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ON t-DESIGNS

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Introduction and preliminaries

An *incidence structure* is a triple $S=(X, \mathcal{A}, \mathcal{I})$ where X and $\mathcal{A}$ are disjoint sets and $\mathcal{I}\subseteq X\times\mathcal{A}$. Elements $x\in X$ are called *points* and elements $A\in\mathcal{A}$ are called *blocks* of S. A point x and a block A are *incident* iff $(x, A)\in\mathcal{I}$. For any block A, (A) will denote the set of points incident with A.

Let v, k, t and λ be integers with $v\geq k\geq t\geq 0$ and $\lambda\geq 1$. An $S_\lambda(t, k, v)$ (*a t-design* on v points with block size k and index λ) is an incidence structure $D=(X, \mathcal{A}, \mathcal{I})$ such that

(i) $|X|=v$,
(ii) $|(A)|=k$ for every $A\in\mathcal{A}$,
(iii) for every t-subset T of X, there are exactly λ blocks $A\in\mathcal{A}$ with $T\subseteq(A)$.

It is well known that every $S_\lambda(t, k, v)$ has exactly $b=\lambda\binom{v}{t}\Big/\binom{k}{t}$ blocks and more generally, for any i-subset I of points $(0\leq i\leq t)$, the number of blocks A of the design with $I\subseteq(A)$ is

$$b_i=\lambda\frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}},$$

independent of the subset I [2].

Abstract: We present the generalization (conjectured by A. Ja. Petrenjuk) of Fisher's Inequality $b\geq v$ for 2–designs and Petrenjuk's Inequality $b\geq\binom{v}{2}$ for 4–designs. The t-designs satisfying the inequality with equality may be considered as generalizations of the symmetric 2–designs $(b=v)$ and have the property that there are exactly $\frac{1}{2}t$ possible values for the size of the intersection of two distinct blocks, these values being computable from the parameters.

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An $S_\lambda(t, k, v)$, say $D=(X, \mathcal{A}, \mathcal{I})$, is *simple* when the mapping $A \mapsto (A)$ from $\mathcal{A}$ into $\mathcal{P}_k(X)$ (the class of all k-element subsets of X) is injective; and D is *trivial* when the mapping $A \mapsto (A)$ is (surjective and) m-to-one for some integer m, *i.e.* each k-subset "occurs as a block" exactly m times. In this latter case, evidently $\lambda = m\binom{v-t}{k-t}$.

The well known Fisher's Inequality (see [2]) asserts that the number b of blocks of an $S_\lambda(2, k, v)$ is at least v, under the assumption $v \geq k+1$. A. Ja. Petrenjuk [4] proved in 1968 that $b \geq \binom{v}{2}$ for any $S_\lambda(4, k, v)$ with $v \geq k+2$ and conjectured that $b \geq \binom{v}{s}$ in any $S_\lambda(2s, k, v)$ with $v \geq k+s$. This conjecture is established in the following section.

This condition shows the nonexistence of certain t-designs. For example, Petrenjuk's Inequality shows that $S_5(4, 22, 79)$ do not exist even though the b_i's $(0 \leq i \leq 4)$ are integral. We might note that a hypothetical $S_2\left(4, k, 2+\frac{1}{2}(k-1)(k-2)\right)$ would satisfy $b=\binom{v}{2}$ (and the b_i's are integral when $k \not\equiv 1 \pmod 4$), but no such designs exist by the corollary of Theorem 5 below. The inequality $b \geq \binom{v}{3}$ rules out the entire family of 6-designs with

$$v = 120m\,,$$
$$k = 60m\,,$$
$$\lambda = (20m-1)(15m-1)(12m-1)\,,$$

(for which the b_i's are integral).

By a *tight* t-desigh (t even, say $t=2s$) we mean an $S_\lambda(t, k, v)$ with $v \geq k+s$ and $b=\binom{v}{s}$. As examples, we have the trivial designs $S_\lambda(2s, k, k+s)$ where $\lambda=\binom{k-s}{k-2s}$. An example of a tight 4-design is the well known $S_1(4, 7, 23)$ where $b=253=\binom{23}{2}$. N. Ito [3] has recently shown, using Theorem 5 below, that the only nontrivial tight 4-desighns are the $S_1(4, 7, 23)$ and its complement, an $S_{52}(4, 16, 23)$. Tight t-designs with $t \geq 4$ seem to be very rare.

Our proof of Petrenjuk's conjecture uses only elementary linear algebra and the observation that the nunber of blocks of an $S_\lambda(t, k, v)$ which are incident with some i points and not incident some other j points is constant (i.e., depends only on i, j, and the parameters; not the particular sets of points) whenever $i+j \leq t$.

Proposition 1. *Let $(X, \mathcal{A}, \mathcal{I})$ be an $S_\lambda(t, k, v)$. Let i and j be nonnegative integers with $i+j \leq t$. Then for any subsets $I, J \subseteq X$ with $|I|=i$, $|J|=j$,*

$I\cap J=\phi$, the number of blocks $A\in\mathcal{A}$ such that $I\subseteq(A)$ and $J\cap(A)=\phi$ is exactly

$$b_i^j=\lambda\frac{\binom{v-i-j}{k-i}}{\binom{v-t}{k-t}}.$$

Proof. By inclusion-exclusion,

$$b_i^j=\sum_{r=0}^{j}(-1)^r\binom{j}{r}b_{i+r}.$$

In view of the above expression for b_i, we have $b_i^j=\lambda c$ where

$$c=\sum_{r=0}^{j}(-1)^r\binom{j}{r}\binom{v-i-r}{t-i-r}\binom{k-i-r}{t-i-r}^{-1}.$$

But in the case of the trivial design $(X, \mathcal{P}_k(X), \in)$, $\lambda=\binom{v-t}{k-t}$ and $b_i^j=\binom{v-i-j}{k-i}$, from which we deduce the simpler expression $c=\binom{v-i-j}{k-i}\binom{v-t}{k-t}^{-1}$.

As a corollary, the *complement* $(X, \mathcal{A}, (X\times\mathcal{A})-\mathcal{I})$ of an $S_\lambda(t, k, v)$ is an $S_{\lambda^*}(t, v-k, v)$ with

$$\lambda^*=b_0^t=\lambda\binom{v-t}{k}\binom{v-t}{k-t}^{-1}$$

(unless $v<k+t$, in which case the original $S_\lambda(t, k, v)$ is evidently trivial).

2. Generalizations of Fisher's inequality

For any set Y, we denote by $V(Y)$ the free vector space over the rationals generated by Y, i.e. $V(Y)$ consists of all formal sums $\alpha=\sum_{y\in Y}a_y y$ with rational coefficients a_y and formal addition and scalar multiplication. The "unit vectors" y, $y\in Y$, by definition provide a basis for $V(Y)$.

Theorem 1. *The existence of an $S_\lambda(t, k, v)$ with t even, say $t=2s$, and $v\geq k+s$ implies*

$$b\geq\binom{v}{s},$$

where b is the number of blocks of the design. In fact, the number of distinct subsets (A) is itself at least $\binom{v}{s}$.

Proof. Let $D=(X,\mathcal{A}, \mathcal{I})$ be an $S_\lambda(t, k, v)$ and put $V_s=V(\mathcal{P}_s(X))$, where $\mathcal{P}_s(X)$ is the class of all s-element subsets of X. For each block A of D, define a vector $\hat{A}\in V_s$ as the "sum" of all s-subsets of (A), i.e.

$$\hat{A} = \sum(S\colon S \in \mathcal{P}_s(X),\ S \subseteq (A))$$

We claim that the set of vectors $\{\hat{A}\colon A \in \mathcal{A}\}$ spans V_s. Since V_s has dimension $\binom{v}{s}$, the theorem follows immediately.

Let $S_0 \in \mathcal{P}_s(X)$. To show S_0 belongs to the span of $\{\hat{A}\colon A \in \mathcal{A}\}$, we introduce the vectors

$$E_i = \sum(S\colon S \in \mathcal{P}_s(X),\ |S \cap S_0| = s-i)$$

(*so* $E_0 = S_0$) and

$$F_i = \sum(\hat{A}\colon A \in \mathcal{A},\ |(A) \cap S_0| = s-i)$$

for $i=0, 1, \cdots, s$. Now for $S_1 \in \mathcal{P}_s(X)$ with $|S_1 \cap S_0| = s-i$, the coefficient of S_1 in the sum F_r is the number of blocks A such that $S_1 \subseteq (A)$ and $|(A) \cap S_0| = s-r$; and this number is $\binom{i}{r} b^r_{s-r+i}$ with the notation of Proposition 1. Thus

$$F_r = \sum_{i=r}^{s} \binom{i}{r} b^r_{s-r+i} E_i \qquad (r = 0, 1, \cdots, s).$$

The above system of linear equations is triangular and the diagonal coefficients b^r_s $(r=0, 1, \cdots, s)$ are all nonzero under our hypothesis $v \geq k+s$. Thus we can solve for the E_i's (in particular, for $E_0 = S_0$) as linear combinations of the F_r's. Since the F_r's are by definition in the span of $\{\hat{A}\colon A \in \mathcal{A}\}$, we have $S_0 \in$ span $\{\hat{A}\colon A \in \mathcal{A}\}$ for every $S_0 \in \mathcal{P}_s(X)$, and our claim is verified.

Corollary. *The existence of an* $S_\lambda(t, k, v)$ *with* t *odd, say* $t = 2s+1$ *and* $(v-1) \geq k+s$ *implies the inequality*

$$b = \frac{\lambda\binom{v}{2s+1}}{\binom{k}{2s+1}} \geq \frac{\lambda\binom{v-1}{2s}}{\binom{k-1}{2s}} + \binom{v-1}{s} \geq 2\binom{v-1}{s}.$$

Proof. Let $D=(X, \mathcal{A}, \mathcal{I})$ be an $S_\lambda(t, k, v)$ and $x \in X$. Let $\mathcal{A}'$ be the class of blocks incident with x and $\mathcal{A}''$ be the class of blocks not incident with x. Observe that both $D'=(X', \mathcal{A}', \mathcal{I} \cap (X' \times \mathcal{A}'))$ and $D''=(X', \mathcal{A}'', \mathcal{I} \cap (X' \times \mathcal{A}''))$, where $X' = X - \{x\}$, are $2s$—designs and apply Theorem 1.

The above inequality also rules out infinitely many parameters for which b_i's are integers, $i=0, 1, \cdots, t$.

Theorem 2. *Let* $D=(X, \mathcal{A}, \mathcal{I})$ *be an* $S_\lambda(t, k, v)$ *where* $t=2s$ *and* $v \geq k+s$. *If there exists a partition* $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r$ *such that each substructure* $(X, \mathcal{A}_i, \mathcal{I} \cap (X \times \mathcal{A}_i))$ *is an* $S_{\lambda_i}(s, k, v)$ *for some positive integers* λ_i, *then*

$$b=|\mathcal{A}|\geq\binom{v}{s}+r-1\,.$$

Proof. With the notation of Theorem 1, the vectors $\{\hat{A}\colon A\in\mathcal{A}\}$ span V. But observe that

$$\sum\{\hat{A}\colon A\in\mathcal{A}_i\}=\lambda_i\sum(S\colon S\in\mathcal{P}_s(X))=\lambda_i\hat{X},\text{ say.}$$

So if we choose one block A_i from each $\mathcal{A}_i$, then $\{\hat{A}\colon A\in\mathcal{A}-\{A_1,\cdots,A_r\}\}\cup\{\hat{X}\}$ spans V. The stated inequality follows.

3. Tight t-designs

Recall that a *tight* t-design ($t=2s$) is an $S_\lambda(t,k,v)$ with $v\geq k+s$ and

$$b=\lambda\binom{v}{t}\Big/\binom{k}{t}=\binom{v}{s}.$$

In view of Theorem 1, tight designs are simple. In this section we extend the well known result that two distinct blocks of a symmetric design (tight 2-design) have exactly λ common incident points (see Theorem 4 below).

Theorem 3. *Let X be a v-set and $\mathcal{A}$ a class of k-subsets of X such that for distinct A, $B\in\mathcal{A}$,*

$$|A\cap B|\in\{\mu_1,\mu_2,\cdots,\mu_s\}$$

where $k>\mu_1>\mu_2>\cdots>\mu_s\geq 0$. Then

$$|\mathcal{A}|\leq\binom{v}{s}.$$

Proof. Let $V=V(\mathcal{A})$. For each $S\in\mathcal{P}_s(X)$, define a vector

$$\bar{S}=\sum(A\colon A\in\mathcal{A},\ A\supseteq S)\,.$$

We claim that the vectors $\{\bar{S}\colon S\in\mathcal{P}_s(X)\}$ span V. Since V has dimension $|\mathcal{A}|$, the theorem will follow.

Write $\mu_0=k$. Let $A_0\in\mathcal{A}$ be given. Define

$$H_i=\sum(B\colon B\in\mathcal{A},\ |B\cap A_0|=\mu_i)$$

for $i=0,1,\cdots,s$ (note $H_0=A_0$). For $r=0,1,\cdots,s$, we see that

$$G_r=\sum(\bar{S}\colon S\in\mathcal{P}_s(X),\ |S\cap A_0|=r)=\sum_{i=0}\binom{\mu_i}{r}\binom{k-\mu_i}{s-r}H_i,$$

by comparing the coefficient of each $A\in\mathcal{A}$ on both sides of the equation. We now show that the coefficient matrix of this system of $s+1$ linear equations is nonsingular, so that we can solve for the H_i's in terms of the G'_rs. In particular, we then have $H_0=A_0\in\text{span}\ \{G_0, G_1, \cdots, G_r\}\subseteq\text{span}\ \{\bar{S}\colon S\in\mathcal{P}_s(X)\}$.

So consider the $s+1$ row vectors

$$v_r=\left(\binom{\mu_0}{r}\binom{k-\mu_0}{s-r},\ \binom{\mu_1}{r}\binom{k-\mu_1}{s-r},\cdots,\binom{\mu_s}{r}\binom{k-\mu_s}{s-r}\right),$$

$r=0, 1, \cdots, s$. Suppose $c_0v_0+c_1v_1+\cdots+c_sv_s=0$. This means that the polynomial

$$p(x)=\sum_{r=0}^{s}c_r\binom{x}{r}\binom{k-x}{s-r}$$

of degree $\leq s$ has $s+1$ distinct roots $\mu_0, \mu_1, \cdots, \mu_s$ and hence is the zero polynomial. Now $p(0)=c_0\binom{k}{s}$, so $c_0=0$; then $p(1)=c_1\binom{k-1}{s-1}$, so $c_1=0$; and, inductively, $c_0=c_1=\cdots=c_s=0$. That is, $v_0, \cdots, v_s$ are linearly independent. This completes the proof.

Theorem 4. *Let $D=(X, \mathcal{A}, \mathcal{I})$ be an $S_\lambda(t, k, v)$ with $t=2s$ and $v\geq k+s$. Then there are at least s distinct elements in the set*

$$\{|(A)\cap(B)|\colon A\in\mathcal{A},\ B\in\mathcal{A},\ A\neq B\}\ ,$$

and there are exactly s distinct elements if and only if D is a tight t-design.

Proof. In view of Theorems 1 and 3, it remains only to show that for any tight t-design, there exist s integers $\mu_1, \mu_2, \cdots, \mu_s$ with $0\leq\mu_i<k$ so that $|(A)\cap(B)|\in\{\mu_1, \cdots, \mu_s\}$ for distinct blocks A and B. Let $D=(X, \mathcal{A}, \mathcal{I})$ be a tight $S_\lambda(t, k, v)$. With the notation of Theorem 1, the $b=\binom{v}{s}$ vectors $\{\hat{A}\colon A\in\mathcal{A}\}$ must, since they span V_s, be a basis for V_s.

Fix $A_0\in\mathcal{A}$ and for $B\in\mathcal{A}$, write $\mu_B=|(B)\cap(A_0)|$. For $i=0, 1, \cdots, s$, define vectors

$$M_i=\sum(S\colon S\in\mathcal{P}_s(X),\ |S\cap(A_0)|=i)\,,$$
$$N_i=\sum\left(\binom{\mu_B}{i}\hat{B}\colon B\in\mathcal{A}\right).$$

Now given $S\in\mathcal{P}_s(X)$ with $|S\cap(A_0)|=i$, the coefficient of S in the sum N_r is

$$\sum\left(\binom{\mu_B}{r}\colon B\in\mathcal{A},\ S\subseteq(B)\right),$$

i.e., the number of ordered pairs (B, R) in $\mathcal{A}\times\mathcal{P}_r(X)$ such that $S\subseteq(B)$ and $R\subseteq(A_0)\cap(B)$. For any r-subset $R\subseteq(A_0)$ with $|R\cap S|=j$, the number of blocks B such that (B, R) satisfies the above conditions is b_{s+r-j}. Thus the coefficient of S in N_r is

$$c_r^i = \sum_{j=0}^{i} \binom{i}{j}\binom{k-i}{r-j} b_{s+r-j}; \text{ and so}$$

$$N_r = \sum_{i=0}^{s} c_r^i M_i \qquad (r = 0, 1, \cdots, s)\,.$$

The $s+1$ vectors $N_r - c_r^s M_s$ are contained in the span of $M_0, M_1, \cdots, M_{s-1}$; hence there exist rationals $a_0, a_1, \cdots, a_s$, not all zero, such that

$$\sum_{r=0}^{s} a_r(N_r - c_r^s M_s) = 0\,, \quad \text{or}$$

$$\sum_{r=0}^{s} a_r \sum_{B\in\mathcal{A}} \left(\binom{\mu_B}{r}\hat{B} - c_r^s \hat{A}_0\right) = 0.$$

Now $\{\hat{A}\colon A\in\mathcal{A}\}$ is a basis for V_s, so for $B\neq A_0$, the coefficient

$$\sum_{r=0}^{s} a_r\binom{\mu_B}{r}$$

of $\hat{B}$ must be 0. That is, for any $B\neq A_0$, the intersection number μ_B is a root of the polynomial

$$f(x) = \sum_{r=0}^{s} a_r\binom{x}{r}$$

of degree at most s. Finally, note that the coefficients c_r^i are (and hence $f(x)$ can be chosen to be) independent of the block A_0: all intersection numbers are roots of $f(x)$.

The polynomials $f(x)$ described in the proof of Theorem 4 have been found explicitly by P. Delsarte [1]. As an example, we consider the case $t=4$. The equations of Theorem 4 are

$$N_0 = b_2M_0 + b_2M_1 + b_2M_2\,,$$
$$N_1 = kb_3M_0 + (b_2+(k-1)b_3)M_1 + (2b_2+(k-2)b_3)M_2\,,$$
$$N_2 = \binom{2}{k}b_4M_0 + \left(\binom{k-1}{2}b_4 + (k-1)b_3\right)M_1 + \left(\binom{k-2}{2}b_4 + 2(k-2)b_3 + b_2\right)M_2.$$

Using the relation $b_2 = \binom{k}{2}$ in a tight 4-design, one verifies that

$$(b_2-b_3)N_2 - (k-1)(b_3-b_4)N_1 + (2b_3(b_3-b_4) - b_4(b_2-b_3))N_0$$

is a scalar multiple of $M_2 = \hat{A}_0$. For a block $B\neq A_0$, the coefficient of $\hat{B}$ in the above expression must be zero, i.e.,

$$\mu_B(\mu_B-1) - \frac{2(k-1)(b_3-b_4)}{(b_2-b_3)}\mu_B + \frac{4b_3(b_3-b_4)}{(b_2-b_3)} - 2b_4 = 0\,.$$

Rewriting the coefficients in terms of v, k, and λ, we have

Theorem 5. *The two "intersection numbers" μ_1, μ_2 of a tight 4-design $S_\lambda(4, k, v)$ are the roots of the polynomial*

$$f(x) = x^2 - \left(\frac{2(k-1)(k-2)}{(v-3)}+1\right)x+\lambda\left(2+\frac{4}{k-3}\right).$$

Application of Theorem 5 yields the well known fact that any two distinct blocks of an $S_1(4, 7, 23)$ meet in 1 or 3 points.

Since $f(x)$ has integral roots, it must have integral coefficients, and we have the

Corollary. *The existence of a tight 4-desigh $S_\lambda(4, k, v)$ implies $v-3$ divides $2(k-1)(k-2)$, and $k-3$ divides 4λ.*

In [1], Delsarte observes that Theorems 4 and 5 are similar to Lloyd's Theorem on perfect codes. Indeed, Delsarte develops a theory of designs and codes (emphasizing a "formal duality") in the context of association schemes. Contained therein are results analogous to the above for orthogonal arrays of strength t, the analogue of Theorem 1 being Rao's bound.

We conclude with the following remarks.

Let $D=(X, \mathcal{A}, \mathcal{I})$ be a tight $S_\lambda(t, k, v)$ with $t=2s$ and $v \geq k+s$. Let $J(s, v)$ denote the association scheme whose points are the s-element subsets of X (see [1]). Let N be a $(0-1)$-matrix whose rows are indexed by elements of $\mathcal{P}_s(X)$ and columus are indexed by the blocks of D. At the row corresponding to S and column corresponding to a block A, the entry of N is 1 iff $S \subseteq (A)$. The matrix NN^T belongs to the Bose-Mesner algebra of the scheme $J(s, v)$. The matrix NN^T is obviously rationally congruent to the identity matrix. Using the properties of the algebra of $J(s, v)$, it is possible to compute the Hasse-Minkowski invariant of NN^T and obtain some more necessary conditions for the existence of tight $2s$-designs. (See also [5].)

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