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SECOND ORDER TYPE-CHANGING EQUATIONS FOR A SCALAR FUNCTION ON A PLANE

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Abstract

In this paper, we consider type-changing equations for one unknown function of two variables by using the theory of differential systems. We give fundamental properties and provide a notion of geometric solutions from a viewpoint of contact geometry of second order. Moreover, we study the structure of associated overdetermined systems and obtain an existence condition of solutions of a special class which are called parabolic solutions of type-changing equations.

1. Introduction

Let $J^2(\mathbb{R}^2, \mathbb{R})$ be the 2-jet space:

$$J^2(\mathbb{R}^2, \mathbb{R}) := \{(x, y, z, p, q, r, s, t)\}. \tag{1}$$

This space has the canonical differential system (or higher order contact system) $C^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ given by the following 1-forms:

$$\varpi_0 := dz - p\,dx - q\,dy,$$
$$\varpi_1 := dp - r\,dx - s\,dy,$$
$$\varpi_2 := dq - s\,dx - t\,dy.$$

In general, by a differential system (M, D), we mean a distribution D on a manifold M, that is, D is a subbundle of the tangent bundle TM of M. Under the canonical system C^2, we have the identification $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$ with respect to the one unknown function $z = z(x, y)$ of two independent variables x, y. On the 2-jet space, we consider PDEs (i.e. partial differential equations) of the form:

$$F(x, y, z, p, q, r, s, t) = 0, \tag{2}$$

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where F is a smooth function on $J^2(\mathbb{R}^2, \mathbb{R})$. We set

$$\Sigma = \{F = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R})$$

and restrict the canonical differential system C^2 to Σ. We denote it by D $(:= C^2|_\Sigma)$. We consider a PDE $\Sigma = \{F = 0\}$ with the condition

$$(F_r, F_s, F_t) \neq (0, 0, 0), \tag{3}$$

which we will call the regularity condition. Then, Σ is a smooth hypersurface, and also the restriction $\pi|_\Sigma \colon \Sigma \to J^1(\mathbb{R}^2, \mathbb{R})$ of the natural projection $\pi \colon J^2(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$ is a submersion. Due to the property, restricted 1-forms $\varpi_i|_\Sigma$ on Σ are linearly independent. Hence, we have the induced differential system $D = \{\varpi_0|_\Sigma = \varpi_1|_\Sigma = \varpi_2|_\Sigma = 0\}$ on Σ. Then, D is a vector bundle of rank 4 on Σ. For brevity, we denote each restricted generator 1-form $\varpi_i|_\Sigma$ of D by ϖ_i in the following. For such an equation $F = 0$, we consider the following discriminant:

$$\Delta := F_r F_t - \frac{1}{4} {F_s}^2. \tag{4}$$

DEFINITION 1.1. Let $\Sigma = \{F = 0\}$ be a smooth hypersurface of $J^2(\mathbb{R}^2, \mathbb{R})$. For the discriminant $\Delta = F_r F_t - (1/4){F_s}^2$ of F, a point $w \in \Sigma$ is said to be hyperbolic or elliptic if $\Delta(w) < 0$ or $\Delta(w) > 0$, respectively. Moreover, a point $w \in \Sigma$ is said to be parabolic if $(F_r, F_s, F_t)_w \neq (0, 0, 0)$ and $\Delta(w) = 0$.

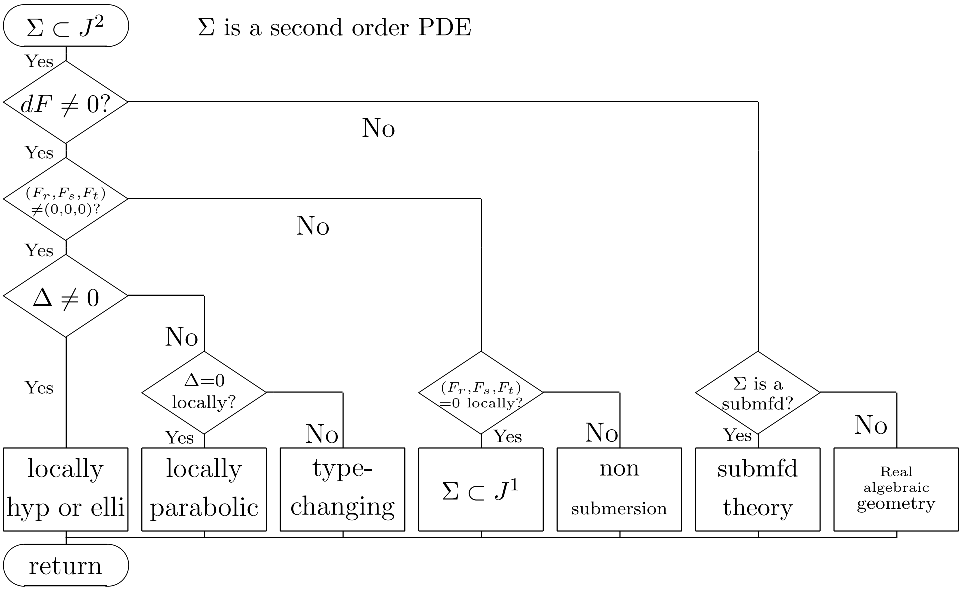
We take the subset $\Sigma_p := \Sigma \cap \{\Delta = 0\}$ of Σ. Then, the condition $\Sigma_p = \Sigma$ means that Σ is locally parabolic, and also the condition $\Sigma_p = \emptyset$ means that Σ is locally hyperbolic or locally elliptic. Then, we define a notion of type-changing equations as follows.

DEFINITION 1.2. Let Σ be a second order regular PDE. If Σ_p is a proper subset of Σ, we call Σ type-changing equation.

In this paper, we consider the following problem.

PROBLEM 1.3. For type-changing equations (2), investigate a local behavior (or degeneration) of associated differential systems around parabolic points.

The notion of these type-changing equations has been already introduced by Clelland, Kossowski and Wilkens in [3]. They considered a special class of Monge–Ampère equations which is called symplectic Monge–Ampère equations, and studied type-changing equations belonging to these Monge–Ampère equations by using a notion of intermediate integrals. Compared with their work, we will study the geometric structure of type-changing equations from a viewpoint of contact geometry of second order. More precisely,



our purpose is to formulate geometry of type-changing equations as a theory of submanifold in the second jet space $J^2(\mathbb{R}^2, \mathbb{R})$.

This paper is organized as follows. In Section 2, we give fundamental results and provide a notion of solutions from a viewpoint of second order. In Section 3, we study regular overdetermined systems associated with type-changing equations to construct solutions of a special class which are called parabolic solutions. Consequently, we obtain a classification of the structure equation of induced regular overdetermined systems (Theorem 3.3). This result involves the involutiveness condition of associated regular overdetermined systems. Hence, we can discuss solutions of original type-changing equations by using this condition. In Section 4, we consider type-changing equations of some special forms, and clarify properties of these special equations (Theorem 4.1, Theorem 4.5, Theorem 4.6).

In the rest of this section, we explain the position of type-changing equations in the category of second order PDEs of single type for two independent one dependent variables. See the above figure. In this figure, categories of right hand side from type-changing equation are still precisely unknown. However, in nonsubmersion category, there exists a result [6] on a characterization of such equations via differential systems. There are uncharted territories for second order PDEs of single type. Our aim is to clarify this figure from a viewpoint of contact geometry of second order.

2. Examples and fundamental properties

In this section, we mention fundamental properties, and give model examples. We first introduce an invariant of a second order PDE $\Sigma = \{F = 0\}$ which is defined in the paper [6] to discuss fundamental properties of type-changing equations. Let $\Sigma = \{F = 0\}$ be a smooth hypersurface of $J^2(\mathbb{R}^2, \mathbb{R})$ with the condition $dF \neq 0$. We fix a base point $w \in \Sigma$. For any open neighborhood U of w in Σ, U is decomposed as follows:

$$U = U_h \cup U_e \cup U_p \cup U_{sing} \quad \text{(disjoint union)}, \tag{5}$$

where the components are given by

$$\begin{aligned} U_h &:= \{v \in U \mid v \text{ is hyperbolic}\}\text{: hyperbolic type},\\ U_e &:= \{v \in U \mid v \text{ is elliptic}\}\text{: elliptic type},\\ U_p &:= \{v \in U \mid v \text{ is parabolic}\}\text{: parabolic type},\\ U_{sing} &:= U \setminus (U_h \cup U_e \cup U_p). \end{aligned}$$

For each component U_i $(i = h, e, p, sing)$, the equivalence relation $w_1 \sim w_2$ $(w_1, w_2 \in U_i)$ is defined as follows:

$$\text{There exist a continuous curve } c\colon [0, 1] \to U_i \quad \text{s.t.} \quad c(0) = w_1,\ c(1) = w_2. \tag{6}$$

We denote the number of elements of the quotient space $U_i/\sim$ consists of path-connected components by $\#(U_i/\sim)$.

Now, we fix a diffeomorphism $J^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^8$. Then, the standard metric on $\mathbb{R}^8$ induce a metric on $J^2(\mathbb{R}^2, \mathbb{R})$ by using the diffeomorphism. By a diffeomorphism $\phi\colon J^2(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^8$, we define the following induced norm $|\cdot|$:

$$|p - q| := \|\phi(p) - \phi(q)\|, \quad p, q \in J^2,$$

where $\|\cdot\|$ is the Euclidean norm. We choose the following neighborhood.

$$U := B_\varepsilon(w) = \{v \in \Sigma \mid |v - w| < \varepsilon\},$$

where $|\cdot|$ is the restriction of the norm on $J^2(\mathbb{R}^2, \mathbb{R})$ to Σ. We decompose U similar to (5):

$$U = B_\varepsilon^H(w) \cup B_\varepsilon^E(w) \cup B_\varepsilon^P(w) \cup B_\varepsilon^{Sing}(w) \quad \text{(disjoint union)}. \tag{7}$$

Then, we consider the numbers:

$$\begin{aligned} H_{|\cdot|}(w) &:= \lim_{\varepsilon \to 0} \#(B_\varepsilon^H(w)/\sim),\\ E_{|\cdot|}(w) &:= \lim_{\varepsilon \to 0} \#(B_\varepsilon^E(w)/\sim),\\ P_{|\cdot|}(w) &:= \lim_{\varepsilon \to 0} \#(B_\varepsilon^P(w)/\sim),\\ S_{|\cdot|}(w) &:= \lim_{\varepsilon \to 0} \#(B_\varepsilon^{Sing}(w)/\sim). \end{aligned}$$

We remark that if a limit does not exist, then we set that the number is ∞. These numbers do not depend on the contact isomorphisms of (Σ, D), but depend on the metric on $J^2(\mathbb{R}^2, \mathbb{R})$ under the identification $J^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^8$. Hence, we define the following invariant of (Σ, D) at w.

DEFINITION 2.1. We set

$$\begin{aligned} H(w) &:= \min_{J^2 \cong \mathbb{R}^8} \Big(\lim_{\varepsilon \to 0} \#(B_\varepsilon^H(w)/\sim) \Big), \\ E(w) &:= \min_{J^2 \cong \mathbb{R}^8} \Big(\lim_{\varepsilon \to 0} \#(B_\varepsilon^E(w)/\sim) \Big), \\ P(w) &:= \min_{J^2 \cong \mathbb{R}^8} \Big(\lim_{\varepsilon \to 0} \#(B_\varepsilon^P(w)/\sim) \Big), \\ S(w) &:= \min_{J^2 \cong \mathbb{R}^8} \Big(\lim_{\varepsilon \to 0} \#(B_\varepsilon^{Sing}(w)/\sim) \Big), \end{aligned}$$

where the minimum is taken over all diffeomorphisms ϕ. Moreover, we set

$$(H, E, P, S)_w := (H(w), E(w), P(w), S(w)).$$

The value $(H, E, P, S)_w$ does not depend on the identification $J^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^8$. Thus, it is an invariant of (Σ, D) at w.

REMARK 2.2. For regular PDEs Σ treated in this paper, values of S are zero.

We start to discuss type-changing equations. The equation $r = 0$ is a normal form of locally parabolic equations. Now, if we consider deformed equations $r = f(x, y, z, p, q, s, t)$, then there exist examples of type-changing equations. Moreover, in fact, all of second order regular PDEs can be written in this form by using the implicit function theorem and contact transformations. Thus, it is sufficient to research the equations of the form $r = f(x,y,z,p,q,s,t)$. In this case, the discriminant is $\Delta = -(f_t + f_s{}^2/4)$, and we have the expression $\Sigma_p = \{ f_s{}^2 + 4 f_t = 0\} \subset \Sigma$. We assume the following essential condition to study type-changing equations via differential systems.

ASSUMPTION 2.3. A subset Σ_p of a type-changing equation Σ is a smooth submanifold.

In general, there are two cases for type-changing equations.

(A) $$d\Delta \neq 0 \quad \text{on} \quad \Sigma_p,$$

(B) $$d\Delta = 0 \quad \text{on} \quad \Sigma_p.$$

We have the following result with respect to the case (A).

Theorem 2.4. *For a second order regular PDE* $\Sigma = \{F = 0\}$, *if* 1*-forms* dF, $d\Delta$ *are linearly independent on* Σ_p, *then* Σ *is a type-changing PDE, and* Σ_p *is a smooth hypersurface of* Σ. *Moreover, for each point* $v \in \Sigma_p$, *type-changing of* Σ *around* v *is given by using our invariant* $(H, E, P, S)_v$ *as follows*:

$$(H, E, P, S)_v = (1, 1, 1, 0) \quad \textit{for} \quad v \in \Sigma_p.$$

Proof. Since 1-forms dF, $d\Delta$ are linearly independent, Σ_p is a smooth hypersurface of Σ, and $\Sigma \setminus \Sigma_p$ has two connected components U_1, U_2 around a point v. Now, signatures of Δ on these components are one of the followings:
(i) Both of U_1, U_2 are hyperbolic.
(ii) Both of U_1, U_2 are elliptic.
(iii) One of the components is hyperbolic, and another is elliptic.
We assume that conditions (i) or (ii) are satisfied for U_1, U_2. We fix any point $v \in \Sigma_p$. Moreover, we take any curve $c\colon [-1, 1] \to \Sigma$ satisfying the following condition. This curve is transverse to Σ_p and satisfies $c(0) = v$. If we regard Δ as a function over this curve $c(t)$, then we have $d\Delta = 0$ at a point $v = c(0)$ from the condition (i) or (ii). However, this fact contradicted the linearly independence of dF and $d\Delta$. □

Now, we introduce regular overdetermined systems which is very important class in overdetermined systems.

DEFINITION 2.5. For smooth functions F, G on $J^2(\mathbb{R}^2, \mathbb{R})$, the system of PDE:

$$F(x, y, z, p, q, r, s, t) = G(x, y, z, p, q, r, s, t) = 0,$$

is called overdetermined system. Moreover, an overdetermined system $\Sigma = \{F = G = 0\}$ is regular if the following condition is satisfied:

Two vectors (F_r, F_s, F_t) and (G_r, G_s, G_t) are linearly independent.

Under this definition, for the case of (A), if vectors (F_r, F_s, F_t) and $(\Delta_r, \Delta_s, \Delta_t)$ are linearly independent, then Σ_p becomes a second order regular overdetermined system.

REMARK 2.6. A geometric theory of second order regular overdetermined systems is developed by E. Cartan, K. Yamaguchi, etc. ([2], [14]). Therefore, we can appropriate their results when Σ_p is a regular overdetermined system. Indeed, we will study regular overdetermined systems Σ_p associated with type-changing equations Σ to construct solutions of Σ which are called parabolic solutions in Section 3.

On the other hand, the case of (B) gives large degeneration. Indeed, there is a possibility that $\dim \Sigma_p = i$ for $0 \le i \le 6$. A submanifold Σ_p is a base point w_0 when

$\dim \Sigma_p = 0$. For the case of (B), type-changing equations are divided into the following subclasses:
(B-i) $\dim \Sigma_p = 6$
(B-i-i) $\Sigma \setminus \Sigma_p$ is hyperbolic, (i.e. $(H, E, P, S)_w = (2, 0, 1, 0)$).
(B-i-ii) $\Sigma \setminus \Sigma_p$ is elliptic, (i.e. $(H, E, P, S)_w = (0, 2, 1, 0)$).
(B-i-iii) $\Sigma \setminus \Sigma_p$ has hyperbolic and elliptic components, (i.e. $(H, E, P, S)_w = (1, 1, 1, 0)$).
(B-ii) $\dim \Sigma_p \leq 5$
(B-ii-i) $\Sigma \setminus \Sigma_p$ is hyperbolic, (i.e. $(H, E, P, S)_w = (1, 0, 1, 0)$).
(B-ii-ii) $\Sigma \setminus \Sigma_p$ is elliptic, (i.e. $(H, E, P, S)_w = (0, 1, 1, 0)$).

Proposition 2.7. *For the above described each class of* (B), *typical examples are the followings. In particular, all classes are not empty.*

EXAMPLE 2.8 (Case A). We consider the equation $\Sigma := \{F := xt + r = 0\}$. This equation is a regular PDE, because $(F_r, F_s, F_t) = (1, 0, x)$. Hence, (Σ, D) is a differential system. Note that, $\Delta = x$, and $\Sigma_p = \{x = 0\}$. For $v \in \Sigma$,

$$v \quad \text{is hyperbolic for} \quad x < 0,$$
$$v \quad \text{is parabolic for} \quad x = 0,$$
$$v \quad \text{is elliptic for} \quad x > 0.$$

EXAMPLE 2.9 (Case B-i-i). Take an integer $n \geq 1$. We consider $\Sigma := \{F := x^n s + r = 0\}$. From a calculation, $(F_r, F_s, F_t) = (1, x^n, 0)$ is satisfied. Hence, Σ is a regular PDE, and (Σ, D) is a differential system. The discriminant is $\Delta = -x^{2n}/4$ and $\Sigma_p = \{x = 0\}$. Moreover we have $d\Delta = 0$ on Σ_p and $\Sigma \setminus \Sigma_p$ is a hyperbolic part.

EXAMPLE 2.10 (Case B-i-ii). Take an integer $n \geq 1$. We consider $\Sigma := \{F := x^{2n} t + r = 0\}$. Since $(F_r, F_s, F_t) = (1, 0, x^{2n})$ is satisfied, (Σ, D) is a differential system. We also have $\Delta = x^{2n}$ and $\Sigma_p = \{x = 0\}$. Moreover we have $d\Delta = 0$ on Σ_p and $\Sigma \setminus \Sigma_p$ is an elliptic part.

EXAMPLE 2.11 (Case B-i-iii). Take an integer $n \geq 1$. We consider $\Sigma := \{F := x^{2n+1} t + r = 0\}$. Since $(F_r, F_s, F_t) = (1, 0, x^{2n+1})$ is satisfied, (Σ, D) is a differential system. We also have $\Delta = x^{2n+1}$ and $\Sigma_p = \{x = 0\}$. Moreover we have $d\Delta = 0$ on Σ_p. By the form of discriminant, type-changing happens associated with a signature of x.

For the case of $\dim \Sigma_p \leq 5$, we give examples which contain examples for the cases of (B-ii-i) and (B-ii-ii).

EXAMPLE 2.12 (Case B-ii-i). Consider the equation $\Sigma := \{F := t(a_1x^2 + a_2y^2 + a_3z^2 + a_4p^2 + a_5q^2 + (1/3)a_6t^2) + a_7s^2 - r = 0\}$, where $(a_1, \dots, a_7) \neq (0, \dots, 0)$ and $a_i \geq 0$. This is also a regular PDE. We have $\Delta = -(a_1x^2 + a_2y^2 + a_3z^2 + a_4p^2 + a_5q^2 + a_6t^2 + a_7{}^2s^2)$ and $\Sigma_p = \{a_1x = a_2y = a_3z = a_4p = a_5q = a_6t = a_7s = 0\}$. We define a number j satisfying $0 \leq j \leq 6$ as follows. Let j be the number of 0 in coefficients a_i. Then, we have $\dim \Sigma_p = j$. For example, if all of a_i are not zero, then $\Sigma_p = \{0 \in J^2\}$ is an only one point. The complement $\Sigma \setminus \Sigma_p$ is a hyperbolic part, and we also have $d\Delta = 0$ on Σ_p.

EXAMPLE 2.13 (Case B-ii-ii). Consider the equation $\Sigma := \{F := t(a_1x^2 + a_2y^2 + a_3z^2 + a_4p^2 + a_5q^2 + (1/3)a_6t^2 + a_7) + r + 2\sqrt{a_7}\sin s = 0\}$, where $(a_1, \dots, a_7) \neq (0, \dots, 0)$ and $a_i \geq 0$. This is also a regular PDE. Moreover, we have $\Delta = (a_1x^2 + a_2y^2 + a_3z^2 + a_4p^2 + a_5q^2 + a_6t^2) - a_7(\cos^2 s - 1)$. For example, if all of a_i are not zero, then $\Sigma_p = \{0 \in J^2\}$ is an origin and the complement is an elliptic part. We also have a relation $d\Delta = 0$ on Σ_p.

Now, we provide a notion of solutions (i.e. integral manifolds) of second order PDEs in the sense of contact geometry of second order. In general, integral manifolds L of differential systems (Σ, D) are defined as 2-dimensional submanifolds L such that $TL \subset D$, that is, pull-back of generator 1-forms of D to L vanish.

DEFINITION 2.14. Let (Σ, D) be a second order regular PDE. For a 2-dimensional integral manifold L of Σ, if a restriction of a natural projection $\pi\colon J^2 \to J^1$ to L is an immersion on an open dense subset in L, then we call L a geometric solution of (Σ, D). If all points of geometric solutions L are immersion point, then we call L regular solutions. On the other hand, geometric solutions L have a nonimmersion point, then we call L singular solutions. In particular, when we consider type-changing equations Σ, if solutions L of Σ are subsets of Σ_p, then we call L parabolic solutions.

REMARK 2.15. From the definition, images $\pi(L)$ of geometric solutions L by the projection π are Legendrian in $J^1(\mathbb{R}^2, \mathbb{R})$, $(\varpi_0|_{\pi(L)} = d\varpi_0|_{\pi(L)} = 0)$.

EXAMPLE 2.16 (regular solution). Consider the regular PDE $\Sigma = \{yt - 2q + (1/3)r + 2xy = 0\}$. From the discriminant $\Delta = (1/3)y$, we have $\Sigma_p = \{y = 0\}$. A 1-form $d\Delta = (1/3)dy$ does not vanish on Σ_p. Hence this equation is a type-changing equation of the case of (A). Then, we consider a submanifold $L = \{(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy})\}$ defined by a function of two variables $z(x, y) = x^3 + y^3 + xy^2 + xy$. This submanifold satisfies an equality $F = yt - 2q + (1/3)r + 2xy = y(6y + 2x) - 2(3y^2 + 2xy + x) + 6x/3 + 2xy = 0$. Moreover, the projection of L to J^1 is an immersion. Hence this solution is a regular solution which is transverse to Σ_p. Thus, this is not a parabolic solution.

EXAMPLE 2.17 (singular solution). Consider the regular PDE $\Sigma = \{r - 2st = 0\}$. From the discriminant $\Delta = -(2s + t^2)$, this is a type-changing equation of the case of (A). Then, we consider a submanifold L given by

$$
\begin{aligned}
&x = x, \quad y = t^3 - 3xt, \quad z = \frac{9}{28}t^7 - \frac{27}{20}xt^5 + \frac{3}{2}x^2t^7,\\
&p = \frac{9}{10}t^5 - \frac{3}{2}xt^3, \quad q = \frac{3}{4}t^4 - \frac{3}{2}xt^2,\\
&r = 3t^3, \quad s = \frac{3}{2}t^3, \quad t = t.
\end{aligned}
$$

This submanifold L is an integral manifold of (Σ, D), and $\pi(L)$ has a singular point at $0 \in J^2$. Thus, this solution which is called cuspidal edge is a singular solution. This is also not a parabolic solution.

3. Submanifolds Σ_p as regular overdetermined systems

In the previous section, we introduced a notion of solutions of second order regular PDEs. In particular, we defined a notion of parabolic solutions for type-changing equations. Parabolic solutions of type-changing equations are obtained by solutions of associated regular overdetermined systems. Thus, we study regular overdetermined systems associated with type-changing equations in this section. By Theorem 2.4, for a type-changing equation $\Sigma = \{F = 0\}$, if 1-forms dF, $d\Delta$ are linearly independent, then $\Sigma_p = \{F = 0, \Delta = 0\}$ is a smooth hypersurface of Σ. Moreover, if we assume that given $F = 0$ is of the form $r = f(x, y, z, p, q, s, t)$ and the function $\Delta = -(f_t + {f_s}^2/4)$ on Σ satisfies $(\Delta_s, \Delta_t) \neq (0, 0)$, then Σ_p is a regular overdetermined system. Recall that results given by Cartan, and we clarify relations between these results and regular overdetermined systems Σ_p associated with type-changing equations.

According to the result given by E. Cartan ([2], [14]), if a rank 3 differential system $D_p := D|_{\Sigma_p}$ on a regular overdetermined system Σ_p does not have torsion, then the structure equation of the system D_p is one of the following three types at each point.
(i) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{22}\}$ such that

$$
\begin{aligned}
d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0,\\
d\varpi_1 &\equiv 0 && \text{mod } \varpi_0, \varpi_1, \varpi_2,\\
d\varpi_2 &\equiv \qquad\qquad\quad \omega_2 \wedge \pi_{22} && \text{mod } \varpi_0, \varpi_1, \varpi_2.
\end{aligned} \tag{8}
$$

(ii) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{12}\}$ such that

$$
\begin{aligned}
d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0,\\
d\varpi_1 &\equiv \qquad\qquad\quad \omega_2 \wedge \pi_{12} && \text{mod } \varpi_0, \varpi_1, \varpi_2,\\
d\varpi_2 &\equiv \omega_1 \wedge \pi_{21} && \text{mod } \varpi_0, \varpi_1, \varpi_2,
\end{aligned} \tag{9}
$$

where $\pi_{12} = \pi_{21}$.

(iii) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{11}\}$ such that

$$
\begin{aligned}
d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0, \\
d\varpi_1 &\equiv \omega_1 \wedge \pi_{11} && \text{mod } \varpi_0, \varpi_1, \varpi_2, \\
d\varpi_2 &\equiv \qquad\qquad \omega_2 \wedge \pi_{22} && \text{mod } \varpi_0, \varpi_1, \varpi_2,
\end{aligned} \tag{10}
$$

where $\pi_{11} = \pi_{22}$.

REMARK 3.1. In the above structure equations, if a term $f\omega_1 \wedge \omega_2 = g\, dx \wedge dy$ for functions f, g appears, then we call the term torsion. The torsion is an obstruction of the existence of solutions. More precisely, the existence of torsions means that there does not exist an integral element of D_p. Here, the integral element is defined as follows:

Let (R, D) be a differential system expressed by

$$D = \{\varpi_1 = \cdots = \varpi_s = 0\}.$$

For $x \in R$, $E \subset T_x R$ is called n-dimensional integral element of D, if E is an n-dimensional subspace in $T_x R$ such that

$$\varpi_1|_E = \cdots = \varpi_s|_E = d\varpi_1|_E = \cdots = d\varpi_s|_E = 0.$$

Namely, integral elements are candidates for tangent spaces of integral manifolds of D. Hence, if a torsion appears, then there does not exist a solution.

Then, it is natural to consider the following problem.

PROBLEM 3.2. Examine when regular overdetermined systems (Σ_p, D_p) associated with type-changing equations Σ correspond to which structure equations.

From now on, we consider this problem. Let

$$\Sigma_p = \{r = f(x, y, z, p, q, s, t),\ \Delta = 0\}$$

be a regular overdetermined system associated with a type-changing equation. We divide a differential system Σ_p into the following two cases:

(I) $\Delta_s \neq 0$,

(II) $\Delta_s = 0$.

We first study the case of (I). By exterior derivation of Δ, we have

$$ds \equiv -\frac{1}{\Delta_s}\left(\frac{d}{dx}\Delta\, dx + \frac{d}{dy}\Delta\, dy + \Delta_t\, dt\right), \quad \text{mod } \varpi_0, \varpi_1, \varpi_2.$$

Here,

$$\frac{d}{dx} := \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q}, \quad \frac{d}{dy} := \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} + t\frac{\partial}{\partial q}.$$

In terms of this expression, the structure equation of D_p is given by

$$\begin{aligned}
d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2, \quad \mathrm{mod}\ \varpi_0,\\
d\varpi_1 &\equiv \left(\frac{d}{dy} f - \frac{f_s}{\Delta_s}\frac{d}{dy}\Delta + \frac{1}{\Delta_s}\frac{d}{dx}\Delta\right) dx \wedge dy\\
&\qquad + \frac{1}{\Delta_s}(f_s \Delta_t - f_t \Delta_s)\, dt \wedge dx + \frac{\Delta_t}{\Delta_s}\, dt \wedge dy, \quad \mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2,\\
d\varpi_2 &\equiv -\left(\frac{1}{\Delta_s}\frac{d}{dy}\Delta\, dx + dt\right) \wedge dy + \frac{\Delta_t}{\Delta_s}\, dt \wedge dx, \quad \mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2,
\end{aligned}$$

where $\omega_1 := dx$, $\omega_2 := dy$. If we set

$$\begin{aligned}
a &:= \frac{df}{dy} - \frac{f_s}{\Delta_s}\frac{d\Delta}{dy} + \frac{1}{\Delta_s}\frac{d\Delta}{dx},\\
b &:= \frac{f_s \Delta_t - f_t \Delta_s}{\Delta_s},\\
c &:= \frac{\Delta_t}{\Delta_s},\\
e &:= -\frac{1}{\Delta_s}\frac{d\Delta}{dy},
\end{aligned}$$

then the structure equation is written as follows:

$$\begin{aligned}
d\varpi_1 &\equiv a\, dx \wedge dy + b\, dt \wedge dx + c\, dt \wedge dy,\\
d\varpi_2 &\equiv e\, dx \wedge dy - dt \wedge dy + c\, dt \wedge dx, \quad \mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2.
\end{aligned}$$

This equation is also expressed by using matrices:

$$d\varpi_0 = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}, \tag{11}$$

$$\begin{aligned}
\begin{pmatrix} d\varpi_1 \\ d\varpi_2 \end{pmatrix} &\equiv \left\{ \begin{pmatrix} b & c \\ c & -1 \end{pmatrix} dt + \begin{pmatrix} 0 & a\,dx \\ 0 & e\,dx \end{pmatrix} \right\} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}\\
&= \left\{ \begin{pmatrix} b & c \\ c & -1 \end{pmatrix} dt + \begin{pmatrix} -a\,dy & 0 \\ 0 & e\,dx \end{pmatrix} \right\} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}.
\end{aligned} \tag{12}$$

Now, we consider the following real symmetric matrix:

$$X := \begin{pmatrix} b & c \\ c & -1 \end{pmatrix}.$$

For signatures of eigenvalues of this matrix X, there are three cases.
(I-I) $(+,+)$ or $(-,-)$ type, (i.e. both of eigenvalues are positive or negative).
(I-II) $(+,-)$ type, (i.e. two eigenvalues have distinct signatures).
(I-III) Degenerated type, (i.e. either of eigenvalues is zero).
For these cases, we consider corresponding normal forms (i.e. (i), (ii), or (iii)) of structure equations. By solving $\det(\lambda E - X) = 0$, we have the description,

$$\lambda = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4(b + c^2)}}{2}.$$

Since eigenvalues of real symmetric matrices are all real numbers, $(b-1)^2 + 4(b+c^2)$ is not negative. Corresponding to the above signatures of eigenvalues, we have the following classification result.

Theorem 3.3. *Let (Σ, D) be a type-changing equation and (Σ_p, D_p) be the regular overdetermined system. Then, D_p has torsion if and only if $b = -c^2$, $a + ce \neq 0$, where*

$$a := \frac{df}{dy} - \frac{f_s}{\Delta_s}\frac{d\Delta}{dy} + \frac{1}{\Delta_s}\frac{d\Delta}{dx},$$
$$b := \frac{f_s\Delta_t - f_t\Delta_s}{\Delta_s},$$
$$c := \frac{\Delta_t}{\Delta_s},$$
$$e := -\frac{1}{\Delta_s}\frac{d\Delta}{dy}.$$

Moreover, the following correspondences hold:

D_p *has normal form* (i) *if and only if* $b = -c^2$, $a + ce = 0$.

D_p *has normal form* (ii) *if and only if* $b > -c^2$, *or* $\Delta_s = 0$.

D_p *has normal form* (iii) *if and only if* $b < -c^2$.

Proof. First of all, we can check easily the following correspondences with respect to signatures of eigenvalues for the case of (I).
(1) Eigenvalues are type of (I-I) if and only if $\Delta_s \neq 0$, $b < -c^2$.
(2) Eigenvalues are type of (I-II) if and only if $\Delta_s \neq 0$ and $b > -c^2$.
(3) Eigenvalues are type of (I-III) if and only if $\Delta_s \neq 0$, $b = -c^2$.
We consider the case of (I-I). It is sufficient to consider $(+,+)$-type. We have the following lemma.

Lemma 3.4. *For any* $P \in GL(2, \mathbb{R})$, *we change* 1*-forms*:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = P^{-1}\begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \begin{pmatrix} \hat{\varpi}_1 \\ \hat{\varpi}_2 \end{pmatrix} = {}^tP\begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}. \tag{13}$$

Then, the structure equation (12) *is transformed in terms of appropriate functions* α, β *as follows*:

$$d\varpi_0 = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \hat{\varpi}_1 \\ \hat{\varpi}_2 \end{pmatrix},$$

(14)

$$\begin{pmatrix} d\hat{\varpi}_1 \\ d\hat{\varpi}_2 \end{pmatrix} \equiv {}^tP\begin{pmatrix} d\varpi_1 \\ d\varpi_2 \end{pmatrix}$$
$$\equiv \left\{ {}^tP\begin{pmatrix} b & c \\ c & -1 \end{pmatrix}P\, dt + \begin{pmatrix} \alpha\omega_2 & 0 \\ 0 & \beta\omega_1 \end{pmatrix} \right\} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \text{mod } \varpi_0, \hat{\varpi}_1, \hat{\varpi}_2.$$

We omit the proof of this lemma. We continue to prove the statement of Theorem 3.3. For any real symmetric matrix X, there exists an orthogonal matrix P such that

$${}^tP\begin{pmatrix} b & c \\ c & -1 \end{pmatrix}P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1,\ \lambda_2 > 0.$$

If we change 1-forms by using this matrix P, we have the following structure equation from the equation (14):

$$d\varpi_0 = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \hat{\varpi}_1 \\ \hat{\varpi}_2 \end{pmatrix},$$
$$\begin{pmatrix} d\hat{\varpi}_1 \\ d\hat{\varpi}_2 \end{pmatrix} \equiv \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} dt + \begin{pmatrix} \alpha\omega_2 & 0 \\ 0 & \beta\omega_1 \end{pmatrix} \right\} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{15}$$

Moreover, if we transform similarly by using a matrix P given by

$$P = \begin{pmatrix} \dfrac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \dfrac{1}{\sqrt{\lambda_2}} \end{pmatrix},$$

then, by taking appropriate 1-forms, we have

$$d\varpi_0 = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \hat{\varpi}_1 \\ \hat{\varpi}_2 \end{pmatrix},$$
$$\begin{pmatrix} d\hat{\varpi}_1 \\ d\hat{\varpi}_2 \end{pmatrix} \equiv \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dt + \begin{pmatrix} \alpha\omega_2 & 0 \\ 0 & \beta\omega_1 \end{pmatrix} \right\} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Here, if we set $\pi := dt + \alpha\omega_2 + \beta\omega_1$, then we have the normal form (iii):

$$d\varpi_0 = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \hat{\varpi}_1 \\ \hat{\varpi}_2 \end{pmatrix},$$

$$\begin{pmatrix} d\hat{\varpi}_1 \\ d\hat{\varpi}_2 \end{pmatrix} \equiv \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

For the case of (I-II), we can take a matrix P satisfying

$${}^tP\begin{pmatrix} b & c \\ c & -1 \end{pmatrix}P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1 > 0,\ \lambda_2 < 0.$$

Then, we have the following structure equation which can be transformed to the normal form (ii):

$$d\varpi_0 = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \hat{\varpi}_1 \\ \hat{\varpi}_2 \end{pmatrix}, \quad \text{mod } \varpi_0,$$

$$\begin{pmatrix} d\hat{\varpi}_1 \\ d\hat{\varpi}_2 \end{pmatrix} \equiv \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \text{mod } \varpi_0, \hat{\varpi}_1, \hat{\varpi}_2.$$

For the case of (I-III), we use the condition $b = -c^2$. Then, the structure equation is written as

$$d\varpi_0 \equiv dx \wedge \varpi_1 + dy \wedge \varpi_2, \quad \text{mod } \varpi_0,$$

$$\begin{pmatrix} d\varpi_1 \\ d\varpi_2 \end{pmatrix} \equiv \left\{\begin{pmatrix} -c^2 & c \\ c & -1 \end{pmatrix} dt + \begin{pmatrix} -a\,dy & 0 \\ 0 & e\,dx \end{pmatrix}\right\} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \text{mod } \varpi_0, \varpi_1, \varpi_2.$$

Here, if we take a 1-form $\hat{\varpi}_1 := \varpi_1 + c\varpi_2$, then we have

$$d\varpi_0 \equiv dx \wedge \hat{\varpi}_1 + (dy - c\,dx) \wedge \varpi_2, \quad \text{mod } \varpi_0,$$

$$\begin{pmatrix} d\hat{\varpi}_1 \\ d\varpi_2 \end{pmatrix} \equiv \left\{\begin{pmatrix} 0 & 0 \\ c & -1 \end{pmatrix} dt + \begin{pmatrix} (-a - ce)\,dy & 0 \\ 0 & e\,dx \end{pmatrix}\right\} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \text{mod } \varpi_0, \hat{\varpi}_1, \varpi_2.$$

If we set $\pi := -dt + e\,dx$, $\omega_1 := dx$, $\omega_2 := dy - c\,dx$, then we have the following:

$$d\varpi_0 \equiv \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \hat{\varpi}_1 \\ \varpi_2 \end{pmatrix}, \quad \text{mod } \varpi_0,$$

$$\begin{pmatrix} d\hat{\varpi}_1 \\ d\varpi_2 \end{pmatrix} \equiv \left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\pi + \begin{pmatrix} 0 & (a + ce)\,dx \\ 0 & 0 \end{pmatrix}\right\} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \text{mod } \varpi_0, \hat{\varpi}_1, \varpi_2.$$

Thus, if $a + ce \neq 0$, then D_p has torsion. On the other hand, if $a + ce = 0$, we have the normal form (i).

Finally, we consider the case of (II). From the condition, we have $\Delta_s = 0$ and $\Delta_t \neq 0$. We have the following relation.

$$dt \equiv -\frac{1}{\Delta_t}\left(\frac{d}{dx}\Delta\, dx + \frac{d}{dy}\Delta\, dy\right), \quad \mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2.$$

By using this relation, the structure equation of D_p is given by

$$\begin{aligned} d\varpi_1 &\equiv \left\{\left(\frac{df}{dy} - \frac{f_t}{\Delta_t}\frac{d\Delta}{dy}\right) dx - ds\right\} \wedge (f_s\, dx + dy), \\ d\varpi_2 &\equiv -\left(\frac{1}{\Delta_t}\frac{d\Delta}{dx}\, dy + ds\right) \wedge dx, \quad \mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

If we set,

$$\begin{aligned} a &:= \frac{df}{dy} - \frac{f_t}{\Delta_t}\frac{d\Delta}{dy}, \\ b &:= f_s, \\ c &:= \frac{1}{\Delta_t}\frac{d\Delta}{dx}, \end{aligned}$$

then we have

$$\begin{pmatrix} d\varpi_1 \\ d\varpi_2 \end{pmatrix} \equiv \begin{pmatrix} -b\, ds & a\, dx - ds \\ -ds & c\, dx \end{pmatrix} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Moreover, we rewrite as follows:

$$\begin{pmatrix} d\varpi_1 \\ d\varpi_2 \end{pmatrix} = \left\{\begin{pmatrix} -b & -1 \\ -1 & 0 \end{pmatrix} ds + \begin{pmatrix} -a\, dy & 0 \\ 0 & c\, dx \end{pmatrix}\right\} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

We consider the real symmetric matrix X given by

$$X = \begin{pmatrix} -b & -1 \\ -1 & 0 \end{pmatrix}.$$

Eigenvalues of this matrix X are given by

$$\frac{-b \pm \sqrt{b^2 + 4}}{2}.$$

Signatures of these eigenvalues are distinct. Thus, we have the normal form (ii) by using the similar argument. □

In fact, Cartan proved that differential systems (Σ_p, D_p) satisfying (i) locally are involutive in general [2]. On the other hand, differential systems satisfying (ii) or (iii) are not involutive. These are equations of finite type [14]. Now, we give examples of associated regular overdetermined systems Σ_p which have the normal forms (ii) or (iii) locally.

EXAMPLE 3.5. Consider the equation $\Sigma = \{r = t^2\}$. From the discriminant $\Delta = 2t$, this equation is type-changing and we have the corresponding regular overdetermined system $\Sigma_p = \{r = t = 0\}$. This is a model example which has the normal form (ii).

EXAMPLE 3.6. Consider the equation $\Sigma = \{r = s^2 + t\}$. From the discriminant $\Delta = 1 - s^2/4$, this equation is type-changing, and we have the corresponding regular overdetermined system $\Sigma_p = \{r = s^2 + t,\ s = \pm 2\} = \{r = t + 4,\ s = \pm 2\}$. This system has the normal form (iii) locally.

In the rest of this section, we discuss associated regular overdetermined systems which have the normal form (i). From the involutiveness of these systems, it is well-known that there exist locally real-analytic solutions for such a system by using the Cartan–Kähler theorem. However, in this case, we have the method of construction of solutions given by Cartan or Yamaguchi in the C^∞-category which is stronger method than the Cartan–Kähler theorem as follows ([2], [14]). First, in fact, Cartan characterized regular overdetermined involutive systems (R, D_R) by the condition that (R, D_R) admits a 1-dimensional Cauchy characteristic system $Ch(D_R)$. Here, the Cauchy characteristic system $Ch(D)$ of a differential system (R, D) is defined by

$$Ch(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \ (\mathrm{mod}\ \omega_1, \ldots, \omega_s) \text{ for } i = 1, \ldots, s\},$$

where $D = \{\omega_1 = \cdots = \omega_s = 0\}$ is defined locally by defining 1-forms $\{\omega_1, \ldots, \omega_s\}$. For these involutive systems R, when we consider corresponding leaf spaces $X := R/Ch(D_R)$, we have differential systems D_X of rank 2 on X. Moreover, for the projection $\rho\colon R \to X$, the relation $D_R = \rho_*^{-1}(D_X)$ is satisfied. Under this relation, we can take an integral curve of D_X, and the lift of the integral curve is a surface on R. By the construction, this surface is required solution (more precisely, see Example 3.7). Regular overdetermined systems treated in this paper are induced by type-changing equations. Hence, solutions of these systems are also parabolic solutions of original type-changing equations. We give such an example, that is, involutive associated regular overdetermined system. First, an overdetermined system described by $\Sigma_p = \{r = f(x, y, z, p, q, s, t),\ \Delta = 0\}$ associated with a type-changing equation Σ is regular if and only if the following condition is satisfied

$$(\Delta_s, \Delta_t) = \left(-\frac{1}{2} f_s f_{ss} - f_{st}, -\frac{1}{2} f_s f_{st} - f_{tt}\right) \neq 0.$$

Next, Σ_p is involutive if and only if $\Delta_s \neq 0$, $b = -c^2$, and $a + ce = 0$ by Theorem 3.3. Here, if type-changing equations are given by the form $r = f(s, t)$, then we have $a =$

$e = 0$. Hence, we study only two conditions $\Delta_s \neq 0$, $b = -c^2$ for this case. From now on, we consider the construction of involutive examples for equations of this form. Then, we write explicitly the above two conditions:

$$\frac{1}{2} f_s f_{ss} + f_{st} \neq 0, \tag{16}$$

$$\Delta_s(f_s\Delta_t - f_t\Delta_s) = -\Delta_t{}^2. \tag{17}$$

By using the condition $\Delta = 0$, the later condition (17) is rewritten as follows:

$$f_{tt} + f_s f_{st} + \frac{1}{4} f_s{}^2 f_{ss} = 0. \tag{18}$$

We consider the existence problem of functions $f(s, t)$ satisfying the conditions (16) and (18). We give a very interesting example which satisfies these conditions.

EXAMPLE 3.7. Consider the equation $\Sigma = \{r = 2st - (2/3)t^3\}$. From the discriminant $\Delta = -2s + t^2$, Σ is type-changing. The corresponding regular overdetermined system is given by $\Sigma_p = \{r = 2st - (2/3)t^3,\ s = t^2/2\} = \{r = t^3/3,\ s = t^2/2\}$. It is well-known that this system Σ_p has infinitesimal symmetry G_2 which is 14-dimensional exceptional simple Lie algebra [2]. We calculate the structure equation of the system (Σ_p, D_p) of rank 3. A differential system $D_p = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is given by

$$\begin{aligned}
\varpi_0 &= dz - p\,dx - q\,dy,\\
\varpi_1 &= dp - \frac{t^3}{3}\,dx - \frac{t^2}{2}\,dy,\\
\varpi_2 &= dq - \frac{t^2}{2}\,dx - t\,dy.
\end{aligned}$$

The structure equation is given by

$$\begin{aligned}
d\varpi_0 &\equiv dx \wedge dp + dy \wedge dq, && \bmod\ \varpi_0,\\
d\varpi_1 &\equiv -t^2 dt \wedge dx - t\,dt \wedge dy, && \bmod\ \varpi_0, \varpi_1, \varpi_2,\\
d\varpi_2 &\equiv -t\,dx \wedge dx - dt \wedge dy, && \bmod\ \varpi_0, \varpi_1, \varpi_2.
\end{aligned} \tag{19}$$

We take a new coframe:

$$\{\varpi_0,\ \hat{\varpi}_1 := \varpi_1 - t\varpi_2,\ \varpi_2, \pi := dt,\ \omega_1 := dx,\ \omega_2 := t\,dx + dy\}.$$

On this coframe, the above structure equation is written as follows:

$$\begin{aligned}
d\varpi_0 &\equiv \omega_1 \wedge \hat{\varpi}_1 + \omega_2 \wedge \varpi_2, && \bmod\ \varpi_0,\\
d\hat{\varpi}_1 &\equiv 0, && \bmod\ \varpi_0, \hat{\varpi}_1, \varpi_2,\\
d\varpi_2 &\equiv \omega_2 \wedge \pi, && \bmod\ \varpi_0, \hat{\varpi}_1, \varpi_2.
\end{aligned} \tag{20}$$

Thus, (Σ_p, D_p) is involutive type. Hence, there exists a solution of (Σ_p, D_p) (i.e. parabolic solution of (Σ, D)). In fact, this solution is constructed explicitly by E. Cartan [2]. Hence, we also describe this solution followed by Cartan or Yamaguchi (refer to [7], [14]). First, the Cauchy characteristic system $Ch(D_p)$ of (Σ_p, D_p) is given by

$$\begin{aligned} Ch(D_p) &= \{\varpi_0 = \hat{\varpi}_1 = \varpi_2 = \omega_2 = \pi = 0\} \\ &= \operatorname{span}\left\{\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + (p - tq)\frac{\partial}{\partial z} - \frac{t^3}{6}\frac{\partial}{\partial p} - \frac{t^2}{2}\frac{\partial}{\partial q}\right\}. \end{aligned}$$

Since this system has constant rank, $Ch(D_p)$ gives a 1-dimensional foliation on Σ_p. Hence a leaf space $B := \Sigma_p/Ch(D_p)$ is a 5-dimensional manifold locally. On this quotient space, We have a local coordinate $(x_1, x_2, x_3, x_4, x_5)$ given by

$$\begin{aligned} &x_1 := z - xp + xqt + \frac{1}{6}x^2t^3, \quad x_2 := p - qt + \frac{1}{2}yt^2 + \frac{1}{6}t^3x, \\ &x_3 := -q + \frac{1}{2}yt, \quad x_4 := y + xt, \quad x_5 := -t. \end{aligned}$$

Conversely, Σ_p is a $\mathbb{R}$-bundle on B locally. If we take a coordinate function λ of $\mathbb{R}$, then the coordinate (x, y, z, p, q, t) is expressed by using the coordinate $(x_1, x_2, x_3, x_4, x_5, \lambda)$:

$$\begin{aligned} &x = \lambda, \quad y = x_4 + \lambda x_5, \\ &z = x_1 + \lambda x_2 - \frac{1}{2}\lambda x_4(x_5)^2 - \frac{1}{6}\lambda^2(x_5)^3, \\ &p = x_2 + x_3x_5 + \frac{1}{6}\lambda(x_5)^3, \quad q = -x_3 - \frac{1}{2}x_4x_5 - \frac{1}{2}\lambda(x_5)^2, \\ &t = -x_5. \end{aligned} \tag{21}$$

On the base space B, we consider a rank 2 differential system $D_B = \{\alpha_1 = \alpha_2 = \alpha_3 = 0\}$ given by

$$\begin{aligned} \alpha_1 &= dx_1 + \left(x_3 + \frac{1}{2}x_4x_5\right) dx_4, \\ \alpha_2 &= dx_2 + \left(x_3 - \frac{1}{2}x_4x_5\right) dx_5, \\ \alpha_3 &= dx_3 + \frac{1}{2}(x_4\, dx_5 - x_5\, dx_4). \end{aligned}$$

It is well-known that this system D_B is a flat model of (2, 3, 5)-distributions [14]. Indeed, we can check this fact by calculating derived systems. Hence, D_B has also infinitesimal automorphism G_2. For a projection $p\colon \Sigma_p \to B$, generator 1-forms of D_p

and D_B are related as follows:

$$\varpi_0 := p^*\alpha_1 + xp^*\alpha_2,$$
$$\varpi_1 := p^*\alpha_2 - xp^*\alpha_3,$$
$$\varpi_2 := -p^*\alpha_3.$$

Thus, (B, D_B) is a retracting space of (Σ_p, D_p), that is, $(\Sigma_p, D_p) = p^*(B, D_B)$. By using this correspondence, solutions of (Σ_p, D_p) are constructed by solutions of (B, D_B). We consider integral curves $c(\tau)$ of D_B given by

$$x_1 = \varphi(\tau), \quad x_2 = -\varphi''(\tau)\left(\varphi'(\tau) - \frac{1}{2}\tau\varphi''(\tau)\right) + \frac{1}{2}\int(\varphi'')^2(\tau)\,d\tau,$$
$$x_3 = -\varphi'(\tau) + \frac{1}{2}\tau\varphi''(\tau), \quad x_4 = \tau, \quad x_5 = -\varphi''(\tau),$$

where τ is a parameter of curves, and $\varphi(\tau)$ is an arbitrary smooth function of τ. By using these integral curves, we construct integral surfaces of (Σ_p, D_p). Projections of integral surfaces S of D_p correspond to integral curves $c(\tau)$. Thus, integral surfaces $S := S(x, \tau)$ of (Σ_p, D_p) are given by the coordinate function $x = \lambda$ of the fiber:

$$x := x, \quad y = \tau - x\tau\varphi''(\tau),$$
$$z = \varphi(\tau) - x\varphi'(\tau)\varphi''(\tau) + \frac{1}{6}x^2(\varphi'')^3(\tau) + \frac{1}{2}x\int(\varphi'')^2(\tau)\,d\tau.$$

Here, we omit the explicit description of p, q, t. By eliminating λ, we can obtain solutions $z = z(x, y)$ of (Σ_p, D_p). From these discussions, we have regular parabolic solutions S of the type-changing equation $r = 2st - (2/3)t^3$.

On the other hand, we give a singular parabolic solution as follows. We first take integral curves $\hat{c}(\tau)$ of D_B which are different from $c(\tau)$ given by

$$x_1 = \frac{1}{2}\left\{\int(\varphi - \tau\varphi')\,d\tau - \varphi\varphi'\tau\right\},$$
$$x_2 = \frac{1}{2}\left\{\int(\varphi - \tau\varphi')\,d\tau + \frac{1}{2}\varphi\tau\right\},$$
$$x_3 = -\frac{1}{2}\int(\varphi - \tau\varphi')\,d\tau, \quad x_4 = \varphi(\tau), \quad x_5 = \tau.$$

Surfaces in Σ_p obtained from these integral curves $\hat{c}(\tau)$ by the correspondence (21) are singular solutions if and only if

$$rank\begin{pmatrix} x_\lambda & y_\lambda & z_\lambda & p_\lambda & q_\lambda \\ x_\tau & y_\tau & z_\tau & p_\tau & q_\tau \end{pmatrix} = 1.$$

If we put $x_4 := f(\tau) = 0$, then we have a singular solution of special type:

$$x = \lambda, \quad y = \lambda t, \quad z = -\frac{1}{6}\lambda^2 t^3,$$
$$p = \frac{1}{6}\lambda t^3, \quad q = -\frac{1}{2}\lambda t^2, \quad t = t.$$

This is a singular parabolic solution of the original type-changing equation.

In the above discussions, we treated examples of type-changing equations Σ whose corresponding regular overdetermined systems Σ_p have the normal forms (i), (ii), (iii). On the other hand, there exist type-changing equations their regular overdetermined system have torsions. Such an example is obtained by modifying involutive systems (i). We give a typical example by modifying Example 3.7.

EXAMPLE 3.8. Consider the equation $\Sigma = \{r = 2st - (2/3)t^3 + y\}$. From the discriminant $\Delta = -2s + t^2$, Σ is type-changing. The corresponding regular overdetermined system is given by $\Sigma_p = \{r = 2st - (2/3)t^3 + y,\ s = t^2/2\} = \{r = t^3/3 + y,\ s = t^2/2\}$. A differential system (Σ_p, D_p) has torsion. Indeed, the structure equation can be written in the form:

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \hat{\varpi}_1 + \omega_2 \wedge \hat{\varpi}_2, && \mod \varpi_0, \\ d\hat{\varpi}_1 &\equiv \frac{1}{\sqrt{t^2+1}}\omega_1 \wedge \omega_2, && \mod \varpi_0, \hat{\varpi}_1, \hat{\varpi}_2, \\ d\hat{\varpi}_2 &\equiv -(t^2+1)\, dt \wedge \omega_2 + \frac{t}{\sqrt{t^2+1}}\omega_1 \wedge \omega_2, && \mod \varpi_0, \hat{\varpi}_1, \hat{\varpi}_2, \end{aligned} \tag{22}$$

where 1-forms $\hat{\varpi}_i, \omega_i$ $(i = 1, 2)$ are given by the transformations (13) for the matrix P:

$$P = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}.$$

REMARK 3.9. Summarizing these discussions, we showed that *all classes* (*i.e. torsion*, (i), (ii), (iii)) *of regular overdetermined systems in* Theorem 3.3 *can be realized from type-changing equations.*

4. Type-changing equations of special types

In this section, we consider type-changing equations of special types. We first consider equations of the following form:

$$\Sigma := \{r = f(x, y, z, p, q, t)\}. \tag{23}$$

For these equations, we state as follows.

Theorem 4.1. *Let $\Sigma = \{r = f(x, y, z, p, q, t)\}$ be a second order regular PDE. Then, Σ is a type-changing PDE around a point w if and only if*

$$f_t(w) = 0 \quad \text{and} \quad f_t \not\equiv 0 \quad \text{around} \quad w \in \Sigma,$$

and the corresponding overdetermined system

$$\Sigma_p = \{r = f,\ f_t = 0\}$$

is regular if and only if $f_{tt} \neq 0$. Moreover, regular overdetermined systems (Σ_p, D_p) have always the normal form (ii).

Proof. The discriminant is given by $\Delta = -f_t$. Hence, Σ is a type-changing equation if there is a point w such that

$$f_t(w) = 0 \quad \text{and} \quad f_t \not\equiv 0 \quad \text{around} \quad w \in \Sigma.$$

Moreover, an overdetermined system Σ_p induced by Σ is regular if and only if $(\Delta_s, \Delta_t) = (0, -f_{tt}) \neq (0, 0)$. Since $\Delta_s = 0$, $\Delta_t \neq 0$ is satisfied on Σ_p. Thus, (Σ_p, D_p) has the normal form (ii) by Theorem 3.3. □

Corollary 4.2. *Type-changing equations of the form $r = f(x, y, z, p, q, t)$ do not induce involutive regular overdetermined systems.*

EXAMPLE 4.3. Consider the equation $\Sigma = \{r = c_1 t^2 + c_2 t + c_3 \mid c_i \in \mathbb{R},\ c_1 \neq 0\}$. Since the discriminant is given by $\Delta = -(2c_1 t + c_2)$, this is a type-changing equation. Moreover, $\Sigma_p = \{r = a,\ t = b \mid a, b \in \mathbb{R}\}$ is a regular overdetermined system which has the normal form (ii).

We next consider equations of the following form:

$$\Sigma := \{r = f(x, y, z, p, q, s)\}. \tag{24}$$

These equations satisfy the regularity condition, and the discriminant is given by $\Delta := -{f_s}^2/4$. Hence Σ is a type-changing equation if and only if there exists a point $w \in \Sigma$ such that

$$f_s(w) = 0 \quad \text{and} \quad f_s \not\equiv 0 \quad \text{around} \quad w.$$

For equations satisfying this condition, when we consider $\Sigma_p = \{r = f,\ \Delta = 0\}$, then we have $d\Delta = 0$ on Σ_p by using $d\Delta = -f_s\, df_s/2$. Hence, Σ_p is belong to the case of (B). Now, we study induced overdetermined systems given by

$$\hat{\Sigma}_p = \{r = f,\ f_s = 0\}. \tag{25}$$

REMARK 4.4. This equation is locally isomorphic to Σ_p as a manifold. However, *we need to distinguish* $\hat{\Sigma}_p$ *and* Σ_p *as regular overdetermined systems associated with type-changing equations* Σ.

From the description of two vectors $(1, f_s, 0)$, $(0, f_{ss}, 0)$, induced equations $\hat{\Sigma}_p$ are regular overdetermined systems if and only if $f_{ss} \neq 0$ on $\hat{\Sigma}_p$. From now on, we consider these regular overdetermined systems. Let $\hat{D}_p$ be the differential system on $\hat{\Sigma}_p$. By exterior derivation of $f_s = 0$, we have

$$ds \equiv -\frac{1}{f_{ss}}\left(\frac{d}{dx} f_s\, dx + \frac{d}{dy} f_s\, dy\right), \quad \text{mod } \hat{D}_p.$$

By using this expression, the structure equation of $\hat{D}_p$ is given by

$$\begin{aligned} d\varpi_1 &\equiv \left(\frac{d}{dy} f - \frac{f_s}{f_{ss}}\frac{d}{dy} f_s + \frac{1}{f_{ss}}\frac{d}{dx} f_s\right) dx \wedge dy, \\ d\varpi_2 &\equiv -\left(\frac{1}{f_{ss}}\frac{d}{dy} f_s\, dx + dt\right) \wedge dy. \end{aligned}$$

If we set

$$a := \frac{d}{dy} f - \frac{f_s}{f_{ss}}\frac{d}{dy} f_s + \frac{1}{f_{ss}}\frac{d}{dx} f_s, \quad b := \frac{1}{f_{ss}}\frac{d}{dy} f_s,$$

then this structure equation can be written as follows:

$$\begin{aligned} d\varpi_1 &\equiv a\, dx \wedge dy, \\ d\varpi_2 &\equiv -(b\, dx + dt) \wedge dy. \end{aligned}$$

If $a = 0$, then this structure equation is of involutive type (i), and also if $a \neq 0$, then this structure equation has torsion. Summarizing these discussions, we obtain the following statements.

Theorem 4.5. *Let* $\Sigma = \{r = f(x, y, z, p, q, s)\}$ *be a second order regular PDE. Then,* Σ *is a type-changing PDE around a point* w *if and only if*

$$f_s(w) = 0 \quad \textit{and} \quad f_s \not\equiv 0 \quad \textit{around} \quad w \in \Sigma.$$

Moreover, the corresponding type-changing equation Σ *is belong to the case of* (B).

Hence, we can not treat Σ_p as a regular overdetermined system, but we have the following theorem.

Theorem 4.6. *Let $\Sigma = \{r = f(x, y, z, p, q, s)\}$ be a type-changing PDE. Then, we can associate an induced regular overdetermined system*

$$\hat{\Sigma}_p = \{r = f,\ f_s = 0\},$$

when $f_{ss} \neq 0$. Moreover, the regular overdetermined system $(\hat{\Sigma}_p, \hat{D}_p)$ are of involutive type (i) *if and only if $a = 0$, where*

$$a := \frac{d}{dy} f - \frac{f_s}{f_{ss}} \frac{d}{dy} f_s + \frac{1}{f_{ss}} \frac{d}{dx} f_s.$$

Corollary 4.7. *Let $\Sigma = \{r = f(x, y, z, p, q, s)\}$ be a type-changing equation satisfying the assumption of* Theorem 4.6. *If the corresponding overdetermined system $\hat{\Sigma}_p$ satisfies $a = 0$ locally, then there exists locally parabolic solution of Σ.*

EXAMPLE 4.8. Consider the equation $\Sigma = \{r = s^n \mid n \geq 2\}$. Since the discriminant is given by $\Delta = -n^2 s^{2(n-1)}/4$, this equation is a type-changing, and $\hat{\Sigma}_p = \{r = 0, s = 0\}$ is an involutive regular overdetermined system.

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References

[1] R.L. Bryant, S.S. Chern, R. Gardner, H. Goldscmidt and P. Griffiths: Exterior Differential Systems, Mathematical Sciences Research Institute Publications **18**, Springer, New York, 1991.

[2] E. Cartan: *Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. École Norm. Sup. (3) **27** (1910), 109–192.

[3] J. Clelland, M. Kossowski and G.R. Wilkens: *Second-order type-changing evolution equations with first-order intermediate equations*, J. Differential Equations **244** (2008), 242–273.

[4] A. Hayakawa, G. Ishikawa, S. Izumiya and K. Yamaguchi: *Classification of generic integral diagrams and first order ordinary differential equations*, Internat. J. Math. **5** (1994), 447–489.

[5] T.A. Ivey and J.M. Landsberg: Cartan for Beginners, Amer. Math. Soc., Providence, RI, 2003.

[6] T. Noda and K. Shibuya, *On implicit second order PDE of a scalar function on a plane*, submitted.

[7] H. Sato: *Contact geometry of second order partial differential equations*: *from Darboux and Goursat*, *through Cartan*, *to modern mathematics*, Sugaku Expositions **20** (2007), 137–148.

[8] K. Shibuya: *On the prolongation of* 2*-jet space of* 2 *independent and* 1 *dependent variables*, Hokkaido Math. J. **38** (2009), 587–626.

[9] N. Tanaka: *On generalized graded Lie algebras and geometric structures* I, J. Math. Soc. Japan **19** (1967), 215–254.

[10] N. Tanaka: *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. **8** (1979), 23–84.

[11] K. Yamaguchi: *Contact geometry of higher order*, Japan. J. Math. (N.S.) **8** (1982), 109–176.

[12] K. Yamaguchi: *Geometrization of jet bundles*, Hokkaido Math. J. **12** (1983), 27–40.

[13] K. Yamaguchi: *Differential systems associated with simple graded Lie algebras*; in Progress in Differential Geometry, Adv. Stud. Pure Math. **22**, Math. Soc. Japan, Tokyo, 1993, 413–494.

[14] K. Yamaguchi: *Contact geometry of second order* I; in Differential Equations: Geometry, Symmetries and Integrability, Abel Symp. **5**, Springer, Berlin, 2009, 335–386.

[15] K. Yamaguchi: G_2*-geometry of overdetermined systems of second order*; in Analysis and Geometry in Several Complex Variables (Katata, 1997), Trends Math, Birkhäuser, Boston, MA, 1999, 289–314.

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