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SELBERG'S TRACE FORMULA AND SPECTRUM

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In this paper we study the spectrum associated with discontinuous groups operating on the upper half plane.

I.M. Gelfand conjectured that if two discontinuous groups with compact fundamental domains have the same spectrum, then they are conjugate. We have no proof for his conjecture, however we can show that the numbers of conjugate classes with same eigenvalues coincide each other for such groups (§4). In §5, we shall give a formulation and a proof of an announced result[1] of I. M. Gelfand which is a weaker version of his conjecture.

Our second object is to give the asymptotic formula for the discrete spectrum associated with the modular group (§6). Our result can be described by Weyl type formula. In the case the fundamental domain is compact, I.M. Gelfand and I. Pyatetzki-Shapiro announced the asymptotic formula of the spectrum (connected with representations of class 1) for general semi-simple Lie groups. The definitions and summary of the known results are give in §1 and §2.

The author expresses his hearty thanks to Professors H. Yoshizawa and A. Orihara for their kind advices. A summary of our results was announced earlier in Proc. Japan Acad. **42** (1966), 327–329.

1. Recapituration

1.1. General situations

Let G be a semi-simple Lie group and Γ be its discrete subgroup. We denote by $\mathfrak{H}$ the space of functions $f(g)$ which satisfy

$$f(\gamma g) = f(g) \qquad \text{for} \quad \gamma \in \Gamma, \ g \in G .$$

and

$$\int_F |f(g)|^2 dg < \infty ,$$

where F is a fundamental domain in G of transformations $g \to \gamma g$, $\gamma \in \Gamma$. $\mathfrak{H}$ is a Hilbert space with the scalar product

$$\langle f_1, f_2 \rangle = \int_F f_1(g)\overline{f_2(g)}\, dg .$$

1) Added in proof: After the preparation of this work, a proof wad publised in, I.M. Gelfand, M.I. Graev, I. Pyatezkki-Shapiro, Theory of representations and automorphic functions (in the Series 'Generalized Functions', Vol. 6, in Russian), Moscow (1966).

The representation $\{T_g, \mathfrak{H}\}$ is unitary and the problem is to decompose this representation into irreducible ones. The first general result is

Theorem ([2]). *If the factor space $\Gamma\backslash G$ is compact, the representation $\{T_g, \mathfrak{H}\}$ can be decomposed into countable discrete direct sum of irreducible representations of G, where the multiplicity of each irreducible representation is finite.*

Irreducible representations of G can be parametrized by indeces ρ. Suppose that the decomposition mentioned above is written as follows.

$$\{T_g, \mathfrak{H}\} = \sum_k \{T_g^{(\rho_k)}, \mathfrak{H}^{(\rho_k)}\} . \tag{1}$$

The main tool of studying the decomposition is the trace formula. We follow here the argument of [2]. For $\varphi(g) \in L^1(G, dg)$, the operator

$$T_\varphi = \int_G \varphi(g)\, T_g\, dg$$

is defined and bounded in $\mathfrak{H}$. We have

$$T_\varphi f(g_1) = \int_F K(g_1, g_2) f(g_2)\, dg_2 ,$$
$$K(g_1, g_2) = \sum_{\gamma \in \Gamma} \varphi(g_1^{-1} \gamma g_2) .$$

Let $\pi_\rho(g)$ be the trace of the irreducible repsrentation $\{T_g^{(\rho)}, \mathfrak{H}^{(\rho)}\}$ and we will put

$$h(\rho) = \int_G \varphi(g) \pi_\rho(g)\, dg .$$

With suitable condition on $\varphi(g)$, the above calculation is justified and T_φ is a trace operator. Calculating the trace of T_φ in two ways, we have

$$\sum_k h(\rho_k) = \int_F \Big(\sum_{\gamma \in \Gamma} \varphi(g^{-1} \gamma g)\Big)\, dg .$$

From now on let us consider not all of irreducible representation of G, but only representations of class 1 of G. A class 1 representation is defined as an irreducible representation of G which has in its representation space a U-invariant vector unique up to constant factor, where U is a maximal compact subgroup of G. If we take as $\varphi(g)$ a two sided U-invariant function, then $h(\rho)=0$ unless $\{T_g^{(\rho)}, \mathfrak{H}^{(\rho)}\}$ is of class 1. If $\{T_g^{(\rho)}, \mathfrak{H}^{(\rho)}\}$ is of class 1, then

$$h(\rho) = \int_G \varphi(g)\, \omega_\rho(g)\, dg ,$$

where $\omega_\rho(g)$ is the zonal spherical function corresponding to this representation.

Class 1 representations can be parametrized by the set of continuous parameters. When the factor space $\Gamma\backslash G$ is compact, I. M. Gelfand and I. Pyatetzki-Shapiro anounced the asymptotic formula for the parameters of class 1

representations which are contained in the decomposition (1) (see [3]). §4.1 and §6 are concerned with this type of problem for $G=SL(2, \boldsymbol{R})$.

1.2 Some facts about $G=SL(2, \boldsymbol{R})$

From now on we shall be concerned with $G=SL(2, \boldsymbol{R})$. Here we describe the class 1 representations of $G=SL(2, \boldsymbol{R})$.

i) Principal series.

These representations are realized in the function spaces on the real line. Operators of representations are determined by formulas

$$T_g^{(r)} f(x) = f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right) |\beta x+\delta|^{2ir-1} \qquad \text{for } g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

where r is a non-negative real number. Scalar product is defined as follows:

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\, dx\,.$$

ii) Supplementary series

Operators of representations are given by the formulas as above, but in this case r is pure imaginary and $0 \leqslant Im\, r < \frac{1}{2}$. The scalar product is defined by

$$(f_1, f_2) = \int_{-\infty}^{\infty}\int f_1(x)\overline{f_2(x)}\, |x_1-x_2|^{-1+2ir} dx_1 dx_2\,.$$

Now let g, $h \in G=SL(2, \boldsymbol{R})$ be matrices $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $h=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. The mapping

$$h \to \frac{\alpha-i\beta}{\gamma-i\delta} = \tau = x+iy$$

induces the identification of G/U with the upper half plane H. Then the image of gh by the above mapping is $\frac{a\tau+b}{c\tau+d}$. So we define the action of G on H by the formula,

$$g\tau = \frac{a\tau+b}{c\tau+d} \qquad \text{for } g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

The measure $d\tau=dxdy/y^2$ on H is induced by the Haar measure of G. $\Gamma\backslash G$ is compact (or, of finite measure) if and only if $\Gamma\backslash H$ is compact (or, of finite measure).

$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $-g=\begin{pmatrix}-a & -b\\ -c & -d\end{pmatrix}$ have the same effect as transformations of H. Considered as a transformation group of H, G is reduced to $G_0=G/\{\pm e\}$, where $e=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

We denote with $\mathfrak{H}_0$ the space of functions $f(\tau)$ which satisfy

$$f(\gamma\tau)=f(\tau) \qquad \text{for} \quad \gamma\in\Gamma,\ \tau\in H$$

and

$$\int_D |f(\tau)|^2 d\tau<\infty\ ,$$

where D is a fundamental domain in H of Γ. $\mathfrak{H}_0$ is a Hilbert space with the scalar product

$$\langle f_1, f_2\rangle=\int_D f_1(\tau)\overline{f_2(\tau)}\,d\tau\ .$$

The Laplace-Beltrami operator

$$\Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

considered in $\mathfrak{H}_0$ is self-adjoint. By the well-known principle, the decomposition of $\{T_g, \mathfrak{H}\}$ into class 1 representations is equivalent to the spectral decomposition of Δ in $\mathfrak{H}_0$. Δ has non-negative spectrum and we denote the discrete part of the spectrum with Λ_Γ. Spectrum λ of Δ and the parameter r of the corresponding representation are connected by the equation $\lambda=\frac{1}{4}+r^2$.

1.3. A conjecture of I. M. Gelfand

In [1] I. M. Gelfand conjuectured that, when $\Gamma_1\backslash H$ and $\Gamma_2\backslash H$ are compact and $\Lambda_{\Gamma_1}=\Lambda_{\Gamma_2}$, then Γ_1 and Γ_2 would be conjugate in G_0. He announced the following weaker result. A continuous deformation of Γ that does not change Λ_Γ is trivial. We shall discuss some aspect of this problem in §4.2 and §5.

2. Selberg's trace formula

The trace formula for class 1 representations of $G=SL(2, \boldsymbol{R})$ can be transformed further and written as (2) when $\Gamma\backslash H$ is compact (A. Selberg [8]. A detailed proof of (2) is given in [4]). A. Selberg states an analogous formula also in the case when $\Gamma\backslash H$ is not compact. This formula are called Selberg's trace formula. Since we need those results in our discussion, we shall write them explicitly in this section.

Let $\Gamma\backslash H$ be compact. If $h(r)$ satisfies conditions (H) that is,

(H.1) $h(r)=h(-r)$,

(H.2) $h(r)$ is regular analytic in a strip $|Imr|<\frac{1}{2}+\varepsilon$, $\varepsilon<0$,

(H.3) $h(r)=0(1+|r|^2)^{-1-\varepsilon}$ in this strip,

then

$$\sum_{\lambda=(1/4)+r^2\in\Delta_\Gamma} h(r) = \frac{A(D)}{2\pi}\int_{-\infty}^{\infty} r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}h(r)\,dr$$

$$+\frac{1}{2}\sum_{\beta=1}^{s}\sum_{k=1}^{m_\beta-1}\frac{1}{m_\beta \sin\frac{k\pi}{m_\beta}}\int_{-\infty}^{\infty}\frac{e^{\pi r-2\pi r(k/m_\beta)}+e^{-\pi r+2\pi r(k/m_\beta)}}{e^{\pi r}+e^{-\pi r}}h(r)\,dr \tag{2}$$

$$+2\sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty}\frac{\log N\{P_\alpha\}}{N\{P_\alpha\}^{k/2}-N\{P_\alpha\}^{-k/2}}g\,(k\log N\{P_\alpha\})\,,$$

where $A(D)=\int_D d\tau$ and

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iru}h(r)\,dr\,.$$

We count both values of r that give the same λ (and if $\lambda=\frac{1}{4}$, $r=0$ with double multiplicity). m_β ($\beta=1,\cdots,s$) are orders of R_β, the representatives of primitive elliptic classes in Γ (considered in G_0). $N\{P_\alpha\}$ ($\alpha=1,2,\cdots$) are the squares of larger eigenvalues (norms) of P_α, the representatives of primitive hyperbolic classes in Γ (considered in G_0). Both sides of (2) converge absolutely.

We introduce some additional notations. It is konwn that $N\{P_\alpha\}$ ($\alpha=1,2,\cdots$) do not accumulate to a finite value. We denote with $a_1, a_2, \cdots$ the smallest, next, $\cdots$ values of $N\{P_\alpha\}$, $\alpha=1,2,\cdots$. Obviously $1<a_1<a_2\cdots\uparrow\infty$. Put $\varepsilon_i=\log a_i$ and let n_i be the number of representatives of primitive hyperbolic classes whose norms are equal to a_i. We put

$$E(r)=\sum_{\beta=1}^{s}\sum_{k=1}^{m_\beta-1}\frac{1}{m_\beta\sin\frac{k\pi}{m_\beta}}\frac{e^{\pi r-2\pi r(k/m_\beta)}+e^{-\pi r+2\pi r(k/m_\beta)}}{e^{\pi r}+e^{-\pi r}}$$

Then the Selberg's trace formula can be written as follows.

$$\sum_{\lambda=(1/4)+r^2\in\Delta_\Gamma} h(r) = \frac{A(D)}{2\pi}\int_{-\infty}^{\infty} r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}h(r)\,dr$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}E(r)h(r)\,dr+2\sum_{i=1}^{\infty}n_i\sum_{k=1}^{\infty}\frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}}g\,(k\varepsilon_i) \tag{2'}$$

When Γ has a non-compact fundamental domain in H with a parabolic cusp, A. Selberg state the following trace formula which differs from (2) (or (2')) in that on the right-hand side there are the new term

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)h(r)\,dr-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+ir)h(r)\,dr$$

$$-2\log 2.\ g(0)+\frac{1}{2}\left(1-\phi\left(\frac{1}{2}\right)\right)h(0)\,. \tag{3}$$

$\phi(s)$ is the meromorphic functions which appears in the functional equation of Eilsenstein series. $h(r)$ is a function which satisfies the condition (H). Both sides of the formula converge absolutely.

REMARK. A. Selberg sketched out the theory in [8]. The author has learned the proof of the formula (3) (Lectures delivered by A. Selberg, Institute for Advanced Study, 1954–1955) from Professor M. Kuga. The description of the proof has some omissions concerning the analytic natures of Eisenstein series. But for the case of the modular group, which we shall consider in §6, the details of the proof of (3) can be made rigorous, since Eisenstein series is well-known in this case (see for example [6], [7]).[2)]

3. Some auxiliary estimates

This section contains some estimates connected with Selberg's trace formula. We use them in the following section. We assume that $\Gamma\backslash H$ is compact or Γ is the modular group.

Let n be a non-negative integer and t be a positive number. Function $h_{n,t}(r)=r^{2n}e^{-tr^2}$ satisfies the condition (H). We put

$$g_{n,t}(r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iru}h_{n,t}(r)\,dr\,.$$

By the formula

$$(4)\qquad \int_{-\infty}^{\infty}x^{2n}e^{-tx^2}x^{-ixy}\,dx=(-1)^n\sqrt{\pi}\,\frac{d^n}{dt^n}(t^{-1/2}e^{-y^2/4t})\,,$$

we have

$$g_{n,t}(u)=P(t^{-1/2},u^2)e^{-u^2/4t}$$

where $P(x,y)$ is a polynomial (of degree N).

Lemma 1. (i) *Series*

$$S_n(t)=\sum_{i=1}^{\infty}n_i\sum_{k=1}^{\infty}\frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}}(k\varepsilon_i)^{2n}e^{-(k^2\varepsilon_i)^2/4t}$$

converges.

$$(ii)\qquad \lim_{t\to 0}\sum_{i=1}^{\infty}n_i\sum_{k=1}^{\infty}\frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}}g_{n,t}(k\varepsilon_i)=0$$

Proof. By (4) $u^{2n}e^{-u^2/4t}$ is the Fourier transform of a function satisfying (H), so (i) follows from the absolute convergence of Selberg's trace formula.

$$\sum_{i=1}^{\infty}n_i\sum_{k=1}^{\infty}\frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}}|g_{n,t}(k\varepsilon_i)|$$

$$\leqslant \text{constant} \times \sum_{m+n\leqslant N} t^{-m/2} \sum_{i=1}^{\infty} n_i \sum_{k=1}^{\infty} \frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}} (k\varepsilon_i)^{2n} e^{-k^2\varepsilon_i^2/4t}$$

$$\leqslant \text{constant} \times \Big(\sum_{m+n\leqslant N} t^{-m/2} e^{-\varepsilon_1^2/8t}\Big) S_n(2t) < \infty .$$

$S_n(2t)$ decreases when $t \downarrow 0$. So (ii) is established.

Now we put

$$S(j, t) = \sum_{i=j}^{\infty} n_i \sum_{k=1}^{\infty} \frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}} e^{-(k\varepsilon_i)^2/4t}$$

for $j=1, 2, \cdots$.

Lemma 2.

$$\lim_{t\to 0} e^{\varepsilon_j^2/4t} S(j, t) = n_j \frac{\varepsilon_j}{a_j^{1/2}-a_j^{-1/2}}$$

Proof. Let c_1, c_2 be positive and $c_1+c_2=1$.

$$S(j, t) = n_j \frac{\varepsilon_j}{a_j^{1/2}-a_j^{-1/2}} e^{-\varepsilon_j^2/4t} + R'(j, t) + R''(j, t) ,$$

where

$$R'(j, t) = n_j \sum_{k=2}^{\infty} \frac{\varepsilon_j}{a_j^{k/2}-a_j^{-k/2}} e^{-k^2\varepsilon_j^2/4t}$$

and

$$R''(j, t) = \sum_{i=j+1}^{\infty} n_i \sum_{k=1}^{\infty} \frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}} e^{-k^2\varepsilon_i^2/4t} .$$

They are estimated as follows.

$$R'(j, t) \leqslant n_j e^{-4c_1\varepsilon_j^2/4t} \sum_{k=2}^{\infty} \frac{\varepsilon_j}{a_j^{k/2}-a_j^{-k/2}} e^{-c_2k^2\varepsilon_j^2/4t}$$

$$\leqslant e^{-4c_1\varepsilon_j^2/4t} S_0(t/c_2) ;$$

$$R''(j, t) \leqslant e^{-c_1\varepsilon_{j+1}^2/4t} \sum_{i=j+1}^{\infty} n_i \sum_{k=1}^{\infty} \frac{\varepsilon_i}{a_i^{k/2}-a_i^{-k/2}} e^{-c_2k^2\varepsilon_i^2/4t}$$

$$\leqslant e^{-c_1\varepsilon_{j+1}^2/4t} S_0(t/c_2) .$$

So we have

$$R'(j, t)+R''(j, t) \leqslant (e^{-4c_1\varepsilon_j^2/4t}+e^{-c_1\varepsilon_{j+1}^2/4t}) S_0(t/c_2) .$$

Now choose c_1 such that $4c_1<1$ and $c_1\varepsilon_{j+1}^2>\varepsilon_j^2$. Then

$$\lim_{t\to 0} e^{-\varepsilon_j^2/4t}\{R'(j, t)+R''(j, t)\} = 0 .$$

So the lemma is established.

Lemma 3. *The moments of $E(r)$ satisfy a sufficient condition for the unique-*

ness of Hamburgar moment problem (see, for instance, [10]).

Proof. $E(r)$ is the sum of terms of type

$$\frac{e^{c'r}+e^{-c'r}}{e^{cr}+e^{-cr}}, \qquad 0<c'<c .$$

Lemma 3 follows from the following estimates.

$$\begin{aligned} m_{2n} &= \int_{-\infty}^{\infty} \frac{e^{c'r}+e^{-c'r}}{e^{cr}+e^{-cr}} r^{2n}\, dr \\ &\leqslant 4\int_0^{\infty} e^{(c'-c)r} r^{2n}\, dr \\ &= \text{constant} \times (c'-c)^{-2n-1}(2n)! . \\ &\lim_{n\to\infty} \frac{1}{n} m_{2n}{}^{1/2n} < \infty . \end{aligned}$$

4. Selberg's trace formula and spectrum

4.1. The asymptotic formula for the spectrum. We assume $\Gamma\backslash H$ is compact. Let $\alpha(\lambda)$ be the number of those elements of Λ_Γ which are smaller than λ. The asymptotic formula of I. M. Gelfand and I. Pyatetzki-Shapiro for this case is

Lemma 4.

$$\lim_{\lambda\to\infty} \alpha(\lambda)/\lambda = A(D)/4\pi .$$

Proof. The first term of the right-hand side of Selberg's trace formula (2′) is modified as follows.

$$\begin{aligned} &\frac{A(D)}{2\pi}\int_{-\infty}^{\infty} r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}} h(r)\, dr \\ &= \frac{A(D)}{2\pi}\cdot 2\int_0^{\infty} rh(r)\, dr - \frac{A(D)}{2\pi}\cdot 4\int_0^{\infty} \frac{e^{-\pi r}}{e^{\pi r}+e^{-\pi r}} h(r)\, dr . \end{aligned}$$

Putting $h(r)=e^{-(1/4+r^2)t}$ in (2′), we have

$$\begin{aligned} 2\int_0^{\infty} e^{-t\lambda}\, d\alpha(\lambda) &= \frac{A(D)}{2\pi}\cdot 2\int_0^{\infty} re^{-(1/4+r^2)t}\, dr \\ &+ \int_0^{\infty}\left\{-\frac{A(D)}{2\pi}\cdot 4\cdot\frac{e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}+E(r)\right\} e^{-(1/4+r^2)t}\, dr \\ &+ e^{-(1/4)t}\sum_{i=1}^{\infty} n_i \sum_{k=1}^{\infty} \frac{\varepsilon_i}{a_i{}^{k/2}-a_i{}^{-k/2}}\, g_{0,t}(k\varepsilon_i) . \end{aligned}$$

By Lemma 1. (ii) and the fact that

$$-\frac{A(D)}{2\pi}4\frac{e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}+E(r)$$

is integrable in $[0, \infty)$, we have

$$\lim_{t\to 0} t\int_0^\infty e^{-t\lambda}d\alpha(\lambda) = A(D)/4\pi\,.$$

So if we apply Hardy-Littlewood-Karamata theorem (see for instance [12]), the lemma follows.

4.2. The partial determination of Γ from Λ_Γ. We will consider two discrete subgroups Γ' and Γ'' such that $\Gamma'\backslash H$ and $\Gamma''\backslash H$ are compact. The symbols and quantities connected with $\Gamma'(\Gamma'')$ are denoted with Λ', D', a_i', n_i' etc. (Λ'' etc.) Now we can prove the following statement.

If $\Lambda'=\Lambda''$, *then* Γ' *and* Γ'' (*considered as transformation groups of* H) *are isomorphic and* $a_i'=a_i''$, $n_i'=n_i''$ ($i=1, 2, \cdots$).

Proof. By lemma 4, $A(D')=A(D'')$. Then by Selberg's trace formula (2'), we have

$$\begin{aligned}(5)\qquad &\frac{1}{2}\int_{-\infty}^{\infty}E'(r)h_{n,t}(r)\,dr+2\sum_{i=1}^{\infty}n_i'\sum_{k=1}^{\infty}\frac{\varepsilon_i'}{a_i'^{k/2}-a_i'^{-k/2}}g_{n,t}(k\varepsilon_i')\\ &=\frac{1}{2}\int_{-\infty}^{\infty}E''(r)h_{n,t}(r)\,dr+2\sum_{i=1}^{\infty}n_i''\sum_{k=1}^{\infty}\frac{\varepsilon_i''}{a_i''^{k/2}-a_i''^{-k/2}}g_{n,t}(k\varepsilon_i'')\,.\end{aligned}$$

Now let t tend to 0. By lemma 1 (ii) we get the equations

$$\int_{-\infty}^{\infty}E'(r)r^{2n}\,dr=\int_{-\infty}^{\infty}E''(r)r^{2n}\,dr \qquad \text{for} \quad n=0, 1, \cdots\,.$$

So from Lemma 3, we have

$$(6)\qquad E'(r)=E''(r)\,.$$

Putting $n=0$ in (5), we have

$$(7)\qquad S'(1, t)=S''(1, t)\,.$$

This implies that

$$(8)\qquad e^{\varepsilon_1'^2/4t-\varepsilon_1''^2/4t}=e^{\varepsilon_1'^2/4t}S'(1, t)/e^{\varepsilon_1''^2/4t}S''(1, t)\,.$$

The right-hand side of (8) has non-zero limit if we let t tend to 0 (by Lemma 2). So we have $\varepsilon_1'=\varepsilon_1''$. Using (7) and Lemma 2 again, we have $n_1'=n_1''$. Then (7) is reduced to

$$S'(2, t) = S''(2, t)$$

and we can repeat the same procedure.
Returning to (6), by a simple argument concerning the order of each terms of $E(r)$ when $r \to \infty$, we have $s'=s''$ and $\{m_1', \cdots, m'_{s'}\} = \{m_1'', \cdots, m''_{s''}\}$. Using the formula

$$A(D) = 2\pi\{2g-2+\sum_{\beta=1}^{s}(1-1/m_\beta)\}$$

(g is the genus of the Rieman surface corresponding to Γ), we have $g'=g''$. Summing up, Γ' and Γ'' are isomorphic.

5. Discussion of a result of I. M. Gelfand

Our result in §4.2 can be applied to formulate and prove a result of I. M. Gelfand announced in [1]. We reproduce here some result of A. Selberg (see [9]).

DEFINITION. Let G be a locally compact group. A family of its discrete subgroups $\Gamma^{(t)}$ $(0 \leqslant t \leqslant t_0)$ with elements $A^{(t)}$ such that $\Gamma^{(t)}\backslash G$ is compact (or, of finite measure) is called a (*continuous*) *deformation* of $\Gamma=\Gamma^{(0)}$, if elements $A^{(t)} \in \Gamma^{(t)}$ depend continuously on t and $A^{(0)} \to A^{(t)}$ gives isomorphism between $\Gamma^{(0)}$ and $\Gamma^{(t)}$: A deformation is called trivial if there exist $T^{(t)} \in G$ depending continuously on t such that $A^{(t)}=T^{(t)}A^{(0)}T^{(t)-1}$.

Lemma 5 (A. Selberg). *Let G be $SL(n, \boldsymbol{R})$. A continuous deformation of Γ (with $\Gamma\backslash G$ compact or of finite measure) which preserves the traces of all element of Γ is a trivial deformation.*

Now we will prove

Result of Gelfand. *Let $G=SL(2, \boldsymbol{R})$ and Γ be a discrete subgroup of G with $\Gamma\backslash G$ compact. A deformation of Γ which does not change Λ_Γ (i.e., $\Lambda_{\Gamma^{(t)}}=\Lambda_\Gamma$, $0 \leqslant t \leqslant t_0$) is trivial.*

Proof. The order of an elliptic element is preserved under any deformation. So absolute value of its trace is preserved. By continuity its trace does not change sign.

Consider a subset of positive numbers consisting of norms of all hyperbolic elements of $\Gamma^{(t)}$ and denote this set with $A^{(t)}$. $A^{(t)}=A^{(0)}$ by §4.2. Let $P^{(0)} \in \Gamma^{(0)}$ be a hyperbolic element. By discretness of $A^{(0)}$ and by continuity, $N\{P^{(t)}\}$ remainss invariant during the deformation. Therefore the absolute value of the trace of $P^{(t)}$ is preserved and it does not change sign by continuity. So the hypothesis of Lemma 5 is satisfied.

6. Asymptotic formula for the modular group

For the modular group, $\phi(s)$ in (3) can be written as

$$\phi(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)},$$

where $\zeta(s)$ is the Riemann Zeta function (see for instance [6]). Using the functional equation of $\zeta(s)$, we have

$$\phi(s) = \pi^{2s-1}\frac{\Gamma(-s+1)}{\Gamma(s)}\frac{\zeta(-2s+2)}{\zeta(2s)}. \tag{9}$$

Let $\alpha(\lambda)$ be the number of elements of the discrete part of the spectrum Λ_Γ smaller than λ.

Asymptotic formula for the modular group

$$\lim_{\lambda\to\infty}\alpha(\lambda)/\lambda = A(D)/4\pi .$$

Proof. We will show

$$\lim_{t\to\infty} t\int_0^\infty e^{-t\lambda}\alpha(\lambda) = A(D)/4\pi . \tag{10}$$

To prove this it is sufficient to show that for $h(r)=e^{-(1/4+r^2)t}$ (3) is of order of $o(t^{-1})$ when $t\to 0$. The main terms we have to consider are (by the expression (9) for $\phi(s)$)

$$\int_{-\infty}^{\infty}\frac{\zeta'}{\zeta}(1+2ir)e^{-(1/4+r^2)t}\,dr$$

and

$$\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(\sigma+ir)e^{-(1/4+r^2)t}\,dr \qquad \left(\sigma=\frac{1}{2},\ 1\right).$$

It is known for large $|r|$,

$$\frac{\zeta'}{\zeta}(1+2ir) = O(\log^9|r|)$$

(see for instance [11], p. 44 (3.6.6.)) and

$$\frac{\Gamma'}{\Gamma}(\sigma+ir) = O(\log|r|) \qquad \text{uniformly in} \quad 0\leqslant\sigma\leqslant 2$$

(see for instance [5], p. 317).
On the other hand, if $A(r)$ is a continouus function satisfying $A(r)=O(|r|^\alpha)$

$(\alpha<1)$ for large $|r|$, then we have

$$\int_{-\infty}^{\infty} A(r)e^{-(1/4+r^2)t}dr = o(t^{-1}) \qquad \text{for} \quad t \to 0 .$$

So (10) is proved.

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