Nishimura, Toshio
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Note on Axiomatic Set Theory I.
The Independence of Zermelo's "Aussonderungsaxiom" from Other Axioms of Set Theory

By Toshio NISHIMURA

In this paper axioms of set theory mean Gödel's axioms of set theory in [2], which are modified so that the axiom C 2 postulates directly the existence of all elements of a given set and C 3 the existence of power set.

In what follows we prove that Zermelo's "Aussonderungsaxiom" is independent of other axioms of set theory. A fortiori, the independence of Gödel's axiom C 4 (Fraenkel's "axiom of substitution"), is proved since the axiom of substitution implies the "Aussonderungsaxiom".

The proof is carried out by constructing an inner model Λ for the axioms A, B, C 1, C 2, C 3, D and E under the axioms A, B, C 1, C 2 and C 4, which does not satisfy the "Aussonderungsaxiom". The idea appears already in [1]. However the proof in [1] is not formal.

In § 1, we give the axioms of set theory.

In § 2, we construct an inner model Λ under the axioms A, B, C 1, C 2 and C 4. Here we notice that the axioms do not imply the existence of power set of any set.

In § 3, we prove some preliminary results with respect to the model Λ.

In § 4, we prove that the model Λ satisfies the axioms A, B, C 1, 2, 3, D and E.

In § 5, we prove that the model Λ does not satisfy the "Aussonderungsaxiom".

§ 1. The axiom system of set theory

In what follows we apply Gödel's notations in [2] in most cases. For logical notations we use following symbols, $\vee$ (or), $\cdot$ (and), $\sim$ (not), $\supset$ (implies), $\equiv$ (equivalence), $=$ (identity), $(\exists X)$ (there is an X), (X) (for all X) and $(\exists ! X)$ (there is exactly one X). The system has three primitive notions: class, denoted by $\mathfrak{Cls}$; set, denoted by $\mathfrak{M}$; and the diadic relation $\in$ between class and class, class and set, set and class, or set and set. Variables for classes are denoted by capital Latin letters A, B, X, Y etc. and those for sets by small Latin letters a, b, x, y etc. The axioms fall into five groups A, B, C, D, E.

Group A.

1. $(x)\,\mathfrak{Cls}\,(x)$
2. $(X)\,(Y)\,[X \in Y . \supset . \mathfrak{M}(X)]$
3. $(X)\,(Y)\,[(u)\,(u \in X . \equiv . u \in Y) . \supset . X = Y]$
4. $(x)\,(y)\,(\exists z)\,(u)\,[u \in z . \equiv : u = x . \vee . u = y]$

Group B.

1. $(\exists A)\,(x)\,(y)\,[\langle xy \rangle \in A . \equiv . x \in y]$
2. $(A)\,(B)\,(\exists C)\,(u)\,[u \in C . \equiv : u \in A . u \in B]$
3. $(A)\,(\exists B)\,(u)\,[u \in B . \equiv . \sim (u \in A)]$
4. $(A)\,(\exists B)\,(x)\,[x \in B . \equiv . (\exists y)\,(\langle yx \rangle \in A)]$
5. $(A)\,(\exists B)\,(x)\,(y)\,[\langle xy \rangle \in B . \equiv . x \in A]$
6. $(A)\,(\exists B)\,(x)\,(y)\,[\langle yx \rangle \in B . \equiv . \langle yx \rangle \in A]$
7. $(A)\,(\exists B)\,(x)\,(y)\,(z)\,[\langle xyz \rangle \in B . \equiv . \langle yzx \rangle \in A]$
8. $(A)\,(\exists B)\,(x)\,(y)\,(z)\,[\langle xyz \rangle \in B . \equiv . \langle xzy \rangle \in A]$

where $\langle xy \rangle$ and $\langle xyz \rangle$ mean the ordered pair of x and y and the ordered triple of x, y and z (cf. [2], p. 4, 1. 12 and 1. 14).

Group C.

1. $(\exists a)\,[\sim \mathfrak{Em}(a) . (x)\,(x \in a . \supset . (\exists y)\,(y \in a . x \subset y))]$
2. $(x)\,(\exists y)\,(u)\,[u \in y . \equiv . (\exists v)\,(u \in v . v \in x)]$
3. $(x)\,(\exists y)\,(u)\,[u \in y . \equiv . (w)\,(w \in u . \supset . w \in x)]$
4. $(x)\,(A)\,[\mathfrak{Un}(A) . \supset . (\exists y)\,(u)\,(u \in y . \equiv . (\exists v)\,(v \in x . \langle uv \rangle \in A))]$

where the notions $a \subset b$, $\mathfrak{Em}(a)$ and $\mathfrak{Un}(A)$ mean "a is a proper subset of b", "a is empty" and "A is unique" respectively (cf. [2], p. 4, 1. 2, 1. 22 and p. 5, 1. 3).

Zermelo's "Aussonderungsaxiom" is postulated as follows:

4′. $$(x)\,(A)\,(\exists y)\,(u)\,[u \in y . \equiv : u \in x . u \in A]$$

Axiom D.

$$(A)\,[\sim \mathfrak{Em}(A) . \supset . (\exists u)\,(u \in A . \mathfrak{Ex}(u, A))]$$

where the notion $\mathfrak{Ex}(u, A)$ means "u and A are exclusive" (cf. [2], p. 4, (1. 23).

Axiom E.

$$(\exists A)\,[\mathfrak{Un}(A)\,.\,(x)\,(\sim\mathfrak{Em}(x)\,.\supset.\,(\exists y)\,(y\in x\,.\,\langle yx\rangle\in A))]$$

§ 2. The model Λ

In this section we construct the model Λ under the axioms A, B, C 1, 2 and 4.

1. PRELIMINARY DEFINITIONS

We develop the theory of ordinal numbers under the axioms A, B, C 1, 2 and 4. (It is well known that is possible.) Ordinal numbers are denoted by small Greek letters α, β, γ etc. When α and β are ordinal numbers, $\alpha\circledast\beta$ and α^{β} are the product and power in arithmetic of ordinal numbers. The existence of these functions are easily ensured. ω is the smallest infinite ordinal number.

We can define the function j such that the following conditions are satisfied:

(1) $j\mathfrak{Fn}3\times\omega^{(\omega^{\omega})}\times\omega^{(\omega^{\omega})}$,

where 1, 2, and 3 are $0+\{0\}$, $1+\{1\}$ and $2+\{2\}$ respectively, and $A\times B$ is the direct product of A and B (cf. [2], p. 14, 4. 1).

(2) $\mu,\ \nu<3\,.\supset.\,(\langle\mu\alpha\beta\rangle S\langle\nu\gamma\delta\rangle\,.\supset.\,j`\langle\mu\alpha\beta\rangle<j`\langle\nu\gamma\delta\rangle)$

where S is the relation given in [2], p. 36, 9. 2, in which 9 is replaced by 3. And $A`x$ is the function given in [2] p. 16, 4. 65.

(3) $\mathfrak{W}(j)$ is an ordinal number, where $\mathfrak{W}(j)$ is the set of all values of j (cf. [2] p. 15, 4. 44).

Then it is clear that the function j is a set.

Then in the similar way as in [2] p. 36, 9. 24 we can easily give the functions (to be a set) k_0, k_1 and k_2 which satisfy the following: if $\mu,\ \nu<4$, then

$$k_0`j`\langle\mu\alpha\beta\rangle=\mu\,,\quad k_1`j`\langle\mu\alpha\beta\rangle=\alpha\quad\text{and}\quad k_2`j`\langle\mu\alpha\beta\rangle=\beta\,,$$
$$k_0`\alpha=0\vee k_0`\alpha=1\vee k_0`\alpha=2\,,$$
$$j`\langle k_0`\alpha,\ k_1`\alpha,\ k_2`\alpha\rangle=\alpha\,,$$

and if $0<\mu<3$ then

$$k_i`j`\langle\mu\alpha\beta\rangle<j`\langle\mu\alpha\beta\rangle\qquad i=1,\ 2\,.$$

The existence of such functions are easily ensured.

2. First we define the function 'f' by transfinite induction simultaneously, which is defined over ω^{ω}.

$$k_0{}^{\backprime}\alpha = 1 . \supset . f^{\backprime}\alpha = \{f^{\backprime}k_1{}^{\backprime}\alpha,\ f^{\backprime}k_2{}^{\backprime}\alpha\}$$
$$k_0{}^{\backprime}\alpha = 2 . \supset . f^{\backprime}\alpha = \mathfrak{S}(f^{\backprime}k_1{}^{\backprime}\alpha)$$
$$k_0{}^{\backprime}\alpha = 0 . \alpha < \omega . \supset : f^{\backprime}\alpha = 0$$
$$k_0{}^{\backprime}\alpha = 0 . \alpha \geqq \omega . \supset : f^{\backprime}\alpha = \omega$$

where $\{a, b\}$ is the non-ordered pair of a and b, and $\mathfrak{S}(a)$ the sum of all elements of a.

Now we give the model Λ. The universal class of the model Λ is $\mathfrak{W}(f)$ which is a set denoted by v_λ, i.e.

$$v_\lambda = \mathfrak{W}(f) .$$

$\mathfrak{M}_\lambda(a)$ means that a is a set of the model Λ and is given by the formula

$$\mathfrak{M}_\lambda(a) . \equiv . a \in v_\lambda .$$

$\mathfrak{Cls}_\lambda(A)$ means that A is a class of the model Λ and is given by the formula

$$\mathfrak{Cls}_\lambda(A) . \equiv . A \subseteq v_\lambda .$$

$\in_\lambda$ means the $\in$-relation in the model Λ and is defined by the formula

$$A \in_\lambda B . \equiv : A \in B . \mathfrak{M}_\lambda(A) . \mathfrak{Cls}_\lambda(B) .$$

Then operations and notions can be relativized for the model Λ in the similar way as in [2], p. 41 and p. 42. The relativization of an operation A is denoted by A_λ and that of a notion B by B_λ.

§ 3. Preliminary results

In this section we prove some lemmas with respect to the model Λ.

1. Following lemmas are concerning arithmetic of ordinal numbers.

1. 1. Lemma. *Let η be an ordinal number such that $\omega \leqq \eta$ and n such that $n < \omega$. Then*

$$(\eta \dot{+} 1)^2 = \eta^2 \dot{+} \eta \dot{+} 1$$
$$(\eta \dot{+} 1)^3 = \eta^3 \dot{+} \eta^2 \dot{+} \eta \dot{+} 1 .$$

And in case that $\eta \in K_{\mathrm{II}}$, $n \circledast \eta = \eta$ and in case that $\eta = \xi \dot{+} m$ where $\xi \in K_{\mathrm{II}}$ and $m < \omega$, $n \circledast \eta = \xi \dot{+} mn < \eta \circledast 2$.

1. 2. Lemma. *Let α and β be ordinal numbers such that $\alpha, \beta < \omega^{(\omega^\omega)}$ and $\eta (= \mathrm{may}\,(\{\alpha, \beta\})) \geqq \omega$. If $\eta = \xi \dot{+} 1$ and $\mu < 3$, then*

$$j'\langle\mu\alpha\beta\rangle \leqq j'\langle 2\xi\xi\rangle \dot{+} 3\circledast\eta \dot{+} 3\circledast\eta \dot{+} \mu\,.$$

If η is a limit number and $\mu < 3$, then

$$j'\langle\mu\alpha\beta\rangle \leqq \underset{\xi<\eta}{\mathfrak{Lim}}\, j'\langle 3\xi\xi\rangle \dot{+} 4\circledast\eta \dot{+} 4\circledast\eta \dot{+} \mu\,,$$

where $\underset{\xi<\eta}{\mathfrak{Lim}}\, j'\langle 2\xi\xi\rangle$ is the limit of values $j'\langle 2\xi\xi\rangle$ for $\xi(<\eta)$.

1. 3. Lemma. *Let α and β be ordinal numbers such that $\alpha, \beta < \omega^{(\omega^\omega)}$ and $\eta(=\max(\{\alpha\beta\})) \geqq \omega$. Then*

$$j'\langle\mu\alpha\beta\rangle < (\eta \dot{+} 1)^3\,.$$

Proof. In case that $\eta=\omega$, $j'\langle\mu\alpha\beta\rangle \leqq \omega\circledast 3 \dot{+} \mu < (\omega \dot{+} 1)^3$. If $\eta=\xi \dot{+} 1$, then by Lemma 1. 1 and Lemma 1. 2

$$j'\langle\mu\alpha\beta\rangle \leqq j'\langle 2\xi\xi\rangle \dot{+} 3\circledast\eta \dot{+} 3\circledast\eta \dot{+} \mu < \eta^3 + \eta\circledast 3 < (\eta \dot{+} 1)^3\,.$$

If η is a limit ordinal number, then by Lemma 1. 1 and Lemma 1. 2

$$j'\langle\mu\alpha\beta\rangle \leqq \underset{\xi<\eta}{\mathfrak{Lim}}\,(j'\langle 2\xi\xi\rangle) \dot{+} 3\circledast\eta \dot{+} 3\circledast\eta \dot{+} \mu \leqq \underset{\xi<\eta}{\mathfrak{Lim}}\,((\xi \dot{+} 1)^3)$$
$$\dot{+}\,\eta\circledast 4 \leqq \eta^3 + \eta\circledast 4 < (\eta \dot{+} 1)^3\,.$$

1. 4. Lemma. *Let α and β be ordinal numbers such that $\alpha, \beta < \omega^{(\omega^n)}$, $n=0, 1, 2, \cdots$. Then $j'\langle\mu\alpha\beta\rangle < \omega^{(\omega^n)}$.*

Proof. In case that $n=0$, we have Lemma in the same way in [2] p. 37, 9. 26. Let $\eta(=\mathrm{Max}(\{\alpha, \beta\})) \geqq \omega$ and η be of the form

$$\omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_l}\,, \quad \omega^n > \gamma_1 \geqq \gamma_2 \geqq \cdots \geqq \gamma_l\,.$$

Then $\eta \dot{+} 1 \leqq \omega^{\gamma_1}\circledast(l \dot{+} 1)$ and so

$$(\eta \dot{+} 1)^3 \leqq (\omega^{\gamma_1}\circledast(l \dot{+} 1))\circledast(\omega^{\gamma_1}\circledast(l \dot{+} 1))\circledast(\omega^{\gamma_1}\circledast(l \dot{+} 1))$$
$$\leqq \omega^{\gamma_1 \dot{+} 1}\circledast\omega^{\gamma_1 \dot{+} 1}\circledast\omega^{\gamma_1 \dot{+} 1} = \omega^{\gamma_1 \dot{+} 1 \dot{+} \gamma_1 \dot{+} 1 \dot{+} \gamma_1 \dot{+} 1}\,.$$

However $\omega^n > \gamma_1$. Hence $\gamma_1 \dot{+} 1 \dot{+} \gamma_1 \dot{+} 1 \dot{+} \gamma_1 \dot{+} 1 < \omega^n$. Therefore $(\eta \dot{+} 1)^3 < \omega^{\omega^n}$ and so we obtain Lemma by Lemma 1. 3, q.e.d.

2. Following lemmas are derived from the definition of the function f and lemmas in 1 in this section.

2. 1. Lemma. $\omega \subseteq f''\omega$.

Proof. We prove by induction. $0=f'0 \in f''\omega$. We prove that $n \dot{+} 1 \in f''\omega$ under the assumption $n \in f''\omega$. $n \dot{+} 1 = n + \{n\} = \mathfrak{S}(\{n, \{n\}\})$. When $n=f'm$, $\{n\}=f'j'\langle 1mm\rangle$. Hence $\{n, \{n\}\}=f'j'\langle 1mj'\langle 1mm\rangle\rangle$. Therefore

$\mathfrak{S}(\{n, \{n\}\}) = f`j`\langle 2j`\langle 1mj`\langle 1mm\rangle\rangle, 0\rangle$, where $j`\langle 2j`\langle 1mj`\langle 1mm\rangle\rangle, 0\rangle < \omega$ by Lemma 1.4, q.e.d.

2.2. Lemma. $\mathfrak{Comp}(f``\alpha)$ *for every ordinal number* $\alpha$ *such that* $\alpha < \omega^\omega$. *That is* $(u)(u \in f``\alpha . \supset . u \subseteq f``\alpha)$.

Proof. It is sufficient to prove that $f`\alpha \subseteq f``\alpha$ for every ordinal number $\alpha$ such that $\alpha < \omega^\omega$. Let $\alpha$ be the smallest ordinal number such that $f`\alpha \not\subseteq f``\alpha$.

When $k_0`\alpha = 0$ and $\alpha < \omega$, $f`\alpha = 0 \subseteq f``\alpha$. When $\alpha \geqq \omega$, $f`\alpha = \omega \subseteq f``\omega$ by Lemma 2.1.

In case that $k_0`\alpha = 1$, $f`\alpha = \{f`k_1`\alpha, f`k_2`\alpha\}$. However $f`k_1`\alpha$, $f`k_2`\alpha \in f``\alpha$ from $k_1\alpha$, $k_2\alpha < \alpha$. Hence $f`\alpha \subseteq f``\alpha$.

Let $k_0`\alpha = 2$. Then $f`\alpha = \mathfrak{S}(f`k_1`\alpha)$. As $k_1`\alpha < \alpha$, $f`k_1`\alpha \subseteq f``k_1`\alpha$ by the assumption of the induction. Therefore $v \in f`k_1`\alpha . \supset . (\exists\gamma)(v = f`\gamma . \gamma < k_1`\alpha)$. Hence $v = f`\gamma \subseteq f``\gamma \subseteq f``k_1`\alpha$ by the assumption of the induction. Hence $f`k_1`\alpha = \mathfrak{S}(f`k_1`\alpha) \subseteq f``k_1`\alpha \subseteq f``\alpha$.

2.3. Lemma. $\mathfrak{Comp}(v_\lambda)$, *i.e.* $(u)(u \in v_\lambda . \supset . u \subseteq v_\lambda)$.

Proof. If $u \in v_\lambda$, we have an ordinal number α such that $\alpha < \omega^\omega$ and $u = f`\alpha$. From Lemma 2.2

$$u = f`\alpha \subseteq f``\alpha \subseteq v_\lambda, \qquad \text{q.e.d.}$$

2.4. Lemma. $\{a, b\} \in v_\lambda . \equiv : a \in v_\lambda . b \in v_\lambda$.

Proof. $\{a, b\} \in v_\lambda . \supset : a \in v_\lambda . b \in v_\lambda$ is followed from Lemma 2.3. Now let $a \in v_\lambda$ and $b \in v_\lambda$ i.e. $a = f`\alpha$, $b = f`\beta$ and $\alpha, \beta < \omega^\omega$. Then $\{a, b\} = f`j`\langle 1\alpha\beta\rangle \in v_\lambda$, since $j`\langle 1\alpha\beta\rangle < \omega^\omega$ from Lemma 1.4, q.e.d.

2.5. Lemma. $\langle ab\rangle \in v_\lambda . \equiv : a \in v_\lambda . b \in v_\lambda$.

This follows from Lemma 2.4.

2.6. Lemma. $v_\lambda^2 \subseteq v_\lambda$.

This follows from Lemma 2.5.

2.7. Definition. The function *od* is defined by the following postulate. $\langle yx\rangle \in od . \equiv : \langle xy\rangle \in f . (z)[z \in y . \supset . \sim\langle xz\rangle \in f] . od \subseteq v_\lambda^2$.

2.8. Lemma. *If* $x \in y$ *and* $x, y \in v_\lambda$, *then* $od`x < od`y$.

Proof. Let $\alpha = od`x$ $\beta = od`y$. From Lemma 2.2 $f`\alpha \in f`\beta \subseteq f``\beta$. Hence $\alpha < \beta$, q.e.d.

2.9. Lemma. *If* $x \in v_\lambda$, *then* $\sim(x \in x)$.

Proof. If $x \in x$, then $od`x < od`x$ by Lemma 2.8, q.e.d.

2.10. Lemma. *If* $\mathfrak{M}_\lambda(x)$ *and* $\mathfrak{M}_\lambda(y)$, *then* $\{xy\}_\lambda = \{xy\}$ *and* $\langle xy\rangle_\lambda = \langle xy\rangle$. *If* $\mathfrak{Cls}_\lambda(A)$ *and* $\mathfrak{Cls}_\lambda(B)$, *then* $\mathfrak{Em}_\lambda(A) \equiv \mathfrak{Em}(A)$, $\mathfrak{Un}_\lambda(A) \equiv \mathfrak{Un}(A)$ *and* $\mathfrak{Ex}_\lambda(A, B) \equiv \mathfrak{Ex}(A, B)$.

3. We can obtain the following lemmas from discussions in §5.

3.1. Lemma. *$f`\alpha$ is the form 0,* $\{f`\alpha_1, \cdots, f`\alpha_n\}$, $f``\omega$ or $f``\omega + \{f`\alpha_1` \cdots, f`\alpha_n\}$.

3.2. Lemma. *If* $f`\alpha = f``\omega$ *and* $f`\beta \leq f`\alpha$, *then* $f`\beta \in f``\omega$ *or* $f`\beta = f``\omega$. *If* $f`\alpha = f``\omega + \{f`\alpha_1, \cdots, f`\alpha_n\}$, *and* $f`\beta \leq f`\alpha$, *then* $f`\beta \in f``\omega$, $f`\beta = f``\omega$ *or* $f`\beta$ *is of the form* $f``\omega + \{f`\alpha_{i_1}, \cdots, f`\alpha_{i_m}\}$.

§4. In this section we prove that the model Λ satisfies the axiom groups A, B, D, E and C 1, C 2 and C 3.

Group A_λ

1_λ. $(x)[\mathfrak{M}_\lambda(x) \supset \mathfrak{Cls}_\lambda(x)]$ is obtained from Lemma 2.3.

2_λ. $(x)(y)[\mathfrak{Cls}_\lambda(x) . \mathfrak{Cls}_\lambda(y) . x \in_\lambda y : \supset . \mathfrak{M}_\lambda(x)]$

3_λ. $(A)(B)[\mathfrak{Cls}_\lambda(A) . \mathfrak{Cls}_\lambda(B) . \supset : (u)(\mathfrak{M}_\lambda(u) \supset (u \in_\lambda A \equiv u \in_\lambda B))] : \supset : . A = B]$ is obtained from the facts that $A \leq v_\lambda$ and $B \leq v_\lambda$

which are derived from $\mathfrak{Cls}_\lambda(A)$ and $\mathfrak{Cls}_\lambda(B)$.

4_λ. $(x)(y)[\mathfrak{M}_\lambda(x) . \mathfrak{M}_\lambda(y) . \supset : (\exists z)(\mathfrak{M}_\lambda(z) . (u)(\mathfrak{M}_\lambda(u) . \supset : u \in_\lambda z . \equiv . (u = x \vee u = y)))]$

Proof. When $x = f`\alpha$ and $y = f`\beta$, $f`j`\langle 1\alpha\beta\rangle$ satisfies the formula, q.e.d.

Group B_λ

1_λ. $(\exists A)[\mathfrak{Cls}_\lambda(A) . (x)(y)(\mathfrak{M}_\lambda(x) . \mathfrak{M}_\lambda(y) . \supset : (\langle xy\rangle_\lambda \in_\lambda A . \equiv . x \in_\lambda y))]$

Proof. $E \cdot v_\lambda$ satisfies the formula, where E is the class by the axiom B_1, q.e.d.

2_λ. $(A)(B)[\mathfrak{Cls}_\lambda(A) . \mathfrak{Cls}_\lambda(B) . \supset . (\exists C)(\mathfrak{Cls}_\lambda(C) . (u)(\mathfrak{M}_\lambda(u) . \supset : (u \in_\lambda C . \equiv : u \in_\lambda A . u \in_\lambda B)))]$

Proof. $A \cdot B$ satisfies B 2_λ.

3_λ. $(A)[\mathfrak{Cls}_\lambda(A) \supset (\exists B)(\mathfrak{Cls}_\lambda(B) . (u)(\mathfrak{M}_\lambda(u) . \supset : (u \in_\lambda B . \equiv . \sim(u \in_\lambda A)))]$

Proof. Consider $v_\lambda - A$ as B, where $v_\lambda - A$ is the complement of A with respect to v_λ. Then $v_\lambda - A$ satisfies B 3_λ, q.e.d.

4_λ. $(A)\,[\mathfrak{Cls}_\lambda(A)\,.\supset.\,(\exists B)\,(\mathfrak{Cls}_\lambda(B)\,.\,(x)\,(\mathfrak{Mt}_\lambda(x)\,.\supset.\,(x\in_\lambda B\,.\equiv:$
$(\exists y)(\mathfrak{Mt}_\lambda(y)\,.\,\langle yx\rangle_\lambda\in_\lambda A))))]$

Proof. This is equivalent to

$$(A)\,[\mathfrak{Cls}_\lambda(A)\,.\supset.\,(\exists B)\,(\mathfrak{Cls}_\lambda(B)\,.\,(x)\,(\mathfrak{Mt}_\lambda(x)\,.\supset.\,(x\in B\,.\equiv:$$
$$(\exists y)\,(\mathfrak{Mt}_\lambda(y)\,.\,\langle yx\rangle\in A))))]$$

Consider $\mathfrak{D}(A\cdot v_\lambda^2)$ as B. Then $A\cdot v_\lambda^2\subseteq v_\lambda^2\,.\,\mathfrak{D}(A\cdot v_\lambda^2)\subseteq v_\lambda$. Therefore $\mathfrak{Cls}_\lambda(\mathfrak{D}(A\cdot v_\lambda^2))$. Moreover

$$\begin{aligned} x\in\mathfrak{D}(A\cdot v_\lambda^2)\,.&\equiv.\,(\exists y)\,(\langle yx\rangle\in A\cdot v_\lambda^2)\\ &\equiv.\,(\exists y)\,(\langle yx\rangle\in A\,.\,\langle yx\rangle\in v_\lambda^2)\\ &\equiv.\,(\exists y)\,(\mathfrak{Mt}_\lambda(y)\,.\,\langle yx\rangle\in A)\,. \end{aligned}$$
q.e.d.

5_λ. $(A)\,[\mathfrak{Cls}_\lambda(A)\,.\supset.\,(\exists B)\,(\mathfrak{Cls}_\lambda(B)\,.\,(x)\,(y)\,(\mathfrak{Mt}_\lambda(x)\,.\,\mathfrak{Mt}_\lambda(y)\,.\supset.$
$(\langle yx\rangle_\lambda\in_\lambda B\,.\equiv.\,x\in_\lambda A)))]$.

Proof. This is equivalent to

$$(A)\,[\mathfrak{Cls}_\lambda(A)\,.\supset.\,(\exists B)\,(\mathfrak{Cls}_\lambda(B)\,.\,(x)\,(y)\,(\mathfrak{Mt}_\lambda(x)\,.\,\mathfrak{Mt}_\lambda(y)\,.\supset.$$
$$(\langle yx\rangle_\lambda\in_\lambda B\,.\equiv.\,x\in_\lambda A)))].$$

Consider $v_\lambda\times A$ as B. Since $A\subseteq v_\lambda$ and $v_\lambda^2\subseteq v_\lambda$, $v_\lambda\times A\subseteq v_\lambda$ and so $\mathfrak{Cls}_\lambda(v_\lambda\times A)$. $\langle yx\rangle\in v_\lambda\times A$ is equivalent to $y\in v_\lambda\,.\,x\in A$. Therefore if $\mathfrak{Mt}_\lambda(x)$ and $\mathfrak{Mt}_\lambda(y)$, then $\langle yx\rangle\in v_\lambda\times A$ is equivalent to $x\in A$, q.e.d.

6_λ. $(A)\,[\mathfrak{Cls}_\lambda(A)\,.\supset.\,(\exists B)\,(\mathfrak{Cls}_\lambda(B)\,.\,(x)\,(y)\,(\mathfrak{Mt}_\lambda(x)\,.\,\mathfrak{Mt}_\lambda(y)\,.\supset.$
$(\langle xy\rangle_\lambda\in_\lambda B\,.\equiv.\,\langle yx\rangle_\lambda\in_\lambda A)))]$.

Proof. Take A^{-1} (cf. [2], p. 15, 4. 4) as B. $A^{-1}\subseteq v_\lambda$ and so $\mathfrak{Cls}_\lambda(A^{-1})$. Then it is clear that A^{-1} satisfies B 6_λ. q.e.d.

7_λ. $(A)\,[\mathfrak{Dls}_\lambda(A)\,.\supset.\,(\exists B)\,(\mathfrak{Cls}_\lambda(B)\,.\,(x)\,(y)\,(z)\,(\mathfrak{Mt}_\lambda(x)\,.$
$\mathfrak{Mt}_\lambda(y)\,.\,\mathfrak{Mt}_\lambda(z)\,.\supset.\,(\langle xyz\rangle_\lambda\in_\lambda B\,.\equiv.\,\langle yzx\rangle_\lambda\in_\lambda A)))]$.

Proof. Take $\mathfrak{Cnv}_2(A)$ (cf. [2], p. 15, 4. 41) as B. $\mathfrak{Cnv}_2(A)\subseteq v_\lambda$ and so $\mathfrak{Cls}_\lambda(\mathfrak{Cnv}_2(A))$. Then $\mathfrak{Cnv}_2(A)$ satisfies B 7_λ, q.e.d.

8_λ. $(A)\,[\mathfrak{Cls}_\lambda(A)\,.\supset.\,(\exists B)\,(\mathfrak{Cls}_\lambda(B)\,.\,(x)\,(y)\,(z)\,(\mathfrak{Mt}_\lambda(x)\,.\,\mathfrak{Mt}_\lambda(y)$
$\mathfrak{Mt}_\lambda(z)\,.\supset.\,(\langle xyz\rangle_\lambda\in_\lambda B\,.\equiv.\,\langle xzy\rangle_\lambda\in_\lambda A)))]$.

Proof. Take $\mathfrak{Cod}_3(A)$ (cf. [2], p. 15, 4.411) as B. $\mathfrak{Cod}_3(A) \subseteq v_\lambda$ and so $\mathfrak{Cls}_\lambda(\mathfrak{Cod})_3(A))$. Then $\mathfrak{Cod}_3(A)$ satisfies B 8_λ, q.e.d.

Group C_λ.

1_λ. $(\exists a)[\mathfrak{M}_\lambda(a) . \sim \mathfrak{Em}_\lambda(a) . (x)(\mathfrak{M}_\lambda(x) . x \in_\lambda a : \supset . (\exists y)(\mathfrak{M}_\lambda(y) . y \in_\lambda a . x \subset_\lambda y))]$.

Proof. This is equivelent to

$$(\exists a)[\mathfrak{M}_\lambda(a) . \sim \mathfrak{Em}(a) . (x)(\mathfrak{M}_\lambda(x) . x \in a : \supset . (\exists y)(\mathfrak{M}_\lambda(y) . y \in a . x \subset y))] .$$

Take ω as $a . f`\omega = \omega$ and so $\mathfrak{M}_\lambda(\omega)$. Then ω satisfies C 1_λ, q.e.d.

2_λ. $(x)[\mathfrak{M}_\lambda(x) . \supset . (\exists y)(\mathfrak{M}_\lambda(y) . (u)(\mathfrak{M}_\lambda(u) . \supset : u \in_\lambda y . \equiv . (\exists v)(\mathfrak{M}_\lambda(v) . u \in_\lambda v . v \in_\lambda x)))]$.

Proof. Consider $\mathfrak{S}(x)$ as y. Then $\mathfrak{M}_\lambda(x) . \supset . \mathfrak{M}_\lambda(\mathfrak{S}(x))$. For u such that $\mathfrak{M}_\lambda(u)$

$$u \in_\lambda \mathfrak{S}(x) . \equiv . u \in \mathfrak{S}(x) . \equiv . (\exists y)(u \in y . y \in x) .$$

When $u \in y$, $y \in x \in v_\lambda . \supset . y \in v_\lambda$. Therefore

$$\begin{aligned} u \in_\lambda \mathfrak{S}(x) . &\equiv . (\exists y)(\mathfrak{M}_\lambda(y) . u \in y . y \in x) \\ &. \equiv . (\exists y)(\mathfrak{M}_\lambda(y) . u \in_\lambda y . y \in_\lambda x) , \end{aligned}$$ q.e.d.

3_λ. $(x)[\mathfrak{M}_\lambda(x) . \supset . (\exists y)(\mathfrak{M}_\lambda(y) . (u)(\mathfrak{M}_\lambda(u) . \supset . (u \in y . \equiv . u \subseteq_\lambda y)))]$

We easily obtain C 3_λ from Lemma 3.1 and 3.2 in § 3.

Axiom D_λ. $(A)[\mathfrak{Cls}_\lambda(A) . \sim \mathfrak{Em}_\lambda(A) : \supset . (\exists u)(\mathfrak{M}_\lambda(u) . u \in_\lambda A . \mathfrak{Ex}_\lambda(u . A))]$

Proof. Since $\mathfrak{Cls}_\lambda(A)$ and $\sim \mathfrak{Em}(A)$, $A \subseteq v_\lambda$ and $(\exists y)(y \in A \subseteq v_\lambda)$. Hence we consider that with the smallest order of such y. Let it be u. Then $\mathfrak{M}_\lambda(u)$ as $u \in A \subseteq v_\lambda$. Now if $x \in u . A$, then $od`x < od`u$. This contradicts the definition of u, q.e.d.

Axiom E_λ. $(\exists A)[\mathfrak{Cls}_\lambda(A) . (x)(\mathfrak{M}_\lambda(x) . \sim \mathfrak{Em}_\lambda(x) . \supset : (\exists y)(\mathfrak{M}_\lambda(y) . y \in_\lambda x . \langle yx \rangle_\lambda \in_\lambda A))]$.

Proof. We define the relation As by the formula: $\langle yx \rangle \in As . \equiv : y \in x . (z)[od`z < od`y . \subset . \sim z \in x] : \mathfrak{Rel}(As) . As \subseteq v_\lambda^2$. Then As satisfies axiom E_λ, q.e.d.

§ **5.** In this section we prove that the model Λ does not satisfy the axiom C 4_λ (Zermelo's 'Aussonderungsaxiom'). This is implied by C $4'_\lambda$ (Fraenkel's axiom of substitution). Therefore, of course, the model Λ does not satisfy the axiom C 4_λ.

1. Some Lemmas.

1.1. Lemma. $\mathfrak{S}(\omega)=\omega$.

1.2. Lemma. *If* $\omega \subseteq f`\alpha$, *then* $\omega \subseteq \mathfrak{S}(f`\alpha)$.

Proof. If $\omega \subseteq f`\alpha$, then $\mathfrak{S}(f`\alpha)=\mathfrak{S}(f`\alpha-\omega)+\mathfrak{S}(\omega)=\mathfrak{S}(f`\alpha-\omega)+\omega \supseteq \omega$ by Lemma 1.1, q.e.d.

1.3. Lemma. $(u)[\mathfrak{M}_\lambda(u) . \supset : ((\exists i)(i<\omega . i \simeq u) \vee \omega \subseteq u)]$, *where* $a \simeq b$ *means that sets a and b are equivalent* (*cf.* [2], *p.* 30, 8.13).

Proof. Let $u=f`\alpha$. We prove by transfinite induction on $\alpha$. If $\alpha=0$, then it is clear. If $k_0`\alpha=0$ and $\alpha<\omega$, then it is clear. In case that $k_0`\alpha=0$ and $\alpha \geqq \omega$, then $f`\alpha=\omega \supseteq \omega$. In case that $k_0`\alpha=1$, $f`\alpha=\{f`k_1`\alpha, f`k_1`\alpha\}$ and so $f`\alpha \simeq 2$. However

$$2=\{0, \{0\}\}=j`\langle 1, 0, j`\langle 100\rangle\rangle$$

In case that $k_0`\alpha=2$, $f`\alpha=\mathfrak{S}(f`k_1`\alpha)$ and $k_1`\alpha<\alpha$. Now $(x)(x \in f`k_1`\alpha . \supset . \mathfrak{M}_\lambda(x))$.

First if $\omega \subseteq f`k_1`\alpha$, then $\omega \subseteq f`\alpha$. Second let $\sim \omega \subseteq f`k_1`\alpha$. Then $(\exists i)(i<\omega . i \simeq f`k_1`\alpha)$.

If $(x)(x \in f`k_1`\alpha \supset . \sim \omega \subseteq x)$, then $(x)(\exists j)(x \in f`k_1`\alpha . \supset . j \simeq x)$ and so $(Ej)(j<\omega . j \simeq f`\alpha)$.

If $(\exists x)(x \in f`k_1`\alpha . \omega \subseteq x)$, then $\omega \subseteq \mathfrak{S}(f`k_1``\alpha)=f``\alpha$, q.e.d.

1.4. Lemma. *The* $\omega-\{0\}$ *does not belong to the model* Λ.

Proof. From Lemma 1.4, if $\mathfrak{M}_\lambda(\omega-\{0\})$, then

$$(\exists i)(i<\omega . i \simeq \omega-\{0\}) \vee \omega \subseteq \omega-\{0\} .$$

However $(i)(i<\omega . \supset . \sim(i \simeq \omega-\{0\}))$ and $\sim(\omega \subseteq \omega-\{0\})$. Therefore $\sim \mathfrak{M}_\lambda(\omega-\{0\})$, q.e.d.

2. Theorem. *The axiom* C $4'_\lambda$ *does not hold.*

Proof. We have the formula

$$(x)[\mathfrak{M}_\lambda(x) . \supset . (x \in \omega-\{0\} . \equiv . x \in \omega . \sim x=0)] .$$

Therefore, if the axiom C $4'_\lambda$ holds, then $\mathfrak{M}_\lambda(\omega-\{o\})$ and this contradicts Lemma 1.4, q.e.d.

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