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As in our previous paper [2] we say that a ring $\Lambda$ with 1 is a separable extension of a subring $\Gamma$ which contains the same 1 if the map $\pi: \Lambda \otimes_\Gamma \Lambda \to \Lambda$ such that $\pi(x \otimes y) = xy$ splits as two sided $\Lambda$-module. There has been a problem whether a separable extension is a Frobenius extension. Recently, K. Nakane has given an affirmative answer to this problem in [8] under the condition that $\Lambda$ is centrally projective over $\Gamma$ in the sense of K. Hirata [4] and $m_\Gamma \neq \Gamma$ holds for every maximal ideal $m$ of a central subring $R$ of $\Gamma$ such that $\Lambda = R \otimes_R \Omega$ with $\Omega$ finitely generated projective over $R$. He also proved that if $\Lambda$ is $\Gamma$-centrally projective and separable over $\Gamma$, $\Lambda$ is a quasi-Frobenius extension of $\Gamma$. In this paper we shall show that the last condition can be omitted (Theorem 2).

Next we consider the opposite situation, that is, $\Lambda \otimes_\Gamma \Lambda$ is $\Lambda$-centrally projective and $\Gamma$ is a $\Gamma$-$\Gamma$-direct summand of $\Lambda$. In this case we can also see that $\Lambda$ is a Frobenius extension of $\Gamma$ if we assume the finitely generated projectivity of $\Lambda_\Gamma$ or $\Gamma_\Lambda$ (Theorem 4).

1. Separable extensions

Throughout this paper we assume that all rings have the identity elements and all subrings contain the same 1 as the over ring. Furthermore whenever we say that $M$ is a $\Gamma$-$\Gamma$-module or a two sided $\Gamma$-module for a ring $\Gamma$, we assume that $M$ is unitary and associative, that is, $(xm)y = x(my)$ for all $x, y \in \Gamma$ and $m \in M$.

Let $\Gamma$ be a ring and $M$ a $\Gamma$-$\Gamma$-module. Then, according to K. Hirata [4] we say that $M$ is centrally projective over $\Gamma$, if $M$ is isomorphic to a direct summand of a finite direct sum of the copies of $\Gamma$ as two sided $\Gamma$-module. The next lemma is due to K. Hirata. But since we need it in this paper so often, we shall state here.

**Lemma 1 (Prop. 5.2 [4]).** If a two sided $\Gamma$-module $M$ is centrally projective over $\Gamma$, $M^\Gamma$ is finitely generated projective over $C$ and $M \cong \Gamma \otimes_C M^\Gamma$ by the map: $x \otimes m \mapsto xm$ and $\text{Hom}(\Gamma M_\Gamma, \Gamma M) \cong \text{Hom}(C M^\Gamma, C M^\Gamma)$, where $M^\Gamma = \{m \in M | xm = mx \text{ for every } x \in \Gamma\}$ and $C$ is the center of $\Gamma$. 
The next theorem is an immediate consequence of Lemma 1. But it attracts our interests to itself.

**Theorem 1.** Let $M$ be an arbitrary centrally projective $\Gamma$-$\Gamma$-module. Then, $\Omega=\text{Hom}(rM, rM)$ is an H-separable extension of $\Gamma/\alpha\Gamma$, where $\alpha$ is the annihilator ideal of $M^\Gamma$ in $C$.

Proof. Since $M$ is isomorphic to a direct summand of $\Gamma\oplus\cdots\oplus\Gamma$ as $\Gamma$-$\Gamma$-module, $\text{Hom}(rM, rM)$ is also a direct summand of a finite direct sum of the copies of $\text{Hom}(r\Gamma, rM)$, which is isomorphic to $M$ as $\Gamma$-$\Gamma$-module. Hence, $\Omega$ is centrally projective over $\Gamma$, as $M$ is so. Then, $\Omega=\Omega^\Gamma\otimes C\Gamma$, where $\Omega^\Gamma=\text{Hom}(r\Gamma, rM)^\Gamma\approx\text{Hom}(C^M, C^M)$. But $\text{Hom}(C^M, C^M)$ is central separable over $C/\alpha\Gamma$, since $M^\Gamma$ is $C$-finitely generated projective. Thus $\Omega$ is H-separable over $C/\alpha\otimes C\Gamma$, as $\Omega=\Omega^\Gamma\otimes C\alpha\otimes C\Gamma$.

**Lemma 2.** A two sided $\Gamma$-module $M$ is centrally projective over $\Gamma$ if and only if there exist $f_j\in\text{Hom}(\Gamma M, \Gamma M)$ and $m_j\in M^\Gamma$, $j=1, 2, \ldots, n$, such that $m=\Sigma f_j(m)m_j$ for every $m\in M$.

Proof. $M$ is centrally projective over $\Gamma$ if and only if there exist $\Gamma$-$\Gamma$-homomorphisms $f$ of $M$ to $\Gamma\oplus\cdots\oplus\Gamma$, the direct sum of $n$ copies of $\Gamma$ for some $n$, and $g$ of $\Gamma\oplus\cdots\oplus\Gamma$ to $M$ such that $gf=1_M$. Assume that such $f$ and $g$ exist, and let $f_j=\pi_j f$, where $\pi_j$ is the $j$th projection of $\Gamma\oplus\cdots\oplus\Gamma$ to $\Gamma$, and $g_j$ the restriction of $g$ to the $j$th direct summand $\Gamma$ of $\Gamma\oplus\cdots\oplus\Gamma$. Then, $g_j$ is given by the multiplication of some $m_j$ in $M^\Gamma$, since $g_j$ is in $\text{Hom}(r\Gamma, r\Gamma M)$, which is isomorphic to $M^\Gamma$. Then $\Sigma f_j(m)m_j=\Sigma g_j f_j(m)=gf(m)=1_M(m)=m$. Conversely, assume that there exist such $f_j\in\text{Hom}(\Gamma M, \Gamma M)$ and $m_j\in M^\Gamma$. Then, if we define $f$ and $g$ as follows:

$$f(m) = (f_1(m), f_2(m), \ldots, f_n(m)),$$

$$g((x_1, x_2, \ldots, x_n)) = \Sigma x_j m_j$$

then $f$ is a $\Gamma$-$\Gamma$-map of $M$ to $\Gamma\oplus\cdots\oplus\Gamma$ and $g$ is a $\Gamma$-$\Gamma$-map of $\Gamma\oplus\cdots\oplus\Gamma$ to $M$ such that $gf=1_M$. Hence $M$ is centrally projective over $\Gamma$.

Let $R$ be a commutative ring, $\Gamma$ an $R$-algebra and $A$ a finitely generated projective $R$-module. Denote $M=\Gamma\otimes_R A$. Then $M$ is a centrally projective $\Gamma$-$\Gamma$-module. Let $f_j\in\text{Hom}(rA, r\Gamma R)$ and $a_j\in A$ be such that $a=\Sigma f_j(a_j)a_j$ for every $a\in A$. Then, clearly $f_j=1_\Gamma\otimes f_j$ and $1\otimes a_j$ satisfy the condition of Lemma 2. Let $m$ be an arbitrary in $M^\Gamma$. Then for any $x\in \Gamma$ and every $j$, $xf_j(m)=f_j(xm)=f_j(mx)=f_j(m)x$, and we see that $f_j(m)\in C$, the center of $\Gamma$. Thus we see that $M^\Gamma=C\otimes_R A$. By this remark, we get.

**Lemma 3.** Let $R$ be a commutative ring. Then if $A$ is a finitely generated projective $R$-module and $\Gamma$ is an $R$-algebra with its center $C$, $\Gamma\otimes_R A$ is centrally projective over $\Gamma$ and $(\Gamma\otimes_R A)^\Gamma=C\otimes_R A$.
Proposition 1. Let $\Lambda$ be a ring and $\Gamma$ a subring of $\Lambda$. Then $\Lambda$ is an H-separable extension of $\Gamma$ if and only if $1 \otimes 1 \in \Delta(\Lambda \otimes \Gamma, \Lambda)^\Lambda$ in $\Lambda \otimes \Gamma \Lambda$ where $\Delta = V_\Lambda(\Gamma)$, the commutor subring of $\Gamma$ in $\Lambda$.

Proof. $\Lambda$ is H-separable over $\Gamma$ if and only if $\Lambda \otimes \Gamma \Lambda$ is centrally projective over $\Lambda$. This is the case if and only if there exist $\varphi_j \in \text{Hom}(\Lambda \Lambda \otimes \Gamma, \Lambda \Lambda)$ and $\delta_j \in (\Lambda \otimes \Gamma, \Lambda)^\Lambda$ such that $\Sigma \varphi_j(1 \otimes 1)\delta_j = 1 \otimes 1$, since $1 \otimes 1$ generates $\Lambda \otimes \Gamma \Lambda$ as a two-sided $\Lambda$-module. On the other hand, since $\text{Hom}(\Lambda \Lambda \otimes \Gamma, \Lambda \Lambda)$ is isomorphic to $\Delta$ by the map: $\varphi \mapsto \varphi(1 \otimes 1)$, each $\Lambda$-$\Lambda$-map $\varphi$ of $\Lambda \otimes \Gamma \Lambda$ to $\Lambda$ is given by the multiplication of some $d \in \Delta$. Hence the above $\varphi_j$ and $\delta_j$ exist if and only if there exist $d_j \in \Delta$ and $\delta_j \in (\Lambda \otimes \Gamma, \Lambda)^\Lambda$ such that $1 \otimes 1 = \Sigma d_j \delta_j$, i.e., $1 \otimes 1 \in \Delta(\Lambda \otimes \Gamma, \Lambda)^\Lambda$.

Now, let a ring $\Lambda$ be left finitely generated projective over a subring $\Gamma$ of it. Then there exist $f_j \in \text{Hom}(\Gamma \Lambda, \Gamma \Gamma)$ and $z_j \in \Lambda$, $j = 1, 2, \ldots, n$, such that $x = \Sigma f_j(x)z_j$ for every $x \in \Lambda$. On the other hand, we have $\Lambda$-$\Lambda$-isomorphisms

$$\Lambda \otimes \Gamma \Lambda \rightarrow \text{Hom}(\Gamma \Lambda, \Lambda \Lambda \otimes \Gamma \Lambda) \rightarrow \text{Hom}(\text{Hom}(\Gamma \Lambda, \Lambda \Lambda \otimes \Gamma \Lambda), \Lambda \Lambda \otimes \Gamma \Lambda)$$

such that the composition $\sigma$ of them is given by $\sigma(x \otimes y)(f) = x f(y)$ for every $f \in \text{Hom}(\Gamma \Lambda, \Gamma \Gamma)$. Then we have a commutative diagram of $\Lambda$-$\Lambda$-maps

$$\Lambda \otimes \Gamma \Gamma \rightarrow \text{Hom}(\Gamma \Lambda, \Lambda \Lambda \otimes \Gamma \Lambda) \rightarrow \text{Hom}(\text{Hom}(\Gamma \Lambda, \Lambda \Lambda \otimes \Gamma \Lambda), \Lambda \Lambda \otimes \Gamma \Lambda)$$

with $\Psi(\psi) = \Sigma \varphi f_j(x)z_j$, $\pi(x \otimes y) = xy$, for $\varphi \in \text{Hom}(\Gamma \Lambda, \Lambda \Lambda \otimes \Gamma \Lambda)$ and $x, y \in \Lambda$, because $\Psi \sigma(x \otimes y) = \Sigma \sigma(x \otimes y)(f_j)z_j = x \Sigma f_j(y)z_j = xy = \pi(x \otimes y)$.

From this fact, we obtain

Proposition 2. Let a ring $\Lambda$ be left finitely generated projective over a subring $\Gamma$, and $f_j$ and $z_j$ be as above. Then, $\Lambda$ is a separable extension of $\Gamma$ if and only if there exists a $\Lambda$-$\Gamma$-homomorphism $h$ of $\text{Hom}(\Gamma \Lambda, \Gamma \Gamma)$ to $\Lambda$ such that $\Sigma h(f_j)z_j = 1$.

Proof. $\Lambda$ is separable over $\Gamma$ if and only if there exists $x_i \otimes y_i$ in $(\Lambda \otimes \Gamma, \Lambda)^\Lambda$ such that $\pi(x_i \otimes y_i) = 1$. But $\sigma$ is an isomorphism and induces a one to one correspondence between $(\Lambda \otimes \Gamma, \Lambda)^\Lambda$ and $\text{Hom}(\Lambda \Lambda \otimes \Gamma, \Lambda \Lambda \otimes \Gamma, \Lambda \Lambda \otimes \Gamma)$. Hence there exists $x_i \otimes y_i \in (\Lambda \otimes \Gamma, \Lambda)^\Lambda$ with $\pi(x_i \otimes y_i) = 1$ if and only if there exists an $h \in \text{Hom}(\Lambda \Lambda \otimes \Gamma, \Lambda \Lambda \otimes \Gamma, \Lambda \Lambda \otimes \Gamma)$ with $\Psi(h) = 1$, i.e., $\Sigma h(f_j)z_j = 1$.

Let $\Lambda$ be a separable extension of $\Gamma$ such that $\Lambda$ is centrally projective over $\Gamma$. Then, there exist $f_j \in \text{Hom}(\Gamma \Lambda, \Gamma \Gamma)$ and $d_j \in \Delta$ as Lemma 2 and
$h \in \text{Hom}_r(\Lambda \Lambda_r, \Lambda_r)$ with $\sum h(f_j)d_j = 1$ by Proposition 2. Then we see that $h(\text{Hom}_r(\Lambda \Lambda_r, \Lambda_r)) \subseteq \Delta$. In fact, let $f$ be an arbitrary in Hom $(\Lambda \Lambda_r, \Lambda_r)$ and $r$ in $\Gamma$. Since $(r \circ f)(x) = f(xr) = f(x)r = (fr)(x)$ for every $x \in \Lambda$, $r \circ f = fr$. Then, $rh(f) = h(r \circ f) = h(fr) = h(f)r$ for any $r \in \Gamma$, since $h$ is a $\Lambda$-$\Gamma$-map. Therefore $h(f) \in \Delta$. Thus $h$ induces a left $\Delta$-map of $\text{Hom}_r(\Lambda \Lambda_r, \Lambda_r)$ to $\Delta$, if we restrict $h$ to Hom $(\Lambda \Lambda_r, \Lambda_r)$. Clearly, $\sum h(f_j)d_j = \sum h(f_j)d_j = 1$. On the other hand, $\Lambda = \Gamma \otimes C \Delta$ by Lemma 1, where $C$ is the center of $\Gamma$. Then, since

$$\text{Hom}_r(\Lambda \otimes \Delta, \Lambda_r) \approx \text{Hom}_r(\Lambda \otimes C \Delta, \Lambda_r) \approx \text{Hom}_r(\Lambda \otimes C, \Lambda)$$

as $\Delta$-$\Delta$-map, we have a $\Delta$-$\Delta$-isomorphism $\nu$ of $\text{Hom}_r(\Lambda \otimes C, \Lambda)$ to $\text{Hom}_r(\nu \Lambda \otimes \Gamma)$ such that $\nu(f)(rd) = rf(d)$ for $r \in \Gamma$ and $d \in \Delta$. Let $\tilde{f}_j = \nu^{-1}(f_j)$ for every $j$. Then, $\sum \tilde{f}_j(d)d_j = d$ for any $d \in \Delta$. Let $h' = h \nu$. Then $h'$ is a left $\Delta$-map of $\text{Hom}_r(\Lambda \otimes C, \Lambda)$ to $\Delta$, and $\sum h'(f_j)d_j = \sum h(f_j)d_j = 1$. This implies that $\Delta$ is a separable $C$-algebra by virtue of Proposition 2.

From this remark we obtain

**Theorem 2.** Let $\Lambda$ be a separable extension of $\Gamma$ such that $\Lambda$ is centrally projective over $\Gamma$. Then we have

1) $\Delta$ is a separable $C$-algebra where $C$ is the center of $\Gamma$, and $\Lambda$ is a Frobenius extension of $\Gamma$.

2) $\Lambda$ is a centrally projective $H$-separable extension of $\Gamma'$ and $\Gamma'$ is a separable extension of $\Gamma$, where $\Gamma' = V_\Lambda(V_\Lambda(\Gamma))$.

**Proof.** 1) $\Delta$ is a separable $C$-algebra by the above remark. Hence, $\Delta$ is a Frobenius $C$-algebra by Theorem 4.2 [1]. Then, since $\Lambda = \Gamma \otimes \Delta$, $\Lambda$ is a Frobenius extension of $\Gamma$ (see Theorem 3 [9]). 2) Let $C'$ be the center of $\Delta$. Then, since $V_\Lambda(C') \approx \Gamma' \otimes C', V_\Lambda(C') = \Gamma' \Delta \supset \Delta = \Lambda$, and $\Lambda = V_\Lambda(C')$. Then we see that $C'$ is the center of $\Lambda$. Then $\Lambda = \Gamma' \otimes C', \Lambda$ is centrally projective and $H$-separable over $\Gamma'$. Next, since $\Lambda = \Gamma' \otimes C', \Gamma'$ is a $\Gamma'$-$\Gamma'$-direct summand, consequently, a $\Gamma$-$\Gamma'$-direct summand of $\Lambda$, which is centrally projective over $\Gamma$. Thus $\Gamma'$ is centrally projective over $\Gamma$, and $\Gamma' = V_\Gamma(\Gamma) \otimes C$ by Lemma 1. But $V_\Gamma(\Gamma) = \Gamma' \cap V_\Lambda(\Gamma) = V_\Lambda(\Delta) \cap \Delta = C'$, which is a separable $C$-algebra as $\Delta$ is a separable $C$-algebra. Hence $\Gamma'$ is a separable extension of $\Gamma$.

Now, we can see that Nakane's theorem in [8] can be obtained under a weaker condition concerning separability.

**Corollary 1.** Let $\Gamma$ and $\Omega$ be $R$-algebras with $\Omega$ finitely generated projective over $R$ and $C$ the center of $\Gamma$. Suppose $mC = C$ holds for every maximal ideal $m$ of $R$. Then, $\Lambda = \Gamma \otimes R \Omega$ is a separable extension of $\Gamma$ if and only if $\Omega$ is a separable $R$-algebra.
Proof. The 'if' part is clear by Prop. 2.7 [2]. Suppose \( \Lambda \) is separable over \( \Gamma \). Then, \( \mathcal{C} \otimes_R \Omega \) is separable over \( \mathcal{C} \) by Lemma 3 and Theorem 2. Then, \( \Omega \) is separable over \( R \) by Nakane's results (see Theorem [8]).

2. Strong Frobenius and symmetric extensions

In case \( \Lambda \) is an algebra over a commutative ring \( R \), \( \Lambda \) is called a symmetric \( R \)-algebra if \( \Lambda \) is \( \Lambda \)-\( \Lambda \)-isomorphic to \( \text{Hom}(\mathcal{R} \Lambda, \mathcal{R} \Gamma) \). In case of ring extension it is impossible to introduce such a notion. But we can consider the case where \( \Lambda | \Gamma \) has the next condition;

\[
\Lambda \Lambda \cong \Lambda \text{Hom}(r \Lambda, r \Gamma)_{r \Lambda} \quad \text{and} \quad r \Lambda \text{ is finitely generated projective.}
\]

In this case we shall call that \( \Lambda \) is a strong Frobenius extension of \( \Gamma \). This condition is equivalent to

\[
\Lambda \Lambda \cong \Lambda \text{Hom}(r \Lambda, r \Gamma)_{r \Lambda} \quad \text{and} \quad r \Lambda \text{ is finitely generated projective.}
\]

The above equivalence can be deduced if we take the dual modules again. In case \( \Lambda \) is an \( R \)-algebra, \( \Lambda \) is a strong Frobenius \( R \)-algebra if and only if \( \Lambda \) is a symmetric \( R \)-algebra. Moreover, if \( \Lambda \) is centrally projective over \( \Gamma \), the condition (s.F.1) (resp. (s.F.r)) implies

\[
\Lambda \Lambda \cong \Lambda \text{Hom}(r \Lambda, r \Gamma)_{r \Lambda} \quad \text{and} \quad r \Lambda \text{ is finitely generated projective.}
\]

The above equivalence can be deduced if we take the dual modules again. In case \( \Lambda \) is an \( R \)-algebra, \( \Lambda \) is a strong Frobenius \( R \)-algebra if and only if \( \Lambda \) is a symmetric \( R \)-algebra. Moreover, if \( \Lambda \) is centrally projective over \( \Gamma \), the condition (s.F.1) (resp. (s.F.r)) implies

\[
\Lambda \Lambda \cong \Lambda \text{Hom}(r \Lambda, r \Gamma)_{r \Lambda} \quad \text{and} \quad r \Lambda \text{ is finitely generated projective.}
\]

Most parts of the next Lemma is well known (see Theorem 3 [9] and Theorem 35 [7] for example).

**Lemma 4.** If \( \Omega \) is a symmetric (resp. Frobenius or quasi-Frobenius) algebra over a commutative ring \( R \), then \( \Lambda = \Gamma \otimes_R \Omega \) is a symmetric (resp. Frobenius or quasi-Frobenius) extension of \( \Gamma \) for any \( R \)-algebra \( \Gamma \).

Proof. We shall prove in the case of symmetric algebra. Suppose \( \Omega \) is \( \Omega \)-\( \Omega \)-isomorphic to \( \text{Hom}(r \Omega, r R) \). Then

\[
\text{Hom}(r \Omega, r \Gamma)_{r \Omega} \cong \text{Hom}(r \Omega, r \Gamma)_{r \Omega} \cong \text{Hom}(r \Omega, r R, r \Gamma)_{r \Omega}
\]

since \( \Omega \) is \( R \)-finitely generated projective. Hence, we see \( \Lambda \text{Hom}(r \Lambda, r \Gamma)_{r \Lambda} \cong \Lambda \text{Hom}(r \Lambda, r \Gamma)_{r \Lambda} \) is finitely generated projective. Thus \( \Lambda \) is a symmetric extension of \( \Gamma \). By the same method we can prove in the case of Frobenius algebra.

**Remark.** If we use Lemma 1.1 [11], we can prove that if \( \Lambda_i \) are \( R \)-algebras
and left quasi-Frobenius extensions of $R$-subalgebras $\Gamma_i$ respectively, and if
the natural map: $\Gamma_1 \otimes_R \Gamma_2 \to \Lambda_1 \otimes_R \Lambda_2$ is a monomorphism, then $\Lambda_1 \otimes_R \Lambda_2$ is
also a left quasi-Frobenius extension of $\Gamma_1 \otimes_R \Gamma_2$.

The most parts of the next theorem are immediate consequences of Lemma 2
and Theorem 35 [5].

**Theorem 3.** Let a ring $\Lambda$ be centrally projective over a subring $\Gamma$. Then,
$\Lambda$ is a symmetric (resp. Frobenius or left (or right) quasi-Frobenius) extension of
$\Gamma$ if and only if $\Delta = V_\Lambda(\Gamma)$ is a symmetric (resp. Frobenius or quasi-Frobenius)
algebra over $C$, the center of $\Gamma$.

Proof. Since $\Lambda = \Gamma \otimes_C \Delta$, the 'if' parts have been proved in Lemma 4.
Suppose $\Lambda$ is a symmetric extension of $\Gamma$, and let $h$ be a $\Lambda$-$\Lambda$-isomorphism
\[ h: \Lambda \text{Hom}(r\Lambda, r\Gamma) \rightarrow \Lambda \Lambda r\Gamma. \]
Then, $h$ induces a $\Delta$-$\Delta$-map $\bar{h}$ of $\text{Hom}(r\Lambda, r\Gamma)$ to $\Delta$ as is shown in the
previous section. Clearly $\bar{h}$ is a monomorphism since $h$ is so. Let $d$ be an
arbitrary in $\Delta$. Then there exists an $f$ in $\text{Hom}(r\Lambda, r\Gamma)$ with $h(f) = d$. Then
$r \circ f = r^2$ for every $r \in \Gamma$, since $h$ is a $\Lambda$-$\Gamma$-map and $d$ is in $\Delta$. Hence $f$ is in
\[ \text{Hom}(r\Lambda, r\Gamma), \]
and $\bar{h}$ is an epimorphism. Thus we see that $\text{Hom}(r\Lambda, r\Gamma)$ is
$\Delta$-$\Delta$-isomorphic to $\Delta$. On the other hand, as is shown before $\text{Hom}(r\Lambda, r\Gamma)$
is $\Delta$-$\Delta$-isomorphic to $\text{Hom}(\Delta, \Delta)$. Thus we see that $\text{Hom}(\Delta, \Delta)$ is
$\Delta$-$\Delta$-isomorphic to $\Delta$, and $\Delta$ is a symmetric $C$-algebra. The same method
as above proves in the case of Frobenius extension. Next, suppose that $\Lambda$ is a
left quasi-Frobenius extension of $\Gamma$. Then, by Satz 2 [5] there exist $\Lambda$-$\Gamma$-
maps $\varphi_k$ of $\text{Hom}(r\Lambda, r\Gamma)$ to $\Lambda$ and $\Gamma$-$\Gamma$-maps $\alpha_k$ of $\Lambda$ to $\Gamma$ with $\Sigma \varphi_k(\alpha_k) = 1$.
But each map $\varphi_k$ induces a left $\Delta$-map $\varphi_k'$ of $\text{Hom}(r\Lambda, r\Gamma)$ to $\Delta$, and there
exists a left $\Delta$-isomorphism $\nu$ of $\text{Hom}(\Delta, \Delta)$ to $\text{Hom}(r\Lambda, r\Gamma)$. Let $\varphi_k'
= \varphi_k \nu$ and $\alpha_k = \nu^{-1}(\alpha_k)$. Then $\Sigma \varphi_k(\alpha_k) = \Sigma \varphi_k(\nu^{-1}(\alpha_k)) = \Sigma \varphi_k(\alpha_k) = 1$. Therefore,$\Delta$ is a quasi-Frobenius $C$-algebra. (See Theorem 35 [5] and the
Bemerkung under it).

### 3. Application of Morita's results

In sections 1 and 2 we considered the case where $\Lambda$ is centrally projective
over $\Gamma$, but in this section we shall consider the case where $\Delta \otimes_R \Lambda$ is cen-
trally projective over $\Lambda$, i.e., $\Lambda$ is an $H$-separable extension of $\Gamma$. To do this we
shall apply the results of Morita [7].

**Lemma 5.** Let $\Lambda$ be an $H$-separable extension of $\Gamma$, $\Delta = V_\Lambda(\Gamma)$ and
$\Omega = \text{Hom}(\Lambda, \Lambda)$. Then, we have

1) $\text{Hom}(\Omega \Lambda, \Lambda \Lambda) = V_\Lambda(\Delta)$, thus if $V_\Lambda(V_\Lambda(\Gamma)) = \Gamma$, $\Lambda$ has the double
centralizer property,
2) If $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$, then $\Gamma$ is $\Gamma$-finitely generated projective.

Proof. Let $\Lambda'$ be an $H$-separable extension of $\Gamma$ with $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$. Consider the ring isomorphism $\psi: \Lambda \otimes \Delta \rightarrow \Omega$ such that $\psi(x \otimes d)(y) = xyd$, where $C$ is the center of $\Lambda$. Then we have

$$\psi \circ \Hom(\Lambda, \Lambda) \cong \Hom(\Lambda, \Lambda) \otimes \Delta$$

Hence, we see

$$\Hom(\Lambda, \Lambda) \cong \Hom(\Lambda, \Lambda) \otimes \Delta$$

Then by Lemma 5 and the above isomorphism, we have

$$\Gamma \cong \Hom(\Lambda, \Lambda) \otimes \Delta$$

Thus $\Lambda$ is a strong Frobenius extension of $\Gamma$. For the rest of the proof, see Theorem 6.1 [7].

**Theorem 4.** Let $\Lambda$ be an $H$-separable extension of $\Gamma$ with $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ and $\Omega$, $\Delta$, and $C$ be as in Lemma 5. Then, if $\Delta$ is a symmetric (resp. Frobenius or quasi-Frobenius) C-algebra, the following conditions are equivalent:

1) $\Lambda$ is left $\Gamma$-finitely generated projective.
2) $\Lambda$ is right $\Gamma$-finitely generated projective.
3) $\Lambda$ is a strong Frobenius (resp. Frobenius or quasi-Frobenius) extension of $\Gamma$.

Proof. Suppose $\Delta$ is a symmetric C-algebra. Then $\Omega$ is a symmetric extension of $\Lambda$, since $\Omega \cong \Lambda \otimes C \Delta$ and $\Delta$ is C-symmetric. Hence we have

$$\psi \circ \Hom(\Lambda, \Lambda) \otimes \Delta \cong \psi \circ \Omega \otimes \Delta$$

Then by Lemma 5 and the above isomorphism, we have

$$\Gamma \cong \Hom(\Lambda, \Lambda) \otimes \Delta$$

Thus $\Lambda$ is a strong Frobenius extension of $\Gamma$. For the rest of the proof, see Theorem 6.1 [7].

**Corollary 2.** Let $\Lambda$ be an $H$-separable extension of $\Gamma$ such that $\Gamma$ is a $\Gamma$-$\Gamma$-direct summand of $\Lambda$. Then the following conditions are equivalent.

1) $\Lambda$ is left $\Gamma$-finitely generated projective.
2) $\Lambda$ is right $\Gamma$-finitely generated projective.
3) $\Lambda$ is a strong Frobenius extension of $\Gamma$.

Proof. Since $\Gamma$ is a $\Gamma$-$\Gamma$-direct summand of $\Lambda$, $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ by Prop. 1.2 [10] and $\Delta$ is C-separable by Prop. 4.7 [4]. Thus $\Delta$ is C-symmetric by Theorem 4.2 [1], and we can apply Theorem 4.

The converse of Theorem 4 holds for $H$-separable extension as follows.
Theorem 5. Let \( \Lambda \) be an \( H \)-separable extension of \( \Gamma \). Then if \( \Lambda \) is a strong Frobenius (resp. Frobenius or left or right quasi-Frobenius) extension of \( \Gamma \), \( \Delta \) is a symmetric (resp. Frobenius or quasi-Frobenius) \( C \)-algebra, where \( \Delta = V_\Lambda(\Gamma) \) and \( C \) is the center of \( \Lambda \).

Proof. Suppose \( \Lambda \) is a strong Frobenius extension of \( \Gamma \). Then there exists a \( \Lambda - (\Delta - \Gamma) \)-isomorphism \( \Lambda \Lambda_{\Delta - \Gamma} \cong \Lambda \text{Hom}(\Gamma \Lambda, \Gamma \Gamma)_{\Delta - \Gamma} \). Then, since \( \Gamma \Lambda \) is finitely generated projective,

\[
\Lambda_{\Delta} \otimes_{\Gamma} \Lambda_{\Delta - \Delta} \cong \Lambda_{\Delta} \text{Hom}(\Gamma_{\Gamma}, \Lambda_{\Gamma}) \otimes_{\Gamma} \Lambda_{\Delta - \Delta} \cong \Lambda_{\Delta} \text{Hom}(\text{Hom}(\Gamma_{\Gamma}, \Lambda_{\Gamma}), \Lambda_{\Gamma})_{\Delta - \Delta}
\]

On the other hand, since \( \Lambda \) is \( H \)-separable over \( \Gamma \), there exist \( \Lambda - (\Delta - \Gamma) \)-(\( \Lambda - (\Delta - \Gamma) \))-isomorphisms

\[
\xi: \otimes_{\Gamma} \Lambda \to \text{Hom}(\Delta, \Lambda) \quad \xi(x \otimes y)(d) = xdy \quad \text{for} \quad x, y \in \Lambda \text{ and } d \in \Delta
\]

\[
\eta: \otimes_{\Gamma} \Delta \to \text{Hom}(\Lambda_{\Gamma}, \Lambda_{\Gamma}) \quad \eta(x \otimes d)(y) = xyd \quad \text{for} \quad x, y \in \Lambda \text{ and } d \in \Delta
\]

Hence we have \( \Lambda - (\Delta - \Gamma) \)-(\( \Lambda - (\Delta - \Gamma) \))-isomorphisms

\[
\text{Hom}(\Delta, \Lambda) \cong \otimes_{\Gamma} \Lambda \cong \text{Hom}(\Lambda_{\Gamma}, \Lambda_{\Gamma}) \cong \otimes_{\Gamma} \Delta
\]

Then, taking \( \text{Hom}(\ast, \Delta \Lambda) \), we obtain \( \Delta - \Delta \)-isomorphisms

\[
\Delta \text{Hom}(\Delta, \Lambda) \cong \Delta \otimes_{\Gamma} \Delta \cong \Delta \text{Hom}(\Lambda_{\Gamma}, \Lambda_{\Gamma}) \cong \Delta \otimes_{\Gamma} \Delta
\]

since \( \Delta \) is \( C \)-finitely generated projective, and

\[
\Delta \text{Hom}(\Delta \otimes \Delta, \Lambda \Delta) \cong \Delta \text{Hom}(\Delta, \text{cHom}(\Lambda \Lambda, \Lambda \Delta)) \cong \Delta \text{Hom}(\Delta, \Delta)
\]

Thus we see \( \Delta \text{Hom}(\Delta, \Delta) \cong \Delta \Delta \), which means that \( \Delta \) is a symmetric \( C \)-algebra. In case of Frobenius extension, \( \Lambda_{\Gamma} = \Delta \text{Hom}(\Gamma \Lambda, \Gamma \Gamma)_{\Gamma} \) induces

\[
\Lambda \otimes_{\Gamma} \Lambda_{\Delta - \Delta} \cong \Lambda \text{Hom}(\text{Hom}(\Gamma \Lambda, \Gamma \Gamma)_{\Gamma}, \Lambda_{\Gamma})_{\Delta - \Delta} \cong \Lambda \text{Hom}(\Lambda_{\Gamma}, \Lambda_{\Gamma})_{\Delta - \Delta}
\]

where \( \Lambda \otimes_{\Gamma} \Lambda_{\Delta - \Delta} \) is induced by \( \Lambda \Lambda_{\Delta - \Delta} \) and \( \Gamma \Lambda \), while in case of right quasi-Frobenius extension \( \Delta \text{Hom}(\Gamma \Lambda, \Gamma \Gamma)_{\Gamma} \otimes_{\Gamma} (\Delta \otimes \Lambda) \) induces

\[
\Lambda \otimes_{\Gamma} \Lambda_{\Delta - \Delta} \cong \Lambda \text{Hom}(\text{Hom}(\Gamma \Lambda, \Gamma \Gamma)_{\Gamma}, \Lambda_{\Gamma})_{\Delta - \Delta} \otimes_{\Gamma} (\text{cHom}(\Lambda_{\Gamma}, \Lambda_{\Gamma}))_{\Delta - \Delta}
\]

Then the same argument as in the case of strong Frobenius extension proves the theorem in both cases.

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References


