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UNIVERSAL COEFFICIENT SEQUENCES FOR COHOMOLOGY THEORIES OF CW-SPECTRA

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Kainen [5] showed that there exists a cohomology theory $k^*(?)$ and a natural short exact sequence

$$0 \rightarrow \text{Ext}(h_{*-1}(X), G) \rightarrow k^*(X; G) \rightarrow \text{Hom}(h_*(X), G) \rightarrow 0$$

for any based CW -complex X if h_* is an (additive) homology theory and G is an abelian group. On the other hand, for an (additive) cohomology theory k^* such that $k^*(\text{point})$ has finite type Anderson [3] constructed a homology theory Dk_* and a natural exact sequence

$$0 \rightarrow \text{Ext}(Dk_{*-1}(F), Z) \rightarrow k^*(F) \rightarrow \text{Hom}(Dk_*(F), Z) \rightarrow 0$$

for any finite CW -complex whose extension to arbitrary CW -complexes is given in a form of a four term exact sequence. He then determined homology theories Dk_* in the special cases $k^*=H^*$, K^* and KO^* . Ordinary cohomology theory and complex K -theory are both self-dual and real K -theory is the dual of symplectic K -theory, i.e., $DH_*=H_*$, $DK_*=K_*$ and $DKSp_*=KO_*$. Moreover he asserted that D^2 is the identity, i.e., $D(Dk)_*=k_*$.

In this note we shall construct a CW -spectrum $\hat{E}(G)$ for every CW -spectrum E and abelian group G by Kainen's method involving an injective resolution of G , and state a relation between E and $\hat{E}(G)$ in a form of a universal coefficient sequence

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \rightarrow \hat{E}(G)^*(X) \rightarrow \text{Hom}(E_*(X), G) \rightarrow 0$$

for any CW -spectrum X . And we shall study some properties of $\hat{E}(G)$. For example, under a certain finiteness assumption on $\pi_*(E)$ we show that $\hat{E}(R)(R)$ has the same homotopy type of ER where $J?$ is a subring of the rationals Q (Theorem 2). The above universal coefficient sequence combined with Theorem 2 gives us a new criterion for $ER^*(X)$ being Hausdorff (Theorem 3). Also we shall discuss uniqueness of $\hat{E}(G)$ (Theorem 4). Furthermore, using Anderson's technique we investigate the homotopy type of $\hat{E}(G)$ in the special cases $E=H$, K and KSp (Theorem 5). Finally we note that $K^{2n}(K \wedge \cdots \wedge K)$

and $KO^m(KO \wedge \cdots \wedge KO)$, $m \equiv 1 \pmod{4}$, are both Hausdorff (Theorem 6).

1. Duality maps

1.1. Let $u: X' \wedge X \rightarrow W$ be a pairing of CW -spectra. Such a pairing defines a homomorphism

$$T = T(u)_E: \{Y, E \wedge X\} \rightarrow \{Y \wedge X, E \wedge W\}$$

by the relation $T(f) = (\lambda u)(f \wedge 1)$ for any CW -spectra Y and E . A pairing $u: X' \wedge X \rightarrow W$ is called an E -duality map provided $T(u)_E$ is an isomorphism for E fixed and $Y = \sum^k$ for all k . If u is an E -duality map, then $T(u)_E$ becomes an isomorphism for any CW -spectrum Y .

Fix CW -spectra X and W and consider the cohomology functor $\{- \wedge X, W\}$ defined on the category of CW -spectra. By the representability theorem, there exists a function spectrum $F(X, W)$ such that $T: \{Y, F(X, W)\} \rightarrow \{Y \wedge X, W\}$ is a natural isomorphism for all Y . So we see that the evaluation map

$$e: F(X, W) \wedge X \rightarrow W$$

is a S -duality map.

Let $u: X' \wedge X \rightarrow W$, $v: Y' \wedge Y \rightarrow W$, $f: X \rightarrow Y$, $g: Y' \rightarrow X'$ be maps such that $v(1 \wedge f)$ and $u(g \wedge 1)$ are homotopic. Consider the cofiber sequences

$$X \xrightarrow{f} Y \rightarrow Z, \quad Z' \rightarrow Y' \xrightarrow{g} X'.$$

We have a CW -spectrum Q and maps $p: X' \wedge X \rightarrow Q$, $q: Y' \wedge Y \rightarrow Q$, and $r: Z' \wedge Z \rightarrow Q$ giving rise to the diagram below homotopy commutative (up to sign)

$$\begin{array}{ccccc}
 & Y' \wedge X & \Sigma^{-1}(X' \wedge Z) & & Z' \wedge Y \\
 & \swarrow & \swarrow & \swarrow & \swarrow \\
 Y' \wedge Y & & X' \wedge X & & Z' \wedge Z & & Y' \wedge Y \\
 & \searrow q & \searrow p & \searrow r & \searrow q & & \\
 & & Q & & & &
 \end{array}$$

Since $v(1 \wedge f)$ and $u(g \wedge 1)$ are homotopic, an easy diagram chase shows that there exists a map $s: Q \rightarrow W$ with $s \circ p = u$ and $s \circ q = v$ (see [7, Proof of Theorem 13.1]). So we obtain a map

$$w: Z' \wedge Z \rightarrow W$$

making the diagram

$$(1.1) \quad \begin{array}{ccccc} \Sigma^{-1}(X' \wedge Z) & \rightarrow & Z' \wedge Z & \leftarrow & Z' \wedge Y \\ \downarrow I & & \downarrow w & & \downarrow I \\ X' \wedge X & \xrightarrow{u} & W & \xleftarrow{v} & Y' \wedge Y \end{array}$$

homotopy commutative (up to sign).

By use of (1.1) and “five lemma” we have

Lemma 1. *Let $u: X \wedge X \rightarrow W$, $v: Y' \wedge Y \rightarrow W$ be E -duality maps and assume that maps $f: X \rightarrow Y$ and $g: Y' \rightarrow X'$ satisfy the property that $v(\wedge f)$ and $u(g \wedge 1)$ are homotopic. Then the above map $w: Z' \wedge Z \rightarrow W$ is an E -duality map. (Cf., [6, Theorem 6.10]).*

Let $\mathcal{C} = \{X_n, f_n\}$ and $\mathcal{C}' = \{X'_n, g_n\}$ be a direct and an inverse sequence of CW -spectra respectively. Pairings $u_n: X'_n \wedge X_n \rightarrow W$ induce the homomorphism

$$\begin{aligned} T\{u_n\}: \{Y, E \wedge (\prod X'_n)\} &\rightarrow \{Y, \prod(E \wedge X'_n)\} \cong \prod \{Y, E \wedge X'_n\} \\ &\xrightarrow{\prod T(u_n)} \prod \{Y \wedge X_n, E \wedge W\} \cong \{\vee(Y \wedge X_n), E \wedge W\} \\ &\cong \{Y \wedge (\vee X_n), E \wedge W\}. \end{aligned}$$

Taking $Y = \prod X'_n$ and $E = S$, there is a map

$$u: (\prod X'_n) \wedge (\vee X_n) \rightarrow W$$

with the homotopy commutative square

$$(1.2) \quad \begin{array}{ccc} (\prod X'_n) \wedge X_n & \rightarrow & X'_n \wedge X_n \\ \downarrow & & \downarrow u_n \\ (\prod X'_n) \wedge (\vee X_n) & \xrightarrow{u} & W. \end{array}$$

Under the assumption that the canonical morphism $E \wedge (\prod X'_n) \rightarrow \prod(E \wedge X'_n)$ is a homotopy equivalence, we see that

(1.3) u is an E -duality map if so are all u_n .

Define maps $f: \vee X_n \rightarrow \vee X$ and $g: \prod X'_n \rightarrow \prod X'_n$ by

$$i_n - i_{n+1} f_n = f \cdot i_n, \quad p_n - g_n \cdot p_{n+1} = p_n \cdot g$$

where $i_n: X_n \rightarrow \vee X_n$, $p_n: \prod X'_n \rightarrow X'_n$ are the canonical maps. And, construct the telescope TC and the cotelescope T^*C' so that we have the cofiber sequences

$$\vee X_n \xrightarrow{f} \vee X \rightarrow TC, \quad T^*C' \rightarrow \prod X'_n \rightarrow tX'_n.$$

Proposition 2. *Let $\mathcal{C} = \{X_n, f_n\}$ and $\mathcal{C}' = \{X'_n, g_n\}$ be a direct and an inverse sequence of CW -spectra, and $u_n: X'_n \wedge X_n \rightarrow W$ be pairings such that $u_{n+1}(1 \wedge f_n)$*

and $u_n(g_n \wedge 1)$ are homotopic. Then there exists a map $u: T^*C' \wedge TC \rightarrow W$ such that the following diagram is homotopy commutative (up to sign):

$$\begin{array}{ccccc} \Sigma^{-1}(\Pi X_n' \wedge TC) & \rightarrow & T^*C' \wedge TC & \leftarrow & T^*C' \wedge (\vee X_n) \\ \downarrow & & \downarrow \tilde{u} & & \downarrow \\ (\Pi X_n') \wedge (\vee X_n) & \xrightarrow{u} & W & \xleftarrow{u} & (\Pi X_n') \wedge (\vee X_n). \end{array}$$

Moreover, assuming that the canonical morphism $E \wedge (\Pi X_n') \rightarrow \Pi(\bigwedge X_n')$ is a homotopy equivalence, \tilde{v} is an E -duality map if so are all u_n .

Proof. An easy diagram chase shows that $u(l \wedge f)$ and $u(g \wedge 1)$ are homotopic. We apply Lemma 1 and (1.3) to obtain the required map.

1.2. Let G be an abelian group and $\Gamma: 0 \rightarrow P_1 \xrightarrow{\phi} P_0 \rightarrow G \rightarrow 0$ a free resolution. We realize P_i and ϕ by wedges MP_i of sphere spectra and a map $M\phi: MP_1 \rightarrow MP_0$. The mapping cone $M\Gamma$ of $M\phi$ forms a Moore spectrum of type G . Then there exists a universal coefficient sequence

$$0 \rightarrow \text{Ext}(G, \pi_{*+1}(X)) \rightarrow \{M\Gamma, X\}_* \xrightarrow{\kappa} \text{Hom}(G, \pi_*(X)) \rightarrow 0$$

where κ associates to a map / the induced homomorphism f_* in 0-th homotopy (see [4]). Therefore a Moore spectrum of type G is uniquely determined up to homotopy type. For any CW -spectrum E we define the corresponding spectrum with coefficient group G

$$EG = E \wedge MG$$

where MG is a Moore spectrum of type G .

Let $/$ be a set of primes which may be empty, and denote by I_l the multiplicative set generated by the primes not in $/$. It is a directed set which is ordered by divisibility. If R is a subring of the rationals \mathbb{Q} (with unit), it is just "the integers localized at l " where $/$ is the set of primes which are not invertible in R . Thus $R = Z_l = I_l^{-1}Z$. Let l^c denote the set of primes $p_k (p_k < p_{k+1})$ not in $/$, i.e., $l \cap l^c = \{\phi\}$ and $l \cup l^c = \{\text{all primes}\}$. Putting $l_n = p_1^n \cdots p_n^n$, we choose a cofinal sequence $J_l = \{l_n\}$ in I_l .

Fix a CW -spectrum W . $\mathcal{C}_l = \{X_n = W, f_n = l_{n+1}/l_n\}$ and $\mathcal{C}_l^* = \{X_n' = Wg_n = l_{n+1}/l_n\}$ form respectively a direct and an inverse sequence (indexed by J_l). Denote by W_l, W_l^* the telescope of \mathcal{C}_l and the cotelescope of \mathcal{C}_l^* i.e.,

$$W_l = T\{Wf_n = l_{n+1}/l_n\}, \quad W_l^* = T^*\{Wg_n = l_{n+1}/l_n\}.$$

Notice that W_l is homotopy equivalent to $W \wedge S_l$. Since S_l is a Moore spectrum of type Z_l , an easy computation shows that

$$(1.4) \quad \text{tf} Z \wedge S \wedge Z \wedge n / Z, / \quad \text{and} \quad HZ_l^n(S_l) = 0 \quad \text{for} \quad n \neq 1$$

where l' is any set of primes with $l' \cap l^c \neq \{\phi\}$.

Define by ι_n and ρ_n the composite maps $\bar{W} \xrightarrow{i_n} \bigvee W \rightarrow W_l, W_l^* \rightarrow \prod W \xrightarrow{\rho_n} W$ and consider the cofiber sequences

$$W \xrightarrow{\iota_1} W_l \rightarrow \bar{W}_l, \quad \bar{W}_l^* \rightarrow Wf \xrightarrow{\rho_1} W.$$

\bar{S}_l is obviously a Moore spectrum of type Z_l/Z , and in addition

$$(1.5) \quad HZ_l^1(\bar{S}_l) \cong \hat{Z}_{l' \cap l} \text{ and } \text{ff} Z_l^?(S_l) = 0, \quad n \neq 1$$

for any l' with $l' \cap l^c \neq \{\phi\}$.

1.3. Here we construct two useful duality maps.

Proposition 3. *We have maps $\bar{u}: W_l^* \wedge S_l \rightarrow W$ and $\bar{w}: \bar{W}_l^* \wedge \bar{S}_l \rightarrow W$ such that the following diagram is commutative (up to sign) for all CW-spectra X and E :*

$$\begin{array}{ccccc} \{X, E \wedge W\} & \nearrow & \{X, E \wedge \bar{W}_l^*\} & \rightarrow & \{X, E \wedge W_l^*\} \\ & & \downarrow T(\bar{w}) & & \downarrow T(\bar{u}) \\ \{X \wedge \bar{S}_l, E \wedge W\} & \rightarrow & \{X \wedge S_l, E \wedge W\} & \rightarrow & \{X, E \wedge W\} \end{array}$$

Proof. Take as $u_n: W \wedge S \rightarrow W$ the canonical identification. From (1.2) and Proposition 2 we obtain maps $u: (\prod W) \wedge (\bigvee S) \rightarrow W, \bar{u}: Wf \wedge S_l \rightarrow W$ with the homotopy commutative squares

$$\begin{array}{ccccc} (\prod W) \wedge S \xrightarrow{\rho_n \wedge 1} W \wedge S & & W_l^* \wedge (\bigvee S) \rightarrow (\prod W) \wedge (\bigvee S) \\ 1 \wedge i_n \downarrow & \parallel & \downarrow & & \downarrow u \\ (\prod W) \wedge (\bigvee S) \xrightarrow{u} W & & W_l^* \wedge S_l \xrightarrow{u} W \end{array}$$

Putting the above two squares together we see that $\rho_n \wedge 1$ and $u(1 \wedge \iota_n)$ are homotopic. By (1.1) there exists a map $W: Wf \wedge \bar{S}_l \rightarrow W$ making the diagram below homotopy commutative (up to sign)

$$\begin{array}{ccccc} \Sigma^{-1}(W \wedge \bar{S}_l) \rightarrow Wf \wedge \bar{S}_l \rightarrow \bar{W}_l^* \wedge S_l \\ \downarrow & & \downarrow \bar{w} & & \downarrow \\ W \wedge S & = & W \xleftarrow{u} & & W_l^* \wedge S_l. \end{array}$$

Now we need the following result in order to apply Proposition 2.

Lemma 4. *Let G be a direct product of R -modules G_ω and M_ω a Moore spectrum of type G_ω . Then $\prod M_\omega$ becomes a Moore spectrum of type G , and the canonical morphism $E \wedge (\prod M_\omega) \rightarrow \prod (E \wedge M_\omega)$ is a homotopy equivalence if $\pi_*(E)$ has finite type as an R -module.*

Proof. The result of Adams [1, Theorem 15.2] asserts that $HR \wedge \prod M_\omega \rightarrow$

$\Pi(HR \wedge M_\alpha)$ is a homotopy equivalence. Thus ΠM_α becomes a Moore spectrum of type G because $\pi_*(\Pi M_\alpha)$ is an R -module and hence so is $H_*(\Pi M_\alpha)$. In the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \pi_*(E) \otimes G & \rightarrow & \pi_*(E \wedge \Pi M_\alpha) & \rightarrow & \text{Tor}_1^R(\pi_{*-1}(E), G) & \rightarrow & 0 \\ & \downarrow * & \downarrow & & \downarrow \text{I} & & \\ 0 \rightarrow \Pi \pi_*(E) \otimes_R G_\alpha & \rightarrow & \Pi \pi_*(E \wedge M_\alpha) & \rightarrow & \Pi \text{Tor}_1^R(\pi_{*-1}(E), G_\alpha) & \rightarrow & 0 \end{array}$$

involving the universal coefficient sequences, the left and right arrows are isomorphisms. The result follows from "five lemma".

Obviously the canonical identification $u_n: W \wedge S \rightarrow W$ is an E -duality map for every E . Using Propositions 2, 3 and Lemma 4 we obtain

Theorem 1. *Let G be an R -module and M a Moore spectrum of type G . Assume that $\pi_*(E)$ is of finite type as an R -module. Then the maps $\bar{u}: M_i^* \wedge S_i \rightarrow M$ and $\bar{w}: M_f \wedge \bar{S}_i \rightarrow M$ given in Proposition 3 are both E -duality maps.*

Remark that $\pi_*((S_i)_\phi^*)$ and $\pi_*((\bar{S}_i)_\phi^*)$ are Z_i -modules. Taking S_i as M and the empty ϕ as $/$ in the above theorem, we compute that

$$H_*((S_i)_\phi^*) \cong HZ_i^*(S_\phi), \quad H_*((\bar{S}_i)_\phi^*) \cong HZ_i^*(\bar{S}_\phi).$$

Thus $\Sigma(S_i)_\phi^*$ and $\Sigma(\bar{S}_i)_\phi^*$ are Moore spectra of type \hat{Z}_i/Z_i and of type \hat{Z}_i where $i \neq \{\phi\}$, because of (1.4) and (1.5). So we get

Corollary 5. *Assume that $\pi_*(E)$ is of finite type as an R -module where R is a proper subring of Q . Then there exist natural isomorphisms $T(\bar{w}): E\hat{R}^*(X) \rightarrow E^{*+1}(X \wedge \bar{S}_\phi)$, $T(\bar{u}): E\hat{R}/Z^*(X) \rightarrow E^{*+1}(X \wedge S_\phi)$ with the commutative (up to sign) diagram*

$$\begin{array}{ccccc} & E\hat{R}^*(X) & \longrightarrow & E\hat{R}/Z^*(X) & \\ & \uparrow & & \downarrow & \\ E^*(X) & & \text{I } T(\bar{w}) & & E^{*+1}(X) \\ & \downarrow & & \uparrow & \\ & E^{*+1}(X \wedge \bar{S}_\phi) & \longrightarrow & E^{*+1}(X \wedge S_\phi) & \end{array}$$

2. Universal coefficient sequences

2.1. Following Kainen [5] we shall construct a universal coefficient sequence for a generalized cohomology theory. Fix a CW -spectrum E . For every injective abelian group $/$ $\text{Hom}(E_*(-), /)$ forms a cohomology theory defined on the category of CW -spectra. The representability theorem gives us a CW -spectrum $\hat{E}(I)$ and a natural isomorphism

$$T_I: \{X, \hat{E}(I)\} \rightarrow \text{Hom}(E_*(X), I)$$

for any CW -spectrum X . Let G be an abelian group and $\Gamma: 0 \rightarrow G \rightarrow I \xrightarrow{\psi} J \rightarrow 0$

an injective resolution. Then there exists a unique (up to homotopy) map $\psi: \hat{E}(I) \rightarrow \hat{E}(J)$ whose induced homomorphism coincides with the natural transformation $T_{\hat{J}}^{-1} \cdot \psi_* \cdot T_I$. Denote by $\Sigma \hat{E}(\Gamma)$ the mapping cone of $\hat{\psi}$, i.e.,

$$\hat{E}(\Gamma) \rightarrow \hat{E}(I) \rightarrow \hat{E}(J)$$

is a cofiber sequence. By homological algebra we obtain a natural exact sequence

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \rightarrow \hat{E}(\Gamma)^*(X) \rightarrow \text{Hom}(E_*(X), G) \rightarrow 0$$

for all X .

Let $\phi: G \rightarrow G'$ be a homomorphism and $\Gamma: 0 \rightarrow G \rightarrow I \rightarrow J \rightarrow 0$, $\Gamma': 0 \rightarrow G' \rightarrow I' \rightarrow J' \rightarrow 0$ be injective resolutions. For a morphism $\mu: \Gamma \rightarrow \Gamma'$ which is a lift of ϕ , we may choose a map

$$A: \hat{E}(\Gamma) \rightarrow \hat{E}(\Gamma')$$

making the diagram with cofiber sequences

$$\begin{array}{ccccccc} \hat{E}(\Gamma) & \rightarrow & \hat{E}(I) & \rightarrow & \hat{E}(J) & \rightarrow & \Sigma \hat{E}(\Gamma) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{E}(\Gamma') & \rightarrow & \hat{E}(I') & \rightarrow & \hat{E}(J') & \rightarrow & \Sigma \hat{E}(\Gamma') \end{array}$$

homotopy commutative. However μ is not uniquely determined (up to homotopy). The map μ yields the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \text{Ext}(E_{*-1}(X), G) & \rightarrow & \hat{E}(\Gamma)^*(X) & \rightarrow & \text{Hom}(E_*(X), G) & \rightarrow & 0 \\ & \downarrow \phi_* & \downarrow \hat{\mu}_* & & \downarrow \phi_* & & \\ 0 \rightarrow \text{Ext}(E_{*-1}(X), G') & \rightarrow & \hat{E}(\Gamma')^*(X) & \rightarrow & \text{Hom}(E_*(X), G') & \rightarrow & 0. \end{array}$$

With an application of “five lemma” we find that $\hat{\mu}: \hat{E}(\Gamma) \rightarrow \hat{E}(\Gamma')$ is a homotopy equivalence if $\phi: G \rightarrow G'$ is an isomorphism. Thus the homotopy type of $\hat{E}(\Gamma)$ is independent of the choice of an injective resolution Γ of G . So we may put

$$\hat{E}(G) = \hat{E}(\Gamma), \quad \hat{\phi} = \hat{\mu}.$$

Consequently we get

Proposition 6. *Let E be a CW-spectrum and G an abelian group. Then there exists a CW-spectrum $\hat{E}(G)$ so that*

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \xrightarrow{\eta} \hat{E}(G)^*(X) \rightarrow \text{Hom}(E_*(X), G) \rightarrow 0$$

is a natural exact sequence for any CW-spectrum X . Moreover a homomorphism $\phi: G \rightarrow G'$ induces a (non-unique) map $\hat{\phi}: \hat{E}(G) \rightarrow \hat{E}(G')$ with the commutative diagram

$$\begin{array}{ccccccc}
0 \rightarrow \text{Ext}(E_{*-1}(X), G) & \rightarrow & \hat{G}^*(X) & \rightarrow & \text{Hom}(E_*(X), G) & \rightarrow & 0 \\
& & \downarrow \phi_* & & \downarrow \hat{\phi}_* & & \downarrow \phi_* \\
0 \rightarrow \text{Ext}(E_{*-1}(X), G') & \rightarrow & \hat{E}(G')^*(X) & \rightarrow & \text{Hom}(E_*(X), G') & \rightarrow & 0.
\end{array}$$

(Cf., [5]).

If Y is a finite CW -spectrum, then the function dual $Y^* = F(Y, S)$ can be taken finite and $E_*(Y) \cong E^{-*}(Y^*)$, $E^*(Y) \cong E_{-*}(Y^*)$. We notice that

(2.1) *there exists a natural exact sequence*

$$0 \rightarrow \text{Ext}(E^{*+1}(Y), G) \rightarrow \hat{E}(G)_*(Y) \rightarrow \text{Hom}(E^*(Y), G) \rightarrow 0$$

for all finite Y .

Let $/: E \rightarrow F$ be a map of CW -spectra. Then $/$ induces a (non-unique) map $/: F(G) \rightarrow E(G)$ such that the diagram

$$\begin{array}{ccccccc}
0 \rightarrow \text{Ext}(F_{*-1}(X), G) & \rightarrow & \hat{F}(G)^*(X) & \rightarrow & \text{Hom}(F_*(X), G) & \rightarrow & 0 \\
(2.2) \quad & & \downarrow f^* & & \downarrow \hat{f}_* & & \downarrow f^* \\
0 \rightarrow \text{Ext}(E_{*-1}(X), G) & \rightarrow & \hat{E}(G)^*(X) & \rightarrow & \text{Hom}(E_*(X), G) & \rightarrow & 0
\end{array}$$

is commutative. Remark that $/$ becomes a homotopy equivalence if so is $/$. Hence we find that

(2.3) *the homotopy type of $\hat{E}(G)$ depends only on that of E and the isomorphism class of G .*

2.2. For simplicity we write \hat{E} instead of $\hat{E}(Z)$. We shall now show that $\hat{E}(G)$ and $\hat{E}G$ have the same homotopy type under some finiteness assumptions on E and G . First we require the following

Lemma 7. i) *Let G be a direct product of abelian groups G_α , i.e., $G = \prod G_\alpha$. Then $\hat{E}(G)$ is homotopy equivalent to $\prod \hat{E}(G_\alpha)$.*
 ii) *Let G be a direct sum of R -modules G_α , i.e., $G = \sum G_\alpha$, and assume that $\pi_*(E)$ is of finite type as an R -module. Then $\hat{E}(G)$ is homotopy equivalent to $\vee \hat{E}(G_\alpha)$.*

Proof. i) Denote by p_α the canonical projection from G onto G_α . The map $\prod \hat{p}_\alpha: \hat{E}(G) \rightarrow \prod \hat{E}(G_\alpha)$ induces the composite homomorphism

$$\hat{E}(G)^*(X) \rightarrow \prod \hat{E}(G_\alpha)^*(X) \cong (\prod \hat{E}(G_\alpha))^*(X)$$

for any CW -spectrum X . In the commutative diagram

$$\begin{array}{ccccccc}
0 \rightarrow \text{Ext}(E_{*-1}(X), G) & \rightarrow & \hat{E}(G)^*(X) & \rightarrow & \text{Hom}(E_*(X), G) & \rightarrow & 0 \\
& & \downarrow \text{I} & & \downarrow \text{I} & & \downarrow \\
0 \rightarrow \prod \text{Ext}(E_{*-1}(X), G_\alpha) & \rightarrow & \prod \hat{E}(G_\alpha)^*(X) & \rightarrow & \prod \text{Hom}(E_*(X), G_\alpha) & \rightarrow & 0
\end{array}$$

involving the universal coefficient sequences, the left and right arrows are isomorphisms. By "five lemma" the center becomes an isomorphism, and hence the map $\prod \hat{p}_\alpha$ is a homotopy equivalence.

ii) The canonical injections $i_\alpha: G_\alpha \rightarrow G$ induce the composite homomorphism

$$(\bigvee \hat{E}(G_\alpha))^*(Y) \leftarrow \sum E(G_\alpha)^*(Y) \rightarrow E(G)^*(Y)$$

for any finite Y . Consider the commutative diagram

$$\begin{array}{ccccccc} \mathbf{0} & \rightarrow & \sum \text{Ext}_R^1(E_{*-1}(Y), G_\alpha) & \rightarrow & \sum^{\wedge} (G_\alpha)^*(Y) & \rightarrow & \sum \text{Hom}_R(E_*(Y), G_\alpha) \rightarrow \mathbf{0} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Ext}_R^1(E_{*-1}(Y), G) & \rightarrow & \hat{E}(G)^*(Y) & \rightarrow & \text{Hom}_R(E_*(Y), G) \rightarrow 0. \end{array}$$

The vertical arrows on both sides are isomorphisms whenever Y is finite. So the map $\bigvee i_\alpha: \bigvee \hat{E}(G_\alpha) \rightarrow \hat{E}(G)$ becomes a homotopy equivalence.

Fix a subring R of Q and assume that $\pi_*(E)$ has finite type as an R -module. For any subrings $R', R'', R' \subset R \subset R''$, the composite maps

$$(2.4) \quad e(R'): \hat{E}(R)R' \rightarrow E(R)R \leftarrow \hat{E}(R), \quad e(R''): E(R)R'' \rightarrow E(R'')R'' \leftarrow \hat{E}(R'')$$

become homotopy equivalences because all arrows induce isomorphisms in homotopy. So we consider the diagram

$$\begin{array}{ccccccc} E(R)R' & \rightarrow & E(R)Q & \rightarrow & \hat{E}(R)Q/R' & \rightarrow & \sum \hat{E}(R)R' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{E}(R) & \rightarrow & \hat{E}(Q) & \rightarrow & \hat{E}(Q/R) & \rightarrow & \sum \hat{E}(R) \end{array}$$

such that the rows are cofiber sequences and the left square is homotopy commutative. Then there exists a homotopy equivalence

$$(2.5) \quad e(Q/R'): \hat{E}(R)Q/R' \rightarrow \hat{E}(Q/R)$$

(denoted by a dotted arrow in the above diagram) which makes the diagram into a morphism of cofiber sequences. Moreover we get a map

$$(2.6) \quad e(Z_q): \hat{E}(R)Z_q \rightarrow E(Z_q)$$

for the R -module Z_q , which becomes also a homotopy equivalence. This gives rise to a homotopy commutative diagram

$$\begin{array}{ccccccc} \hat{E}(R)Z_q & \rightarrow & E(R)Q/R & \rightarrow & E(R)Q/R & \rightarrow & \sum \hat{E}(R)Z_q \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{E}(Z_q) & \rightarrow & \hat{E}(Q/R) & \rightarrow & \hat{E}(Q/R) & \rightarrow & \sum \hat{E}(Z_q) \end{array}$$

where the rows are cofiber sequences associated with the injective resolution

$$0 \rightarrow Z_q \rightarrow Q/R \xrightarrow{q} Q/R \rightarrow 0.$$

Proposition 8. *Let G be a direct sum or product of finitely generated R -modules G_α . If $\pi_*(E)$ is of finite type as an R -module, then $\hat{E}(G)$ has the same homotopy type of $\hat{E}(R)G$.*

Proof. We may put $G_\alpha = R$ or Z_q . Using (2.4), (2.6) and Lemmas 4, 7 we find that the composite maps

$$E(R)G \leftarrow \vee \hat{E}(R)G_\alpha \rightarrow \vee \hat{E}(G_\alpha) \rightarrow \hat{E}(G), \quad E(R)G \rightarrow \prod \hat{E}(R)G_\alpha \rightarrow \prod \hat{E}(G_\alpha) \leftarrow \hat{E}(G)$$

are homotopy equivalences.

2.3. Let $S \xrightarrow{\iota_1} S_\phi \xrightarrow{\tilde{i}} \bar{S}_\phi$ be the cofiber sequence constructed in §1. Assume that for all finite CW -spectra Y we have natural homomorphisms

$$\phi': E^*(Y \wedge \bar{S}_\phi) \rightarrow F^*(Y \wedge \bar{S}_\phi), \quad \phi'': E^*(Y \wedge S_\phi) \rightarrow F^*(Y \wedge S_\phi)$$

which satisfy the relation that $\phi''(1 \wedge \tilde{i})^* = (1 \wedge \tilde{i})^* \phi'$. Moreover we assume that $\pi_*(E)$ and $\pi_*(F)$ are R -modules where R is a proper subring of Q . If $\pi_*(F)$ is of finite type, then $F\hat{R}^*(X)$ and $FR/Z^*(X)$ are always Hausdorff for all X [8, III]. Thus $FR^*(X) \cong \varprojlim F\hat{R}^*(X^\lambda)$ and $F\hat{R}/Z^*(X) \cong \varprojlim FR/Z^*(X^\lambda)$ where $\{X^\lambda\}$ runs over the set of all finite subspectra of X . Applying Corollary 5 (and Proposition 3) we obtain natural homomorphisms

$$\psi': E\hat{R}^*(X) \rightarrow FR^*(X), \quad \psi'': E\hat{R}/Z^*(X) \rightarrow F\hat{R}/Z^*(X)$$

for arbitrary X which gives us the commutative square

$$\begin{array}{ccc} E\hat{R}^*(X) & \rightarrow & ER/Z^*(X) \\ \downarrow & & \downarrow \\ F\hat{R}^*(X) & \rightarrow & F\hat{R}/Z^*(X). \end{array}$$

Putting $f' = \psi'(1_{E\hat{R}})$ and $f'' = \psi''(1_{E\hat{R}/Z})$ we get the diagram

$$\begin{array}{ccccccc} E & \rightarrow & E\hat{R} & \rightarrow & E\hat{R}/Z & \rightarrow & \Sigma E \\ & & f' \downarrow & & \downarrow f'' & & \\ F & \rightarrow & F\hat{R} & \rightarrow & F\hat{R}/Z & \rightarrow & \Sigma F \end{array}$$

with cofiber sequences and a homotopy commutative square. Then there exists a map

$$(2.7) \quad \quad \quad /: E \rightarrow F$$

making the above diagram into a morphism of cofiber sequences. In particular, we obtain the following result which is a useful tool in studying properties of $\hat{E}(G)$.

Lemma 9. Assume that $\pi_*(E)$ and $\pi_*(F)$ have finite type as R -modules. If for any finite CW-spectrum Y we have natural isomorphisms

$$\phi': E^*(Y \wedge \bar{S}_\phi) \rightarrow F^*(Y \wedge \bar{S}_\phi), \quad \phi'': E^*(Y \wedge S_\phi) \rightarrow F^*(Y \wedge S_\phi)$$

such that $\phi''(1 \wedge \bar{t})^* - (1 \wedge \bar{t})^* \phi'$, then E is homotopy equivalent to F .

Now we study the homotopy type of $\hat{E}(R)$ by use of Lemma 9.

Theorem 2. If $\pi_*(E)$ is of finite type as an R' -module, then $\hat{E}(R')(R)$ has the same homotopy type of ER where $R' \subset R \subset Q$.

Proof. By [8, (II. 1.10)], (2.1) and Proposition 6 the composite homomorphism

$$E^*(Y) \otimes Q \rightarrow \text{Hom}(\text{Hom}(E^*(Y), R'), Q) \xrightarrow{\kappa^*} \text{Hom}(\hat{E}(R')_*(Y), Q) \xleftarrow{\kappa} \hat{E}(R')(Q)^*(Y)$$

is a natural isomorphism for all finite Y . In particular the coefficient $\pi_*(\hat{E}(R')(Q))$ is equal to the Q -module $\pi_*(EQ)$. Therefore $\hat{E}(R')(Q)$ becomes homotopy equivalent to EQ .

So we may assume that R is a proper subring of Q . For any finite Y we consider the following diagram

$$\begin{array}{ccccc} ER^{*+1}(Y \wedge \bar{S}_\phi) & \xrightarrow{T(\bar{w})} & (ER)\hat{R}^*(Y) & \leftarrow & E^*(Y) \otimes \text{Ext}(Q/R', R) \\ \downarrow & & \downarrow & & \downarrow \\ ER^{*+1}(Y \wedge S_\phi) & \xleftarrow{T(\bar{u})} & (ER)\hat{R}/Z^*(Y) & \leftarrow & E^*(Y) \otimes \text{Ext}(Q, R) \\ & & & & \downarrow \\ & & & & \text{Ext}(\text{Hom}(E^*(Y), Q/R'), R) \xrightarrow{\kappa^*} \text{Ext}(\hat{E}(Q/R')_*(Y), R) \\ & & & & \downarrow \\ & & & & \text{Ext}(\text{Hom}(E^*(Y), Q), R) \xrightarrow{\kappa^*} \text{Ext}(\hat{E}(Q)_*(Y), R) \\ & & & & \downarrow \\ & & & & \text{Ext}(\hat{E}(R')Q/Z_*(Y), R) \xrightarrow{\eta} \hat{E}(R')(R)^*(Y \wedge S_\phi) \\ & & & & \downarrow \\ & & & & \text{Ext}(\hat{E}(R')Q_*(Y), R) \xrightarrow{\eta} \hat{E}(R')(R)^*(Y \wedge S_\phi) \end{array}$$

(in which we drop the subscript R' on the functors \otimes , $\text{Hom}_{R'}$, $\text{Ext}_{R'}$). Note that $\text{Ext}(Q, R) \cong \hat{R}/R \cong R \otimes \hat{R}/\mathbb{Z}$ and $\text{Ext}(Q/R', R) \cong \hat{R}$. All squares are commutative by Corollary 5, (2.1), (2.5) and Proposition 6. In addition all horizontal arrows are isomorphisms because of Corollary 5, [8, (II. 1.10)], (2.1), (2.4), (2.5) and Proposition 6. Applying Lemma 9 to the above diagram the desirable result is obtained.

Theorem 2 asserts that we have a natural exact sequence

$$(2.8) \quad 0 \rightarrow \text{Ext}(\hat{E}(R')_{*-1}(X), R) \rightarrow ER^*(X) \rightarrow \text{Hom}(\hat{E}(R')_*(X), R) \rightarrow 0$$

for any X and $R' \subset R \subset Q$ if $\pi_*(E)$ is of finite type as an R' -module.

Using the above universal coefficient sequence we give a new criterion for $ER^*(X)$ being Hausdorff.

Theorem 3. Assume that $\pi_*(E)$ has finite type as an R' -module. $(ER)^{n+1}(X)$ is Hausdorff if and only if $\text{Ext}(\hat{E}(R')_n(X)/\hat{T}E(R')_n(\lambda R) = 0$ where $R' \subset R \subset Q$ and TG denotes the torsion subgroup of G .

Proof. The proof is similar to that of [8, Theorem IV. 4]. Assume that R is a proper subring. Recall that $ER^{n+1}(X)$ is Hausdorff if and only if the boundary homomorphism $\delta: ER\hat{R}/Z^n(X) \rightarrow ER^{n+1}(X)$ is trivial (cf., [8, Theorem III.1]). Then Corollary 5 implies that $ER^{n+1}(X)$ is Hausdorff if and only if $(1 \wedge \iota_1)^*: ER^{n+1}(X \wedge S_\phi) \rightarrow ER^{n+1}(X)$ is trivial. In the commutative diagram

$$\begin{array}{ccccccc} \text{Ext}(\hat{E}(R')Q_n(X), R) & \rightarrow & ER^{n+1}(X \wedge S_\phi) & & & & \\ \downarrow & & \downarrow & & & & \\ 0 \rightarrow \text{Ext}(\hat{E}(R')_n(X), R) & \rightarrow & ER^{n+1}(X) & \rightarrow & \text{Hom}(\hat{E}(R')_{n+1}(X), R) & \rightarrow & 0 \end{array}$$

the upper arrow is an isomorphism and the lower row is exact by (2.8). On the other hand, the left vertical arrow admits a factorization

$$\text{Ext}(\hat{E}(R')Q_n(X), R) \rightarrow \text{Ext}(\hat{E}(R')_n(X)/T\hat{E}(R')_n(\lambda R), R) \rightarrow \text{Ext}(\hat{E}(R')_n(X), R)$$

such that the former is an epimorphism but the latter is a monomorphism. An easy diagram chase shows that $(1 \wedge \iota_1)^*$ is the zero map if and only if $\text{Ext}(\hat{E}(R')_n(X)/T\hat{E}(R')_n(\lambda R) = 0$. So the result follows immediately.

2.4. Now we discuss uniqueness of $\hat{E}(G)$ under some restrictions on E and G .

Theorem 4. Let G be a finitely generated R -module with $\text{Tor}(\pi_*(E), G) = 0$, and assume that $\pi_*(E)$ is of finite type as an R -module. If F satisfies the property that there exists a natural exact sequence

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \rightarrow F^*(X) \rightarrow \text{Hom}(E_*(X), G) \rightarrow 0$$

for any CW-spectrum X , then F has the same homotopy type of $\hat{E}(G)$.

Proof. Assume that R is a proper subring of O . The torsion subgroup $T = TG$ is a direct summand of G and the quotient $P = G/TG$ is a free R -module. Consider the commutative diagram

$$\begin{array}{ccccccc} \hat{E}(P)^{*+1}(X \wedge \bar{S}_\phi) & \xrightarrow{\sim} & \text{Ext}(EQ/Z_*(X), P) & \rightarrow & \text{Ext}(EQ/Z_*(X), G) & \rightarrow & F^{*+1}(X \wedge \bar{S}_\phi) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{E}(P)^{*+1}(X \wedge S_\phi) & \xleftarrow{\cong} & \text{Ext}(EQ_*(X), P) & \rightarrow & \text{Ext}(EQ_*(X), G) & \rightarrow & F^{*+1}(X \wedge S_\phi) \end{array}$$

for any X . (2.7) gives rise to a map $f: \hat{E}(P) \rightarrow F$ with the commutative diagram

$$\begin{array}{ccccc} \hat{E}(P)^*(X) & \rightarrow & \hat{E}(P)\hat{R}^*(X) & \rightarrow & \hat{E}(P)\hat{R}/Z^*(X) \\ \downarrow & & \downarrow & & \downarrow \\ F^*(X) & \rightarrow & FR^*(X) & \rightarrow & F\hat{R}/Z^*(X). \end{array}$$

Looking at the previous diagram we find that in the above the central arrow is a monomorphism and the right is an isomorphism. So $f^*: \hat{E}(P)^*(X) \rightarrow F^*(X)$ becomes a monomorphism whenever $\hat{E}(P)^*(X)$ is Hausdorff, and in addition f induces an isomorphism $f_*: \pi_*(\hat{E}(P)) \otimes Q \rightarrow \pi_*(F) \otimes Q$. Denote by F_T the mapping cone of f , thus

$$\hat{E}(P) \xrightarrow{f} F \xrightarrow{g} F_T$$

is a cofiber sequence. $\hat{E}(P)^*(F_T)$ is Hausdorff as $\pi_*(F_T) \otimes Q = 0$ [8, Theorem III. 2]. Therefore we have a short exact sequence

$$0 \rightarrow \hat{E}(P)^*(F_T) \rightarrow F^*(F_T) \rightarrow F_T^*(F_T) \rightarrow 0.$$

Then we may choose a map $h: F_T \rightarrow F$ such that the composite map $g \circ h$ is homotopic to the identity. This means that the sequence

$$0 \rightarrow \hat{E}(P)^*(X) \rightarrow F^*(X) \rightarrow F_T^*(X) \rightarrow 0$$

is split exact. F is obviously homotopic to the wedge of $\hat{E}(P)$ and F_T .

We are now left to show that F_T has the same homotopy type of $\hat{E}(T)$ under the assumption that $\text{Tor}(\pi_*(E), G) = 0$. Consider the commutative exact diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow \text{Ext}(EQ/Z_*(X), P) & \rightarrow & \hat{E}(P)^{**+1}(X \wedge \bar{S}_\phi) & \rightarrow & \text{Hom}(EQ/Z_{**+1}(X), P) & \rightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow \text{Ext}(EQ/Z_*(X), G) & \rightarrow & F^{**+1}(X \wedge \bar{S}_\phi) & \rightarrow & \text{Hom}(EQ/Z_{**+1}(X), G) & \rightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \text{Ext}(EQ/Z_*(X), T) & & F_T^{**+1}(X \wedge \bar{S}_\phi) & & \text{Hom}(EQ/Z_{**+1}(X), T) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

in which $\text{Hom}(EQ/Z_{**+1}(X), P) = 0$. With an application of "3 x 3 lemma" we get a natural exact sequence

$$0 \rightarrow \text{Ext}(EQ/Z_*(X), T) \rightarrow F_T^{**+1}(X \wedge \bar{S}_\phi) \rightarrow \text{Hom}(EQ/Z_{**+1}(X), T) \rightarrow 0$$

for any X . Take a free resolution $0 \rightarrow P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} T \rightarrow 0$ consisting of finitely generated R -modules. The composite homomorphisms

$$\begin{aligned} \text{Ext}(EQ/Z_*(X), P_i) &\leftarrow \text{Ext}(EQ/Z_*(X), R) \otimes_R P_i \\ &\rightarrow \hat{E}(R)^{**+1}(X \wedge \bar{S}_\phi) \bigoplus_R P_i \rightarrow \hat{E}(R)P_i^{**+1}(X \wedge \bar{S}_\phi) \end{aligned}$$

are isomorphisms. The R -free resolution yields the following commutative exact diagram

$$\begin{array}{ccccccc} & & & & & & 0 \\ 0 \rightarrow & \text{Ext}(EQ/Z_{*-1}(X), T) & \xrightarrow{\eta'} & F_T^*(X \wedge \bar{S}_\phi) & \xrightarrow{\kappa'} & \text{Hom}(EQ/Z_*(X), T) & \rightarrow 0 \\ & \searrow & & \swarrow \partial & & & \\ & \text{Ext}(EQ/Z_*(X), P_1) & \xrightarrow[\eta]{\cong} & \hat{E}(R)P_1^{**+1}(X \wedge \bar{S}_\phi) & & & \\ & \downarrow \phi_* & & \downarrow \phi_* & & & \\ & \text{Ext}(EQ/Z_*(X), P_0) & \xrightarrow[\eta]{\cong} & \hat{E}(R)P_0^{**+1}(X \wedge \bar{S}_\phi) & & & \\ & \downarrow \psi_* & & & & & \\ 0 \rightarrow & \text{Ext}(EQ/Z_*(X), T) & \xrightarrow{\eta'} & F_T^{**+1}(X \wedge \bar{S}_\phi) & \xrightarrow{\kappa} & \text{Hom}(EQ/Z_{*+1}(X), T) & \rightarrow 0 \\ & \downarrow & & & & & \\ & 0 & & & & & \end{array}$$

Define homomorphisms

$$\bar{\psi}: \hat{E}(R)P_0^*(X \wedge \bar{S}_\phi) \rightarrow F_T^*(X \wedge \bar{S}_\phi), \quad \bar{\delta}: F_T^*(X \wedge \bar{S}_\phi) \rightarrow \hat{E}(R)P_1^{**+1}(X \wedge \bar{S}_\phi)$$

by the composite maps $\bar{\psi} = \eta' \psi_* \eta^{-1}$, $\bar{\delta} = \eta \kappa'$. By an easy diagram chase we show that the long sequence

$$\rightarrow \hat{E}(R)P_1^*(X \wedge \bar{S}_\phi) \rightarrow \hat{E}(R)P_0^*(X \wedge \bar{S}_\phi) \xrightarrow{\bar{\psi}} F_T^*(X \wedge \bar{S}_\phi) \xrightarrow{\bar{\delta}} \hat{E}(R)P_1^{**+1}(X \wedge \bar{S}_\phi) \rightarrow$$

is exact for all X .

Next we consider the commutative exact diagram

$$\begin{array}{ccccc} & \downarrow & & \downarrow & \\ E(R)P_1^*(X) & \xrightarrow{\phi_*} & \hat{E}(R)P_0^*(X) & & F_T^*(X) \\ & \downarrow \iota & \downarrow \iota & & \downarrow \iota' \\ \rightarrow \hat{E}(R)P_1\hat{R}^*(X) & \xrightarrow{\psi_*} & E(R)P_0R^*(X) & \xrightarrow{\psi^*} & F_T R^*(X) \rightarrow \\ & \downarrow & \downarrow & & \\ \hat{E}(R)P_1\hat{R}/Z^*(X) & \xrightarrow[\phi_*]{\cong} & \hat{E}(R)P_0\hat{R}/Z^*(X) & & \\ & \downarrow & \downarrow & & \end{array}$$

in which the middle row is rewritten the previous long exact sequence by the aid of Corollary 5. As is easily seen, we get an exact sequence

$$E(R)P_1^*(X) \xrightarrow{\phi_*} E(R)P_0^*(X) \xrightarrow{\rho} F_T^*(X)$$

for any X . Taking $e' = \rho(1_{\hat{E}(R)P_0})$ the composite map $e' \phi$ becomes homotopic

to the zero map. Therefore e' admits a factorization (up to homotopy)

$$\hat{E}(R)P_0 \rightarrow \hat{E}(R)T \xrightarrow{e} F_T.$$

This yields the commutative triangle

$$\begin{array}{ccc} \hat{E}(R)^*(X) \otimes_R T & \nearrow & \hat{E}(R)T^*(X) \\ & \searrow \rho' & \downarrow e_* \\ & & F_T^*(X) \end{array}$$

If $\text{Tor}(\pi_*(E), T) = 0$, then $\text{Tor}(\pi_*(\hat{E}(R)), T) = 0$ and hence the above e_* is an isomorphism. So F_T becomes homotopy equivalent to $\hat{E}(T)$ because of Proposition 8. Putting this and the previous result together, the required result is obtained from Lemma 7.

3. Complex and real K -theories

3.1. First we shall construct an injective resolution

$$\Gamma(G): 0 \rightarrow G \rightarrow I_G \rightarrow J_G \rightarrow 0$$

for every abelian group G which is functorial in G (see [5]). Let $A(G)$ denote the direct sum of copies A_g of A which runs over the set of all elements g of G , where $A = \mathbb{Z}$, \mathbb{Q} or \mathbb{Z}/\mathbb{Q} . G admits the canonical free resolution $0 \rightarrow P \rightarrow Z(G) \rightarrow G \rightarrow 0$. Consider the commutative exact diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & & \\ & \downarrow & & \downarrow & & & \\ & p & = & P & & & \\ & \downarrow & & \downarrow & & & \\ 0 & \rightarrow & Z(G) & \rightarrow & Q(G) & \rightarrow & Q/Z(G) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & G & \rightarrow & I_G & \rightarrow & J_G \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

and take the lower row in the above diagram as $\Gamma(G)$. Note that J_G is isomorphic to $Q/Z(G)$.

Let $\mu: E \wedge F \rightarrow W$ be a pairing of CW -spectra with $\pi_0(W) \cong \mathbb{Z}$ and $\pi_{-1}(W)$ torsion free. This gives rise to the natural homomorphism

$$\bar{\mu}: FG^*(X) \rightarrow \text{Hom}(E_*(X), \pi_0(WG))$$

for any G .

We shall require the following result in studying the duals of K -theories.

Lemma 10. *If μ induces isomorphisms $\overline{\mu}: \pi_*(FG') \rightarrow \text{Hom}(\pi_{-*}(E), G')$ in the cases $G' = I_G, J_G$, then we have a homotopy equivalence $f: FG \rightarrow \hat{E}(G)$ with $\overline{\mu} = \kappa f_*$.*

Proof. In the commutative diagram

$$\begin{array}{ccccc} FG^*(X) & \rightarrow & \text{Hom}(E_*(X), G) & \xleftarrow{\quad} & \hat{E}(G)^*(X) \\ \downarrow & & \downarrow & & \downarrow \\ FI_G^*(X) & \rightarrow & \text{Hom}(E_*(X), I_G) & \xleftarrow{\quad} & \hat{E}(I_G)^*(X) \\ \downarrow & & \downarrow & & \downarrow \\ FJ_G^*(X) & \rightarrow & \text{Hom}(E_*(X), J_G) & \xleftarrow{\quad} & \hat{E}(J_G)^*(X), \end{array}$$

the last two left-hand arrows are isomorphisms. So the above diagram yields the homotopy commutative diagram

$$\begin{array}{ccccccc} FG & \rightarrow & FI_G & \rightarrow & FJ_G & \rightarrow & \Sigma FG \\ & & & & \downarrow & & \\ \hat{E}(G) & \rightarrow & \hat{E}(I_G) & \rightarrow & \hat{E}(J_G) & \rightarrow & \Sigma \hat{E}(G) \end{array}$$

with cofiber sequences. Choose a map

$$f: FG \rightarrow \hat{E}(G)$$

making the above diagram homotopy commutative. Then it becomes a homotopy equivalence from our hypothesis. The composite map $\kappa f_*: FG^*(X) \rightarrow \hat{E}(G)^*(X) \rightarrow \text{Hom}(E_*(X), G)$ coincides with the homomorphism $\overline{\mu}$ induced by the pairing μ , because $\text{Hom}(E_*(X), G) \rightarrow \text{Hom}(E_*(X), I_G)$ is a monomorphism.

3.2. Let us denote by H, K, KO and KSp the Eilenberg-MacLane spectrum, the BU -, BO - and BSp -spectrum respectively. We now investigate the homotopy types of $\hat{H}(G)$, $\hat{K}(G)$ and $\widehat{KSp}(G)$.

Theorem 5. *For any abelian group G $\hat{H}(G)$, $\hat{K}(G)$ and $\widehat{KSp}(G)$ have the same homotopy types of HG , KG and KOG respectively (cf., [3]).*

Proof. The proof is essentially due to Anderson [3].

The \hat{H} and K cases: Let E denote either H or K , and $\mu_E: E \wedge E \rightarrow E$ the usual pairing. As is well known, $\overline{\mu}_E: \pi_*(E) \rightarrow \text{Hom}(\pi_{-*}(E), \mathbb{Z})$ is an isomorphism. This implies that $\overline{\mu}_E: \pi_*(EG) \rightarrow \text{Hom}(\pi_{-*}(E), G)$ is an isomorphism for all G . The result follows immediately from Lemma 10.

The \widehat{KSp} case: There is a well known pairing $\mu_{KSp}: KSp \wedge KO \rightarrow KSp$. We see easily that $\overline{\mu}_{KSp}: \pi_n(KO) \rightarrow \text{Hom}(\pi_{-n}(KSp), \mathbb{Z})$ is an isomorphism except $n \equiv 1, 2 \pmod{8}$, and hence $\overline{\mu}_{KSp}: \pi_n(KOA) \rightarrow \text{Hom}(\pi_{-n}(KSp), A)$ is so for all n and Q -modules A . Fix a Q -module A . For any subgroup B of A we define

homomorphisms λ_n by the composite maps

$$\begin{aligned} \pi_n(KO) \otimes A/B &\xrightarrow{\cong} \pi_n(KOA/B) \rightarrow \text{Hom}(\pi_{-n}(KSp), A/B) \quad \text{when } n \not\equiv 2, 3 \pmod{8} \\ \pi_{n-1}(KO) \otimes B &\xrightarrow{\cong} \pi_{n-1}(KOB) \xleftarrow{\cong} \pi_n(KOA/B) \rightarrow \text{Hom}(\pi_{-n}(KSp), A/B) \\ &\quad \text{when } n \equiv 2, 3 \pmod{8} \end{aligned}$$

which are natural with respect to B . Let η_1 be the generator of $\pi_1(KO) \cong Z_2$ and define as the multiplications by η_1 $\phi: \pi_1(KO) \rightarrow \pi_2(KO)$ and $\phi: \pi_{-3}(KSp) \rightarrow \pi_{-2}(KSp)$. Then we remark that ϕ 's are isomorphisms and $\phi^* \lambda_2 = \lambda_3(\phi \otimes 1)$. The simplification $\varepsilon_{Sp}: K \rightarrow KSp$ induces a natural transformation $K^*(Y) \rightarrow KSp^*(Y)$ of $KO^*(\quad)$ -modules for all finite Y . So we get a weak homotopy commutative diagram

$$\begin{array}{ccccc} K \wedge KO & \rightarrow & K \wedge K & \rightarrow & K \\ \downarrow & & & & \downarrow \\ KSp \wedge KO & \longrightarrow & & \rightarrow & KSp. \end{array}$$

(In fact this diagram is homotopy commutative by use of Corollary 13 below). This yields the commutative diagram

$$\begin{array}{ccc} \pi_2(KOA/B) & \longrightarrow & \text{Hom}(\pi_{-2}(KSp), \pi_0(KSpA/B)) \\ \downarrow & & \downarrow \\ \pi_2(KA/B) \rightarrow \text{Hom}(\pi_{-2}(K), \pi_0(KA/B)) & \rightarrow & \text{Hom}(\pi_{-2}(K), \pi_0(KSpA/B)). \end{array}$$

The left vertical arrow is a monomorphism because $\pi_1(KOA/B) = 0$, and the lower horizontal ones are isomorphisms. Therefore the upper becomes a monomorphism, and hence so are both λ_2 and λ_3 . Let $\{B_\lambda\}$ be the set of all finitely generated subgroups of B . As is easily checked, λ_n are isomorphisms for all n and B_λ because B_λ is free. On the other hand, A/B is isomorphic to the direct limit of A/B_λ and $\text{Hom}(\pi_{-n}(KSp), A/B) \xrightarrow{\cong} \lim \text{Hom}(\pi_{-n}(KSp), A/B_\lambda)$. So we see immediately that λ_n are isomorphisms for any subgroup B . Thus

$$\bar{\mu}_{KSp}: \pi_n(KOG') \rightarrow \text{Hom}(\pi_{-n}(KSp), G')$$

is an isomorphism for any quotient group G' of a Q -module. Taking I_G and J_G as the above G' and applying Lemma 10 we get the desirable result.

In other words, Theorem 5 says that there exist universal coefficient sequences

$$(3.1) \quad \begin{aligned} 0 &\rightarrow \text{Ext}(H_{n-1}(X), G) \rightarrow HG^n(X) \rightarrow \text{Hom}(H_n(X), G) \rightarrow 0 \\ 0 &\rightarrow \text{Ext}(K_{n-1}(X), G) \rightarrow KG^n(X) \rightarrow \text{Hom}(K_n(X), G) \rightarrow 0 \\ 0 &\rightarrow \text{Ext}(KO_{n+3}(X), G) \rightarrow KOG^n(X) \rightarrow \text{Hom}(KO_{n+4}(X), G) \rightarrow 0 \end{aligned}$$

for any CW -spectrum X .

Theorem 3 combined with (3.1) implies the following

- Corollary 11.** i) $HR^{n+1}(X)$ is Hausdorff and only if $\text{Ext}(H_n(X)/TH_n(X), R)=0$.
 ii) $KR^{n+1}(X)$ is Hausdorff if and only if $\text{Ext}(K_n(X)/TK_n(XR)=0$.
 iii) $KOR^{n+1}(X)$ is Hausdorff if and only if $\text{Ext}(KO_{n+4}(X)/TKO_{n+4}(XR)=0$.

3.3. Finally we shall make a comment for Hausdorff-ness of K -theories.

Proposition 12. Let E be a CW -spectrum such that $\pi_*(E)$ is of finite type as an R -module and fix a degree n . If $\pi_k(X) \otimes \pi_{k-n}(E) \otimes Q = 0$ for all k , then $E^{n+1}(X)$ is Hausdorff. (Cf., [8, Theorem III, 2]).

Proof. Under our assumptions we compute

$$E\hat{Z}/Z^n(X) \cong \prod H^k(X; \pi_{k-n}(E) \otimes Z/Z) \cong \prod \text{Hom}(H_k(X), \pi_{k-n}(E) \otimes Z/Z) = 0.$$

Then the result is immediate from [8, Theorem III. 1].

For CW -spectra E and X whose rational homotopy groups are sparse we have

Corollary 13. Let E be a CW -spectrum such that $\pi_*(E)$ is of finite type as an R -module. Assume that $\pi_m(E) \otimes Q = \pi_m(X) \otimes Q = 0$ unless $m = 0 \bmod n$. Then $E^{m+1}(X)$ is Hausdorff whenever $m \not\equiv 0 \bmod n$.

As is well known, $\pi_{2n+1}(K) = 0$ and $\pi_m(KO) \otimes Q = 0$ if $m \not\equiv 0 \bmod 4$. Therefore Corollary 13 implies

- Theorem 6.** i) $K^{2n}(K \wedge \cdots \wedge K)$ is Hausdorff.
 ii) $KO^m(KO \wedge \cdots \wedge KO)$ is Hausdorff whenever $m \not\equiv 1 \bmod 4$.

REMARK. Informations on $K_*(K)$ and $KO_*(KO)$ have been obtained by Adams, Harris and Switzer [2].

As an immediate corollary we have

Corollary 14. Complex and real K -theories K^* , KO^* (defined on the category of CW -spectra) possess an associative and commutative multiplication.

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