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ON t-DESIGNS

DIJEN K. RAY-CHAUDHURI* AND RICHARD M. WILSON**

(Received January 14, 1975)

Introduction and preliminaries

An incidence structure is a triple $S=(X, \mathcal{A}, \mathcal{B})$ where $X$ and $\mathcal{A}$ are disjoint sets and $\mathcal{B} \subseteq X \times \mathcal{A}$. Elements $x \in X$ are called points and elements $A \in \mathcal{A}$ are called blocks of $S$. A point $x$ and a block $A$ are incident iff $(x, A) \in \mathcal{B}$. For any block $A$, $(A)$ will denote the set of points incident with $A$.

Let $v$, $k$, $t$ and $\lambda$ be integers with $v \geq k \geq t \geq 0$ and $\lambda \geq 1$. An $S_\lambda(t, k, v)$ (a $t$-design on $v$ points with block size $k$ and index $\lambda$) is an incidence structure $D=(X, \mathcal{A}, \mathcal{B})$ such that

(i) $|X|=v$,
(ii) $|(A)|=k$ for every $A \in \mathcal{A}$,
(iii) for every $t$-subset $T$ of $X$, there are exactly $\lambda$ blocks $A \in \mathcal{A}$ with $T \subseteq (A)$.

It is well known that every $S_\lambda(t, k, v)$ has exactly $b=\lambda \binom{v}{t} / \binom{k}{t}$ blocks and more generally, for any $i$-subset $I$ of points $(0 \leq i \leq t)$, the number of blocks $A$ of the design with $I \subseteq (A)$ is

$$b_i = \lambda \binom{v-i}{t-i} / \binom{k-i}{t-i},$$

independent of the subset $I$ [2].

Abstract: We present the generalization (conjectured by A. Ja. Petrenjuk) of Fisher’s Inequality $b \geq v$ for 2-designs and Petrenjuk’s Inequality $b \geq \binom{v}{2}$ for 4-designs. The $t$-designs satisfying the inequality with equality may be considered as generalizations of the symmetric 2-designs ($b=v$) and have the property that there are exactly $\frac{1}{2} t$ possible values for the size of the intersection of two distinct blocks, these values being computable from the parameters.

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An $S_\lambda(t, k, v)$, say $D=(X, \mathcal{A}, \mathcal{F})$, is simple when the mapping $A\mapsto(A)$ from $\mathcal{A}$ into $\mathcal{P}_k(X)$ (the class of all $k$-element subsets of $X$) is injective; and $D$ is trivial when the mapping $A\mapsto(A)$ is (surjective and) $m$-to-one for some integer $m$, i.e. each $k$-subset "occurs as a block" exactly $m$ times. In this latter case, evidently $\lambda=m\binom{v-t}{k-t}$.

The well known Fisher's Inequality (see [2]) asserts that the number $b$ of blocks of an $S_\lambda(2, k, v)$ is at least $v$, under the assumption $v\geq k+1$. A. Ja. Petrenjuk [4] proved in 1968 that $b\geq\binom{v}{2}$ for any $S_\lambda(4, k, v)$ with $v\geq k+2$ and conjectured that $b\geq\binom{v}{s}$ in any $S_\lambda(2s, k, v)$ with $v\geq k+s$. This conjecture is established in the following section.

This condition shows the nonexistence of certain $t$-designs. For example, Petrenjuk's Inequality shows that $S_\lambda(4, 22, 79)$ do not exist even though the $b_i$'s ($0\leq i\leq 4$) are integral. We might note that a hypothetical $S_\lambda\left(4, k, 2+\frac{1}{2}(k-1)(k-2)\right)$ would satisfy $b=\binom{v}{2}$ (and the $b_i$'s are integral when $k\equiv 1 \pmod{4}$), but no such designs exist by the corollary of Theorem 5 below. The inequality $b\geq\binom{v}{3}$ rules out the entire family of 6-designs with

$$v = 120m,$$

$$k = 60m,$$

$$\lambda = (20m-1)(15m-1)(12m-1),$$

(for which the $b_i$'s are integral).

By a tight $t$-design ($t$ even, say $t=2s$) we mean an $S_\lambda(t, k, v)$ with $v\geq k+s$ and $b=\binom{v}{s}$. As examples, we have the trivial designs $S_\lambda(2s, k, k+s)$ where

$$\lambda=\binom{k-s}{k-2s}.$$  

An example of a tight 4-design is the well known $S_\lambda(4, 7, 23)$ where $b=253=\binom{23}{2}$. N. Ito [3] has recently shown, using Theorem 5 below, that the only nontrivial tight 4-designs are the $S_\lambda(4, 7, 23)$ and its complement, an $S_{2s}(4, 16, 23)$. Tight $t$-designs with $t\geq 4$ seem to be very rare.

Our proof of Petrenjuk's conjecture uses only elementary linear algebra and the observation that the number of blocks of an $S_\lambda(t, k, v)$ which are incident with some $i$ points and not incident some other $j$ points is constant (i.e., depends only on $i, j$, and the parameters; not the particular sets of points) whenever $i+j\leq t$.

**Proposition 1.** Let $(X, \mathcal{A}, \mathcal{F})$ be an $S_\lambda(t, k, v)$. Let $i$ and $j$ be nonnegative integers with $i+j\leq t$. Then for any subsets $I, J \subseteq X$ with $|I|=i, |J|=j,$
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$I \cap J = \emptyset$, the number of blocks $A \in \mathcal{A}$ such that $I \subseteq (A)$ and $J \cap (A) = \emptyset$ is exactly

$$b_i^t = \lambda \frac{\binom{\nu - i - j}{k - t}}{\binom{\nu - t}{t}}.$$ 

Proof. By inclusion-exclusion,

$$b_i^t = \sum_{r=0}^{i} (-1)^r \binom{j}{r} b_{i+r}.$$ 

In view of the above expression for $b_i$, we have $b_i^t = \lambda c$ where

$$c = \sum_{r=0}^{i} (-1)^r \binom{j}{r} \binom{\nu - i - r}{t - i - r} \binom{k - i - r}{t - i - r}^{-1}.$$ 

But in the case of the trivial design $(X, \mathcal{A}, (X \times \mathcal{A}) - \mathcal{D})$ of an $S_\lambda(t, k, \nu)$ is an $S_\lambda^*(t, \nu - k, \nu)$ with

$$\lambda^* = b_\nu^t = \lambda \binom{\nu - t}{k} \binom{\nu - t}{k - t}^{-1}$$ 

(unless $\nu < k + t$, in which case the original $S_\lambda(t, k, \nu)$ is evidently trivial).

2. Generalizations of Fisher's inequality

For any set $Y$, we denote by $V(Y)$ the free vector space over the rationals generated by $Y$, i.e. $V(Y)$ consists of all formal sums $\alpha = \sum_{y \in Y} a_y y$ with rational coefficients $a_y$ and formal addition and scalar multiplication. The “unit vectors” $y, y^\prime, \ldots, Y$ by definition provide a basis for $V(Y)$.

**Theorem 1.** The existence of an $S_\lambda(t, k, \nu)$ with $t$ even, say $t=2s$, and $\nu \geq k+s$ implies

$$b \geq \binom{\nu}{s},$$

where $b$ is the number of blocks of the design. In fact, the number of distinct subsets $(A)$ is itself at least $\binom{\nu}{s}$.

Proof. Let $D=(X, \mathcal{A}, \mathcal{D})$ be an $S_\lambda(t, k, \nu)$ and put $V_s = V(\mathcal{P}_s(X))$, where $\mathcal{P}_s(X)$ is the class of all $s$-element subsets of $X$. For each block $A$ of $D$, define a vector $\hat{A} \in V_s$ as the “sum” of all $s$-subsets of $(A)$, i.e.
We claim that the set of vectors \( \{ A : A \in \mathcal{A} \} \) spans \( V_s \). Since \( V_s \) has dimension \( \binom{v}{s} \), the theorem follows immediately.

Let \( S_0 \in \mathcal{P}_s(X) \). To show \( S_0 \) belongs to the span of \( \{ A : A \in \mathcal{A} \} \), we introduce the vectors

\[
E_i = \sum(S : S \in \mathcal{P}_s(X), |S \cap S_0| = s-i)
\]

(so \( E_0 = S_0 \)) and

\[
F_i = \sum(\hat{A} : A \in \mathcal{A}, |(A) \cap S_0| = s-i)
\]

for \( i = 0, 1, \ldots, s \). Now for \( S_i \in \mathcal{P}_s(X) \) with \( |S_i \cap S_0| = s-i \), the coefficient of \( S_i \) in the sum \( F_r \) is the number of blocks \( A \) such that \( S_i \subseteq (A) \) and \( |(A) \cap S_0| = s-r \); and this number is \( \binom{i}{r} b_{s-r+i} \) with the notation of Proposition 1. Thus

\[
F_r = \sum_{i=r}^{s} \binom{i}{r} b_{s-r+i} E_i \quad (r = 0, 1, \ldots, s).
\]

The above system of linear equations is triangular and the diagonal coefficients \( b_r \) (\( r = 0, 1, \ldots, s \)) are all nonzero under our hypothesis \( v \geq k+s \). Thus we can solve for the \( E_i \)'s (in particular, for \( E_0 = S_0 \)) as linear combinations of the \( F_r \)'s. Since the \( F_r \)'s are by definition in the span of \( \{ \hat{A} : A \in \mathcal{A} \} \), we have \( S_0 \in \text{span} \{ \hat{A} : A \in \mathcal{A} \} \) for every \( S_0 \in \mathcal{P}_s(X) \), and our claim is verified.

**Corollary.** The existence of an \( S_\lambda(t, k, v) \) with \( t \) odd, say \( t = 2s+1 \) and \( (v-1) \geq k+s \) implies the inequality

\[
b = \frac{\lambda}{(2s+1)} \left( \frac{v}{k} \right) \geq \frac{\lambda(v-1)}{k-1} + \binom{v-1}{s} \geq 2\binom{v-1}{s}.
\]

**Proof.** Let \( D = (X, \mathcal{A}, \mathcal{J}) \) be an \( S_\lambda(t, k, v) \) and \( x \in X \). Let \( \mathcal{A}' \) be the class of blocks incident with \( x \) and \( \mathcal{A}'' \) be the class of blocks not incident with \( x \). Observe that both \( D' = (X', \mathcal{A}', \mathcal{J} \cap (X' \times \mathcal{A}')) \) and \( D'' = (X', \mathcal{A}'', \mathcal{J} \cap (X' \times \mathcal{A}'')) \), where \( X' = X - \{x\} \), are \( 2s \)-designs and apply Theorem 1.

The above inequality also rules out infinitely many parameters for which \( b_i \)'s are integers, \( i = 0, 1, \ldots, t \).

**Theorem 2.** Let \( D = (X, \mathcal{A}, \mathcal{J}) \) be an \( S_\lambda(t, k, v) \) where \( t = 2s \) and \( v \geq k+s \). If there exists a partition \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r \) such that each substructure \( (X, \mathcal{A}_i, \mathcal{J} \cap (X \times \mathcal{A}_i)) \) is an \( S_\lambda(s, k, v) \) for some positive integers \( \lambda_i \), then
Proof. With the notation of Theorem 1, the vectors \( \{A: A \in \mathcal{A}\} \) span \( V \). But observe that
\[
\sum \{\tilde{A}: A \in \mathcal{A}_i\} = \lambda_i \sum (S: S \in \mathcal{P}_s(X)) = \lambda_i \tilde{X}, \text{ say.}
\]
So if we choose one block \( A_i \) from each \( \mathcal{A}_i \), then \( \{\tilde{A}: A \in \mathcal{A} - \{A_1, \ldots, A_s\} \} \cup \{\tilde{X}\} \) spans \( V \). The stated inequality follows.

3. Tight t-designs

Recall that a tight \( t \)-design \( (t = 2s) \) is an \( S_\lambda(t, k, v) \) with \( v \geq k + s \) and

\[
b = \lambda \binom{v}{t} / \binom{k}{t} = \binom{v}{s}.
\]

In view of Theorem 1, tight designs are simple. In this section we extend the well known result that two distinct blocks of a symmetric design (tight 2-design) have exactly \( \lambda \) common incident points (see Theorem 4 below).

**Theorem 3.** Let \( X \) be a \( v \)-set and \( \mathcal{A} \) a class of \( k \)-subsets of \( X \) such that for distinct \( A, B \in \mathcal{A} \),

\[
|A \cap B| \in \{\mu_1, \mu_2, \ldots, \mu_s\}
\]

where \( k > \mu_1 > \mu_2 > \cdots > \mu_s \geq 0 \). Then

\[
|\mathcal{A}| \leq \binom{v}{s}.
\]

Proof. Let \( V = V(\mathcal{A}) \). For each \( S \in \mathcal{P}_s(X) \), define a vector

\[
\tilde{S} = \sum (A: A \in \mathcal{A}, A \supseteq S).
\]

We claim that the vectors \( \{\tilde{S}: S \in \mathcal{P}_s(X)\} \) span \( V \). Since \( V \) has dimension \( |\mathcal{A}| \), the theorem will follow.

Write \( \mu_0 = k \). Let \( A_0 \in \mathcal{A} \) be given. Define

\[
H_i = \sum (B: B \in \mathcal{A}, \ |B \cap A_0| = \mu_i)
\]

for \( i = 0, 1, \ldots, s \) (note \( H_0 = A_0 \)). For \( r = 0, 1, \ldots, s \), we see that

\[
G_r = \sum (\tilde{S}: S \in \mathcal{P}_s(X), \ |S \cap A_0| = r) = \sum \binom{\mu_i}{r} \binom{k - \mu_i}{s - r} H_i,
\]

by comparing the coefficient of each \( A \in \mathcal{A} \) on both sides of the equation. We now show that the coefficient matrix of this system of \( s+1 \) linear equations is
nonsingular, so that we can solve for the $H_i$'s in terms of the $G'_r$s. In particular, we then have $H_o=A_o \in \text{span } \{G_o, G_1, \ldots, G_r\} \subseteq \text{span } \{S: S \in \mathcal{P}_s(X)\}$.

So consider the $s+1$ row vectors
\[ v_r = \left( \binom{\mu_0}{r}, \binom{k-\mu_1}{s-r}, \binom{k-\mu_2}{s-r}, \ldots, \binom{k-\mu_s}{s-r} \right), \]
$r=0, 1, \ldots, s$. Suppose $c_0v_0 + c_1v_1 + \cdots + c_sv_s = 0$. This means that the polynomial
\[ p(x) = \sum_{r=0}^{s} c_r \binom{x}{r}\left(\frac{k-x}{s-r}\right) \]
of degree $\leq s$ has $s+1$ distinct roots $\mu_0, \mu_1, \ldots, \mu_s$ and hence is the zero polynomial. Now $p(0) = c_0 \binom{k}{s}$, so $c_0 = 0$; then $p(1) = c_1 \binom{k-1}{s-1}$, so $c_1 = 0$; and, inductively, $c_0 = c_1 = \cdots = c_s = 0$. That is, $v_0, \ldots, v_s$ are linearly independent. This completes the proof.

**Theorem 4.** Let $D=(X, \mathcal{A}, \mathcal{B})$ be an $S_\lambda(t, k, v)$ with $t=2s$ and $v \geq k+s$. Then there are at least $s$ distinct elements in the set
\[ \{(A) \cap (B)|: A \in \mathcal{A}, B \in \mathcal{A}, A \neq B\}, \]
and there are exactly $s$ distinct elements if and only if $D$ is a tight $t$-design.

**Proof.** In view of Theorems 1 and 3, it remains only to show that for any tight $t$-design, there exist $s$ integers $\mu_1, \mu_2, \ldots, \mu_s$ with $0 \leq \mu_i < k$ so that $|(A) \cap (B)| \in \{\mu_1, \ldots, \mu_s\}$ for distinct blocks $A$ and $B$. Let $D=(X, \mathcal{A}, \mathcal{B})$ be a tight $S_\lambda(t, k, v)$. With the notation of Theorem 1, the $b=\binom{v}{s}$ vectors
\[ \{A: A \in \mathcal{A}\} \]
must, since they span $V_s$, be a basis for $V_s$.

Fix $A_0 \in \mathcal{A}$ and for $B \in \mathcal{A}$, write $\mu_B = |(B) \cap (A_0)|$. For $i=0, 1, \ldots, s$, define vectors
\[ M_i = \sum(S: S \in \mathcal{P}_s(X), |S \cap (A_0)| = i), \]
\[ N_i = \sum(\binom{\mu_B}{i}B: B \in \mathcal{A}). \]

Now given $S \in \mathcal{P}_s(X)$ with $|S \cap (A_0)| = i$, the coefficient of $S$ in the sum $N_r$ is
\[ \sum(\binom{\mu_B}{i}B: B \in \mathcal{A}, S \subseteq (B)), \]
i.e., the number of ordered pairs $(B, R)$ in $\mathcal{A} \times \mathcal{P}_s(X)$ such that $S \subseteq (B)$ and $R \subseteq (A_0) \cap (B)$. For any $r$-subset $R \subseteq (A_0)$ with $|R \cap S| = j$, the number of blocks $B$ such that $(B, R)$ satisfies the above conditions is $b_{s+r-j}$. Thus the coefficient of $S$ in $N_r$ is
The $s+1$ vectors $N_r - c_r^* M_s$ are contained in the span of $M_0, M_1, \ldots, M_{s-1}$; hence there exist rations $a_0, a_1, \ldots, a_s$, not all zero, such that

$$\sum_{r=0}^{s} a_r (N_r - c_r^* M_s) = 0, \quad \text{or}$$

$$\sum_{r=0}^{s} a_r \sum_{A \in \mathcal{A}} \left( \binom{\mu_B}{r} \right) B - c_r^* A_0 = 0.$$

Now \{$A : A \in \mathcal{A}$\} is a basis for $V_s$, so for $B \neq A_0$, the coefficient

$$\sum_{r=0}^{s} a_r \left( \binom{\mu_B}{r} \right)$$

of $B$ must be 0. That is, for any $B \neq A_0$, the intersection number $\mu_B$ is a root of the polynomial

$$f(x) = \sum_{r=0}^{s} a_r \left( \binom{x}{r} \right)$$

of degree at most $s$. Finally, note that the coefficients $c_r^*$ are (and hence $f(x)$ can be chosen to be) independent of the block $A_0$: all intersection numbers are roots of $f(x)$.

The polynomials $f(x)$ described in the proof of Theorem 4 have been found explicitly by P. Delsarte [1]. As an example, we consider the case $t=4$. The equations of Theorem 4 are

$$N_0 = b_2 M_0 + b_2 M_1 + b_2 M_2,$$

$$N_1 = k b_2 M_0 + (b_2 + (k-1)b_3) M_1 + (2b_2 + (k-2)b_3) M_2,$$

$$N_2 = \left( \frac{2}{k} \right) b_2 M_0 + \left( \left( \frac{k-1}{2} \right) b_4 + (k-1)b_3 \right) M_1 + \left( \left( \frac{k-2}{2} \right) b_4 + 2(k-2)b_3 + b_3 \right) M_2.$$

Using the relation $b_2 = \left( \frac{k}{2} \right)$ in a tight 4-design, one verifies that

$$(b_2 - b_3) N_3 - (k-1)(b_3-b_4) N_1 + (2b_4(b_3-b_4) - b_4(b_3-b_4)) N_0$$

is a scalar multiple of $M_3 = A_0$. For a block $B \neq A_0$, the coefficient of $B$ in the above expression must be zero, i.e.,

$$\mu_B \left( \mu_B - 1 \right) - \frac{2(k-1)(b_3-b_4)}{(b_2-b_3)} \mu_B + \frac{4b_4(b_3-b_4)}{(b_2-b_3)} - 2b_4 = 0.$$

Rewriting the coefficients in terms of $v$, $k$, and $\lambda$, we have
Theorem 5. The two "intersection numbers" $\mu_1, \mu_2$ of a tight 4-design $S_4(4, k, v)$ are the roots of the polynomial

$$f(x) = x^2 - \left( \frac{2(k-1)(k-2)}{v-3} + 1 \right) x + \lambda \left( \frac{2 + \frac{4}{k-3}}{k-3} \right).$$

Application of Theorem 5 yields the well known fact that any two distinct blocks of an $S_4(4, 7, 23)$ meet in 1 or 3 points.

Since $f(x)$ has integral roots, it must have integral coefficients, and we have the

**Corollary.** The existence of a tight 4-design $S_4(4, k, v)$ implies $v - 3$ divides $2(k-1)(k-2)$, and $k - 3$ divides $4\lambda$.

In [1], Delsarte observes that Theorems 4 and 5 are similar to Lloyd's Theorem on perfect codes. Indeed, Delsarte develops a theory of designs and codes (emphasizing a "formal duality") in the context of association schemes. Contained therein are results analogous to the above for orthogonal arrays of strength $t$, the analogue of Theorem 1 being Rao's bound.

We conclude with the following remarks.

Let $D = (X, \mathcal{A}, J)$ be a tight $S_4(t, k, v)$ with $t = 2s$ and $v \geq k + s$. Let $J(s, v)$ denote the association scheme whose points are the $s$-element subsets of $X$ (see [1]). Let $N$ be a $(0-1)$-matrix whose rows are indexed by elements of $\mathcal{P}_s(X)$ and columns are indexed by the blocks of $D$. At the row corresponding to $S$ and column corresponding to a block $A$, the entry of $N$ is 1 iff $S \subseteq(A)$. The matrix $NN^T$ belongs to the Bose-Mesner algebra of the scheme $J(s, v)$. The matrix $NN^T$ is obviously rationally congruent to the identity matrix. Using the properties of the algebra of $J(s, v)$, it is possible to compute the Hasse-Minkowski invariant of $NN^T$ and obtain some more necessary conditions for the existence of tight $2s$-designs. (See also [5].)

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**References**


