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Osaka University
Note on Axiomatic Set Theory I.

The Independence of Zermelo’s “Aussonderungsaxiom” from Other Axioms of Set Theory

By Toshio Nishimura

In this paper axioms of set theory mean Gödel’s axioms of set theory in [2], which are modified so that the axiom C2 postulates directly the existence of all elements of a given set and C3 the existence of power set.

In what follows we prove that Zermelo’s “Aussonderungsaxiom” is independent of other axioms of set theory. A fortiori, the independence of Gödel’s axiom C4 (Fraenkel’s “axiom of substitution”), is proved since the axiom of substitution implies the “Aussonderungsaxiom”.

The proof is carried out by constructing an inner model Λ for the axioms A, B, C1, C2, C3, D and E under the axioms A, B, C1, C2 and C4, which does not satisfy the “Aussonderungsaxiom”. The idea appears already in [1]. However the proof in [1] is not formal.

In §1, we give the axioms of set theory.

In §2, we construct an inner model Λ under the axioms A, B, C1, C2 and C4. Here we notice that the axioms do not imply the existence of power set of any set.

In §3, we prove some preliminary results with respect to the model Λ.

In §4, we prove that the model Λ satisfies the axioms A, B, C1, 2, 3, D and E.

In §5, we prove that the model Λ does not satisfy the “Aussonderungsaxiom”.

§1. The axiom system of set theory

In what follows we apply Gödel’s notations in [2] in most cases. For logical notations we use following symbols, ∨ (or), · (and), ∼ (not), ⊃ (implies), ≡ (equivalence), = (identity), (∃X) (there is an X), (X) (for all X) and (∃!X) (there is exactly one X). The system has three primitive notions: class, denoted by C; set, denoted by M; and the diadic relation ∈ between class and class, class and set, set and class, or
set and set. Variables for classes are denoted by capital Latin letters $A, B, X, Y$ etc. and those for sets by small Latin letters $a, b, x, y$ etc. The axioms fall into five groups $A, B, C, D, E$.

**Group A.**

1. $(x) \in \mathcal{S}(x)$
2. $(X) (Y) [X \in Y \supset \mathcal{W}(X)]$
3. $(X) (Y) [(u) (u \in X \equiv u \in Y) \supset X = Y]$
4. $(x) (y) (\exists z) (u) [u \in z \implies u = x \lor u = y]$

**Group B.**

1. $(\exists A) (x) (y) [\langle xy \rangle \in A \equiv x \in y]$
2. $(A) (B) (\exists C) (u) [u \in C \equiv u \in A \land u \in B]$
3. $(A) (\exists B) (u) [u \in B \equiv \neg (u \in A)]$
4. $(A) (\exists B) (x) [x \in B \equiv (\exists y) (\langle yx \rangle \in A)]$
5. $(A) (\exists B) (x) (y) [\langle xy \rangle \in B \equiv x \in A]$
6. $(A) (\exists B) (x) (y) [\langle yx \rangle \in B \equiv \langle xy \rangle \in A]$
7. $(A) (\exists B) (x) (y) (z) [\langle xyz \rangle \in B \equiv \langle yzx \rangle \in A]$
8. $(A) (\exists B) (x) (y) (z) [\langle xyz \rangle \in B \equiv \langle xzy \rangle \in A]$

where $\langle xy \rangle$ and $\langle xyz \rangle$ mean the ordered pair of $x$ and $y$ and the ordered triple of $x, y$ and $z$ (cf. [2], p. 4, 1.12 and 1.14).

**Group C.**

1. $(\exists a) [\neg \mathcal{E}m(a) \cdot (x) (x \in a \supset (\exists y) (y \in a \cdot x \subset y))]$
2. $(x) (\exists y) (u) [u \in y \equiv (\exists v) (u \in v \cdot v \in x)]$
3. $(x) (\exists y) (u) [u \in y \equiv (w) (w \in u \supset w \in x)]$
4. $(x) (A) [\mathcal{U}n(A) \supset (\exists y) (u) (u \in y \equiv (\exists v) (v \in x \cdot \langle uv \rangle \in A))]$

where the notions $a \subset b$, $\mathcal{E}m(a)$ and $\mathcal{U}n(A)$ mean "$a$ is a proper subset of $b"$, "$a$ is empty" and "$A$ is unique" respectively (cf. [2], p. 4, 1.2, 1.22 and p. 5, 1.3).

Zermelo's "Aussonderungsaxiom" is postulated as follows:

$4'$. 

$(x) (A) (\exists y) (u) [u \in y \equiv u \in x \lor u \in A]$

**Axiom D.**

$(A) [\neg \mathcal{E}m(A) \cdot (\exists u) (u \in A \cdot \mathcal{G}x(u, A))]$

where the notion $\mathcal{G}x(u, A)$ means "$u$ and $A$ are exclusive" (cf. [2], p. 4, (1.23).
Axiom E.

\[(\exists A) [\forall n(A) \cdot (x) (\sim \mathcal{B}(n(x)) \cdot (\exists y) (y \in x \cdot \langle yx \rangle \in A))]\]

§ 2. The model Λ

In this section we construct the model Λ under the axioms A, B, C 1, 2 and 4.

1. Preliminary Definitions

We develop the theory of ordinal numbers under the axioms A, B, C 1, 2 and 4. (It is well known that is possible.) Ordinal numbers are denoted by small Greek letters α, β, γ etc. When α and β are ordinal numbers, α * β and α β are the product and power in arithmetic of ordinal numbers. The existence of these functions are easily ensured. ω is the smallest infinite ordinal number.

We can define the function j such that the following conditions are satisfied:

1. \( j : \mathbb{N}_0 \times \alpha^{(\omega_0)} \times \alpha^{(\omega_0)} \rightarrow \alpha^{(\omega_0)} \)

where 1, 2, and 3 are \( 0 + \{0\}, 1 + \{1\} \) and \( 2 + \{2\} \) respectively, and \( A \times B \) is the direct product of A and B (cf. [2], p. 14, 4.1).

2. \( \mu, \nu < \beta \rightarrow \langle \langle \mu \alpha \beta \rangle S(\nu \gamma \delta) \cdot \rangle \cdot j^i \alpha \langle \mu \alpha \beta \rangle \langle \beta \gamma \delta \rangle \rangle \)

where S is the relation given in [2], p. 36, 9.2, in which \( \beta \) is replaced by \( \beta \). And \( A'x \) is the function given in [2] p. 16, 4.65.

3. \( \mathcal{B}(j) \) is an ordinal number, where \( \mathcal{B}(j) \) is the set of all values of \( j \) (cf. [2] p. 15, 4.44).

Then it is clear that the function \( j \) is a set.

Then in the similar way as in [2] p. 36, 9.24 we can easily give the functions (to be a set) \( k_0, k_1 \) and \( k_2 \) which satisfy the following:

if \( \mu, \nu < 4 \), then

\[
\begin{align*}
k_0 j^i \langle \mu \alpha \beta \rangle & = \mu, \quad k_1 j^i \langle \mu \alpha \beta \rangle = \alpha \quad \text{and} \quad k_2 j^i \langle \mu \alpha \beta \rangle = \beta, \\
k_0 ^i \alpha & = 0 \vee k_0 ^i \alpha = 1 \vee k_0 ^i \alpha = 2, \\
j^i \langle k_1 ^i \alpha, k_1 ^i \alpha, k_2 ^i \alpha \rangle & = \alpha,
\end{align*}
\]

and if \( 0 < \mu < 3 \) then

\[
k_1 ^i \langle \mu \alpha \beta \rangle \langle \beta \gamma \delta \rangle = 1, 2.
\]

The existence of such functions are easily ensured.

2. First we define the function 'f' by transfinite induction simultaneously, which is defined over \( \omega \).
where \( \{a, b\} \) is the non-ordered pair of \( a \) and \( b \), and \( \mathfrak{H}(a) \) the sum of all elements of \( a \).

Now we give the model \( A \). The universal class of the model \( A \) is \( \mathfrak{W}(f) \) which is a set denoted by \( v_\lambda \), i.e.

\[
v_\lambda = \mathfrak{W}(f).
\]

\( \mathfrak{M}_\lambda(a) \) means that \( a \) is a set of the model \( A \) and is given by the formula

\[
\mathfrak{M}_\lambda(a) \iff a \in v_\lambda.
\]

\( \mathfrak{G}_\lambda(A) \) means that \( A \) is a class of the model \( A \) and is given by the formula

\[
\mathfrak{G}_\lambda(A) \iff A \subseteq v_\lambda.
\]

\( \in_\lambda \) means the \( \epsilon \)-relation in the model \( A \) and is defined by the formula

\[
A \in_\lambda B \iff A \in B \cdot \mathfrak{M}_\lambda(A) \cdot \mathfrak{G}_\lambda(B).
\]

Then operations and notions can be relativized for the model \( A \) in the similar way as in [2], p. 41 and p. 42. The relativization of an operation \( A \) is denoted by \( A_\lambda \) and that of a notion \( B \) by \( B_\lambda \).

§ 3. Preliminary results

In this section we prove some lemmas with respect to the model \( A \).

1. Following lemmas are concerning arithmetic of ordinal numbers.

1.1. Lemma. Let \( \eta \) be an ordinal number such that \( \omega \leq \eta \) and \( n < \omega \). Then

\[
(\eta + 1)^2 = \eta^2 + \eta + 1
\]

\[
(\eta + 1)^3 = \eta^3 + \eta^2 + \eta + 1.
\]

And in case that \( \eta \in K_\Pi \), \( n \cdot \eta = \eta \) and in case that \( \eta = \xi + m \) where \( \xi \in K_\Pi \) and \( m < \omega \), \( n \cdot \eta = \xi + mn < \eta \cdot 2 \).

1.2. Lemma. Let \( \alpha \) and \( \beta \) be ordinal numbers such that \( \alpha, \beta < \omega^{\omega(\omega)} \) and \( \eta(=\text{\text{may}}(\{\alpha, \beta\})) \geq \omega \). If \( \eta = \xi + 1 \) and \( \mu < 3 \), then
If \( \eta \) is a limit number and \( \mu < 3 \), then

\[
f^j (\mu\alpha\beta) \leq \lim_{\xi < \eta} f^j (2^\xi) + 3^\varepsilon + 3^\varepsilon + \mu,
\]

where \( \lim_{\xi < \eta} f^j (2^\xi) \) is the limit of values \( f^j (2^\xi) \) for \( \xi (\xi < \eta) \).

1.3. Lemma. Let \( \alpha \) and \( \beta \) be ordinal numbers such that \( \alpha, \beta < \omega^{(\omega n)} \) and \( \eta (= \max (\{\alpha\beta\})) \geq \omega \). Then

\[
f^j (\mu\alpha\beta) < (\eta + 1)^\varepsilon.
\]

Proof. In case that \( \eta = \omega \), \( f^j (\mu\alpha\beta) \leq \omega \cdot 3 + \mu < (\omega + 1)^\varepsilon \). If \( \eta = \varepsilon + 1 \), then by Lemma 1.1 and Lemma 1.2

\[
f^j (\mu\alpha\beta) \leq \lim_{\xi < \eta} f^j (2^\xi) + 3^\varepsilon + 3^\varepsilon + \mu < \eta^\varepsilon + \eta \cdot 3 < (\eta + 1)^\varepsilon.
\]

If \( \eta \) is a limit ordinal number, then by Lemma 1.1 and Lemma 1.2

\[
f^j (\mu\alpha\beta) \leq \lim_{\xi < \eta} (f^j (2^\xi) + 3^\varepsilon + 3^\varepsilon + \mu) \leq \lim_{\xi < \eta} (\xi + 1)^\varepsilon + \eta \cdot 4 < (\eta + 1)^\varepsilon.
\]

1.4. Lemma. Let \( \alpha \) and \( \beta \) be ordinal numbers such that \( \alpha, \beta < \omega^{(\omega n)} \), \( n = 0, 1, 2, \ldots \). Then \( f^j (\mu\alpha\beta) < \omega^{(\omega n)} \).

Proof. In case that \( n = 0 \), we have Lemma in the same way in [2] p. 37, 9.26. Let \( \eta (= \max (\{\alpha, \beta\})) \geq \omega \) and \( \eta \) be of the form \( \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_l} \), \( \omega^n > \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_l \).

Then \( \eta + 1 \leq \omega^{\gamma_1} \cdot (l + 1) \) and so

\[
(\eta + 1)^\varepsilon \leq (\omega^{\gamma_1} \cdot (l + 1))^{\varepsilon + 1} = \omega^{\gamma_1 + 1} \cdot (l + 1)^{\gamma_1 + 1}.
\]

However \( \omega^n > \gamma_1 \). Hence \( \gamma_1 + 1 > \gamma_1 \cdot 1 + \gamma_1 = \omega^n \). Therefore \( (\eta + 1)^\varepsilon < \omega^{(\omega n)} \) and so we obtain Lemma by Lemma 1.3, q.e.d.

2. Following lemmas are derived from the definition of the function \( f \) and lemmas in 1 in this section.

2.1. Lemma. \( \omega \leq f^{(\omega)} \).

Proof. We prove by induction. \( 0 = f^0 \cdot 0 \in f^{(\omega)} \). We prove that \( n + 1 \in f^{(\omega)} \) under the assumption \( n \in f^{(\omega)} \). \( n + 1 = n + \{n\} = \omega \cdot (\{n\}, \{n\}) \). When \( n = f^m, \{n\} = f^j (1^m) \). Hence \( \{n\} \in f^j (1^m) \). Therefore
$\mathcal{S}(\{n, \{n\})=f^j j^\langle 2j^\langle 1mj^\langle 1mm\rangle, 0\rangle, \text{where } j^\langle 2j^\langle 1mj^\langle 1mm\rangle, 0\rangle \subsetneq \omega$
by Lemma 1.4.

2.2. Lemma. $\text{Comp}(f^\alpha)$ for every ordinal number $\alpha$ such that $\alpha \subsetneq \omega$. That is $(u)\ (u \in f^\alpha \cdot \supset u \subseteq f^\alpha)$.

Proof. It is sufficient to prove that $f^\alpha \subseteq f^\alpha$ for every ordinal number $\alpha$ such that $\alpha \subsetneq \omega$. Let $\alpha$ be the smallest ordinal number such that $f^\alpha \subseteq f^\alpha$.

When $k^\alpha=0$ and $\alpha \subsetneq \omega$, $f^\alpha=0 \subseteq f^\alpha$. When $\alpha \geq \omega$, $f^\alpha=\omega \subseteq f^\alpha$ by Lemma 2.1.

In case that $k^\alpha=1$, $f^\alpha\{f^k^1 \alpha, f^k^2 \alpha\}$. However $f^k^1 \alpha, f^k^2 \alpha \in f^\alpha$ from $k^\alpha, k^\alpha \subsetneq \alpha$. Hence $f^k^1 \alpha \subseteq f^\alpha$.

Let $k^\alpha=2$. Then $f^\alpha=\mathcal{S}(f^k^1 \alpha)$. As $k^\alpha \alpha \subsetneq \alpha$, $f^k^1 \alpha \subseteq f^\alpha \subseteq f^\alpha$ by the assumption of the induction. Therefore $v \in f^k^1 \alpha \cdot \supset (\exists \gamma)\ (v=f^\gamma \cdot \gamma \subsetneq k^\alpha)$. Hence $v=f^\gamma \subseteq f^\alpha \subseteq f^\alpha$ by the assumption of the induction. Hence $f^k^1 \alpha \subseteq \mathcal{S}(f^k^1 \alpha) \subseteq f^\alpha$.

2.3. Lemma. $\text{Comp}(v^\lambda)$, i.e. $(u \in v^\lambda \cdot \supset u \subseteq v^\lambda)$.

Proof. If $u \in v^\lambda$, we have an ordinal number $\alpha$ such that $\alpha \subsetneq \omega$ and $u=f^\alpha$. From Lemma 2.2

$$u=f^\alpha \subseteq f^\alpha \subseteq v^\lambda,$$
q.e.d.

2.4. Lemma. $\{a, b\} \in v^\lambda \iff a \in v^\lambda \cdot b \in v^\lambda$.

Proof. $\{a, b\} \in v^\lambda \cdot \supset a \in v^\lambda \cdot b \in v^\lambda$ is followed from Lemma 2.3. Now let $a \in v^\lambda$ and $b \in v^\lambda$ i.e. $a=f^\alpha$, $b=f^\beta$ and $\alpha, \beta \subsetneq \omega$. Then $\{a, b\}=f^j j^\langle 1\alpha \beta\rangle \in v^\lambda$, since $j^\langle 1\alpha \beta\rangle \subsetneq \omega$ from Lemma 1.4, q.e.d.

2.5. Lemma. $\langle ab\rangle \in v^\lambda \iff a \in v^\lambda \cdot b \in v^\lambda$.

This follows from Lemma 2.4.

2.6. Lemma. $v^2 \subseteq v^\lambda$.

This follows from Lemma 2.5.

2.7. Definition. The function $od$ is defined by the following postulate. $\langle xy\rangle \in od \iff \langle xy\rangle \in f^z [z \in y \cdot \supset z \in f \cdot od \subseteq v^2]$.

2.8. Lemma. If $x \in y$ and $x$, $y \in v^\lambda$, then $od^\alpha x \subsetneq od^\alpha y$.

Proof. Let $\alpha=od^\alpha x \beta=od^\alpha y$. From Lemma 2.2 $f^\alpha \alpha \subsetneq f^\alpha \beta \subseteq f^\alpha \beta$. Hence $\alpha \beta$, q.e.d.

2.9. Lemma. If $x \in v^\lambda$, then $\sim(x \in x)$.
Proof. If \( x \in x \), then \( \od x \prec \od x \) by Lemma 2.8, q.e.d.

2.10. Lemma. If \( \mathcal{M}_\lambda(x) \) and \( \mathcal{M}_\lambda(y) \), then \( \{xy\}_\lambda = \{xy\} \) and \( \langle xy \rangle_\lambda = \langle xy \rangle \). If \( \mathcal{G}_\lambda(A) \) and \( \mathcal{G}_\lambda(B) \), then \( \mathcal{G}_\lambda(A) = \mathcal{G}_\lambda(A) \), \( \mathcal{U}_\lambda(A) = \mathcal{U}_\lambda(A) \), and \( \mathcal{H}_\lambda(A, B) = \mathcal{H}_\lambda(A, B) \).

We can obtain the following lemmas from discussions in §5.

3.1. Lemma. \( f^\alpha \) is the form \( 0, \{f^\alpha_1, \ldots, f^\alpha_n\}, f^\omega \) or \( f^\omega + \{f^\alpha_1, \ldots, f^\alpha_n\} \).

3.2. Lemma. If \( f^\alpha = f^\omega \) and \( f^\beta \leq f^\alpha \), then \( f^\beta \in f^\omega \) or \( f^\beta = f^\omega \). If \( f^\alpha = f^\omega + \{f^\alpha_1, \ldots, f^\alpha_n\} \) and \( f^\beta \leq f^\alpha \), then \( f^\beta \in f^\omega \), \( f^\beta = f^\omega \) or \( f^\beta \) is of the form \( f^\omega + \{f^\alpha_1, \ldots, f^\alpha_n\} \).

§4. In this section we prove that the model \( \Lambda \) satisfies the axiom groups A, B, D, E and C1, C2 and C3.

Group \( A_\lambda \)

1. \((x) \mathcal{M}_\lambda(x) \supset \mathcal{C}_\lambda(x) \) is obtained from Lemma 2.3.
2. \((x)(y) \mathcal{C}_\lambda(x) \cdot \mathcal{C}_\lambda(y) \cdot x \in \lambda y : \supset \mathcal{M}_\lambda(x) \)
3. \((A)(B) \mathcal{C}_\lambda(A) \cdot \mathcal{C}_\lambda(B) \supset : (u) \mathcal{M}_\lambda(u) \supset : (u \in \lambda A = u \in \lambda B) \) : \( \supset \) : \( \mathcal{A} = B \) is obtained from the facts that \( \mathcal{A} \leq v_\lambda \) and \( \mathcal{B} \leq v_\lambda \), which are derived from \( \mathcal{C}_\lambda(A) \) and \( \mathcal{C}_\lambda(B) \).
4. \((x)(y) \mathcal{M}_\lambda(x) \cdot \mathcal{M}_\lambda(y) \supset : (\exists z) \mathcal{M}_\lambda(z) \cdot (u) \mathcal{M}_\lambda(u) \supset : u \in \lambda z \cdot \\
   \equiv : (u = x \lor u = y ))) \)

Proof. When \( x = f^\alpha \) and \( y = f^\beta \), \( f^\alpha \langle 1 \alpha \beta \rangle \) satisfies the formula, q.e.d.

Group \( B_\lambda \)

1. \( (\exists A) \mathcal{C}_\lambda(A) \cdot \mathcal{C}_\lambda(x) \cdot \mathcal{M}_\lambda(x) \supset : (\langle xy \rangle \in \lambda A \equiv \cdot \\
   x \in \lambda y) ) \)

Proof. \( E \cdot v_\lambda \) satisfies the formula, where \( E \) is the class by the axiom \( B_1 \), q.e.d.

2. \((A)(B) \mathcal{C}_\lambda(A) \cdot \mathcal{C}_\lambda(B) \supset : (\exists C) \mathcal{C}_\lambda(C) \cdot \mathcal{M}_\lambda(u) \supset : \\
   (u \in \lambda C \equiv : u \in \lambda A \cdot u \in \lambda B )) \)

Proof. \( A \cdot B \) satisfies \( B_2 \).

3. \((A) \mathcal{C}_\lambda(A) \supset (\exists B) \mathcal{C}_\lambda(B) \cdot \mathcal{M}_\lambda(u) \supset : (u \in \lambda B \equiv \cdot \\
   \sim(u \in \lambda A ))) \)
Proof. Consider $v_\lambda - A$ as $B$, where $v_\lambda - A$ is the complement of $A$ with respect to $v_\lambda$. Then $v_\lambda - A$ satisfies $B_3$, q.e.d.

4. (A) $[[\xi_\lambda(A) \supset (\exists B) (\xi_\lambda(B) \cdot (x) (\mathcal{I}_\lambda(x) \supset (x \in B \iff (\exists y)(\mathcal{I}_\lambda(y \cdot \langle yx \rangle_\lambda \in A))))]]$

Proof. This is equivalent to

$(A) [[\xi_\lambda(A) \supset (\exists B) (\xi_\lambda(B) \cdot (x) (\mathcal{I}_\lambda(x) \supset (x \in B \iff (\exists y)(\mathcal{I}_\lambda(y \cdot \langle yx \rangle \in A))))]]$

Consider $\mathcal{D}(A \cdot v^\lambda_\lambda)$ as $B$. Then $A \cdot v^\lambda_\lambda \subseteq v^\lambda_\lambda \cdot \mathcal{D}(A \cdot v^\lambda_\lambda) \subseteq v_\lambda$. Therefore $\xi_\lambda(\mathcal{D}(A \cdot v^\lambda_\lambda))$. Moreover

$$x \in \mathcal{D}(A \cdot v^\lambda_\lambda) \iff (\exists y) (\langle yx \rangle \in A \cdot v^\lambda_\lambda)$$
$$\iff (\exists y) (\langle yx \rangle \in A \iff (\exists y)(\mathcal{I}_\lambda(y \cdot \langle yx \rangle \in A)))$$
$$\iff (\exists y) (\mathcal{I}_\lambda(y \cdot \langle yx \rangle \in A))$$

q.e.d.

5. (A) $[[\xi_\lambda(A) \supset (\exists B) (\xi_\lambda(B) \cdot (x) (y) (\mathcal{I}_\lambda(x) \cdot \mathcal{I}_\lambda(y) \supset (\langle xy \rangle_\lambda \in A \iff x \in A)))]$

Proof. This is equivalent to

$(A) [[\xi_\lambda(A) \supset (\exists B) (\xi_\lambda(B) \cdot (x) (y) (\mathcal{I}_\lambda(x) \cdot \mathcal{I}_\lambda(y) \supset (\langle xy \rangle_\lambda \in A))]]$

Consider $v_\lambda \times A$ as $B$. Since $A \subseteq v_\lambda$ and $v^\lambda_\lambda \subseteq v_\lambda$, $v_\lambda \times A \subseteq v_\lambda$ and so $\xi_\lambda(v_\lambda \times A)$. $\langle xy \rangle \in v_\lambda \times A$ is equivalent to $y \in v_\lambda \cdot x \in A$. Therefore if $\mathcal{I}_\lambda(x)$ and $\mathcal{I}_\lambda(y)$, then $\langle xy \rangle \in v_\lambda \times A$ is equivalent to $x \in A$, q.e.d.

6. (A) $[[\xi_\lambda(A) \supset (\exists B) (\xi_\lambda(B) \cdot (x) (y) (\mathcal{I}_\lambda(x) \cdot \mathcal{I}_\lambda(y) \supset (\langle xy \rangle_\lambda \in A \iff x \in A))]]$

Proof. Take $A^{-1}$ (cf. [2], p. 15, 4.4) as $B$. $A^{-1} \subseteq v_\lambda$ and so $\xi_\lambda(A^{-1})$. Then it is clear that $A^{-1}$ satisfies $B_6$, q.e.d.

7. (A) $[[\xi_\lambda(A) \supset (\exists B) (\xi_\lambda(B) \cdot (x) (y) (z) (\mathcal{I}_\lambda(x) \cdot \mathcal{I}_\lambda(y) \cdot \mathcal{I}_\lambda(z) \supset (\langle xyz \rangle_\lambda \in A \iff \langle yzx \rangle_\lambda \in A))]]$

Proof. Take $\text{Cov}_x(A)$ (cf. [2], p. 15, 4.41) as $B$. $\text{Cov}_x(A) \subseteq v_\lambda$ and so $\xi_\lambda(\text{Cov}_x(A))$. Then $\text{Cov}_x(A)$ satisfies $B_7$, q.e.d.

8. (A) $[[\xi_\lambda(A) \supset (\exists B) (\xi_\lambda(B) \cdot (x) (y) (z) (\mathcal{I}_\lambda(x) \cdot \mathcal{I}_\lambda(y) \cdot \mathcal{I}_\lambda(z) \supset (\langle xyz \rangle_\lambda \in A \iff \langle xzy \rangle_\lambda \in A))]]$
Proof. Take $\text{Cov}_3(A)$ (cf. [2], p. 15, 4.411) as $B$. Then $\text{Cov}_3(A) \subseteq v_\lambda$ and so $\text{Cov}_3(\text{Cov}_3(A))$. Then $\text{Cov}_3(A)$ satisfies $B_\lambda$, q.e.d.

Group $C_\lambda$.

1. $\langle 3a \rangle [\langle 3a \rangle \cdot \text{Cov}_3(a) \cdot (x) (\langle 3a \rangle) \cdot x \in \lambda a \cdot \geq (\exists y) (\langle 3a \rangle) \cdot y \in \lambda a \cdot x <_\lambda y]]$.

Proof. This is equivalent to

$$\langle 3a \rangle [\langle 3a \rangle \cdot \text{Cov}_3(a) \cdot (x) (\langle 3a \rangle) \cdot x \in \lambda a \cdot \geq (\exists y) (\langle 3a \rangle) \cdot y \in \lambda a \cdot x < y]]].$$

Take $\omega$ as $a \cdot f^\omega = \omega$ and so $\langle 3a \rangle (\omega)$. Then $\omega$ satisfies $C_1\lambda$, q.e.d.

2. $\langle x \rangle [\langle 3a \rangle \cdot \geq (\exists y) (\langle 3a \rangle) \cdot u (\langle 3a \rangle) \cdot \geq : u \in \lambda y \cdot \equiv . (\exists v) (\langle 3a \rangle) \cdot u \in \lambda v \cdot v \in \lambda x]]].$

Proof. Consider $\langle x \rangle$ as $y$. Then $\langle 3a \rangle \cdot \geq (\exists y) (\langle 3a \rangle) \cdot u$ for $u$ such that $\langle 3a \rangle (u)$

$$u \in \lambda \langle x \rangle \cdot \equiv . u \in \langle x \rangle \cdot \equiv . (\exists y) (u \in y \cdot y \in x) .$$

When $u \in y$, $y \in x \in v_\lambda \cdot \geq . y \in v_\lambda$. Therefore

$$u \in \lambda \langle x \rangle \cdot \equiv . (\exists y) (\langle 3a \rangle) \cdot u \in \lambda y \cdot y \in x ,$$

$q.e.d.$

3. $\langle x \rangle [\langle 3a \rangle \cdot \geq (\exists y) (\langle 3a \rangle) \cdot u (\langle 3a \rangle) \cdot \geq : u \in \lambda y \cdot \equiv . u \subseteq \lambda y)]].$

We easily obtain $C_3\lambda$ from Lemma 3.1 and 3.2 in § 3.

Axiom $D_\lambda$. (A) $[\langle 3a \rangle \cdot \text{Cov}_3(a) \cdot \geq (\exists u) (\langle 3a \rangle) \cdot u \in \lambda A \cdot \text{Cov}_3(u) \cdot (u, A)]]$.

Proof. Since $\text{Cov}_3(A)$ and $\text{Cov}_3(A)$, $A \subseteq x_\lambda$ and $\langle 3y \rangle (y \in A \subseteq v_\lambda)$. Hence we consider that with the smallest order of such $y$. Let it be $u$. Then $\langle 3a \rangle (u)$ as $u \in A \subseteq v_\lambda$. Now if $x \in u \cdot A$, then $\text{od}^d x \subseteq \text{od}^d u$. This contradicts the definition of $u$.

$q.e.d.$

Axiom $E_\lambda$. (A) $[\langle 3a \rangle \cdot (x) (\langle 3a \rangle) \cdot \geq (\exists y) (\langle 3a \rangle) .$

$$y \in \lambda x \cdot \langle yx \rangle \in \lambda A)]].$$

Proof. We define the relation $\text{As}$ by the formula: $\langle yx \rangle \in \text{As} . \equiv .$$

$$y \in x \cdot (z) [\text{od}^d z \subseteq \text{od}^d y \subseteq . \sim z \in x] : \text{Rel}(\langle x \rangle) \cdot \text{As} \subseteq v_\lambda^2.$$

Then $\text{As}$ satisfies axiom $E_\lambda$, q.e.d.

§ 5. In this section we prove that the model $\Lambda$ does not satisfy the axiom $C_4\lambda$ (Zermelo’s ‘Aussonderungssaxiom’). This is implied by $C_4\lambda'$ (Fraenkel’s axiom of substitution). Therefore, of course, the model $\Lambda$ does not satisfy the axiom $C_4\lambda$. 
1. Some Lemmas.

1.1. Lemma. \( \mathfrak{S}(\omega) = \omega \).

1.2. Lemma. If \( \omega \subseteq f^i\alpha \), then \( \omega \subseteq \mathfrak{S}(f^i\alpha) \).

Proof. If \( \omega \subseteq f^i\alpha \), then \( \mathfrak{S}(f^i\alpha) = \mathfrak{S}(f^i\alpha - \omega) + \mathfrak{S}(\omega) = \mathfrak{S}(f^i\alpha - \omega) + \omega \geq \omega \) by Lemma 1.1, q.e.d.

1.3. Lemma. \( (u) (\exists \lambda \leq \omega . \lambda > (3 \iota \leq \omega . \iota ^< \omega . \iota ^\leq \omega \land t) \), where \( a = b \) means that sets \( a \) and \( b \) are equivalent (cf. [2], p. 30, 8.13).

Proof. Let \( u = f^i\alpha \). We prove by transfinite induction on \( \alpha \). If \( \alpha = 0 \), then it is clear. If \( k^i\alpha = 0 \) and \( \alpha < \omega \), then it is clear. In case that \( k^i\alpha = 0 \) and \( \alpha \geq \omega \), then \( f^i\alpha = \omega \geq \omega \). In case that \( k^i\alpha = 1 \), \( f^i\alpha = \{ f^i k^i\alpha \} \) and so \( f^i \alpha = 2 \). However
\[
2 = \{ 0, \{ 0 \} \} = \langle 1, 0, \langle 1, 00 \rangle \rangle
\]

In case that \( k^i\alpha = 2 \), \( f^i\alpha = \mathfrak{S}(f^i k^i\alpha) \) and \( k^i\alpha < \alpha \). Now \( (x) (x \in f^i k^i\alpha \land \lambda > (\exists \iota \leq \omega . \iota ^< \omega \land \iota ^\leq \omega \land t) \).

First if \( \omega \subseteq f^i k^i\alpha \), then \( \omega \subseteq f^i\alpha \). Second let \( \sim \omega \subseteq f^i k^i\alpha \). Then \( (\exists \iota \leq \omega . \iota = f^i k^i\alpha) \).

If \( (x) (x \in f^i k^i\alpha \land \sim \omega \subseteq x) \), then \( (x) (\exists j) (x \in f^i k^i\alpha \land j \leq x) \) and so \( (E \iota) (i \leq \omega . j = f^i\alpha) \).

If \( (\exists x) (x \in f^i k^i\alpha \land \omega \subseteq x) \), then \( \omega \subseteq \mathfrak{S}(f^i k^i\alpha) = f^i\alpha \), q.e.d.

1.4. Lemma. The \( \omega - \{ 0 \} \) does not belong to the model \( \Lambda \).

Proof. From Lemma 1.4, if \( \mathfrak{M}_\lambda (\omega - \{ 0 \}) \), then
\[ (\exists \iota \leq \omega . \iota \leq \omega - \{ 0 \}) \lor \omega \leq \omega - \{ 0 \} . \]

However \( (i) (i \leq \omega . \lambda > (i \leq \omega - \{ 0 \}) \) and \( (\omega \leq \omega - \{ 0 \}) \). Therefore \( \sim \mathfrak{M}_\lambda (\omega - \{ 0 \}) \), q.e.d.

2. Theorem. The axiom \( C4'_\lambda \) does not hold.

Proof. We have the formula
\[ (x) [\mathfrak{M}_\lambda (x) \land (x \in \omega - \{ 0 \} . \implies x \in \omega . \sim x = 0) . \]

Therefore, if the axiom \( C4'_\lambda \) holds, then \( \mathfrak{M}_\lambda (\omega - \{ 0 \}) \) and this contradicts Lemma 1.4, q.e.d.

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References

