<table>
<thead>
<tr>
<th>Title</th>
<th>Universal coefficient sequences for cohomology theories of CW-spectra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yosimura, Zen-ichi</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 12(2) P.305-P.323</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1975</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/7318">https://doi.org/10.18910/7318</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/7318</td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive : OUKA
https://ir.library.osaka-u.ac.jp/repo/ouka/all/
UNIVERSAL COEFFICIENT SEQUENCES FOR COHOMOLOGY THEORIES OF CW-SPECTRA

ZEN-ICHI YOSIMURA

(Received April 24, 1974)

Kainen [5] showed that there exists a cohomology theory \( k^* \) (\( ? \)) and a natural short exact sequence

\[ 0 \to \text{Ext}(h^*(X), G) \to k^*(X; G) \to \text{Hom}(h^*(X), G) \to 0 \]

for any based \( CW \)-complex \( X \) if \( h^* \) is an (additive) homology theory and \( G \) is an abelian group. On the other hand, for an (additive) cohomology theory \( k^* \) such that \( k^*(\text{point}) \) has finite type Anderson [3] constructed a homology theory \( Dk_\ast \) and a natural exact sequence

\[ 0 \to \text{Ext}(Dk_{\ast-1}(F), Z) \to k^*(F) \to \text{Hom}(Dk_{\ast}(F), Z) \to 0 \]

for any finite \( CW \)-complex whose extension to arbitrary \( CW \)-complexes is given in a form of a four term exact sequence. He then determined homology theories \( Dk_\ast \) in the special cases \( k^* = H^*, K^* \) and \( KO^* \). Ordinary cohomology theory and complex \( K \)-theory are both self-dual and real \( K \)-theory is the dual of sympletic \( K \)-theory, i.e., \( DH_\ast = H_\ast, DK_\ast = K_\ast \) and \( DKSp_\ast = KO_\ast \). Moreover he asserted that \( D^2 \) is the identity, i.e., \( D(Dk_\ast) = k_\ast \).

In this note we shall construct a \( CW \)-spectrum \( E(G) \) for every \( CW \)-spectrum \( E \) and abelian group \( G \) by Kainen’s method involving an injective resolution of \( G \), and state a relation between \( E \) and \( E(G) \) in a form of a universal coefficient sequence

\[ 0 \to \text{Ext}(E_{\ast-1}(X), G) \to E(G)^{\ast}(X) \to \text{Hom}(E_{\ast}(X), G) \to 0 \]

for any \( CW \)-spectrum \( X \). And we shall study some properties of \( E(G) \). For example, under a certain finiteness assumption on \( \pi_{\ast}(E) \) we show that \( E(R) \) (\( R \)) has the same homotopy type of \( ER \) where \( J? \) is a subring of the rationals \( Q \) (Theorem 2). The above universal coefficient sequence combined with Theorem 2 gives us a new criterion for \( ER^\ast(X) \) being Hausdorff (Theorem 3). Also we shall discuss uniqueness of \( E(G) \) (Theorem 4). Furthermore, using Anderson’s technique we investigate the homotopy type of \( E(G) \) in the special cases \( E = H, K \) and \( KSp \) (Theorem 5). Finally we note that \( K^{2n}(K_\wedge \cdots \wedge K) \)
and \( KO^m(KO \wedge \cdots \wedge KO), m \equiv 1 \mod 4 \), are both Hausdorff (Theorem 6).

1. Duality maps

1.1. Let \( u: X' \wedge X \to W \) be a pairing of CW-spectra. Such a pairing defines a homomorphism

\[
T = T(u)_E: \{Y, E \wedge X'\} \to \{Y \wedge X, E \wedge W\}
\]

by the relation \( T(f) = (\wedge u)(f \wedge 1) \) for any CW-spectra \( Y \) and \( E \). A pairing \( u: X' \wedge X \to W \) is called an E-duality map provided \( T(u)_E \) is an isomorphism for \( E \) fixed and \( Y = \sum k \) for all \( k \). If \( u \) is an E-duality map, then \( T(u)_E \) becomes an isomorphism for any CW-spectrum \( Y \).

Fix CW-spectra \( X \) and \( W \) and consider the cohomology functor \( \{- \wedge X, W\} \) defined on the category of CW-spectra. By the representability theorem, there exists a function spectrum \( F(X, W) \) such that \( T: \{Y, F(X,W)\} \to \{Y \wedge X, W\} \) is a natural isomorphism for all \( Y \). So we see that the evaluation map

\[
e: F(X, W) \wedge X \to W
\]

is a \( S \)-duality map.

Let \( u: X' \wedge X \to W, v: Y' \wedge Y \to W, f: X \to Y, g: Y' \to X' \) be maps such that \( v(1 \wedge f) \) and \( u(g \wedge 1) \) are homotopic. Consider the cofiber sequences

\[
x \overset{f}{\to} y \to z, \quad z \overset{r}{\to} y \overset{g}{\to} x'.
\]

We have a CW-spectrum \( Q \) and maps \( p: X' \wedge X \to Q, \ q: Y' \wedge Y \to Q, \) and \( r: Z' \wedge Z \to Q \) giving rise to the diagram below homotopy commutative (up to sign)

\[
\begin{array}{cccccc}
Y' \wedge X & \rightarrow & \sum^{-1}(X' \wedge Z) & \rightarrow & Z' \wedge Y \\
\downarrow & & \downarrow & & \downarrow \\
Y' \wedge Y & \rightarrow & X' \wedge X & \rightarrow & Z' \wedge Z & \rightarrow & Y' \wedge Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
p & & p & & q & & q \\
\end{array}
\]

Since \( v(1 \wedge f) \) and \( u(g \wedge 1) \) are homotopic, an easy diagram chase shows that there exists a map \( s: Q \to W \) with \( s \ p = u \) and \( s \ q = v \) (see [7, Proof of Theorem 13.1]). So we obtain a map

\[
w: Z' \wedge Z \to W
\]

making the diagram
homotopy commutative (up to sign).
By use of (1.1) and "five lemma" we have

**Lemma 1.** Let \( u: X \rightarrow W, \ v: Y \rightarrow W \) be \( E \)-duality maps and assume that maps \( f: X \rightarrow Y \) and \( g: Y' \rightarrow X' \) satisfy the property that \( v(1 \wedge f) \) and \( u(g \wedge 1) \) are homotopic. Then the above map \( \omega: Z \wedge Z \rightarrow W \) is an \( E \)-duality map.
(Cf., [6, Theorem 6.10]).

Let \( C=\{X_n,f_n\} \) and \( C'=\{X_n',g_n\} \) be a direct and an inverse sequence of \( CW \)-spectra respectively. Pairings \( u_n: X_n \wedge X_n \rightarrow W \) induce the homomorphism

\[
T\{u_n\} : \{Y,E \wedge (\coprod X_n)\} \rightarrow \{Y,\coprod (E \wedge X_n)\} \cong \{Y,\coprod (E \wedge X_n)\}
\]

Under the assumption that the canonical morphism \( E \wedge (\coprod X_n) \rightarrow \coprod (E \wedge X_n') \) is a homotopy equivalence, we see that

\[
(1.3) \quad u \text{ is an } E \text{-duality map if so are all } u_n.
\]

Define maps \( f: \bigvee X \rightarrow \bigvee X \) and \( g: \coprod X_n \rightarrow \coprod X_n \) by

\[
i_n - i_{n+1}, f_n = f \cdot i_n, \quad p_n - g \cdot p_{n+1} = p_n \cdot g
\]

where \( i_n: X_n \rightarrow \bigvee X_n, p_n: \coprod X_n \rightarrow X_n \) are the canonical maps. And, construct the telescope \( TC \) and the cotelescope \( T^*C' \) so that we have the cofiber sequences

\[
(1.3) \quad \bigvee X_n \rightarrow \bigvee X_n \rightarrow TC, \ T^*C' \rightarrow \coprod X_n \rightarrow \Sigma X_n'.
\]

**Proposition 2.** Let \( C=\{X_n,f_n\} \) and \( C'=\{X_n',g_n\} \) be a direct and an inverse sequence of \( CW \)-spectra, and \( u_n: X_n \wedge X_n \rightarrow W \) be pairings such that \( u_{n+1}(1 \wedge f_n) \)}
and $u_\Lambda(g_\Lambda \Lambda^1)$ are homotopic. Then there exists a map $u: T^*C' \Lambda TC \to W$ such that the following diagram is homotopy commutative (up to sign):
\[
\begin{array}{c}
\sum^{-1}((\Pi X_n') \Lambda TC) \to T^*C' \Lambda TC \leftarrow T^*C' \Lambda (\vee X_n) \\
\downarrow \quad \uparrow \hat{u} \\
(\Pi X_n') \Lambda (\vee X_n) \to W \leftarrow (\Pi X_n') \Lambda (\vee X_n).
\end{array}
\]
Moreover, assuming that the canonical morphism $E \Lambda ((\Pi X_n') \to (\vee X_n)$ is a homotopy equivalence, $\hat{u}$ is an $E$-duality map if so are all $u_n$.

Proof. An easy diagram chase shows that $u(l \Lambda f)$ and $u(g \Lambda 1)$ are homotopic. We apply Lemma 1 and (1.3) to obtain the required map.

1.2. Let $G$ be an abelian group and $\Gamma: 0 \to P_1 \to P_0 \to G \to 0$ a free resolution. We realize $P_1$ and $\phi$ by wedges $MP_1$ of sphere spectra and a map $M\phi: MP_1 \to MP_0$. The mapping cone $M\Gamma$ of $M\phi$ forms a Moore spectrum of type $G$. Then there exists a universal coefficient sequence
\[
0 \to \text{Ext}(G, \pi_{*+1}(X)) \to \{M\Gamma, X\}_* \xrightarrow{\kappa} \text{Hom}(G, \pi_*(X)) \to 0
\]
where $\kappa$ associates to a map $/ \phi$ the induced homomorphism $f_\*$ in 0-th homotopy (see [4]). Therefore a Moore spectrum of type $G$ is uniquely determined up to homotopy type. For any $CW$-spectrum $E$ we define the corresponding spectrum with coefficient group $G$.

\[EG = E \Lambda MG\]

where $MG$ is a Moore spectrum of type $G$.

Let $/ \phi$ be a set of primes which may be empty, and denote by $I_\phi$ the multiplicative set generated by the primes not in $/$. It is a directed set which is ordered by divisibility. If $R$ is a subring of the rationals $Q$ (with unit), it is just "the integers localized at $l$" where $/ \phi$ is the set of primes which are not invertible in $R$. Thus $R=Z/I=L^Z$. Let $l^e$ denote the set of primes $p_k(p_k < p_{k+1})$ not in $/$, i.e., $l \cap l^e = \{\phi\}$ and $l \cup l^e = \{\text{all primes}\}$. Putting $l_n = p_1 \cdots p_n$, we choose a cofinal sequence $J_l = \{l_n\}$ in $I_\phi$.

Fix a $CW$-spectrum $W$. $C_l = \{X_n = W, f_n = l_{n+1}/l_n\}$ and $C^e_l = \{X'_n = Wg_n = l_{n+1}/l_n\}$ form respectively a direct and an inverse sequence (indexed by $J_l$). Denote by $W, W^f$ the telescope of $C_l$ and the cotelescope of $C^e_l$, i.e.,
\[
W = T\{Wf_n = l_{n+1}/l_n\}, \quad W^f = T^*\{Wg_n = l_{n+1}/l_n\}.
\]
Notice that $W^f$ is homotopy equivalent to $W \Lambda S_1$. Since $S_1$ is a Moore spectrum of type $Z_1$, an easy computation shows that
\[tfZ^\Lambda S^\Lambda Z^\Lambda n^f/\Lambda Z_\Lambda, \quad \text{and } HZ_\Lambda^f(S_1) = 0 \text{ for } n \neq 1\]
where \( l' \) is any set of primes with \( l' \cap l = \emptyset \).

Define by \( \iota_n \) and \( \rho_n \) the composite maps \( \hat{W} \to W \to W^* \to \prod W^* \to \tilde{W}^* \) and consider the cofiber sequences

\[
W \xrightarrow{\iota_n} W_l \to \tilde{W}_l, \quad \tilde{W}_l^* \to W^* \Omega^l W.
\]

\( \tilde{S}_l \) is obviously a Moore spectrum of type \( Z_l|Z \), and in addition

\[
(1.5) \quad HZ_1(\tilde{S}_l) \cong \mathbb{Z}/n, \text{ and } \mathbb{F}Z/(S_n) = 0, \quad n \neq 1
\]

for any \( l' \) with \( l' \cap l = \emptyset \).

1.3. Here we construct two useful duality maps.

**Proposition 3.** We have maps \( \bar{u}: W^* \wedge S_l \to W \) and \( \bar{w}: \tilde{W}^* \wedge \tilde{S}_l \to W \) such that the following diagram is commutative (up to sign) for all CW-spectra \( X \) and \( E\):

\[
\begin{array}{ccc}
\{X, E \wedge W^*\} & \to & \{X, E \wedge \tilde{W}^*\} \\
\downarrow T(w) & & \downarrow T(u) \\
\{X \wedge \tilde{S}_l, E \wedge W\} & \to & \{X \wedge S_l, E \wedge W\} \\
\end{array}
\]

**Proof.** Take as \( u_n: W \wedge S \to W \) the canonical identification. From (1.2) and Proposition 2 we obtain maps \( u: (\prod W) \wedge (\vee S) \to W, \bar{u}: W^* \wedge \tilde{S}_l \to W \) with the homotopy commutative squares

\[
\begin{array}{ccc}
(\prod W) \wedge S & \xrightarrow{\rho_n \wedge 1} & W \wedge S \\
\downarrow 1 \wedge \iota_n & & \downarrow \downarrow \downarrow \downarrow \downarrow \\
(\prod W) \wedge (\vee S) & \xrightarrow{u} & W \wedge \tilde{S}_l \\
\end{array}
\]

Putting the above two squares together we see that \( \rho_n \wedge 1 \) and \( u(1 \wedge \iota_n) \) are homotopic. By (1.1) there exists a map \( W: W^* \wedge \tilde{S}_l \to W \) making the diagram below homotopy commutative (up to sign)

\[
\begin{array}{ccc}
\Sigma^{-1}(W \wedge \tilde{S}_l) & \to & W^* \wedge \tilde{S}_l \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \downarrow \\
W \wedge S & \xrightarrow{\bar{w}} & W \wedge S_l.
\end{array}
\]

Now we need the following result in order to apply Proposition 2.

**Lemma 4.** Let \( G \) be a direct product of R-modules \( G_a \) and \( M_a \) a Moore spectrum of type \( G_a \). Then \( \prod M_a \) becomes a Moore spectrum of type \( G \), and the canonical morphism \( E \wedge (\prod M_a) \to \prod (E \wedge M_a) \) is a homotopy equivalence if \( \pi_* (E) \) has finite type as an R-module.

**Proof.** The result of Adams [1, Theorem 15.2] asserts that \( HR \wedge \prod M_a \to \prod \)
\[ \Pi(\text{HR} \wedge M_a) \] is a homotopy equivalence. Thus \( \Pi M_a \) becomes a Moore spectrum of type \( G \) because \( \pi_*(\Pi M_a) \) is an \( R \)-module and hence so is \( H_*(\Pi M_a) \). In the commutative diagram
\[
\begin{array}{c}
0 \to \pi_*(E) \otimes G \to \pi_*(E \wedge \Pi M_a) \to \text{Tor}^I_{\pi_{*-1}}(E, G) \to 0 \\
\downarrow \ast \downarrow & \Pi \pi_*(E) \otimes G_a \to \Pi \pi_*(E \wedge M) \to \Pi \text{Tor}^I_{\pi_{*-1}}(E, G_a) \to 0
\end{array}
\]

involving the universal coefficient sequences, the left and right arrows are isomorphisms. The result follows from "five lemma".

Obviously the canonical identification \( u_a: W \wedge S \to W \) is an \( E \)-duality map for every \( E \). Using Propositions 2, 3 and Lemma 4 we obtain

**Theorem 1.** Let \( G \) be an \( R \)-module and \( M \) a Moore spectrum of type \( G \). Assume that \( \pi_*(E) \) is of finite type as an \( R \)-module. Then the maps \( \check{u}: M^\# \wedge S \to M \) and \( \check{w}: M^\# \wedge \tilde{S} \to M \) given in Proposition 3 are both \( E \)-duality maps.

Remark that \( \pi_*(\tilde{S}) \) and \( \pi_*(\tilde{S}) \) are \( Z \)-modules. Taking \( S \) as \( M \) and the empty \( \phi \) as \( / \) in the above theorem, we compute that

\[
H_*(\tilde{S}) = \text{HZ}^*(S), \quad H_*(\tilde{S}) = \text{HZ}^*(S).
\]

Thus \( \sum(S) \) and \( \sum(\tilde{S}) \) are Moore spectra of type \( \tilde{Z} \) and of type \( \tilde{Z} \), where \( I = \{ \phi \} \), because of (1.4) and (1.5). So we get

**Corollary 5.** Assume that \( \pi_*(E) \) is of finite type as an \( R \)-module where \( R \) is a proper subring of \( Q \). Then there exist natural isomorphisms \( T(\check{w}): \check{E}^*(X) \to E^*(X \wedge \tilde{S}) \), \( T(\check{u}): \check{E}^*(X) \to E^*(X \wedge S) \) with the commutative (up to sign) diagram
\[
\begin{array}{c}
\check{E}^*(X) \xrightarrow{1T(\check{w})} \check{E}^*(X) \xrightarrow{1T(\check{u})} E^*(X) \to E^*(X).
\end{array}
\]

2. Universal coefficient sequences

2.1. Following Kainen [5] we shall construct a universal coefficient sequence for a generalized cohomology theory. Fix a \( CW \)-spectrum \( E \). For every injective abelian group \( \text{Hom}(E_*(-), /) \) forms a cohomology theory defined on the category of \( CW \)-spectra. The representability theorem gives us a \( CW \)-spectrum \( \check{E}(I) \) and a natural isomorphism

\[
T_I: \{X, \check{E}(I)\} \to \text{Hom}(E_*(X), I)
\]

for any \( CW \)-spectrum \( X \). Let \( G \) be an abelian group and \( \Gamma: 0 \to G \to I \to F \to 0 \)
an injective resolution. Then there exists a unique (up to homotopy) map \( \psi: \hat{E}(I) \to \hat{E}(J) \) whose induced homomorphism coincides with the natural transformation \( T_f^* \cdot \psi_* \cdot T_I \). Denote by \( \Sigma \hat{E}(\Gamma) \) the mapping cone of \( \hat{\psi} \), i.e.,

\[
\hat{E}(\Gamma) \to \hat{E}(I) \to \hat{E}(J)
\]

is a cofiber sequence. By homological algebra we obtain a natural exact sequence

\[
0 \to \text{Ext}(E^*(X), G) \to \hat{E}(\Gamma)^*(X) \to \text{Hom}(E^*(X), G) \to 0
\]

for all \( X \).

Let \( \phi: G \to G' \) be a homomorphism and \( \Gamma: 0 \to G \to I \to J \to 0 \), \( \Gamma': 0 \to G' \to I' \to J' \to 0 \) be injective resolutions. For a morphism \( \mu: \Gamma \to \Gamma' \) which is a lift of \( \phi \), we may choose a map

\[
A: \hat{E}(\Gamma) \to \hat{E}(\Gamma')
\]

making the diagram with cofiber sequences

\[
\begin{array}{ccc}
\hat{E}(\Gamma) & \to & \hat{E}(I) \\
\downarrow & & \downarrow \\
\hat{E}(\Gamma') & \to & \hat{E}(I') \\
\end{array}
\]

homotopy commutative. However \( \mu \) is not uniquely determined (up to homotopy). The map \( \mu \) yields the commutative diagram

\[
0 \to \text{Ext}(E^*(-)(X), G) \to \hat{E}(\Gamma)^*(X) \to \text{Hom}(E^*(X), G) \to 0
\]

\[
0 \to \text{Ext}(E^*(-)(X), G) \to \hat{E}(\Gamma)^*(X) \to \text{Hom}(E^*(X), G) \to 0.
\]

With an application of "five lemma" we find that \( \mu: \hat{E}(\Gamma) \to \hat{E}(\Gamma') \) is a homotopy equivalence if \( \phi: G \to G' \) is an isomorphism. Thus the homotopy type of \( \hat{E}(\Gamma) \) is independent of the choice of an injective resolution \( \Gamma \) of \( G \). So we may put

\[
\hat{E}(G) = \hat{E}(\Gamma), \quad \hat{\phi} = \hat{\mu}.
\]

Consequently we get

**Proposition 6.** Let \( E \) be a CW-spectrum and \( G \) an abelian group. Then there exists a CW-spectrum \( \hat{E}(G) \) so that

\[
0 \to \text{Ext}(E^*(-)(X), G) \to \hat{E}(G)^*(X) \to \text{Hom}(E^*(X), G) \to 0
\]

is a natural exact sequence for any CW-spectrum \( X \). Moreover a homomorphism \( \phi: G \to G' \) induces a (non-unique) map \( \phi: \hat{E}(G) \to \hat{E}(G') \) with the commutative diagram
0 → Ext \( (E_\ast X)_G \) → \( \tilde{G}(X) \) → Hom \( (E_\ast X)_G \) → 0
\[ \downarrow \phi_\ast \downarrow \tilde{\phi}_\ast \downarrow \phi_\ast \]
0 → Ext \( (E_\ast X)G' \) → \( \tilde{G}'(X) \) → Hom \( (E_\ast X)G' \) → 0.

(Cf., [5]).

If \( Y \) is a finite CW-spectrum, then the function dual \( Y^\ast = F(Y,S) \) can be taken finite and \( E_\ast(Y) \cong E^\ast(Y^\ast) \cong E^\ast_\ast(Y^\ast) \). We notice that

(2.1) there exists a natural exact sequence

0 → Ext \( (E^{\ast+1} Y, G) \) → \( \tilde{E}(Y) \) → Hom \( (E^\ast(Y), G) \) → 0

for all finite \( Y \).

Let \( / : E \to F \) be a map of CW-spectra. Then \( / \) induces a (non-unique) map \( / : F(G) \to E(G) \) such that the diagram

\[
\begin{array}{ccc}
0 & \to & \text{Ext} \left( E^\ast +1, G \right) \\
\downarrow f^\ast & & \downarrow \tilde{f}^\ast \\
0 & \to & \text{Ext} \left( E^\ast X, G \right)
\end{array}
\]

is commutative. Remark that \( / \) becomes a homotopy equivalence if so is \( / \).

Hence we find that

(2.3) the homotopy type of \( \tilde{E}(G) \) depends only on that of \( E \) and the isomorphism class of \( G \).

2.2. For simplicity we write \( \tilde{E} \) instead of \( \tilde{E}(Z) \). We shall now show that \( \tilde{E}(G) \) and \( \tilde{E}G \) have the same homotopy type under some finiteness assumptions on \( E \) and \( G \). First we require the following

Lemma 7. i) Let \( G \) be a direct product of abelian groups \( G_\alpha \), i.e., \( G = \prod G_\alpha \). Then \( \tilde{E}(G) \) is homotopy equivalent to \( \prod \tilde{E}(G_\alpha) \).

ii) Let \( G \) be a direct sum of \( R \)-modules \( G_\alpha \), i.e., \( G = \sum G_\alpha \), and assume that \( \pi_\ast(E) \) is of finite type as an \( R \)-module. Then \( \tilde{E}(G) \) is homotopy equivalent to \( \bigvee \tilde{E}(G_\alpha) \).

Proof. i) Denote by \( p_\alpha \) the canonical projection from \( G \) onto \( G_\alpha \). The map \( \prod p_\alpha : \tilde{E}(G) \to \prod \tilde{E}(G_\alpha) \) induces the composite homomorphism

\[ \tilde{E}(G)^\ast(X) \to \prod \tilde{E}(G_\alpha)^\ast(X) \cong (\prod \tilde{E}(G_\alpha))^\ast(X) \]

for any CW-spectrum \( X \). In the commutative diagram

\[
\begin{array}{ccc}
0 & \to & \text{Ext} \left( E^\ast X, G \right) \\
\downarrow 1 & & \downarrow 1 \\
\prod \text{Ext} \left( E^\ast X, G_\alpha \right) & \to & \prod \tilde{E}(G_\alpha)^\ast(X) \\
\downarrow & & \downarrow \\
0 & \to & \prod \text{Hom} \left( E^\ast X, G_\alpha \right)
\end{array}
\]
involving the universal coefficient sequences, the left and right arrows are isomorphisms. By “five lemma” the center becomes an isomorphism, and hence the map $\prod \hat{p}_a$ is a homotopy equivalence.

ii) The canonical injections $i_a: G_a \to G$ induce the composite homomorphism

$$\left( \bigvee \hat{E}(G_a) \right)^*(Y) \leftarrow \sum E(G_a)^*(Y) \to E(G)^*(Y)$$

for any finite $Y$. Consider the commutative diagram

$$0 \to \Sigma \text{Ext}^1_k\left( E_{a-1}(Y), G_a \right) \to \Sigma \hat{\text{Hom}}_R(\hat{E}(Y), G_a) \to 0$$

The vertical arrows on both sides are isomorphisms whenever $Y$ is finite. So the map $\bigvee i_a: \bigvee \hat{E}(G_a) \to \hat{E}(G)$ becomes a homotopy equivalence.

Fix a subring $R$ of $Q$ and assume that $\pi*(E)$ has finite type as an $R$-module. For any subrings $R', R''$, $R' \subset R \subset R''$, the composite maps

$$e(R') : \hat{E}(R)R' \to E(R)R \leftarrow \hat{E}(R), \quad e(R'') : E(R)R'' \to E(R'')R'' \leftarrow \hat{E}(R'')$$

become homotopy equivalences because all arrows induce isomorphisms in homotopy. So we consider the diagram

$$
\begin{array}{c}
E(R)R' \to E(R)Q \to \hat{E}(R)Q/R' \\
\downarrow \downarrow \downarrow \\
\hat{E}(R) \to \hat{E}(Q) \to \hat{E}(Q|R) \\
\end{array}
$$

such that the rows are cofiber sequences and the left square is homotopy commutative. Then there exists a homotopy equivalence

$$e(Q|R') : \hat{E}(R)Q/R' \to \hat{E}(Q|R)$$

(denoted by a dotted arrow in the above diagram) which makes the diagram into a morphism of cofiber sequences. Moreover we get a map

$$e(Q_q) : \hat{E}(R)Z_q \to E(Z_q)$$

for the $R$-module $Z_q$, which becomes also a homotopy equivalence. This gives rise to a homotopy commutative diagram

$$
\begin{array}{ccc}
\hat{E}(R)Z_q & \to & E(R)Q/R \\
\downarrow & & \downarrow \\
\hat{E}(Z_q) & \to & \hat{E}(Q|R) \\
\end{array}
$$

where the rows are cofiber sequences associated with the injective resolution
Proposition 8. Let $G$ be a direct sum or product of finitely generated $R$-modules $G_\alpha$. If $\pi_*(E)$ is of finite type as an $R$-module, then $\hat{E}(G)$ has the same homotopy type of $\hat{E}(R)G$.

Proof. We may put $G_\alpha=R$ or $Z_\alpha$. Using (2.4), (2.6) and Lemmas 4, 7 we find that the composite maps

$$E(R)G \leftarrow \vee \hat{E}(R)G_\alpha \rightarrow \hat{E}(G_\alpha) \rightarrow \hat{E}(G), \ E(R)G \rightarrow \prod \hat{E}(R)G_\alpha \rightarrow \prod \hat{E}(G_\alpha) \leftarrow \hat{E}(G)$$

are homotopy equivalences.

2.3. Let $S^T \rightarrow S_\delta \rightarrow \tilde{S}_\delta$ be the cofiber sequence constructed in §1. Assume that for all finite $CW$-spectra $Y$ we have natural homomorphisms

$$\phi': E^*(Y \wedge \tilde{S}_\delta) \rightarrow F^*(Y \wedge S_\delta), \ \phi'': E^*(Y \wedge S_\delta) \rightarrow F^*(Y \wedge S_\delta)$$

which satisfy the relation that $\phi'/(1 \wedge \tilde{t})^*=(1 \wedge \tilde{t})^*\phi'$. Moreover we assume that $\pi_*(E)$ and $\pi_*(F)$ are $R$-modules where $R$ is a proper subring of $Q$. If $\pi_*(F)$ is of finite type, then $FR^*(X)$ and $FR/Z^*(X)$ are always Hausdorff for all $X$ [8, III]. Thus $FR^*(X) \simeq \lim FR^*(X^\lambda)$ and $FR/Z^*(X) \simeq \lim FR/Z^*(X^\lambda)$ where $\{X^\lambda\}$ runs over the set of all finite subspectra of $X$. Applying Corollary 5 (and Proposition 3) we obtain natural homomorphisms

$$\psi': ER^*(X) \rightarrow FR^*(X), \ \psi'': ER/Z^*(X) \rightarrow FR/Z^*(X)$$

for arbitrary $X$ which gives us the commutative square

$$
\begin{array}{c}
E \rightarrow ER \rightarrow ER/Z \rightarrow \Sigma E \\
f' \downarrow \quad \downarrow f'' \\
F \rightarrow FR \rightarrow FR/Z \rightarrow \Sigma F
\end{array}
$$

Putting $f'=\psi'(1_{ER})$ and $f''=\psi'(1_{ER/Z})$ we get the diagram

$$
E \rightarrow ER \rightarrow ER/Z \rightarrow \Sigma E \\
F \rightarrow FR \rightarrow FR/Z \rightarrow \Sigma F
$$

with cofiber sequences and a homotopy commutative square. Then there exists a map

$$(2.7) \quad \vdash: E \rightarrow F$$

making the above diagram into a morphism of cofiber sequences. In particular, we obtain the following result which is a useful tool in studying properties of $\hat{E}(G)$.
Lemma 9. Assume that $\pi_*(E)$ and $\pi_*(F)$ have finite type as $R$-modules. If for any finite CW-spectrum $Y$ we have natural isomorphisms

$$\phi': E^*(Y \wedge S_\delta) \to F^*(Y \wedge S_\delta), \quad \phi'': E^*(Y \wedge S_\delta) \to F^*(Y \wedge S_\delta)$$

such that $\phi''(1 \wedge i)^* - (1 \wedge i)^* \phi'$, then $E$ is homotopy equivalent to $F$.

Now we study the homotopy type of $\hat{E}(R)$ by use of Lemma 9.

Theorem 2. If $\pi_*(E)$ is of finite type as an $R'$-module, then $\hat{E}(R')(R)$ has the same homotopy type of $ER$ where $R' \subseteq R \subseteq Q$.

Proof. By $[8, (II, 1.10)]$, (2.1) and Proposition 6 the composite homomorphism

$$E^*(Y) \otimes Q \to \text{Hom}(\text{Hom}(E^*(Y), R'), Q) \rightarrow \text{Hom}(\hat{E}(R')_*(Y), Q) \leftarrow \hat{E}(R')(Q)_*(Y)$$

is a natural isomorphism for all finite $Y$. In particular the coefficient $\pi_*(\hat{E}(R')(Q))$ is equal to the $Q$-module $\pi_*(EQ)$. Therefore $\hat{E}(R')(Q)$ becomes homotopy equivalent to $EQ$.

So we may assume that $R$ is a proper subring of $Q$. For any finite $Y$ we consider the following diagram

\[
\begin{array}{c}
ER^* \leftarrow (ER)R^*(Y) \leftarrow E^*(Y) \otimes \text{Ext}(Q/R', R) \\
\downarrow \quad \downarrow \quad \downarrow \\
E^*(Y) \otimes \text{Ext}(Q, R) \\
\end{array}
\]

Theorem 2 asserts that we have a natural exact sequence
Using the above universal coefficient sequence we give a new criterion for $E(R^*(X))$ being Hausdorff.

**Theorem 3.** Assume that $\pi_*(E)$ has finite type as an $R'$-module. $(E(R)^{n+1}(X)$ is Hausdorff if and only if $\text{Ext}(E(R')^{n+1}(X), R)\to 0$ where $R'\subset R\subset \mathbb{Q}$ and $\Gamma G$ denotes the torsion subgroup of $G$.

**Proof.** The proof is similar to that of [8, Theorem IV. 4]. Assume that $R$ is a proper subring. Recall that $E(R^{n+1}(X))$ is Hausdorff if and only if the boundary homomorphism $\partial: E(R')^n(X)\to E(R')^{n+1}(X)$ is trivial (cf., [8, Theorem III.1]). Then Corollary 5 implies that $E(R^{n+1}(X))$ is Hausdorff if and only if $(1\wedge \iota_1)^*: E(R^{n+1}(X\wedge S\phi))\to E(R^{n+1}(X))$ is trivial. In the commutative diagram

$$\text{Ext}(\hat{E}(R')^{n+1}(X), R)\to E(R^{n+1}(X))\to \text{Hom}(E(R')^n(X), R)\to 0,$$

the upper arrow is an isomorphism and the lower row is exact by (2.8). On the other hand, the left vertical arrow admits a factorization

$$\text{Ext}(\hat{E}(R')^{n+1}(X), R)\to \text{Ext}(\hat{E}(R')^{n}(X), R)\to \text{Ext}(\hat{E}(R')^{n}(X), R)$$

such that the former is an epimorphism but the latter is a monomorphism. An easy diagram chase shows that $(1\wedge \iota_1)^*$ is the zero map if and only if $\text{Ext}(\hat{E}(R')^{n+1}(X\wedge S\phi))\cong \text{Ext}(\hat{E}(R')^{n+1}(X\wedge S\phi), R)$. So the result follows immediately.

**2.4.** Now we discuss uniqueness of $\hat{E}(G)$ under some restrictions on $E$ and $G$.

**Theorem 4.** Let $G$ be a finitely generated $R$-module with $\text{Tor}(\pi_*(E), G)\to 0$, and assume that $\pi_*(E)$ is of finite type as an $R$-module. If $F$ satisfies the property that there exists a natural exact sequence

$$0\to \text{Ext}(\hat{E}(R')^{n+1}(X\wedge S\phi), G)\to F^{n+1}(X)\to \text{Hom}(\hat{E}(R')^{n+1}(X), G)\to 0$$

for any CW-spectrum $X$, then $F$ has the same homotopy type of $\hat{E}(G)$.

**Proof.** Assume that $R$ is a proper subring of $O$. The torsion subgroup $T=TG$ is a direct summand of $G$ and the quotient $P=G/TG$ is a free $R$-module. Consider the commutative diagram

$$\text{Ext}(\hat{E}(R')^{n+1}(X\wedge S\phi), P)\to \text{Ext}(\hat{E}(R')^{n+1}(X\wedge S\phi), G)\to F^{n+1}(X\wedge S\phi)$$

$$\text{Ext}(\hat{E}(R')^{n+1}(X\wedge S\phi), P)\to \text{Ext}(\hat{E}(R')^{n+1}(X\wedge S\phi), G)\to F^{n+1}(X\wedge S\phi)$$
for any $X$. (2.7) gives rise to a map $f: \hat{E}(P) \to F$ with the commutative diagram

\[
\begin{array}{c}
\hat{E}(P)*(X) \to \hat{E}(P)\hat{R}*(X) \to \hat{E}(P)\hat{R}/Z*(X) \\
\downarrow \quad \downarrow \quad \downarrow \\
F*(X) \to FR*(X) \to FR/Z*(X).
\end{array}
\]

Looking at the previous diagram we find that in the above the central arrow is a monomorphism and the right is an isomorphism. So $f*: E(P)*(X) \to F*(X)$ becomes a monomorphism whenever $E(P)*(X)$ is Hausdorff, and in addition $/\$ induces an isomorphism $f*_*: \pi_*(E(P)) \otimes Q \to \pi_*(F) \otimes Q$. Denote by $F_T$ the mapping cone of $f$, thus

\[ E(P) \xrightarrow{f} F \xrightarrow{g} F_T \]

is a cofiber sequence. $E(P)*(F_T)$ is Hausdorff as $\pi_*(F_T) \otimes Q = 0$ [8, Theorem III. 2]. Therefore we have a short exact sequence

\[ 0 \to E(P)*(F_T) \to F*(F_T) \to F_T*(F_T) \to 0. \]

Then we may choose a map $h: F_T \to F$ such that the composite map $g \circ h$ is homotopic to the identity. This means that the sequence

\[ 0 \to E(P)*(X) \to F*(X) \to F_T*(X) \to 0 \]

is split exact. $F$ is obviously homotopic to the wedge of $E(P)$ and $F_T$.

We are now left to show that $F_T$ has the same homotopy type of $E(P)$ under the assumption that Tor($\pi_*(E)$, $G$)=0. Consider the commutative exact diagram

\[
\begin{array}{c}
0 \to \text{Ext}(E(\mathbb{Z}/Z*(X)), P) \to \hat{E}(P)*^{+1}(X \wedge \mathbb{S}_\phi) \to \text{Hom}(E(\mathbb{Z}/Z*(X)), P) \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to \text{Ext}(E(\mathbb{Z}/Z*(X)), G) \to F*^{+1}(X \wedge \mathbb{S}_\phi) \to \text{Hom}(E(\mathbb{Z}/Z*(X)), G) \to 0 \\
\downarrow \quad \downarrow \\
\text{Ext}(E(\mathbb{Z}/Z*(X), T) \quad F_T^{*+1}(X \wedge \mathbb{S}_\phi) \quad \text{Hom}(E(\mathbb{Z}/Z*(X), T) \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \quad 0 \quad 0
\end{array}
\]

in which $\text{Hom}(E(\mathbb{Z}/Z*(X), P)=0$. With an application of "3 x 3 lemma" we get a natural exact sequence

\[ 0 \to \text{Ext}(E(\mathbb{Z}/Z*(X), T) \to F_T^{*+1}(X \wedge \mathbb{S}_\phi) \to \text{Hom}(E(\mathbb{Z}/Z*(X), T) \to 0 \]

for any $X$. Take a free resolution $0 \to P_1^{\phi} \to P_0^{\psi} \to T \to 0$ consisting of finitely generated $R$-modules. The composite homomorphisms
are isomorphisms. The $R$-free resolution yields the following commutative exact diagram

$$
0 \rightarrow \text{Ext}(EQ/Z_*(X), T) \rightarrow \Phi_*(X \wedge S_\phi) \rightarrow \text{Hom}(EQ/Z_*(X), T) \rightarrow 0
$$

Define homomorphisms \( \Phi: \hat{E}(R)P_*(X \wedge S_\phi) \rightarrow F_*(X \wedge S_\phi), \ S: F_*(X \wedge S_\phi) \rightarrow \wedge S_\phi \)

by the composite maps \( \Phi = \gamma' \psi_\phi \eta', \delta = \eta \circ \eta' \). By an easy diagram chase we show that the long sequence

$$
\rightarrow \hat{E}(R)P_*(X \wedge S_\phi) \rightarrow E(R)P_*(X \wedge S_\phi) \rightarrow F_*(X \wedge S_\phi) \rightarrow E(R)P_*(X \wedge S_\phi) \rightarrow
$$

is exact for all $X$.

Next we consider the commutative exact diagram

$$
\begin{array}{ccc}
\downarrow & \phi_\phi & \downarrow \\
E(R)P_*(X) & \hat{E}(R)P_*(X) & F_*(X) \\
\rightarrow \hat{E}(R)P_*(X) \rightarrow & E(R)P_*(X) \rightarrow & F_*(X) \\
\downarrow & \phi_\phi & \downarrow \\
\hat{E}(R)P_*(X) \rightarrow & E(R)P_*(X) \rightarrow & F_*(X) \\
\downarrow & \phi_\phi & \downarrow \\
\hat{E}(R)P_*(X) \rightarrow & E(R)P_*(X) \rightarrow & F_*(X)
\end{array}
$$

in which the middle row is rewritten the previous long exact sequence by the aid of Corollary 5. As is easily seen, we get an exact sequence

$$
E(R)P_*(X) = E(R)P_*(X) \rightarrow F_*(X)
$$

for any $X$. Taking $e' = \rho(1_{\hat{E}(R)P_\phi})$, the composite map $e' \phi$ becomes homotopic
to the zero map. Therefore $e'$ admits a factorization (up to homotopy)

$$
\hat{E}(R)P_{\ast} \rightarrow \hat{E}(R)T \xrightarrow{e} F_T.
$$

This yields the commutative triangle

$$
\hat{E}(R)^{\ast}(X) \otimes R T \xrightarrow{\rho^\ast} \hat{E}(R)T^{\ast}(X)
$$

If $\text{Tor}(\pi_{\ast}(E), T) = 0$, then $\text{Tor}(\pi_{\ast}(\hat{E}(R)), T) = 0$ and hence the above $e_{\ast}$ is an isomorphism. So $F_T$ becomes homotopy equivalent to $\hat{E}(T)$ because of Proposition 8. Putting this and the previous result together, the required result is obtained from Lemma 7.

3. **Complex and real $K$-theories**

3.1. First we shall construct an injective resolution

$$
\Gamma(G): 0 \rightarrow G \rightarrow I_G \rightarrow J_G \rightarrow 0
$$

for every abelian group $G$ which is functorial in $G$ (see [5]). Let $A(G)$ denote the direct sum of copies $A_g$ of $A$ which runs over the set of all elements $g$ of $G$, where $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{Z}/Q$. $G$ admits the canonical free resolution $0 \rightarrow P \rightarrow \mathbb{Z}(G) \rightarrow G \rightarrow 0$. Consider the commutative exact diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z}(G) \\
\downarrow & & \downarrow \\
0 & \rightarrow & G \\
\downarrow & & \downarrow \\
0 & \rightarrow & I_G \\
\downarrow & & \downarrow \\
0 & \rightarrow & J_G \\
\end{array}
$$

and take the lower row in the above diagram as $\Gamma(G)$. Note that $J_G$ is isomorphic to $\mathbb{Q}/\mathbb{Z}(G)$.

Let $\mu: E \wedge F \rightarrow W$ be a pairing of $CW$-spectra with $\pi_0(W) \cong \mathbb{Z}$ and $\pi_1(W)$ torsion free. This gives rise to the natural homomorphism

$$
\overline{\mu}: FG^{\ast}(X) \rightarrow \text{Hom}(E_\ast(X), \pi_0(WG))
$$

for any $G$.

We shall require the following result in studying the duals of $K$-theories.
Lemma 10. If $\mu$ induces isomorphisms $\overline{\mu}: \pi_*(FG') \to \text{Hom}(\pi_*(E), G')$ in the cases $G' = I_G, J_G$, then we have a homotopy equivalence $f: FG \to E(G)$ with $\overline{\mu} = \kappa f_*$.

Proof. In the commutative diagram

$$
\begin{array}{ccc}
FG^*(X) & \to & \text{Hom}(E^*(X), G) \\
\downarrow & & \downarrow \\
FI_G^*(X) & \to & \text{Hom}(E^*(X), I_G) \\
\downarrow & & \downarrow \\
FJ_G^*(X) & \to & \text{Hom}(E^*(X), J_G)
\end{array}
$$

the last two left-hand arrows are isomorphisms. So the above diagram yields the homotopy commutative diagram

$$
FG \to FI_G \to FJ_G \to \Sigma FG
$$

with cofiber sequences. Choose a map

$$
f: FG \to \hat{E}(G)
$$

making the above diagram homotopy commutative. Then it becomes a homotopy equivalence from our hypothesis. The composite map $\kappa f_*: FG^*(X) \to \hat{E}(G)^*(X) \to \text{Hom}(E^*(X), G)$ coincides with the homomorphism $\overline{\mu}$ induced by the pairing $\mu$, because $\text{Hom}(E^*(X), G) \to \text{Hom}(E^*(X), I_G)$ is a monomorphism.

3.2. Let us denote by $H, K, KO$ and $KSp$ the Eilenberg-MacLane spectrum, the $BU-$, $BO-$ and $BSp-$spectrum respectively. We now investigate the homotopy types of $H(G), K(G)$ and $KSp(G)$.

Theorem 5. For any abelian group $G$ $\hat{H}(G), \hat{K}(G)$ and $\hat{KSp}(G)$ have the same homotopy types of $HG, KG$ and $KOG$ respectively (cf., [3]).

Proof. The proof is essentially due to Anderson [3].

The $\hat{H}$ and $K$ cases: Let $E$ denote either $H$ or $K$, and $\mu_E: E \wedge E \to E$ the usual pairing. As is well known, $\overline{\mu}_E: \pi_*(E) \to \text{Hom}(\pi_*(E), Z)$ is an isomorphism. This implies that $\overline{\mu}_E: \pi_*(EG) \to \text{Hom}(\pi_*(E), G)$ is an isomorphism for all $G$. The result follows immediately from Lemma 10.

The $\hat{KSp}$ case: There is a well known pairing $\mu_{KSp}: KSp \wedge KO \to KSp$. We see easily that $\overline{\mu}_{KSp}: \pi_n(KO) \to \text{Hom}(\pi_{-n}(KSp), Z)$ is an isomorphism except $n \equiv 1, 2 \mod 8$, and hence $\overline{\mu}_{KSp}: \pi_n(KO, A) \to \text{Hom}(\pi_{-n}(KSp), A)$ is so for all $n$ and $Q$-modules $A$. Fix a $Q$-module $A$. For any subgroup $B$ of $A$ we define
homomorphisms \( \lambda_n \) by the composite maps

\[
\begin{align*}
\pi_n(KO) \otimes A/B & \cong \pi_n(KOA/B) \to \text{Hom}(\pi_{-n}(KSp), A/B) \quad \text{when } n \equiv 2, 3 \mod 8 \\
\pi_{n-1}(KO) \otimes B & \cong \pi_{n-1}(KOB) \cong \pi_n(KOA/B) \to \text{Hom}(\pi_{-n}(KSp), A/B) \\
& \quad \text{when } n \equiv 2, 3 \mod 8
\end{align*}
\]

which are natural with respect to \( B \). Let \( \eta_1 \) be the generator of \( \pi_1(KO) \cong \mathbb{Z} \), and define as the multiplications by \( \eta_1 \phi : \pi_1(KO) \to \pi_1(KO) \) and \( \phi : \pi_2(KSp) \to \pi_{-2}(KSp) \). Then we remark that \( \phi \)'s are isomorphisms and \( \phi^* \lambda_2 = \lambda_3(\phi \otimes 1) \).

The simplification \( \varepsilon_{SP} : K \to KSp \) induces a natural transformation \( K^*(Y) \to KSp^*(Y) \) of \( KO^*(\cdot) \)-modules for all finite \( Y \). So we get a weak homotopy commutative diagram

\[
\begin{array}{ccc}
K \wedge KO & \to & KSp \\
\downarrow & & \downarrow \\
KSp \wedge KO & \to & KSp
\end{array}
\]

(In fact this diagram is homotopy commutative by use of Corollary 13 below). This yields the commutative diagram

\[
\begin{array}{ccc}
\pi_2(KOA/B) & \to & \text{Hom}(\pi_{-2}(KSp), \pi_0(KSpA/B)) \\
\downarrow & & \downarrow \\
\pi_2(KA/B) & \to & \text{Hom}(\pi_{-2}(K), \pi_0(KA/B))
\end{array}
\]

The left vertical arrow is a monomorphism because \( \pi_2(KOA/B) = 0 \), and the lower horizontal ones are isomorphisms. Therefore the upper becomes a monomorphism, and hence so are both \( \lambda_2 \) and \( \lambda_3 \). Let \( \{ B_\lambda \} \) be the set of all finitely generated subgroups of \( B \). As is easily checked, \( \lambda_n \) are isomorphisms for all \( n \) and \( B_\lambda \), because \( B_\lambda \) is free. On the other hand, \( A/B \) is isomorphic to the direct limit of \( A/B_\lambda \) and \( \text{Hom}(\pi_{-n}(KSp), A/B) \cong \lim \text{Hom}(\pi_{-n}(KSpA/B_\lambda)) \). So we see immediately that \( \lambda_n \) are isomorphisms for any subgroup \( B \). Thus

\[
\tilde{\mu}_{KSp} : \pi_n(KOG') \to \text{Hom}(\pi_{-n}(KSp), G')
\]

is an isomorphism for any quotient group \( G' \) of a \( Q \)-module. Taking \( I_G \) and \( J_G \) as the above \( G' \) and applying Lemma 10 we get the desirable result.

In other words, Theorem 5 says that there exist universal coefficient sequences

\[
\begin{align*}
0 \to \text{Ext}(H_{n-1}(X), G) & \to H^G_n(X) \to \text{Hom}(H_n(X), G) \to 0 \\
(3.1) & \quad 0 \to \text{Ext}(K_{n-1}(X), G) \to K^G_n(X) \to \text{Hom}(K_n(X), G) \to 0 \\
0 \to \text{Ext}(KO_{n+3}(X), G) & \to KO^G_n(X) \to \text{Hom}(KO_{n+1}(X), G) \to 0
\end{align*}
\]

for any \( CW \)-spectrum \( X \).
Theorem 3 combined with (3.1) implies the following.

**Corollary 11.**

i) $HR^{n+1}(X)$ is Hausdorff if and only if $\text{Ext}(H_n(X)/TH_n(X), R)=0$.

ii) $KR^{n+1}(X)$ is Hausdorff if and only if $\text{Ext}(K_n(X)/TK_n(X) R)=0$.

iii) $KOR^{n+1}(X)$ is Hausdorff if and only if $\text{Ext}(KO_{n+4}(X)/TKO_{n+4}(XR) R)=0$.

3.3. Finally we shall make a comment for Hausdorff-ness of $K$-theories.

**Proposition 12.** Let $E$ be a CW-spectrum such that $\pi_*(E)$ is of finite type as an $R$-module and fix a degree $n$. If $\pi_k(X) \otimes \pi_{k-n}(E) \otimes Q=0$ for all $k$, then $E^{n+1}(X)$ is Hausdorff. (Cf., [8, Theorem III, 2]).

Proof. Under our assumptions we compute

$$E(Z[Z(X)]=\bigoplus H^*(X; \pi_{k-n}(E) \otimes Q) \approx \prod \text{Hom}(H_n(X), \pi_{k-n}(E) \otimes Z[Z)=0.$$ 

Then the result is immediate from [8, Theorem III. 1].

For CW-spectra $E$ and $X$ whose rational homotopy groups are sparse we have

**Corollary 13.** Let $E$ be a CW-spectrum such that $\pi_*(E)$ is of finite type as an $R$-module. Assume that $\pi_m(E) \otimes Q=\pi_m(X) \otimes Q=0$ unless $m \equiv 0 \mod n$. Then $E^{n+1}(X)$ is Hausdorff whenever $m \equiv 0 \mod n$.

As is well known, $\pi_{2n+1}(K)=0$ and $\pi_{m}(KO) \otimes Q=0$ if $m \equiv 0 \mod 4$. Therefore Corollary 13 implies

**Theorem 6.** i) $K^n(K_\Lambda \cdots \Lambda K)$ is Hausdorff.

ii) $KO^m(K_\Lambda \cdots \Lambda KO)$ is Hausdorff whenever $m \equiv 1 \mod 4$.

REMARK. Informations on $K_*(K)$ and $KO_*(KO)$ have been obtained by Adams, Harris and Switzer [2].

As an immediate corollary we have

**Corollary 14.** Complex and real $K$-theories $K^*$, $KO^*$ (defined on the category of CW-spectra) possess an associative and commutative multiplication.

OSAKA CITY UNIVERSITY

References


