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SAMPEI USUI

Torelli theorem for surfaces with \( p_g = c_1^2 = 1 \) and \( K \) ample and with certain type of automorphism

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0. Introduction

The moduli space of isomorphism classes of surfaces with $p_g = c_1^2 = 1$ is studied by Catanese in [2]. Every such surface with the ample canonical divisor can be represented as a smooth weighted complete intersection of type $(6, 6)$ in $\mathbb{P} = \mathbb{P}(1, 2, 2, 3, 3)$ parametrized by a Zariski open set $U \subset \mathbb{A}^6$ (cf. (1.3)). This leads to a universal family

$$\pi': \mathcal{X}' \to U.$$ 

There is an 8-dimensional subgroup $G$ of $\text{Aut}(\mathbb{P})$ (cf. (1.5) and (1.6)) acting on $U$ with finite isotropy groups and

$$M = U/G = \text{the moduli space of canonical surfaces}$$

$$\text{with } p_g = c_1^2 = 1.$$ 

In particular, $\dim_{\mathbb{C}} M = 18$.

The period domain $D$, which parametrizes polarized Hodge structures on the second primitive cohomology groups of the surfaces in question, is isomorphic to

$$\{ [a] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (a, a) = 0, (a, \bar{a}) > 0 \}$$

where $L$ is a free $\mathbb{Z}$-module of rank 20 equipped with a symmetric bilinear form $(\ , \ )$ of signature $(2, 18)$. The group $\Gamma = \text{Aut}(L)$ acts properly discontinuously on $D$. 

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Set

\[ \hat{U} = \{(u, \alpha) \mid u \in U, \alpha \in \text{Isom}(P^2(X_m, \mathbb{Z}), L)\} \quad \text{and} \quad \hat{\mathcal{H}} = \hat{\mathcal{H}}' \times \hat{U} \]

Then we have the universal family

\[
(0.1) \quad \hat{\pi} : \hat{\mathcal{H}} = \hat{\mathcal{H}}'/G \to \hat{M} = \hat{U}/G
\]

of marked canonical surfaces with \( p_g = c_1^2 = 1 \) (cf. Proposition (2.24) in [11]). \( \hat{M} \) and \( \hat{\mathcal{H}} \) are complex manifolds and this family serves as a universal family of the deformations of the surfaces in question. This gives a period map

\[ \Phi : \hat{M} \to D. \]

Catanese has shown in [2] (cf. also [12]) that \( \Phi \) has non-empty ramification locus \( \Delta \subset \hat{M} \). Thus the local Torelli fails at \( m \in \Delta \). The problem then is to study how badly it can fail. First of all observe that

\[ \dim \text{Ker } d\Phi (m) \leq 2. \]

This directly follows from the exact sequence

\[
0 \to H^0(C_m, \Omega_{X_m}^1 \otimes \mathcal{O}_{C_m}) \to H^1(X_m, T_{X_m}) \xrightarrow{d\Phi(m)} H^1(X_m, \Omega_{X_m}^1)
\]

together with the fact that \( h^0(C_m, \Omega_{X_m}^1 \otimes \mathcal{O}_{C_m}) \leq h^0(C_m, \Omega_{C_m}^1) = 2 \), where \( C_m \) is the canonical curve of \( X_m \). This means that the fibre of \( \Phi \) through \( \bar{m} \in \bar{M} \) has at most dimension 2. Todorov ([9]) and the author ([10]) have shown that this indeed happens for certain surfaces \( X_m \) which are double coverings of K3 surfaces.

We have classified in [11] the automorphisms of the surfaces in question and shown, in particular, that any automorphism of prime order of the surfaces in question is conjugate to one of \( \sigma_1, \sigma_3, \sigma_8, \sigma_{11}, \sigma_{15}, \sigma_0 \in \text{Aut}(\mathcal{P}) \), which are defined respectively by

\[
\begin{align*}
\sigma_1(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, y_2, z_3, -z_4) \\
\sigma_3(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, y_2, -z_3, -z_4) \\
\sigma_8(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, y_2, z_3, z_4) \\
\sigma_{11}(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, \omega y_2, x_3, z_4) \\
\sigma_{15}(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, \omega^2 y_2, z_3, z_4) \\
\sigma_0(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, -y_2, z_4, z_3)
\end{align*}
\]
where $x_0, y_1, y_2, z_3$ and $z_4$ are weighted homogeneous coordinates of $P(1, 2, 2, 3, 3)$ and $\omega = \exp(2\pi i/3)$. By using this classification, we have shown:

\[
\Phi \text{ has the 2-dimensional } \exists \sigma \in \text{Aut}(X_{\tilde{m}}) \text{ which is fibre through } \tilde{m} \in \tilde{M} \leftrightarrow \text{ conjugate to } \sigma_3,
\]

\[
\Phi \text{ has the positive dimensional } \exists \sigma \in \text{Aut}(X_{\tilde{m}}) \text{ which is fibre through } \tilde{m} \in \tilde{M} \leftarrow \text{ conjugate to } \sigma_1 \text{ or } \sigma_8
\]

(see, for detail, [10] and [11]).

In this paper, we investigate those canonical surfaces with $p_g = c_1^2 = 1$ which have automorphisms conjugate to $\sigma_{15}$. Let $M_{15}$ be the set of isomorphism classes of these surfaces. After our classification in [11], we have:

$M_{15} = \text{the set of isomorphism classes of canonical surfaces with } p_g = c_1^2 = 1 \text{ and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.}$

Set $\sigma = \sigma_{15}$ and let us consider smooth weighted complete intersections of type $(6, 6)$ in $P = P(1, 2, 2, 3, 3)$ with defining equations

\[
\begin{align*}
  f &= z_3^2 + f_0z_4x_0^3 + f_{11}y_1y_3 + f_{22}y_2^3 + f_{012}x_0^2y_1y_2 + f_{000}x_0^6, \\
  g &= z_4^2 + g_0z_3x_0^3 + g_{111}y_1y_3 + g_{222}y_2^3 + g_{012}x_0^2y_1y_2 + g_{000}x_0^6.
\end{align*}
\]

These surfaces are stable under the action of $\sigma$. Denote by

\[
\pi_{15} : \mathcal{X}_{15} \to U_{15}
\]

the smooth family of weighted complete intersections of type $(6, 6)$ in $P(1, 2, 2, 3, 3)$ with equations (0.2) parametrized by their 10 coefficients

\[
(f_0, f_{11}, f_{22}, f_{012}, f_{000}, g_0, g_{111}, g_{222}, g_{012}, g_{000}) \in U_{15} \subset \mathbb{A}^{10}.
\]

The automorphism $\sigma \in \text{Aut}(P)$ has the induced action on the family (0.3) which is trivial on the parameter space $U_{15}$. We abuse the notation $\sigma$ for indicating the induced automorphism of each fibre $X_u = \pi_{15}^{-1}(u) \ (u \in U_{15})$.

There exists a 4-dimensional subgroup $H \subset G \subset \text{Aut}(P)$ (cf. (1.12)) and our Proposition (1.14) asserts that

$U_{15}/H \cong M_{15}$ (and hence dim $M_{15} = 6$)
sending $u \in U_{15}$ to the isomorphism class containing $X_u$, and that, for any $X \in M_{15}$ and for any automorphism $\alpha$ of $X$ of order 3 acting trivially on $H^0(X, K_X)$, there exists a point $u \in U_{15}$ and an isomorphism $\tau: X_u \cong X$ such that $\alpha = \tau_0 \tau^{-1}$.

Let $u_k \in U_{15}$ and set $X_k = X_{u_k}$ ($k = 1, 2$). Take a basis $\omega_{X_k}$ of $H^0(X_k, K_{X_k})$. Set

$$H_2(X_k, \mathbb{Z})^o = \ker\{1 - \sigma: H_2(X_k, \mathbb{Z}) \to H_2(X_k, \mathbb{Z})\}.$$

Now our main theorem in the present paper is stated as follows:

**Theorem (3.4):** Let $u_k \in U_{15}$ ($k = 1, 2$). Suppose that there exists a path $\tilde{\tau}$ in $U_{15}$ joining $u_1$ and $u_2$ which induces an isometry

$$\tau_*: H_2(X_1, \mathbb{Z})^o \to H_2(X_2, \mathbb{Z})^o$$

preserving the periods of integrals of the holomorphic 2-forms $\omega_{X_k}$ on $X_k$, i.e.

$$\int_{\tau_* \gamma} \omega_{X_2} = (\text{constant}) \int_{\gamma} \omega_{X_1} \quad \text{for all } \gamma \in H_2(X_1, \mathbb{Z}),$$

where (constant) is independent of $\gamma$.

Then, there exists an isomorphism $\tau: X_1 \to X_2$ inducing the given isometry $\tau_*$ and such $\tau$ is uniquely determined up to composition with an element of the group $\langle \sigma \rangle$ generated by $\sigma$. We have also $\tau_0 \tau^{-1} = \sigma$ or $\sigma^2$.

Roughly speaking, Theorem (3.4) is proved by applying the Strong Torelli Theorem for algebraic K3 surfaces (cf. [8], [1] and [7]) to the K3 surfaces obtained as the desingularizations of $X_u/\langle \sigma \rangle$ ($u \in U_{15}$).

Our present results can be rephrased in the language of period map as follows. Fix a base point $u_0 \in U_{15}$ and identify $P^2(X_{u_0}, \mathbb{Z}) = L$. Set

$$\tilde{U}_{15} = \left\{(u, \tau_*): u \in U_{15}, \tau_* \in \text{Isom}(P^2(X_u, \mathbb{Z}), L) \text{ coming from a path } \tilde{\tau} \text{ joining } u \text{ and } u_0 \text{ in } U_{15}\right\}$$

and

$$\tilde{\mathcal{P}}_{15} = \tilde{\mathcal{P}}_{15} \times \tilde{U}_{15}.$$
Note that the fibre of $\tilde{U}_{15} \to U_{15}$ is the geometric monodromy group $\Gamma_{U_{15}} = \text{Im}\{\pi_1(U_{15}) \to \text{Aut}(L)\}$. Then we have, as in a similar way as (0.1), the universal family

$$\pi_{15}: \tilde{\mathcal{X}}_{15} = \tilde{\mathcal{X}}'_{15}/H \to \tilde{M}_{15} = \tilde{U}_{15}/H$$

and the period map

$$\Phi_{15}: \tilde{M}_{15} \to D.$$

$\Phi_{15}$ induces a set-theoretic map

$$\bar{\Phi}_{15}: M_{15} \to D/\Gamma_{U_{15}}.$$ 

Our Proposition (1.17) and Theorem (3.4) assert that $\Phi_{15}$ is unramified and $\bar{\Phi}_{15}$ is injective.

The following are unknown at present:

(0.4) Whether $\Phi_{15}$ is an immersion.
(0.5) The description of the difference of $\Gamma_{U_{15}}$ and $\Gamma = \text{Aut}(L)$.
(0.6) The determination of the image of $\Phi_{15}$.
(0.7) The study of the surfaces with automorphisms conjugate to $\sigma_{11}$ or to $\sigma_0$.
(0.8) The determination of all the points of $\tilde{M}$ through which $\Phi$ has 1-dimensional fibres.

Every variety in this paper is a variety over the field $C$ of complex numbers.

1. Surfaces with $p_g = c_1^2 = 1$

1.1. F. Catanese showed in [2] that the canonical models of the surfaces with $p_g = c_1^2 = 1$ are represented as weighted complete intersections of type $(6, 6)$ in $\mathbb{P} = \mathbb{P}(1, 2, 2, 3, 3)$. If we assume furthermore that the canonical invertible sheaf $K_X$ of the surface $X$ in question is ample, the canonical model of $X$ is smooth and hence we can identify $X$ with its canonical model.

Let $R = C[x_0, y_1, y_2, z_3, z_4]$ be the weighted polynomial ring with $\deg x_0 = 1$, $\deg y_1 = \deg y_2 = 2$ and $\deg z_3 = \deg z_4 = 3$. Catanese also showed that the defining equations of the canonical models in question are partially normalized as follows (cf. [2]):

$$\begin{cases} f = z_3^2 + f^{(1)}z_4x_0 + f^{(3)}, \\ g = z_4^2 + g^{(1)}z_3x_0 + g^{(3)}, \end{cases}$$

(1.1)
where \( f^{(1)} \) and \( g^{(1)} \) are linear and \( f^{(3)} \) and \( g^{(3)} \) are cubic forms in \( x_0, y_1 \) and \( y_2 \), i.e., by using the notation \( y_0 = x_0^2 \).

\[
\begin{align*}
 f^{(1)} &= \sum_{0 \leq i \leq 2} f_i y_i, &
 g^{(1)} &= \sum_{0 \leq i \leq 2} g_i y_i, \\
 f^{(3)} &= \sum_{0 \leq i \leq 2} f_{ijk} y_i y_j y_k, &
 g^{(3)} &= \sum_{0 \leq i \leq 2} g_{ijk} y_i y_j y_k.
\end{align*}
\]

(1.2)

Varying these 26 coefficients \( f_i, f_{ijk}, g_i \) and \( g_{ijk} \), we get a family of weighted complete intersections in \( P = \mathbb{P}(1, 2, 2, 3, 3) \). Set

\[
U = \left\{ u \in \mathbb{A}^{26} \mid \text{the corresponding surface is a smooth weighted complete intersections of type (6, 6) in } \mathbb{P}(1, 2, 2, 3, 3) \right\}
\]

(1.3)

and let

\[
\mathcal{X}' \to U
\]

(1.4)

be the family of the surfaces in \( \mathbb{P}(1, 2, 2, 3, 3) \). Note that \( U \) is a Zariski open subset of \( \mathbb{A}^{26} \).

Let \( G \) be the group consisting of the non-degenerate matrices over \( C \) of the forms

\[
\begin{bmatrix}
  d_0 &  &  & 0 \\
  d_{10} & d_{11} & d_{12} & \\
  d_{20} & d_{21} & d_{22} & 0 \\
  0 & d_3 & 0 & d_4
\end{bmatrix}
\]

(1.5)

and

\[
\begin{bmatrix}
  d_0 &  &  & 0 \\
  d_{10} & d_{11} & d_{12} & \\
  d_{20} & d_{21} & d_{22} & 0 \\
  0 & 0 & d_3 & d_4
\end{bmatrix}
\]

(1.6)
acting on $\mathbb{P}(1, 2, 2, 3, 3)$ as

$$
\begin{align*}
  x_0 & \mapsto d_0 x_0 \\
  y_i & \mapsto \sum_{0 \leq j \leq 2} d_{ij} y_j \quad (i = 1, 2) \\
  z_i & \mapsto d_i z_i \quad (i = 3, 4)
\end{align*}
$$

in case (1.5), and

$$
\begin{align*}
  x_0 & \mapsto d_0 x_0 \\
  y_i & \mapsto \sum_{0 \leq j \leq 2} d_{ij} y_j \quad (i = 1, 2) \\
  z_3 & \mapsto d_3 z_4 \\
  z_4 & \mapsto d_4 z_3
\end{align*}
$$

in case (1.6).

Since the canonical invertible sheaves of the surfaces $X_u$ ($u \in U$) are isomorphic to $\mathcal{O}_{X_u}(1)$ and their defining equations are partially normalized as (1.1), we can prove easily that every isomorphism between the surfaces $X_u$ ($u \in U$) is induced from some element in $G$ (see, for detail, [2] or [11]). Hence we see, by [4], that

\begin{equation}
(1.7) \quad U/G = \text{the coarse moduli scheme of complete, smooth surfaces with } p_g = c_1^2 = 1 \text{ and } K \text{ ample.}
\end{equation}

1.2. In [11], we classified the automorphisms of the surfaces $X$ with $p_g = c_1^2 = 1$ and $K_X$ ample, and determined the induced action on $H^2(X, \mathbb{C})$, on $H^{2,0}(X)$ and on $H^1(X, T_X)$.

Among these automorphisms we are mainly interested in the present paper in $\sigma_{15}$ in Theorem (2.14) in [11]. We fix, throughout this paper, the notation

\begin{equation}
(1.8) \quad \sigma = \sigma_{15} = (1, \omega, \omega^2, 1, 1) \in G
\end{equation}

which means the diagonal matrix

$$
\sigma = \begin{pmatrix}
  1 & \omega & 0 \\
  \omega & 0 & 1 \\
  0 & \omega^2 & 1
\end{pmatrix}, \text{ where } \omega = \exp(2\pi \sqrt{-1}/3).
$$
Set

\[(1.9) \quad U_{15} = \{ u \in U \mid \sigma u = u \}\]

and denote by

\[(1.10) \quad \pi'_{15}: \mathcal{X}'_{15} \to U_{15}\]

the family induced from (1.4) by \( U_{15} \hookrightarrow U \). More explicitly, the defining equations of the surfaces \( X_u = \pi^{-1}_{15}(u) \) (\( u \in U_{15} \)) have the following forms:

\[(1.11) \begin{cases} f = z_3^2 + f_0z_4x_0^3 + f_{111}y_1^3 + f_{222}y_2^3 + f_{012}x_0^2y_1y_2 + f_{000}x_0^6, \\ g = z_4^2 + g_0z_3x_0^3 + g_{111}y_1^3 + g_{222}y_2^3 + g_{012}x_0^2y_1y_2 + g_{000}x_0^6. \end{cases}\]

Define

\[H = \{ \tau \in G \mid \tau(U_{15}) \cap U_{15} \neq \emptyset \} .\]

By an elementary calculation using (1.11), we can prove that \( H \) consists of the following four types of matrices:

\[
\begin{array}{c|c|c|c|c}
 d_0 & 0 & 0 \\
 \hline
 d_1 & 0 & d_2 \\
 \hline
 0 & d_3 & 0 \\
 \hline
 0 & 0 & d_4 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
 d_0 & 0 & 0 \\
 \hline
 0 & d_1 & 0 \\
 \hline
 d_2 & 0 & 0 \\
 \hline
 0 & 0 & d_4 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
 d_0 & 0 & 0 \\
 \hline
 d_1 & 0 & 0 \\
 \hline
 0 & d_3 & 0 \\
 \hline
 d_2 & d_4 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
 d_0 & 0 & 0 \\
 \hline
 0 & d_1 & 0 \\
 \hline
 d_2 & 0 & 0 \\
 \hline
 0 & 0 & d_3 \\
\end{array}
\]

We can also prove, by using the forms (1.12), that \( H \) is the normalizer of \( \langle \sigma \rangle \) in \( G \), where \( \langle \sigma \rangle \) is the subgroup of \( G \) generated by \( \sigma \) in (1.8).
Set

(1.13) \( M_{15} \) = the set of the isomorphism classes of the complete, smooth surfaces with \( p_g = c_1^2 = 1 \) and \( K \) ample and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

**Proposition (1.14):** We have a natural bijection \( U_{15}/H \cong M_{15} \) as sets and \( U_{15}/H \) is a 6-dimensional irreducible subvariety of the coarse moduli space \( U/G \) in (1.7). Moreover, for any surface \( X \in M_{15} \) and for any automorphism \( \alpha \) of \( X \) of order 3 acting trivially on \( H^0(X, K_X) \), there exist a point \( u \in U_{15} \) and an isomorphism \( \tau : X_u \cong X \) satisfying \( \alpha = \tau \sigma \tau^{-1} \).

**Proof:** This is an immediate consequence of Theorem (2.14) in [11]. Note that “natural” in the statement of the proposition means that \( H \)-orbit of \( u \in U_{15} \) corresponds to the isomorphism class containing \( X_u \). Q.E.D.

1.3. Let \( X = X_u \) for some \( u \in U_{15} \) and let \( S \) be the parameter space of the Kuranishi family of the deformations of \( X = X_{s_0} \) \((s_0 \in S)\).

\( S \) is smooth and the Kuranishi family is universal (see, for detail, [11]). Hence, \( \sigma \in \text{Aut}(X) \) has the induced action on \( S \) via the identification \( X = X_{s_0} \). Set

(1.15) \( S^\sigma = \{ s \in S \mid \sigma s = s \} \).

Note that, since \( \sigma \) is of finite order, \( S^\sigma \) is a submanifold of \( S \). Note also that \( S^\sigma \) is the parameter space of the universal family of the deformations of the pair \((X, \sigma)\) of the surface \( X \) and \( \sigma \in \text{Aut}(X) \).

Let

(1.16) \( \phi : S \to D \)

be the period map, using the Hodge decomposition of the second primitive cohomology group \( P^2(X_s, \mathbb{C}) \) \((s \in S)\), obtained from the Kuranishi family, where \( D \) is the period domain (see, for detail, [5]).

**Proposition (1.17) (Local Torelli theorem for the restricted family):** The restriction

\( \text{res} \ \phi : S^\sigma \to D \)

of the period map \( \phi \) in (1.16) is injective.
PROOF: First of all, note that \( \sigma \) has induced actions on \( S \) as above and also on \( D \) and that \( \phi \) is \( \sigma \)-equivariant with these induced actions. Let

\[
d\phi(s_0): T_S(s_0) \to T_D(\phi(s_0))
\]

be the differential map of the period map \( \phi \) at \( s_0 \in S \). Since \( T_S(s_0) \) (resp. \( T_D(\phi(s_0)) \)) can be identified with \( H^1(X, T_X) \) (resp. \( \text{Hom}(P^{2,0}(X), P^{1,1}(X)) \)), we know, from Theorem (2.14) in [11], that the decomposition of \( T_S(s_0) \) and \( T_D(\phi(s_0)) \) into their eigen spaces under the action of \( \sigma \) are the following:

\[
T_S(s_0) = T_1 \oplus T_\omega \oplus T_\omega^2 \quad \text{with dim } T_1 = \text{dim } T_\omega = \text{dim } T_\omega^2 = 6,
T_D(\phi(s_0)) = T'_1 \oplus T'_\omega \oplus T'_\omega^2 \quad \text{with dim } T'_1 = 8,
\]

\[
\text{dim } T'_\omega = \text{dim } T'_\omega^2 = 5,
\]

where \( T_\lambda \) (resp. \( T'_\lambda \)) is the \( \lambda \)-eigen subspace of \( T_S(s_0) \) (resp. \( T_D(\phi(s_0)) \)).

Since \( d\phi(s_0) \) is also \( \sigma \)-equivariant, \( d\phi(s_0) \) is compatible with the decompositions in (1.18). Hence, from (1.18), \( \text{Ker } d\phi(s_0) \) contains at least 2-dimensional subspace of \( T_\omega \oplus T_\omega^2 \). On the other hand, it can be shown easily (cf. [6], [2] or [11]) that \( \dim \text{Ker } d\phi(s_0) \leq 2 \). Thus, we can conclude that

\[
(1.19) \quad T_1 \cap \text{Ker } d\phi(s_0) = \{0\}.
\]

Since \( T_{S^\sigma}(s_0) = T_1 \), (1.19) means that

\[
\text{res } d\phi(s_0): T_{S^\sigma}(s_0) \to T_D(\phi(s_0))
\]

is injective. This shows that

\[
\text{res } \phi: S^\sigma \to D
\]

is injective, because we consider \( S^\sigma \) as germ.

Q.E.D.

2. Structure theorem

We continue to use the notation in the previous section.

2.1. Let \( X = X_u \) (\( u \in U_{15} \)). Since \( \sigma = (1, \omega, \omega^2, 1, 1) \) (see (1.18)), the fixed points of \( X \) by \( \sigma \) satisfy the equations

\[
x_0 = y_1 = 0,
\]
We can calculate easily that

the intersection number of the curves \((x_0 = 0)\) and \((y_i = 0)\) = 2 \((i = 1, 2)\)
the intersection number of the curves \((y_1 = 0)\) and \((y_2 = 0)\) = 4.

Moreover, since \(\sigma \in \text{Aut}(X)\) is of finite order, the fixed points locus \(X^\sigma\) of \(X\) by \(\sigma\) is smooth. Thus we get that \(X^\sigma\) consists of 8 distinct points. We denote these points by

\[
X = \{D_i, E_i \mid (i = 1, 2, 3, 4)\},
\]
where
\(D_i (i = 1, 2)\) satisfy the equations (2.1),
\(D_i (i = 3, 4)\) satisfy the equations (2.2) and
\(E_i (i = 1, 2, 3, 4)\) satisfy the equations (2.3).

Since we can take \(x_0z_3/y_2^2, y_1/y_2\) (resp. \(x_0z_3/y_1^2, y_2/y_1\); resp. \(y_1/x_0^2, y_2/x_0^2\)) as local coordinates of \(X\) at \(D_i (i = 1, 2)\) (resp. \(D_i (i = 3, 4)\) resp. \(E_i (i = 1, 2, 3, 4)\)), we see that the induced actions of \(\sigma\) on the normal spaces of these points in \(X\) are

\[
(\omega_2, \omega_2^2) \quad \text{at} \quad D_i (i = 1, 2),
\]
\[
(\omega, \omega) \quad \text{at} \quad D_i (i = 3, 4) \quad \text{and}
\]
\[
(\omega, \omega^2) \quad \text{at} \quad E_i (i = 1, 2, 3, 4).
\]

Let

\[
\tilde{X} \to X
\]
be the blowing-up of \(X\) with center \(X^\sigma\). Denote by

\[
\tilde{D}_i \quad \text{and} \quad \tilde{E}_i \quad (i = 1, 2, 3, 4)
\]
the exceptional curves on \(\tilde{X}\) corresponding to the points \(D_i\) and \(E_i\) on \(X\) respectively.

The action of \(\sigma\) extends naturally on \(\tilde{X}\) so that the morphism (2.6) is \(\sigma\)-equivariant. From (2.5), we see that there are 2 distinct points, say

\[
\tilde{E}_{ij} \quad (j = 1, 2).
\]
on each $\tilde{E}_i$ which are fixed by $\sigma$, and the fixed points locus $\tilde{X}^\sigma$ of $\tilde{X}$ by $\sigma$ is

\[(2.9)\quad \tilde{X}^\sigma = \{ \tilde{D}_n, \tilde{E}_{ij} \ (i = 1, 2, 3, 4; j = 1, 2) \}.
\]

We know, also from (2.5), that the induced action of $\sigma$ on the normal bundle of each component of $\tilde{X}^\sigma$ in $\tilde{X}$ is

\[(2.10)\quad \begin{align*}
(\omega^3) \quad & \text{along } \tilde{D}_i \quad (i = 1, 2), \\
(\omega) \quad & \text{along } \tilde{D}_i \quad (i = 3, 4), \\
(\omega, \omega) \quad & \text{at } \tilde{E}_{i1} \quad (i = 1, 2, 3, 4) \quad \text{and} \\
(\omega^2, \omega^2) \quad & \text{at } \tilde{E}_{i2} \quad (i = 1, 2, 3, 4).
\end{align*}
\]

Let

\[(2.11)\quad \tilde{X} \to \tilde{X}
\]

be the blowing-up of $\tilde{X}$ with center $\tilde{X}^\sigma$. Denote by

\[(2.12)\quad \tilde{D}_n, \tilde{E}_i \quad \text{and} \quad \tilde{E}_{ij} \quad (i = 1, 2, 3, 4; j = 1, 2)
\]

the curves on $\tilde{X}$ which are the inverse images of $\tilde{D}_n$, the proper transforms of $\tilde{E}_i$ and the exceptional divisors corresponding to $\tilde{E}_{ij}$ respectively.

The action of $\sigma$ extends again to $\tilde{X}$ and we see, from (2.10), that the fixed points locus $\tilde{X}^\sigma$ of $\tilde{X}$ by $\sigma$ is now a disjoint union of 12 curves, i.e.

\[(2.13)\quad \tilde{X}^\sigma = \{ \tilde{D}_n, \tilde{E}_{ij} \ (i = 1, 2, 3, 4; j = 1, 2) \}.
\]

From (2.10) again, we know that the induced action of $\sigma$ on the normal bundle of each component of $\tilde{X}^\sigma$ in $\tilde{X}$ is the following:

\[(2.14)\quad \begin{align*}
(\omega) \quad & \text{along } \tilde{D}_i \quad (i = 3, 4) \quad \text{and} \quad \tilde{E}_{i1} \quad (i = 1, 2, 3, 4), \\
(\omega^2) \quad & \text{along } \tilde{D}_i \quad (i = 1, 2) \quad \text{and} \quad \tilde{E}_{i2} \quad (i = 1, 2, 3, 4).
\end{align*}
\]

We denote by

\[(2.15)\quad p : \tilde{X} \to X
\]

the composite morphism of (2.11) and (2.6). Note that $p$ is $\sigma$-equivariant.
We can calculate easily the self-intersection numbers of the exceptional curves on $\hat{X}$ of the morphism $p$:

\[(\hat{D}_i)^2 = (\hat{E}_{ij})^2 = -1, \quad (\hat{E}_i)^2 = -3 \quad (i = 1, 2, 3, 4; j = 1, 2).\]

Denote by

\[(2.17) \quad C \text{ and } \hat{C}\]

the canonical divisor of $X$ and its proper transform by $p$ in (2.15). Since $x_0 = 0$ is the homogeneous equation of $C$ in $X$, $C$ contains 4 points $D_i$ $(i = 1, 2, 3, 4)$ in (2.4). From this fact we get that

\[(2.18) \quad (\hat{C})^2 = -3.\]

2.2. Since $\sigma \in \text{Aut}(\hat{X})$ is of order 3 and $\hat{X}^\sigma$ is of pure codimension 1, we get a ramified triple covering

\[(2.19) \quad r : \hat{X} \rightarrow \hat{Y},\]

where $\hat{Y} = \hat{X}/(\sigma)$ is smooth. We denote by $\hat{R}$ the ramification locus and by $\hat{B}$ the branch locus of $r$, i.e.

\[(2.20) \quad \hat{R} = \hat{X}^\sigma = \sum_{1 \leq i \leq 4} \hat{D}_i + \sum_{1 \leq i \leq 4, j = 1, 2} \hat{E}_{ij} \quad \text{and} \quad \hat{B} = r(\hat{R}).\]

We consider $\hat{R}$ and $\hat{B}$ as reduced curves.

We use the notation

\[(2.21) \quad \hat{C}' = r(\hat{C}), \quad \hat{D}'_i = r(\hat{D}_i), \quad \hat{E}'_i = r(\hat{E}_i) \quad \text{and} \quad \hat{E}'_{ij} = r(\hat{E}_{ij}),\]

where all these curves are considered as reduced curves on $\hat{Y}$.

**Lemma (2.22):** All the curves in (2.21) are smooth, irreducible, rational curves with self-intersection numbers

\[(\hat{C}')^2 = (\hat{E}_i')^2 = -1 \quad \text{and} \quad (\hat{D}_i')^2 = (\hat{E}_{ij}')^2 = -3 \quad (i = 1, 2, 3, 4; j = 1, 2).\]

**Proof:** We see easily that $C$ is a smooth curve of genus 2 by the Jacobian criterion and adjunction formula. Hence, so is $\hat{C}$, because $\hat{C}$ is isomorphic to $C$. From the construction, we know that

$\hat{C} \rightarrow \hat{C}'$
is a triple covering ramified at 4 distinct points $\hat{C} \cap (\sum_{1 \leq i \leq 4} \hat{D}_i)$. Hence, we see that $\hat{C}'$ is a smooth, irreducible, rational curve by the Hurwitz formula.

In the same way, by using the fact that $\hat{E}_i \to \hat{E}'_i$

is a triple covering ramified at 2 distinct points $\hat{E}_i \cap (\hat{E}_{i1} + \hat{E}_{i2})$, we can prove that $\hat{E}'_i$ are also smooth, irreducible, rational curves.

The same assertion for the curves $\hat{D}'_i$ and $\hat{E}'_{ij}$ is trivial because they are isomorphic to $\hat{D}_i$ and $\hat{E}_{ij}$ respectively.

As for the statement for the self-intersection numbers, we can obtain immediately from (2.16) and (2.18) by the projection formula. Q.E.D.

2.3. Let

\[(2.23) \quad q : \hat{Y} \to Y\]

be the morphism obtained by blowing-down the exceptional curves of the first kind $\hat{C}'$ and $\hat{E}'_i \ (i = 1, 2, 3, 4)$. Set

\[(2.24) \quad C' = q(\hat{C}'), \quad E'_i = q(\hat{E}_i), \quad D'_i = q(\hat{D}_i) \quad \text{and} \quad E'_{ij} = q(\hat{E}'_{ij})\]

\[(i = 1, 2, 3, 4; j = 1, 2).\]

Then, $C'$ and $E'_i$ are points, and $D'_i$ and $E'_{ij}$ are smooth, irreducible, rational curves with self-intersection number $-2$.

We write down the configurations of the points and the curves appeared in 2.1, 2.2 and 2.3 with their self-intersection numbers:
2.4. Now we can state the relation of our surfaces with K3 surfaces. We use the notation in 2.1, 2.2 and 2.3.

**Proposition (2.26) (Structure theorem):** Set \( X = X_u \ (u \in U^*) \). Then, starting from \( X \), we can construct a diagram

\[
\begin{array}{c}
X \\ \downarrow^p \\
\hat{X} \\ \downarrow^r \\
Y \\ \downarrow^q \\
\hat{Y}
\end{array}
\]

where

(i) \( p \) is the morphism in (2.15), i.e. the morphism obtained by a sequence of blowings-up at the fixed points by \( \sigma \), so that the fixed points locus in \( \hat{X} \) under the induced action of \( \sigma \) is of pure codimension 1,

(ii) \( r \) is the morphism in (2.19), i.e. the natural projection onto the quotient of \( \hat{X} \) by the group \((\sigma)\) generated by \( \sigma \), and

(iii) \( q \) is the morphism in (2.23), i.e. the morphism obtained by blowing-down onto the minimal model \( Y \).

Moreover, we have that

(iv) \( Y \) is a minimal K3 surface,

(v) \( 3(\Sigma_{1 \leq i \leq 4} D_i'') - 2(\Sigma_{1 \leq i \leq 4, j=1,2} E_i'') \) is an ample divisor on \( Y \), and

(vi) \( \pi_1(\hat{X} - \hat{R}) = \{1\} \), where \( \hat{R} \) is the ramification locus of \( r \).

**Proof:** The remaining things to prove are the assertions (iv), (v) and (vi).
First, we will prove (iv). By the construction of $Y$, it is clear that the unique holomorphic 2-form on $X$, vanishing on $C$ and $\sigma$-invariant, gives a nowhere vanishing holomorphic 2-form on $Y$. Combining this with $q(Y) \leq q(X) = 0$, we get (iv).

For the proof of (v), we use the configuration (2.25). First of all, we see that

\begin{equation}
\begin{aligned}
(2.27) \quad & \left(3 \left( \sum_{1 \leq i \leq 4} D'_i \right) - 2 \left( \sum_{1 \leq i \leq 4, j=1,2} E'_{ij} \right) \right)^2 \\
& = 9 \left( \sum D'_i \right)^2 + 4 \left( \sum E'_{ij} \right)^2 = 4 > 0.
\end{aligned}
\end{equation}

By the assumption, $C$ is ample and hence so is

$$p^*(4C) - \left( \sum \hat{D}_i + \sum \hat{E}_i + 2 \left( \sum \hat{E}_{ij} \right) \right)$$

$$= 4\hat{C} - \left( \sum \hat{E}_i \right) + 3 \left( \sum \hat{D}_i \right) - 2 \left( \sum \hat{E}_{ij} \right).$$

Since $r$ is a finite morphism and

$$3 \left( 4\hat{C} - \left( \sum \hat{E}_i \right) + 3 \left( \sum \hat{D}_i \right) - 2 \left( \sum \hat{E}_{ij} \right) \right)$$

$$= r^* \left( 12\hat{C}' - 3 \left( \sum \hat{E}'_i \right) + 3 \left( \sum \hat{D}'_i \right) - 2 \left( \sum \hat{E}'_{ij} \right) \right),$$

we see that

$$12\hat{C}' - 3 \left( \sum \hat{E}'_i \right) + 3 \left( \sum \hat{D}'_i \right) - 2 \left( \sum \hat{E}'_{ij} \right)$$

is an ample divisor on $\hat{Y}$. Denote this divisor by $F$. Since $\hat{C}'$ and $\hat{E}'_i$ are the exceptional curves of the morphism $q$, we see, by the Nakai criterion of ampleness for $F$, that for any integral curve $Z$ on $Y$

\begin{equation}
\begin{aligned}
(2.28) \quad & \left(3 \left( \sum D'_i \right) - 2 \left( \sum E'_{ij} \right), Z \right) \\
& = \left( q^* \left( 3 \left( \sum D'_i \right) - 2 \left( \sum E'_{ij} \right) \right), q^*Z \right) = (F, q^*Z) > 0.
\end{aligned}
\end{equation}

Thus, the assertion (v) follows from (2.27) and (2.28) by the Nakai criterion again.
Finally, we will prove (vi). We use the result in [2]:

$$\pi_1(X) = \{1\}.$$  

Since $X^\sigma$ consists of finite points, we see that

$$(2.29) \quad \pi_1(X - X^\sigma) = \pi_1(X) = \{1\}.$$  

By using (2.29) and the following diagram

$$X - X^\sigma \cong \hat{X} - \left( \hat{\mathcal{R}} + \sum_{1 \leq i \leq 4} \hat{E}_i \right)$$

$$\cap \hat{X} - \hat{k},$$

we get our assertion (vi). Q.E.D.

3. Torelli theorem

In this section, we will prove the Torelli theorem for the surfaces with $p_g = c_1^2 = 1$, with an ample canonical divisor and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

We continue to use the notation in the previous sections.

First, we give an elementary lemma which can be verified easily by a standard argument using the discreteness of integral homology groups.

**Lemma (3.1):** Let $\psi$ be a morphism of smooth families $\{V_t\}_{t \in T}$ and $\{W_t\}_{t \in T}$ of compact, complex manifolds over a complex manifold $T$ and suppose we are given a path $\alpha$ in $T$ joining two points $t$ and $t'$ in $T$.

Then, we have a commutative diagram

$$
\begin{array}{ccc}
H_n(V_t, \mathbb{Z}) & \xrightarrow{\phi_*} & H_n(W_t, \mathbb{Z}) \\
\downarrow{\alpha_*} & & \downarrow{\alpha_*} \\
H_n(V_{t'}, \mathbb{Z}) & \xrightarrow{\phi_*} & H_n(W_{t'}, \mathbb{Z})
\end{array}
$$

for all $n$, where $\alpha_*$ is the isomorphism obtained by a $C^\infty$-trivialization along the path $\alpha$, and this $\alpha_*$ is compatible with intersection products.
Let $\pi_{15}: \mathcal{X}_{15} \to U_{15}$ be the family in (1.10). For any two points $u_k \in U_{15}$ ($k = 1, 2$), taking a path $\bar{\tau}$ in $U_{15}$ joining $u_1$ and $u_2$ and applying Lemma (3.1), we get a commutative diagram

$$
\begin{array}{c}
H_2(X_1, \mathbb{Z}) \xrightarrow{1-\sigma} H_2(X_1, \mathbb{Z}) \\
\downarrow \tau_* \downarrow \tau_* \\
H_2(X_2, \mathbb{Z}) \xrightarrow{1-\sigma} H_2(X_2, \mathbb{Z})
\end{array}
$$

where $X_k = \pi_{15}^{-1}(u_k)$ and $\tau_*$ is the isometry obtained from the path $\bar{\tau}$. Hence, we get the induced isometry

$$\tau_* : H_2(X_1, \mathbb{Z})^\sigma \to H_2(X_2, \mathbb{Z})^\sigma$$

of the kernels of $1 - \sigma$ in (3.2).

**Theorem (3.4):** Suppose we are given two points $u_k \in U_{15}$ ($k = 1, 2$) and a path $\bar{\tau}$ in $U_{15}$ joining $u_1$ and $u_2$, and suppose the induced isometry $\tau_*$ in (3.3) preserves the periods of integrals of the holomorphic 2-forms $\omega_{X_k}$ on $X_k = \pi_{15}^{-1}(u_k)$ ($k = 1, 2$), i.e.

$$\int_{\gamma} \omega_{X_2} = (\text{constant}) \int_{\gamma} \omega_{X_1}$$

for all $\gamma \in H_2(X_1, \mathbb{Z})^\sigma$, where (constant) is independent of $\gamma$.

Then, there exists an isomorphism

$$\tau : X_1 \simeq X_2$$

inducing the given $\tau_*$ and such $\tau$ is uniquely determined up to composition with an element of the group $\langle \sigma \rangle$ generated by $\sigma$. We have also $\tau \sigma \tau^{-1} = \sigma$ or $\sigma^2$.

**Proof:** Starting from the family (1.10), we can construct, in a similar way as in the section 2, a commutative diagram

$$
\begin{array}{c}
\mathcal{X}'_{15} \xrightarrow{\hat{p}} \mathcal{X} \xrightarrow{\bar{\tau}} \mathcal{Y} \xrightarrow{\hat{q}} \mathcal{Y} \\
\downarrow \pi_{15} \downarrow \hat{\pi} \downarrow \hat{\pi}' \\
U_{15}
\end{array}
$$

(3.5)
whose fibre over every point of $U_{15}$ satisfies the properties (i) to (vi) in Proposition (2.26). In fact, $\tilde{p}$ and $\tilde{r}$ in (3.5) can be constructed just in the same way as $p$ and $r$ in the section 2, and the construction of $\tilde{q}$ in (3.5) is justified by the result in [3].

For $k = 1, 2$, set $X_k = \pi^{-1}(u_k)$, $\hat{Y}_k = \pi^{-1}(u_k)$, and $Y_k = \pi^{-1}(u_k)$, and let $p_k: \hat{X}_k \to X_k$, $r_k: \hat{X}_k \to \hat{Y}_k$ and $q_k: \hat{Y}_k \to Y_k$ be the restrictions to the fibres of the morphisms $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ in (3.5) respectively. We denote by $\tilde{D}_i^{(k)}, \tilde{E}_i^{(k)}$ and $\tilde{E}_j^{(k)}$ the corresponding curves on $\hat{X}_k$ and by $C_i^{(k)}, D_i^{(k)}, E_i^{(k)}$ and $E_j^{(k)}$ the corresponding points and curves on $Y_k$ ($k = 1, 2$) constructed in the section 2. Denote also by $\hat{R}_k$ and $\hat{B}_k$ the ramification locus and the branch locus of the triple covering $r_k: \hat{X}_k \to \hat{Y}_k$ ($k = 1, 2$). For a divisor $F$ on a surface, we denote by $[F]$ the integral homology class represented by $F$.

Then, by Lemma (3.1), we get, from (3.5), the commutative diagram of homology groups:

$$
\begin{array}{cccccccc}
H_2(X_1, \mathbb{Z}) & \xleftarrow{p_{1*}} & H_2(\hat{X}_1, \mathbb{Z}) & \xrightarrow{\tau_1} & H_2(\hat{Y}_1, \mathbb{Z}) & \xrightarrow{q_{1*}} & H_2(Y_1, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(X_2, \mathbb{Z}) & \xleftarrow{p_{2*}} & H_2(\hat{X}_2, \mathbb{Z}) & \xrightarrow{\tau_2} & H_2(\hat{Y}_2, \mathbb{Z}) & \xrightarrow{q_{2*}} & H_2(Y_2, \mathbb{Z})
\end{array}
$$

(3.6)

$$
\begin{array}{cccccccc}
H_2(X_1, \mathbb{Z}) & \xleftarrow{p_{1*}} & H_2(\hat{X}_1, \mathbb{Z}) & \xrightarrow{\tau_1} & H_2(\hat{Y}_1, \mathbb{Z}) & \xrightarrow{q_{1*}} & H_2(Y_1, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(X_2, \mathbb{Z}) & \xleftarrow{p_{2*}} & H_2(\hat{X}_2, \mathbb{Z}) & \xrightarrow{\tau_2} & H_2(\hat{Y}_2, \mathbb{Z}) & \xrightarrow{q_{2*}} & H_2(Y_2, \mathbb{Z})
\end{array}
$$

(3.6)

where $\hat{\tau}_*, \hat{\tau}'_*$ and $\tau'_*$ are the induced isometries, like $\tau_*$, from the path $\hat{\tau}$. By our construction of (3.5), we see that

$$
\begin{aligned}
\hat{\tau}_*([\tilde{D}_i^{(1)}]) &= [\tilde{D}_i^{(2)}], & \hat{\tau}_*([\tilde{E}_i^{(1)}]) &= [\tilde{E}_i^{(2)}], & \hat{\tau}_*([\tilde{E}_j^{(1)}]) &= [\tilde{E}_j^{(2)}], \\
\hat{\tau}'_*([\hat{B}_i]) &= [\hat{B}_2], & \tau'_*([D_i^{(1)}]) &= [D_i^{(2)}], & \tau'_*([E_i^{(1)}]) &= [E_i^{(2)}].
\end{aligned}
$$

(3.7)

Note also that $p_{k*}p_k^* = id$, $q_{k*}q_k^* = id$, $r_{k*}r_k^* = 3id$ and $r_k^*r_{k*} = 3id$ ($k = 1, 2$).

Let $\omega_{\hat{X}_k}$ (resp. $\omega_{\hat{Y}_k}, \omega_{Y_k}$) be the holomorphic 2-form on $\hat{X}_k$ (resp.
\( \hat{Y}_k, Y_k \) induced from \( \omega_{X_k} \) \((k = 1, 2)\). Since

\[
\int_\gamma \omega_{Y_k} = \int_{\hat{\gamma}_k} \omega_{\hat{Y}_k} = 3 \int_{\hat{\gamma}_k} \omega_{\hat{X}_k} = 3 \int_{p_{k*} \hat{\gamma}_k} \omega_{X_k}
\]

for any \( \gamma \in H_2(Y_k, \mathbb{Z}) \), we can deduce, by (3.6), the property

\[
\int_{\tau_* \gamma} \omega_{Y_2} = (\text{constant}) \int_\gamma \omega_{Y_1} \quad \text{for all } \gamma \in H_2(Y_1, \mathbb{Z})
\]

from that on \( X_k \).

Since

\[
\tau'_* \left( \left[ 3 \left( \sum_i D_i^{(1)} \right) - 2 \left( \sum_{ij} E_{i}^{(1)} \right) \right] \right) = \left[ 3 \left( \sum_i E_i^{(2)} \right) - 2 \left( \sum_{ij} E_{ij}^{(2)} \right) \right]
\]

from (3.7), we see, by (v) in Proposition (2.26), that \( \tau'_* \) sends some ample divisor class on \( Y_1 \) to an ample divisor class on \( Y_2 \).

Hence, we can apply the Strong Torelli Theorem for algebraic K3 surfaces proved and supplemented in [8], [1] and [7] to our case, and we see that there exists uniquely the isomorphism

\( \tau': Y_1 \cong Y_2 \)

inducing the isometry \( \tau'_* \) in (3.6).

Considering (3.7) and intersection numbers, we can observe easily

\( \tau'(D_i^{(1)}) = D_i^{(2)} \) and \( \tau'(E_{ij}^{(1)}) = E_{ij}^{(2)} \)

and hence, in particular,

\( \tau'(C_i^{(1)}) = C_i^{(2)} \) and \( \tau'(E_i^{(1)}) = E_i^{(2)} \).

Therefore, by the construction of \( q_k: \hat{Y}_k \to Y_k \), \( \tau' \) can be lifted uniquely to an isomorphism

\( \hat{\tau}': \hat{Y}_1 \cong \hat{Y}_2 \)

inducing the isometry \( \hat{\tau}'_* \) in (3.6).

Considering (3.7) and intersection numbers again, we see

\( \hat{\tau}'(\hat{B}_1) = \hat{B}_2 \).
Since we know that \( r_k : \tilde{X}_k - k \to \tilde{Y}_k - k \) are universal coverings by (vi) in Proposition (2.26), there exists an isomorphism
\[
\hat{\tau} : \tilde{X}_1 - \tilde{R}_1 \to \tilde{X}_2 - \tilde{R}_2
\]
compatible with \( \tau' \). Such \( \hat{\tau} \) are unique up to the covering transformation group \( \langle \sigma \rangle \). Now, by the Riemann Extension Theorem, \( \hat{\tau} \) extends uniquely to an isomorphism
\[
\hat{\tau} : \tilde{X}_1 \to \tilde{X}_2,
\]
where we abuse the notation \( \hat{\tau} \). \( \hat{\tau} \) is compatible with \( \hat{\tau}' \) and hence induces the isometry \( \hat{\tau}_* \) in (3.6).

By the argument on intersection numbers, we get, from (3.7), that
\[
\hat{\tau}(\tilde{D}_i^{(1)}) = \tilde{D}_i^{(2)}, \quad \hat{\tau}(\tilde{E}_j^{(1)}) = \tilde{E}_j^{(2)} \quad \text{and} \quad \hat{\tau}(\tilde{E}_j^{(1)}) = \tilde{E}_j^{(2)}.
\]
Hence, \( \hat{\tau} \) descends uniquely to an isomorphism
\[
\tau : X_1 \to X_2
\]
inducing the given isometry \( \tau_* \).

The other assertion follows easily. Q.E.D.

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