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THE REAL K-GROUPS OF SO(n) FOR n=2 MOD 4

Dedicated to Professor Shôrô Araki on his sixtieth birthday

HARUO MINAMI

(Received May 6, 1988)

In [9], [10] we studied the algebra $KO^*(SO(n))$ for $n=0, 1, 3 \mod 4$ using an idea of [7]. We first showed that a map from $P^{n-1} \times \text{Spin}(n)$ to $SO(n)$ introduced in [7] to compute $K^*(SO(n))$ also induces a monomorphism in $KO$-theory

$$I: KO^*(SO(n)) \to KO^*(P^{n-1} \times \text{Spin}(n)).$$

As in [7] using this embedding enabled us to compute $KO^*(SO(n))$ from $KO^*(P^{n-1} \times \text{Spin}(n))$ whose structure can be obtained from the results of [1], [6], [12], [11].

The purpose of this note is to consider the remaining case, that is, $KO^*(SO(n))$ for $n=2 \mod 4$. However, in the present case, the analogous homomorphism $I$ is not a monomorphism. This must come from the fact that the simple spin representations of Spin($n$) are neither real nor quaternionic representations. To determine the kernel and image of $I$ so we make use of our results on the algebra structure of $KO^*(SO(n))$ for $n=1 \mod 4$.

1. $KO^*(P^{n-1} \times \text{Spin}(n))$

Throughout this note we regard $KO$ and $K$ as $\mathbb{Z}_2$-graded cohomology functors using the Bott periodicity. Let $\eta_1 \in KO^{-1}(+) \text{ and } \eta_4 \in KO^4(+)$ be generators of $KO^*(+)$ satisfying the relations $2\eta_1 = \eta_1^2 = \eta_1 \eta_4 = 0$, $\eta_4^2 = 4$ and $\mu \in K^{-2}(+)$ denote the Bott class satisfying the relation $\mu^4 = 1$ ($+$ = point).

Let $c$ and $r$ denote the complexification and realification homomorphisms. According to [3] we then have a useful exact sequence

$$\cdots \to KO^{-\gamma}(X) \xrightarrow{\chi} KO^{-\gamma}(X) \xrightarrow{c} K^{-\gamma}(X) \xrightarrow{\delta} KO^{\gamma}(X) \xrightarrow{\delta} \cdots$$

which connects $KO$ with $K$ where $\chi$ is multiplication by $\eta_1$ and $\delta$ is given by $\delta(\mu x) = r(x)$ for $x \in K^{2-\gamma}(X)$.

We also assume that $n \equiv 2 \mod 4 \text{ and } a = \frac{n-2}{2}$.
throughout this note.

To determine $KO^*(P^{n-1} \times \text{Spin}(n))$ we first deal with $KO^*(P^{n-1})$ where $P^{n-1}$ is the real projective $(n-1)$-space. For the additive structure of $KO^*(P^n)$ needed below we refer to [6]. Referring also to [4] for the structure of $K^*(P^{n-1})$ and using (1.1) we can find elements $v_1 \in KO^{-3}(P^{n-1})$ and $v_2 \in KO^{-7}(P^{n-1})$ such that

\begin{equation}
    c(v_1) = \mu v_1 \quad \text{and} \quad c(v_2) = \mu^3 v_2
\end{equation}

and we can readily show that $KO^*(P^{n-1})$ is generated by $\gamma = \gamma' - 1$, $v_1$ and $v_2$ as follows. Here $\nu$ denotes the generator $\nu_{n-1}$ of $K^{-1}(P^{n-1})$ as in [9], Proposition 2.1 and $\gamma'$ the canonical non-trivial real line bundle over $P^{n-1}$.

**Proposition 1.3.**

\begin{align*}
    KO^0(P^{n-1}) &= \mathbb{Z}_{2^{n+1}} \cdot \gamma, \\
    KO^{-1}(P^{n-1}) &= \mathbb{Z}_2 \cdot \gamma_1 \gamma, \\
    KO^{-2}(P^{n-1}) &= \mathbb{Z}_2 \cdot \gamma_1^2 \gamma, \\
    KO^{-3}(P^{n-1}) &= \mathbb{Z} \cdot v_1 \\
    KO^{-4}(P^{n-1}) &= \mathbb{Z}_2 \cdot \gamma_4 \gamma, \\
    KO^{-5}(P^{n-1}) &= KO^{-6}(P^{n-1}) = 0, \\
    KO^{-7}(P^{n-1}) &= \mathbb{Z} \cdot v_3
\end{align*}

with the relations

\begin{align*}
    \gamma^2 &= -2 \gamma, \quad \gamma v_1 = \gamma v_2 = v_1^2 = v_2^2 = v_1 v_2 = 0, \quad \eta_4 v_1 = 2^{n-1} \eta_4 \gamma, \\
    \eta_4 v_3 &= 2 \gamma, \quad \eta_4 v_1 = 2 v_3, \quad \eta_4 v_3 = 2 v_1 .
\end{align*}

Let $\Delta^+$ and $\Delta^-$ be the even and odd half-spin representations of $\text{Spin}(n)$. According to [8], §13 these are neither real nor quaternionic and can be viewed as continuous homomorphisms

\[ \Delta^+, \Delta^- : \text{Spin}(n) \to GL(2^n, \mathbb{C}) \]

These maps give rise to the elements of $K^{-1}(\text{Spin}(n))$, denoted by $\beta(\Delta^+)$ and $\beta(\Delta^-)$ as usual, in a canonical manner.

Since each of $\Delta^+$ and $\Delta^-$ is complex conjugate to the other, so that $\beta(\Delta^-) = \beta(\Delta^+) ^*$, by [11], Proposition 4.6 we have an element $\lambda \in KO(\text{Spin}(n))$ such that

\[ c(\lambda) = \mu \beta(\Delta^+) \beta(\Delta^-) . \]

Here $*$ is the operation on $K^*(X)$ induced by the assignment which sends a complex vector bundle to its complex conjugate bundle.

Set

\[ \lambda_i = r(\mu^i \beta(\Delta^+)) \quad \text{in} \quad KO^{-2i-1}(\text{Spin}(n)) \]

where $i$ is reduced mod 4. Note that using (1.1) when $X = \text{Spin}(n)$ gives
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$$r(\mu_i\beta(\Delta^*)) = (-1)^i\lambda_i$$

because $\mu* = -\mu$ and $cr = 1 + *$.

Let $\rho: SO(n) \subseteq GL(n, R)$ be the evident inclusion and let us denote by the same letter $\rho$ the composite of this with the covering map $\pi: Spin(n) \rightarrow SO(n)$. Then we obtain the elements

$$\beta(\lambda^i\rho) \ (1 \leq i \leq n) \quad \text{in} \quad KO^{-1}(Spin(n))$$
in a similar way where $\lambda^i\rho$ denotes the $i$-th exterior power of $\rho$. Using these elements, by [13], Theorem 5.6 we have

**Proposition 1.4.** $KO^*(Spin(n))$ is generated by $\lambda, \lambda_1, \lambda_2, \lambda_3$ and $\beta(\lambda^i\rho)$ $(1 \leq k \leq a-1)$ as a $KO^* (\cdot)$-algebra and there hold the relations

$$\lambda^2 = \lambda \lambda_i = \eta_1 \lambda_i = 0, \eta_4 \lambda_{i+2} = 2\lambda_i,$$
$$\lambda_i \lambda_j = \eta^2 \lambda \quad \text{if} \quad i+j \equiv 0 \mod 4,$$
$$= (-1)^i\eta_4 \lambda \quad \text{if} \quad i+j \equiv 1 \mod 4,$$
$$= 0 \quad \text{if} \quad i+j \equiv 2 \mod 4,$$
$$= (-1)^i2\lambda \quad \text{if} \quad i+j \equiv 3 \mod 4,$$

$$\beta(\lambda^i\rho)^2 = \eta_i(\beta(\lambda^2(\lambda^i\rho)) + \binom{n}{i} \beta(\lambda^i\rho)).$$

The last relation in the above proposition is due to [5], §6 and the others can be found in [11]. In proving the relations $\eta_4$ is assumed to be chosen so that $r(\mu^2) = \eta_4$ and also hereafter is done so. To complete the last relation we must give the explicit form of $\beta(\lambda^2(\lambda^i\rho))$. But we only show how this can be described in terms of the given generators. It is clear that this can be expressed as a polynomial in $\beta(\lambda^1\rho), \ldots, \beta(\lambda^a\rho)$ and $\beta(\lambda^{a+1}\rho) = \beta(\lambda^{a+1}\rho)$ for $2 \leq i \leq a+2$. Hence it suffices to check $\eta_i\beta(\lambda^i\rho)$ and $\eta_i\beta(\lambda^{a+1}\rho)$. We have

$$\eta_i(\beta(\lambda^i\rho) + \beta(\lambda^{a+1}\rho) + \cdots) = \eta_i^2 \lambda \quad \text{and} \quad \eta_i\beta(\lambda^{a+1}\rho) = 0$$

which are proved in the last section.

For our calculation we need a result of [2] further. Let $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the $i$-th position and let us consider $e_i, \ldots, e_n$ as multiplicative generators of the Clifford algebra $C_n$ satisfying the relations $e_i^2 = -1, e_ie_j + e_je_i = 0$ ($i \neq j$). Let $S^{n-1}$ be the unit sphere in $R^n \subset C_n$. Then we set

$$S_+ = S^{n-1} \cap \{(x_1, \ldots, x_n); x_n \geq 0\},$$
$$S_- = S^{n-1} \cap \{(x_1, \ldots, x_n); x_n \leq 0\},$$
$$S^{n-2} = S_+ \cap S_-.$$
and $\text{Spin}(n-1)$ as the isotropy subgroup at $e_n$. Thus $\text{Spin}(n)/\text{Spin}(n-1)=S^{n-1}$ and so we have the principal $\text{Spin}(n-1)$-bundle

$$
\phi: \text{Spin}(n) \to S^{n-1}.
$$

Let $G=\{\pm 1\}$ be the multiplicative subgroup of $\text{Spin}(n-1)$ and let us view as $\text{SO}(n)=\text{Spin}(n)/G$ and $\text{SO}(n-1)=\text{Spin}(n-1)/G$. Analogously we then have the principal $\text{SO}(n-1)$-bundle

$$
\phi: \text{SO}(n) \to S^{n-1}.
$$

We parametrize $S_+^n$ and $S_-^n$ by use of polar coordinates as follows.

$$(x, t) = \cos t \cdot e_n + \sin t \cdot x$$

and

$$(x, t) = -\cos t \cdot e_n + \sin t \cdot x$$

for $x \in S^{n-1}$ and $0 \leq t \leq \pi/2$. Define maps

$$
j_1: S_+ \times \text{Spin}(n-1) \to \phi^{-1}(S_+),
j_2: S_- \times \text{Spin}(n-1) \to \phi^{-1}(S_-)
$$

by

$$
nj_1(x, t, g) = (-\cos t/2 + \sin t/2 \cdot x) g,$$ 

$$
nj_2(x, t, e_1 x g) = (\cos t/2 \cdot x - \sin t/2) g.$$

Then it is clear that these maps become $\text{Spin}(n-1)$-bundle isomorphisms. Since $j_1$ and $j_2$ are compatible with the action of $G$ these maps induce also $\text{SO}(n-1)$-bundle isomorphisms

$$
j_1: S_+ \times \text{SO}(n-1) \to \phi^{-1}(S_+),
j_2: S_- \times \text{SO}(n-1) \to \phi^{-1}(S_-).
$$

Therefore we get

**Lemma 1.5** ([2], Proposition 13.2). Let $G(l)=\text{Spin}(l)$ or $\text{SO}(l)$ for $l=n-1, n$. Then the principal $G(n-1)$-bundle $\phi: G(n)\to S^{n-1}$ is isomorphic to the bundle obtained from the two product bundles

$$
S_+ \times G(n-1) \to S_+, S_- \times G(n-1) \to S_-
$$

by the identification

$$(x, g) \leftrightarrow (x, e_1 x g) \quad \text{or} \quad (x, \pi(g)) \leftrightarrow (x, \pi(e_1 x g))$$

for $x \in S^{n-2}, g \in \text{Spin}(n-1)$ according as $G(l)=\text{Spin}(l)$ or $\text{SO}(l)$.

Denote the map which gives the identification in the above lemma by

$$
d: S^{n-2} \times G(n-1) \to S^{n-2} \times G(n-1).
$$
Namely $d$ is given by

$$d(x, g) = (x, e, xg) \quad \text{or} \quad d(x, \pi(g)) = (x, \pi(e, xg))$$

for $x \in S^{n-2}$, $g \in \text{Spin}(n-1)$ according as $G(l) = \text{Spin}(l)$ or $SO(l)$. We consider the Mayer-Vietoris exact sequence of $(G(n), \phi^{-1}(S_+), \phi^{-1}(S_-))$ in $KO$-theory. Then by using Lemma 1.5 we obtain the following exact sequence

$$(1.6) \quad \cdots \rightarrow h^*(X \times S^{n-2} \times G(n-1)) \rightarrow h^*(X \times G(n)) \xrightarrow{\delta} h^*(X \times G(n-1)) \oplus h^*(X \times S^{n-2} \times G(n-1)) \rightarrow \cdots$$

for $h = KO, K$. Here

$$\varphi = ((1 \times i)^*, (1 \times i)^*), \quad \psi = (1 \times p)^* - (1 \times pd)^*$$

where $i: G(n-1) \subset G(n)$ is the inclusion above and $p: S^{n-2} \times G(n-1) \rightarrow G(n-1)$ the obvious projection. Note that there holds the relation

$$\delta(x(1 \times ip)^*(y)) = \delta(x)y$$

for $x \in h^*(X \times S^{n-2} \times G(n-1))$, $y \in h^*(X \times G(n))$.

Let us denote by $\rho$ also the composite $\rho i$ and by $\Delta$ the simple spin-representation of $\text{Spin}(n-1)$ which is real or quaternionic according as $n \equiv 2$ or $6 \mod 8$ ([8], §13). From [11], Theorem 5.6 (also see [9], Prop. 2.4 and [10], Prop. 3.5) again it follows that

$$KO^*(\text{Spin}(n-1)) = \bigwedge_{\text{Kop}(+)\beta(\lambda^1\rho), \cdots, \beta(\lambda^{n-1}\rho)}, \bar{\kappa}$$

as a $KO^*(+)$-module. Here $\bar{\kappa} = \beta(\Delta)$ or $\bar{\kappa}_{a-1}$ as in [10] according as $n \equiv 2$ or $6 \mod 8$ so that

$$c(\bar{\kappa}) = \mu^a c(\beta(\Delta))$$

where we denote by $c$ two kinds of the complexification homomorphisms $KO(X) \rightarrow K(X)$ and $KH(X) \rightarrow K(X)$.

We now consider behavior of $\delta, \varphi$ and $\psi$ in (1.6) when $X =$ point, $G(l) = \text{Spin}(l)$ ($l = n-1, n$) and $h = KO$. Clearly

$$\varphi(\beta(\lambda^i\rho)) = (\beta(\lambda^i\rho) + \beta(\lambda^{i-1}\rho), \beta(\lambda^i\rho) + \beta(\lambda^{i-1}\rho)) \quad (1 \leq i \leq a-1)$$

and since $i^*(\Delta^*) = c(\Delta)$ it is easy to see that

$$\varphi(\lambda_j) = 2\bar{\kappa}, \eta^2_\kappa, \eta_\kappa \text{ or } 0$$

according as $j \equiv 0, 1, 2$ or $3 \mod 4$.

We have a commutative diagram with $\delta$ as in (1.6) when $h = K$. 

\begin{figure}
\centering
\begin{tikzpicture}
\node (A) at (0,0) {$h^*(X \times S^{n-2} \times G(n-1))$};
\node (B) at (2,0) {$h^*(X \times G(n))$};
\node (C) at (4,0) {$h^*(X \times G(n-1)) \oplus h^*(X \times S^{n-2} \times G(n-1))$};
\node (D) at (6,0) {$\cdots$};

\draw[->] (A) -- node[above] {$\delta$} (B);
\draw[->] (B) -- node[below] {$\psi$} (C);
\draw[->] (C) -- (D);
\end{tikzpicture}
\end{figure}
\[ K^{2-n}(S^{n-2} \times \text{Spin}(n-1)) \xrightarrow{\delta} \tilde{K}^{3-n}(\text{Spin}(n)) \]
\[ K^{2-n}(S^{n-2}) \xrightarrow{q^*} \tilde{K}^{3-n}(S^{n-1}) \]

where the lower \( \delta \) is an isomorphism and \( q \) denotes the evident projection. Choose a generator \( t \in \tilde{KO}^{-n}(S^{n-2}) \cong \mathbb{Z} \) so that
\[ \mu^{n+1} \delta c(t) = \beta(\delta) \in \tilde{K}^{3-n}(S^{n-1}) \cong \mathbb{Z} \]

which is a generator of \( \tilde{K}^{3-n}(S^{n-1}) \), where \( \delta: S^{n-1} \to GL(2^n, \mathbb{C}) \) is a map defined by \( \delta(\phi(g)) = \Delta^+(g) \Delta^-(g)^{-1} \) for \( g \in \text{Spin}(n) \). Then the commutativity of the diagram above yields
\[ \delta(c(t) \times 1) = \mu^{n+1}(\beta(\Delta^+) - \beta(\Delta^-)) \]

Hence we have
\[ c \delta(t \times \bar{\kappa}) = \mu^3 \beta(\Delta^+) \beta(\Delta^-) \]

because of \( i^*(\beta(\Delta^+)) = \beta(\Delta) \). So we may take
\[ \lambda = \delta(t \times \bar{\kappa}) \quad \text{so that} \quad \varphi(\lambda) = 0. \]

By observing \( (pd)^*(\beta(\Delta)) \) we can check that \( (pd)^*(\bar{\kappa}) \) takes the form of
\[ (pd)^*(\bar{\kappa}) = 1 \times \bar{\kappa} + x \times 1 \quad \text{for} \quad x \in \tilde{KO}^{-n}(S^{n-2}) = \mathbb{Z}_2 \cdot \eta_1 t. \]

Then \( \psi(\bar{\kappa}, \bar{\kappa}) = x \times 1 \). Hence if \( x=0 \), there is an element \( y \in KO^*(\text{Spin}(n)) \) such that \( \varphi(y) = (\bar{\kappa}, \bar{\kappa}) \), that is, \( i^*(y) = \bar{\kappa} \). Using this we have \( \lambda = \delta(t \times 1) y \) and so applying \( c \) to both sides of this we get \( \mu^3 \beta(\Delta^+) \beta(\Delta^-) = \mu^{n+1}(\beta(\Delta^+) - \beta(\Delta^-)) \circ c(y) \).

This implies that \( c(y) = \mu^3 \beta(\Delta^+) \) or \( \mu^{n+1} \beta(\Delta^-) \), because \( K^*(\text{Spin}(n)) \) is the exterior algebra over \( K^*(+) \) generated by \( \beta(\lambda^i \rho), \ldots, \beta(\lambda^{n-1} \rho), \beta(\Delta^+), \beta(\Delta^-) \). By exactness of \( (1.1) \) when \( X = \text{Spin}(n) \) we hence have \( \lambda_{n+3} = 0 \). This is a contradiction because \( \lambda_{n+3} \neq 0 \) by Proposition 1.4. Therefore \( x \neq 0 \), that is, \( x = \eta_1 t \) and so we have
\[ (pd)^*(\bar{\kappa}) = 1 \times \bar{\kappa} + \eta_1 t \times 1. \]

Consequently we have
\[ \psi(\bar{\kappa}, 0) = 1 \times \bar{\kappa}, \quad \psi(0, \bar{\kappa}) = -1 \times \bar{\kappa} + \eta_1 t \times 1. \]

Since \( \pi^*: \tilde{KO}^{-1}(P^{n-2}) \to \tilde{KO}^{-1}(S^{n-2}) \) is a zero map it is clear that
\[ \psi(\beta(\lambda^i \rho), 0) = -\psi(0, \beta(\lambda^i \rho)) = \beta(\lambda^i \rho) \quad (1 \leq i \leq a - 1). \]

Finally we consider \( \delta(t \times 1) \). As shown above \( c \delta(t \times 1) = \mu^{n+1}(\beta(\Delta^+) - \beta(\Delta^-)) \).
\( \beta(\Delta^-) \) which means \( c(\delta(t \times 1) - \lambda_{-2}) = 0 \) since \( a \) is even. Using the exactness of (1.1) when \( X = \text{Spin}(n) \) we have an element \( x \in \text{KO}^*(\text{Spin}(n)) \) such that \( \eta_i x = \delta(t \times 1) - \lambda_{-2} \). Hence \( \eta_i x = \delta(\eta_i t \times 1) = \delta(\psi_k, k) = 0 \). So by observing the structure of \( \text{KO}^*(\text{Spin}(n)) \) we see that \( x \) must be zero. This implies
\[
\delta(t \times 1) = \lambda_{-2}.
\]

From these facts we obtain

**Lemma 1.7.**

\[
\text{KO}^*(P^{s-1} \times \text{Spin}(n)) = (\text{KO}^*(P^{s-1}) \otimes_{\text{KO}^*(\text{+})} \text{KO}^*(\text{Spin}(n))) / \mathcal{J}
\]

where \( \mathcal{J} \) is the ideal generated by
\[
\begin{align*}
\varphi_1 \otimes \lambda_0 - \varphi_2 \otimes \lambda_2, & \quad \varphi_3 \otimes \lambda_2 - \varphi_4 \otimes \lambda_0, \\
\varphi_1 \otimes \lambda_1 - \varphi_3 \otimes \lambda_2, & \quad \varphi_2 \otimes \lambda_3 - \varphi_3 \otimes \lambda_1.
\end{align*}
\]

**Proof.** Consider (1.6) when \( X = P^{s-1}, G(l) = \text{Spin}(l) \) \((l=n-1, n)\) and \( h = \text{KO} \). Since \( \text{KO}^*(\text{Spin}(n-1)) \) is \( \text{KO}^*(\text{+}) \)-free as mentioned above, we have a canonical isomorphism
\[
\text{KO}^*(X \times \text{Spin}(n-1)) \cong \text{KO}^*(X) \otimes_{\text{KO}^*(\text{+})} \text{KO}^*(\text{Spin}(n-1))
\]
for any finite \( CW \)-complex \( X \). Applying this fact to (1.6) in the present case we can easily get the lemma from the above results on \( \varphi, \psi \) and \( \delta \). Now the relations can be shown as follows. For example,
\[
\begin{align*}
\varphi_1 \otimes \lambda_0 &= r(c(\varphi_1 \times 1)(1 \times \beta(\Delta^+)))) \\
&= r(\mu \nu \times \beta(\Delta^+)) \\
&= r(\mu^2 \nu \times \mu^2 \beta(\Delta^+)) \\
&= r(\epsilon(\varphi_2 \times 1)(1 \times \mu^2 \beta(\Delta^+)) \\
&= \varphi_3 \otimes \lambda_2
\end{align*}
\]
The others are analogous.

**2. The module structure of \( \text{KO}^*(\text{SO}(n)) \)**

Let \( \xi' \) be the canonical non-trivial real line bundle over \( \text{SO}(n) \) and set
\[
\xi = \xi' - 1 \quad \text{in} \quad \text{KO}(\text{SO}(n)).
\]

Define maps
\[
\delta, \varepsilon: \text{SO}(n) \to \text{GL}(2^s, C)
\]
by \( \delta(\pi(g)) = \Delta^-(g)^{-1} \Delta^+(g), \varepsilon(\pi(g)) = \Delta^+(g)^2 \) for \( g \in \text{Spin}(n) \). Then we have the elements \( \beta(\varepsilon), \beta(\delta) \) of \( K^{-1}(\text{SO}(n)) \). So we set

\[
\delta(t \times 1) = \lambda_{-2}.
\]
\( e_i = r(\mu^i \beta(\delta)), \delta_i = r(\mu^i \beta(\delta)) \) in \( KO^{-2i-1}(SO(n)) \) where \( i \) is of course reduced mod 4. Clearly there hold the relations

\[
\eta_i e_i = 2e_{i+2}, \quad \eta_i \delta_i = 2\delta_{i+2}.
\]

For the standard representation \( \rho \) of \( SO(n) \) as in §1 we also have the elements

\[
\beta(\lambda^j \rho) (1 \leq j \leq n) \) in \( KO^{-1}(SO(n)) \).
\]

Let \( G = \{ \pm 1 \} \) act on \( Spin(n) \) as a subgroup of \( Spin(n) \) and let \( R^{p,q} \) be the \( R^{p,q} \) with a \( G \)-action such that \( -1 \) reverses the first \( p \) coordinates and fixes the last \( q \). Let \( S^{p,q} \) and \( B^{p,q} \) be the unit sphere and ball in \( R^{p,q} \) and \( \Sigma^{p,q} = B^{p,q}/S^{p,q} \) with the collapsed \( S^{p,q} \) as base point.

By [7] we have a homeomorphism

\[
S^{n,0} \times_G Spin(n) \to P^{n-1} \times Spin(n)
\]

which is induced by the assignment

\[(x, g) \mapsto (\pi(x), x \epsilon_1 g)\]

for \( x \in S^{n,0}, g \in Spin(n) \) where \( \pi: S^{n,0} \to P^{n-1} \) denotes the canonical projection. Using this, from the exact sequence of \( (B^{n,0} \times Spin(n), S^{n,0} \times Spin(n)) \) in the equivariant \( KO \) (or \( K \))-theory associated with \( G \) we have an exact sequence

\[
\ldots \to h^*(SO(n)) \to h^*(P^{n-1} \times Spin(n)) \to h^*_G(\Sigma^{n,0} \wedge Spin(n)_+ ) \to \ldots
\]

for \( h = KO \) or \( K \). Here there holds the relation

\[
\delta(x I(y)) = \delta(x) y
\]

for \( x \in h^*(P^{n-1} \times Spin(n)), y \in h^*(SO(n)) \).

In the case when \( h = KO \) we have

\[
I(\xi) = \gamma \times 1,
I(\beta(\lambda^i \rho)) = 1 \times \beta(\lambda^i \rho) + (\frac{n-2}{i-1}) \eta_i \gamma \times 1 \quad (1 \leq i \leq n),
I(\delta_0) = I(\delta_0) = 0,
I(\delta_i) = 2(1 \times \lambda_1 - \nu_i \times 1),
I(\delta_2) = 2(1 \times \lambda_3 - \nu_2 \times 1),
I(\xi_0) = (\gamma + 2) \times \lambda_0,
I(\xi_i) = (\gamma + 2) \times \lambda_i - 2\nu_i \times 1,
I(\xi_2) = (\gamma + 2) \times \lambda_2,
I(\xi_3) = (\gamma + 2) \times \lambda_3 - 2\nu_3 \times 1.
\]
The first equality is clear, the second one can be verified in the same way as in [10] and the others follow from [9], Lemma 3.3, iii), iv) immediately.

We consider the image of

\[ J: \widetilde{KO}_n^g(\Sigma^{n,0} \wedge \text{Spin}(n)_+^0) \to KO^*(SO(n)) \]

Let \( \omega_i^* \in \widetilde{KO}_G(\Sigma^{n,0}) \), \( \tau_i^* \in \widetilde{K}_G(\Sigma^{2n,0}) \) be the Bott elements mentioned in [9] such that \( j^*(\omega_i^*) = 2^{n-1}(1 - R^{1,0}) \), \( j^*(\tau_i^*) = 2^{n-1}(1 - R^{1,0} \otimes C) \) where \( j \) denotes the inclusions of \( \Sigma^{n,0} \) in \( \Sigma^{2n,0} \) and \( \Sigma^{2n} \). Put \( n = 8k + 2 \) or \( 8k + 6 \). Clearly then any element of \( \widetilde{KO}_n^g(\Sigma^{n,0} \wedge \text{Spin}(n)_+^0) \) can be written in the form \( \omega_i^* x \) where \( x \in \widetilde{KO}_n^g(\Sigma^{2n,0} \wedge \text{Spin}(n)_+^0) \) (\( t = 1 \) or \( 3 \)). Moreover if we put \( c(x) = \tau_i^* y \) for \( y \in K^*(SO(n)) \), then we obtain

\[
(a) \quad J(\omega_i^* x) = 2^{n-2} \xi \tau_{\xi}(yc(\xi)).
\]

According to [9], Theorem 3.5

\[
(b) \quad K^*(SO(n)) = \wedge_{i=0}^{\infty} (c(\beta(\lambda^1 \rho)), \cdots, c(\beta(\lambda^{n-1} \rho)), \beta(\xi), \beta(\delta))
\]

\[
\otimes_{\omega} (Z \cdot 1 + Z_{\omega} \cdot c(\xi))
\]

with the relations

\[
c(\xi)^2 = -2c(\xi), \beta(\xi) \otimes c(\xi) = 0.
\]

If we set \( \delta(1 \times \lambda) = \omega_i^* x \), then we have

\[
c(\omega_i^* x) = \tau_i^* \tau_i^* \mu^3 c(\xi + 1) (\beta(\delta) - \beta(\xi))
\]

by using [9], Lemma 3.4, iv), because of \( c(\lambda) = \mu^3 \beta(\Delta^+) \beta(\Delta^-) \). Hence using the relation \( c(\xi) \otimes \beta(\xi) = 0 \) gives

\[
(c) \quad 2^{n-1} \xi \delta_{3} = J\delta(1 \times \lambda) = 0.
\]

Since \( \beta(\Delta^+) = \beta(\Delta^-) \) and \( \nu^* = - \nu \) by definition of \( \nu \), we have \( \beta(\delta)^* = - \beta(\delta) \) by [9], Lemma 3.3, iii). So, from exactness of (1.1) when \( X = SO(n) \) it follows that

\[
(d) \quad 2\xi \delta_{2i} = r(\mu^{2i} c(\xi) \cdot 2 \beta(\delta))
\]

\[
= \delta(\mu^{2i+1} c(\xi) (\beta(\delta) - \beta(\delta)^*))
\]

\[
= \delta c(r(\mu^{2i+1} c(\xi) \beta(\delta)))
\]

\[
= 0
\]

for \( i = 0, 1 \).

Calculate the right-hand side of (a) making use of (b), (c) and (d). Then we see that \( J(\omega_i^* x) \) can be written as
\[ J(\omega x) = 2^s \xi P_1 + 2^{s-1} \eta_4 \xi P_2 + 2^{s-1} \xi \delta P_3 \]

where \( P_i \) is a polynomial in \( \beta(\lambda^i \rho) \), \( \ldots \), \( \beta(\lambda^{s-1} \rho) \) with integers as coefficients for \( i = 1, 2, 3 \). So apply \( I \) to both sides of such an expression of \( J(\omega x) \) and estimate this by using (2.2). Since \( II = 0 \) it then follows from Lemma 1.7 that the first two terms of \( J(\omega x) \) are zero. Thus we have

\begin{equation}
(2.3) \text{Im } J \text{ is generated by elements of the form } 2^{s-1} \xi \delta P \text{ where } P \text{ is a polynomial in } \beta(\lambda^1 \rho), \ldots, \beta(\lambda^{s-1} \rho) \text{ with integers as coefficients, and } \eta_4 \text{ Im } J = 0.
\end{equation}

We now observe the exact sequence

\begin{equation}
(2.4) \cdots \rightarrow KO^*(S^{s-2} \times SO(n-1)) \xrightarrow{\delta} KO^*(SO(n)) \xrightarrow{\varphi} KO^*(SO(n-1)) \oplus KO^*(SO(n-1)) \xrightarrow{\psi} KO^*(S^{s-2} \times SO(n-1)) \rightarrow \cdots
\end{equation}

which follows from (1.6).

Denote by \( \xi \) also the restriction \( i^*(\xi) \) to \( SO(n-1) \) and by \( \rho \) the composite \( \rho i \) as before. By [9] and [10] we then have

\begin{equation}
(2.5) \text{As a } KO^*(-)\text{-module, } KO^*(SO(n-1)) \text{ is generated by the elements in the form } P, \xi P, \kappa P \text{ and } \nu P \text{ where } \kappa \text{ denotes } \beta(\xi_{-1}) \text{ or } \kappa_{-1} \text{ of } KO^{s-n}(SO(n-1)) \text{ and } \nu \text{ denotes } \nu_{-1} \text{ or } \nu_{-1} \text{ of } KO^{-n}(SO(n-1)) \text{ as in } [9], [10] \text{ according as } n \equiv 2 \text{ or } 6 \text{ mod } 8 \text{ and } P \text{ denotes a polynomial in } \beta(\lambda^1 \rho), \ldots, \beta(\lambda^{s-1} \rho). \text{ Also there hold the relations}
\end{equation}

\[ \kappa^2 = \nu^2 = \xi \kappa = \eta_4 \nu = 2 \nu = 0, \kappa \nu = \eta_4 ^2 \beta(\lambda^2 \Delta), \]

\[ \eta_4 \kappa = \xi \nu, \eta_4 \nu = 2^{s-1} \theta \eta_4, 2^{s-2} \theta \kappa \xi = 0 \]

where \( \theta = \eta_4 \) or 2 according as \( n \equiv 2 \) or 6 mod 8.

Let \( tr: h^*(Spin(n-1)) \rightarrow h^*(SO(n-1)) \) be the transfer where \( h = KO \) or \( K \). Then observation of the definitions of \( \kappa \) and \( \kappa \) ([9], [10]) gives

\[ tr(\kappa) = \kappa \]

because of \( tr(\beta(\Delta)) = \beta(\xi) \) and

\[ tr(1) = \xi + 2. \]

Therefore we have from the formula on \( \kappa \) given in §1

\begin{equation}
(2.6) \psi(\kappa, 0) = 1 \times \kappa, \psi(0, \kappa) = -1 \times \kappa + \eta_4 \times \xi.
\end{equation}

We now show that

\begin{equation}
(2.7) \psi(\nu, 0) = 1 \times \nu, \psi(0, \nu) = 1 \times \nu + \eta_4 ^2 \times (l \xi + 1) \quad (l = 0, 1).
\end{equation}
The first equality is clear. To prove the second one we define maps

\[ \begin{align*}
m &: S^{n-2} \times SO(n-1) \to SO(n-1), \\
m' &: S^{n-1} \times Spin(n-1) \to Spin(n-1), \\
m_0 &: P^{n-2} \times Spin(n-1) \to SO(n-1), \\
m_1 &: S^{n-2} \times P^{n-2} \times Spin(n-1) \to P^{n-2} \times Spin(n-1), \\
m_2 &: S^{n-2} \times P^{n-2} \to P^{n-2}, \\
m_3 &: S^{n-2} \times S^{n-2} \to S^{n-2}, \\
m_4 &: Spin(n-1) \times Spin(n-1) \to Spin(n-1)
\end{align*} \]

by

\[ \begin{align*}
m(x, \pi(g)) &= \pi(e, xg), \\
m'(x, g) &= e_1 xg, \\
m_0(\pi(x), g) &= \pi(e_1 xg), \\
m_1(x, \pi(y), g) &= (m_2(x, \pi(y)), xe_1 g), \\
m_2(x, \pi(y)) &= \pi(xe_1 ye_1 x), \\
m_3(x, y) &= xe_1 ye_1 x, \\
m_4(g, g') &= gg'
\end{align*} \]

for \( x, y \in S^{n-2}, g, g' \in Spin(n-1) \). Here by \( \pi \) we denote the obvious projection. Moreover we define embeddings

\[ \begin{align*}
i &: S^{n-2} \to Spin(n-1), \\
i &: P^{n-2} \to SO(n-1)
\end{align*} \]

by \( i(x) = xe_1, i(\pi(x)) = \pi(xe_1) \).

According to [9] and [10], \( m_0 \) yields a monomorphism

\[ I: KO^*(SO(n-1)) \to KO^*(P^{n-2} \times Spin(n-1)) \]

and by [9], (4.17) and [10], (4.20) we have

\[ I(\nu) = 1 \times \eta_1 \kappa + \nu \times 1 \]

where \( \nu \) denotes \( \nu_{n-2} \) or \( \mu_{n-2} \) of \( KO^{-n}(P^{n-2}) \) as in [9] or [10] according as \( n \equiv 2 \) or \( 6 \) mod 8. From this equality it follows readily that

\[ \pi^*(\nu) = \eta_1 \kappa \quad \text{and} \quad i^*(\nu) = \nu. \]

Let

\[ \delta: KO^{-n}(P^{n-2}) = KO^{-n}_c(S^{n-1,0}) \to KO^{-n}_c(S^{n-1,0}) \]

be the coboundary homomorphism appeared in the exact sequence of \((B^{n-1,0}, S^{n-1,0})\). Furthermore we then see that \( \delta(\nu) \) is a generator of \( KO^{-n}_c(S^{n-1,0}) \approx Z_2 \) and the forgetful homomorphism \( KO^{-n}_c(S^{n-1,0}) \to KO^{-n}_c(S^{n-1}) \) becomes an isomorphism. From these facts we obtain

(a) \[ \pi^*(\nu) = \eta_1^* t, \quad \text{so that} \quad i^*(\eta_1 \kappa) = \eta_1^* t. \]

Since \( m_4(\beta(\Delta)) = 2\beta(\Delta) \times 1 + 1 \times \beta(\Delta) \) in \( KO \) or \( KH \)-theory, we have
(b) \[ m^{\sharp}(\eta_i r) = 1 \times \eta_1 r. \]

By (a), (b) we get

\[ m^{\sharp}_i(\eta_i^2 t) = 1 \times \eta_1^2 t. \]

So, using (a) again gives

\[ (1 \times \pi)^* m^{\sharp}_i(\bar{\nu}) = 1 \times \eta_1^2 t. \]

This and (a) imply

\[ m^{\sharp}_i(\bar{\nu}) = 1 \times \bar{\nu} + t \times x \quad \text{for some} \quad x \in KO^{-2}(P^{n-2}). \]

Since degree \( \bar{\nu} = -n \) and degree \( t = 2 - n \), we can infer from the structure of \( KO^{-2}(P^{n-2}) \) that

\[ x = 0 \quad \text{or} \quad \eta_1^2 \gamma \]

where \( \gamma \) denotes also the restriction \( \iota^*(\gamma) \) to \( P^{n-2} \). Therefore

\[ m^{\sharp}_i(\bar{\nu}) = 1 \times \bar{\nu} + t \times \eta_1^2 \gamma \quad (l = 0, 1), \]

so that

\[ m^{\sharp}_i(\bar{\nu} \times 1) = 1 \times \bar{\nu} \times 1 + \eta_1^2 t \times \eta_1 \gamma \times 1 \quad (l = 0, 1). \]

On the other hand, the argument parallel to that about \((pd)^*\) in §1 yields

\[ m^{\ast}(\bar{\rho}) = 1 \times \bar{\rho} + \eta_1 t \times 1. \]

Hence

\[ m^{\ast}(1 \times \bar{\rho}) = 1 \times 1 \times \bar{\rho} + \eta_1 t \times 1 \times 1. \]

From this and (c) it follows that

\[ m^{\ast} I(\nu) = 1 \times 1 \times \eta_1 \bar{r} + \eta_1^2 t \times (l \gamma + 1) \times 1 + 1 \times \bar{\nu} \times 1 \quad (l = 0, 1) \]

and so

\[ (1 \times m_o)^* m^*(\nu) = (1 \times m_o)^* (1 \times \nu + \eta_1^2 t \times (l \xi + 1)) \quad (l = 0, 1). \]

Since \( KO^*(SO(n-1)) \) is \( KO^*(+) \)-free, we see from the injectivity of \( I \) that

\[ (1 \times m_o)^* \]

is a monomorphism. Therefore

\[ m^*(\nu) = 1 \times \nu + \eta_1^2 t \times (l \xi + 1) \quad (l = 0, 1), \]

which is the required result because \( m = pd \). This completes the proof of (2.7).

Further, clearly we have

\[ \varphi(\xi) = (\xi, \xi), \]

\[ \varphi(\beta(\lambda^i r)) = \beta(\lambda^i r) + \beta(\lambda^{i-1} r), \beta(\lambda^i r) + \beta(\lambda^{i-1} r)) \quad (1 \leq i \leq n). \]
Using (2.5), (2.6), (2.7) and these formulas, we obtain easily the following result concerning \(\psi\) and \(\varphi\) of (2.4)

(2.8) As \(KO^*(-\mathbb{Z})\)-modules, \(\text{Coker } \psi\) is generated by elements of the form \(t \times P, t \times \varepsilon P, t \times X^P, t \times \eta_1 P, t \times \eta_3 P, t \times \eta_4 P, t \times \eta_5 P, t \times \eta_6 P, t \times \eta_7 P\) and \(t \times \eta_8 v P\), and \(\text{Im } \varphi\) by elements of the form \((P, P), (\kappa P, \kappa P), \eta_1^2(v P, v P), \eta_1^3(\kappa P, \kappa P)\) and \(\eta_1(\kappa P, \kappa P)\). Here \(P\) denotes a polynomial as in (2.5).

Now we add some generators for \(KO^*(SO(n))\) to the ones given at beginning of this section. Since \(\lambda = \delta(t \times \varepsilon)\), we have

\[
\text{tr}((\psi)) = \delta(t \times \kappa) \quad \text{in} \quad KO^{2n}(SO(n)),
\]

for which we write \(\text{tr } \lambda\) simply.

By (2.7) and exactness of (2.4) there is an element \(\nu_1 \in KO^{*-1}(SO(n))\) such that

\[
\varphi(\nu_1) = \eta_1(v_1, v).
\]

But we need to choose such an element so that

(2.9) \[
I(\nu_1) = \nu_{a+1} \times 1 - 1 \times \lambda_{a+1},
\]

where \(a+1\) is reduced mod 4. The equality \(\varphi(\nu_1) = \eta_1(v_1, v)\) follows from (2.9). Because \(i^* (\nu_{a+1}) = \eta_1 \nu_1, i^* (\lambda_{a+1}) = \eta_1^2 \varepsilon\) and \(I(\nu) = 1 \times \eta_1 \varepsilon + \nu \times 1\) where \(i\) denotes the inclusions \(P^{n-2} \subset P^{n-1}\), \(\text{Spin} (n-1) \subset \text{Spin} (n)\). We construct such an element actually. Let \(\delta\) be as in (2.1) and set \(n = 8k + 2\) where \(s = 1\) or 3. Then by [9], Lemma 3.4 we have \(\delta(1 \times \mu^{*+1} \beta(\Delta^+)) = \tau_4^k \tau_3^* \mu^{*+1} c(\xi + 1)\) and so

\[
\delta(1 \times \lambda_{a+1}) = \omega^k \tau_3^* \mu^{*+1} (\xi + 1).
\]

Also, we have \(\delta(\mu^{*+1} \nu_1 \times 1) = \tau_4^k \tau_3^* \mu^{*+1} c(\xi + 2)\) and hence we get

\[
\delta(\nu_{a+1} \times 1) = \omega^k \tau_3^* \mu^{*+1}
\]

by using the facts that \(KO^0(\Sigma^{n-1}) = Z \cdot r(\tau_3^* \mu^{*+1})\) and \(\tau_3^* = -(R^{n-d} \otimes C) \tau_4^*\). From this and the formula of (2.1) we have \(r(\tau_3^* \mu^{*+1}) \xi = 0\) since \(\gamma \nu_{a+1} = 0\) and so we have

\[
\delta(\nu_{a+1} \times 1 - 1 \times \lambda_{a+1}) = 0.
\]

This and using (2.1) give rise to the required element.

Define \(\tau \in KO^{-1}(SO(n))\) and \(\nu_3 \in KO^{*-n}(SO(n))\) as

\[
\tau = \delta(t \times \nu) \quad \text{and} \quad \nu_3 = -\delta(t \times (\xi + 1)).
\]

Here let \(\delta\) be as in (2.4). Then using the formula after (1.6) we have
\[ \delta(t \times t^*(P)) = -(\xi + 1) \nu_3 P, \quad \delta(t \times \xi t^*(P)) = \xi \nu_3 P, \]
\[ \delta(t \times \xi t^*(P)) = (tr \lambda) P, \quad \delta(t \times \nu t^*(P)) = \tau P \]

where \( P \) is a polynomial as in (2.3). Moreover as stated above
\[ \varphi(v_i) = \eta_i(v, v) \]
and by definition we have
\[ \varphi(\varepsilon_i) = 2(\kappa, \kappa), \eta_i(\kappa, \kappa), \eta_0(\kappa, \kappa) \quad \text{or} \quad 0 \]
according as \( i \equiv -a, 1-a, 2-a \) or \( 3-a \) mod 4. From (2.8) and these equalities we obtain immediately

(2.10) As a \( KO^*(+) \)-module, \( KO^*(SO(n)) \) is generated by elements of the form \( P, (tr \lambda) P, \tau P, \nu P, \nu_3 P, \varepsilon_{-a} P, \varepsilon_{-a} P \) and \( \varepsilon_{1-a} P \) where \( P \) denotes a polynomial in \( \xi, \beta(\lambda^1\rho), \ldots, \beta(\lambda^{a-1}\rho) \) and the indices of \( \varepsilon \) are reduced mod 4.

In (2.10) we find that \( \varepsilon_{1-a} \) can be expressed by the other generators. To show this we need some results. Define a map \( m: P_{n-1} \times S_{n-1} \rightarrow S_{n-1} \) by
\[ m(\pi(x), \phi(g)) = \phi(\varepsilon(x, g)) \quad \text{for} \quad x \in S_{n-1}, g \in \text{Spin}(n-1). \]
Then from construction of \( \beta(\delta) \) and \( \nu \) it follows that
\[ m^*(\beta(\delta)) = c(\gamma + 1) \times \beta(\delta) - \nu \times 1. \]
This implies that
\[ c(m^*\delta(t)) = c((\gamma + 1) \times \delta(t) - \nu_{-a-1} \times 1) \]
because \( c\delta(t) = \mu^{-\nu_3} \beta(\delta) \) and so using (1.1) we have
\[ m^*\delta(t) = (\gamma + 1) \times \delta(t) - \nu_{-a-1} \times 1 + \eta_1(x \times \delta(t) + y \times 1) \]
for some \( x \in \widetilde{KO}^{-7}(P_{n-1}), y \in \widetilde{KO}^{-8}(P_{n-1}). \) Since \( I(\delta(t \times 1)) = (1 \times \phi)^* m^*\delta(t), \phi^* \delta(t) = \lambda_{-a-1} \) by the result just before Proposition 1.7, \( \eta_1 \lambda_{-a-1} = 0 \) and \( \phi^*(y) = 0 \) for the reason of dimension, we obtain
\[ I(\delta(t \times 1)) = (\gamma + 1) \times \lambda_{-a-1} - \nu_{-a-1} \times 1, \]
so that
\[ I(\nu_3) = \nu_{-a-1} \times 1 - 1 \times \lambda_{-a-1} \]
because of \( \gamma \nu_{-a-1} = 0 \) where also \( a + 1 \) is reduced mod 4.

By [9], Theorem 3.5
\[ 2^a c(\xi) = 0, \quad \text{so that} \quad 2^{a+1} \xi = 2^a \eta \xi = 0. \]
On the other hand \( \nu^*(\xi) = \gamma \) and \( \nu^*(\eta \xi) = \eta \gamma \) are the generators of \( \widetilde{KO}^0(P_{n-1}) \approx \)
\(Z_{2}^{*+1}\) and \(\widetilde{KO}^{-t}(P_{n-1})\cong Z_{2}^{*}\) respectively where \(\iota\) is an embedding of \(P_{n-1}\) in \(SO(n)\). Hence we get

(2.12) The orders of \(\xi\) and \(\eta_{2}\xi\) are \(2^{*+1}\) and \(2^{*}\) respectively.

From (2.2), (2.9) and (2.11) it follows that

\[I(\delta_{1}+2n_{*+1})=I(\delta_{1}+\eta_{2}n_{*+3})=0\]

because of \(\eta_{2}n_{*+3}=2n_{*+1}, \eta_{2}n_{*+3}=2n_{*+1}\). So, by (2.3)

\[\delta_{1}+2n_{*+1} = 2^{*+1}\xi \delta_{1}, P, \delta_{1}+\eta_{2}n_{*+3} = 2^{*+1}\xi \delta_{1}, P'\]

for some polynomials \(P, P'\) as in (2.3). This and (2.12) mean that

(2.13) \[2^{*+1}\xi \delta_{1} = -2^{*+1}\xi \delta_{1} = -2^{*+1}\eta_{2}n_{*+3} .\]

Again by (2.2), (2.9) and (2.11) we have

\[I(\xi_{1}+(\xi+2)\nu_{1}) = 0 \quad \text{or} \quad I(\xi_{2}+(\xi+2)\nu_{1}) = 0\]

according as \(n\equiv 2\) or \(6 \mod 8\), because \(\gamma_{1} \equiv \gamma_{2} \equiv 0\).

In any case, by (2.3) and (2.13) we therefore see that \(\xi_{1} \equiv \eta_{2}\) can be described by \(\xi, \nu_{1}, \nu_{3}\). Thus, by (2.10) we obtain

**Lemma 2.14.** As a \(KO^{*}(\pm\)-module, \(KO^{*}(SO(n))\) is generated by elements in the form \(P, (tr \lambda) P, \tau P, \nu_{1} P, \nu_{2} P, \xi_{2-a} P\) and \(\xi_{2-a} P\) where \(P\) is a polynomial as in (2.10) and the indices of \(\xi\) are reduced \(\mod 4\).

Further we provide a lemma. Because of \(\nu_{3} = -\delta(t \times (\xi+1))\), (2.13) yields

\[2^{*+1}\xi \delta_{1} = -\delta(t \times 2^{*+1} \theta \xi) ,\]

that is, \(2^{*+1}\xi \delta_{1} \in \text{Im} \delta\) where \(\delta\) is as in (2.4) and \(\theta\) as in (2.5). Clearly \(Coker \psi \cong \text{Im} \delta\) and this isomorphism sends \(-t \times 2^{*+1} \theta \xi t^{*}(P)\) to \(2^{*+1}\xi \delta_{1} P\) where \(P\) is a polynomial as in (2.3). From (2.3), (2.8) and (2.13) we therefore have

**Lemma 2.15.** As a \(KO^{*}(\pm\)-module

\[\text{Im} J = \wedge z_{k}(\beta(\lambda^{1} \rho), \ldots, \beta(\lambda^{*+1} \rho)) \{2^{*} \xi \nu_{*+1}\},\]

and \(z \text{Im} J = 0\) for \(z = \xi, \eta_{1}\) and \(\eta_{4}\) where the index of \(\nu\) is reduced \(\mod 4\).

3. The algebra structure of \(KO^{*}(SO(n))\)

For our aim we need the formulas for \(I(tr \lambda)\) and \(I(\tau)\) similar to those of (2.2). We begin with calculating \(I(tr \lambda)\). Since \(c(\lambda) = \mu \beta(\Delta^{+}) \beta(\Delta^{-})\) and \(\pi^{*}(\beta(\xi) - \beta(\delta)) = \beta(\Delta^{+}) + \beta(\Delta^{-})\) by construction of \(\beta(\xi)\) and \(\beta(\delta)\), it follows that \(c(\lambda) = \mu \beta(\Delta^{+}) \pi^{*}(\beta(\xi) - \beta(\delta))\), so that we have \(c(tr \lambda) = \mu \beta(\delta) \beta(\xi)\) because
$tr(\beta(\Delta^*)) = \beta(\varepsilon)$ and $\beta(\varepsilon)^2 = 0$. From this and [9], Lemma 3.2, iii), iv) we get
$$c(I(tr \lambda) - ((\gamma + 2) \times \lambda - p_3 \times \lambda_0)) = 0.$$ So, by (1.1) and Lemma 1.7 we can write

(a) $$I(tr \lambda) = (\gamma + 2) \times \lambda - p_3 \times \lambda_0 + \eta 0 \alpha$$ and
$$\alpha = 1 \times x_1 + \gamma \times x_2 + p_1 \times x_3 + p_2 \times x_4$$

for some $x_i \in KO^*(Spin(n))$.

Let $S^{n-2,0} = S^{n,0} \cap \{(x_1, \ldots, x_n); x_1 = x_n = 0\}$ and $P^{n-3} = S^{n-2,0}/G$. Define a map
$$m: S^{n-2} \times P^{n-3} \times Spin(n-1) \to S^{n-2} \times SO(n-1)$$
by $m(x, \pi(y), g) = (e_1 y x e_1, \pi(e_1 y g))$ for $x \in S^{n-2}, y \in S^{n-2,0}, g \in Spin(n-1)$. Then the following diagram with $\delta$ as in (1.6) is commutative.

Also, obviously $m^* = (j \times 1)^* I$ where $j$ denotes the inclusion of $P^{n-3}$ in $P^{n-1}$. Apply $(j \times 1)^*$ to both sides of the first equality of (a). Then considering the order of $\gamma$ we have

(b) $$m^*(tr \lambda) = (\gamma + 2) \times \lambda + \eta_1 \times x_1 + \eta_2 \gamma \times x_2$$
where $\gamma$ denotes $j^*(\gamma)$. On the other hand by discussion similar to that about $(pd)^*$ in §1 we get

(c) $$m^*(t \times 1) = t \times (\gamma + 1) \times 1 + x$$
for some $x \in (1 \times 2 \gamma \times 1) KO^*(S^{n-2} \times P^{n-3} \times Spin(n))$. Moreover, by [9], Lemma 4.14, iii) and [10], Lemma 4.18, iii) we have $I(\kappa) = (\gamma + 2) \times \kappa$. From this and (c) we have $m^*(t \times \kappa) = t \times (\gamma + 2) \times \kappa$. Since $tr \lambda = \delta(t \times \kappa)$ and $\lambda = \delta(t \times \kappa)$, it therefore follows from the commutativity of the above diagram and (b) that
$$\eta_1 \times x_1 + \eta_1 \gamma \times x_2 = 0.$$ Hence we may put
$$\alpha = p_1 \times x_2 + p_3 \times x_4,$$ so that we have

(3.1) $$I(tr \lambda) = (\gamma + 2) \times \lambda - p_3 \times \lambda_0 + \eta_1 \alpha$$
and there hold the relations $\eta_1^2 \alpha = \gamma \alpha = \alpha^2 = 0.$
Since \( I(\nu) = 1 \times \eta \bar{r} + \nu \times 1 \) and \( c(\nu) = 2^{a-1} \mu^{e+1} c(\gamma) \) we get \( c(\nu) = 2^{a-1} \mu^{e+1} c(\xi) \). Also, by (2.11) and [9], Lemma 3.3, iii) we have \( c(\nu_3) = -\mu^{e+2} \beta(\delta) \). Using these facts we obtain
\[
c(I(\tau) - 2^{a-1} \gamma \times \lambda_4) = 0.
\]
Analogously from this equality we can show that
\[
(3.2) \quad I(\tau) = (\gamma + 1) \times \eta \lambda + 2^{a-1} \gamma \times \lambda_0 + \eta_1 \beta
\]
and there hold the relations \( \eta_1 \beta = \gamma \beta = \beta^2 = 0 \).

We are now ready to obtain

**Theorem 3.3.** As a KO*(+) module

\[
KO^*(SO(n)) = \bigwedge_{\Lambda^O} (\beta(\lambda^p), \cdots, \beta(\lambda^{a-1}), \xi, \xi_1, \xi_2, \nu_1, \nu_2)
\]
\[
\otimes \mathbb{Z}(Z \cdot 1 \oplus Z \cdot 1 + 1 \oplus \mathbb{Z} \cdot 1 \otimes 1 \cdot \tau) \lambda
\]
in which the following relations hold:

\[
\xi^2 = -2\xi, \beta(\lambda^p)^3 = \eta_1 (\beta(\lambda^p) + \frac{n}{k}) \beta(\lambda^p) \quad (1 \leq k \leq a - 1),
\]
\[
\eta_1 \xi_i = 0, \eta_1 \nu_{s+1} = 2^s \xi, \eta_1 \nu_{s+1} = 2^{s-1} \eta_1^2, \eta_1 \xi_i = 2 \xi_i + 1,
\]
\[
\nu_j \tau = 2 \nu_{j+1}, \eta_4 \tau = 0, \xi_i \tau = (tr \lambda)^2 = \tau^2 = 0,
\]
\[
\xi \xi_i = \xi \tau \lambda = \xi_i \tau \lambda = \xi_i \tau = \nu_j \tau \lambda = \nu_j \tau = \eta_2 \xi_2 = \tau \tau \lambda = 0,
\]
\[
\nu_1 \nu_3 = \eta_1 (\xi + 1) \tau, \xi \tau = \eta_1 \tau \lambda, \eta_2 \nu_{s+1} = \eta_2 \nu_{s+3} = \eta_4 \tau \lambda,
\]
\[
\nu_1 \nu_{s+3} = \xi_2 \nu_{s+1} = 2 \tau \lambda
\]
for \( i = 0, 2, j = 1, 3 \) if the indices of \( \xi \) and \( \nu \) are reduced mod 4 and \( \otimes \mathbb{Z} \) is left out.

**Proof.** From Lemma 2.15 we see that \( I \) induces a monomorphism

\[
KO^*(SO(n)) / (2^a \xi \nu_{s+1}) \rightarrow KO^*(P^{e-1} \times Spin(n))
\]

Let \( R \) denote the right-hand side of the equality stated in the theorem. Then a computation, using (2.2), (2.9), (2.11), (3.1), (3.2), Lemmas 1.7 and 2.14, shows that as a KO*(+) module

\[
KO^*(SO(n)) / (2^a \xi \nu_{s+1}) = R / (2^a \xi \nu_{s+1})
\]
in which there hold the above relations reduced mod \( (2^a \xi \nu_{s+1}) \). So, if it is shown that in \( KO^*(SO(n)) \) these relations hold, then the theorem follows immediately.

We now consider the relations. The first relation is clear. The second one and the relations \( \nu_j^2 = 0 \) are due to [5], §6.

\[
\eta_1 \xi_i = \eta_1 r(\mu^l \beta(\delta))
\]
\[
= \chi \delta(\mu^{l+1} \beta(\delta)) = 0 \quad \text{since} \quad \chi \delta = 0 \quad \text{in (1.1)}.
\]
By definition \( \eta_1 v_1 = \delta c(v_2) = 0 \). So, by exactness of (1.1) there is an element \( x \in K^*(SO(n)) \) such that

\[
\eta_1 v_1 = r(x).
\]

Then \( rI(x) = 2^{s-1} \theta \gamma \times 1 \) by Proposition 1.3 where \( \theta \) is as in (2.5). Observing \( \text{Im } rI \), we get \( I(x) = 2^{s-2} c(\theta \gamma) \times 1 \). Since \( I \) in complex case is injective, we have

\[
x = 2^{s-2} c(\theta \gamma)
\]

and so

\[
\eta_1 v_1 = 2^{s-1} \theta \gamma.
\]

By arguing as above we get also another relation \( \eta_1 v_3 = 2^{s-2} \theta \eta_4 \xi \).

\[
\eta_4 x = r(c(\eta_4) \mu^j \beta(e)) = r(2\mu^{i+2} \beta(e)) = 2\xi_{i+2}.
\]

\[
\eta_4 v_j = r(\mu^2 c(v_j)) = r(\nu_{j+2}) = 2\nu_{j+2} \quad \text{since} \quad c(v_j) = -\mu^{s+1} \beta(\delta).
\]

\[
\eta_4 \tau = \delta(t \times \eta_4 v) = 0 \quad \text{by (2.5)}.
\]

\[
\xi_1^j = r(c(\xi_1) \mu^i \beta(e)) = (-1)^j 2\delta(\mu^{2i+1} \beta(e) \beta(\delta)) \quad \text{since} \quad \beta(e)^* = \beta(e) - c(\xi + 2) \beta(\delta)
\]

\[
= (-1)^{i+1} \delta c(\xi_{i+2} v_i) = 0 \quad \text{since} \quad \delta c = 0 \text{ in (1.1)}.
\]

\[
\tau^2 = \delta(t \times u^*(\tau)) = 0 \quad \text{since} \quad i^*(\tau) = 0.
\]

\[
(tr \lambda)^2 = tr(\pi^*(tr \lambda) \lambda) = 2tr \lambda^2 = 0 \quad \text{since} \quad \lambda^2 = 0.
\]

Similarly the others can be shown, so we omit the proof of them. Thus the theorem follows.

Finally we show how we can get the explicit description of \( \eta_1 \beta(\lambda^3(\lambda^4 \rho)) \) appeared in the second relation of Theorem 3.3. Analogously to the case of \( KO^*(\text{Spin}(n)) \), also in the present case it suffices to check \( \eta_1 \beta(\lambda^4 \rho) \) and \( \eta_1 \beta(\lambda^4 \rho) \). We now prove the following

\[
\eta_1 \beta(\lambda^4 \rho) = 0 \quad \text{in } KO^*(SO(n)) \text{ or } KO^*(\text{Spin}(n))
\]

and

\[
\eta_1(\beta(\lambda^3 \rho) + \beta(\lambda^4 \rho) + \cdots) = \eta_1 \tau + \eta_1^2 tr \lambda \quad \text{in } KO^*(SO(n)) \text{ or } KO^*(\text{Spin}(n))
\]

according as \( \rho \) is viewed as a representation of \( SO(n) \) or \( \text{Spin}(n) \).

As shown in [10] we have

\[
\beta(\lambda^4 \rho) = 2^s \theta \kappa - \beta(\lambda^3 \rho) - \cdots - \beta(\lambda \rho) \quad \text{in } KO^*(SO(n+1)),
\]

\[
\beta(\lambda^4 \rho) = 2^{s+1} \theta \kappa - \beta(\lambda^3 \rho) - \cdots - \beta(\lambda \rho) \quad \text{in } KO^*(\text{Spin}(n+1)).
\]

Here \( \theta \) is as in (2.5), \( \kappa = \kappa_{n+1} \) or \( \beta(\xi_{n+1}) \) and \( \kappa = \kappa_{n+1} \) or \( \beta(\Delta_{n+1}) \) as in [10] according as \( n \equiv 2 \) or 6 mod 8 and \( \rho \) denotes also the \( (n+1) \)-dimensional stan-
standard representations of $SO(n+1)$ and Spin($n+1$). So it follows that in either case
\[ \eta_i(\beta(\lambda^{*+1}p)+\beta(\lambda^{*}p)+\cdots+\beta(\lambda^{1}p)) = 0. \]
By restricting this to $SO(n)$ or Spin($n$) according as we consider $p$ as a representation of $SO(n+1)$ or Spin($n+1$) we get readily
\[ \eta_i\beta(\lambda^{*+1}p) = 0. \]

By Proposition 1.4 $\eta_i^2 \lambda = \lambda^2 = \beta(r(\Delta^+))^2$ and so from the square formula of [5] it follows that
\[ \eta_i^2 \lambda = \eta_i\beta(\lambda^2(r(\Delta^+))). \]
Considering the character of $\Delta^+$ on a maximal torus of Spin($n$) ([8], §13, Prop. 9.4) we see that
\[ \lambda^2(r(\Delta^+)) = (\lambda^0p+\lambda^{*2}p+\cdots)+2s(\lambda^{*3}p+\lambda^{*4}p+\cdots) \]
for some integer $s$. Hence we have
\[ \eta_i^2 \lambda = \eta_i(\beta(\lambda^0p)+\beta(\lambda^{*2}p)+\cdots) \text{ in } KO^*(\text{Spin}(n)). \]

To show the remaining case we recall the equality $\Delta^+ \otimes c \Delta^- = c(\lambda^0p+\lambda^{*2}p+\cdots)$ from [8]. This gives $c((\beta(\lambda^0p)+\beta(\gamma^{*2}p)+\cdots)-2^s \lambda_0)=0$. Therefore we may put
\[ \beta(\lambda^0p)+\beta(\gamma^{*2}p)+\cdots = 2^s \lambda_0+\eta_i(P+\lambda Q)+\eta_i(P'+\lambda Q') \]
where $P, P', Q$ and $Q'$ are polynomials in $\beta(\lambda^1p), \cdots, \beta(\lambda^{*1}p)$ as in (2.3). Since, by [10], $\beta(\lambda^0p)+\beta(\lambda^{*1}p)+\cdots=2^s \theta \mathfrak{R}$ in $KO^*(\text{Spin}(n-1))$, comparing this equality with the restriction of the above to Spin($n-1$) yields $P=P'=0$ and so the previous result implies $Q=1$. Hence
\[ \beta(\lambda^0p)+\beta(\lambda^{*2}p)+\cdots = 2^s \lambda_0+\eta_i \lambda+\eta_i^2 \lambda Q' \text{ in } KO^*(\text{Spin}(n)). \]
Also we have
\[ c((\beta(\lambda^0p)+\beta(\lambda^{*2}p)+\cdots)-2^{s-1} \epsilon_0-\tau) = 0 \text{ in } KO^*(SO(n)). \]
So we can set
\[ \beta(\lambda^0p)+\beta(\lambda^{*2}p)+\cdots = 2^{s-1} \epsilon_0+\tau+\eta_1 x \]
for some $x \in KO^*(SO(n))$. Apply $\pi^*$ to both sides of (b) and compare this with (a), then we have
\[ \pi^*(x) = \eta_1 \lambda Q'. \]
On the other hand, applying $I$ to both sides of (b) again and using (a) yield
$I(\eta_1 x + \eta_1 t^r \lambda + \eta_1 (t^r + 1) t^Q') = \eta_1 \beta$ where $\beta$ is as in (3.2). Since $\eta_1^2 \beta = 0$ and $\text{Ker } I = (\eta_1 (t^r + 1) t^r)$ by Theorem 3.4, it follows that $I(\eta_1^2 (x + t^r \lambda)) = 0$, so that we can set
\[ \eta_1^2 (x + t^r \lambda + (t^r + 1) t^r R) = 0 \]
for some polynomial $R$ in $\beta(t^r \lambda), \ldots, \beta(t^{r-1} \lambda)$ as above. By observing the relations of Theorem 3.4 we therefore see that $x + t^r \lambda + (t^r + 1) t^r R$ is described in terms of $\xi_0, \xi_2, \nu_1$ and $\nu_3$ and so $\eta_1 \lambda^Q' + 2 \lambda + \eta_1 \lambda R$ in terms of $\lambda_i (i = 0, 1, 2, 3)$ because of $\pi^*(x) = \eta_1 \lambda^Q, \pi^*(t^r \lambda) = 2 \lambda, \pi^*(\nu_1) = \eta_1 \lambda, \pi^*(\xi_0) = 2 \lambda_0, \pi^*(\xi_2) = 2 \lambda_2, \pi^*(\nu_3) = -\lambda_{r+1}$ and $\pi^*(\nu_3) = -\lambda_{r+1}$. Hence, from the relations of Proposition 1.4 we infer that $Q'$ and $R$ are divisible by $\eta_1$. This implies $\eta_1^2 x = \eta_1^2 t^r \lambda$. Thus by (b) we have
\[ \eta_1 (\beta(t^r \lambda) + \beta(t^{r-2} \lambda) + \cdots) = \eta_1 \tau + \eta_1^2 t^r \lambda \text{ in } KO^*(SO(n)). \]

References

[10] ———: The real K-groups of SO(n) for n≡3, 4 and 5 mod 8, Osaka J. Math. 25 (1988), 185–211.

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