array}{c}
\\ \\ \hline R_1 \\ R'_1 \\ R_2 \\ R'_2 \\ R_3 \\ R'_3 \\ \hline S_1 \\ S_2 \\ S_3
\end{array}
\left(
\begin{array}{c|ccc|ccc|c}
 & x_{i_12} & x_{i_22} & y_{j2} & x' & z' & y' & a\ c\ b \\
\alpha & & * & & & 0 & & * \\
\hline
 & \cdots\cdots & & & & & & \\
0\cdots 0 & -1 & 0 & 0 & 0 & 0 & 0 & \\
 & \cdots\cdots & & & & & & \\
0\cdots\cdots 0 & & -1 & 0 & 0 & 0 & 0 & * \\
 & \cdots\cdots & & & & & & \\
0\cdots\cdots 0 & & 0 & -1 & 0 & 0 & 0 & \\
\hline
 & & & & & & & \\
 & & 0 & & & & & * \\
 & & & & & & &
\end{array}
\right)
\sim
\begin{array}{c}
\\ \\ \hline R_1 \\ R_2 \\ R_3 \\ \hline S_1 \\ S_2 \\ S_3
\end{array}
\left(
\begin{array}{c|c|c}
 & x'\ y'\ z' & a\ c\ b \\
\alpha & 0 & * \\
\hline
 & & \\
* & M_2 & * \\
 & & \\
\hline
 & & \\
 & 0 & M_1 \\
 & &
\end{array}
\right),
$$

where $M_1=\left(\dfrac{\partial S_\iota}{\partial w}\right)$ ($\iota=1, 2, 3$, $w=a, b, c$) and $M_2=\left(\dfrac{\partial R_\iota}{\partial w}\right)$ ($\iota=1,2,3$, $w=x',y',z'$).

To complete the proof of the special case, it suffices to show that if $\det M_1 \doteq F(x, y)$, then $\det M_2 \doteq F(x^{-1}, y^{-1})$ and $|F(1, 1)|=1$, since the first non-zero polynomial of the above matrix is a product of the first non-zero polynomial of (α), $\det M_1$ and $\det M_2$. Therefore, consider $\frac{\partial R_\iota}{\partial w}$ ($\iota=1, 2, 3$, $w=x'$, y', z') and $\frac{\partial S_\iota}{\partial w}$ ($\iota=1, 2, 3$, $w=a$, b, c).

Since the words of a_i's, b_j's and c_k's are the conjugates of a, b and c, we obtain the following:

Proposition 6.

$$\frac{\partial R_1}{\partial w}=\frac{\partial(W_1(x_i, y_j, 1, 1, 1, x', y', z')\cdot x'\cdot W_1^{-1}(x_i, \cdots, z')x_{i_1 1}^{-1}}{\partial w}$$
$$\frac{\partial R_2}{\partial w}=\frac{\partial(W_2(x_i, y_j, 1, 1, 1, x', y', z')\cdot z'\cdot W_2^{-1}(x_i, \cdots, z')x_{i_2 1}^{-1}}{\partial w}$$
$$\frac{\partial R_3}{\partial w}=\frac{\partial(W_3(x_i, y_j, 1, 1, 1, x', y', z')\cdot y'\cdot W_3^{-1}(x_i, \cdots, z')y_{j 1}^{-1}}{\partial w},$$

where $w=x'$, y' or z'.

Let us consider the words $\widetilde{W}_\iota=W_\iota(x_i, y_j, 1, 1, 1, x', y', z')$ ($\iota=1, 2, 3$). Since the relators $R_\iota(\iota=1, 2, 3)$ are obtained from the edges of the attaching bands, the length of $\widetilde{W}_\iota$ are related to the indices n_a, n_b and n_c. The indices of a_*, b_* and c_* are changed when the attaching bands pass under the edges of $O_1\cup O_2\cup O_3\cup L$, at the same time the length of $\widetilde{W}_\iota$ increases by just one letter.

Assume that

$$\begin{aligned}\widetilde{W}_1 &= w_1 w_2 \cdots w_n\\ \widetilde{W}_2 &= v_1\, v_2 \cdots v_m \qquad (w_*, v_*, u_* = x_i^\varepsilon, y_j^\varepsilon, x'^\varepsilon, y'^\varepsilon, z'^\varepsilon, \varepsilon=\pm 1)\\ \widetilde{W}_3 &= u_1\, u_2 \cdots u_l\,,\end{aligned}$$

where $n=n_a-1$, $m=n_c-1$ $l=n_b-1$.

Since a_{n_a}, b_{n_b} and c_{n_c} are obtained by the paths of the attaching bands, it follows that

$$\begin{aligned}a_{n_a} &= w_n^{-1} \cdots w_1^{-1}\cdot a\cdot w_1 \cdots w_n\\ b_{n_b} &= u_l^{-1} \cdots u_1^{-1}\cdot b\cdot u_1 \cdots u_l\\ c_{n_c} &= v_m^{-1} \cdots v_1^{-1}\cdot c\cdot v_1 \cdots v_m\,.\end{aligned}$$

Similarly,

$$\begin{aligned}a_{i*} &= w_{i*-1}^{-1} \cdots w_1^{-1}\cdot a\cdot w_1 \cdots w_{i*-1}\\ b_{j*} &= u_{j*-1}^{-1} \cdots u_1^{-1}\cdot b\cdot u_1 \cdots u_{j*-1}\\ c_{k*} &= v_{k*-1}^{-1} \cdots v_1^{-1}\cdot c\cdot v_1 \cdots v_{k*-1}\,.\end{aligned}$$

Proposition 7.

$$\frac{\partial S_1}{\partial a}=-w_n^{-1}\cdots w_1^{-1}+(1-x^{-1})\,(\varepsilon_{i_1}w_{i_1-1}^{-1}\cdots w_1^{-1}+\cdots+\varepsilon_{i_l}w_{i_l-1}^{-1}\cdots w_1^{-1})$$

$$\frac{\partial S_1}{\partial b}=(1-x^{-1})\,(\varepsilon_{k_1}u_{k_1-1}^{-1}\cdots u_1^{-1}+\cdots+\varepsilon_{k_n}u_{k_n-1}^{-1}\cdots u_1^{-1})$$

$$\frac{\partial S_1}{\partial c}=(1-x^{-1})\,(\varepsilon_{j_1}v_{j_1-1}^{-1}\cdots v_1^{-1}+\cdots+\varepsilon_{j_m}v_{j_m-1}^{-1}\cdots v_1^{-1})$$

$$\frac{\partial S_2}{\partial a}=(1-x^{-1})\,(\varepsilon_{i_1'}w_{i_1'-1}^{-1}\cdots w_1^{-1}+\cdots+\varepsilon_{i_l'}w_{i_l'-1}^{-1}\cdots w_1^{-1})$$

$$\frac{\partial S_2}{\partial b}=(1-x^{-1})\,(\varepsilon_{k_1'}u_{k_1'-1}^{-1}\cdots u_1^{-1}+\cdots+\varepsilon_{k_m'}u_{k_m'-1}^{-1}\cdots u_1^{-1})$$

$$\frac{\partial S_2}{\partial c}=-v_m^{-1}\cdots v_1^{-1}+(1-x^{-1})\,(\varepsilon_{j_1'}v_{j_1'-1}^{-1}\cdots v_1^{-1}+\cdots+\varepsilon_{j_m'}v_{j_m'-1}^{-1}\cdots v_1^{-1})$$

$$\frac{\partial S_3}{\partial a}=(1-y^{-1})\,(\varepsilon_{i_1''}w_{i_1''-1}^{-1}\cdots w_1^{-1}+\cdots+\varepsilon_{i_l''}w_{i_l''-1}^{-1}\cdots w_1^{-1})$$

$$\frac{\partial S_3}{\partial b}=-u_l^{-1}\cdots u_1^{-1}+(1-y^{-1})\,(\varepsilon_{k_1''}u_{k_1''-1}^{-1}\cdots u_1^{-1}+\cdots+\varepsilon_{k_n''}u_{k_n''-1}^{-1}\cdots u_1^{-1})$$

$$\frac{\partial S_3}{\partial c}=(1-y^{-1})\,(\varepsilon_{i_1''}v_{j_1''-1}^{-1}\cdots v_1^{-1}+\cdots+\varepsilon_{j_m''}v_{j_m''-1}^{-1}\cdots v_1^{-1})\,.$$

Proof. These are deduced from the forms of S_ι.

To calculate $\dfrac{\partial R_\iota}{\partial w}$ ($\iota=1,\,2,\,3,\ w=x',\,y',\,z'$), we check where x', y' and z' appear. When the attaching bands cross under $O_1\cup O_2\cup O_3$, then x', y' or z' appears in $\widetilde{W}_\iota$.

Let us consider $a_{i_*}^{\varepsilon_{i_*}}$ in S_1. There are two cases (see, Fig. 5).

(IV)

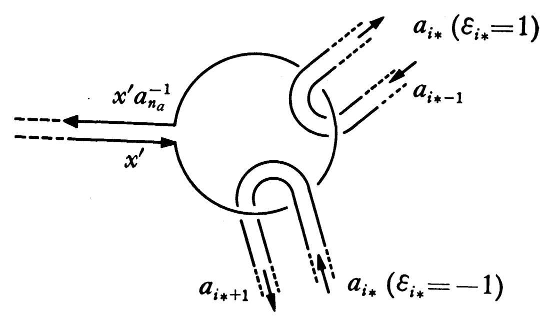


Fig. 5

Case (I). If a_{i_*} crosses over O_1 from left to right, then $\varepsilon_{i_*}=1$ and $a_{i_*}=x'a_{i_*-1}x'^{-1}$. So there exists w_{i_*-1} in R_1 ($1\leq i_*-1\leq n$), such that $w_{i_*-1}=x'^{-1}$.

Case (II). If a_{i_*} crosses over O_1 from right to left, then $\varepsilon_{i_*}=-1$ and $a_{i_*+1}=$

$x'^{-1}a_{i*}x'$. So there exists w_{i*} in R_1 $(1 \leq i_* \leq n)$, such that $w_{i*}=x'$.

Then, for a_{i*} in S_1, there exists index i_*-1 or i_*, and a letter w_{i*-1} or w_{i*} in R_1, such that $w_{i*-1}=x'^{-1}$ or $w_{i*}=x'$. Corresponding to this letter,

$$\begin{aligned}\frac{\partial R_1}{\partial x'} &= \begin{cases}\cdots + w_1 \cdots w_{i*-2}(x'^{-1}) + \cdots & (\varepsilon_{i*}=1)\\ \cdots + w_1 \cdots w_{i*-1}(\ 1\ ) + \cdots & (\varepsilon_{i*}=-1)\end{cases}\\ &= \cdots + (-\varepsilon_{i*}w_1 \cdots w_{i*-1}) + \cdots .\end{aligned}$$

By the same reasoning, corresponding to the letters b_{k*} and c_{j*} in S_1, we obtain the equations

$$\frac{\partial R_3}{\partial x'} = \cdots + (-\varepsilon_{k*}u_1 \cdots u_{k*-1}) + \cdots$$

and

$$\frac{\partial R}{\partial x'} = \cdots + (-\varepsilon_{j*}v_1 \cdots v_{j*-1}) + \cdots ,$$

respectively.

Using these equations, we can prove Proposition 8.

Proposition 8.

$$\frac{\partial R_1}{\partial x'} = w_1 \cdots w_n - (1-x)(\varepsilon_{i_1}w_1 \cdots w_{i_1-1} + \cdots + \varepsilon_{i_l}w_1 \cdots w_{i_l-1})$$

$$\frac{\partial R_1}{\partial y'} = -(1-x)(\varepsilon_{i_1}''w_1 \cdots w_{i_1''-1} + \cdots + \varepsilon_{i_l}''w_1 \cdots w_{i_l''-1})$$

$$\frac{\partial R_1}{\partial z'} = -(1-x)(\varepsilon_{i_1}'w_1 \cdots w_{i_1'-1} + \cdots + \varepsilon_{i_l}'w_1 \cdots w_{i_l'-1})$$

$$\frac{\partial R_2}{\partial x'} = -(1-x)(\varepsilon_{j1}v_1 \cdots v_{j_1-1} + \cdots + \varepsilon_{jm}v_1 \cdots v_{jm-1})$$

$$\frac{\partial R_2}{\partial y'} = -(1-x)(\varepsilon_{j_1}''v_1 \cdots v_{j_1''-1} + \cdots + \varepsilon_{j_m}''v_1 \cdots v_{jm''-1})$$

$$\frac{\partial R_2}{\partial z'} = v_1 \cdots v_m - (1-x)(\varepsilon_{j_1}'v_1 \cdots v_{j_1'-1} + \cdots + \varepsilon_{j_m}'v_1 \cdots v_{jm'-1})$$

$$\frac{\partial R_3}{\partial x'} = -(1-y)(\varepsilon_{k_1}u_1 \cdots u_{k_1-1} + \cdots + \varepsilon_{k_n}u_1 \cdots u_{k_n-1})$$

$$\frac{\partial R_3}{\partial y'} = u_1 \cdots u_l - (1-y)(\varepsilon_{k_1}''u_1 \cdots u_{k_1''-1} + \cdots + \varepsilon_{k_n}''u_1 \cdots u_{k_n''-1})$$

$$\frac{\partial R_3}{\partial z'} = -(1-y)(\varepsilon_{k_1}'u_1 \cdots u_{k_1'-1} + \cdots + \varepsilon_{k_n}'u_1 \cdots u_{k_n'-1}) .$$

Proof. For example, consider the form of R_1. Except for the letter x' in the center of R_1, all letters x' appear in $\widetilde{W}_1$ corresponding to the parts of the attaching bands crossing over O_1. Then, it is not difficult to get the desired equation of $\frac{\partial R_1}{\partial x'}$. And all y' (or z') appear in $\widetilde{W}_1$ corresponding to the parts of the attaching bands crossing over $O_3(O_2)$.

Using Propositions 7 and 8, let $\frac{\partial S_1}{\partial a}=f_1(x, y)$, $\frac{\partial S_1}{\partial b}=(1-x^{-1})f_2(x, y)$, $\frac{\partial S_1}{\partial c}=(1-x^{-1})f_3(x, y)$, $\frac{\partial S_2}{\partial a}=(1-x^{-1})g_1(x, y)$, $\frac{\partial S_2}{\partial b}=(1-x^{-1})g_2(x, y)$, $\frac{\partial S_2}{\partial c}=g_3(x, y)$, $\frac{\partial S_3}{\partial a}=(1-y^{-1})h_1(x, y)$, $\frac{\partial S_3}{\partial b}=h_2(x, y)$ and $\frac{\partial S_3}{\partial c}=(1-y^{-1})h_3(x, y)$. Then,

$$\begin{aligned}
&\frac{\partial R_1}{\partial x'} = -\bar{f}_1 & &\frac{\partial R_2}{\partial x'} = -(1-x)\bar{f}_3, & &\frac{\partial R_3}{\partial x'} = -(1-y)\bar{f}_2,\\
&\frac{\partial R_1}{\partial z'} = -(1-x)\bar{g}_1, & &\frac{\partial R_2}{\partial z'} = -\bar{g}_3, & &\frac{\partial R_3}{\partial z'} = -(1-y)\bar{g}_2,\\
&\frac{\partial R_1}{\partial y'} = -(1-x)\bar{h}_1, & &\frac{\partial R_2}{\partial y'} = -(1-x)\bar{h}_3, & &\frac{\partial R_3}{\partial y'} = -\bar{h}_2\,,
\end{aligned}$$

where $\bar{f}_i$ means $f_i(x^{-1}, y^{-1})$ and so on.

We have

$$M_1 \sim \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & c & b \end{array} \\ \begin{pmatrix} f_1 & (1-x^{-1})f_3 & (1-x^{-1})f_2 \\ (1-x^{-1})g & g_3 & (1-x^{-1})g_2 \\ (1-y^{-1})h_1 & (1-y^{-1})h_3 & h_2 \end{pmatrix} \end{array}$$

and

$$M_2 \sim \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x' & z' & y' \end{array} \\ \begin{pmatrix} -\bar{f}_1 & -(1-x)\bar{g}_1 & -(1-x)\bar{h}_1 \\ -(1-x)\bar{f}_3 & -\bar{g}_3 & -(1-x)\bar{h}_3 \\ -(1-y)\bar{f}_2 & -(1-y)\bar{g}_2 & -\bar{h}_2 \end{pmatrix} \end{array}.$$

Thus, $F(x, y) = \det M_1 = -(1-x^{-1})(1-y^{-1})f_1g_2h_3-(1-x^{-1})(1-y^{-1})f_2g_3h_1$
$-(1-x^{-1})^2f_3g_1h_2+(1-x^{-1})^2(1-y^{-1})f_3g_2h_1$
$+f_1g_3h_2+(1-x^{-1})^2(1-y^{-1})f_2g_1h_3\,,$

and $\det M_2 = (1-x)(1-y)\bar{f}_1\bar{g}_2\bar{h}_3+(1-x)(1-y)\bar{f}_2\bar{g}_3\bar{h}_1+(1-x)^2\bar{f}_3\bar{g}_1\bar{h}_2$
$-(1-x)^2(1-y)\bar{f}_3\bar{g}_2\bar{h}_1-\bar{f}_1\bar{g}_3\bar{h}_2-(1-x)^2(1-y)\bar{f}_2\bar{g}_1\bar{h}_3$
$= -F(x^{-1}, y^{-1})\,.$

It is immediate that

$$|F(1, 1)| = |f_1(1, 1, 1)\cdot g_3(1, 1, 1)\cdot h_2(1, 1, 1)| = 1\,.$$

For general cases of μ and ν, it is sufficient only to check the matrices M_1 and M_2 as in the previous step. These matrices are related to the trivial link $O_1 \cup \cdots \cup O_\nu$ and the attaching bands.

Instead of $a, b, c, x', y', z', R_\iota, S_\iota$ ($\iota=1, 2, 3$), we need generators a_ι, x'_ι ($\iota= 1, 2, \cdots, \nu$) and relators R_ι, S_ι ($\iota=1, 2, \cdots, \nu$). Since the situation is just the same as in the previous case,

$$M_1 \sim \begin{array}{c} \\ S_1 \\ \vdots \\ S_\nu \end{array} \begin{array}{c} a_1 \cdots\cdots a_\nu \\ \left(\begin{array}{ccc} & & \\ & \dfrac{\partial S_\iota}{\partial a_\iota} & \\ & & \end{array} \right) \end{array}$$

and

$$M_2 \sim \begin{array}{c} \\ R_1 \\ \vdots \\ R_\nu \end{array} \begin{array}{c} x'_1 \cdots\cdots x'_\nu \\ \left(\begin{array}{ccc} & & \\ & \dfrac{\partial R_\iota}{\partial x'_{\iota'}} & \\ & & \end{array} \right) \end{array}.$$

Let $\dfrac{\partial S_\iota}{\partial a_\iota} = f_{\iota\iota}(x, \cdots, x_\mu) \qquad (\iota = 1, \cdots, \nu)$

$\dfrac{\partial S_\iota}{\partial a_\rho} = (1-x_\iota'^{-1}) f_{\iota\rho}(x_1, \cdots, x_\mu) \qquad (\iota = 1, \cdots, \nu, \rho \neq \iota)$,(*)

then

$$\frac{\partial R_\iota}{\partial x'_\iota} = -\bar{f}_{\iota\iota} \qquad (\iota = 1, \cdots, \nu)$$

$$\frac{\partial R_\iota}{\partial x'_\iota} = -(1-x'_\iota)\bar{f}_{\rho\iota} \qquad (\iota = 1, \cdots, \nu, \rho \neq \iota).$$

So, $\det M_1 = \begin{vmatrix} f_{11} & (1-x_1'^{-1})f_{12} & \cdots & (1-x_1'^{-1})f_{1\nu} \\ (1-x_2'^{-1})f_{21} & f_{22} & & \vdots \\ (1-x_3'^{-1})f_{31} & & & \vdots \\ \cdots & \cdots & \cdots & \vdots \\ (1-x_\nu'^{-1})f_{\nu 1} & & & f_{\nu\nu} \end{vmatrix}$

$\det M_2 = \begin{vmatrix} -\bar{f}_{11} & -(1-x'_2)\bar{f}_{21} & & -(1-x'_\nu)\bar{f}_{\nu 1} \\ -(1-x'_1)\bar{f}_{12} & \vdots & \cdots & \vdots \\ \vdots & & & \vdots \\ -(1-x'_1)\bar{f}_{1\nu} & \vdots & & -\bar{f}_{\nu\nu} \end{vmatrix}$

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* Here, x'_ι denotes a suitable letter in $\{x_1, \cdots, x_\mu\}$.

$$= (-1)^{\nu} \begin{vmatrix} \bar{f}_{11} & (1-x_2')\bar{f}_{21} & & (1-x_\nu')\bar{f}_{\nu 1} \\ (1-x_1')\bar{f}_{12} & \vdots & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ (1-x_1')\bar{f}_{1\nu} & \vdots & & \bar{f}_{\nu\nu} \end{vmatrix}$$

$$= (-1)^{\nu}\overline{\det^t}\, M_1 = (-1)^{\nu}\overline{\det}\, M_1.$$

Thus, there exists a polynomial $F(x_1, \cdots, x_\mu)$ such that

$$\det M_1 \doteq F(x_1, \cdots, x_\mu)\,,$$
$$\det M_2 \doteq F(x_1^{-1}, \cdots, x_\mu^{-1})$$

and $|F(1, \cdots, 1)| = 1$.

This completes the proof of Lemma 2.

REMARK. In the proof of Lemma 2, we can also find that the integer $\beta(L)$ is the invariant of PL cobordant links [5]. To see this, let L_i, $i=1, 2$, be PL cobordant links. L_1 is cobordant to a link L_1', where each component of L_2' is obtained from a component of L_2 by tying a knot in a small 3-cell. We have $\beta(L_1)=\beta(L_2')$, since $\det M_1 \neq 0$ and $\det M_2 \neq 0$ in the proof of Lemma 2 imply that $\beta(L)$ is the cobordism invariant. $\beta(L_2')=\beta(L_2)$ easily follows from a direct use of Fox's free calculus. Hence $\beta(L_1)=\beta(L_2)$.

3. Proof of Theorem 2

Theorem 2. *For a given polynomial $F(t_1, \cdots, t_\mu)$ with $|F(1, \cdots, 1)|=1$, there exists a slice link L with μ components in the strong sense whose Alexander polynomial is $F(t_1, \cdots, t_\mu)\cdot F(t_1^{-1}, \cdots, t_1^{-1})$.*

To avoid unnecessary complexity, let us consider the case that $\mu=3$, but the construction of a slice link L with μ components in the strong sense are completely done by the same way.

Theorem 2′. *For a given polynomial $F(x, y, z)$ with $|F(1, 1, 1)|=1$, there exists a slice link L with 3 components in the strong sense whose Alexander polynomial is $F(x, y, z)\cdot F(x^{-1}, y^{-1}, z^{-1})$.*

Proof. Since $|F(1, 1, 1)|=1$, we can assume that $F(x, y, z)$ will be splitted into the form

$$F(x, y, z) = 1-(1-x)f_1(x, y, z)-(1-y)f_2(x, y, z)$$
$$-(1-z)f_3(x, y, z)\,.$$

In order to construct a slice link L, it's enough to get the informations of attaching bands. So we need relators R_i and S_i $(i=1, 2, 3)$. Since the relators S_i can be automatically obtained from R_i, let us consider R_i. Therefore, we have to consider a part of the Alexander matrix $M_2=\left(\frac{\partial R_\iota}{\partial w}\right)$, $(w=x', y', z')$.

To consider the matrix M_2, let us deform the polynomial $F(x, y, z)$ as follows;

$$\begin{aligned}
&F(x, y, z)\\
&\quad=\{1-(1-x)f_1(x, y, z)\}\{1-(1-y)f_2(x, y, z)\}\{1-(1-z)f_3(x, y, z)\}\\
&\qquad-(1-x)(1-y)f_1(x, y, z)f_2(x, y, z)-(1-y)(1-z)f_2(x, y, z)f_3(x, y, z)\\
&\qquad-(1-z)(1-x)f_3(x, y, z)f_1(x, y, z)\\
&\qquad+(1-x)(1-y)(1-z)f_1(x, y, z)\cdot f_2(x, y, z)\cdot f_3(x, y, z)\,.
\end{aligned}$$

It's easy to check that this form is the determinant of the following matrix M;

$$M\sim\begin{pmatrix} 1-(1-x)f_1 & -(1-x)f_1 & -(1-x)f_1 \\ -(1-y)f_2 & 1-(1-y)f_2 & -(1-y)f_2 \\ -(1-z)f_3 & -(1-z)f_3 & 1-(1-z)f_3 \end{pmatrix}.$$

Let us take the matrix M as M_2; i.e.,

$$\begin{aligned}
&\frac{\partial R_1}{\partial x'}=1-(1-x)f_1, && \frac{R\partial_1}{\partial y'}=-(1-x)f_1, && \frac{\partial R_1}{\partial z'}=-(1-x)f_1,\\
&\frac{\partial R_2}{\partial x'}=-(1-y)f_2, && \frac{\partial R_2}{\partial y'}=1-(1-y)f_2, && \frac{\partial R_2}{\partial z'}=-(1-y)f_2,\\
&\frac{\partial R_3}{\partial x'}=-(1-z)f_3, && \frac{\partial R_3}{\partial y'}=-(1-z)f_3, && \frac{\partial R_3}{\partial z'}=1-(1-z)f_3\,.
\end{aligned}$$

Instead of the relator R_i is a word of $x, x', y, y', z, z', a_\iota, b_\iota$ and c_ι it's enough to construct R_i as a word of $x, x', y, y', z, z', a_{n_a}, b_{n_b}$ and c_{n_c}.

$$\begin{aligned}
\text{Since}\quad R_1&=w_1w_2\cdots w_nx'a_{n_a}^{-1}w_n^{-1}\cdots w^{-1}x_*\\
R_2&=w_1'\cdots w_m'y'b_{n_b}^{-1}w_m'^{-1}\cdots w_1'^{-1}y_*\\
R_3&=w_1''\cdots w_l''z'c_{n_c}^{-1}w_l''^{-1}\cdots w_1''^{-1}z_*\,,
\end{aligned}$$

we will make the words $w_1\cdots w_n$, $w_1'\cdots w_m'$ and $w_1''\cdots w_l''$.

For example let us assume that

$$f_1(x, y, z)=\varepsilon_1x^{\alpha_1}y^{\beta_1}z^{\gamma_1}+\varepsilon_2x^{\alpha_2}y^{\beta_2}z^{\gamma_2}+\cdots+\varepsilon_rx^{\alpha_r}y^{\beta_r}z^{\gamma_r}\,.$$

Then, $w_1\cdots w_n$ contains x', y', z' in r places, respectively.

Let us put these $3r$ letters as the following manner;

$$z'^{-\varepsilon_1}\cdots z'^{-\varepsilon_i}\cdots z'^{-\varepsilon_r}\cdots y'^{-\varepsilon_1}\cdots y'^{-\varepsilon_r}\cdots x'^{-\varepsilon_1}\cdots x'^{-\varepsilon_r}\,.$$

Decide $w_1\cdots w_*$ to satisfy the equation

$$\frac{\partial(w_1\cdots w_*)}{\partial z'}=\varepsilon_1x^{\alpha_1}y^{\beta_1}z^{\gamma_1}\,.$$

Depending on $\alpha_1>0$ or $\alpha_1<0, \alpha_1$ letters among $w_1 \cdots w_*$ are x or x^{-1}, and β_1 letters among $w_1 \cdots w_*$ are y or y^{-1}. If $\varepsilon_1>0$ and $\gamma_1>0$, there must be γ_1+1 letters z among $w_1 \cdots w_*$. Then,

$$w_1 \cdots w_* z'^{-1} = \underbrace{x^{\pm 1} \cdots x^{\pm 1}}_{\alpha_1} \underbrace{y^{\pm 1} \cdots y^{\pm 1}}_{\beta_1} \underbrace{z \cdots z}_{\gamma_1+1} \cdot z'^{-1} .$$

Let us repeat this step $3r-1$ times more. After these steps, the word $w_1 \cdots w_n$ can be obtained. By the same way, $w'_1 \cdots w'_m$ and $w''_1 \cdots w''_l$ can also be obtained.

By using these words, the following relators will be read;

$$\begin{aligned} a_{n_a} &= w_n^{-1} \cdots w_1^{-1} \cdot a_1 \cdot w_1 \cdots w_n \\ b_{n_b} &= w_m'^{-1} \cdots w_1'^{-1} \cdot b_1 \cdot w'_1 \cdots w'_m \\ c_{n_c} &= w_l''^{-1} \cdots w_1''^{-1} \cdot c_1 \cdot w''_1 \cdots w''_l . \end{aligned}$$

Since the indices of a, b and c are changed when the attaching bands cross under the overpasses of the link, the relators above give us the informations about the attaching bands;

$$\begin{array}{lll} a_2 = w_1^{-1} a_1 w_1 & b_2 = w_1'^{-1} b_1 w'_1 & c_2 = w_1''^{-1} c_1 w''_1 \\ a_3 = w_2^{-1} a_2 w_2 & \vdots & \vdots \\ \vdots & & \\ a_{n_a} = w_n^{-1} a_{n_a-1} w_n & b_{n_b} = w_m'^{-1} b_{m_b-1} w'_m & c_{n_c} = w_l''^{-1} c_{n_c-1} w''_l . \end{array}$$

To construct a link L, put the trivial link with 6 components, representing the generators x, x', y, y', z and z', and contact two components representing x and x' by a attaching band which goes over according as relators a_ι, and so on. Since the attaching bands can freely go over the other parts of attaching bands, it is possible to combine two components to represent the relators a_ι, b_ι and c_ι.

This completes the proof.

Note that there may be many different links satisfying conditions for the given polynomial.

4. Some examples

EXAMPLE 1. Let $F(x, y)=x-xy+y$. Then, this polynomial can be deformed into

$$FF(x, y) \doteq 1-y+x^{-1}y = 1-(1-x)(-x^{-1}y) .$$

This is the determinant of a matrix M_2;

$$M_2 \sim \begin{pmatrix} 1-(1-x)(-x^{-1}y) & 0 \\ 0 & 1 \end{pmatrix}$$

Then $\dfrac{\partial R_1}{\partial x'} = 1-(1-x)(-x^{-1}y),\ \dfrac{\partial R_1}{\partial y'} = 0$,

$\dfrac{\partial R_2}{\partial x'} = 0 \qquad\qquad ,\ \dfrac{\partial R_2}{\partial y'} = 1$.

So there are three x' but no y' among letters of R_1. Since $\dfrac{\partial(w_1 \cdots w_*)}{\partial x'} = -x^{-1}y$, $\varepsilon_i = -1$, $w_1 = x^{-1}$, $w_2 = y$, $w_3 = x'$, and since $\dfrac{\partial(w_4 \cdots w_n)}{\partial x'} = 1$, $n=4$ and $w_4 = y^{-1}$.

Then, we get

$$R_1 = x^{-1}yx'y^{-1} \cdot x'x_{n_a}^{-1} \cdot yx'^{-1}y^{-1}x \cdot x_i^{-1},$$

and

$$a_2 = w_1^{-1}a_1w_1 = xa_1x^{-1},$$
$$a_3 = w_2^{-1}a_2w_2 = y^{-1}a_2y,$$
$$a_4 = w_3^{-1}a_3w_3 = x'^{-1}a_3x',$$
$$a_5 = w_4^{-1}a_4w_4 = ya_4y^{-1}.$$

Similarly, $R_2 = y'b_{n_b}y_{j_1}^{-1}$.

By using these relators, it's possible to construct a link L with 2 components.

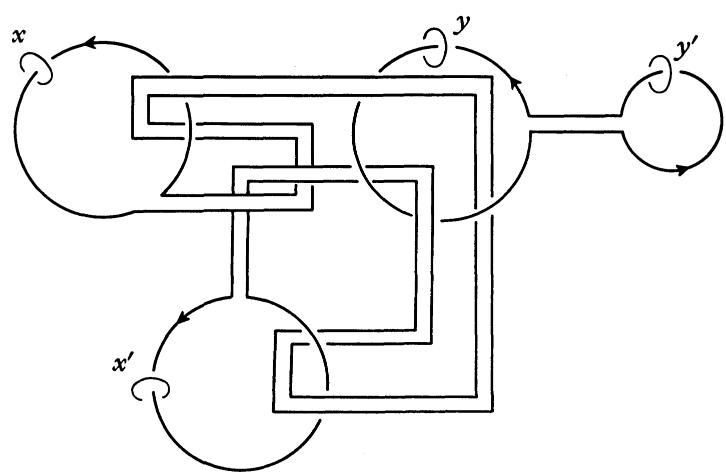

Fig. 9

EXAMPLE 2*. Let $F(x, y, z) = -x + yz + xyz$. Then

$$F(x, y, z)$$
$$\doteq 1+x-x\bar{y}\bar{z} = 1-(1-x)(1-\bar{y}\bar{z})-(1-y)\bar{y}-(1-z)\bar{y}\bar{z}$$
$$= (1-(1-x)(1-\bar{y}\bar{z}))(1-(1-y)\bar{y})(1-(1-z)\bar{y}\bar{z})$$
$$-(1-x)(1-y)(1-\bar{y}\bar{z})\cdot\bar{y}-(1-y)(1-z)\cdot\bar{y}\cdot\bar{y}\bar{z}$$
$$-(1-z)(1-x)\cdot\bar{y}\bar{z}\cdot(1-\bar{y}\bar{z})+(1-x)(1-y)(1-z)\cdot(1-\bar{y}\bar{z})\cdot\bar{y}\cdot\bar{y}\bar{z}.$$

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* In example 2, $\bar{w}$ means w^{-1}.

Then,

$$M_2 \sim \begin{pmatrix} 1-(1-x)(1-\bar{y}\bar{z}) & -(1-x)(1-\bar{y}\bar{z}) & -(1-x)(1-\bar{y}\bar{z}) \\ -(1-y)\bar{y} & 1-(1-y)\bar{y} & -(1-y)\bar{y} \\ -(1-z)\bar{y}\bar{z} & -(1-z)\bar{y}\bar{z} & 1-(1-z)\bar{y}\bar{z} \end{pmatrix}.$$

Let us construct R_1;

$$\begin{array}{c|c|c|c|c|c|c|c} w_1\cdots w_* & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots\ w_{**} & \\ \underbrace{\cdots\cdots\cdots}_{1} & \bar{z}'\ \underbrace{\cdots}_{\bar{y}\bar{z}} & z'\underbrace{\cdots\ \bar{y}'}_{1} & \underbrace{\cdots\cdots}_{\bar{y}\bar{z}} & y'\underbrace{\cdots\ \bar{x}'}_{1} & \underbrace{\cdots\cdots}_{\bar{y}\bar{z}} & x'\underbrace{\cdots\cdots\cdots}_{1} & x'. \end{array}$$

Since $\dfrac{\partial(w_1\cdots\bar{z}')}{\partial z'}=1$, $w_1=z$ and $w_2=\bar{z}'$ and $z\bar{z}'\dfrac{\partial(w_3\cdots z')}{\partial z'}=-\bar{y}\bar{z}$, $w_3=\bar{y}$, $w_4=\bar{z}$ and $w_5=z'$. Since $z\bar{z}'\bar{y}\bar{z}z'\cdot\dfrac{\partial(w_6\cdots\bar{y}')}{\partial y'}=1$, $w_6=y$, $w_7=y$ and $w_8=y'$. By the similar way, we can get

$$\begin{aligned} R_1 = {} & z\bar{z}'\,|\,\bar{y}\bar{z}z'\,|\,yy\bar{y}'\,|\,\bar{y}\bar{z}y'\,|\,zx\bar{x}'\,|\,\bar{y}\bar{z}x'\,|\,\bar{x}yzx'a_{n_a}^{-1} \\ & \times\bar{z}\bar{y}x\bar{x}'zyx'\bar{x}\bar{z}\bar{y}'zyy'\bar{y}\bar{y}\bar{z}'zyz'\bar{z}\cdot x_i^{-1}, \end{aligned}$$

and $n_a=21$

$$\begin{array}{lll} a_2=\bar{z}a_1z & a_9=y'a_8\bar{y}' & a_{16}=ya_{15}\bar{y} \\ a_3=z'a_2\bar{z}' & a_{10}=ya_9\bar{y} & a_{17}=za_{16}\bar{z} \\ a_4=ya_3\bar{y} & a_{11}=za_{10}\bar{z} & a_{18}=\bar{x}'a_{17}x' \\ a_5=za_4\bar{z} & a_{12}=\bar{y}'z_{11}y' & a_{19}=xa_{18}\bar{x} \\ a_6=\bar{z}'a_5z' & a_{13}=\bar{z}a_{12}z & a_{20}=\bar{y}a_{19}y \\ a_7=\bar{y}a_6y & a_{14}=\bar{x}a_{13}x & a_{21}=\bar{z}a_{20}z\,. \\ a_8=\bar{y}a_7y & a_{15}=x'a_{14}\bar{x}' & \end{array}$$

YAMAGUCHI WOMEN'S UNIVERSITY

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