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SEPARABLE EXTENSIONS AND FROBENIUS EXTENSIONS

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As in our previous paper [2] we say that a ring Λ with 1 is a separable extension of a subring Γ which contains the same 1 if the map $\pi: \Lambda \otimes_{\Gamma} \Lambda \rightarrow \Lambda$ such that $\pi(x \otimes y) = xy$ splits as two sided Λ -module. There has been a problem whether a separable extension is a Frobenius extension. Recently, K. Nakane has given an affirmative answer to this problem in [8] under the condition that Λ is centrally projective over Γ in the sense of K. Hirata [4] and $m\Gamma \neq \Gamma$ holds for every maximal ideal m of a central subring R of Γ such that $\Lambda = \Gamma \otimes_R \Omega$ with Ω finitely generated projective over R . He also proved that if Λ is Γ -centrally projective and separable over Γ , Λ is a quasi-Frobenius extension of Γ . In this paper we shall show that the last condition can be omitted (Theorem 2). Next we consider the opposite situation, that is, $\Lambda \otimes_{\Gamma} \Lambda$ is Λ -centrally projective and Γ is a Γ - Γ -direct summand of Λ . In this case we can also see that Λ is a Frobenius extension of Γ if we assume the finitely generated projectivity of Λ_{Γ} or ${}_{\Gamma}\Lambda$ (Theorem 4).

1. Separable extensions

Throughout this paper we assume that all rings have the identity elements and all subrings contain the same 1 as the over ring. Furthermore whenever we say that M is a Γ - Γ -module or a two sided Γ -module for a ring Γ , we assume that M is unitary and associative, that is, $(xm)y = x(my)$ for all $x, y \in \Gamma$ and $m \in M$.

Let Γ be a ring and M a Γ - Γ -module. Then, according to K. Hirata [4] we say that M is centrally projective over Γ , if M is isomorphic to a direct summand of a finite direct sum of the copies of Γ as two sided Γ -module. The next lemma is due to K. Hirata. But since we need it in this paper so often, we shall state here.

Lemma 1 (*Prop. 5.2 [4]*). *If a two sided Γ -module M is centrally projective over Γ , M^{Γ} is finitely generated projective over C and $M \cong \Gamma \otimes_C M^{\Gamma}$ by the map: $x \otimes m \rightarrow xm$ and $\text{Hom}({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}M_{\Gamma}) \cong \text{Hom}({}_C M^{\Gamma}, {}_C M^{\Gamma})$, where $M^{\Gamma} = \{m \in M \mid xm = mx \text{ for every } x \in \Gamma\}$ and C is the center of Γ .*

The next theorem is an immediate consequence of Lemam 1. But it attracts our interests to itself.

Theorem 1. *Let M be an arbitrary centrally projective Γ - Γ -module. Then, $\Omega = \text{Hom}({}_{\Gamma}M, {}_{\Gamma}M)$ is an H-separable extension of $\Gamma/\alpha\Gamma$, where α is the annihilator ideal of M^{Γ} in C .*

Proof. Since M is isomorphic to a direct summand of $\Gamma \oplus \cdots \oplus \Gamma$ as Γ - Γ -module, $\text{Hom}({}_{\Gamma}M, {}_{\Gamma}M)$ is also a direct summand of a finite direct sum of the copies of $\text{Hom}({}_{\Gamma}\Gamma, {}_{\Gamma}\Gamma)$, which is isomorphic to M as Γ - Γ -module. Hence, Ω is centrally projective over Γ , as M is so. Then, $\Omega = \Omega^{\Gamma} \otimes_C \Gamma$, where $\Omega^{\Gamma} = \text{Hom}({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}M_{\Gamma}) \cong \text{Hom}({}_C M^{\Gamma}, {}_C M^{\Gamma})$. But $\text{Hom}({}_C M^{\Gamma}, {}_C M^{\Gamma})$ is central separable over C/α , since M^{Γ} is C -finitely generated projective. Thus Ω is H-separable over $C/\alpha \otimes_C \Gamma$, as $\Omega = \Omega^{\Gamma} \otimes_{C/\alpha} C/\alpha \otimes_C \Gamma$.

Lemma 2. *A two sided Γ -module M is centrally projective over Γ if and only if there exist $f_j \in \text{Hom}({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}\Gamma)$ and $m_j \in M^{\Gamma}$, $j=1, 2, \dots, n$, such that $m = \sum f_j(m)m_j$ for every $m \in M$.*

Proof. M is centrally projective over Γ if and only if there exist Γ - Γ -homomorphisms f of M to $\Gamma \oplus \cdots \oplus \Gamma$, the direct sum of n copies of Γ for some n , and g of $\Gamma \oplus \cdots \oplus \Gamma$ to M such that $gf=1_M$. Assume that such f and g exist, and let $f_j = \pi_j f$, where π_j is the j th projection of $\Gamma \oplus \cdots \oplus \Gamma$ to Γ , and g_j the restriction of g to the j th direct summand Γ of $\Gamma \oplus \cdots \oplus \Gamma$. Then, g_j is given by the multiplication of some m_j in M^{Γ} , since g_j is in $\text{Hom}({}_{\Gamma}\Gamma_{\Gamma}, {}_{\Gamma}M_{\Gamma})$, which is isomorphic to M^{Γ} . Then $\sum f_j(m)m_j = \sum g_j f_j(m) = gf(m) = 1_M(m) = m$. Conversely, assume that there exist such $f_j \in \text{Hom}({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}\Gamma)$ and $m_j \in M^{\Gamma}$. Then, if we define f and g as follows;

$$f(m) = (f_1(m), f_2(m), \dots, f_n(m)), \quad g((x_1, x_2, \dots, x_n)) = \sum x_j m_j$$

then f is a Γ - Γ -map of M to $\Gamma \oplus \cdots \oplus \Gamma$ and g is a Γ - Γ -map of $\Gamma \oplus \cdots \oplus \Gamma$ to M such that $gf=1_M$. Hence M is centrally projective over Γ .

Let R be a commutative ring, Γ an R -algebra and A a finitely generated projective R -module. Denote $M = \Gamma \otimes_R A$. Then M is a centrally projective Γ - Γ -module. Let $f_j \in \text{Hom}({}_{R}A, {}_{R}R)$ and $a_j \in A$ be such that $a = \sum f_j(a)a_j$ for every $a \in A$. Then, clearly $f_j = 1_{\Gamma} \otimes f_j$ and $1 \otimes a_j$ satisfy the condition of Lemma 2. Let m be an arbitrary in M^{Γ} . Then for any $x \in \Gamma$ and every j , $xf_j(m) = f_j(xm) = f_j(mx) = f_j(m)x$, and we see that $f_j(m) \in C$, the center of Γ . Thus we see that $M^{\Gamma} = C \otimes_R A$. By this remark, we get.

Lemma 3. *Let R be a commutative ring. Then if A is a finitely generated projective R -module and Γ is an R -algebra with its center C , $\Gamma \otimes_R A$ is centrally projective over Γ and $(\Gamma \otimes_R A)^{\Gamma} = C \otimes_R A$.*

Proposition 1. *Let Λ be a ring and Γ a subring of Λ . Then Λ is an H -separable extension of Γ if and only if $1 \otimes 1 \in \Delta(\Lambda \otimes_{\Gamma} \Lambda)^{\Delta}$ in $\Lambda \otimes_{\Gamma} \Lambda$ where $\Delta = V_{\Lambda}(\Gamma)$, the commutator subring of Γ in Λ .*

Proof. Λ is H -separable over Γ if and only if $\Lambda \otimes_{\Gamma} \Lambda$ is centrally projective over Λ . This is the case if and only if there exist

$$\varphi_j \in \text{Hom}({}_{\Lambda} \Lambda \otimes_{\Gamma} \Lambda_{\Lambda}, {}_{\Lambda} \Lambda_{\Lambda}) \quad \text{and} \quad \delta_j \in (\Lambda \otimes_{\Gamma} \Lambda)^{\Delta} \quad j = 1, 2, \dots, n$$

such that $\sum \varphi_j(1 \otimes 1) \delta_j = 1 \otimes 1$, since $1 \otimes 1$ generates $\Lambda \otimes_{\Gamma} \Lambda$ as two sided Λ -module. On the other hand, since $\text{Hom}({}_{\Lambda} \Lambda \otimes_{\Gamma} \Lambda_{\Lambda}, {}_{\Lambda} \Lambda_{\Lambda})$ is isomorphic to Δ by the map: $\varphi \rightarrow \varphi(1 \otimes 1)$, each Λ - Λ -map φ of $\Lambda \otimes_{\Gamma} \Lambda$ to Λ is given by the multiplication of some $d \in \Delta$. Hence the above φ_j and δ_j exist if and only if there exist $d_j \in \Delta$ and $\delta_j \in (\Lambda \otimes_{\Gamma} \Lambda)^{\Delta}$ such that $1 \otimes 1 = \sum d_j \delta_j$, i.e., $1 \otimes 1 \in \Delta(\Lambda \otimes_{\Gamma} \Lambda)^{\Delta}$.

Now, let a ring Λ be left finitely generated projective over a subring Γ of it. Then there exist $f_j \in \text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)$ and $z_j \in \Lambda, j=1, 2, \dots, n$, such that $x = \sum f_j(x) z_j$ for every $x \in \Lambda$. On the other hand, we have Λ - Λ -isomorphisms

$$\Lambda \otimes_{\Gamma} \Lambda \rightarrow \text{Hom}({}_{\Gamma} \Lambda, \Lambda_{\Gamma}) \otimes_{\Gamma} \Lambda \rightarrow \text{Hom}(\text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)_{\Gamma}, \Lambda_{\Gamma})$$

such that the composition σ of them is given by $\sigma(x \otimes y)(f) = xf(y)$ for every $f \in \text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)$. Then we have a commutative diagram of Λ - Λ -maps

$$\begin{array}{ccc} \Lambda \otimes_{\Gamma} \Lambda & \xrightarrow{\sigma} & \text{Hom}(\text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)_{\Gamma}, \Lambda_{\Gamma}) \\ \pi \searrow & & \swarrow \Psi \\ & \Lambda & \end{array}$$

with $\Psi(\psi) = \sum \psi(f_j) z_j, \pi(x \otimes y) = xy$, for $\psi \in \text{Hom}(\text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)_{\Gamma}, \Lambda_{\Gamma})$ and $x, y \in \Lambda$, because $\Psi \sigma(x \otimes y) = \sum \sigma(x \otimes y)(f_j) z_j = x \sum f_j(y) z_j = xy = \pi(x \otimes y)$.

From this fact, we obtain

Proposition 2. *Let a ring Λ be left finitely generated projective over a subring Γ , and f_j and z_j be as above. Then, Λ is a separable extension of Γ if and only if there exists a Λ - Γ -homomorphism h of $\text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)$ to Λ such that $\sum h(f_j) z_j = 1$.*

Proof. Λ is separable over Γ if and only if there exists $\sum x_i \otimes y_i$ in $(\Lambda \otimes_{\Gamma} \Lambda)^{\Delta}$ such that $\pi(\sum x_i \otimes y_i) = 1$. But σ is an isomorphism and induces a one to one correspondence between $(\Lambda \otimes_{\Gamma} \Lambda)^{\Delta}$ and $\text{Hom}({}_{\Lambda} \text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)_{\Gamma}, {}_{\Lambda} \Lambda_{\Gamma})$. Hence there exists $\sum x_i \otimes y_i \in (\Lambda \otimes_{\Gamma} \Lambda)^{\Delta}$ with $\pi(\sum x_i \otimes y_i) = 1$ if and only if there exists an $h \in \text{Hom}({}_{\Lambda} \text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)_{\Gamma}, {}_{\Lambda} \Lambda_{\Gamma})$ with $\Psi(h) = 1$, i.e., $\sum h(f_j) z_j = 1$.

Let Λ be a separable extension of Γ such that Λ is centrally projective over Γ . Then, there exist $f_j \in \text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)$ and $d_j \in \Delta$ as Lemma 2 and

$h \in \text{Hom}({}_{\Delta} \text{Hom}({}_{\Gamma} \Lambda, {}_{\Gamma} \Gamma)_{\Gamma}, {}_{\Delta} \Lambda_{\Gamma})$ with $\sum h(f_j)d_j=1$ by Proposition 2. Then we see that $h(\text{Hom}({}_{\Gamma} \Lambda_{\Gamma}, {}_{\Gamma} \Gamma_{\Gamma})) \subset \Delta$. In fact, let f be an arbitrary in $\text{Hom}({}_{\Gamma} \Lambda_{\Gamma}, {}_{\Gamma} \Gamma_{\Gamma})$ and r in Γ . Since $(r \circ f)(x) = f(xr) = f(x)r = (fr)(x)$ for every $x \in \Lambda$, $r \circ f = fr$. Then, $rh(f) = h(r \circ f) = h(fr) = h(f)r$ for any $r \in \Gamma$, since h is a Λ - Γ -map. Therefore $h(f) \in \Delta$. Thus h induces a left Δ -map \bar{h} of $\text{Hom}({}_{\Gamma} \Lambda_{\Gamma}, {}_{\Gamma} \Gamma_{\Gamma})$ to Δ , if we restrict h to $\text{Hom}({}_{\Gamma} \Lambda_{\Gamma}, {}_{\Gamma} \Gamma_{\Gamma})$. Clearly, $\sum \bar{h}(f_j)d_j = \sum h(f_j)d_j = 1$. On the other hand, $\Lambda = \Gamma \otimes_C \Delta$ by Lemma 1, where C is the center of Γ . Then, since

$$\text{Hom}({}_{\Gamma} \Gamma \otimes_C \Delta_{\Gamma}, {}_{\Gamma} \Gamma_{\Gamma}) \cong \text{Hom}({}_C \Delta, {}_C \text{Hom}({}_{\Gamma} \Gamma_{\Gamma}, {}_{\Gamma} \Gamma_{\Gamma})) \cong \text{Hom}({}_C \Delta, {}_C C)$$

as Δ - Δ -map, we have a Δ - Δ -isomorphism ν of $\text{Hom}({}_C \Delta, {}_C C)$ to $\text{Hom}({}_{\Gamma} \Lambda_{\Gamma}, {}_{\Gamma} \Gamma_{\Gamma})$ such that $\nu(f)(rd) = rf(d)$ for $r \in \Gamma$ and $d \in \Delta$. Let $\bar{f}_j = \nu^{-1}(f_j)$ for every j . Then, $\sum \bar{f}_j(d)d_j = d$ for any $d \in \Delta$. Let $h' = h\nu$. Then h' is a left Δ -map of $\text{Hom}({}_C \Delta, {}_C C)$ to Δ , and $\sum h'(\bar{f}_j)d_j = \sum h(\nu(\bar{f}_j))d_j = \sum h(f_j)d_j = 1$. This implies that Δ is a separable C -algebra by virtue of Proposition 2.

From this remark we obtain

Theorem 2. *Let Λ be a separable extension of Γ such that Λ is centrally projective over Γ . Then we have*

- 1) Δ is a separable C -algebra where C is the center of Γ , and Λ is a Frobenius extension of Γ .
- 2) Λ is a centrally projective H -separable extension of Γ' and Γ' is a separable extension of Γ , where $\Gamma' = V_{\Lambda}(V_{\Lambda}(\Gamma))$.

Proof. 1). Δ is a separable C -algebra by the above remark. Hence, Δ is a Frobenius C -algebra by Theorem 4.2 [1]. Then, since $\Lambda \cong \Gamma \otimes_C \Delta$, Λ is a Frobenius extension of Γ (see Theorem 3 [9]). 2). Let C' be the center of Δ . Then, since $V_{\Lambda}(C') \cong \Gamma' \otimes_{C'} \Delta$, $V_{\Lambda}(C') = \Gamma' \Delta \supset \Gamma \Delta = \Lambda$, and $\Lambda = V_{\Lambda}(C')$. Then we see that C' is the center of Λ . Then $\Lambda \cong \Gamma' \otimes_{C'} \Delta$, Λ is centrally projective and H -separable over Γ' . Next, since $\Lambda = \Gamma' \otimes_{C'} \Delta$, Γ' is a Γ' - Γ' -direct summand, consequently, a Γ - Γ -direct summand of Λ , which is centrally projective over Γ . Thus Γ' is centrally projective over Γ , and $\Gamma' = V_{\Gamma'}(\Gamma) \otimes_C \Gamma$ by Lemma 1. But $V_{\Gamma'}(\Gamma) = \Gamma' \cap V_{\Lambda}(\Gamma) = V_{\Lambda}(\Delta) \cap \Delta = C'$, which is a separable C -algebra as Δ is a separable C -algebra. Hence Γ' is a separable extension of Γ .

Now, we can see that Nakane's theorem in [8] can be obtained under a weaker condition concerning separability.

Corollary 1. *Let Γ and Ω be R -algebras with Ω finitely generated projective over R and C the center of Γ . Suppose $mC \neq C$ holds for every maximal ideal m of R . Then, $\Lambda = \Gamma \otimes_R \Omega$ is a separable extension of Γ if and only if Ω is a separable R -algebra.*

Proof. The ‘if’ part is clear by Prop. 2.7 [2]. Suppose Λ is separable over Γ . Then, $C \otimes_R \Omega$ is separable over C by Lemma 3 and Theorem 2. Then, Ω is separable over R by Nakane’s results (see Theorem [8]).

2. Strong Frobenius and symmetric extensions

In case Λ is an algebra over a commutative ring R , Λ is called a symmetric R -algebra if Λ is Λ - Λ -isomorphic to $\text{Hom}({}_R\Lambda, {}_R R)$. In case of ring extension it is impossible to introduce such a notion. But we can consider the case where $\Lambda | \Gamma$ has the next condition;

$$(s.F.1) \quad {}_{\Delta}\Lambda_{\Gamma-\Delta} \cong {}_{\Delta}\text{Hom}({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)_{\Gamma-\Delta} \text{ and } {}_{\Gamma}\Lambda \text{ is finitely generated projective.}$$

In this case we shall call that Λ is a strong Frobenius extension of Γ . This condition is equivalent to

$$(s.F.r) \quad {}_{\Delta-\Gamma}\Lambda_{\Delta} = {}_{\Delta-\Gamma}\text{Hom}(\Lambda_{\Gamma}, \Gamma_{\Gamma})_{\Delta} \text{ and } \Lambda_{\Gamma} \text{ is finitely generated projective.}$$

The above equivalence can be deduced if we take the dual modules again. In case Λ is an R -algebra, Λ is a strong Frobenius R -algebra if and only if Λ is a symmetric R -algebra. Moreover, if Λ is centrally projective over Γ , the condition (s.F.1) (resp. (s.F.r)) implies

$${}_{\Delta}\Lambda_{\Delta} \cong {}_{\Delta}\text{Hom}({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)_{\Delta} \text{ (resp. } {}_{\Delta}\Lambda_{\Delta} \cong {}_{\Delta}\text{Hom}(\Lambda_{\Gamma}, \Gamma_{\Gamma})_{\Delta})$$

where $\text{Hom}({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)_{\Gamma \otimes \Delta}$ is given by $(f(r \otimes d))(x) = f(dx)r$ for $r \in \Gamma, d \in \Delta, x \in \Lambda$ and $f \in \text{Hom}({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)$. Hence in this case, we shall call that Λ is a symmetric extension of Γ .

Most parts of the next Lemma is well known (see Theorem 3 [9] and Theorem 35 [7] for example).

Lemma 4. *If Ω is a symmetric (resp. Frobenius or quasi-Frobenius) algebra over a commutative ring R , then $\Lambda = \Gamma \otimes_R \Omega$ is a symmetric (resp. Frobenius or quasi-Frobenius) extension of Γ for any R -algebra Γ .*

Proof. We shall prove in the case of symmetric algebra. Suppose Ω is Ω - Ω -isomorphic to $\text{Hom}({}_R\Omega, {}_R R)$. Then

$$\begin{aligned} {}_{\Gamma \otimes \Omega}\text{Hom}({}_{\Gamma}\Gamma \otimes_R \Omega, {}_{\Gamma}\Gamma)_{\Gamma-\Delta} &\cong {}_{\Gamma \otimes \Omega}\text{Hom}({}_R\Omega, {}_R\text{Hom}({}_{\Gamma}\Gamma, {}_{\Gamma}\Gamma))_{\Gamma-\Omega} \\ &\cong {}_{\Gamma \otimes \Omega}\text{Hom}({}_R\Omega, {}_R\Gamma)_{\Gamma-\Omega} \cong {}_{\Gamma \otimes \Omega}\text{Hom}({}_R\text{Hom}({}_R\Omega, {}_R R), {}_R\Gamma)_{\Gamma-\Omega} \\ &\cong {}_{\Gamma \otimes \Omega}\text{Hom}({}_R R, {}_R\Gamma) \otimes_R \Omega_{\Gamma-\Omega} \cong {}_{\Gamma \otimes \Omega}\Lambda_{\Gamma-\Omega} \end{aligned}$$

since Ω is R -finitely generated projective. Hence, we see ${}_{\Delta}\text{Hom}({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)_{\Delta} \cong {}_{\Delta}\Lambda_{\Delta}$, ${}_{\Gamma}\Lambda$ is finitely generated projective. Thus Λ is a symmetric extension of Γ . By the same method we can prove in the case of Frobenius algebra.

REMARK. If we use Lemma 1.1 [11], we can prove that if Λ_i are R -algebras

and left quasi-Frobenius extensions of R -subalgebras Γ_i respectively, and if the natural map: $\Gamma_1 \otimes_R \Gamma_2 \rightarrow \Lambda_1 \otimes_R \Lambda_2$ is a monomorphism, then $\Lambda_1 \otimes_R \Lambda_2$ is also a left quasi-Frobenius extension of $\Gamma_1 \otimes_R \Gamma_2$.

The most parts of the next theorem are immediate consequences of Lemma 2 and Theorem 35 [5].

Theorem 3. *Let a ring Λ be centrally projective over a subring Γ . Then, Λ is a symmetric (resp. Frobenius or left (or right) quasi-Frobenius) extension of Γ if and only if $\Delta = V_\Lambda(\Gamma)$ is a symmetric (resp. Frobenius or quasi-Frobenius) algebra over C , the center of Γ .*

Proof. Since $\Lambda = \Gamma \otimes_C \Delta$, the ‘if’ parts have been proved in Lemma 4. Suppose Λ is a symmetric extension of Γ , and let h be a Λ - Λ -isomorphism

$$h: {}_\Lambda \text{Hom}({}_\Gamma \Lambda, {}_\Gamma \Gamma)_{\Gamma-\Delta} \rightarrow {}_\Delta \Lambda_{\Gamma-\Delta}$$

Then, h induces a Δ - Δ -map \bar{h} of $\text{Hom}({}_\Gamma \Lambda_\Gamma, {}_\Gamma \Gamma_\Gamma)$ to Δ as is shown in the previous section. Clearly \bar{h} is a monomorphism since h is so. Let d be an arbitrary in Δ . Then there exists an f in $\text{Hom}({}_\Gamma \Lambda, {}_\Gamma \Gamma)$ with $h(f) = d$. Then $r \circ f = fr$ for every $r \in \Gamma$, since h is a Λ - Γ -map and d is in Δ . Hence f is in $\text{Hom}({}_\Gamma \Lambda_\Gamma, {}_\Gamma \Gamma_\Gamma)$, and \bar{h} is an epimorphism. Thus we see that $\text{Hom}({}_\Gamma \Lambda_\Gamma, {}_\Gamma \Gamma_\Gamma)$ is Δ - Δ -isomorphic to Δ . On the other hand, as is shown before $\text{Hom}({}_\Gamma \Lambda_\Gamma, {}_\Gamma \Gamma_\Gamma)$ is Δ - Δ -isomorphic to $\text{Hom}({}_C \Delta, {}_C C)$. Thus we see that $\text{Hom}({}_C \Delta, {}_C C)$ is Δ - Δ -isomorphic to Δ , and Δ is a symmetric C -algebra. The same method as above proves in the case of Frobenius extension. Next, suppose that Λ is a left quasi-Frobenius extension of Γ . Then, by Satz 2 [5] there exist Λ - Γ -maps φ_k of $\text{Hom}({}_\Gamma \Lambda, {}_\Gamma \Gamma)$ to Λ and Γ - Γ -maps α_k of Λ to Γ with $\sum \varphi_k(\alpha_k) = 1$. But each map φ_k induces a left Δ -map φ'_k of $\text{Hom}({}_\Gamma \Lambda_\Gamma, {}_\Gamma \Gamma_\Gamma)$ to Δ , and there exists a left Δ -isomorphism ν of $\text{Hom}({}_C \Delta, {}_C C)$ to $\text{Hom}({}_\Gamma \Lambda_\Gamma, {}_\Gamma \Gamma_\Gamma)$. Let $\bar{\varphi}_k = \varphi'_k \nu$ and $\bar{\alpha}_k = \nu^{-1}(\alpha_k)$. Then $\sum \bar{\varphi}_k(\bar{\alpha}_k) = \sum \varphi_k \nu(\nu^{-1}(\alpha_k)) = \sum \varphi_k(\alpha_k) = 1$. Therefore, Δ is a quasi-Frobenius C -algebra. (See Theorem 35 [5] and the Bemerkung under it).

3. Application of Morita’s results

In sections 1 and 2 we considered the case where Λ is centrally projective over Γ , but in this section we shall consider the case where $\Lambda \otimes_\Gamma \Lambda$ is centrally projective over Λ , i.e., Λ is an H -separable extension of Γ . To do this we shall apply the results of Morita [7].

Lemma 5. *Let Λ be an H -separable extension of Γ , $\Delta = V_\Lambda(\Gamma)$ and $\Omega = \text{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$. Then, we have*

1) $\text{Hom}({}_\Omega \Lambda, {}_\Omega \Lambda) = V_\Delta(\Delta)$, thus if $V_\Delta(V_\Delta(\Gamma)) = \Gamma$, Λ_Γ has the double centralizer property,

2) If $V_\Delta(V_\Delta(\Gamma))=\Gamma$, ${}_{\Gamma-\Delta}\text{Hom}({}_\Omega\Lambda, {}_\Omega\Omega)_\Delta \cong {}_{\Gamma-\Delta}\text{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Delta$

Proof. 1). By Prop. 3.3 [4], there exist a ring isomorphism η of $\Lambda \otimes_C \Delta^\circ$ to $\text{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$ such that $\eta(x \otimes d^\circ)(y) = xyd$, where C is the center of Λ . Thus ${}_\Omega\Lambda$ is equivalent to ${}_\Delta\Lambda_\Delta$. Hence, we see

$$\text{Hom}({}_\Omega\Lambda, {}_\Omega\Lambda) \cong \text{Hom}({}_\Delta\Lambda_\Delta, {}_\Delta\Lambda_\Delta) \cong V_\Delta(\Delta).$$

2). By Theorem 1.1 [7] ${}_\Gamma\text{Hom}({}_\Omega\Lambda, {}_\Omega\Omega)_\Omega \cong {}_\Gamma\text{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Omega$, and we have

$${}_{\Gamma-\Delta}\text{Hom}({}_\Omega\Lambda, {}_\Omega\Omega)_\Delta \cong {}_{\Gamma-\Delta}\text{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Delta$$

The next theorem is almost due to Theorem 6.1 [7].

Theorem 4. *Let Λ be an H -separable extension of Γ with $V_\Delta(V_\Delta(\Gamma))=\Gamma$ and Ω, Δ and C be as in Lemma 5. Then, if Δ is a symmetric (resp. Frobenius or quasi-Frobenius) C -algebra, the following conditions are equivalent;*

- 1) Λ is left Γ -finitely generated projective.
- 2) Λ is right Γ -finitely generated projective.
- 3) Λ is a strong Frobenius (resp. Frobenius or quasi-Frobenius) extension of Γ .

Proof. Suppose Δ is a symmetric C -algebra. Then Ω is a symmetric extension of Λ , since $\Omega = \Lambda \otimes_C \Delta^\circ$ and Δ° is C -symmetric. Hence we have

$${}_\Omega\text{Hom}({}_\Delta\Omega, {}_\Delta\Lambda)_{\Delta \otimes \Delta^\circ} \cong {}_\Omega\Omega_{\Delta \otimes \Delta^\circ}, \text{ i.e., } {}_{\Omega-\Delta}\text{Hom}({}_\Delta\Omega, {}_\Delta\Lambda)_\Delta \cong {}_{\Omega-\Delta}\Omega_\Delta$$

Then by Lemma 5 and the above isomorphism, we have

$$\begin{aligned} {}_{\Gamma-\Delta}\text{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Delta &\cong {}_{\Gamma-\Delta}\text{Hom}({}_\Omega\Lambda, {}_\Omega\Omega)_\Delta \\ &\cong {}_{\Gamma-\Delta}\text{Hom}({}_\Omega\Lambda, {}_\Omega\text{Hom}({}_\Delta\Omega, {}_\Delta\Lambda))_\Delta \\ &\cong {}_{\Gamma-\Delta}\text{Hom}({}_\Delta\Omega \otimes {}_\Omega\Lambda, {}_\Delta\Lambda)_\Delta \\ &\cong {}_{\Gamma-\Delta}\text{Hom}({}_\Delta\Lambda, {}_\Delta\Lambda)_\Delta \cong {}_{\Gamma-\Delta}\Lambda_\Delta \end{aligned}$$

Thus Λ is a strong Frobenius extension of Γ . For the rest of the proof, see Theorem 6.1 [7].

Corollary 2. *Let Λ be an H -separable extension of Γ such that Γ is a Γ - Γ -direct summand of Λ . Then the following conditions are equivalent.*

- 1) Λ is left Γ -finitely generated projective.
- 2) Λ is right Γ -finitely generated projective.
- 3) Λ is a strong Frobenius extension of Γ .

Proof. Since Γ is a Γ - Γ -direct summand of Λ , $V_\Delta(V_\Delta(\Gamma))=\Gamma$ by Prop. 1.2 [10] and Δ is C -separable by Prop. 4.7 [4]. Thus Δ is C -symmetric by Theorem 4.2 [1], and we can apply Theorem 4.

The converse of Theorem 4 holds for H -separable extension as follows.

Theorem 5. *Let Λ be an H -separable extension of Γ . Then if Λ is a strong Frobenius (resp. Frobenius or left or right quasi-Frobenius) extension of Γ , Δ is a symmetric (resp. Frobenius or quasi-Frobenius) C -algebra, where $\Delta = V_\Lambda(\Gamma)$ and C is the center of Λ .*

Proof. Suppose Λ is a strong Frobenius extension of Γ . Then there exists a Λ - $(\Delta - \Gamma)$ -isomorphism ${}_\Lambda\Lambda_{\Delta - \Gamma} \cong {}_\Lambda\text{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_{\Delta - \Gamma}$. Then, since ${}_\Gamma\Lambda$ is finitely generated projective,

$$\begin{aligned} {}_{\Lambda - \Delta}\Lambda \otimes_\Gamma \Lambda_{\Lambda - \Delta} &\cong {}_{\Lambda - \Delta}\text{Hom}({}_\Gamma\Lambda, \Lambda_\Gamma) \otimes_\Gamma \Lambda_{\Lambda - \Delta} \cong {}_{\Lambda - \Delta}\text{Hom}(\text{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma)_{\Lambda - \Delta} \\ &\cong {}_{\Lambda - \Delta}\text{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)_{\Lambda - \Delta} \end{aligned}$$

On the other hand, since Λ is H -separable over Γ , there exist $(\Lambda - \Delta)$ - $(\Lambda - \Delta)$ -isomorphisms

$$\begin{aligned} \xi: \Lambda \otimes_\Gamma \Lambda &\rightarrow \text{Hom}({}_C\Delta, {}_C\Lambda) & \xi(x \otimes y)(d) &= xdy \text{ for } x, y \in \Lambda \text{ and } d \in \Delta \\ \eta: \Lambda \otimes_C \Delta &\rightarrow \text{Hom}(\Lambda_\Gamma, \Lambda_\Gamma) & \eta(x \otimes d)(y) &= xyd \text{ for } x, y \in \Lambda \text{ and } d \in \Delta \end{aligned}$$

Hence we have $(\Lambda - \Delta)$ - $(\Lambda - \Delta)$ -isomorphisms

$$\text{Hom}({}_C\Delta, {}_C\Lambda) \cong \Lambda \otimes_\Gamma \Lambda \cong \text{Hom}(\Lambda_\Gamma, \Lambda_\Gamma) \cong \Lambda \otimes_C \Delta$$

Then, taking $\text{Hom}(*, {}_\Lambda\Lambda_\Lambda)$, we obtain Δ - Δ -isomorphisms

$${}_\Delta\text{Hom}({}_\Lambda\text{Hom}({}_C\Delta, {}_C\Lambda)_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Delta \cong {}_\Delta\text{Hom}({}_\Lambda\Lambda_\Lambda, {}_\Lambda\Lambda_\Lambda) \otimes_C \Delta \cong {}_\Delta C \otimes_C \Delta \cong {}_\Delta\Delta$$

since Δ is C -finitely generated projective, and

$${}_\Delta\text{Hom}({}_\Lambda\Lambda \otimes_C \Delta_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Delta \cong {}_\Delta\text{Hom}({}_C\Delta, {}_C\text{Hom}({}_\Lambda\Lambda_\Lambda, {}_\Lambda\Lambda_\Lambda))_\Delta \cong {}_\Delta\text{Hom}({}_C\Delta, {}_CC)_\Delta$$

Thus we see ${}_\Delta\text{Hom}({}_C\Delta, {}_CC)_\Delta \cong {}_\Delta\Delta$, which means that Δ is a symmetric C -algebra. In case of Frobenius extension, ${}_\Lambda\Lambda_\Gamma \cong {}_\Lambda\text{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma$ induces

$${}_\Lambda\Lambda \otimes_\Gamma \Lambda_{\Lambda - \Delta} \cong {}_\Lambda\text{Hom}(\text{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma)_{\Lambda - \Delta} \cong {}_\Lambda\text{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)_{\Lambda - \Delta}$$

where ${}_\Lambda\Lambda \otimes_\Gamma \Lambda_{\Lambda - \Delta}$ is induced by ${}_\Lambda\Lambda_{\Gamma - \Delta}$ and ${}_\Gamma\Lambda_\Lambda$, while in case of right quasi-Frobenius extension ${}_\Lambda\text{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_{\Gamma \ltimes_\Lambda (\Sigma^\oplus \Lambda)_\Gamma}$ induces

$${}_\Lambda\Lambda \otimes_\Gamma \Lambda_{\Lambda - \Delta} \cong {}_\Lambda\text{Hom}(\text{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma)_{\Lambda - \Delta} \ltimes_\Lambda (\Sigma^\oplus \text{Hom}(\Lambda_\Gamma, \Lambda_\Gamma))_{\Lambda - \Delta}$$

Then the same argument as in the case of strong Frobenius extension proves the theorem in both cases.

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