Minami, H.
Osaka J. Math.
26 (1989), 299–318

THE REAL K-GROUPS OF SO(n) FOR n≡2 MOD 4

Dedicated to Professer Shôrô Araki on his sixtieth birthday

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(Received May 6, 1988)

In [9], [10] we studied the algebra $KO^*(SO(n))$ for $n\equiv 0$, 1, 3 mod 4 using an idea of [7]. We first showed that a map from $P^{n-1}\times \mathrm{Spin}(n)$ to $SO(n)$ introduced in [7] to compute $K^*(SO(n))$ also induces a monomorphism in KO-theory

$$I\colon KO^*(SO(n)) \to KO^*(P^{n-1}\times \mathrm{Spin}(n)).$$

As in [7] using this embedding enabled us to compute $KO^*(SO(n))$ from $KO^*(P^{n-1}\times \mathrm{Spin}(n))$ whose structure can be obtained from the results of [1], [6], [12], [11].

The purpose of this note is to consider the remaining case, that is, $KO^*(SO(n))$ for $n\equiv 2$ mod 4. However, in the present case, the analogous homomorphism I is not a monomorphism. This must come from the fact that the simple spin representations of $\mathrm{Spin}(n)$ are neither real nor quaternionic representations. To determine the kernel and image of I so we make use of our results on the algebra structure of $KO^*(SO(n))$ for $n\equiv 1$ mod 4.

1. $KO^*(P^{n-1}\times \mathrm{Spin}(n))$

Throughout this note we regard KO and K as Z_8-graded cohomology functors using the Bott periodicity. Let $\eta_1\in KO^{-1}(+)$ and $\eta_4\in KO^{-4}(+)$ be generators of $KO^*(+)$ satisfying the relations $2\eta_1=\eta_1^3=\eta_1\eta_4=0$, $\eta_4^2=4$ and $\mu\in K^{-2}(+)$ denote the Bott class satisfying the relation $\mu^4=1$ ($+$=point).

Let c and r denote the complexification and realification homomorphisms. According to [3] we then have a useful exact sequence

$$(1.1)\qquad \cdots\to KO^{1-q}(X)\overset{\chi}{\to} KO^{-q}(X)\overset{c}{\to} K^{-q}(X)\overset{\delta}{\to} KO^{2-q}(X)\to\cdots$$

which connects KO with K where χ is multiplication by η_1 and δ is given by $\delta(\mu x)=r(x)$ for $x\in K^{2-q}(X)$.

We also assume that

$$n\equiv 2 \bmod 4 \quad\text{and}\quad a=\frac{n-2}{2}$$

throughout this note.

To determine $KO^*(P^{n-1}\times \mathrm{Spin}(n))$ we first deal with $KO^*(P^{n-1})$ where P^{n-1} is the real projective $(n-1)$-space. For the additive structure of $KO^*(P^l)$ needed below we refer to [6]. Referring also to [4] for the structure of $K^*(P^{n-1})$ and using (1.1) we can find elements $\tilde{\nu}_1\in KO^{-3}(P^{n-1})$ and $\tilde{\nu}_3\in KO^{-7}(P^{n-1})$ such that

$$(1.2)\qquad c(\tilde{\nu}_1)=\mu\nu \quad\text{and}\quad c(\tilde{\nu}_3)=\mu^3\nu$$

and we can readily show that $KO^*(P^{n-1})$ is generated by $\gamma=\gamma'-1$, $\tilde{\nu}_1$ and $\tilde{\nu}_3$ as follows. Here ν denotes the generator ν_{n-1} of $K^{-1}(P^{n-1})$ as in [9], Proposition 2.1 and γ' the canonical non-trivial real line bundle over P^{n-1}.

Proposition 1.3.
$$\begin{aligned}
\widetilde{KO}^0(P^{n-1}) &= Z_{2^{a+1}}\cdot\gamma\,,\\
\widetilde{KO}^{-1}(P^{n-1}) &= Z_2\cdot\eta_1\gamma\,,\\
\widetilde{KO}^{-2}(P^{n-1}) &= Z_2\cdot\eta_1^2\gamma\,,\\
\widetilde{KO}^{-3}(P^{n-1}) &= Z\cdot\tilde{\nu}_1\\
\widetilde{KO}^{-4}(P^{n-1}) &= Z_{2^a}\cdot\eta_4\gamma\,,\\
\widetilde{KO}^{-5}(P^{n-1}) &= \widetilde{KO}^{-6}(P^{n-1})=0\,,\\
\widetilde{KO}^{-7}(P^{n-1}) &= Z\cdot\tilde{\nu}_3
\end{aligned}$$

with the relations

$$\gamma^2=-2\gamma,\ \gamma\tilde{\nu}_1=\gamma\tilde{\nu}_3=\tilde{\nu}_1^2=\tilde{\nu}_3^2=\tilde{\nu}_1\tilde{\nu}_3=0,\ \eta_1\tilde{\nu}_1=2^{a-1}\eta_4\gamma\,,$$
$$\eta_1\tilde{\nu}_3=2^a\gamma,\ \eta_4\tilde{\nu}_1=2\tilde{\nu}_3,\ \eta_4\tilde{\nu}_3=2\tilde{\nu}_1\,.$$

Let Δ^+ and Δ^- be the even and odd half-spin representations of $\mathrm{Spin}(n)$. According to [8], §13 these are neither real nor quaternionic and can be viewed as continuous homomorphisms

$$\Delta^+,\ \Delta^-\colon \mathrm{Spin}(n)\to GL(2^a, \boldsymbol{C})$$

These maps give rise to the elements of $K^{-1}(\mathrm{Spin}(n))$, denoted by $\beta(\Delta^+)$ and $\beta(\Delta^-)$ as usual, in a canonical manner.

Since each of Δ^+ and Δ^- is complex conjugate to the other, so that $\beta(\Delta^-)=\beta(\Delta^+)^*$, by [11], Proposition 4.6 we have an element $\lambda\in KO(\mathrm{Spin}(n))$ such that

$$c(\lambda)=\mu^3\beta(\Delta^+)\,\beta(\Delta^-)\,.$$

Here $*$ is the operation on $K^*(X)$ induced by the assignment which sends a complex vector bundle to its complex conjugate bundle.

Set

$$\lambda_i=r(\mu^i\beta(\Delta^+))\quad\text{in}\quad KO^{-2i-1}(\mathrm{Spin}(n))$$

where i is reduced mod 4. Note that using (1.1) when $X=\mathrm{Spin}(n)$ gives

$$r(\mu^i\beta(\Delta^-)) = (-1)^i\lambda_i$$

because $\mu^* = -\mu$ and $cr = 1+*$.

Let ρ: $SO(n) \subset GL(n, \boldsymbol{R})$ be the evident inclusion and let us denote by the same letter ρ the composite of this with the covering map π: Spin$(n) \to SO(n)$. Then we obtain the elements

$$\beta(\lambda^i\rho)\ (1 \leqq i \leqq n) \quad \text{in} \quad KO^{-1}(\mathrm{Spin}(n))$$

in a similar way where $\lambda^i\rho$ denotes the i-th exterior power of ρ. Using these elements, by [13], Theorem 5.6 we have

Proposition 1.4. *$KO^*(\mathrm{Spin}(n))$ is generated by λ, λ_1, λ_2, λ_3 and $\beta(\lambda^k\rho)$ $(1 \leqq k \leqq a-1)$ as a $KO^*(+)$-algebra and there hold the relations*

$$\begin{aligned}
&\lambda^2 = \lambda\lambda_i = \eta_1\lambda_i = 0,\ \eta_4\lambda_{i+2} = 2\lambda_i\,,\\
&\lambda_i\lambda_j = \eta_1^2\lambda \qquad\quad\ \textit{if} \quad i+j \equiv 0 \bmod 4\,,\\
&\qquad\ = (-1)^j\eta_4\lambda \quad \textit{if} \quad i+j \equiv 1 \bmod 4\,,\\
&\qquad\ = 0 \qquad\qquad\ \ \textit{if} \quad i+j \equiv 2 \bmod 4\,,\\
&\qquad\ = (-1)^j 2\lambda \quad\ \ \textit{if} \quad i+j \equiv 3 \bmod 4\,,\\
&\beta(\lambda^k\rho)^2 = \eta_1(\beta(\lambda^2(\lambda^k\rho)) + \tbinom{n}{k}\beta(\lambda^k\rho))\,.
\end{aligned}$$

The last relation in the above proposition is due to [5], §6 and the others can be found in [11]. In proving the relations η_4 is assumed to be chosen so that $r(\mu^2) = \eta_4$ and also hereafter is done so. To complete the last relation we must give the explicit form of $\beta(\lambda^2(\lambda^k\rho))$. But we only show how this can be described in terms of the given generators. It is clear that this can be expressed as a polynomial in $\beta(\lambda^1\rho), \cdots, \beta(\lambda^a\rho)$ and $\beta(\lambda^{a+l}\rho) = \beta(\lambda^{n-a-l}\rho)$ for $2 \leqq l \leqq a+2$. Hence it suffices to check $\eta_1\beta(\lambda^a\rho)$ and $\eta_1\beta(\lambda^{a+1}\rho)$. We have

$$\eta_1(\beta(\lambda^a\rho) + \beta(\lambda^{a-2}\rho) + \cdots) = \eta_1^2\lambda \quad \text{and} \quad \eta_1\beta(\lambda^{a-1}\rho) = 0$$

which are proved in the last section.

For our calculation we need a result of [2] further. Let $e_i = (0, \cdots, 1, \cdots, 0)$ with 1 in the i-th position and let us consider $e_1, \cdots, e_n$ as multiplicative generators of the Clifford algebra C_n satisfying the relations $e_i^2 = -1$, $e_ie_j + e_je_i = 0$ $(i \neq j)$. Let S^{n-1} be the unit sphere in $\boldsymbol{R}^n \subset C_n$. Then we set

$$\begin{aligned}
S_+ &= S^{n-1} \cap \{(x_1, \cdots, x_n);\ x_n \geqq 0\}\,,\\
S_- &= S^{n-1} \cap \{(x_1, \cdots, x_n);\ x_n \leqq 0\}\,,\\
S^{n-2} &= S_+ \cap S_-\,.
\end{aligned}$$

We view S^{n-1} as the orbit space of e_n for Spin$(n) \subset C_n$ acting on $\boldsymbol{R}^n$ through π

and Spin$(n-1)$ as the isotropy subgroup at e_n. Thus Spin(n)/Spin$(n-1)=S^{n-1}$ and so we have the principal Spin$(n-1)$-bundle

$$\phi\colon \mathrm{Spin}(n) \to S^{n-1}\,.$$

Let $G=\{\pm 1\}$ be the multiplicative subgroup of Spin$(n-1)$ and let us view as $SO(n)=\mathrm{Spin}(n)/G$ and $SO(n-1)=\mathrm{Spin}(n-1)/G$. Analogously we then have the principal $SO(n-1)$-bundle

$$\phi\colon SO(n) \to S^{n-1}\,.$$

We parametrize S_+ and S_- by use of polar coordinates as follows.

$$(x, t) = \cos t\cdot e_n + \sin t\cdot x \quad \text{and} \quad (x, t) = -\cos t\cdot e_n + \sin t\cdot x$$

for $x\in S^{n-1}$ and $0\leqq t\leqq \pi/2$. Define maps

$$j_1\colon S_+\times \mathrm{Spin}(n-1) \to \phi^{-1}(S_+)\,,$$
$$j_2\colon S_-\times \mathrm{Spin}(n-1) \to \phi^{-1}(S_-)$$

by

$$j_1(x, t, g) = (-\cos t/2 + \sin t/2\cdot xe_n)\, g\,,$$
$$j_2(x, t, e_1\, xg) = (\cos t/2\cdot xe_n - \sin t/2)\, g\,.$$

Then it is clear that these maps become Spin$(n-1)$-bundle isomorphisms. Since j_1 and j_2 are compatible with the action of G these maps induces also $SO(n-1)$-bundle isomorphisms

$$j_1\colon S_+\times SO(n-1) \to \phi^{-1}(S_+)\,,$$
$$j_2\colon S_-\times SO(n-1) \to \phi^{-1}(S_-)\,.$$

Therefore we get

Lemma 1.5 ([2], Proposition 13.2). *Let $G(l)=\mathrm{Spin}(l)$ or $SO(l)$ for $l=n-1, n$. Then the principal $G(n-1)$-bundle $\phi\colon G(n)\to S^{n-1}$ is isomorphic to the bundle obtained from the two product bundles*

$$S_+\times G(n-1) \to S_+,\; S_-\times G(n-1) \to S_-$$

by the identification

$$(x, g) \leftrightarrow (x, e_1\, xg) \quad or \quad (x, \pi(g)) \leftrightarrow (x, \pi(e_1\, xg))$$

for $x\in S^{n-2}$, $g\in \mathrm{Spin}(n-1)$ according as $G(l)=\mathrm{Spin}(l)$ or $SO(l)$.

Denote the map which gives the identification in the above lemma by

$$d\colon S^{n-2}\times G(n-1) \to S^{n-2}\times G(n-1)\,.$$

Namely d is given by

$$d(x,g)=(x,e_1xg) \quad \text{or} \quad d(x,\pi(g))=(x,\pi(e_1xg))$$

for $x\in S^{n-2}$, $g\in \mathrm{Spin}(n-1)$ according as $G(l)=\mathrm{Spin}(l)$ or $SO(l)$. We consider the Mayer-Vietoris exact sequence of $(G(n), \phi^{-1}(S_+), \phi^{-1}(S_-))$ in KO(or K)-theory. Then by using Lemma 1.5 we obtain the following exact sequence

$$(1.6)\qquad \cdots\to h^*(X\times S^{n-2}\times G(n-1))\overset{\delta}{\to} h^*(X\times G(n))\overset{\varphi}{\to}$$
$$h^*(X\times G(n-1))\oplus h^*(X\times G(n-1))\overset{\psi}{\to} h^*(X\times S^{n-2}\times G(n-1))\to\cdots$$

for $h=KO, K$. Here

$$\varphi=((1\times i)^*, (1\times i)^*),\quad \psi=(1\times p)^*-(1\times pd)^*$$

where i: $G(n-1)\subset G(n)$ is the inclusion above and p: $S^{n-2}\times G(n-1)\to G(n-1)$ the obvious projection. Note that there holds the relation

$$\delta(x(1\times ip)^*(y))=\delta(x)y$$

for $x\in h^*(X\times S^{n-2}\times G(n-1))$, $y\in h^*(X\times G(n))$.

Let us denote by ρ also the composite ρi and by Δ the simple spin-representation of $\mathrm{Spin}(n-1)$ which is real or quaternionic according as $n\equiv 2$ or 6 mod 8 ([8], §13). From [11], Theorem 5.6 (also see [9], Prop. 2.4 and [10], Prop. 3.5) again it follows that

$$KO^*(\mathrm{Spin}(n-1))=\wedge_{KO^*(+)}(\beta(\lambda^1\rho),\cdots,\beta(\lambda^{a-1}\rho)),\tilde{\kappa})$$

as a $KO^*(+)$-module. Here $\tilde{\kappa}=\beta(\Delta)$ or $\tilde{\kappa}_{n-1}$ as in [10] according as $n\equiv 2$ or 6 mod 8 so that

$$c(\tilde{\kappa})=\mu^a c(\beta(\Delta))$$

where we denote by c two kinds of the complexification homomorphisms $KO(X)\to K(X)$ and $KH(X)\to K(X)$.

We now consider behavior of δ, φ and ψ in (1.6) when X=point, $G(l)=\mathrm{Spin}(l)$ $(l=n-1, n)$ and $h=KO$. Clearly

$$\varphi(\beta(\lambda^i\rho))=(\beta(\lambda^i\rho)+\beta(\lambda^{i-1}\rho), \beta(\lambda^i\rho)+\beta(\lambda^{i-1}\rho))\quad (1\leqq i\leqq a-1)$$

and since $i^*(\Delta^+)=c(\Delta)$ it is easy to see that

$$\varphi(\lambda_j)=2\tilde{\kappa},\ \eta_1^2\tilde{\kappa},\ \eta_4\tilde{\kappa}\quad \text{or}\quad 0$$

according as $j\equiv 0$, 1, 2 or 3 mod 4.

We have a commutative diagram with δ as in (1.6) when $h=K$

$$\begin{array}{ccc} \tilde{K}^{2-n}(S^{n-2}\times \mathrm{Spin}(n-1)) & \xrightarrow{\delta} & \tilde{K}^{3-n}(\mathrm{Spin}(n)) \\ q^* \uparrow & & \uparrow \phi^* \\ \tilde{K}^{2-n}(S^{n-2}) & \xrightarrow{\delta} & \tilde{K}^{3-n}(S^{n-1}) \end{array}$$

where the lower δ is an isomorphism and q denotes the evident projection. Choose a generator $t\in\widetilde{KO}^{2-n}(S^{n-2})\cong Z$ so that

$$\mu^{a+1}\,\delta c(t) = \beta(\bar{\delta})\in\tilde{K}^{3-2n}(S^{n-1})\cong Z$$

which is a generator of $\tilde{K}^{3-2n}(S^{n-1})$, where $\bar{\delta}\colon S^{n-1}\to GL(2^a, \boldsymbol{C})$ is a map defined by $\bar{\delta}(\phi(g))=\Delta^+(g)\,\Delta^-(g)^{-1}$ for $g\in\mathrm{Spin}(n)$. Then the commutativity of the diagram above yields

$$\delta(c(t)\times 1) = \mu^{-a-1}(\beta(\Delta^+)-\beta(\Delta^-))\,.$$

Hence we have

$$c\delta(t\times\tilde{\kappa}) = \mu^3\beta(\Delta^+)\,\beta(\Delta^-)$$

because of $i^*(\beta(\Delta^+))=\beta(\Delta)$. So we may take

$$\lambda = \delta(t\times\tilde{\kappa}) \quad \text{so that} \quad \varphi(\lambda) = 0\,.$$

By observing $(pd)^*(\beta(\Delta))$ we can check that $(pd)^*(\tilde{\kappa})$ takes the form of

$$(pd)^*(\tilde{\kappa}) = 1\times\tilde{\kappa}+x\times 1 \quad \text{for} \quad x\in\widetilde{KO}^{1-n}(S^{n-2}) = Z_2\cdot\eta_1\,t\,.$$

Then $\psi(\tilde{\kappa},\tilde{\kappa})=x\times 1$. Hence if $x=0$, there is an element $y\in KO^*(\mathrm{Spin}(n))$ such that $\varphi(y)=(\tilde{\kappa},\tilde{\kappa})$, that is, $i^*(y)=\tilde{\kappa}$. Using this we have $\lambda=\delta(t\times 1)y$ and so applying c to both sides of this we get $\mu^3\beta(\Delta^+)\,\beta(\Delta^-)=\mu^{-a-1}(\beta(\Delta^+)-\beta(\Delta^-))\,c(y)$. This implies that $c(y)=\mu^{a+4}\beta(\Delta^+)$ or $\mu^{a+4}\beta(\Delta^-)$, because $K^*(\mathrm{Spin}(n))$ is the exterior algebra over $K^*(+)$ generated by $\beta(\lambda^1\rho), \cdots, \beta(\lambda^{a-1}\rho), \beta(\Delta^+), \beta(\Delta^-)$. By exactness of (1.1) when $X=\mathrm{Spin}(n)$ we hence have $\lambda_{a+3}=0$. This is a contradiction because $\lambda_{a+3}\neq 0$ by Proposition 1.4. Therefore $x\neq 0$, that is, $x=\eta_1 t$ and so we have

$$(pd)^*(\tilde{\kappa}) = 1\times\tilde{\kappa}+\eta_1\,t\times 1\,.$$

Consequently we have

$$\psi(\tilde{\kappa}, 0) = 1\times\tilde{\kappa}\,, \quad \psi(0, \tilde{\kappa}) = -1\times\tilde{\kappa}+\eta_1\,t\times 1\,.$$

Since $\pi^*\colon \widetilde{KO}^{-1}(P^{n-2})\to\widetilde{KO}^{-1}(S^{n-2})$ is a zero map it is clear that

$$\psi(\beta(\lambda^i\rho), 0) = -\psi(0, \beta(\lambda^i\rho)) = \beta(\lambda^i\rho) \quad (1\leqq i\leqq a-1)\,.$$

Finally we consider $\delta(t\times 1)$. As shown above $c\delta(t\times 1)=\mu^{-a-1}(\beta(\Delta^+)-$

$\beta(\Delta^-)$) which means $c(\delta(t\times 1)-\lambda_{-a-1})=0$ since a is even. Using the exactness of (1.1) when $X=\mathrm{Spin}(n)$ we have an element $x\in KO^*(\mathrm{Spin}(n))$ such that $\eta_1 x=\delta(t\times 1)-\lambda_{-a-1}$. Hence $\eta_1^2 x=\delta(\eta_1 t\times 1)=\delta\psi(\tilde{\kappa},\tilde{\kappa})=0$. So by observing the structure of $KO^*(\mathrm{Spin}(n))$ we see that x must be zero. This implies

$$\delta(t\times 1)=\lambda_{-a-1}.$$

From these facts we obtain

Lemma 1.7.

$$KO^*(P^{n-1}\times \mathrm{Spin}(n))=(KO^*(P^{n-1})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(n)))/\mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by

$$\tilde{\nu}_1\otimes\lambda_0-\tilde{\nu}_3\otimes\lambda_2\,,\quad \tilde{\nu}_1\otimes\lambda_2-\tilde{\nu}_3\otimes\lambda_0\,,$$
$$\tilde{\nu}_1\otimes\lambda_1-\tilde{\nu}_3\otimes\lambda_3\,,\quad \tilde{\nu}_1\otimes\lambda_3-\tilde{\nu}_3\otimes\lambda_1\,.$$

Proof. Consider (1.6) when $X=P^{n-1}$, $G(l)=\mathrm{Spin}(l)$ ($l=n-1$, n) and $h=KO$. Since $KO^*(\mathrm{Spin}(n-1))$ is $KO^*(+)$-free as mentioned above, we have a canonical isomorphism

$$KO^*(X\times \mathrm{Spin}(n-1))\cong KO^*(X)\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(n-1))$$

for any finite CW-complex X. Applying this fact to (1.6) in the present case we can easily get the lemma from the above results on φ, ψ and δ. Now the relations can be shown as follows. For example,

$$\begin{aligned}\tilde{\nu}_1\times\lambda_0 &= r(c(\tilde{\nu}_1\times 1)(1\times\beta(\Delta^+))\\ &= r(\mu\nu\times\beta(\Delta^+))\\ &= r(\mu^3\nu\times\mu^2\beta(\Delta^+))\\ &= r(c(\tilde{\nu}_2\times 1)(1\times\mu^2\beta(\Delta^+))\\ &= \tilde{\nu}_3\times\lambda_2\,.\end{aligned}$$

The others are analogous.

2. The module structure of $KO^*(SO(n))$

Let ξ' be the canonical non-trivial real line bundle over $SO(n)$ and set

$$\xi=\xi'-1 \quad\text{in}\quad KO(SO(n))\,.$$

Define maps

$$\delta,\ \varepsilon\colon SO(n)\to GL(2^a, \boldsymbol{C})$$

by $\delta(\pi(g))=\Delta^-(g)^{-1}\Delta^+(g)$, $\varepsilon(\pi(g))=\Delta^+(g)^2$ for $g\in\mathrm{Spin}(n)$. Then we have the elements $\beta(\varepsilon)$, $\beta(\delta)$ of $K^{-1}(SO(n))$. So we set

$$\varepsilon_i = r(\mu^i\beta(\varepsilon)),\ \delta_i = r(\mu^i\beta(\delta)) \quad \text{in} \quad KO^{-2i-1}(SO(n))$$

where i is of course reduced mod 4. Clearly there hold the relations

$$\eta_4\varepsilon_i = 2\varepsilon_{i+2}\,, \quad \eta_4\delta_i = 2\delta_{i+2}\,.$$

For the standard representation ρ of $SO(n)$ as in §1 we also have the elements

$$\beta(\lambda^j\rho)\ (1\leqq j\leqq n) \quad \text{in} \quad KO^{-1}(SO(n))\,.$$

Let $G=\{\pm 1\}$ act on Spin(n) as a subgroup of Spin(n) and let $R^{p,q}$ be the $\boldsymbol{R}^{p+q}$ with a G-action such that -1 reverses the first p coordinates and fixes the last q. Let $S^{p,q}$ and $B^{p,q}$ be the unit sphere and ball in $R^{p,q}$ and $\Sigma^{p,q}=B^{p,q}/S^{p,q}$ with the collapsed $S^{p,q}$ as base point.

By [7] we have a homeomorphism

$$S^{n,0}\times_G \mathrm{Spin}(n) \to P^{n-1}\times \mathrm{Spin}(n)$$

which is induced by the assignment

$$(x, g) \mapsto (\pi(x), xe_1 g)$$

for $x\in S^{n,0}$, $g\in \mathrm{Spin}(n)$ where π: $S^{n,0}\to P^{n-1}$ denotes the canonical projection. Using this, from the exact sequence of $(B^{n,0}\times \mathrm{Spin}(n),\ S^{n,0}\times \mathrm{Spin}(n))$ in the equivariant KO(or K)-theory associated with G we have an exact sequence

$$\text{(2.1)} \qquad \cdots\to h^*(SO(n)) \xrightarrow{I} h^*(P^{n-1}\times \mathrm{Spin}(n)) \xrightarrow{\delta} \tilde{h}^*_G(\Sigma^{n,0}\wedge \mathrm{Spin}(n)_+) \xrightarrow{J} h^*(SO(n)) \to\cdots$$

for $h=KO$ or K. Here there holds the relation

$$\delta(xI(y)) = \delta(x)y$$

for $x\in h^*(P^{n-1}\times \mathrm{Spin}(n))$, $y\in h^*(SO(n))$.

In the case when $h=KO$ we have

$$\text{(2.2)} \qquad \begin{aligned}
&I(\xi) = \gamma\times 1\,,\\
&I(\beta(\lambda^i\rho)) = 1\times\beta(\lambda^i\rho)+\binom{n-2}{i-1}\eta_1\gamma\times 1 \quad (1\leqq i\leqq n)\,,\\
&I(\delta_0) = I(\delta_2) = 0\,,\\
&I(\delta_1) = 2(1\times\lambda_1-\tilde{\nu}_1\times 1)\,,\\
&I(\delta_3) = 2(1\times\lambda_3-\tilde{\nu}_3\times 1)\,,\\
&I(\varepsilon_0) = (\gamma+2)\times\lambda_0\,,\\
&I(\varepsilon_1) = (\gamma+2)\times\lambda_1-2\tilde{\nu}_1\times 1\,,\\
&I(\varepsilon_2) = (\gamma+2)\times\lambda_2\,,\\
&I(\varepsilon_3) = (\gamma+2)\times\lambda_3-2\tilde{\nu}_3\times 1\,.
\end{aligned}$$

The first equality is clear, the second one can be verified in the same way as in [10] and the others follows from [9], Lemma 3.3, iii), iv) immediately.

We consider the image of

$$J\colon \widetilde{KO}_G^*(\Sigma^{n,0}\wedge \mathrm{Spin}(n)_+) \to KO^*(SO(n)).$$

Let $\omega_s^+\in\widetilde{KO}_G(\Sigma^{8s,0})$, $\tau_s^+\in\tilde{K}_G(\Sigma^{2s,0})$ be the Bott elements mentioned in [9] such that $j^*(\omega_s^+)=2^{4s-1}(1-R^{1,0})$, $j^*(\tau_s^+)=2^{s-1}(1-R^{1,0}\otimes \boldsymbol{C})$ where j denotes the inclusions of $\Sigma^{0,0}$ in $\Sigma^{8s,0}$ and $\Sigma^{2s,0}$. Put $n=8k+2$ or $8k+6$. Clearly then any element of $\widetilde{KO}_G^*(\Sigma^{n,0}\wedge \mathrm{Spin}(n)_+)$ can be written in the form ω_k^+x where $x\in\widetilde{KO}_G^*(\Sigma^{2t,0}\wedge \mathrm{Spin}(n)_+)$ ($t=1$ or 3). Moreover if we put $c(x)=\tau_t^+ y$ for $y\in K^*(SO(n))$, then we obtain

(a) $$J(\omega_k^+x) = 2^{a-2}\,\xi r(yc(\xi)).$$

According to [9], Theorem 3.5

(b) $$K^*(SO(n))=\wedge_{K^*(+)}(c(\beta(\lambda^1\rho)),\cdots,c(\beta(\lambda^{a-1}\rho)),\beta(\varepsilon),\beta(\delta))\otimes_Z(Z\cdot 1\oplus Z_{2^a}\cdot c(\xi))$$

with the relations

$$c(\xi)^2=-2c(\xi),\ \beta(\varepsilon)\otimes c(\xi)=0.$$

If we set $\delta(1\times\lambda)=\omega_k^+x$, then we have

$$c(\omega_k^+x)=\tau_{4k}^+\tau_t^+\,\mu^3 c(\xi+1)\,(\beta(\delta)-\beta(\varepsilon))$$

by using [9], Lemma 3.4, iv), because of $c(\lambda)=\mu^3\beta(\Delta^+)\,\beta(\Delta^-)$. Hence using the relation $c(\xi)\otimes\beta(\varepsilon)=0$ gives

(c) $$\begin{aligned}2^{a-1}\,\xi\delta_3 &= J\delta(1\times\lambda)\\ &=0.\end{aligned}$$

Since $\beta(\Delta^+)^*=\beta(\Delta^-)$ and $\nu^*=-\nu$ by definition of ν, we have $\beta(\delta)^*=-\beta(\delta)$ by [9], Lemma 3.3, iii). So, from exactness of (1.1) when $X=SO(n)$ it follows that

(d) $$\begin{aligned}2\xi\delta_{2i} &= r(\mu^{2i}c(\xi)\cdot 2\beta(\delta))\\ &=\delta(\mu^{2i+1}c(\xi)\,(\beta(\delta)-\beta(\delta)^*))\\ &=\delta c(r(\mu^{2i+1}c(\xi)\,\beta(\delta)))\\ &=0\end{aligned}$$

for $i=0$, 1.

Calculate the right-hand side of (a) making use of (b), (c) and (d). Then we see that $J(\omega_k^+x)$ can be written as

$$J(\omega_k^+ x) = 2^a \xi P_1 + 2^{a-1} \eta_4 \xi P_2 + 2^{a-1} \xi \delta_1 P_3$$

where P_i is a polynomial in $\beta(\lambda^1\rho), \cdots, \beta(\lambda^{a-1}\rho)$ with integers as coefficients for $i=1, 2, 3$. So apply I to both sides of such an expression of $J(\omega_k^+ x)$ and estimate this by using (2.2). Since $IJ=0$ it then follows from Lemma 1.7 that the first two terms of $J(\omega_k^+ x)$ are zero. Thus we have

(2.3) Im J *is generated by elements of the form* $2^{a-1}\xi\delta_1 P$ *where* P *is a polynomial in* $\beta(\lambda^1\rho), \cdots, \beta(\lambda^{a-1}\rho)$ *with integers as coefficients, and* η_4 Im $J=0$.

We now obseve the exact sequence

$$(2.4) \quad \cdots \to KO^*(S^{n-2}\times SO(n-1)) \overset{\delta}{\to} KO^*(SO(n)) \overset{\varphi}{\to}$$
$$KO^*(SO(n-1))\oplus KO^*(SO(n-1)) \overset{\psi}{\to} KO^*(S^{n-2}\times SO(n-1)) \to \cdots$$

which follows from (1.6).

Denote by ξ also the restriction $i^*(\xi)$ to $SO(n-1)$ and by ρ the composite ρi as before. By [9] and [10] we then have

(2.5) *As a* $KO^*(+)$*-module,* $KO^*(SO(n-1))$ *is generated by the elements in the form* P, ξP, κP *and* vP *where* κ *denotes* $\beta(\varepsilon_{n-1})$ *or* κ_{n-1} *of* $KO^{1-n}(SO(n-1))$ *and* v *denotes* v_{n-1} *or* ν_{n-1} *of* $KO^{-n}(SO(n-1))$ *as in* [9], [10] *according as* $n\equiv 2$ *or* 6 mod 8 *and* P *denotes a polynomial in* $\beta(\lambda^1\rho), \cdots, \beta(\lambda^{a-1}\rho)$. *Also there hold the relations*

$$\kappa^2 = v^2 = \xi\kappa = \eta_4 v = 2v = 0,\ \kappa v = \eta_1^2\xi\beta(\lambda^2\Delta)\,,$$
$$\eta_1\kappa = \xi v,\ \eta_1^2 v = 2^{a-1}\theta\eta,\ 2^{a-2}\theta\eta_4\xi = 0$$

where $\theta=\eta_4$ *or* 2 *according as* $n\equiv 2$ *or* 6 mod 8.

Let tr: $h^*(\mathrm{Spin}(n-1))\to h^*(SO(n-1))$ be the transfer where $h=KO$ or K. Then observation of the definitions of $\tilde{\kappa}$ and κ ([9], [10]) gives

$$tr(\tilde{\kappa}) = \kappa$$

because of $tr(\beta(\Delta))=\beta(\varepsilon)$ and

$$tr(1) = \xi+2\,.$$

Therefore we have from the formula on $\tilde{\kappa}$ given in §1

$$(2.6) \qquad \psi(\kappa, 0) = 1\times\kappa,\ \psi(0, \kappa) = -1\times\kappa+\eta_1 t\times\xi\,.$$

We now show that

$$(2.7) \qquad \psi(v, 0) = 1\times v,\ \psi(0, v) = 1\times v+\eta_1^2 t\times(l\xi+1) \quad (l = 0, 1)\,.$$

The first equality is clear. To prove the second one we define maps

$$\begin{aligned}
&m\colon\ S^{n-2}\times SO(n-1)\to SO(n-1)\,,\\
&m'\colon\ S^{n-1}\times \mathrm{Spin}(n-1)\to \mathrm{Spin}(n-1)\,,\\
&m_0\colon\ P^{n-2}\times \mathrm{Spin}(n-1)\to SO(n-1)\,,\\
&m_1\colon\ S^{n-2}\times P^{n-2}\times \mathrm{Spin}(n-1)\to P^{n-2}\times \mathrm{Spin}(n-1)\,,\\
&m_2\colon\ S^{n-2}\times P^{n-2}\to P^{n-2}\,,\\
&m_2'\colon\ S^{n-2}\times S^{n-2}\to S^{n-2}\,,\\
&m_3\colon\ \mathrm{Spin}(n-1)\times \mathrm{Spin}(n-1)\to \mathrm{Spin}(n-1)
\end{aligned}$$

by

$$\begin{aligned}
&m(x,\pi(g))=\pi(e_1xg),\ m'(x,g)=e_1xg\,,\quad m_0(\pi(x),g)=\pi(e_1xg)\,,\\
&m_1(x,\pi(y),g)=(m_2(x,\pi(y)),xe_1g)\,,\quad m_2(x,\pi(y))=\pi(xe_1ye_1x)\,,\\
&m_2'(x,y)=xe_1ye_1x\,,\quad m_3(g,g')=gg'g
\end{aligned}$$

for $x,y\in S^{n-2}$, $g,g'\in\mathrm{Spin}(n-1)$. Here by π we denote the obvious projection. Moreover we define embeddings

$$\bar{\iota}\colon S^{n-2}\to \mathrm{Spin}(n-1)\,,\quad \iota\colon P^{n-2}\to SO(n-1)$$

by $\bar{\iota}(x)=xe_1$, $\iota(\pi(x))=\pi(xe_1)$.

According to [9] and [10], m_0 yields a monomorphism

$$I\colon KO^*(SO(n-1))\to KO^*(P^{n-2}\times\mathrm{Spin}(n-1))$$

and by [9], (4.17) and [10], (4.20) we have

$$I(v)=1\times\eta_1\tilde{\kappa}+\tilde{\nu}\times 1$$

where $\tilde{\nu}$ denotes $\tilde{\nu}_{n-2}$ or μ_{n-2} of $KO^{-n}(P^{n-2})$ as in [9] or [10] according as $n\equiv 2$ or 6 mod 8. From this equality it follows readily that

$$\pi^*(v)=\eta_1\tilde{\kappa}\quad\text{and}\quad \iota^*(v)=\tilde{\nu}\,.$$

Let

$$\delta\colon KO^{-n}(P^{n-2})=KO_G^{-n}(S^{n-1,0})\to \widetilde{KO}_G^{1-n}(\Sigma^{n-1,0})$$

be the coboundary homomorphism appeared in the exact sequence of $(B^{n-1,0}, S^{n-1,0})$. Furthermore we then see that $\delta(\tilde{\nu})$ is a generator of $\widetilde{KO}_G^{1-n}(\Sigma^{n-1,0})\cong Z_2$ and the forgetful homomorphism $KO_G^{1-n}(\Sigma^{n-1,0})\to KO^{1-n}(S^{n-1})$ becomes an isomorphism. From these facts we obtain

(a) $$\pi^*(\tilde{\nu})=\eta_1^2\,t,\quad\text{so that}\quad \bar{\iota}^*(\eta_1\tilde{\kappa})=\eta_1^2\,t\,.$$

Since $m_1^*(\beta(\Delta))=2\beta(\Delta)\times 1+1\times\beta(\Delta)$ in KO or KH-theory, we have

(b) $$m_3^*(\eta_1\tilde{\kappa}) = 1\times\eta_1\tilde{\kappa}\,.$$

By (a), (b) we get

$$m_2'^*(\eta_1^2\, t) = 1\times\eta_1^2\, t\,.$$

So, using (a) again gives

$$(1\times\pi)^*m_2^*(\tilde{\nu}) = 1\times\eta_1^2\, t\,.$$

This and (a) imply

$$m_2^*(\tilde{\nu}) = 1\times\tilde{\nu}+t\times x \quad \text{for some} \quad x\in KO^{-2}(P^{n-2})\,.$$

Since degree $\tilde{\nu}=-n$ and degree $t=2-n$, we can infer from the structure of $KO^{-2}(P^{n-2})$ that

$$x = 0 \quad \text{or} \quad \eta_1^2\gamma$$

where γ denotes also the restriction $\iota^*(\gamma)$ to P^{n-2}. Therefore

$$m_2^*(\tilde{\nu}) = 1\times\tilde{\nu}+t\times l\eta_1^2\gamma \quad (l = 0, 1)\,,$$

so that

(c) $$m_1^*(\tilde{\nu}\times 1) = 1\times\tilde{\nu}\times 1+\eta_1^2\, t\times l\gamma\times 1 \quad (l = 0, 1)\,.$$

On the other hand, the argument parallel to that about $(pd)^*$ in §1 yields

$$m'^*(\tilde{\kappa}) = 1\times\tilde{\kappa}+\eta_1\, t\times 1\,.$$

Hence

$$m_1^*(1\times\tilde{\kappa}) = 1\times 1\times\tilde{\kappa}+\eta_1\, t\times 1\times 1\,.$$

From this and (c) it follows that

$$m_1^*\, I(\nu) = 1\times 1\times\eta_1\,\tilde{\kappa}+\eta_1^2\, t\times(l\gamma+1)\times 1+1\times\tilde{\nu}\times 1 \quad (l = 0, 1)$$

and so

$$(1\times m_0)^*m^*(\nu) = (1\times m_0)^*(1\times\nu+\eta_1^2\, t\times(l\xi+1)) \quad (l = 0, 1)\,.$$

Since $KO^*(SO(n-1))$ is $KO^*(+)$-free, we see from the injectivity of I that $(1\times m_0)^*$ is a monomorphism. Therefore

$$m^*(\nu) = 1\times\nu+\eta_1^2\, t\times(l\xi+1) \quad (l = 0, 1)\,,$$

which is the required result because $m=pd$. This completes the proof of (2.7).

Further, clearly we have

$$\varphi(\xi) = (\xi, \xi)\,,$$
$$\varphi(\beta(\lambda^i\rho)) = \beta(\lambda^i\rho)+\beta(\lambda^{i-1}\rho),\ \beta(\lambda^i\rho)+\beta(\lambda^{i-1}\rho)) \quad (1\leqq i\leqq n)\,.$$

Using (2.5), (2.6), (2.7) and these formulas, we obtain easily the following result concerning ψ and φ of (2.4)

(2.8) *As $KO^*(+)$-modules,* Coker ψ *is generated by elements of the form* $t\times P$, $t\times\xi P$, $t\times\kappa P$, $t\times vP$, $t\times\eta_1 P$, $t\times\eta_1\kappa P$, $t\times\eta_1 vP$, $t\times\eta_1^2\kappa P$, $t\times\eta_4 P$, $t\times\eta_4\xi P$, $t\times\eta_4\kappa P$ *and* $t\times\eta_4 vP$, *and* Im φ *by elements of the form* (P,P), $2(\kappa P,\kappa P)$, $\eta_1(vP,vP)$, η_1^2 $(\kappa P,\kappa P)$ *and* $\eta_4(\kappa P,\kappa P)$. *Here P denotes a polynomial as in* (2.5).

Now we add some generators for $KO^*(SO(n))$ to the ones given at beginning of this section. Since $\lambda=\delta(t\times\tilde{\kappa})$, we have

$$tr(\lambda)=\delta(t\times\kappa)\quad\text{in}\quad KO^{2-n}(SO(n))\,,$$

for which we write $tr\ \lambda$ simply.

By (2.7) and exactness of (2.4) there is an element $\nu_1\in KO^{-n-1}(SO(n))$ such that

$$\varphi(\nu_1)=\eta_1(v,v)\,.$$

But we need to choose such an element so that

$$I(\nu_1)=\tilde{\nu}_{a+1}\times 1-1\times\lambda_{a+1} \tag{2.9}$$

where $a+1$ is reduced mod 4. The equality $\varphi(\nu_1)=\eta_1(v,v)$ follows from (2.9). Because $i^*(\tilde{\nu}_{a+1})=\eta_1\tilde{\nu}$, $i^*(\lambda_{a+1})=\eta_1^2\tilde{\kappa}$ and $I(v)=1\times\eta_1\tilde{\kappa}+\tilde{\nu}\times 1$ where i denotes the inclusions $P^{n-2}\subset P^{n-1}$, $\mathrm{Spin}(n-1)\subset\mathrm{Spin}(n)$. We construct such an element actually. Let δ be as in (2.1) and set $n=8k+2s$ where $s=1$ or 3. Then by [9], Lemma 3.4 we have $\delta(1\times\mu^{a+1}\beta(\Delta^+))=\tau_{4k}^+\tau_s^+\mu^{a+1}c(\xi+1)$ and so

$$\delta(1\times\lambda_{a+1})=\omega_k^+ r(\tau_s^+\mu^{a+1})(\xi+1)\,.$$

Also, we have $\delta(\mu^{a+1}\nu\times 1)=\tau_{4k}^+\tau_s^+\mu^{a+1}c(\xi+2)$ and hence we get

$$\delta(\tilde{\nu}_{a+1}\times 1)=\omega_k^+ r(\tau_s^+\mu^{a+1})$$

by using the facts that $\widetilde{KO}_G^{-s}(\Sigma^{s,0})=Z\cdot r(\tau_s^+\mu^{a+1})$ and $\tau_s^{+*}=-(R^{1,0}\otimes C)\tau_s^+$. From this and the fromula of (2.1) we have $r(\tau_s^+\mu^{a+1})\xi=0$ since $\gamma\tilde{\nu}_{a+1}=0$ and so we have

$$\delta(\tilde{\nu}_{a+1}\times 1-1\times\lambda_{a+1})=0\,.$$

This and using (2.1) give rise to the required element.

Define $\tau\in KO^{-1}(SO(n))$ and $\nu_3\in KO^{3-n}(SO(n))$ as

$$\tau=\delta(t\times v)\quad\text{and}\quad \nu_3=-\delta(t\times(\xi+1))\,.$$

Here let δ be as in (2.4). Then using the formula after (1.6) we have

$$\delta(t\times i^*(P)) = -(\xi+1)\,\nu_3 P\,,\quad \delta(t\times\xi i^*(P)) = \xi\nu_3 P\,,$$
$$\delta(t\times\kappa i^*(P)) = (tr\,\lambda)\,P\,,\quad \delta(t\times\upsilon i^*(P)) = \tau P$$

where P is a ploynomial as in (2.3). Moreover as stated above

$$\varphi(\nu_1) = \eta_1(\upsilon, \upsilon)$$

and by definition we have

$$\varphi(\varepsilon_i) = 2(\kappa, \kappa),\ \eta_1^2(\kappa, \kappa),\ \eta_4(\kappa, \kappa)\quad \text{or}\quad 0$$

according as $i\equiv -a$, $1-a$, $2-a$ or $3-a$ mod 4. From (2.8) and these equalities we obtain immediately

(2.10) *As a $KO^*(+)$-module, $KO^*(SO(n))$ is generated by elements of the form P, $(tr\,\lambda)\,P$, τP, $\nu_1 P$, $\nu_3 P$, $\varepsilon_{-a}P$, $\varepsilon_{1-a}P$ and $\varepsilon_{2-a}P$ where P denotes a polynomial in ξ, $\beta(\lambda^1\rho), \cdots, \beta(\lambda^{a-1}\rho)$ and the indices of ε are reduced* mod. 4.

In (2.10) we find that ε_{1-a} can be expressed by the other generators.

To show this we need some results. Define a map m: $P^{n-1}\times S^{n-1}\rightarrow S^{n-1}$ by $m(\pi(x), \phi(g))=\phi(e_1 xg)$ for $x\in S^{n-1}$, $g\in \mathrm{Spin}(n-1)$. Then from construction of $\beta(\bar{\delta})$ and ν it follows that

$$m^*(\beta(\bar{\delta})) = c(\gamma+1)\times\beta(\bar{\delta})-\nu\times 1\,.$$

This implies that

$$c(m^*\delta(t)) = c((\gamma+1)\times\delta(t)-\tilde{\nu}_{-a-1}\times 1)$$

because $c\delta(t)=\mu^{-a-1}\beta(\bar{\delta})$ and so using (1.1) we have

$$m^*\delta(t) = (\gamma+1)\times\delta(t)-\nu_{-a-1}\times 1+\eta_1(x\times\delta(t)+y\times 1)$$

for some $x\in\widetilde{KO}^{-7}(P^{n-1})$, $y\in\widetilde{KO}^{-n-4}(P^{n-1})$. Since $I(\delta(t\times 1))=(1\times\phi)^*m^*\delta(t)$, $\phi^*\delta(t)=\lambda_{-a-1}$ by the result just before Proposition 1.7, $\eta_1\lambda_{-a-1}=0$ and $\phi^*(y)=0$ for the reason of dimension, we obtain

$$I(\delta(t\times 1)) = (\gamma+1)\times\lambda_{-a-1}-\nu_{-a-1}\times 1\,,$$

so that

(2.11) $$I(\nu_3) = \nu_{-a-1}\times 1-1\times\lambda_{-a-1}$$

because of $\gamma\tilde{\nu}_{-a-1}=0$ where also $a+1$ is reduced mod 4.

By [9], Therofem 3.5

$$2^a c(\xi) = 0\,,\quad \text{so that}\quad 2^{a+1}\xi = 2^a\eta_4\xi = 0\,.$$

On the other hand $\iota^*(\xi)=\gamma$ and $\iota^*(\eta_4\,\xi)=\eta_4\gamma$ are the generators of $\widetilde{KO}^0(P^{n-1})\cong$

$Z_{2^{a+1}}$ and $\widetilde{KO}^{-4}(P^{n-1})\cong Z_{2^a}$ respectively where ι is an embedding of P^{n-1} in $SO(n)$. Hence we get

(2.12) *The orders of* ξ *and* $\eta_4\xi$ *are* 2^{a+1} *and* 2^a *respectively.*

From (2.2), (2.9) and (2.11) it follows that

$$I(\delta_1+2\nu_{a+1}) = I(\delta_1+\eta_4\nu_{a+3}) = 0$$

because of $\eta_4\nu_{a+3} = 2\nu_{a+1}$, $\eta_4\lambda_{a+3}=2\lambda_{a+1}$. So, by (2.3)

$$\delta_1+2\nu_{a+1} = 2^{a-1}\xi\delta_1 P,\ \delta_1+\eta_4\nu_{a+3} = 2^{a-1}\xi\delta_1 P'$$

for some ploynomials P, P' as in (2.3). This and (2.12) mean that

$$2^{a-1}\xi\delta_1 = -2^a\xi\nu_{a+1} = -2^{a-1}\eta_4\xi\nu_{a+3}\,. \tag{2.13}$$

Again by (2.2), (2.9) and (2.11) we have

$$I(\varepsilon_1+(\xi+2)\nu_1) = 0 \quad \text{or} \quad I(\varepsilon_2+(\xi+2)\nu_1) = 0$$

according as $n\equiv 2$ or 6 mod 8, because $\gamma\tilde{\nu}_1=\gamma\tilde{\nu}_3=0$.

In any case, by (2.3) and (2.13) we therefore see that ε_{1-a} can be described by ξ, ν_1, ν_3. Thus, by (2.10) we obtain

Lemma 2.14. *As a* $KO^*(+)$*-module,* $KO^*(SO(n))$ *is generated by elements in the form* P, $(tr\ \lambda)P$, τP, $\nu_1 P$, $\nu_3 P$, $\varepsilon_{-a}P$ *and* $\varepsilon_{2-a}P$ *where* P *is a polynomial as in* (2.10) *and the indices of* ε *are reduced* mod 4.

Further we provide a lemma. Because of $\nu_3=-\delta(t\times(\xi+1))$, (2.13) yields

$$2^{a-1}\xi\delta_1 = -\delta(t\times 2^{a-1}\theta\xi)\,,$$

that is, $2^{a-1}\xi\delta_1\in \operatorname{Im}\delta$ where δ is as in (2.4) and θ as in (2.5). Clearly Coker $\psi\cong\operatorname{Im}\delta$ and this isomorphism sends $-t\times 2^{a-1}\theta\xi i^*(P)$ to $2^{a-1}\xi\delta_1 P$ where P is a polynomial as in (2.3). From (2.3), (2.8) and (2.13) we therefore have

Lemma 2.15. *As a* $KO^*(+)$*-module*

$$\operatorname{Im} J = \wedge_{Z_2}(\beta(\lambda^1\rho), \cdots, \beta(\lambda^{a-1}\rho))\,\{2^a\xi\nu_{a+1}\}$$

and $z\operatorname{Im} J=0$ *for* $z=\xi$, η_1 *and* η_4 *where the index of* ν *is reduced* mod 4.

3. The algebra structure of $KO^*(SO(n))$

For our aim we need the formulas for $I(tr\ \lambda)$ and $I(\tau)$ similar to those of (2.2). We begin with calculating $I(tr\ \lambda)$. Since $c(\lambda)=\mu^3\beta(\Delta^+)\beta(\Delta^-)$ and $\pi^*(\beta(\varepsilon)-\beta(\delta))=\beta(\Delta^+)+\beta(\Delta^-)$ by construction of $\beta(\varepsilon)$ and $\beta(\delta)$, it follows that $c(\lambda)=\mu^3\beta(\Delta^+)\pi^*(\beta(\varepsilon)-\beta(\delta))$, so that we have $c(tr\ \lambda)=\mu^3\beta(\delta)\beta(\varepsilon)$ because

$tr(\beta(\Delta^+))=\beta(\varepsilon)$ and $\beta(\varepsilon)^2=0$. From this and [9], Lemma 3.2, iii), iv) we get

$$c(I(tr\,\lambda)-((\gamma+2)\times\lambda-\mathfrak{v}_3\times\lambda_0))=0\,.$$

So, by (1.1) and Lemma 1.7 we can write

$$\text{(a)}\qquad I(tr\,\lambda)=(\gamma+2)\times\lambda-\mathfrak{v}_3\times\lambda_0+\eta_1\alpha \quad\text{and}$$
$$\alpha=1\times x_1+\gamma\times x_2+\mathfrak{v}_1\times x_3+\mathfrak{v}_3\times x_4$$

for some $x_i\in KO^*(\mathrm{Spin}(n))$.

Let $S^{n-2,0}=S^{n,0}\cap\{(x_1,\cdots,x_n);\ x_1=x_n=0\}$ and $P^{n-3}=S^{n-2,0}/G$. Define a map

$$m\colon S^{n-2}\times P^{n-3}\times\mathrm{Spin}(n-1)\to S^{n-2}\times SO(n-1)$$

by $m(x,\pi(y),g)=(e_1yxye_1,\pi(e_1yg))$ for $x\in S^{n-2}$, $y\in S^{n-2,0}$, $g\in\mathrm{Spin}(n-1)$. Then the following diagram with δ as in (1.6) is commutative.

$$\begin{array}{ccc} KO^*(S^{n-2}\times SO(n-1)) & \xrightarrow{\delta} & KO^*(SO(n)) \\ m^*\downarrow & & m^*\downarrow \\ KO^*(S^{n-2}\times P^{n-3}\times\mathrm{Spin}(n-1)) & \xrightarrow{\delta} & KO^*(P^{n-3}\times\mathrm{Spin}(n)) \end{array}$$

Also, obviously $m^*=(j\times 1)^*I$ where j denotes the inclusion of P^{n-3} in P^{n-1}. Apply $(j\times 1)^*$ to both sides of the first equality of (a). Then considering the order of γ we have

$$\text{(b)}\qquad m^*(tr\,\lambda)=(\gamma+2)\times\lambda+\eta_1\times x_1+\eta_1\gamma\times x_2$$

where γ denotes $j^*(\gamma)$. On the other hand by discussion similar to that about $(pd)^*$ in §1 we get

$$\text{(c)}\qquad m^*(t\times 1)=t\times(\gamma+1)\times 1+x$$

for some $x\in(1\times 2\gamma\times 1)\,KO^*(S^{n-2}\times P^{n-3}\times\mathrm{Spin}(n))$. Moreover, by [9], Lemma 4.14, iii) and [10], Lemma 4,.18, iii) we have $I(\kappa)=(\gamma+2)\times\tilde{\kappa}$. From this and (c) we have $m^*(t\times\kappa)=t\times(\gamma+2)\times\tilde{\kappa}$. Since $tr\,\lambda=\delta(t\times\kappa)$ and $\lambda=\delta(t\times\tilde{\kappa})$, it therefore follows from the commutativity of the above diagram and (b) that

$$\eta_1\times x_1+\eta_1\gamma\times x_2=0\,.$$

Hence we may put

$$\alpha=\mathfrak{v}_1\times x_3+\mathfrak{v}_3\times x_4\,,$$

so that we have

$$\text{(3.1)}\qquad I(tr\,\lambda)=(\gamma+2)\times\lambda-\mathfrak{v}_3\times\lambda_0+\eta_1\alpha$$

and there hold the relations $\eta_1^2\alpha=\gamma\alpha=\alpha^2=0$.

Since $I(\nu)=1\times\eta_1\tilde{\kappa}+\tilde{\nu}\times 1$ and $c(\tilde{\nu})=2^{a-1}\mu^{a+1}c(\gamma)$ we get $c(\nu)=2^{a-1}\mu^{a+1}c(\xi)$. Also, by (2.11) and [9], Lemma 3.3, iii) we have $c(\nu_3)=-\mu^{a+3}\beta(\delta)$. Using these facts we obtain

$$c(I(\tau)-2^{a-1}\gamma\times\lambda_0)=0\,.$$

Analogously from this equality we can show that

$$(3.2)\qquad I(\tau)=(\gamma+1)\times\eta_1\lambda+2^{a-1}\gamma\times\lambda_0+\eta_1\beta$$

and there hold the relations $\eta_1^2\beta=\gamma\beta=\beta^2=0$.

We are now ready to obtain

Theorem 3.3. *As a $KO^*(+)$-module*

$$KO^*(SO(n))=\wedge_{KO^*(+)}(\beta(\lambda^1\rho),\cdots,\beta(\lambda^{a-1}\rho),\varepsilon_0,\varepsilon_2,\nu_1,\nu_3)$$
$$\otimes_Z(Z\cdot 1\oplus Z_{2^{a+1}}\cdot\xi\oplus Z_2\cdot\tau\oplus Z\cdot tr\,\lambda)$$

in which the following relations hold:

$$\xi^2=-2\xi,\ \beta(\lambda^k\rho)^2=\eta_1(\beta(\lambda^2(\lambda^k\rho))+\binom{n}{k}\beta(\lambda^k\rho))\quad(1\leqq k\leqq a-1)\,,$$
$$\eta_1\varepsilon_i=0,\ \eta_1\nu_{a+3}=2^a\xi,\ \eta_1\nu_{a+1}=2^{a-1}\eta_4\xi,\ \eta_4\varepsilon_i=2\varepsilon_{i+2}\,,$$
$$\eta_4\nu_j=2\nu_{j+2},\ \eta_4\tau=0,\ \varepsilon_i^2=\nu_j^2=(tr\,\lambda)^2=\tau^2=0\,,$$
$$\xi\varepsilon_i=\xi\,tr\,\lambda=\varepsilon_i\,tr\,\lambda=\varepsilon_i\tau=\nu_j\,tr\,\lambda=\nu_j\tau=\varepsilon_0\varepsilon_2=\tau\,tr\,\lambda=0\,,$$
$$\nu_1\nu_3=\eta_1(\xi+1)\tau,\ \xi\tau=\eta_1\,tr\,\lambda,\ \varepsilon_0\nu_{a+1}=\varepsilon_2\nu_{a+3}=\eta_4\,tr\,\lambda\,,$$
$$\varepsilon_0\nu_{a+3}=\varepsilon_2\nu_{a+1}=2tr\,\lambda$$

for $i=0,2$, $j=1,3$ if the indices of ε and ν are reduced mod 4 *and $\otimes_Z$ is left out.*

Proof. From Lemma 2.15 we see that I induces a monomorphism

$$KO^*(SO(n))/(2^a\xi\nu_{a+1})\to KO^*(P^{n-1}\times\mathrm{Spin}(n))\,.$$

Let R denote the right-hand side of the equality stated in the theorem. Then a computation, using (2.2), (2.9), (2.11), (3.1), (3.2), Lemmas 1.7 and 2.14, shows that as a $KO^*(+)$-module

$$KO^*(SO(n))/(2^a\xi\nu_{a+1})=R/(2^a\xi\nu_{a+1})$$

in which there hold the above relations reduced mod $(2^a\xi\nu_{a+1})$. So, if it is shown that in $KO^*(SO(n))$ these relations hold, then the theorem follows immediately.

We now consider the relations. The first relation is clear. The second one and the relations $\nu_j^2=0$ are due to [5], §6.

$$\eta_1\varepsilon_i=\eta_1 r(\mu^i\beta(\delta))$$
$$=\chi\delta(\mu^{i+1}\beta(\varepsilon))=0\quad\text{since}\quad\chi\delta=0\quad\text{in (1.1)}\,.$$

By definition $\eta_1^2 \nu_1 = \delta c(\nu_3) = 0$. So, by exactness of (1.1) there is an element $x \in K^*(SO(n))$ such that

$$\eta_1 \nu_1 = r(x).$$

Then $rI(x) = 2^{a-1}\theta\gamma \times 1$ by Proposition 1.3 where θ is as in (2.5). Observing $\operatorname{Im} rI$, we get $I(x) = 2^{a-2} c(\theta\gamma) \times 1$. Since I in complex case is injective, we have

$$x = 2^{a-2} c(\theta\xi)$$

and so

$$\eta_1 \nu_1 = 2^{a-1}\theta\xi.$$

By arguing as above we get also another relation $\eta_1 \nu_3 = 2^{a-2}\theta\eta_4\xi$.

$$\eta_4 \varepsilon_i = r(c(\eta_4)\mu^i\beta(\varepsilon)) = r(2\mu^{i+2}\beta(\varepsilon)) = 2\varepsilon_{i+2}.$$

$$\eta_4 \nu_j = r(\mu^2 c(\nu_j)) = rc(\nu_{j+2}) = 2\nu_{j+2} \quad \text{since} \quad c(\nu_j) = -\mu^{a+j}\beta(\delta).$$

$$\eta_4 \tau = \delta(t \times \eta_4 v) = 0 \quad \text{by (2.5)}.$$

$$\begin{aligned} \varepsilon_i^2 &= r(c(\varepsilon_i)\mu^i\beta(\varepsilon)) \\ &= (-1)^i 2\delta(\mu^{2i+1}\beta(\varepsilon)\beta(\delta)) \quad \text{since} \quad \beta(\varepsilon)^* = \beta(\varepsilon) - c(\xi+2)\beta(\delta) \\ &= (-1)^{i+1}\delta c(\varepsilon_{2i-a}\nu_1) = 0 \qquad \text{since} \quad \delta c = 0 \text{ in (1.1)}. \end{aligned}$$

$$\tau^2 = \delta(t \times v i^*(\tau)) = 0 \quad \text{since} \quad i^*(\tau) = 0.$$

$$(tr\,\lambda)^2 = tr(\pi^*(tr\,\lambda)\lambda) = 2tr\,\lambda^2 = 0 \quad \text{since} \quad \lambda^2 = 0.$$

Similarly the others can be shown, so we omit the proof of them. Thus the theorem follows.

Finally we show how we can get the explicit description of $\eta_1\beta(\lambda^2(\lambda^k\rho))$ appeared in the second relation of Theorem 3.3. Analogously to the case of $KO^*(\mathrm{Spin}(n))$, also in the present case it suffices to check $\eta_1\beta(\lambda^a\rho)$ and $\eta_1\beta(\lambda^{a+1}\rho)$. We now prove the following

(3.4) $\quad \eta_1\beta(\lambda^{a+1}\rho) = 0$ *in* $KO^*(SO(n))$ *or* $KO^*(\mathrm{Spin}(n))$

and $\eta_1(\beta(\lambda^a\rho) + \beta(\lambda^{a-2}\rho) + \cdots) = \eta_1\tau + \eta_1^2\, tr\,\lambda$ *in* $KO^*(SO(n))$ *or*
$\qquad = \eta_1^2\lambda$ *in* $KO^*(\mathrm{Spin}(n))$

according as ρ *is viewed as a representation of* $SO(n)$ *or* $\mathrm{Spin}(n)$.

As shown in [10] we have

$$\beta(\lambda^{a+1}\rho) = 2^a\theta\kappa - \beta(\lambda^a\rho) - \cdots - \beta(\lambda^1\rho) \quad \text{in} \quad KO^*(SO(n+1)),$$

$$\beta(\lambda^{a+1}\rho) = 2^{a+1}\theta\tilde{\kappa} - \beta(\lambda^a\rho) - \cdots - \beta(\lambda^1\rho) \quad \text{in} \quad KO^*(\mathrm{Spin}(n+1)).$$

Here θ is as in (2.5), $\kappa = \kappa_{n+1}$ or $\beta(\varepsilon_{n+1})$ and $\tilde{\kappa} = \tilde{\kappa}_{n+1}$ or $\beta(\Delta_{n+1})$ as in [10] according as $n \equiv 2$ or 6 mod 8 and ρ denotes also the $(n+1)$-dimensional standard representations of $SO(n+1)$ and $\mathrm{Spin}(n+1)$. So it follows that in either case

$$\eta_1(\beta(\lambda^{a+1}\rho)+\beta(\lambda^a\rho)+\cdots+\beta(\lambda^1\rho))=0\,.$$

By restricting this to $SO(n)$ or $\mathrm{Spin}(n)$ according as we consider ρ as a representation of $SO(n+1)$ or $\mathrm{Spin}(n+1)$ we get readily

$$\eta_1\beta(\lambda^{a+1}\rho)=0\,.$$

By Proposition 1.4 $\eta_1^2\lambda=\lambda^2=\beta(r(\Delta^+))^2$ and so from the square formula of [5] it follows that

$$\eta_1^2\lambda=\eta_1\beta(\lambda^2(r(\Delta^+)))\,.$$

Considering the character of Δ^+ on a maximal torus of $\mathrm{Spin}(n)$ ([8], §13, Prop. 9.4) we see that

$$\lambda^2(r(\Delta^+))=(\lambda^a\rho+\lambda^{a-2}\rho+\cdots)+2s(\lambda^{a-3}\rho+\lambda^{a-5}\rho+\cdots)$$

for some integer s. Hence we have

$$\eta_1^2\lambda=\eta_1(\beta(\lambda^a\rho)+\beta(\lambda^{a-2}\rho)+\cdots)\quad\text{in}\quad KO^*(\mathrm{Spin}(n))\,.$$

To show the remaining case we recall the equality $\Delta^+\otimes_C\Delta^-=c(\lambda^a\rho+\lambda^{a-2}\rho+\cdots)$ from [8]. This gives $c((\beta(\lambda^a\rho)+\beta(\gamma^{a-2}\rho)+\cdots)-2^a\lambda_0)=0$. Therefore we may put

$$\beta(\lambda^a\rho)+\beta(\gamma^{a-2}\rho)+\cdots=2^a\lambda_0+\eta_1(P+\lambda Q)+\eta_1^2(P'+\lambda Q')$$

where P, P', Q and Q' are polynomials in $\beta(\lambda^1\rho),\cdots,\beta(\lambda^{a-1}\rho)$ as in (2.3). Since, by [10], $\beta(\lambda^a\rho)+\beta(\lambda^{a-1}\rho)+\cdots=2^a\theta\tilde{\kappa}$ in $KO^*(\mathrm{Spin}(n-1))$, comparing this equality with the restriction of the above to $\mathrm{Spin}(n-1)$ yields $P=P'=0$ and so the previous result implies $Q=1$. Hence

$$\text{(a)}\qquad \beta(\lambda^a\rho)+\beta(\lambda^{a-2}\rho)+\cdots=2^a\lambda_0+\eta_1\lambda+\eta_1^2\lambda Q'\quad\text{in}\quad KO^*(\mathrm{Spin}(n))\,.$$

Also we have

$$c((\beta(\lambda^a\rho)+\beta(\lambda^{a-2}\rho)+\cdots)-2^{a-1}\varepsilon_0-\tau)=0\quad\text{in}\quad KO^*(SO(n))\,.$$

So we can set

$$\text{(b)}\qquad \beta(\lambda^a\rho)+\beta(\lambda^{a-2}\rho)+\cdots=2^{a-1}\varepsilon_0+\tau+\eta_1 x$$

for some $x\in KO^*(SO(n))$. Apply π^* to both sides of (b) and compare this with (a), then we have

$$\pi^*(x)=\eta_1\lambda Q'\,.$$

On the other hand, applying I to both sides of (b) again and using (a) yield $I(\eta_1 x+\eta_1 tr\,\lambda+\eta_1(\xi+1)\tau Q')=\eta_1\beta$ where β is as in (3.2). Since $\eta_1^2\beta=0$ and Ker $I=(\eta_1^2(\xi+1)\tau)$ by Theorem 3.4, it follows that $I(\eta_1^2(x+tr\,\lambda))=0$, so that we can set

$$\eta_1^2(x+tr\,\lambda+(\xi+1)\tau R)=0$$

for some polynomial R in $\beta(\lambda^1\rho),\cdots,\beta(\lambda^{a-1}\rho)$ as above. By observing the relations of Theorem 3.4 we therefore see that $x+tr\,\lambda+(\xi+1)\tau R$ is described in terms of ε_0, ε_2, ν_1 and ν_3 and so $\eta_1\lambda Q'+2\lambda+\eta_1\lambda R$ in terms of λ_i ($i=0, 1, 2, 3$) because of $\pi^*(x)=\eta_1\lambda Q'$, $\pi^*(tr\,\lambda)=2\lambda$, $\pi^*(\tau)=\eta_1\lambda$, $\pi^*(\varepsilon_0)=2\lambda_0$, $\pi^*(\varepsilon_2)=2\lambda_2$, $\pi^*(\nu_1)=-\lambda_{a+1}$ and $\pi^*(\nu_3)=-\lambda_{-a-1}$. Hence, from the relations of Proposition 1.4 we infer that Q' and R are divisible by η_1. This implies $\eta_1^2 x=\eta_1^2\,tr\,\lambda$. Thus by (b) we have

$$\eta_1(\beta(\lambda^a\rho)+\beta(\lambda^{a-2}\rho)+\cdots)=\eta_1\tau+\eta_1^2\,tr\,\lambda \quad \text{in} \quad KO^*(SO(n)).$$

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