

Title	Torelli theorem for surfaces with $p_g = c^2_1 = 1$ and K ample and with certain type of automorphism
Author(s)	Usui, Sampei
Citation	Compositio Mathematica. 1982, 45(3), p. 293-314
Version Type	VoR
URL	https://hdl.handle.net/11094/73373
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

COMPOSITIO MATHEMATICA

SAMPEI USUI

Torelli theorem for surfaces with $p_g = c_1^2 = 1$ and K ample and with certain type of automorphism

Compositio Mathematica, tome 45, n° 3 (1982), p. 293-314

http://www.numdam.org/item?id=CM_1982__45_3_293_0

© Foundation Compositio Mathematica, 1982, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

TORELLI THEOREM FOR SURFACES WITH $p_g = c_1^2 = 1$ AND K AMPLE AND WITH CERTAIN TYPE OF AUTOMORPHISM

Sampei Usui

0. Introduction

The moduli space of isomorphism classes of surfaces with $p_g = c_1^2 = 1$ is studied by Catanese in [2]. Every such surface with the ample canonical divisor can be represented as a smooth weighted complete intersection of type (6, 6) in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$ parametrized by a Zariski open set $U \subset \mathbf{A}^{26}$ (cf. (1.3)). This leads to a universal family

$$\pi': \mathcal{X}' \rightarrow U.$$

There is an 8-dimensional subgroup G of $\text{Aut}(\mathbf{P})$ (cf. (1.5) and (1.6)) acting on U with finite isotropy groups and

$$M = U/G = \begin{array}{l} \text{the moduli space of canonical surfaces} \\ \text{with } p_g = c_1^2 = 1. \end{array}$$

In particular, $\dim_{\mathbf{C}} M = 18$.

The period domain D , which parametrizes polarized Hodge structures on the second primitive cohomology groups of the surfaces in question, is isomorphic to

$$\{[a] \in \mathbf{P}(L \otimes \mathbf{C}) \mid (a, a) = 0, (a, \bar{a}) > 0\}$$

where L is a free \mathbf{Z} -module of rank 20 equipped with a symmetric bilinear form $(\ , \)$ of signature $(2, 18)$. The group $\Gamma = \text{Aut}(L)$ acts properly discontinuously on D .

Set

$$\tilde{U} = \{(u, \alpha) \mid u \in U, \alpha \in \text{Isom}(P^2(X_u, \mathbb{Z}), L)\} \quad \text{and} \quad \tilde{\mathcal{X}}' = \mathcal{X}' \times_U \tilde{U}.$$

Then we have the universal family

$$(0.1) \quad \tilde{\pi} : \tilde{\mathcal{X}} = \tilde{\mathcal{X}}'/G \rightarrow \tilde{M} = \tilde{U}/G$$

of marked canonical surfaces with $p_g = c_1^2 = 1$ (cf. Proposition (2.24) in [11]). \tilde{M} and $\tilde{\mathcal{X}}$ are complex manifolds and this family serves as a universal family of the deformations of the surfaces in question. This gives a period map

$$\Phi : \tilde{M} \rightarrow D.$$

Catanese has shown in [2] (cf. also [12]) that Φ has non-empty ramification locus $\tilde{\Delta} \subset \tilde{M}$. Thus the local Torelli fails at $\tilde{m} \in \tilde{\Delta}$. The problem then is to study how badly it can fail. First of all observe that

$$\dim \text{Ker } d\Phi(\tilde{m}) \leq 2.$$

This directly follows from the exact sequence

$$0 \longrightarrow H^0(C_{\tilde{m}}, \Omega_{X_{\tilde{m}}}^1 \otimes \mathcal{O}_{C_{\tilde{m}}}) \longrightarrow H^1(X_{\tilde{m}}, T_{X_{\tilde{m}}}) \xrightarrow{d\Phi(m)} H^1(X_{\tilde{m}}, \Omega_{X_{\tilde{m}}}^1)$$

together with the fact that $h^0(C_{\tilde{m}}, \Omega_{X_{\tilde{m}}}^1 \otimes \mathcal{O}_{C_{\tilde{m}}}) \leq h^0(C_{\tilde{m}}, \Omega_{C_{\tilde{m}}}^1) = 2$, where $C_{\tilde{m}}$ is the canonical curve of $X_{\tilde{m}}$. This means that the fibre of Φ through $\tilde{m} \in \tilde{M}$ has at most dimension 2. Todorov ([9]) and the author ([10]) have shown that this indeed happens for certain surfaces $X_{\tilde{m}}$ which are double coverings of K3 surfaces.

We have classified in [11] the automorphisms of the surfaces in question and shown, in particular, that any automorphism of prime order of the surfaces in question is conjugate to one of $\sigma_1, \sigma_3, \sigma_8, \sigma_{11}, \sigma_{15}, \sigma_9 \in \text{Aut}(\mathbf{P})$, which are defined respectively by

$$\begin{aligned} \sigma_1(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, y_2, z_3, -z_4) \\ \sigma_3(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, y_2, -z_3, -z_4) \\ \sigma_8(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, y_2, z_3, z_4) \\ \sigma_{11}(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, \omega y_2, x_3, z_4) \\ \sigma_{15}(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, \omega^2 y_2, z_3, z_4) \\ \sigma_9(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, -y_2, z_4, z_3) \end{aligned}$$

where x_0, y_1, y_2, z_3 and z_4 are weighted homogeneous coordinates of $\mathbf{P}(1, 2, 2, 3, 3)$ and $\omega = \exp(2\pi i/3)$. By using this classification, we have shown:

Φ has the 2-dimensional fibre through $\tilde{m} \in \tilde{M}$ $\Leftrightarrow \exists \sigma \in \text{Aut}(X_{\tilde{m}})$ which is conjugate to σ_3 ,

Φ has the positive dimensional fibre through $\tilde{m} \in \tilde{M}$ $\Leftarrow \exists \sigma \in \text{Aut}(X_{\tilde{m}})$ which is conjugate to σ_1 or σ_8

(see, for detail, [10] and [11]).

In this paper, we investigate those canonical surfaces with $p_g = c_1^2 = 1$ which have automorphisms conjugate to σ_{15} . Let M_{15} be the set of isomorphism classes of these surfaces. After our classification in [11], we have:

M_{15} = the set of isomorphism classes of canonical surfaces with $p_g = c_1^2 = 1$ and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

Set $\sigma = \sigma_{15}$ and let us consider smooth weighted complete intersections of type (6, 6) in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$ with defining equations

$$(0.2) \quad \begin{cases} f = z_3^2 + f_0 z_4 x_0^3 + f_{111} y_1^3 + f_{222} y_2^3 + f_{012} x_0^2 y_1 y_2 + f_{000} x_0^6, \\ g = z_4^2 + g_0 z_3 x_0^3 + g_{111} y_1^3 + g_{222} y_2^3 + g_{012} x_0^2 y_1 y_2 + g_{000} x_0^6. \end{cases}$$

These surfaces are stable under the action of σ . Denote by

$$(0.3) \quad \pi'_{15}: \mathcal{X}'_{15} \rightarrow U_{15}$$

the smooth family of weighted complete intersections of type (6, 6) in $\mathbf{P}(1, 2, 2, 3, 3)$ with equations (0.2) parametrized by their 10 coefficients

$$(f_0, f_{111}, f_{222}, f_{012}, f_{000}, g_0, g_{111}, g_{222}, g_{012}, g_{000}) \in U_{15} \subset \mathbf{A}^{10}.$$

The automorphism $\sigma \in \text{Aut}(\mathbf{P})$ has the induced action on the family (0.3) which is trivial on the parameter space U_{15} . We abuse the notation σ for indicating the induced automorphism of each fibre $X_u = \pi'^{-1}_{15}(u)$ ($u \in U_{15}$).

There exists a 4-dimensional subgroup $H \subset G \subset \text{Aut}(\mathbf{P})$ (cf. (1.12)) and our Proposition (1.14) asserts that

$$U_{15}/H \xrightarrow{\sim} M_{15} \quad (\text{and hence } \dim M_{15} = 6)$$

sending $u \in U_{15}$ to the isomorphism class containing X_u , and that, for any $X \in M_{15}$ and for any automorphism α of X of order 3 acting trivially on $H^0(X, K_X)$, there exists a point $u \in U_{15}$ and an isomorphism $\tau: X_u \xrightarrow{\sim} X$ such that $\alpha = \tau\sigma\tau^{-1}$.

Let $u_k \in U_{15}$ and set $X_k = X_{u_k}$ ($k = 1, 2$). Take a basis ω_{X_k} of $H^0(X_k, K_{X_k})$. Set

$$H_2(X_k, \mathbf{Z})^\sigma = \text{Ker}\{1 - \sigma: H_2(X_k, \mathbf{Z}) \rightarrow H_2(X_k, \mathbf{Z})\}.$$

Now our main theorem in the present paper is stated as follows:

THEOREM (3.4): *Let $u_k \in U_{15}$ ($k = 1, 2$). Suppose that there exists a path $\tilde{\tau}$ in U_{15} joining u_1 and u_2 which induces an isometry*

$$\tau_*: H_2(X_1, \mathbf{Z})^\sigma \rightarrow H_2(X_2, \mathbf{Z})^\sigma$$

preserving the periods of integrals of the holomorphic 2-forms ω_{X_k} on X_k , i.e.

$$\int_{\tau_*\gamma} \omega_{X_2} = (\text{constant}) \int_{\gamma} \omega_{X_1} \quad \text{for all } \gamma \in H_2(X_1, \mathbf{Z}),$$

where (constant) is independent of γ .

Then, there exists an isomorphism

$$\tau: X_1 \rightarrow X_2$$

inducing the given isometry τ_ and such τ is uniquely determined up to composition with an element of the group $\langle \sigma \rangle$ generated by σ . We have also $\tau\sigma\tau^{-1} = \sigma$ or σ^2 .*

Roughly speaking, Theorem (3.4) is proved by applying the Strong Torelli Theorem for algebraic K3 surfaces (cf. [8], [1] and [7]) to the K3 surfaces obtained as the desingularizations of $X_u/\langle \sigma \rangle$ ($u \in U_{15}$).

Our present results can be rephrased in the language of period map as follows. Fix a base point $u_0 \in U_{15}$ and identify $P^2(X_{u_0}, \mathbf{Z}) = L$. Set

$$\tilde{U}_{15} = \left\{ (u, \tau_*) \left| \begin{array}{l} u \in U_{15}, \tau_* \in \text{Isom}(P^2(X_u, \mathbf{Z}), L) \text{ coming from a path} \\ \tilde{\tau} \text{ joining } u \text{ and } u_0 \text{ in } U_{15} \end{array} \right. \right\}$$

and

$$\tilde{\mathcal{X}}'_{15} = \mathcal{X}'_{15} \times_{U_{15}} \tilde{U}_{15}.$$

Note that the fibre of $\tilde{U}_{15} \rightarrow U_{15}$ is the geometric monodromy group $\Gamma_{U_{15}} = \text{Im}\{\pi_1(U_{15}) \rightarrow \text{Aut}(L)\}$. Then we have, as in a similar way as (0.1), the universal family

$$\tilde{\pi}_{15}: \tilde{\mathcal{X}}_{15} = \tilde{\mathcal{X}}'_{15}/H \rightarrow \tilde{M}_{15} = \tilde{U}_{15}/H$$

and the period map

$$\Phi_{15}: \tilde{M}_{15} \rightarrow D.$$

Φ_{15} induces a set-theoretic map

$$\bar{\Phi}_{15}: M_{15} \rightarrow D/\Gamma_{U_{15}}.$$

Our Proposition (1.17) and Theorem (3.4) assert that Φ_{15} is unramified and $\bar{\Phi}_{15}$ is injective.

The following are unknown at present:

- (0.4) Whether Φ_{15} is an immersion.
- (0.5) The description of the difference of $\Gamma_{U_{15}}$ and $\Gamma = \text{Aut}(L)$.
- (0.6) The determination of the image of Φ_{15} .
- (0.7) The study of the surfaces with automorphisms conjugate to σ_{11} or to σ_0 .
- (0.8) The determination of all the points of \tilde{M} through which Φ has 1-dimensional fibres.

Every variety in this paper is a variety over the field \mathbb{C} of complex numbers.

1. Surfaces with $p_g = c_1^2 = 1$

1.1. F. Catanese showed in [2] that the canonical models of the surfaces with $p_g = c_1^2 = 1$ are represented as weighted complete intersections of type (6, 6) in $\mathbb{P} = \mathbb{P}(1, 2, 2, 3, 3)$. If we assume furthermore that the canonical invertible sheaf K_X of the surface X in question is ample, the canonical model of X is smooth and hence we can identify X with its canonical model.

Let $R = \mathbb{C}[x_0, y_1, y_2, z_3, z_4]$ be the weighted polynomial ring with $\deg x_0 = 1$, $\deg y_1 = \deg y_2 = 2$ and $\deg z_3 = \deg z_4 = 3$. Catanese also showed that the defining equations of the canonical models in question are partially normalized as follows (cf. [2]):

$$(1.1) \quad \begin{cases} f = z_3^2 + f^{(1)}z_4x_0 + f^{(3)}, \\ g = z_4^2 + g^{(1)}z_3x_0 + g^{(3)}, \end{cases}$$

where $f^{(1)}$ and $g^{(1)}$ are linear and $f^{(3)}$ and $g^{(3)}$ are cubic forms in x_0^2, y_1 and y_2 , i.e., by using the notation $y_0 = x_0^2$,

(1.2)

$$\begin{aligned} f^{(1)} &= \sum_{0 \leq i \leq 2} f_i y_i, & f^{(3)} &= \sum_{0 \leq i \leq j \leq k \leq 2} f_{ijk} y_i y_j y_k, \\ g^{(1)} &= \sum_{0 \leq i \leq 2} g_i y_i, & g^{(3)} &= \sum_{0 \leq i \leq j \leq k \leq 2} g_{ijk} y_i y_j y_k. \end{aligned}$$

Varying these 26 coefficients f_i, f_{ijk}, g_i and g_{ijk} , we get a family of weighted complete intersections in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$. Set

(1.3)

$$U = \left\{ u \in \mathbf{A}^{26} \left| \begin{array}{l} \text{the corresponding surface is a} \\ \text{smooth weighted complete intersections} \\ \text{of type } (6, 6) \text{ in } \mathbf{P}(1, 2, 2, 3, 3) \end{array} \right. \right\}$$

and let

(1.4)

$$\mathcal{X}' \rightarrow U$$

be the family of the surfaces in $\mathbf{P}(1, 2, 2, 3, 3)$. Note that U is a Zariski open subset of \mathbf{A}^{26} .

Let G be the group consisting of the non-degenerate matrices over \mathbf{C} of the forms

(1.5)

d_0					
d_{10} d_{20}		d_{11}	d_{12}	0	
		d_{21}	d_{22}		
0				d_3	0
				0	d_4

and

(1.6)

d_0					
d_{10} d_{20}		d_{11}	d_{12}	0	
		d_{21}	d_{22}		
0				0	d_3
				d_4	0

acting on $\mathbf{P}(1, 2, 2, 3, 3)$ as

$$\begin{cases} x_0 \mapsto d_0 x_0 \\ y_i \mapsto \sum_{0 \leq j \leq 2} d_{ij} y_j & (i = 1, 2) \\ z_i \mapsto d_i z_i & (i = 3, 4) \end{cases}$$

in case (1.5), and

$$\begin{cases} x_0 \mapsto d_0 x_0 \\ y_i \mapsto \sum_{0 \leq j \leq 2} d_{ij} y_j & (i = 1, 2) \\ z_3 \mapsto d_3 z_4 \\ z_4 \mapsto d_4 z_3 \end{cases}$$

in case (1.6).

Since the canonical invertible sheaves of the surfaces X_u ($u \in U$) are isomorphic to $\mathcal{O}_{X_u}(1)$ and their defining equations are partially normalized as (1.1), we can prove easily that every isomorphism between the surfaces X_u ($u \in U$) is induced from some element in G (see, for detail, [2] or [11]). Hence we see, by [4], that

(1.7) U/G = the coarse moduli scheme of complete, smooth surfaces with $p_g = c_1^2 = 1$ and K ample.

1.2. In [11], we classified the automorphisms of the surfaces X with $p_g = c_1^2 = 1$ and K_X ample, and determined the induced action on $H^2(X, \mathbb{C})$, on $H^{2,0}(X)$ and on $H^1(X, T_X)$.

Among these automorphisms we are mainly interested in the present paper in σ_{15} in Theorem (2.14) in [11]. We fix, throughout this paper, the notation

$$(1.8) \quad \sigma = \sigma_{15} = (1, \omega, \omega^2, 1, 1) \in G$$

which means the diagonal matrix

$$\sigma = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \text{ where } \omega = \exp(2\pi\sqrt{-1}/3).$$

Set

$$(1.9) \quad U_{15} = \{u \in U \mid \sigma u = u\}$$

and denote by

$$(1.10) \quad \pi'_{15}: \mathcal{X}'_{15} \rightarrow U_{15}$$

the family induced from (1.4) by $U_{15} \hookrightarrow U$. More explicitly, the defining equations of the surfaces $X_u = \pi'^{-1}_{15}(u)$ ($u \in U_{15}$) have the following forms:

$$(1.11) \quad \begin{cases} f = z_3^2 + f_0 z_4 x_0^3 + f_{111} y_1^3 + f_{222} y_2^3 + f_{012} x_0^2 y_1 y_2 + f_{000} x_0^6, \\ g = z_4^2 + g_0 z_3 x_0^3 + g_{111} y_1^3 + g_{222} y_2^3 + g_{012} x_0^2 y_1 y_2 + g_{000} x_0^6. \end{cases}$$

Define

$$H = \{\tau \in G \mid \tau(U_{15}) \cap U_{15} \neq \emptyset\}.$$

By an elementary calculation using (1.11), we can prove that H consists of the following four types of matrices:

$$(1.12) \quad \begin{array}{cc} \begin{array}{|c|c|c|} \hline d_0 & & \\ \hline & \begin{array}{|c|c|} \hline d_1 & 0 \\ 0 & d_2 \\ \hline \end{array} & 0 \\ \hline & 0 & \begin{array}{|c|c|} \hline d_3 & 0 \\ 0 & d_4 \\ \hline \end{array} \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline d_0 & & \\ \hline & \begin{array}{|c|c|} \hline 0 & d_1 \\ d_2 & 0 \\ \hline \end{array} & 0 \\ \hline & 0 & \begin{array}{|c|c|} \hline d_3 & 0 \\ 0 & d_4 \\ \hline \end{array} \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|} \hline d_0 & & \\ \hline & \begin{array}{|c|c|} \hline d_1 & 0 \\ 0 & d_2 \\ \hline \end{array} & 0 \\ \hline & 0 & \begin{array}{|c|c|} \hline 0 & d_3 \\ d_4 & 0 \\ \hline \end{array} \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline d_0 & & \\ \hline & \begin{array}{|c|c|} \hline 0 & d_1 \\ d_2 & 0 \\ \hline \end{array} & 0 \\ \hline & 0 & \begin{array}{|c|c|} \hline 0 & d_3 \\ d_4 & 0 \\ \hline \end{array} \\ \hline \end{array} \end{array}$$

We can also prove, by using the forms (1.12), that H is the normalizer of $\langle \sigma \rangle$ in G , where $\langle \sigma \rangle$ is the subgroup of G generated by σ in (1.8).

Set

- (1.13) M_{15} = the set of the isomorphism classes of the complete, smooth surfaces with $p_g = c_1^2 = 1$ and K ample and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

PROPOSITION (1.14): *We have a natural bijection $U_{15}/H \simeq M_{15}$ as sets and U_{15}/H is a 6-dimensional irreducible subvariety of the coarse moduli space U/G in (1.7). Moreover, for any surface $X \in M_{15}$ and for any automorphism α of X of order 3 acting trivially on $H^0(X, K_X)$, there exist a point $u \in U_{15}$ and an isomorphism $\tau: X_u \xrightarrow{\sim} X$ satisfying $\alpha = \tau\sigma\tau^{-1}$.*

PROOF: This is an immediate consequence of Theorem (2.14) in [11]. Note that “natural” in the statement of the proposition means that H -orbit of $u \in U_{15}$ corresponds to the isomorphism class containing X_u . Q.E.D.

1.3. Let $X = X_u$ for some $u \in U_{15}$ and let S be the parameter space of the Kuranishi family of the deformations of $X = X_{s_0}$ ($s_0 \in S$).

S is smooth and the Kuranishi family is universal (see, for detail, [11]). Hence, $\sigma \in \text{Aut}(X)$ has the induced action on S via the identification $X = X_{s_0}$. Set

$$(1.15) \quad S^\sigma = \{s \in S \mid \sigma s = s\}.$$

Note that, since σ is of finite order, S^σ is a submanifold of S . Note also that S^σ is the parameter space of the universal family of the deformations of the pair (X, σ) of the surface X and $\sigma \in \text{Aut}(X)$.

Let

$$(1.16) \quad \phi: S \rightarrow D$$

be the period map, using the Hodge decomposition of the second primitive cohomology group $P^2(X_s, \mathbb{C})$ ($s \in S$), obtained from the Kuranishi family, where D is the period domain (see, for detail, [5]).

PROPOSITION (1.17) (Local Torelli theorem for the restricted family):
The restriction

$$\text{res } \phi: S^\sigma \rightarrow D$$

of the period map ϕ in (1.16) is injective.

PROOF: First of all, note that σ has induced actions on S as above and also on D and that ϕ is σ -equivariant with these induced actions.

Let

$$d\phi(s_0): T_S(s_0) \rightarrow T_D(\phi(s_0))$$

be the differential map of the period map ϕ at $s_0 \in S$. Since $T_S(s_0)$ (resp. $T_D(\phi(s_0))$) can be identified with $H^1(X, T_X)$ (resp. $\text{Hom}(P^{2,0}(X), P^{1,1}(X))$), we know, from Theorem (2.14) in [11], that the decomposition of $T_S(s_0)$ and $T_D(\phi(s_0))$ into their eigen spaces under the action of σ are the following:

$$(1.18) \quad \begin{aligned} T_S(s_0) &= T_1 \oplus T_\omega \oplus T_{\omega^2} \quad \text{with } \dim T_1 = \dim T_\omega = \dim T_{\omega^2} = 6, \\ T_D(\phi(s_0)) &= T'_1 \oplus T'_\omega \oplus T'_{\omega^2} \quad \text{with } \dim T'_1 = 8, \\ \dim T'_\omega &= \dim T'_{\omega^2} = 5, \end{aligned}$$

where T_λ (resp. T'_λ) is the λ -eigen subspace of $T_S(s_0)$ (resp. $T_D(\phi(s_0))$).

Since $d\phi(s_0)$ is also σ -equivariant, $d\phi(s_0)$ is compatible with the decompositions in (1.18). Hence, from (1.18), $\text{Ker } d\phi(s_0)$ contains at least 2-dimensional subspace of $T_\omega \oplus T_{\omega^2}$. On the other hand, it can be shown easily (cf. [6], [2] or [11]) that $\dim \text{Ker } d\phi(s_0) \leq 2$. Thus, we can conclude that

$$(1.19) \quad T_1 \cap \text{Ker } d\phi(s_0) = \{0\}.$$

Since $T_{S^\sigma}(s_0) = T_1$, (1.19) means that

$$\text{res } d\phi(s_0): T_{S^\sigma}(s_0) \rightarrow T_D(\phi(s_0))$$

is injective. This shows that

$$\text{res } \phi: S^\sigma \rightarrow D$$

is injective, because we consider S^σ as germ.

Q.E.D.

2. Structure theorem

We continue to use the notation in the previous section.

2.1. Let $X = X_u$ ($u \in U_{15}$). Since $\sigma = (1, \omega, \omega^2, 1, 1)$ (see (1.18)), the fixed points of X by σ satisfy the equations

$$(2.1) \quad x_0 = y_1 = 0,$$

$$(2.2) \quad x_0 = y_2 = 0 \quad \text{or}$$

$$(2.3) \quad y_1 = y_2 = 0.$$

We can calculate easily that

the intersection number of the curves $(x_0 = 0)$ and $(y_i = 0) = 2$ ($i = 1, 2$)
the intersection number of the curves $(y_1 = 0)$ and $(y_2 = 0) = 4$.

Moreover, since $\sigma \in \text{Aut}(X)$ is of finite order, the fixed points locus X^σ of X by σ is smooth. Thus we get that X^σ consists of 8 distinct points. We denote these points by

$$(2.4) \quad \begin{aligned} X &= \{D_i, E_i \ (i = 1, 2, 3, 4)\}, \quad \text{where} \\ D_i \ (i = 1, 2) &\text{ satisfy the equations (2.1),} \\ D_i \ (i = 3, 4) &\text{ satisfy the equations (2.2) and} \\ E_i \ (i = 1, 2, 3, 4) &\text{ satisfy the equations (2.3).} \end{aligned}$$

Since we can take $x_0 z_3 / y_2^2$, y_1 / y_2 (resp. $x_0 z_3 / y_1^2$, y_2 / y_1 ; resp. y_1 / x_0^2 , y_2 / x_0^2) as local coordinates of X at D_i ($i = 1, 2$) (resp. D_i ($i = 3, 4$) resp. E_i ($i = 1, 2, 3, 4$)), we see that the induced actions of σ on the normal spaces of these points in X are

$$(2.5) \quad \begin{aligned} (\omega^2, \omega^2) &\quad \text{at } D_i \ (i = 1, 2), \\ (\omega, \omega) &\quad \text{at } D_i \ (i = 3, 4) \text{ and} \\ (\omega, \omega^2) &\quad \text{at } E_i \ (i = 1, 2, 3, 4). \end{aligned}$$

Let

$$(2.6) \quad \tilde{X} \rightarrow X$$

be the blowing-up of X with center X^σ . Denote by

$$(2.7) \quad \tilde{D}_i \text{ and } \tilde{E}_i \quad (i = 1, 2, 3, 4)$$

the exceptional curves on \tilde{X} corresponding to the points D_i and E_i on X respectively.

The action of σ extends naturally on \tilde{X} so that the morphism (2.6) is σ -equivariant. From (2.5), we see that there are 2 distinct points, say

$$(2.8) \quad \tilde{E}_{ij} \quad (j = 1, 2),$$

on each \tilde{E}_i which are fixed by σ , and the fixed points locus \tilde{X}^σ of \tilde{X} by σ is

$$(2.9) \quad \tilde{X}^\sigma = \{\tilde{D}_i, \tilde{E}_{ij} \ (i = 1, 2, 3, 4; j = 1, 2)\}.$$

We know, also from (2.5), that the induced action of σ on the normal bundle of each component of \tilde{X}^σ in \tilde{X} is

$$(2.10) \quad \begin{array}{lll} (\omega^2) & \text{along} & \tilde{D}_i \ (i = 1, 2), \\ (\omega) & \text{along} & \tilde{D}_i \ (i = 3, 4), \\ (\omega, \omega) & \text{at} & \tilde{E}_{i1} \ (i = 1, 2, 3, 4) \text{ and} \\ (\omega^2, \omega^2) & \text{at} & \tilde{E}_{i2} \ (i = 1, 2, 3, 4). \end{array}$$

Let

$$(2.11) \quad \hat{X} \rightarrow \tilde{X}$$

be the blowing-up of \tilde{X} with center \tilde{X}^σ . Denote by

$$(2.12) \quad \hat{D}_i, \hat{E}_i \text{ and } \hat{E}_{ij} \ (i = 1, 2, 3, 4; j = 1, 2)$$

the curves on \hat{X} which are the inverse images of \tilde{D}_i , the proper transforms of \tilde{E}_i and the exceptional divisors corresponding to \tilde{E}_{ij} respectively.

The action of σ extends again to \hat{X} and we see, from (2.10), that the fixed points locus \hat{X}^σ of \hat{X} by σ is now a disjoint union of 12 curves, i.e.

$$(2.13) \quad \hat{X}^\sigma = \{\hat{D}_i, \hat{E}_{ij} \ (i = 1, 2, 3, 4; j = 1, 2)\}.$$

From (2.10) again, we know that the induced action of σ on the normal bundle of each component of \hat{X}^σ in \hat{X} is the following:

$$(2.14) \quad \begin{array}{ll} (\omega) & \text{along } \hat{D}_i \ (i = 3, 4) \text{ and along } \hat{E}_{i1} \ (i = 1, 2, 3, 4). \\ (\omega^2) & \text{along } \hat{D}_i \ (i = 1, 2) \text{ and along } \hat{E}_{i2} \ (i = 1, 2, 3, 4). \end{array}$$

We denote by

$$(2.15) \quad p : \hat{X} \rightarrow X$$

the composite morphism of (2.11) and (2.6). Note that p is σ -equivariant.

We can calculate easily the self-intersection numbers of the exceptional curves on \hat{X} of the morphism p :

$$(2.16) \quad (\hat{D}_i)^2 = (\hat{E}_{ij})^2 = -1, \quad (\hat{E}_i)^2 = -3 \quad (i = 1, 2, 3, 4; j = 1, 2).$$

Denote by

$$(2.17) \quad C \text{ and } \hat{C}$$

the canonical divisor of X and its proper transform by p in (2.15). Since $x_0 = 0$ is the homogeneous equation of C in X , C contains 4 points D_i ($i = 1, 2, 3, 4$) in (2.4). From this fact we get that

$$(2.18) \quad (\hat{C})^2 = -3.$$

2.2. Since $\sigma \in \text{Aut}(\hat{X})$ is of order 3 and \hat{X}^σ is of pure codimension 1, we get a ramified triple covering

$$(2.19) \quad r: \hat{X} \rightarrow \hat{Y},$$

where $\hat{Y} = \hat{X}/\langle \sigma \rangle$ is smooth. We denote by \hat{R} the ramification locus and by \hat{B} the branch locus of r , i.e.

$$(2.20) \quad \hat{R} = \hat{X}^\sigma = \sum_{1 \leq i \leq 4} \hat{D}_i + \sum_{1 \leq i \leq 4, j=1,2} \hat{E}_{ij} \quad \text{and} \quad \hat{B} = r(\hat{R}).$$

We consider \hat{R} and \hat{B} as reduced curves.

We use the notation

$$(2.21) \quad \hat{C}' = r(\hat{C}), \quad \hat{D}'_i = r(\hat{D}_i), \quad \hat{E}'_i = r(\hat{E}_i) \quad \text{and} \quad \hat{E}'_{ij} = r(\hat{E}_{ij}),$$

where all these curves are considered as reduced curves on \hat{Y} .

LEMMA (2.22): *All the curves in (2.21) are smooth, irreducible, rational curves with self-intersection numbers*

$$(\hat{C}')^2 = (\hat{E}'_{ij})^2 = -1 \quad \text{and} \quad (\hat{D}'_i)^2 = (\hat{E}'_i)^2 = -3 \quad (i = 1, 2, 3, 4; j = 1, 2).$$

PROOF: We see easily that C is a smooth curve of genus 2 by the Jacobian criterion and adjunction formula. Hence, so is \hat{C} , because \hat{C} is isomorphic to C . From the construction, we know that

$$\hat{C} \rightarrow \hat{C}'$$

is a triple covering ramified at 4 distinct points $\hat{C} \cap (\sum_{1 \leq i \leq 4} \hat{D}_i)$. Hence, we see that \hat{C}' is a smooth, irreducible, rational curve by the Hurwitz formula.

In the same way, by using the fact that

$$\hat{E}_i \rightarrow \hat{E}'_i$$

is a triple covering ramified at 2 distinct points $\hat{E}_i \cap (\hat{E}_{i1} + \hat{E}_{i2})$, we can prove that \hat{E}'_i are also smooth, irreducible, rational curves.

The same assertion for the curves \hat{D}'_i and \hat{E}'_{ij} is trivial because they are isomorphic to \hat{D}_i and \hat{E}_{ij} respectively.

As for the statement for the self-intersection numbers, we can obtain immediately from (2.16) and (2.18) by the projection formula. Q.E.D.

2.3. Let

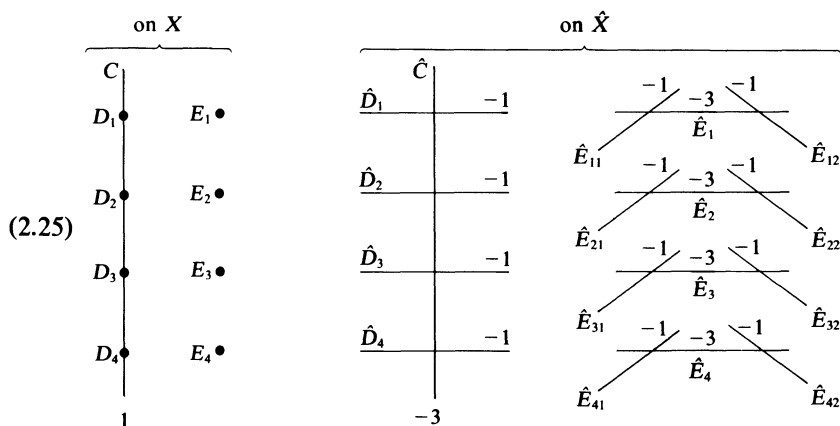
$$(2.23) \quad q: \hat{Y} \rightarrow Y$$

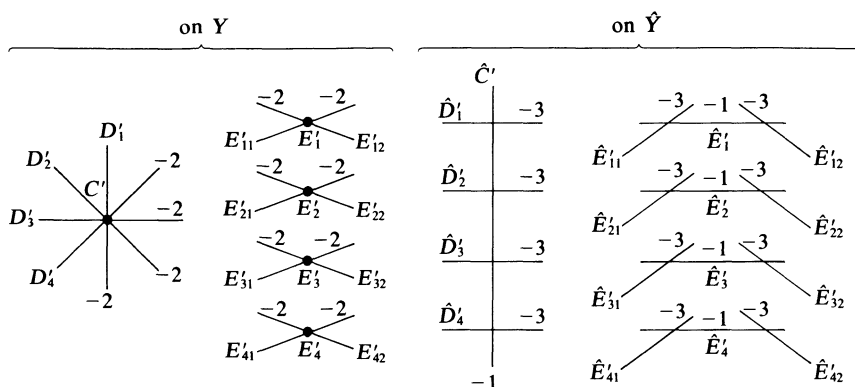
be the morphism obtained by blowing-down the exceptional curves of the first kind \hat{C}' and \hat{E}'_i ($i = 1, 2, 3, 4$). Set

$$(2.24) \quad C' = q(\hat{C}'), \quad E'_i = q(\hat{E}'_i), \quad D'_i = q(\hat{D}'_i) \quad \text{and} \quad E'_{ij} = q(\hat{E}'_{ij}) \\ (i = 1, 2, 3, 4; j = 1, 2).$$

Then, C' and E'_i are points, and D'_i and E'_{ij} are smooth, irreducible, rational curves with self-intersection number -2 .

We write down the configurations of the points and the curves appeared in 2.1, 2.2 and 2.3 with their self-intersection numbers:





2.4. Now we can state the relation of our surfaces with K3 surfaces. We use the notation in 2.1, 2.2 and 2.3.

PROPOSITION (2.26) (Structure theorem): Set $X = X_u$ ($u \in U^\sigma$). Then, starting from X , we can construct a diagram

$$\begin{array}{ccc}
 X & \xleftarrow{p} & \hat{X} \\
 & & \downarrow r \\
 Y & \xleftarrow{q} & \hat{Y}
 \end{array}$$

where

(i) p is the morphism in (2.15), i.e. the morphism obtained by a sequence of blowings-up at the fixed points by σ , so that the fixed points locus in \hat{X} under the induced action of σ is of pure codimension 1,

(ii) r is the morphism in (2.19), i.e. the natural projection onto the quotient of \hat{X} by the group $\langle \sigma \rangle$ generated by σ , and

(iii) q is the morphism in (2.23), i.e. the morphism obtained by blowing-down onto the minimal model Y .

Moreover, we have that

(iv) Y is a minimal K3 surface,

(v) $3(\sum_{1 \leq i \leq 4} D'_i) - 2(\sum_{1 \leq i \leq 4, j=1,2} E'_{ij})$ is an ample divisor on Y , and

(vi) $\pi_1(\hat{X} - \hat{R}) = \{1\}$, where \hat{R} is the ramification locus of r .

PROOF: The remaining things to prove are the assertions (iv), (v) and (vi).

First, we will prove (iv). By the construction of Y , it is clear that the unique holomorphic 2-form on X , vanishing on C and σ -invariant, gives a nowhere vanishing holomorphic 2-form on Y . Combining this with $q(Y) \leq q(X) = 0$, we get (iv).

For the proof of (v), we use the configuration (2.25). First of all, we see that

$$(2.27) \quad \left(3 \left(\sum_{1 \leq i \leq 4} D'_i \right) - 2 \left(\sum_{1 \leq i \leq 4, j=1,2} E'_{ij} \right) \right)^2 \\ = 9 \left(\sum D'_i \right)^2 + 4 \left(\sum E'_{ij} \right)^2 = 4 > 0.$$

By the assumption, C is ample and hence so is

$$p^*(4C) - \left(\sum \hat{D}_i + \sum \hat{E}_i + 2 \left(\sum \hat{E}_{ij} \right) \right) \\ = 4\hat{C} - \left(\sum \hat{E}_i \right) + 3 \left(\sum \hat{D}_i \right) - 2 \left(\sum \hat{E}_{ij} \right).$$

Since r is a finite morphism and

$$3 \left(4\hat{C} - \left(\sum \hat{E}_i \right) + 3 \left(\sum \hat{D}_i \right) - 2 \left(\sum \hat{E}_{ij} \right) \right) \\ = r^* \left(12\hat{C}' - 3 \left(\sum \hat{E}'_i \right) + 3 \left(\sum \hat{D}'_i \right) - 2 \left(\sum \hat{E}'_{ij} \right) \right),$$

we see that

$$12\hat{C}' - 3 \left(\sum \hat{E}'_i \right) + 3 \left(\sum \hat{D}'_i \right) - 2 \left(\sum \hat{E}'_{ij} \right)$$

is an ample divisor on \hat{Y} . Denote this divisor by F . Since \hat{C}' and \hat{E}'_i are the exceptional curves of the morphism q , we see, by the Nakai criterion of ampleness for F , that for any integral curve Z on Y

$$(2.28) \quad \left(3 \left(\sum D'_i \right) - 2 \left(\sum E'_{ij} \right), Z \right) \\ = \left(q^* \left(3 \left(\sum D'_i \right) - 2 \left(\sum E'_{ij} \right) \right), q^*Z \right) = (F, q^*Z) > 0.$$

Thus, the assertion (v) follows from (2.27) and (2.28) by the Nakai criterion again.

Finally, we will prove (vi). We use the result in [2]:

$$\pi_1(X) = \{1\}.$$

Since X^σ consists of finite points, we see that

$$(2.29) \quad \pi_1(X - X^\sigma) = \pi_1(X) = \{1\}.$$

By using (2.29) and the following diagram

$$X - X^\sigma \curvearrowright \hat{X} - \left(\hat{R} + \sum_{1 \leq i \leq 4} \hat{E}_i \right)$$

$$\bigcap$$

$$\hat{X} - \hat{R},$$

we get our assertion (vi).

Q.E.D.

3. Torelli theorem

In this section, we will prove the Torelli theorem for the surfaces with $p_g = c_1^2 = 1$, with an ample canonical divisor and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

We continue to use the notation in the previous sections.

First, we give an elementary lemma which can be verified easily by a standard argument using the discreteness of integral homology groups.

LEMMA (3.1): *Let ψ be a morphism of smooth families $\{V_t\}_{t \in T}$ and $\{W_t\}_{t \in T}$ of compact, complex manifolds over a complex manifold T and suppose we are given a path α in T joining two points t and t' in T .*

Then, we have a commutative diagram

$$\begin{array}{ccc} H_n(V_t, \mathbf{Z}) & \xrightarrow{\psi_{t*}} & H_n(W_t, \mathbf{Z}) \\ \alpha_* \downarrow \wr & & \alpha_* \downarrow \wr \\ H_n(V_{t'}, \mathbf{Z}) & \xrightarrow{\psi_{t'*}} & H_n(W_{t'}, \mathbf{Z}) \end{array}$$

for all n , where α_ is the isomorphism obtained by a C^∞ -trivialization along the path α , and this α_* is compatible with intersection products.*

Let $\pi'_{15}: \mathcal{X}'_{15} \rightarrow U_{15}$ be the family in (1.10). For any two points $u_k \in U_{15}$ ($k = 1, 2$), taking a path $\tilde{\tau}$ in U_{15} joining u_1 and u_2 and applying Lemma (3.1), we get a commutative diagram

$$(3.2) \quad \begin{array}{ccc} H_2(X_1, \mathbf{Z}) & \xrightarrow{1-\sigma} & H_2(X_1, \mathbf{Z}) \\ \tau_* \downarrow \wr & & \tau_* \downarrow \wr \\ H_2(X_2, \mathbf{Z}) & \xrightarrow{1-\sigma} & H_2(X_2, \mathbf{Z}) \end{array}$$

where $X_k = \pi'^{-1}_{15}(u_k)$ and τ_* is the isometry obtained from the path $\tilde{\tau}$. Hence, we get the induced isometry

$$(3.3) \quad \tau_*: H_2(X_1, \mathbf{Z})^\sigma \xrightarrow{\sim} H_2(X_2, \mathbf{Z})^\sigma$$

of the kernels of $1 - \sigma$ in (3.2).

THEOREM (3.4): Suppose we are given two points $u_k \in U_{15}$ ($k = 1, 2$) and a path $\tilde{\tau}$ in U_{15} joining u_1 and u_2 , and suppose the induced isometry τ_* in (3.3) preserves the periods of integrals of the holomorphic 2-forms ω_{X_k} on $X_k = \pi'^{-1}_{15}(u_k)$ ($k = 1, 2$), i.e.

$$\int_{\tau_*\gamma} \omega_{X_2} = (\text{constant}) \int_{\gamma} \omega_{X_1}$$

for all $\gamma \in H_2(X_1, \mathbf{Z})^\sigma$, where (constant) is independent of γ .

Then, there exists an isomorphism

$$\tau: X_1 \xrightarrow{\sim} X_2$$

inducing the given τ_* and such τ is uniquely determined up to composition with an element of the group $\langle \sigma \rangle$ generated by σ . We have also $\tau\sigma\tau^{-1} = \sigma$ or σ^2 .

PROOF: Starting from the family (1.10), we can construct, in a similar way as in the section 2, a commutative diagram

$$(3.5) \quad \begin{array}{ccccccc} & & \hat{\mathcal{X}} & \xrightarrow{\tilde{r}} & \hat{\mathcal{Y}} & \xrightarrow{\tilde{q}} & \mathcal{Y} \\ & \nwarrow \pi'_{15} & \downarrow \hat{\pi} & & \downarrow \hat{\pi}' & \swarrow \pi' & \\ & & U_{15} & & & & \end{array}$$

whose fibre over every point of U_{15} satisfies the properties (i) to (vi) in Proposition (2.26). In fact, \tilde{p} and \tilde{r} in (3.5) can be constructed just in the same way as p and r in the section 2, and the construction of \tilde{q} in (3.5) is justified by the result in [3].

For $k = 1, 2$, set $\hat{X}_k = \hat{\pi}^{-1}(u_k)$, $\hat{Y}_k = \hat{\pi}'^{-1}(u_k)$, and $Y_k = \pi'^{-1}(u_k)$, and let $p_k: \hat{X}_k \rightarrow X_k$, $r_k: \hat{X}_k \rightarrow \hat{Y}_k$ and $q_k: \hat{Y}_k \rightarrow Y_k$ be the restrictions to the fibres of the morphisms \tilde{p} , \tilde{q} and \tilde{r} in (3.5) respectively. We denote by $\hat{D}_i^{(k)}$, $\hat{E}_{ij}^{(k)}$ and $\hat{E}_{ij}'^{(k)}$ the corresponding curves on \hat{X}_k and by $C_i^{(k)}$, $D_i'^{(k)}$, $E_i^{(k)}$ and $E_{ij}'^{(k)}$ the corresponding points and curves on Y_k ($k = 1, 2$) constructed in the section 2. Denote also by \hat{R}_k and \hat{B}_k the ramification locus and the branch locus of the triple covering $r_k: \hat{X}_k \rightarrow \hat{Y}_k$ ($k = 1, 2$). For a divisor F on a surface, we denote by $[F]$ the integral homology class represented by F .

Then, by Lemma (3.1), we get, from (3.5), the commutative diagram of homology groups:

$$(3.6) \quad \begin{array}{ccccccc} H_2(X_1, \mathbb{Z})^\sigma & \xleftarrow{p_{1*}} & H_2(\hat{X}_1, \mathbb{Z})^\sigma & \xrightarrow{r_{1*}} & H_2(\hat{Y}_1, \mathbb{Z}) & \xrightarrow{q_{1*}} & H_2(Y_1, \mathbb{Z}) \\ \tau_* \downarrow \wr & & \hat{\tau}_* \downarrow \wr & & \hat{\tau}'_* \downarrow \wr & & \tau'_* \downarrow \wr \\ H_2(X_2, \mathbb{Z})^\sigma & \xleftarrow{p_{2*}} & H_2(\hat{X}_2, \mathbb{Z})^\sigma & \xrightarrow{r_{2*}} & H_2(\hat{Y}_2, \mathbb{Z}) & \xrightarrow{q_{2*}} & H_2(Y_2, \mathbb{Z}) \end{array}$$

$$\begin{array}{ccccccc} H_2(X_1, \mathbb{Z})^\sigma & \xleftarrow{p_1^*} & H_2(\hat{X}_1, \mathbb{Z})^\sigma & \xrightarrow{r_1^*} & H_2(\hat{Y}_1, \mathbb{Z}) & \xrightarrow{q_1^*} & H_2(Y_1, \mathbb{Z}) \\ \tau_* \downarrow \wr & & \hat{\tau}_* \downarrow \wr & & \hat{\tau}'_* \downarrow \wr & & \tau'_* \downarrow \wr \\ H_2(X_2, \mathbb{Z})^\sigma & \xleftarrow{p_2^*} & H_2(\hat{X}_2, \mathbb{Z})^\sigma & \xrightarrow{r_2^*} & H_2(\hat{Y}_2, \mathbb{Z}) & \xrightarrow{q_2^*} & H_2(Y_2, \mathbb{Z}) \end{array}$$

where $\hat{\tau}_*$, $\hat{\tau}'_*$ and τ'_* are the induced isometries, like τ_* , from the path $\tilde{\tau}$. By our construction of (3.5), we see that

$$(3.7) \quad \begin{aligned} \hat{\tau}_*([\hat{D}_i^{(1)}]) &= [\hat{D}_i^{(2)}], & \hat{\tau}_*([\hat{E}_i^{(1)}]) &= [\hat{E}_i^{(2)}], & \hat{\tau}_*([\hat{E}_{ij}^{(1)}]) &= [\hat{E}_{ij}^{(2)}], \\ \hat{\tau}'_*([\hat{B}_1]) &= [\hat{B}_2], & \tau'_*([D_i^{(1)}]) &= [D_i^{(2)}], & \tau'_*([E_{ij}^{(1)}]) &= [E_{ij}^{(2)}]. \end{aligned}$$

Note also that $p_{k*}p_k^* = id$, $q_{k*}q_k^* = id$, $r_{k*}r_k^* = 3id$ and $r_k^*r_{k*} = 3id$ ($k = 1, 2$).

Let $\omega_{\hat{X}_k}$ (resp. $\omega_{\hat{Y}_k}$, ω_{Y_k}) be the holomorphic 2-form on \hat{X}_k (resp.

\hat{Y}_k, Y_k) induced from ω_{X_k} ($k = 1, 2$). Since

$$\int_{\gamma} \omega_{Y_k} = \int_{q_k^* \gamma} \omega_{\hat{Y}_k} = 3 \int_{r_k^* q_k^* \gamma} \omega_{\hat{X}_k} = 3 \int_{p_k^* r_k^* q_k^* \gamma} \omega_{X_k}$$

for any $\gamma \in H_2(Y_k, \mathbf{Z})$, we can deduce, by (3.6), the property

$$\int_{\tau'_* \gamma} \omega_{Y_2} = (\text{constant}) \int_{\gamma} \omega_{Y_1} \quad \text{for all } \gamma \in H_2(Y_1, \mathbf{Z})$$

from that on X_k .

Since

$$\tau'_* \left(\left[3 \left\{ \sum_i D_i^{(1)} \right\} - 2 \left(\sum_{i,j} E_{ij}^{\prime(1)} \right) \right] \right) = \left[3 \left(\sum_i E_i^{\prime(2)} \right) - 2 \left(\sum_{i,j} E_{ij}^{\prime(2)} \right) \right]$$

from (3.7), we see, by (v) in Proposition (2.26), that τ'_* sends some ample divisor class on Y_1 to an ample divisor class on Y_2 .

Hence, we can apply the Strong Torelli Theorem for algebraic K3 surfaces proved and supplemented in [8], [1] and [7] to our case, and we see that there exists uniquely the isomorphism

$$\tau': Y_1 \xrightarrow{\sim} Y_2$$

inducing the isometry τ'_* in (3.6).

Considering (3.7) and intersection numbers, we can observe easily

$$\tau'(D_i^{(1)}) = D_i^{\prime(2)} \quad \text{and} \quad \tau'(E_{ij}^{\prime(1)}) = E_{ij}^{\prime(2)}$$

and hence, in particular,

$$\tau'(C^{\prime(1)}) = C^{\prime(2)} \quad \text{and} \quad \tau'(E_i^{\prime(1)}) = E_i^{\prime(2)}.$$

Therefore, by the construction of $q_k: \hat{Y}_k \rightarrow Y_k$, τ' can be lifted uniquely to an isomorphism

$$\hat{\tau}': \hat{Y}_1 \xrightarrow{\sim} \hat{Y}_2$$

inducing the isometry $\hat{\tau}'_*$ in (3.6).

Considering (3.7) and intersection numbers again, we see

$$\hat{\tau}'(\hat{B}_1) = \hat{B}_2.$$

Since we know that $r_k: \hat{X}_k - \hat{R}_k \rightarrow \hat{Y}_k - \hat{B}_k$ are universal coverings by (vi) in Proposition (2.26), there exists an isomorphism

$$\hat{\tau}: \hat{X}_1 - \hat{R}_1 \xrightarrow{\sim} \hat{X}_2 - \hat{R}_2$$

compatible with $\hat{\tau}'$. Such $\hat{\tau}$ are unique up to the covering transformation group $\langle \sigma \rangle$. Now, by the Riemann Extension Theorem, $\hat{\tau}$ extends uniquely to an isomorphism

$$\hat{\tau}: \hat{X}_1 \xrightarrow{\sim} \hat{X}_2,$$

where we abuse the notation $\hat{\tau}$. $\hat{\tau}$ is compatible with $\hat{\tau}'$ and hence induces the isometry $\hat{\tau}_*$ in (3.6).

By the argument on intersection numbers, we get, from (3.7), that

$$\hat{\tau}(\hat{D}_i^{(1)}) = \hat{D}_i^{(2)}, \quad \hat{\tau}(\hat{E}_{ij}^{(1)}) = \hat{E}_{ij}^{(2)} \quad \text{and} \quad \hat{\tau}(\hat{E}_i^{(1)}) = \hat{E}_i^{(2)}.$$

Hence, $\hat{\tau}$ descends uniquely to an isomorphism

$$\tau: X_1 \xrightarrow{\sim} X_2$$

inducing the given isometry τ_* .

The other assertion follows easily.

Q.E.D.

REFERENCES

- [1] D. BURNS and M. RAPOPORT: On the Torelli problems for Kählerian K-3 surfaces. *Ann. scient. Éc. Norm. Sup.* 4^e sér. 8-2 (1975) 235-274.
- [2] F. CATANESE: Surfaces with $K^2 = p_g = 1$ and their period mapping. Proc. Summer Meeting on Algebraic Geometry, Copenhagen 1978, *Lecture Notes in Math.* No 732, Springer Verlag, 1-29.
- [3] A. FUJIKI and S. NAKANO; Supplement to "On the inverse of Monoidal Transformation", *Publ. R.I.M.S. Kyoto Univ.* 7 (1972) 637-644.
- [4] D. GIESEKER: Global moduli for surfaces of general type. *Invent. Math.* 43 (1977) 233-282.
- [5] P. GRIFFITHS: Periods of integrals on algebraic manifolds I, II, III: *Amer. J. Math.* 90 (1968) 568-626; 805-865; *Publ. Math. I.H.E.S.* 38 (1970) 125-180.
- [6] F.I. KİNEV: A simply connected surface of general type for which the local Torelli theorem does not hold (Russian). *Cont. Ren. Acad. Bulgare des Sci.* 30-3 (1977) 323-325.
- [7] E. LOOIJENGA and C. PETERS: Torelli theorems for Kähler K3 surfaces, *Comp. Math.* 42-2 (1981) 145-186.
- [8] I. PIATECKIĬ-ŠAPIRO and I.R. ŠAFAREVIČ: A Torelli theorem for algebraic surfaces of type K-3, *Izv. Akad. Nauk.* 35 (1971) 530-572.

- [9] A.N. TODOROV: Surfaces of general type with $p_g = 1$ and $(K, K) = 1$. I, *Ann. scient. Éc. Norm. Sup. 4^e sér.* 13-1 (1980) 1-21.
- [10] S. USUI: Period map of surfaces with $p_g = c_1^2 = 1$ and K ample. *Mem. Fac. Sci. Kochi Univ. (Math.)* 2 (1981) 37-73.
- [11] S. USUI: Effect of automorphisms on variation of Hodge structure. *J. Math. Kyoto Univ.* 21-4 (1981).
- [12] F. CATANESE: The moduli and the global period mapping of surfaces with $K^2 = p_g = 1$: A counterexample to the global Torelli problem, *Comp. Math.* 41-3 (1980) 401-414.

(Oblatum 27-III-1981, 30-VI-1981)

Kochi University
 Department of Mathematics
 Faculty of Sciences
 Kochi, 780 Japan
Univ. 21-4 (1981).