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# UNTWISTING NUMBER AND BLANCHFIELD PAIRINGS

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## Abstract

In this note we use Blanchfield forms to study knots that can be turned into an unknot using a single  $\bar{t}_{2k}$  move.

## 1. Overview

Let  $K \subset S^3$  be a knot and  $k \in \mathbb{Z} \setminus \{0\}$ . In this paper by a  $k$ -twisting move we mean a move depicted in Fig. 1, that is, a full right  $k$ -twist on two strands of  $K$  going in the opposite direction (in [16] this move is called a  $\bar{t}_{2k}$ -move). We will call a knot  $k$ -simple, if it can be unknotted by a single  $k$ -untwisting move. A knot is algebraically  $k$ -simple, if a single  $k$ -untwisting move turns it into a knot with Alexander polynomial 1.

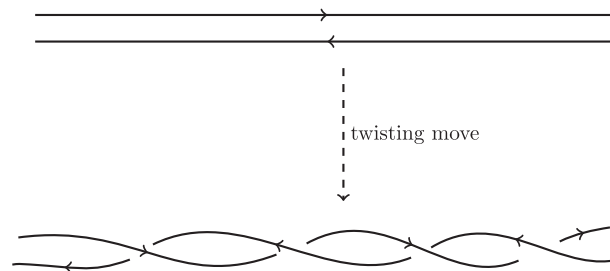


Fig. 1. A  $k$ -twisting move for  $k = 2$ . Note that the strands in the picture go in different directions.

Our first result gives an obstruction to the untwisting move in terms of the algebraic unknotting number [7, 15, 20].

**Theorem 1.1.** *Suppose  $K$  is an algebraically  $k$ -simple knot. If  $k$  is odd, then  $K$  can be turned into a knot with Alexander polynomial 1 using at most two crossing changes. If  $k$  is even, then at most three crossing changes are enough to turn  $K$  into a knot with Alexander polynomial 1.*

Our second result restricts the homology of the double branched cover of an algebraically  $k$ -simple knot.

**Theorem 1.2.** *Suppose  $K$  is an algebraically  $k$ -simple knot. Denote by  $\Sigma(K)$  the double branched cover of  $K$ . Then  $H_1(\Sigma(K); \mathbb{Z})$  is cyclic.*

Both Theorem 1.1 and Theorem 1.2 follow from the following result, which is the main technical result of this paper.

**Theorem 1.3.** *Suppose  $K$  is an algebraically  $k$ -simple knot. Then there exists a polynomial  $\alpha(t) \in \mathbb{Z}[t, t^{-1}]$  satisfying  $\alpha(1) = 0$ ,  $\alpha(t^{-1}) = \alpha(t)$ , such that the matrix*

$$\begin{pmatrix} \alpha(t) & 1 \\ 1 & -k \end{pmatrix}$$

*represents the Blanchfield pairing for  $K$ .*

Theorem 1.3 can be regarded as a generalization of [16, Theorem 3.2(b)].

It is possible to generalize the techniques used in this paper to study knots that are untwisted with several  $t_{2k}$  moves, possibly with varying the twisting coefficients  $k$ . This generalization is straightforward, we omit it to make the paper shorter and more concise.

Proof of Theorem 1.3 is given in Section 3. Proof of Theorem 1.1 is given in Section 4. Section 5 contains the proof of a stronger version of Theorem 1.2.

## 2. Blanchfield pairing

Let  $K \subset S^3$  be a knot and let  $M_K$  denote its zero-framed surgery. Denote by  $\widetilde{M}_K$  the universal abelian cover of  $M_K$ . The chain complex  $C_*(\widetilde{M}_K; \mathbb{Z})$  admits the action of the deck transform and thus it has a structure of a  $\Lambda$ -module, where  $\Lambda = \mathbb{Z}[t, t^{-1}]$ . The homology of this complex, regarded as a  $\Lambda$ -module, is denoted by  $H_*(M_K; \Lambda)$ . The module  $H_1(M_K; \Lambda)$  is called the *Alexander module* of the knot  $K$ .

**REMARK 2.1.** Usually the Alexander module is defined using knot complements instead of zero-framed surgeries, but the two definitions are equivalent; see e.g. [10].

The ring  $\Lambda$  has a naturally defined convolution  $t \mapsto t^{-1}$ . The Blanchfield pairing defined in [1] for  $K$  is a sesquilinear symmetric pairing  $H_1(M_K; \Lambda) \times H_1(M_K; \Lambda) \rightarrow Q/\Lambda$ , where  $Q$  is the field of fractions for  $\Lambda$ . We refer to [10, 13] for a precise and detailed construction of the Blanchfield pairing and [5, 6] for generalizations.

**DEFINITION 2.2.** We say that an  $n \times n$  matrix  $A$  with entries in  $\Lambda$  *represents* the Blanchfield pairing if  $H_1(M_K; \Lambda) \cong \Lambda^n / A\Lambda^n$  as a  $\Lambda$ -module, under this identification the Blanchfield pairing has form  $(a, b) \mapsto \bar{a}^T A^{-1} b$  and moreover  $A(1)$  is diagonalizable over  $\mathbb{Z}$ .

It is known, see [14], that every Blanchfield pairing can be represented by a finite matrix. The minimal size of a matrix representing the Blanchfield pairing of a knot is denoted by  $n(K)$ . It is equal to the algebraic unknotting number  $u_a(K)$ ; see [2, 4].

The invariant  $n(K)$  can also be generalized for other coefficient rings. In this paper we restrict to rings that are subrings of  $\mathbb{C}$ . If  $R$  is such a ring, we denote by  $n_R(K)$  the minimal size of a matrix over  $R[t, t^{-1}]$  representing the Blanchfield pairing over  $R[t, t^{-1}]$ . We have that  $n_R(K) \leq n_{R'}(K)$  if  $R'$  is a subring of  $R$ . Often  $n_R(K)$  is easier to compute than  $n(K) = n_{\mathbb{Z}}(K)$ , for example the value of  $n_{\mathbb{R}}$  can be calculated from the Tristram–Levine signature [3]. One motivation of this paper is to give a geometric interpretation of  $n_R(K)$  for some rings  $R$ .

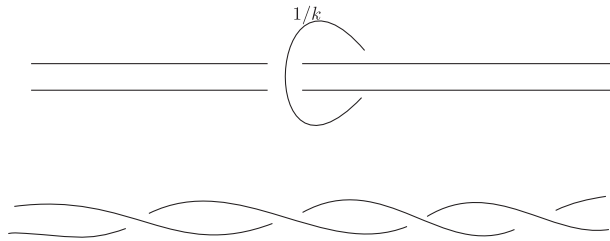


Fig. 2. The  $1/k$  surgery on the circle in the top picture induces  $k$  full left twists of the two strands passing through the circle.

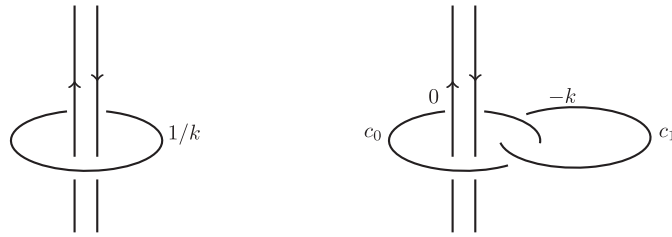


Fig. 3. Changing a  $1/k$  surgery on a circle to a surgery on a two-component link with framings  $0$  and  $-k$ .

**3. Proof of Theorem 1.3**

The main ingredient in the proof of Theorem 1.3 is the following.

**Theorem 3.1** (see [4, Theorem 2.6]). *Suppose  $W_K$  is a topological four-manifold such that  $\partial W_K = M_K$ ,  $\pi_1(W_K) = \mathbb{Z}$  and the inclusion induced map  $H_1(M_K; \mathbb{Z}) \rightarrow H_1(W_K; \mathbb{Z})$  is an isomorphism. Then  $H_2(W_K; \Lambda)$  is free of rank  $b_2(W_K)$ . Moreover if  $A$  is matrix over  $\Lambda$  representing the twisted intersection form on  $H_2(W_K; \Lambda)$  in some basis of  $H_2(W_K; \Lambda)$ , then  $A$  also represents the Blanchfield pairing on  $M_K$  (in the sense of Definition 2.2).*

In the light of Theorem 3.1, the proof of Theorem 1.3 consists of constructing an appropriate manifold  $W_K$  and applying Theorem 3.1. The construction begins with noticing that the twisting move can be realized by a surgery. Namely we have the following well-known fact.

**Proposition 3.2.** *A  $k$ -twisting move can be realized by a  $-1/k$  surgery on a knot. That is, if  $K_2$  arises from  $K_1$  by a  $k$ -twisting move, then there is a simple closed circle  $C$  disjoint from  $K_1$ , such that  $C$  bounds a smooth disk intersecting  $K_2$  at two points with opposite signs and such that the  $-1/k$  surgery on  $C$  transforms  $K_1$  into  $K_2$ ; see Fig. 2*

**REMARK 3.3.** The move described in Fig. 2 is a special case of the Rolfsen twist, see [12, Figure 5.27]. It can be seen on [21, Figure 3.12] that the surgery with a positive coefficient (i.e. the  $1/k$  surgery if  $k > 0$ ) gives rise to a left  $k$ -twist and the surgery with a negative coefficient (i.e. the  $-1/k$  surgery with  $k > 0$ ) gives rise to a right  $k$ -twist.

The surgery in Fig. 2 can be changed into a surgery with integer coefficients as in Fig. 3 by a ‘slam-dunk’ operation, see [12, Section 5.3].

Suppose  $J$  is a knot with Alexander polynomial  $1$  and  $K$  is a knot resulting from  $J$  by

applying a full left  $k$ -twist (so  $J$  is obtained from  $K$  by a full right  $k$ -twist). Let  $M_J$  be the zero-surgery on  $J$  and  $M_K$  the zero-surgery on  $K$ . By [11, Theorem 117B]  $M_J$  is a boundary of a topological four-manifold that is a homotopy  $D^3 \times S^1$ . Denote this four-manifold by  $W_J$ .

A full left  $k$ -twist on  $J$  can be realized as a surgery on a two-component link with framings 0 and  $-k$  as in Fig. 3. Let  $c_0$  and  $c_1$  denote the components of this link. The curve  $c_0$  has framing 0,  $c_1$  has framing  $k$ . Both  $c_0$  and  $c_1$  are curves disjoint from  $J$ , so we can and will assume that they are separated from a small neighborhood of  $J$  in  $S^3$ . Performing a 0-surgery on  $J$  does not affect these curves, therefore  $c_0$  and  $c_1$  can also be viewed as curves on  $M_J$ . Now performing surgery on  $c_0$  and  $c_1$  with coefficients 0 and  $-k$ , respectively, produces  $M_K$ .

The trace of the surgery on  $c_0$  and  $c_1$  yields a cobordism between  $M_J$  and  $M_K$ . Call this cobordism  $W_{JK}$ . Define now

$$W_K = W_J \cup W_{JK}$$

so that  $\partial W_K = M_K$ . We have the following fact.

**Lemma 3.4.** *We have  $\pi_1(W_K) \cong \mathbb{Z}$ ,  $H_1(W_K; \mathbb{Z}) \cong \mathbb{Z}$  and the inclusion of  $M_K$  to  $W_K$  induces an isomorphism on the first homology. Moreover  $H_2(W_K; \mathbb{Z}) \cong \mathbb{Z}^2$  and there are generators of  $H_2(W_K; \mathbb{Z})$  that are represented by immersed spheres.*

*Proof.* The homology groups of  $W_K$  are calculated using the Mayer-Vietoris sequence. The manifold  $W_K$  is obtained from  $W_J$  by adding two-handles along null-homologous curves  $c_0$  and  $c_1$ . This shows that  $H_1(W_K; \mathbb{Z}) \cong \mathbb{Z}$  and  $H_2(W_K; \mathbb{Z}) \cong \mathbb{Z}^2$ .

To compute  $\pi_1$  we observe that  $\pi_1(W_J) \cong \mathbb{Z}$ . Hence  $c_0, c_1$  being null-homologous are also null-homotopic. Van Kampen theorem implies that  $\pi_1(W_K) \cong \mathbb{Z}$ .

To show that the generators of  $H_2(W_K; \mathbb{Z})$  can be chosen to be spherical we again use the fact that  $c_0$  and  $c_1$  are null-homotopic in  $W_J$ . This implies that  $c_0$  and  $c_1$  bound disks  $D_0$  and  $D_1$  in  $W_J$ . The disk  $D_1$  can be chosen to be the obvious disk on  $M_J$ , but  $D_0$  is in general only an immersed disk and it cannot lie on  $M_J$  (because in general  $c_0$  is not null-homotopic on  $M_J$ ). We can form spheres  $\Sigma_0$  and  $\Sigma_1$  by adding to  $D_0$  and  $D_1$  the cores of the two-handles that are attached. It is clear that the homology classes  $[\Sigma_0]$  and  $[\Sigma_1]$  generate  $H_2(W_K; \mathbb{Z})$ . Moreover, by construction,  $\Sigma_1$  is a smoothly embedded sphere and  $\Sigma_0$  can be chosen to intersect  $\Sigma_1$  precisely at one point.

Finally, in order to prove that the inclusion induced map  $H_1(M_K; \mathbb{Z}) \rightarrow H_1(W_K; \mathbb{Z})$  is an isomorphism, invert the cobordism  $W_{JK}$ , that is, present  $W_{JK}$  as  $M_K \times [0, 1]$  with two two-handles attached. The attaching curves of these handles are homologically trivial (but not necessarily homotopy trivial,  $\pi_1(M_K)$  can be complicated), hence the boundary inclusion induces an isomorphism  $H_1(M_K; \mathbb{Z}) \cong H_1(W_{JK}; \mathbb{Z})$ . Clearly  $H_1(W_{JK}; \mathbb{Z}) \cong H_1(W_K; \mathbb{Z})$ .  $\square$

Lemma 3.4 gives us two spheres  $\Sigma_0, \Sigma_1 \subset W_K$ , which are the generators of  $H_2(W_K; \mathbb{Z})$ . Choose a basepoint  $x_0 = \Sigma_0 \cap \Sigma_1$ . This choice allows us to consider  $\Sigma_0$  and  $\Sigma_1$  as elements of  $\pi_2(W_K, x_0)$ .

**Lemma 3.5.** *The group  $\pi_2(W_K, x_0)$  is freely generated as a  $\Lambda = \mathbb{Z}[\pi_1(W_K, x_0)]$ -module by classes of  $\Sigma_0$  and  $\Sigma_1$ . In particular  $\pi_2(W_K, x_0) \cong \Lambda^2$ .*

Proof. The space  $W_K$  is obtained from  $W_J$  by attaching two two–handles along null-homotopic curves  $c_0$  and  $c_1$ . We have that  $\pi_1(W_J) = \mathbb{Z}$  and  $\pi_2(W_J) = 0$  by definition. The statement follows from [18, Proposition 3.30].  $\square$

We will use Lemma 3.5 in connection with the following well-known result.

**Lemma 3.6.** *We have an isomorphism of  $\Lambda$ –modules  $\pi_2(W_K, x_0) \cong \pi_2(\widetilde{W}_K, \widetilde{x}_0) \cong H_2(\widetilde{W}_K; \mathbb{Z}) \cong H_2(W_K; \Lambda)$ .*

Proof. The first isomorphism in the lemma is the isomorphism of higher homotopy groups under the covering map. The second is the Hurewicz isomorphism because  $\widetilde{W}_K$  is simply connected. The third isomorphism is the definition of the twisted homology groups.  $\square$

In particular, Lemma 3.5 together with Lemma 3.6 gives a simple and independent argument that  $H_2(W_K; \Lambda)$  is a free  $\Lambda$ –module, compare [4, Lemma 2.7].

**Corollary 3.7.** *The (classes of the) lifts of  $\Sigma_0$  and  $\Sigma_1$  to  $\widetilde{W}_K$  generate  $H_2(W_K; \Lambda)$  as a  $\Lambda$ –module.*

Recall that  $A(t)$  is a matrix over  $\Lambda$  representing the intersection form on  $H_2(W_K; \Lambda)$ . The following result together with Theorem 3.1 gives the proof of Theorem 1.3 from the introduction.

**Theorem 3.8.** *The matrix  $A(t)$  has form*

$$\begin{pmatrix} \alpha(t) & 1 \\ 1 & -k \end{pmatrix},$$

where  $\alpha(t) \in \Lambda$  is such that  $\alpha(1) = 0$  and  $\alpha(t^{-1}) = \alpha(t)$ .

Proof. By Corollary 3.7 the entries of  $A(t)$  are twisted intersection indices of  $\Sigma_0$  and  $\Sigma_1$ . For example, the bottom-right entry of  $A(t)$  is equal to the twisted intersection index of  $\Sigma_1$  and  $\Sigma'_1$ , where  $\Sigma'_1$  is a small perturbation of  $\Sigma_1$  intersecting  $\Sigma_1$  in finitely many points.

To compute the twisted intersection index of  $\Sigma_1$  and  $\Sigma'_1$ , choose a basing for  $\Sigma_1, \Sigma'_1$ , that is a path  $\gamma$  from  $x_0$  to  $\Sigma_1$  and a path  $\gamma'$  from  $x_0$  to  $\Sigma'_1$ . Let  $x, x'$  be the end points of  $\gamma$  and  $\gamma'$ .

For any intersection point  $y \in \Sigma_1$  and  $\Sigma'_1$  we choose a smooth path  $\rho_y$  from  $x$  to  $y$  on  $\Sigma_1$  and a path  $\rho'_y$  from  $x'$  to  $y$  on  $\Sigma'_1$ ; see Fig. 4.

Let  $\theta_y$  be the loop  $\gamma\rho_y(\rho'_y)^{-1}(\gamma')^{-1}$ . Define  $m_y \in \mathbb{Z}$  to be the homology class of  $\theta_y$  in  $H_1(W_K; \mathbb{Z}) \cong \mathbb{Z}$ . Finally, let  $\epsilon_y$  be the sign of the intersection point  $y$  assigned in the usual way, that is, if  $T_y\Sigma_1 \oplus T_y\Sigma'_1 = T_yW_K$  agrees with the orientation, we set  $\epsilon_y = +1$ , otherwise we set  $\epsilon_y = -1$ .

Given these definitions, the twisted intersection index of  $\Sigma_1$  and  $\Sigma'_1$  is equal to

$$(3.9) \quad \sum_{y \in \Sigma_1 \cap \Sigma'_1} \epsilon_y t^{m_y} \in \mathbb{Z}[t, t^{-1}].$$

In general this sum might depend on the choice of  $\rho_y$  and  $\rho'_y$ . However if any smooth closed curve on  $\Sigma_1$  and on  $\Sigma'_1$  is homologically trivial in  $W_K$  (in the language of [2, Section 3.2] this means that  $\Sigma_1$  and  $\Sigma'_1$  are homologically invisible in  $W_K$ ), the definition does not depend on paths  $\rho_y$  and  $\rho'_y$ . In the present situation  $\Sigma_1$  and  $\Sigma'_1$  are immersed (and even embedded)

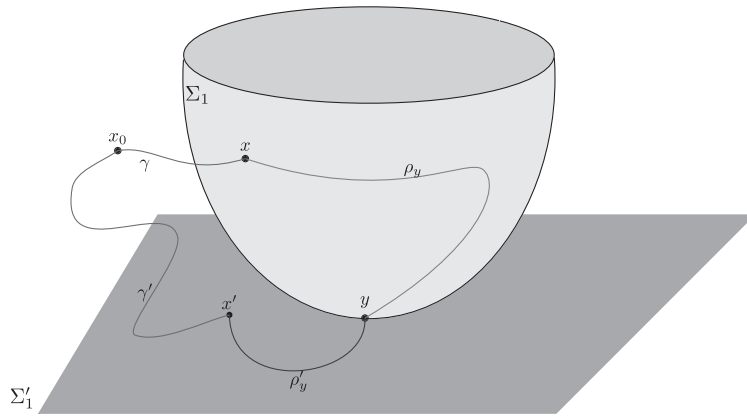


Fig. 4. Notation in proof of Theorem 3.8. In the four-dimensional situation the intersection of  $\Sigma$  and  $\Sigma'$  at  $y$  is transverse.

spheres, so they are homologically invisible, in particular (3.9) is a well-defined Laurent polynomial.

As  $\Sigma_1$  and  $\Sigma'_1$  are embedded spheres, we claim more, namely that  $m_y$  does not depend on  $y$ . In fact, suppose  $z$  is another intersection point of  $\Sigma_1$  and  $\Sigma'_1$ . Consider the curve  $\delta = (\rho'_y)^{-1}\rho'_z\rho_z^{-1}\rho_y$  in  $W_K$ . As  $\Sigma'_1$  is a perturbation of  $\Sigma_1$ , the path  $(\rho'_y)^{-1}\rho'_z$  can be pushed by a homotopy (in  $W_K$ ) to a path  $\tilde{\rho}$  on  $\Sigma_1$  having the same endpoints. Then  $\tilde{\rho}\rho_z^{-1}\rho_y$  is a loop homotopically equivalent to  $\delta$ , but this is a loop on a smoothly embedded sphere  $\Sigma_1$ . Hence it is contractible in  $W_K$ . This shows that  $m_y = m_z$ .

We conclude that the twisted intersection index of  $\Sigma_1$  and  $\Sigma'_1$  is equal to the standard intersection number of  $\Sigma_1$  and  $\Sigma'_1$  (which is equal to the self-intersection of  $\Sigma_1$ , that is  $-k$ ) multiplied by  $t^{m_y}$ . We can choose a basing for  $\Sigma'_1$  in such a way that  $m_y = 0$ .

An analogous, but simpler argument shows that  $\Sigma_0 \cdot \Sigma_1 = \pm 1$ . Indeed by construction  $\Sigma_0 \cap \Sigma_1$  consists of a single point. It follows that the twisted intersection between  $\Sigma_0$  and  $\Sigma_1$  is  $\pm t^m$  for some  $m$ . We choose a basing for  $\Sigma_0$  in such a way that  $m = 0$ . We can also choose an orientation of  $\Sigma_0$  in such a way that the sign is positive.

It remains to discuss the properties of  $\alpha(t)$ , that is, the top-left entry of  $A(t)$ . First we notice that as the twisted intersection pairing is sesquilinear, the matrix  $A(t)$  is hermitian and so  $\alpha(t^{-1}) = \alpha(t)$ . Moreover, it follows from the construction of  $A(t)$  sketched above that  $A(t = 1)$  is a matrix of classical intersection indices between  $\Sigma_0$  and  $\Sigma_1$ . Therefore  $\alpha(1)$  is the self-intersection of  $\Sigma_0$ . This is the same number as the framing of the curve  $c_0$ . This concludes the proof that  $\alpha(1) = 0$ . □

REMARK 3.10. There is an alternative calculation of the matrix  $A$  using Rolfsen's argument [19]. However one still has to make some effort proving that  $A$  represents not only the Alexander module, but also the Blanchfield pairing.

#### 4. Proof of Theorem 1.1

We begin with proving Theorem 1.1. The following corollary deals with the first part of this theorem.

**Corollary 4.1.** *Suppose  $K$  is an algebraically  $k$ -simple and  $k$  is odd. Then there are at most two crossing changes that turn  $K$  into a knot with Alexander polynomial 1.*

Proof. We have  $A(1) = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}$ . As  $k$  is odd, this matrix is diagonalizable over  $\mathbb{Z}$ . By [2, Theorem 1.1] we infer that the algebraic unknotting number of  $K$  is at most 2.  $\square$

If  $k$  is even, then  $A(1)$  is not diagonalizable over  $\mathbb{Z}$ , because it is an even symmetric form. However the block sum  $A(1) \oplus (1)$  is diagonalizable. The block matrix  $A(t) \oplus (1)$  is a  $3 \times 3$  matrix over  $\Lambda$  representing the Blanchfield pairing, so the algebraic unknotting number of  $K$  is bounded from above by 3. This shows the second part of Theorem 1.1.

We have the following consequence of Theorem 1.3.

**Theorem 4.2.** *Suppose  $K$  can be algebraically  $k$ -simple. Let  $R_k = \mathbb{Z}[\frac{1}{k}]$ . Then  $n_{R_k}(K) = 1$ .*

Proof. By Theorem 1.3 we know that the Blanchfield pairing over  $\mathbb{Z}$  can be represented by a matrix of form  $\begin{pmatrix} \alpha(t) & 1 \\ 1 & -k \end{pmatrix}$ . The same matrix represents the Blanchfield pairing over  $R_k$ , but over  $R_k$  this matrix is congruent to a matrix  $\begin{pmatrix} \tilde{\alpha}(t) & 0 \\ 0 & 1 \end{pmatrix}$  for  $\tilde{\alpha}(t) \in R_k[t, t^{-1}]$ . By [17, Proposition 1.7.1] (see also [4, Proposition 3.1]) the matrix  $(\tilde{\alpha}(t))$  also represents the Blanchfield pairing over  $R_k[t, t^{-1}]$ .  $\square$

The following corollary is well known, see [16].

**Corollary 4.3.** *If  $K$  is algebraically  $k$ -simple, then its Alexander polynomial is equal to  $\Delta_K(t) = 1 + k\alpha(t)$ , where  $\alpha(t) \in \mathbb{Z}[t, t^{-1}]$ .*

Proof. This follows from Theorem 4.1 because if  $A(t)$  represents the Blanchfield pairing of a knot  $K$ , then  $\Delta_K(t) = \det A(t)$  up to multiplication by a unit in  $\Lambda$ .  $\square$

### 5. Linking forms

An abstract *linking pairing* is a pair  $(H, l)$ , where  $H$  is a finite abelian group of an odd order and  $l$  is a bilinear symmetric pairing  $l: H \times H \rightarrow \mathbb{Q}/\mathbb{Z}$ . As a model example, if  $Y$  is a closed three-manifold with  $b_1(Y) = 0$ , there is defined a linking pairing  $l(Y)$  on  $H = H_1(Y; \mathbb{Z})$ . If  $Y = \Sigma(K)$  is the double branched cover of a knot  $K$ , we denote this pairing by  $l(K)$ . It is known that the linking pairing  $l(K)$  is represented by  $V + V^T$ , where  $V$  is the Seifert matrix for  $K$ . The meaning of ‘represented’ is explained in the following definition.

**DEFINITION 5.1.** Let  $P$  be an  $n \times n$  matrix with integer coefficients and such that  $\det P$  is odd. The *linking form represented by  $P$*  is the pair  $(H(P), l(P))$ , where  $H(P) = \mathbb{Z}^n/P\mathbb{Z}^n$  and  $l(P)$  is the bilinear form defined by

$$\begin{aligned} \mathbb{Z}^n/P\mathbb{Z}^n \times \mathbb{Z}^n/P\mathbb{Z}^n &\rightarrow \mathbb{Q}/\mathbb{Z} \\ (a, b) &\mapsto a^T P^{-1} b \pmod{1}. \end{aligned}$$

We have the following relation between the Blanchfield form for  $K$  and the linking form  $l(K)$ .



**Proposition 5.2** (see [4, Lemma 3.3]). *If  $A$  is a matrix over  $\Lambda$  representing the Blanchfield pairing, then  $l(A(-1)) = 2l(K)$ .*

Here  $2l(K)$  means the linking pairing with the same underlying group as  $l(K)$ , but the linking form is multiplied by 2; compare [4, Section 3].

We can use this result to obtain the following corollary.

**Corollary 5.3.** *Suppose  $K$  is algebraically  $k$ -simple. Then the linking form  $2l(K)$  is isometric to the linking form represented by*

$$(5.4) \quad B = \begin{pmatrix} d & 1 \\ 1 & -k \end{pmatrix},$$

where  $d = \alpha(-1) \in \mathbb{Z}$  is such that  $-(dk + 1)$  is the (signed) determinant of  $K$ .

As in [4, Section 5.2] we can use Corollary 5.3 to obstruct untwisting number 2.

From Corollary 5.3 we immediately recover Theorem 1.2 from the introduction.

**Proposition 5.5.** *If  $K$  is algebraically  $k$ -simple and  $\Sigma(K)$  is the double branched cover, then  $H_1(\Sigma(K); \mathbb{Z})$  is cyclic.*

**REMARK 5.6.** It follows that Wendt's criterion for the unknotting number [22] coming from the double branched covers, does not distinguish between knots that have unknotting number 1 and knots that are algebraically  $k$ -simple for some  $k$ .

**Proof of Proposition 5.5.** By Corollary 5.3 we infer that  $H_1(\Sigma(K); \mathbb{Z}) \cong \mathbb{Z}^2/B\mathbb{Z}^2$ , where  $B$  is as in (5.4). Subtract from the first column of  $B$  the second column multiplied by  $d$  to obtain the matrix  $\begin{pmatrix} 0 & 1 \\ 1+dk & -k \end{pmatrix}$ . Then add to the second row the first one multiplied by  $k$ . We obtain the matrix

$$B' = \begin{pmatrix} 0 & 1 \\ 1+dk & 0 \end{pmatrix}.$$

Row and column operations on matrices do not affect the cokernel, hence  $\mathbb{Z}^2/B'\mathbb{Z}^2 \cong \mathbb{Z}^2/B\mathbb{Z}^2$ . Evidently we have  $\mathbb{Z}^2/B'\mathbb{Z}^2 \cong \mathbb{Z}/|dk + 1|\mathbb{Z}$ .  $\square$

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