ON $t$-DESIGNS

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Introduction and preliminaries

An incidence structure is a triple $S = (X, \mathcal{A}, \mathcal{B})$ where $X$ and $\mathcal{A}$ are disjoint sets and $\mathcal{B} \subseteq X \times \mathcal{A}$. Elements $x \in X$ are called points and elements $A \in \mathcal{A}$ are called blocks of $S$. A point $x$ and a block $A$ are incident if $(x, A) \in \mathcal{B}$. For any block $A$, $(A)$ will denote the set of points incident with $A$.

Let $v, k, t$ and $\lambda$ be integers with $v \geq k \geq t > 0$ and $\lambda \geq 1$. An $S_\lambda(t, k, v)$ (a $t$-design on $v$ points with block size $k$ and index $\lambda$) is an incidence structure $D = (X, \mathcal{A}, \mathcal{B})$ such that

1. $|X| = v$,
2. $|(A)| = k$ for every $A \in \mathcal{A}$,
3. for every $t$-subset $T$ of $X$, there are exactly $\lambda$ blocks $A \in \mathcal{A}$ with $T \subseteq (A)$.

It is well known that every $S_\lambda(t, k, v)$ has exactly $b = \lambda \binom{v}{t} / \binom{k}{t}$ blocks and more generally, for any $i$-subset $I$ of points ($0 \leq i \leq t$), the number of blocks $A$ of the design with $I \subseteq (A)$ is

$$b_i = \lambda \binom{v - i}{t - i} \binom{k - i}{t - i},$$

independent of the subset $I$ [2].

Abstract: We present the generalization (conjectured by A. Ja. Petrenjuk) of Fisher’s Inequality $b \geq v$ for 2-designs and Petrenjuk’s Inequality $b \geq \binom{v}{2}$ for 4-designs. The $t$-designs satisfying the inequality with equality may be considered as generalizations of the symmetric 2-designs ($b = v$) and have the property that there are exactly $\frac{1}{2} t$ possible values for the size of the intersection of two distinct blocks, these values being computable from the parameters.

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An $S_\lambda(t, k, v)$, say $D=(X, \mathcal{A}, \mathcal{J})$, is simple when the mapping $A \mapsto (A)$ from $\mathcal{A}$ into $2^X$ (the class of all $k$-element subsets of $X$) is injective; and $D$ is trivial when the mapping $A \mapsto (A)$ is surjective and $m$-to-one for some integer $m$, i.e. each $k$-subset "occurs as a block" exactly $m$ times. In this latter case, evidently $\lambda=m\left(\frac{v-t}{k-t}\right)$.

The well known Fisher's Inequality (see [2]) asserts that the number $b$ of blocks of an $S_\lambda(2, k, v)$ is at least $v$, under the assumption $v \geq k+1$. A. Ja. Petrenjuk [4] proved in 1968 that $b \geq \left(\frac{v}{2}\right)$ for any $S_\lambda(4, k, v)$ with $v \geq k+2$ and conjectured that $b \geq \left(\frac{v}{s}\right)$ in any $S_\lambda(2s, k, v)$ with $v \geq k+s$. This conjecture is established in the following section.

This condition shows the nonexistence of certain $t$-designs. For example, Petrenjuk's Inequality shows that $S_\lambda(4, 22, 79)$ do not exist even though the $b_i$'s $(0 \leq i \leq 4)$ are integral. We might note that a hypothetical $S_\lambda\left(4, k, 2+\frac{1}{2}(k-1)(k-2)\right)$ would satisfy $b=\left(\frac{v}{2}\right)$ (and the $b_i$'s are integral when $k \equiv 1 \pmod{4}$), but no such designs exist by the corollary of Theorem 5 below. The inequality $b \geq \left(\frac{v}{3}\right)$ rules out the entire family of $6$-designs with

$$v = 120m, \quad k = 60m, \quad \lambda = (20m-1)(15m-1)(12m-1),$$

(for which the $b_i$'s are integral).

By a tight $t$-design ($t$ even, say $t=2s$) we mean an $S_\lambda(t, k, v)$ with $v \geq k+s$ and $b=\left(\frac{v}{s}\right)$. As examples, we have the trivial designs $S_\lambda(2s, k, k+s)$ where

$$\lambda = \left(\frac{k-s}{k-2s}\right).$$

An example of a tight 4-design is the well known $S_\lambda(4, 7, 23)$ where $b=253=\left(\frac{23}{2}\right)$. N. Ito [3] has recently shown, using Theorem 5 below, that the only nontrivial tight 4-designs are the $S_\lambda(4, 7, 23)$ and its complement, an $S_{2\lambda}(4, 16, 23)$. Tight $t$-designs with $t \geq 4$ seem to be very rare.

Our proof of Petrenjuk's conjecture uses only elementary linear algebra and the observation that the number of blocks of an $S_\lambda(t, k, v)$ which are incident with some $i$ points and not incident some other $j$ points is constant (i.e., depends only on $i, j$, and the parameters; not the particular sets of points) whenever $i+j \leq t$.

Proposition 1. Let $(X, \mathcal{A}, \mathcal{J})$ be an $S_\lambda(t, k, v)$. Let $i$ and $j$ be nonnegative integers with $i+j \leq t$. Then for any subsets $I, J \subseteq X$ with $|I|=i$, $|J|=j$, 

I \cap J = \phi, the number of blocks \( A \subseteq \mathcal{A} \) such that \( I \subseteq (A) \) and \( J \cap (A) = \phi \) is exactly

\[
 b_i^I = \lambda \frac{\binom{v-i-j}{k-i}}{\binom{v-t}{k-t}}.
\]

Proof. By inclusion-exclusion,

\[
 b_i^I = \sum_{r=0}^i (-1)^r \binom{j}{r} b_{i+r}.
\]

In view of the above expression for \( b_i \), we have \( b_i^I = \lambda c \) where

\[
 c = \sum_{r=0}^i (-1)^r \binom{j}{r} \binom{v-i-r}{t-i-r} \binom{k-i-r}{t-i-r}^{-1}.
\]

But in the case of the trivial design \((X, \mathcal{P}(X), \subseteq)\), \( \lambda = \binom{v-t}{k-t} \) and \( b_i^I = \binom{v-i-j}{k-i} \), from which we deduce the simpler expression \( c = \binom{v-i-j}{k-i} \binom{v-t}{k-t}^{-1} \).

As a corollary, the complement \((X, \mathcal{A}, (X \times \mathcal{A}) - \mathcal{B})\) of an \( S_\lambda(t, k, v) \) is an \( S_\lambda^*(t, v-k, v) \) with

\[
 \lambda^* = b_0^I = \lambda \binom{v-t}{k} \binom{v-t}{k-t}^{-1}
\]

(unless \( v < k + t \), in which case the original \( S_\lambda(t, k, v) \) is evidently trivial).

2. Generalizations of Fisher's inequality

For any set \( Y \), we denote by \( V(Y) \) the free vector space over the rationals generated by \( Y \), i.e. \( V(Y) \) consists of all formal sums \( \alpha = \sum_{y \in Y} a_y y \) with rational coefficients \( a_y \) and formal addition and scalar multiplication. The "unit vectors" \( y, y^* \in Y \), by definition provide a basis for \( V(Y) \).

**Theorem 1.** The existence of an \( S_\lambda(t, k, v) \) with \( t \) even, say \( t = 2s \), and \( v \geq k+s \) implies

\[
 b \geq \binom{v}{s},
\]

where \( b \) is the number of blocks of the design. In fact, the number of distinct subsets \((A)\) is itself at least \( \binom{v}{s} \).

Proof. Let \( D = (X, \mathcal{A}, \mathcal{B}) \) be an \( S_\lambda(t, k, v) \) and put \( V_s = V(\mathcal{P}_s(X)) \), where \( \mathcal{P}_s(X) \) is the class of all \( s \)-element subsets of \( X \). For each block \( A \) of \( D \), define a vector \( \tilde{A} \in V_s \) as the "sum" of all \( s \)-subsets of \((A)\), i.e.
\[ \hat{A} = \sum (S: S \in P_s(X), S \subseteq (A)) \]

We claim that the set of vectors \( \{\hat{A}: A \in \mathcal{A}\} \) spans \( V_s \). Since \( V_s \) has dimension \( \binom{v}{s} \), the theorem follows immediately.

Let \( S_0 \in P_s(X) \). To show \( S_0 \) belongs to the span of \( \{\hat{A}: A \in \mathcal{A}\} \), we introduce the vectors

\[ E_i = \sum (S: S \in P_s(X), |S \cap S_0| = s-i) \]

(since \( E_0 = S_0 \)) and

\[ F_i = \sum (\hat{A}: A \in \mathcal{A}, |(A) \cap S_0| = s-i) \]

for \( i = 0, 1, \ldots, s \). Now for \( S_i \in P_s(X) \) with \( |S_i \cap S_0| = s-i \), the coefficient of \( S_i \) in the sum \( F_r \) is the number of blocks \( A \) such that \( S_i \subseteq (A) \) and \( |(A) \cap S_0| = s-r \); and this number is \( \binom{i}{r} b_{s-r+i}^i \) with the notation of Proposition 1. Thus

\[ F_r = \sum_{i=r}^{s} \binom{i}{r} b_{s-r+i}^i E_i \quad (r = 0, 1, \ldots, s). \]

The above system of linear equations is triangular and the diagonal coefficients \( b_r^r \) are all nonzero under our hypothesis \( v \geq k+s \). Thus we can solve for the \( E_i \)'s (in particular, for \( E_0 = S_0 \)) as linear combinations of the \( F_r \)'s. Since the \( F_r \)'s are by definition in the span of \( \{\hat{A}: A \in \mathcal{A}\} \), we have \( S_0 \in \text{span} \{\hat{A}: A \in \mathcal{A}\} \) for every \( S_0 \in P_s(X) \), and our claim is verified.

**Corollary.** The existence of an \( S_\lambda(t, k, v) \) with \( t \) odd, say \( t = 2s+1 \) and \( (v-1) \geq k+s \) implies the inequality

\[ b = \frac{\lambda \binom{v}{2s+1}}{k} \geq \frac{\lambda \binom{v-1}{k-1}}{k} + \binom{v-1}{s} \geq 2 \binom{v-1}{s}. \]

**Proof.** Let \( D = (X, \mathcal{A}, \mathcal{J}) \) be an \( S_\lambda(t, k, v) \) and \( x \in X \). Let \( \mathcal{A}' \) be the class of blocks incident with \( x \) and \( \mathcal{A}'' \) be the class of blocks not incident with \( x \). Observe that both \( D' = (X', \mathcal{A}', \mathcal{J} \cap (X' \times \mathcal{A}')) \) and \( D'' = (X', \mathcal{A}'', \mathcal{J} \cap (X' \times \mathcal{A}'')) \), where \( X' = X - \{x\} \), are \( 2s \)-designs and apply Theorem 1.

The above inequality also rules out infinitely many parameters for which \( b_i \)'s are integers, \( i = 0, 1, \ldots, t \).

**Theorem 2.** Let \( D = (X, \mathcal{A}, \mathcal{J}) \) be an \( S_\lambda(t, k, v) \) where \( t = 2s \) and \( v \geq k+s \). If there exists a partition \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r \) such that each substructure \( (X, \mathcal{A}_i, \mathcal{J} \cap (X \times \mathcal{A}_i)) \) is an \( S_\lambda(s, k, v) \) for some positive integers \( \lambda_i \), then
Proof. With the notation of Theorem 1, the vectors \( \{A: A \in \mathcal{A}\} \) span \( V \). But observe that
\[
\sum \{A: A \in \mathcal{A}_i\} = \lambda_i \sum (S: S \in \mathcal{P}_s(X)) = \lambda_i \hat{X}, \text{ say}.
\]
So if we choose one block \( A_i \) from each \( \mathcal{A}_i \), then \( \{A: A \in \mathcal{A} - \{A_1, \ldots, A_s\}\} \cup \{\hat{X}\} \) spans \( V \). The stated inequality follows.

3. Tight t-designs

Recall that a tight t-design \((t=2s)\) is an \( S_\lambda(t, k, v) \) with \( v \geq k+s \) and
\[
b = \lambda \binom{v}{t} / \binom{k}{t} = \binom{v}{s}.
\]
In view of Theorem 1, tight designs are simple. In this section we extend the well known result that two distinct blocks of a symmetric design (tight 2-design) have exactly \( \lambda \) common incident points (see Theorem 4 below).

Theorem 3. Let \( X \) be a \( v \)-set and \( \mathcal{A} \) a class of \( k \)-subsets of \( X \) such that for distinct \( A, B \in \mathcal{A} \),
\[
|A \cap B| \in \{\mu_1, \mu_2, \ldots, \mu_s\}
\]
where \( k > \mu_1 > \mu_2 > \cdots > \mu_s \geq 0 \). Then
\[
|\mathcal{A}| \leq \binom{v}{s}.
\]
Proof. Let \( V = V(\mathcal{A}) \). For each \( S \in \mathcal{P}_s(X) \), define a vector
\[
\tilde{S} = \sum (A: A \in \mathcal{A}, A \supseteq S).
\]
We claim that the vectors \( \{\tilde{S}: S \in \mathcal{P}_s(X)\} \) span \( V \). Since \( V \) has dimension \( |\mathcal{A}| \), the theorem will follow.

Write \( \mu_0 = k \). Let \( A_0 \in \mathcal{A} \) be given. Define
\[
H_i = \sum (B: B \in \mathcal{A}, |B \cap A_0| = \mu_i)
\]
for \( i = 0, 1, \ldots, s \) (note \( H_0 = A_0 \)). For \( r = 0, 1, \ldots, s \), we see that
\[
G_r = \sum (\tilde{S}: S \in \mathcal{P}_s(X), |S \cap A_0| = r) = \sum \binom{k}{r} \binom{k-\mu_i}{s-r} H_i,
\]
by comparing the coefficient of each \( A \in \mathcal{A} \) on both sides of the equation. We now show that the coefficient matrix of this system of \( s+1 \) linear equations is
nonsingular, so that we can solve for the $H_i$'s in terms of the $G'_r$'s. In particular, we then have $H_0= A_0 \subseteq \text{span} \{G_0, G_1, \ldots, G_r\} \subseteq \text{span} \{S: S \subseteq \mathcal{P}_s(X)\}$.

So consider the $s+1$ row vectors

$$v_r = \left(\begin{array}{c}
\left(\frac{\mu_0}{r}\right) (k-\mu_0), & \left(\frac{\mu_1}{r}\right) (k-\mu_1), & \ldots, & \left(\frac{\mu_s}{r}\right) (k-\mu_s)
\end{array}\right),$$

$r=0, 1, \ldots, s$. Suppose $c_0v_0+c_1v_1+\cdots+c_sv_s=0$. This means that the polynomial

$$p(x) = \sum_{r=0}^{s} c_r \left(\frac{x}{r}\right) (k-x)$$

of degree $\leq s$ has $s+1$ distinct roots $\mu_0, \mu_1, \ldots, \mu_s$ and hence is the zero polynomial. Now $p(0)=c_0\left(\frac{k}{s}\right)$, so $c_0=0$; then $p(1)=c_1\left(\frac{k-1}{s-1}\right)$, so $c_1=0$; and, inductively, $c_0=c_1=\cdots=c_s=0$. That is, $v_0, \ldots, v_s$ are linearly independent. This completes the proof.

**Theorem 4.** Let $D=(X, \mathcal{A}, \mathcal{B})$ be an $S_{\lambda}(t, k, v)$ with $t=2s$ and $v \geq k+s$. Then there are at least $s$ distinct elements in the set

$$\{(A) \cap (B) \mid A \subseteq \mathcal{A}, B \subseteq \mathcal{A}, A \neq B\},$$

and there are exactly $s$ distinct elements if and only if $D$ is a tight $t$-design.

Proof. In view of Theorems 1 and 3, it remains only to show that for any tight $t$-design, there exist $s$ integers $\mu_1, \mu_2, \ldots, \mu_s$ with $0 \leq \mu_i < k$ so that $|(A) \cap (B)| \in \{\mu_1, \ldots, \mu_s\}$ for distinct blocks $A$ and $B$. Let $D=(X, \mathcal{A}, \mathcal{B})$ be a tight $S_{\lambda}(t, k, v)$. With the notation of Theorem 1, the $b=\binom{s}{v}$ vectors $\{\bar{A}: A \subseteq \mathcal{A}\}$ must, since they span $V_s$, be a basis for $V_s$.

Fix $A_0 \subseteq \mathcal{A}$ and for $B \subseteq \mathcal{A}$, write $\mu_B=|(B) \cap (A_0)|$. For $i=0, 1, \ldots, s$, define vectors

$$M_i = \sum (S: S \subseteq \mathcal{P}_s(X), |S\cap (A_0)| = i),$$

$$N_i = \sum \left(\begin{array}{c}
\mu_B \\
S \subseteq (B)
\end{array}\right).$$

Now given $S \subseteq \mathcal{P}_s(X)$ with $|S\cap (A_0)| = i$, the coefficient of $S$ in the sum $N_r$ is

$$\sum \left(\begin{array}{c}
\mu_B \\
B \subseteq \mathcal{A}, S \subseteq (B)
\end{array}\right),$$

i.e., the number of ordered pairs $(B, R)$ in $\mathcal{A} \times \mathcal{P}_r(X)$ such that $S \subseteq (B)$ and $R \subseteq (A_0) \cap (B)$. For any $r$-subset $R \subseteq (A_0)$ with $|R\cap S| = j$, the number of blocks $B$ such that $(B, R)$ satisfies the above conditions is $b_{s+r-j}$. Thus the coefficient of $S$ in $N_r$ is
\[ c' \leq \sum_{r=0}^{s} \binom{i}{j} \binom{k-i}{r-j} b_{r+r-j}; \text{ and so} \]

\[ N_r = \sum_{r=0}^{s} c'_r M_r \quad (r = 0, 1, \ldots, s). \]

The \( s+1 \) vectors \( N_r - c'_r M_s \) are contained in the span of \( M_0, M_1, \ldots, M_{s-1} \); hence there exist rationals \( a_0, a_1, \ldots, a_s \), not all zero, such that

\[ \sum_{r=0}^{s} a_r (N_r - c'_r M_s) = 0, \quad \text{or} \]

\[ \sum_{r=0}^{s} a_r \sum_{i \in A} \binom{\mu_B}{r} (B - c'_r A_0) = 0. \]

Now \( \{ \tilde{A}: A \in \mathcal{A} \} \) is a basis for \( V_s \), so for \( B \neq A_0 \), the coefficient

\[ \sum_{r=0}^{s} a_r \binom{\mu_B}{r} \]

of \( \tilde{B} \) must be 0. That is, for any \( B \neq A_0 \), the intersection number \( \mu_B \) is a root of the polynomial

\[ f(x) = \sum_{r=0}^{s} a_r \binom{x}{r} \]

of degree at most \( s \). Finally, note that the coefficients \( c'_r \) are (and hence \( f(x) \) can be chosen to be) independent of the block \( A_0 \): all intersection numbers are roots of \( f(x) \).

The polynomials \( f(x) \) described in the proof of Theorem 4 have been found explicitly by P. Delsarte [1]. As an example, we consider the case \( t=4 \). The equations of Theorem 4 are

\[ N_0 = b_2 M_0 + b_2 M_1 + b_2 M_2, \]

\[ N_1 = kb_3 M_0 + (b_2 + (k-1)b_3) M_1 + (2b_2 + (k-2)b_3) M_2, \]

\[ N_2 = \left( \frac{2}{k} \right) b_4 M_0 + \left( \binom{k-1}{2} b_4 + (k-1)b_3 \right) M_1 + \left( \binom{k-2}{2} b_4 + 2(k-2)b_3 + b_2 \right) M_2. \]

Using the relation \( b_2 = \left( \frac{k}{2} \right) \) in a tight 4-design, one verifies that

\[ (b_2 - b_3) N_0 - (k-1)(b_2 - b_3) N_1 + (2b_2(b_2 - b_3) - b_4(b_2 - b_3) - b_3) N_2 \]

is a scalar multiple of \( M_2 = \tilde{A}_0 \). For a block \( B \neq A_0 \), the coefficient of \( \tilde{B} \) in the above expression must be zero, i.e.,

\[ \mu_B (\mu_B - 1) - \frac{2(k-1)(b_2 - b_3)}{(b_2 - b_3)} \mu_B + \frac{4b_3(b_2 - b_3)}{(b_2 - b_3)} - 2b_4 = 0. \]

Rewriting the coefficients in terms of \( v, k, \) and \( \lambda \), we have
Theorem 5. The two "intersection numbers" $\mu_1, \mu_2$ of a tight 4-design $S_4(4, k, v)$ are the roots of the polynomial

$$f(x) = x^2 - \left(\frac{2(k-1)(k-2)}{v-3} + 1\right)x + \lambda \left(2 + \frac{4}{k-3}\right).$$

Application of Theorem 5 yields the well known fact that any two distinct blocks of an $S_4(4, 7, 23)$ meet in 1 or 3 points.

Since $f(x)$ has integral roots, it must have integral coefficients, and we have the

Corollary. The existence of a tight 4-design $S_4(4, k, v)$ implies $v - 3$ divides $2(k-1)(k-2)$, and $k - 3$ divides $4\lambda$.

In [1], Delsarte observes that Theorems 4 and 5 are similar to Lloyd's Theorem on perfect codes. Indeed, Delsarte develops a theory of designs and codes (emphasizing a "formal duality") in the context of association schemes. Contained therein are results analogous to the above for orthogonal arrays of strength $t$, the analogue of Theorem 1 being Rao's bound.

We conclude with the following remarks.

Let $D=(X, A, \mathcal{J})$ be a tight $S_4(t, k, v)$ with $t = 2s$ and $v \geq k + s$. Let $J(s, v)$ denote the association scheme whose points are the $s$-element subsets of $X$ (see [1]). Let $N$ be a $(0-1)$-matrix whose rows are indexed by elements of $\mathcal{P}_s(X)$ and columns are indexed by the blocks of $D$. At the row corresponding to $S$ and column corresponding to a block $A$, the entry of $N$ is 1 iff $S \subseteq (A)$. The matrix $NN^T$ belongs to the Bose-Mesner algebra of the scheme $J(s, v)$. The matrix $NN^T$ is obviously rationally congruent to the identity matrix. Using the properties of the algebra of $J(s, v)$, it is possible to compute the Hasse-Minkowski invariant of $NN^T$ and obtain some more necessary conditions for the existence of tight $2s$-designs. (See also [5].)

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References