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Studies of closed/open mirror symmetry for quintic threefolds through log mixed Hodge theory

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Dedicated to Kazuhiro Konno and dedicated to James D. Lewis on his sixtieth birthday

ABSTRACT. We correct the definitions and descriptions of the integral structures in [30]. The previous flat basis in [ibid] is characterized by the Frobenius solutions and is integral in the first approximation by mean of the graded quotients of monodromy filtration, but it is not integral in the strict sense. In this article, we use $\hat{\Gamma}$-integral structure of Iritani in [7] for A-model. Using this precise version, we study open mirror symmetry for quintic threefolds through log mixed Hodge theory, especially the recent result on Néron models for admissible normal functions with non-torsion extensions in the joint work [14] with K. Kato and C. Nakayama. We understand asymptotic conditions as values in the fiber over a base point on the boundary of $S^{\log}$.

Contents

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0 Introduction

In a series of joint works with K. Kato and C. Nakayama, we are constructing a fundamental diagram which consists of various kind of partial compactifications of classifying space of mixed Hodge structures and their relations. We try to understand mirror symmetry in this framework of the fundamental diagram. In this paper, we first complete the insufficient results 3.5–3.6 in the previous paper of Usui [30] (see Remark in 2.6 below), and then study open mirror symmetry for quintic threefolds through log mixed Hodge theory, especially the fine moduli of log Hodge structures and Néron models over it.

0.1. Fundamental Diagram

For a classifying space D of Hodge structures of specified type, we have

$$
\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \rightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
\Gamma\backslash D_{\Sigma,\mathrm{val}} & \leftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \xrightarrow{\psi} & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
\Gamma\backslash D_{\Sigma} & \leftarrow & D^{\sharp}_{\Sigma} & & & &
\end{array}
$$

(Γ is a monodromy group) in pure case: [15], [16], [17]. For mixed case, we should extend to an amplified diagram: [9], [10], [12], [13], continuing.

0.2. Mirror symmetry for quintic threefolds

Let V be a quintic threefold in $\mathbf{P}^4$ and V°_ψ be its mirror family (cf. [1], Sect. 4.2). For simplicity, we denote the family V°_ψ simply by V° if there would be no confusions.

Mirror symmetry for the A-model of quintic threefold V and the B-model of its mirror V° was predicted by Candelas–de la Ossa–Green–Parks in the famous paper [2]. We recall two styles of the theorem 0.2.1 and 0.2.2

below. Every statement in the present paper is near the large radius point q_0 of the complexified Kähler moduli $\mathcal{KM}(V)$ and the maximally unipotent monodromy point p_0 of the complex moduli $\mathcal{M}(V^\circ)$.

Let $t := y_1/y_0$, $u := t/2\pi i$ be the canonical parameters and $q := e^t = e^{2\pi i u}$ be the canonical coordinate for B-model in 2.2 below and the respective ones for A-model in 2.3 below.

The following theorem is due to Givental [5] and Lian–Liu–Yau [20].

Theorem 0.2.1. (Potential). *The potentials of the two models coincide:* $\Phi^V_{\mathrm{GW}}(t) = \Phi^{V^\circ}_{\mathrm{GM}}(t)$.

Morrison [22] formulated the following style Theorem 0.2.2 and proved the theorem except integral structure. Iritani [7] defined $\hat{\Gamma}$-integral structure for A-model and proved the theorem completely for wider objects.

Theorem 0.2.2. (Variation of Hodge structure). *The isomorphism* $(q_0 \in \overline{\mathcal{KM}}(V)) \overset{\sim}{\leftarrow} (p_0 \in \overline{\mathcal{M}}(V^\circ))$ *of neighborhoods of the compactifications, by the canonical coordinates* $q = \exp(2\pi i u)$, *lifts to an isomorphism, over the punctured neighborhoods* $\mathcal{KM}(V) \overset{\sim}{\leftarrow} \mathcal{M}(V^\circ)$, *of polarized* $\mathbf{Z}$*-variations of Hodge structure with a specified section*

$$(\mathcal{H}^V, S, \nabla^{\mathrm{even}}, \mathcal{H}^V_{\mathbf{Z}}, F; T^3) \overset{\sim}{\leftarrow} (\mathcal{H}^{V^\circ}, Q, \nabla^{\mathrm{GM}}, \mathcal{H}^{V^\circ}_{\mathbf{Z}}, F; \tilde{\Omega}).$$

Our Theorem 0.2.3 below is equivalent to Theorem 0.2.1 and Theorem 0.2.2 by a log version [17], Theorem 2.5.14 of the nilpotent orbit theorem of Schmid [27] (this part of [30] is valid).

Theorem 0.2.3. (Log Hodge structure, Log pperiod map). *The isomorphism* $(q_0 \in \overline{\mathcal{KM}}(V)) \overset{\sim}{\leftarrow} (p_0 \in \overline{\mathcal{M}}(V^\circ))$ *of neighborhoods of the compactifications uniquely lifts to an isomorphism of B-model log variation of polarized Hodge structure with a specified section* $\tilde{\Omega}$ *for* V° *and A-model log variation of polarized Hodge structure with a specified section* T^3 *for* V, *whose restriction over the punctured* $\mathcal{KM}(V) \overset{\sim}{\leftarrow} \mathcal{M}(V^\circ)$ *coincides with the isomorphism of variations of polarized Hodge structure with specified sections in Theorem 0.2.2.*

This rephrases as follows. Let σ *be the common monodromy cone, transformed by a level structure into* End *of a reference fiber of the local system, for the A-model and for the B-model. Then, we have a commutative diagram of horizontal log period maps*

$$\begin{array}{ccc} (q_0 \in \overline{\mathcal{KM}}(V)) & \overset{\sim}{\leftarrow} & (p_0 \in \overline{\mathcal{M}}(V^\circ)) \\ & \searrow \quad \swarrow & \\ & ([\sigma, \exp(\sigma_{\mathbf{C}})F_0] \in \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma) & \end{array}$$

with extensions of specified sections in Theorem 0.2.2, where $[\sigma, \exp(\sigma_{\mathbf{C}})F_0]$ *is the class of the nilpotent orbit, regarded as a boundary point, and* $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ *is the fine moduli of log Hodge structures of specified type which will be explained in Section 1 below.*

0.3. Open mirror symmetry for quintic threefolds

The following theorem is due to Walcher [31] and Morrison–Walcher [23].

Theorem 0.3.1. (Inhomogenous solutions). *The potentials of the two models coincide: Let* $\mathcal{L}$ *be the Picard–Fuchs differential operator for quintic mirror family (cf. 2.2 below). Let*

$$\mathcal{T}_A = \frac{u}{2} \pm \Big(\frac{1}{4} + \frac{1}{2\pi^2}\sum_{d\,odd} n_d q^{d/2}\Big)$$

be the A-model domainwall tension in [23], where the n_d *are open Gromov–Witten invariants, and*

$$\mathcal{T}_B = \int_{C_-}^{C_+} \Omega$$

be the B-model domainwall tension, where $C_\pm \subset V^\circ$ *are the disjoint smooth curves coming from the two conics in* $\{x_1 + x_2 = x_3 + x_4 = 0\} \cap V_\psi \subset \mathbf{P}^4(\mathbf{C})$ *[ibid].*

Then

$$\mathcal{L}(y_0(z)\mathcal{T}_A(z)) = \mathcal{L}(\mathcal{T}_B(z))\Big(= \frac{15}{16\pi^2}\sqrt{z}\Big) \quad \Big(z = \frac{1}{(5\psi)^5}\Big).$$

Concerning this, we have the following observations.

0.4. Log mixed Hodge structure, Log normal function

We describe for B-model. The same holds for A-model by 0.2–0.3 and the correspondence table in 2.5 below.

Put $\mathcal{H} := \mathcal{H}^{V^\circ}$ and $\mathcal{T} := \mathcal{T}_B$. We use $e^0 \in I^{0,0}$, $e^1 \in I^{1,1}$ which are a part of a basis of $\mathcal{H}_{\mathcal{O}}$ respecting the Deligne decomposition at p_0 (see 2.5 (3B)) and a part of flat basis $s^0 = e^0$, $s^1 = e^1 - (u-1)e^0$ of $\mathcal{H}_{\mathbf{Z}}$ (see 2.5 (7B)). To make the local monodromy of $\mathcal{T}$ unipotent, we take a double cover $z^{1/2} \mapsto z$. Let $L_{\mathbf{Q}}$ be the translated local system comparing to the trivial extension $\mathbf{Q}(-2) \oplus \mathcal{H}_{\mathbf{Q}}$ by $(0, -(\mathcal{T}/y_0)s^0)$ in $\mathcal{E}xt^1(\mathbf{Q}(-2), \mathcal{H}_{\mathbf{Q}})$. Let $J_{L_{\mathbf{Q}}}$ be the Néron model on a neighborhood S of p_0 in the $z^{1/2}$-plane which lies over $L_{\mathbf{Q}}$ in [14] (there is a difference of Tate twist). Then, $J_{L_{\mathbf{Q}}} = \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}(-2), \mathcal{H})$ (extension group of log mixed Hodge structures over S) in the present case (cf. [13], Corollary 6.1.6, and 1.8 below), and we have the following theorem.

Theorem 0.4.1. (Log normal function).

(1) *The normalized tension* $\mathcal{T}/y_0$ *is understood as a multi-valued truncated normal function by* $(\mathcal{T}/y_0)s^0$. *Then it lifts and extends uniquely to a single-valued log normal function* $S \to J_{L_{\mathbf{Q}}}$ *so that the corresponding exact sequence* $0 \to \mathcal{H} \to \tilde{\mathcal{H}} \to \mathbf{Z}(-2) \to 0$ *of log mixed Hodge structures over* S *is given by the liftings* $1_{\mathbf{Z}}$ *and* 1_F *in* $\tilde{\mathcal{H}}$ *of* $(2\pi i)^{-2} \cdot 1 \in \mathbf{Z}(-2)$ *respecting the lattice and the Hodge filtration, respectively, which are defined as follows:* $1_{\mathbf{Z}} := ((2\pi i)^{-2} \cdot 1, -(\mathcal{T}/y_0)s^0)$ *with* $(\mathcal{T}/y_0)s^0 \in \mathcal{H}_{\mathcal{O}^{\log}} = (\mathrm{gr}_3^W)_{\mathcal{O}^{\log}}$, *and* $1_F - 1_{\mathbf{Z}} := (\delta(\mathcal{T}/y_0))e^1 - (\mathcal{T}/y_0)e^0$, *where* $\delta := 2\pi i q d/dq$.

(2) *A splitting of the weight filtration* W *of the local system* $\tilde{\mathcal{H}}_{\mathbf{Z}}$, *i.e., a splitting compatible with the monodromy of the local system* $\tilde{\mathcal{H}}_{\mathbf{Z}}$, *is given by* $1_{\mathbf{Z}}^{spl} = 1_{\mathbf{Z}} + s^1/2$, *and the log normal function over it is given by* $1_F^{spl} - 1_{\mathbf{Z}}^{spl} = (\delta(\mathcal{T}/y_0))e^1 - (\mathcal{T}/y_0)e^0$, *where* δ *is as in (1).*

(3) *Theorem 0.3.1 says that the inverse of the normal function in (1) from its image is given by* $16\pi^2/15$ *times the Picard–Fuchs differential operator* $\mathcal{L}$.

Theorem 0.4.1 is proved in Section 3, and after these proofs some geometric backgrounds are discussed in Section 3.3.

0.5.

The organization of this paper is as follows. Section 1 is a summary of log mixed Hodge theory mainly from [17], [13] and [14], which is used to study mirror symmetry in later sections and also is expected to work as a brief guide of this theory. In Section 2, after preparations including $\hat{\Gamma}$-integral structure in [6] and [7], we give a correspondence table of closed mirror symmetry for quintic threefolds and their mirrors, which is the precision and the expansion of our previous paper [30], Sect. 3. In Section 3, we prove log mixed Hodge theoretic interpretation Theorem 0.4.1. We also give some discussions on the related geometries and local systems in Section 3.3.

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Notation.
Fix $\Lambda := (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$, where
H_0 is a free $\mathbf{Z}$-module of finite rank,

W is an increasing filtration on $H_{0,\mathbf{Q}} := \mathbf{Q} \otimes H_0$,
$\langle\ ,\ \rangle_w$ is a non-degenerate $(-1)^w$-symmetric bilinear form on gr^W_w,
$(h^{p,q})_{p,q}$ is a set of Hodge numbers.
D : the classifying space of graded polarized mixed Hodge structures for the data Λ, consisting of all Hodge filtrations.
$\check{D}$: the "compact dual" of D.
$G_A := \mathrm{Aut}(H_{0,A}, W, (\langle\ ,\ \rangle_w)_w)$, where $H_{0,A} := A \otimes H_0$ $(A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C})$,
$\mathfrak{g}_A := \mathrm{Lie} G_A = \mathrm{End}(H_{0,A}, W, (\langle\ ,\ \rangle_w)_w)$ $(A = \mathbf{Q}, \mathbf{R}, \mathbf{C})$.

1 Log mixed Hodge theory

This section is a summary of log mixed Hodge theory from [17], [13], and [14]. We write a general form of these results as a brief guide for future use. Section 1.8 is adapted for the use in Section 3. The corresponding results in [13] and [14] are written in more general settings.

1.1. Category $\mathcal{B}(\log)$

Let S be a subset of an analytic space Z. The *strong topology of S in Z* is the strongest one among those topologies on S in which, for any analytic space A and any morphism $f : A \to Z$ with $f(A) \subset S$ as sets, $f : A \to S$ is continuous. S is regarded as a local ringed space by the pullback sheaf of $\mathcal{O}_Z$.

Let $\mathcal{B}$ be the category of local ringed spaces S over $\mathbf{C}$ which have an open covering $(U_\lambda)_\lambda$ satisfying the following condition: For each λ, there exist an analytic space Z_λ, and a subset S_λ of Z_λ such that, as local ringed space over $\mathbf{C}$, U_λ is isomorphic to an open subset of S_λ which is endowed with the strong topology in Z_λ and the inverse image of $\mathcal{O}_{Z_\lambda}$.

A *log structure* on a local ringed space S is a sheaf of monoids M on S together with a homomorphisim $\alpha : M \to \mathcal{O}_S$ such that $\alpha^{-1}\mathcal{O}_S^\times \xrightarrow{\sim} \mathcal{O}_S^\times$. *fs* log structure means, locally on the underlying space, the log structure has a chart which is finitely generated, integral and saturated.

Let $\mathcal{B}(\log)$ be the category of objects of $\mathcal{B}$ endowed with an fs log structure.

A log analytic space is called *log smooth* if, locally, it is isomorphic to an open set of a toric variety endowed with the canonical log structure. A *log manifold* is a log local ringed space over $\mathbf{C}$ which has an open covering $(U_\lambda)_\lambda$ satisfying the following condition: For each λ, there exist a log smooth fs log analytic space Z_λ, a finite subset I_λ of global log differential 1-forms $\Gamma(Z_\lambda, \omega^1_{Z_\lambda})$, and an isomorphism of log local ringed spaces over $\mathbf{C}$ between U_λ and an open subset of $S_\lambda := \{z \in Z_\lambda \mid \text{the image of } I_\lambda \text{ in the stalk } \omega^1_z \text{ is zero}\}$ in the strong topology in Z_λ.

1.2. Ringed space $(S^{\log}, \mathcal{O}_S^{\log})$

The ringed space $(S^{\log}, \mathcal{O}_S^{\log})$ was defined for fs log schemes by K. Kato and C. Nakayama in [8]. It was generalized for the category $\mathcal{B}(\log)$ in [15].

Let $S \in \mathcal{B}(\log)$. As a set define

$S^{\log} := \{(s,h) \mid s \in S, h : M_s^{\mathrm{gp}} \to \mathbf{S}^1$ homomorphism s.t. $h(u) = u/|u|$ if $u \in \mathcal{O}_{S,s}^{\times}\}$.

Endow $S^{\log}$ with the weakest topology such that the following two maps are continuous.

(1) $\tau : S^{\log} \to S, (s,h) \mapsto s$.

(2) For any open set $U \subset S$ and any $f \in \Gamma(U, M^{\mathrm{gp}})$, $\tau^{-1}(U) \to \mathbf{S}^1$, $(s,h) \mapsto h(f_s)$.

Then, τ is proper and surjective with fiber $\tau^{-1}(s) = (\mathbf{S}^1)^{r(s)}$, where $r(s)$ is the rank of $(M^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s$ which varies with $s \in S$.

Define a sheaf $\mathcal{L}$ on $S^{\log}$ as the fiber product:

$$\begin{array}{ccccc}
\mathcal{L} & \xrightarrow{\exp} & \tau^{-1}(M^{\mathrm{gp}}) & \ni & (f \text{ at } (s,h)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cont}(*, i\mathbf{R}) & \xrightarrow{\exp} & \mathrm{Cont}(*, \mathbf{S}^1) & \ni & h(f).
\end{array}$$

Let $\iota : \tau^{-1}(\mathcal{O}_S) \to \mathcal{L}$ be a morphism induced from

$$\begin{array}{ccccc}
f & \in & \tau^{-1}(\mathcal{O}_S) & \xrightarrow{\exp} & \tau^{-1}(\mathcal{O}_S^{\times}) \subset \tau^{-1}(M^{\mathrm{gp}}) \\
\downarrow & & \downarrow & & \downarrow \\
(f - \bar{f})/2 & \in & \mathrm{Cont}(*, i\mathbf{R}) & \xrightarrow{\exp} & \mathrm{Cont}(*, \mathbf{S}^1).
\end{array}$$

Define

$$\mathcal{O}_S^{\log} := \frac{\tau^{-1}(\mathcal{O}_S) \otimes \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L})}{(f \otimes 1 - 1 \otimes \iota(f) \mid f \in \tau^{-1}(\mathcal{O}_S))}.$$

Thus $\tau : (S^{\log}, \mathcal{O}_S^{\log}) \to (S, \mathcal{O}_S)$ is a morphism of ringed spaces over $\mathbf{C}$. For $s \in S$ and $t \in S^{\log}$ lying over s, let $t_j \in \mathcal{L}_t$ $(1 \le j \le r(s))$ be elements such that their images in $(M^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s$ of $\exp(t_j)$ form a basis. Then, $\mathcal{O}_{S,t}^{\log} = \mathcal{O}_{S,s}[t_j\ (1 \le j \le r(s))]$ is a polynomial ring.

1.3. Toric variety

Toric varieties offer typical examples of $S^{\log}$ and also they are building blocks of fine moduli spaces of log mixed Hodge structures.

Let $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ be a *nilpotent cone*, i.e., a sharp cone, $\sigma \cap (-\sigma) = \{0\}$, generated by a finite number of mutually commutative nilpotent elements. Assume that the cone generators of σ can be taken from $\mathfrak{g}_{\mathbf{Q}}$. Let Γ be a subgroup of $G_{\mathbf{Z}}$.

Define a monoid $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$ and the dual monoid $P(\sigma) := \Gamma(\sigma)^\vee = \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$. Define a toric variety and a torus by

$$\begin{aligned}\mathrm{toric}_\sigma : &= \mathrm{Spec}(\mathbf{C}[P(\sigma)])(\mathbf{C}) = \mathrm{Hom}\,(P(\sigma), \mathbf{C}^{\mathrm{mult}}) \supset \mathrm{torus}_\sigma : \\ &= \mathrm{Hom}\,(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^\times),\end{aligned}$$

where $\mathbf{C}^{\mathrm{mult}}$ is $\mathbf{C}$ regarded as a monoid by multiplication and $P(\sigma)^{\mathrm{gp}}$ is the group generated by the monoid $P(\sigma)$. The exponential sequence $0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^\times \to 1$ induces the universal covering of the torus

$$0 \to \mathrm{Hom}\,(P(\sigma)^{\mathrm{gp}}, \mathbf{Z}) \to \mathrm{Hom}\,(P(\sigma)^{\mathrm{gp}}, \mathbf{C}) \xrightarrow{\mathbf{e}} \mathrm{Hom}\,(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^\times) \to 1,$$

where $\mathrm{Hom}\,(P(\sigma)^{\mathrm{gp}}, \mathbf{Z}) = \Gamma(\sigma)^{\mathrm{gp}}$ is considered as the fundamental group of torus_σ, and $\mathbf{e}(z \otimes \gamma) := e^{2\pi i z} \otimes \gamma$ $(z \in \mathbf{C},\ \gamma \in \Gamma(\sigma)^{\mathrm{gp}} = \mathrm{Hom}\,(P(\sigma)^{\mathrm{gp}}, \mathbf{Z}))$.

Fix the above cone σ. For a face ρ of σ, define $\tilde{P}(\rho) := \{l \in P(\sigma)^{\mathrm{gp}} \mid l(\rho) \geq 0\}$. Then we have an open covering

$$\mathrm{toric}_\sigma = \mathrm{Spec}(\mathbf{C}[P(\sigma)])(\mathbf{C}) = \bigcup_{\rho \prec \sigma} \mathrm{Spec}(\mathbf{C}[\tilde{P}(\rho)])(\mathbf{C}).$$

We now recall a stratification. Fix the above cone σ and let ρ be a face of the cone σ. Then, we have a homomorphism $P(\rho) \lambda P(\sigma)$ and hence a morphism $\mathrm{toric}_\rho \to \mathrm{toric}_\sigma$. The origin $0_\rho \in \mathrm{toric}_\rho$ is the monoid homomorphism $P(\sigma) \to \mathbf{C}^{\mathrm{mult}}$ sending 1 to 1 and all the other elements of $P(\rho)$ to 0, which is sent to a point of toric_σ by the above morphism. Then, as a set, we have a stratification into torus orbits

$$\mathrm{toric}_\sigma = \{\mathbf{e}(z)0_\rho \mid \rho \prec \sigma, z \in \sigma_{\mathbf{C}}/(\rho_{\mathbf{C}} + \log \Gamma(\sigma)^{\mathrm{gp}})\}.$$

Here $\mathbf{e}(c \log \gamma) := \mathbf{e}(c \otimes \gamma) = e^{2\pi i c} \otimes \gamma$ $(c \in \mathbf{C}, \gamma \in \Gamma(\sigma)^{\mathrm{gp}})$.

For $S := \mathrm{toric}_\sigma$, the polar coordinate $\mathbf{R}_{\geq 0} \times \mathbf{S}^1 \to \mathbf{R}_{\geq 0}\mathbf{S}^1 = \mathbf{C}$ induces $\tau : S^{\log} \to S$ as

$$\tau : S^{\log} = \mathrm{Hom}\,(P(\sigma), \mathbf{R}_{\geq 0}^{\mathrm{mult}} \times \mathrm{Hom}\,(P(\sigma), \mathbf{S}^1)$$

$$= \{(\mathbf{e}(iy)0_\rho, \mathbf{e}(x)) \mid \rho \prec \sigma, x \in \sigma_{\mathbf{R}}/(\rho_{\mathbf{R}} + \log \Gamma(\sigma)^{\mathrm{gp}}), y \in \sigma_{\mathbf{R}}/\rho_{\mathbf{R}}\}$$

$$\to S = \mathrm{Hom}\,(P(\sigma), \mathbf{C}^{\mathrm{mult}}),$$

$$\tau(\mathbf{e}(ib)0_\rho, \mathbf{e}(a)) = \mathbf{e}(a + ib)0_\rho.$$

Since $0 \to \rho_{\mathbf{R}}/\log \Gamma(\rho)^{\mathrm{gp}} \to \sigma_{\mathbf{R}}/\log \Gamma(\sigma)^{\mathrm{gp}} \to \sigma_{\mathbf{R}}/(\rho_{\mathbf{R}} + \log \Gamma(\sigma)^{\mathrm{gp}}) \to 0$ is exact, the fiber of τ, as a set, is described as

$$\tau^{-1}(\mathbf{e}(a + ib)0_\rho) = \{(\mathbf{e}(ib)0_\rho, \mathbf{e}(a + x)) \mid x \in \rho_{\mathbf{R}}/\log \Gamma(\rho)^{\mathrm{gp}}\} \simeq (\mathbf{S}^1)^r,$$

where $r = r(\rho) := \mathrm{rank}\,\rho$ varies with $\rho \prec \sigma$.

Let $H_\sigma = (H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$ be the canonical local system endowed with the weight filtration and the polarizations on graded quotients on $S^{\log}$, which are given by the representation $\pi_1(S^{\log}) = \Gamma(\sigma)^{\mathrm{gp}} \subset G_{\mathbf{Z}} = \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_w)_w)$.

1.4. Local systems on $(S^{\log}, \mathcal{O}_S^{\log})$

We recall three results about local systems on $(S^{\log}, \mathcal{O}_S^{\log}) \in \mathcal{B}(\log)$ from [17], Sect. 2.3.

Let L be a locally constant sheaf of abelian groups on $S^{\log}$. For $s \in S$ and $t \in S^{\log}$ lying over s, we call the action of $\pi_1(s^{\log}) = \pi_1(\tau^{-1}(s))$ on L_t the *local monodromy* of L at t. We say the local monodromy of L is *unipotent* if the local monodromy of L at t is unipotent for any $t \in S^{\log}$.

Let $s \in S$. Let $(q_j)_{1 \le j \le n}$ be a finite family of elements of $M^{\mathrm{gp}}_{S,s}$ whose image in $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ is a free basis, and let $(\gamma_j)_{1 \le j \le n}$ be the dual basis of $\pi_1(s^{\log})$, that is $[\gamma_j, q_k] = (2\pi i)\delta_{jk}$ where $[\ ,\ ]$ is the pairing given by $\pi_1(s^{\log}) \simeq \mathrm{Hom}\,(M_s^{\mathrm{gp}}/\mathcal{O}_s^\times, \mathbf{Z})$.

Let L be a locally constant sheaf on $S^{\log}$ of free $\mathbf{Z}$-modules of finite rank. Let $s \in S$ and $t \in \tau^{-1}(s)$, and assume that the local monodromy of L at t is unipotent. For a fixed t, we denote L_0 the constant sheaf on $S^{\log}$ with fiber L_t. Let $L_{0,\mathbf{Q}} = \mathbf{Q} \otimes_A L_0$, and let $N_j : L_{0,\mathbf{Q}} \to L_{0,\mathbf{Q}}$ be the endomorphism of constant sheaf which is induced by the logarithm of the monodromy action of γ_j on the stalk L_t of the locally constant sheaf L. Lift q_j in $\Gamma(S, M_S^{\mathrm{gp}})$ (by replacing S by an open neighborhood of s), and let

$$\xi = \exp(\sum_{j=1}^{n} (2\pi i)^{-1} \log(q_j) \otimes N_j) : \mathcal{O}_S^{\log} \otimes_{\mathbf{Q}} L_{0,\mathbf{Q}} \xrightarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathbf{Q}} L_{0,\mathbf{Q}}.$$

Note that the operator ξ depends on the choices of the branches of $\log(q_j)$ in $\mathcal{O}_S^{\log}$ locally on $S^{\log}$, but that the subsheaf $\xi^{-1}(1 \otimes L_0)$ of $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} L_0$ is independent of the choices and hence is defined globally on $S^{\log}$.

The following proposition shows that the locally constant sheaf L is embedded in $\mathcal{O}_S^{\log} \otimes L_0$.

Proposition 1.4.1. ([17], Prop. 2.3.2). *Let the situation be as above. If we replace S by some open neighborhood of s, we have an isomorphism of $\mathcal{O}_S^{\log}$-modules*

$$\nu : \mathcal{O}_S^{\log} \otimes_A L \xrightarrow{\sim} \mathcal{O}_S^{\log} \otimes_A L_0$$

satisfying the following condition (1).

(1) *The restriction of ν to $L = 1 \otimes L$ induces an isomorphism of locally constant sheaves $\nu : L \xrightarrow{\sim} \xi^{-1}(1 \otimes L_0)$.*

If we take suitable branches $\log(q_j)_{t,0}$ *in* $\mathcal{O}^{\log}_{S,t}$ *of the germs* $\log(q_j)_t$ *at* t $(1 \le j \le n)$*, we can take an isomorphism* ν *which satisfies above* (1) *and also the following* (2).

(2) *The branch* $\xi_{t,0}$ *of the germ* ξ_t*, defined by the fixed branches* $\log(q_j)_{t,0}$ *of the germs* $\log(q_j)_t$*, satisfies* $\nu(1 \otimes v) = \xi_{t,0}^{-1}(1 \otimes v)$ *for any* $v \in L_t = L_0$.

The following proposition yields a log Hodge theoretic understanding [ibid, Sect. 2.3.6] of the canonical extension of Deligne in [3].

Proposition 1.4.2. ([17], Prop. 2.3.3). *Let* $S \in \mathcal{B}(\log)$ *and let* L *be a locally constant sheaf of finite dimensional* $\mathbf{C}$*-vector spaces on* $S^{\log}$.

(i) *If the local monodromy of* L *is unipotent, the* $\mathcal{O}_S$*-module* $\mathcal{M} := \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{C}} L)$ *is locally free of finite rank, and we have an isomorphism* $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathcal{M} \xrightarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathbf{C}} L$.

(ii) *Conversely, assume that there are a locally free* $\mathcal{O}_S$*-module* $\mathcal{M}$ *of finite rank on* S *and an isomorphism of* $\mathcal{O}_S^{\log}$*-modules* $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathcal{M} \simeq \mathcal{O}_S^{\log} \otimes_{\mathbf{C}} L$*. Then the local monodromy of* L *is unipotent and* $\mathcal{M} \xrightarrow{\sim} \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{C}} L)$.

Proposition 1.4.3. ([17], Prop. 2.3.4). *Let* $S \in \mathcal{B}(\log)$*, let* L *be a locally constant sheaf on* $S^{\log}$ *of free* $\mathbf{Q}$*-modules of finite rank. Assume that the local monodromy of* L *is unipotent.*

(i) *There exists a unique* $\mathbf{Q}$*-homomorphism*

$$\mathcal{N} : L \to (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}) \otimes L$$

satisfying the following condition (1).

(1) *For any* $s \in S$*, any* $t \in S^{\log}$ *lying over* s*, and any* $\gamma \in \pi_1(s^{\log})$*, if* $h_\gamma : (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s \to \mathbf{Z}$ *denotes the homomorphism corresponding to* γ *by* $\pi_1(s^{\log}) \simeq \mathrm{Hom}\,(M_s^{\mathrm{gp}}/\mathcal{O}_s^{\times}, \mathbf{Z})$*, the composition* $L_t \xrightarrow{\mathcal{N}} (M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s \otimes L_t \xrightarrow{h_\gamma} L_t$ *coincides with the logarithm of the action of* γ *on* L_t.

(ii) *Assume that* S *is an fs log point* $\{s\}$*. Let*

$$\mathcal{N}' : L \to \omega_s^1 \otimes L$$

be the composition of $\mathcal{N}$ *and the* $\mathbf{Q}$*-linear map* $M_s^{\mathrm{gp}}/\mathcal{O}_s^{\times} \otimes L \to \omega_s^1 \otimes L, f \otimes v \mapsto (2\pi i)^{-1} d\log(f) \otimes v$*, and let* $1 \otimes \mathcal{N}' : \mathcal{O}_s^{\log} \otimes L \to \omega_s^{1,\log} \otimes L$ *be the* $\mathcal{O}_s^{\log}$*-linear homomorphism induced by* $\mathcal{N}'$*. Let* $\mathcal{M} := H^0(s^{\log}, \mathcal{O}_s^{\log} \otimes L) = \tau_*(\mathcal{O}_s^{\log} \otimes L)$*. Then the restriction* $\mathcal{M} \to \omega_s^1 \otimes_{\mathbf{C}} \mathcal{M}$ *of* $d \otimes 1_L : \mathcal{O}_s^{\log} \otimes L \to \omega_s^{1,\log} \otimes L$ *coincides with the restriction of* $1 \otimes \mathcal{N}'$ *to* $\mathcal{M}$.

$\mathcal{N}$ in the above proposition is described as follows. Assume $L = \xi^{-1}(1 \otimes L_0)$ as in the first proposition. Then $\mathcal{N}(\xi^{-1}(1 \otimes v)) := \sum_{j=1}^n q_j \otimes \xi^{-1}(1 \otimes N_j v)$ for $v \in L_0$.

1.5. Graded polarized log mixed Hodge structure

Let $S \in \mathcal{B}(\log)$. A *pre-graded polarized log mixed Hodge structure on* S is a tuple $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ consisting of a local system of $\mathbf{Z}$-free modules $H_{\mathbf{Z}}$ of finite rank on $S^{\log}$, an increasing filtration W of $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$, a non-degenerate $(-1)^w$-symmetric $\mathbf{Q}$-bilinear form $\langle\ ,\ \rangle_w$ on gr^W_w, a locally free $\mathcal{O}_S$-module $H_{\mathcal{O}}$ on S, a specified isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ (*log Riemann-Hilbert correspondence*), and a specified decreasing filtration $FH_{\mathcal{O}}$ of $H_{\mathcal{O}}$ such that $F^pH_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^pH_{\mathcal{O}}$ are locally free. Put $F^p := \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^pH_{\mathcal{O}}$. Then $\tau_* F^p = F^pH_{\mathcal{O}}$. For each integer w, the orthogonality condition $\langle F^p(\mathrm{gr}^W_w), F^q(\mathrm{gr}^W_w)\rangle_w = 0$ $(p+q > w)$ is imposed.

A pre-graded polarized log mixed Hodge structure on S is a *graded polarized log mixed Hodge structure on* S if its pullback to each $s \in S$ is a graded polarized log mixed Hodge structure on s in the following sense.

Let $(H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ be a pre-graded polarized log mixed Hodge structure on a log point s. It is a *graded polarized log mixed Hodge structure* if it satisfies the following three conditions.

(1) (Admissibility). For each logarithm N of the local monodromy of the local system $(H_{\mathbf{R}}, W, (\langle\ ,\ \rangle_w)_w)$, there exists a W-relative N-filtration $M(N, W)$.

(2) (Griffiths transversality). For any integer p, $\nabla F^p \subset \omega_s^{1,\log} \otimes F^{p-1}$ is satisfied, where $\omega_s^{1,\log}$ is the sheaf of $\mathcal{O}_s^{\log}$-module of log differential 1-forms on $(s^{\log}, \mathcal{O}_s^{\log})$, and $\nabla = d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_s^{\log} \otimes H_{\mathbf{Z}} \to \omega_s^{1,\log} \otimes H_{\mathbf{Z}}$ is the log Gauss-Manin connection.

(3) (Positivity). For a point $t \in s^{\log}$ and a $\mathbf{C}$-algebra homomorphism $a : \mathcal{O}_{s,t}^{\log} \to \mathbf{C}$, define a filtration $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ on $H_{\mathbf{C},t}$. Then, $(H_{\mathbf{Z},t}(\mathrm{gr}^W_w), \langle\ ,\ \rangle_w, F(a))$ is a polarized Hodge structure of weight w in the usual sense if a is sufficiently twisted, i.e., for $(q_j)_{1 \le j \le n} \subset M_s$ inducing generators of $M_s/\mathcal{O}_s^\times$, $|\exp(a(\log q_j))| \ll 1$ for any j.

1.6. Nilpotent orbit

Let $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ be a nilpotent cone (see 1.3). A subset $Z \subset \check{D}$ is σ*-nilpotent orbit* if the following (1)–(4) hold for $F \in Z$.

(1) $Z = \exp(\sigma_{\mathbf{C}})F$.
(2) For any $N \in \sigma$, there exists W-relative N-filtration $M(N, W)$.
(3) For any $N \in \sigma$ any p, $NF^p \subset F^{p-1}$.
(4) If $N_1, \dots, N_n$ generate σ and $y_j \gg 0$ for any j, then $\exp(\sum_j iy_jN_j)F \in D$.

A *weak fan* Σ *in* $\mathfrak{g}_{\mathbf{Q}}$ is a set of nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$, defined over $\mathbf{Q}$, which satisfies the following three conditions.

(5) Every $\sigma \in \Sigma$ is admissible relative to W.
(6) If $\sigma \in \Sigma$ and $\tau \prec \sigma$, then $\tau \in \Sigma$.

(7) If $\sigma, \sigma' \in \Sigma$ have a common interior point and if there exists $F \in \check{D}$ such that (σ, F) and (σ', F) generate nilpotent orbits, then $\sigma = \sigma'$.

Let Σ be a weak fan and Γ be a subgroup of $G_{\mathbf{Z}}$. Σ and Γ are *strongly compatible* if the following two conditions are satisfied.

(8) If $\sigma \in \Sigma$ and $\gamma \in \Gamma$, then $\mathrm{Ad}(\gamma)\sigma \in \Sigma$.
(9) For any $\sigma \in \Sigma$, σ is generated by $\log \Gamma(\sigma)$, where $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$.

1.7. Moduli of log mixed Hodge structures of type Φ

Let $\Phi := (\Lambda, \Sigma, \Gamma)$ be a data consisting of a Hodge data Λ (in Notation), a weak fan Σ and a subgroup Γ of $G_{\mathbf{Z}}$ such that Σ and Γ are strongly compatible (1.6).

Let $\sigma \in \Sigma$ and $S := \mathrm{toric}_\sigma$. Let $H_\sigma = (H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$ be the canonical local system $H_{\sigma,\mathbf{Z}}$ endowed with the weight filtration W and the polarizations $\langle\ ,\ \rangle_w$ on the graded quotients gr_w^W $(w \in \mathbf{Z})$ over $S^{\log}$, which is determined by the representation $\Gamma \subset G_{\mathbf{Z}} = \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_w)_w)$.

Let $\check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D}$. The *universal pre-graded polarized log mixed Hodge structure H on $\check{E}_\sigma$* is given by H_σ together with the isomorphism $\mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathbf{Z}} H_{\sigma,\mathbf{Z}} = \mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathcal{O}_{\check{E}_\sigma}} H_{\mathcal{O}}$ (1.5), where $H_{\mathcal{O}} := \mathcal{O}_{\check{E}_\sigma} \otimes H_0$ is the free $\mathcal{O}_{\check{E}_\sigma}$-module coming from that on $\check{D}$ endowed with the universal Hodge filtration F.

Let $E_\sigma := \{x \in \check{E}_\sigma \mid H|_x$ is a graded polarized log mixed Hodge structure on $x\}$. Note that slits appear in E_σ because of log-pointwise Griffiths transversality 1.5 (2) and positivity 1.5 (3), or equivalently 1.6 (3) and 1.6 (4) respectively.

As a set, define $D_\Sigma := \{(\sigma, Z) \mid \text{nilpotent orbit}, \sigma \in \Sigma, Z \subset \check{D}\}$. Let $\sigma \in \Sigma$. Assume that Γ is neat. A structure as an object of $\mathcal{B}(\log)$ on $\Gamma \backslash D_\Sigma$ is introduced by a diagram:

$$
\begin{array}{ccc}
E_\sigma & \overset{\text{GPLMH}}{\subset} & \check{E} := \mathrm{toric}_\sigma \times \check{D} \\
\Big\downarrow \scriptstyle{\sigma_{\mathbf{C}}\text{-torsor}} & & \\
\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma & & \\
\Big\downarrow \scriptstyle{\text{loc. isom.}} & & \\
\Gamma \backslash D_\Sigma & &
\end{array}
$$

The action of $h \in \sigma_{\mathbf{C}}$ on $(\mathbf{e}(a)0_\rho, F) \in E_\sigma$ is $(\mathbf{e}(h+a)0_\rho, \exp(-h)F)$, and the projection is $(\mathbf{e}(a)0_\rho, F) \mapsto [\rho, \exp(\rho_{\mathbf{C}} + a)F]$.

Let $S \in \mathcal{B}(\log)$. A *log mixed Hodge structure of type Φ on S* is a pre-graded polarized log mixed Hodge structure $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ endowed with Γ-level structure $\mu \in H^0(S^{\log}, \Gamma \backslash \mathcal{I}som((H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w), (H_0, W, (\langle\ ,\ \rangle_w)_w)))$ satisfying the following condition: For any point $s \in S$, any point $t \in \tau^{-1}(s) = s^{\log}$ and any representative $\tilde{\mu}_t : H_{\mathbf{Z},t} \xrightarrow{\sim} H_0$, there exists $\sigma \in \Sigma$ such that σ

contains $\tilde{\mu}_t \pi_1^+(s^{\log})\tilde{\mu}_t^{-1}$ and $(\sigma, \tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}_{S,t}^{\log}} F_t))$ generates a nilpotent orbit. Here $\pi_1^+(s^{\log}) := \mathrm{Image}(\mathrm{Hom}((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}) \hookrightarrow \pi_1(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},t}))$ is the local monodromy monoid of $H_{\mathbf{Z}}$ at s (cf. [17], Sect. 3.3.2). (Then, the smallest such σ exists.)

Theorem 1.7.1. *For a given data* Φ, *we have the following.*
(i) $\Gamma \backslash D_\Sigma \in \mathcal{B}(\log)$, *which is Hausdorff. If* Γ *is neat,* $\Gamma \backslash D_\Sigma$ *is a log manifold.*
(ii) *On* $\mathcal{B}(\log)$, $\Gamma \backslash D_\Sigma$ *represents a functor* LMH_Φ *of log mixed Hodge structures of type* Φ.

Log period map. *Given* Φ. *Let* $S \in \mathcal{B}(\log)$. *Then we have an isomorphism*

$$\mathrm{LMH}_\Phi(S) \xrightarrow{\sim} \mathrm{Map}(S, \Gamma \backslash D_\Sigma),$$
$$H \mapsto \big(S \ni s \mapsto [\sigma, \exp(\sigma_{\mathbf{C}})\tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}_{S,t}^{\log}} F_t)]\big)(t \in \tau^{-1}(s)),$$

which is functorial in S.

A log period map is a unified compactification of a period map and a normal function of Griffiths.

The above $\Gamma \backslash D_\Sigma$ is the fine moduli of log mixed Hodge structures of type Φ, whose underlying coarse moduli, in the sense of log points, is the set of equivalence classes of all nilpotent orbits of specified type.

1.8. Néron model for admissible normal function

We review some results from [14], Theorem 1.3, [13], Section 6.1, and [11], Section 8 adapted to the situation 0.4 in Introduction.

For a pure case $h^{p,q} = 1$ $(p+q=3,\ p,q \geq 0)$ and $h^{p,q} = 0$ otherwise, a complete fan is constructed in [17], Section 12.3. For a mixed case $h^{p,q} = 1$ (the above (p,q), plus $(p,q) = (2,2)$) and $h^{p,q} = 0$ otherwise, over the above fan, a fan of Néron model for given admissible normal function is constructed in [14], Theorem 3.1, and we have a Néron model in the following sense.

Let $S \in \mathcal{B}(\log)$, $U := S_{\mathrm{triv}} \subset S$ (consisting of those points with trivial log structure), $H_{(-1)}$ be a polarized variation of Hodge structure of weight -1 (Tate-twisted by 2 for $\mathcal{H}$ in Introduction 0.4) on U and $L_{\mathbf{Q}}$ be a local system of $\mathbf{Q}$-vector spaces which is an extension of $\mathbf{Q}$ by $H_{(-1),\mathbf{Q}}$. An admissible normal function over U for $H_{(-1)}$ underlain by the local system $L_{\mathbf{Q}}$ can be regarded as an admissible variation of mixed Hodge structure which is an extension of $\mathbf{Z}$ by $H_{(-1)}$ and lies over local system $L_{\mathbf{Q}}$.

For any given unipotent admissible normal function over U as above, $H_{(-1)}$ and $L_{\mathbf{Q}}$ extend to a polarized log mixed Hodge structure on S and a local system on $S^{\log}$, respectively, denoted by the same symbols, and there is a relative log manifold $J_{L_{\mathbf{Q}}}$ over S which is strict over S (i.e., endowed with the pullback log structure from S) and which represents the following functor on $\mathcal{B}/S^\circ$ ($S^\circ \in \mathcal{B}$ is the underlying space of S):

$S' \mapsto$ {LMH H on S' satisfying $H(\mathrm{gr}^W_w) = H_{(w)}|_{S'}$ ($w = -1, 0$) and $(*)$ below}/isom.

$(*)$ Locally on S', there is an isomorphism $H_{\mathbf{Q}} \simeq L_{\mathbf{Q}}$ on $(S')^{\log}$ preserving W.

Here $H_{(w)}|_{S'}$ is the pullback of $H_{(w)}$ by the structure morphism $S' \to S^\circ$, and S' is endowed with the pullback log structure from S.

Put $H' := H_{(-1)}$. In the present case, we have $J_{L_{\mathbf{Q}}} = \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, H')$ by [13], Corollary 6.1.6. This is the subgroup of $\tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}}))$ restricted by admissibility condition and log-pointwise Griffiths transversality condition ([11], Section 8, cf. 1.5). Let $\tilde{J}_{L_{\mathbf{Q}}}$ be the pullback of $J_{L_{\mathbf{Q}}}$ by $\tau_*(H'_{\mathcal{O}^{\log}}/F^0) \to \tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}}))$, and $\bar{J}_{L_{\mathbf{Q}}}$ be the image of $\tilde{J}_{L_{\mathbf{Q}}}$ by $\tau_*(H'_{\mathcal{O}^{\log}}/F^0) \to \tau_*(H'_{\mathcal{O}^{\log}}/F^{-1})$. Then, by using the polarization, we have a commutative diagram:

$$\begin{array}{ccccc}
J_{L_{\mathbf{Q}}} = \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, H') & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}})) & \overset{\mathrm{pol}}{\underset{\sim}{\to}} & \tau_*((F^0)^*/H'_{\mathbf{Z}}) \\
\uparrow & & \uparrow & & \uparrow \\
\tilde{J}_{L_{\mathbf{Q}}} & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/F^0) & \overset{\mathrm{pol}}{\underset{\sim}{\to}} & \tau_*((F^0)^*) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{J}_{L_{\mathbf{Q}}} & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/F^{-1}) & \overset{\mathrm{pol}}{\underset{\sim}{\to}} & \tau_*((F^1)^*).
\end{array}$$

2 Quintic threefolds

Let V be a quintic threefold in $\mathbf{P}^4$ and let V°_ψ be its mirror family (cf. [1], Sect. 4.2).

In this section, we give a correspondence table of A-model for V and B-model for V°_ψ. This is a precision and an expansion of our previous [30], Sect. 3 by using $\hat{\Gamma}$-integral structure of Iritani [7]. We will use this table in Section 3 below.

2.1. Quintic mirror family

Following [21], [23], etc., we briefly recall the construction of the mirror family V°_ψ by quotient method. Let $V_\psi : f := \sum_{j=1}^5 x_j^5 - 5\psi \prod_{j=1}^5 x_j = 0$ ($\psi \in \mathbf{P}^1$) be the Dwork pencil of quintics in $\mathbf{P}^4$. Let μ_5 be the group consisting of the fifth roots of the unity in $\mathbf{C}$. Then the group $G := \{(a_j) \in (\mu_5)^5 \mid a_1 \ldots a_5 = 1\}$ acts on V_ψ by $x_j \mapsto a_j x_j$. Let V°_ψ be a crepant resolution of quotient singularity of V_ψ/G (cf. [21], [23]). Divide further by the action $(x_1, \ldots, x_5) \mapsto (a^{-1}x_1, x_2, \ldots, x_5)$ and $\psi \mapsto a\psi$ ($a \in \mu_5$).

2.2. Picard–Fuchs equation on the mirror V°

Let Ω be a 3-form on V°_ψ with a log pole over $\psi = \infty$ induced from

$$\left(\frac{5}{2\pi i}\right)^3 \mathrm{Res}_{V_\psi}\left(\frac{\psi}{f}\sum_{j=1}^{5}(-1)^{j-1}x_j dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_5\right).$$

Let $z := 1/(5\psi)^5$ and $\theta := zd/dz$. Let

$$\mathcal{L} := \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)$$

be the Picard–Fuchs differential operator for Ω, i.e., $\mathcal{L}\Omega = 0$ via the Gauss-Manin connection ∇. There are three special points of the complex moduli:

$z = 0$: maximally unipotent monodromy point,
$z = \infty$: Gepner point,
$z = 1/5^5$: conifold point.

At $z = 0$, the Picard–Fuchs differential equation $\mathcal{L}y = 0$ has the indicial equation $\rho^4 = 0$ (ρ is indeterminate), i.e., maximally unipotent. By the Frobenius method, we have a basis of solutions $y_j(z)$ $(0 \le j \le 3)$ as follows. Let

$$\tilde{y}(-z;\rho) := \sum_{n=0}^{\infty}\frac{\prod_{m=1}^{5n}(5\rho+m)}{\prod_{m=1}^{n}(\rho+m)^5}(-z)^{n+\rho}$$

be a solution of $\mathcal{L}(\tilde{y}(-z;\rho)) = \rho^4(-z)^\rho$, and let

$$\tilde{y}(-z;\rho) = y_0(z)+y_1(z)\rho+y_2(z)\rho^2+y_3(z)\rho^3+\cdots,\quad y_j(z) := \frac{1}{j!}\frac{\partial^j\tilde{y}(-z;\rho)}{\partial\rho^j}\Big|_{\rho=0}$$

be the Taylor expansion at $\rho = 0$. Then, y_j $(0 \le j \le 3)$ form a basis of homogeneous solutions for the linear differential equation $\mathcal{L}y = 0$. We have

$$y_0 = f_0 = \textstyle\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n,$$

$$y_1 = f_0\log z + f_1 = y_0\log z + 5\textstyle\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)z^n,$$

$$2!y_2 = f_0(\log z)^2 + 2f_1\log z + f_2,$$

$$3!y_3 = f_0(\log z)^3 + 3f_1(\log z)^2 + 3f_2\log z + f_3,$$

where all f_j are holomorphic functions in z with $f_0(0) = 1$ and $f_j(0) = 0$ for $j > 0$.

Define the canonical parameters by $t := y_1/y_0$, $u := t/2\pi i$, and the canonical coordinate by $q := e^t = e^{2\pi i u}$ which is a specific chart of the log structure given by the divisor $(z = 0)$ of a disc in $\mathbf{P}^1$ and gives a mirror map.

Write $z = z(q)$ which is holomorphic in q. Then we have

$$\log z = 2\pi iu - \frac{5}{y_0(z(q))}\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\Big(\sum_{j=n+1}^{5n}\frac{1}{j}\Big)z(q)^n.$$

The Gauss-Manin potential of V_z° is

$$\Phi_{\mathrm{GM}}^{V^\circ} = \frac{5}{2}\Big(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\Big).$$

Let $\tilde{\Omega} := \Omega/y_0$ and $\delta := 2\pi iqd/dq = du$. Then, the Yukawa coupling at $z = 0$ is

$$Y := -\int_{V^\circ}\tilde{\Omega}\wedge\nabla_\delta\nabla_\delta\nabla_\delta\tilde{\Omega} = \frac{5}{(1+5^5z)y_0(z)^2}\Big(\frac{qdz}{zdq}\Big)^3.$$

2.3. *A-model of quintic V*

Let V be a general quintic hypersurface in $\mathbf{P}^4$. Let H be the cohomology class of a hyperplane section of V in $\mathbf{P}^4$, $K(V) = \mathbf{R}_{>0}H$ be the Kähler cone of V, and u be the coordinate of $\mathbf{C}H$. Put $t := 2\pi iu$. A complexified Kähler moduli is defined as

$$\mathcal{KM}(V) := (H^2(V,\mathbf{R}) + iK(V))/H^2(V,\mathbf{Z}) \xrightarrow{\sim} \Delta^*, \quad uH \mapsto q := e^{2\pi iu}.$$

Let $C \in H_2(V,\mathbf{Z})$ be the homology class of a line on V.

For $\beta = dC \in H_2(V,\mathbf{Z})$, define $q^\beta := q^d$. The Gromov–Witten potential of V is defined as

$$\Phi_{\mathrm{GW}}^V := \frac{1}{6}\int_V(2\pi iuH)^3 + \sum_{0\neq\beta\in H_2(V,\mathbf{Z})}N_dq^\beta = \frac{5}{6}(2\pi i)^3u^3 + \sum_{d>0}N_dq^d.$$

Here the Gromov–Witten invariant N_d is

$$\overline{M}_{0,0}(\mathbf{P}^4,d) \xleftarrow{\pi_1} \overline{M}_{0,1}(\mathbf{P}^4,d) \xrightarrow{e_1} \mathbf{P}^4,$$

$$N_d := \int_{\bar{M}_{0,0}(\mathbf{P}^4,d)} c_{5d+1}(\pi_{1*}e_1^*\mathcal{O}_{\mathbf{P}^4}(5)).$$

Note that $N_d = 0$ if $d \leq 0$. Let $N_d = \sum_{k|d}n_{d/k}k^{-3}$. Then $n_{d/k}$ is the instanton number. (n_l here is different from n_l in Theorem 0.3.1.)

The differentials of $\Phi = \Phi_{\mathrm{GW}}^V$ are computed easily:

$$\frac{d\Phi}{du} = \frac{5}{2}(2\pi i)^3u^2 + (2\pi i)\sum_{d>0}N_ddq^d, \quad \frac{d^2\Phi}{du^2} = 5(2\pi i)^3u + (2\pi i)^2\sum_{d>0}N_dd^2q^d.$$

2.4. *Integral structure*

As we stated in Introduction, we consider everything near the large radius point q_0 and the maximally unipotent monodromy point p_0. Let S be a neighborhood disc of q_0 (resp. p_0) in $\overline{\mathcal{KM}}(V)$ (resp. $\overline{\mathcal{M}}(V^\circ)$) for A-model of V (resp. for B-model of V°), and let S^* be $S \setminus \{q_0\}$ (resp. $S \setminus \{p_0\}$) for A-model (resp. B-model) (see 2.2, 2.3). Endow S with the log structure associated to the divisor $S \setminus S^*$.

The B-model variation of Hodge structure $\mathcal{H}^{V^\circ}$ is the usual variation of Hodge structure arising from the smooth projective family $f : X \to S^*$ of the quintic mirrors over the punctured neighborhood of p_0. Its integral structure is the usual one $\mathcal{H}^{V^\circ}_{\mathbf{Z}} = R^3 f_* \mathbf{Z}$. This is compatible with the monodromy weight filtration M around p_0. Define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}^{V^\circ}_{\mathbf{Z}}$ for all k.

For the A-model $\mathcal{H}^V$ on S^*, the locally free sheaf on S^*, the Hodge filtration, and the monodromy weight filtration M around q_0 are given by $\mathcal{H}^V_{\mathcal{O}} := \mathcal{O}_{S^*} \otimes (\bigoplus_{0 \le p \le 3} H^{2p}(V))$, $F^p := \mathcal{O}_{S^*} \otimes H^{\le 2(3-p)}(V)$, and $M_{2p} := H^{\ge 2(3-p)}(V)$, respectively. Iritani defined $\hat{\Gamma}$-integral structure in more general setting in [7], Definition 3.6. In the present case, it is characterized as follows. Let H and C be a hyperplane section and a line on V, respectively. Then, in the present case, a basis of the $\hat{\Gamma}$-integral structure is given by $\{s(\mathcal{E}) \mid \mathcal{E} \text{ is } \mathcal{O}_V, \mathcal{O}_H, \mathcal{O}_C, \mathcal{O}_{\mathrm{pt}}\}$ [ibid, Example 6.18], where $s(\mathcal{E})$ is a unique ∇^{even}-flat section satisfying an asymptotic condition

$$s(\mathcal{E}) \sim (2\pi i)^{-3} e^{-2\pi i u H} \cdot \hat{\Gamma}(T_V) \cdot (2\pi i)^{\deg/2} \mathrm{ch}(\mathcal{E})$$

at the large radius point q_0 when $\mathrm{Im}(u) \to \infty$ for each fixed $\mathrm{Re}(u)$. Here, for the Chern roots $c(T_V) = \prod_{j=1}^3 (1+\delta_j)$, the Gamma class $\hat{\Gamma}(T_V)$ is defined by

$$\begin{aligned}\hat{\Gamma}(T_V) := \prod_{j=1}^{3} \Gamma(1+\delta_j) &= \exp(-\gamma\, c_1(V) + \sum_{k \ge 2} (-1)^k (k-1)! \zeta(k) \mathrm{ch}_k(T_V)) \\ &= \exp(\zeta(2)\mathrm{ch}_2(T_V) - 2\zeta(3)\mathrm{ch}_3(T_V))\end{aligned}$$

where γ is the Euler constant, and $\deg|_{H^{2p}(V)} := 2p$. The important point is that this class $\hat{\Gamma}(T_V)$ plays the role of a "square root" of the Todd class in Hirzebruch-Riemann-Roch ([6], 1; [7], 1, (13)). Denote this $\hat{\Gamma}$-integral structure by $\mathcal{H}^V_{\mathbf{Z}}$. This is compatible with the monodromy weight filtration M and we define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}^V_{\mathbf{Z}}$ for all k. The above asymptotic relation is actually computed as

$$\begin{aligned} s^0 :=& s(\mathcal{O}_{\mathrm{pt}}) = \frac{1}{5} H^3, \\ s^1 :=& s(\mathcal{O}_C) = \frac{1}{5} (2\pi i)^{-1} H^2 + \frac{1}{5}(-u+1) H^3, \\ s^2 :=& s(\mathcal{O}_H) \sim (2\pi i)^{-2} H + \frac{5}{2} (2\pi i)^{-1} \Big(-u - \frac{1}{2}\Big) H^2 + \Big(\frac{1}{2}u^2 + \frac{1}{2}u + \frac{7}{12}\Big) H^3, \end{aligned}$$

$$s^3 := s(\mathcal{O}_V) \sim (2\pi i)^{-3} - (2\pi i)^{-2}uH + (2\pi i)^{-1}\Big(\frac{1}{2}u^2 + \frac{5}{12}\Big)H^2 + \Big(-\frac{1}{6}u^3 - \frac{5}{12}u + \frac{5i\zeta(3)}{\pi^3}\Big)H^3.$$

Fixing an isomorphism of VHS in (2) in Introduction, we also use s^p for the corresponding ∇-flat integral basis for the B-model $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ (vanishing cycles are used for B-model in [7], Theorems 6.9, 6.10, Example 6.18).

In both A-model case and B-model case, the integral structures $\mathcal{H}_{\mathbf{Z}}^{V}$ and $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ on S^* extend to the local systems of $\mathbf{Z}$-modules over $S^{\log}$ ([24]; [17], Proposition 2.3.5), still denoted $\mathcal{H}_{\mathbf{Z}}^{V}$ and $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$, respectively.

Consider a diagram:

$$\begin{array}{ccc} \tilde{S}^{\log} := \mathbf{R} \times i(0,\infty] & \supset & \tilde{S}^* := \mathbf{R} \times i(0,\infty) \\ \downarrow & & \downarrow \\ S^{\log} & \supset & S^* \\ {\scriptstyle\tau}\downarrow & & \\ S & & \end{array}$$

The coordinate u of $\tilde{S}^*$ extends over $\tilde{S}^{\log}$. Fix base points as $u_0 = 0 + i\infty \in \tilde{S}^{\log} \mapsto b := \bar{0} + i\infty \in S^{\log} \mapsto q = 0 \in S$, where $q = 0$ corresponds to q_0 for A-model and p_0 for B-model. Note that fixing a base point $u = u_0$ on $\tilde{S}^{\log}$ is equivalent to fixing a base point b on $S^{\log}$ and also a branch of $(2\pi i)^{-1}\log q$.

Let $B := \mathcal{H}_{\mathbf{Z}}^{V}(u_0) = \mathcal{H}_{\mathbf{Z}}^{V}(b)$ for A-model and $B := \mathcal{H}_{\mathbf{Z}}^{V^\circ}(u_0) = \mathcal{H}_{\mathbf{Z}}^{V^\circ}(b)$ for B-model.

2.5. Correspondence table

We use the mirror theorems in Introduction 0.2–0.3. Put $\Phi := \Phi_{\mathrm{GW}}^{V} = \Phi_{\mathrm{GM}}^{V^\circ}$ and fix an isomorphisim of VHS in 0.2.2 in Introduction (cf. 2.4).

(1A) *Polarization of A-model of V.*

$$S(\alpha,\beta) := (-1)^p(2\pi i)^3 \int_V \alpha \cup \beta \quad (\alpha \in H^{p,p}(V), \beta \in H^{3-p,3-p}(V)).$$

(1B) *Polarization of B-model of V°.*

$$Q(\alpha,\beta) := (-1)^{3(3-1)/2} \int_{V^\circ} \alpha \cup \beta = -\int_{V^\circ} \alpha \cup \beta \quad (\alpha,\beta \in H^3(V^\circ)).$$

(2A) ***Z**-basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}_{\mathbf{Z}}^{V}(u_0) = \mathcal{H}_{\mathbf{Z}}^{V}(b)$ be as in Section 2.4. Let $b^3 := s^3(u_0) = s(\mathcal{O}_V)(u_0)$, $b^2 := s^2(u_0) = s(\mathcal{O}_H)(u_0)$, $b^1 := s^1(u_0) = s(\mathcal{O}_C)(u_0)$ and $b^0 := s^0(u_0) = s(\mathcal{O}_{\mathrm{pt}})(u_0)$ be the basis of the fiber B at u_0 coming from ∇-flat integral basis in 2.4.

The endomorphism of $B_{\mathbf{Q}} := \mathbf{Q} \otimes B$ coming from the monodromy logarithm coincides with the cup product with $-2\pi i H$ where H is a hyperplane section of V ([7], Definition 3.6; cf. Proposition 1.4.3). Hence the above basis is compatible with the monodromy weight filtration M.

(2B) **Z**-*basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}_{\mathbf{Z}}^{V^\circ}(u_0) = \mathcal{H}_{\mathbf{Z}}^{V^\circ}(b)$, and b^0, b^1, b^2, b^3 be the basis of B corresponding to that in (2A) by the mirror symmetry 0.2.2 and 0.2.3 in Introduction.

The endomorphism of $B_{\mathbf{Q}}$ coming from the monodromy logarithm is denoted by N, and the above basis is compatible with the monodromy weight filtration M [ibid].

For both cases (2A) and (2B), we regard B as a constant sheaf on $S^{\log}$ and also on S, endowed with the associated filtrations M.

From the asymptotics of the basis s^p $(0 \le p \le 3)$ in 2.4, the matrix of the polarization pairings S in (1A) and Q in (1B) for the basis $b^p = s^p(u_0)$ is computed as

$$(S(b^p, b^q))_{p,q} = (Q(b^p, b^q))_{p,q} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -5 \\ 1 & 1 & 5 & 0 \end{pmatrix}.$$

(3A) *Sections compatible with Deligne decomposition and inducing* **Z**-*basis of* gr^M *for A-model of* V.

Let T^3, T^2, T^1, and T^0 be the basis of $\mathcal{H}_{\mathcal{O}}^V$ corresponding to the e^3, e^2, e^1, and e^0 in (3B) below by the mirror symmetry 0.2.2 and 0.2.3 in Introduction. Then $S(T^3, T^0) = 1$ and $S(T^2, T^1) = -1$. Hence T^3, T^2, $-T^0$, T^1 form a symplectic base for S in (1A).

Note that on gr^M they are

$$\mathrm{gr}_3^M(T^3) = 1 \in H^0(V, \mathbf{Z}), \quad \mathrm{gr}_2^M(T^2) = H \in H^2(V, \mathbf{Z}),$$

$$\mathrm{gr}_1^M(T^1) = C \in H^4(V, \mathbf{Z}), \quad \mathrm{gr}_0^M(T^0) = [\mathrm{pt}] \in H^6(V, \mathbf{Z}),$$

where H and C are the cohomology classes of a hyperplane section and a line on V, respectively. Abusing notation, we mean by C the Poincaré dual class of the homology class in 2.3.

(3B) *Sections compatible with Deligne decomposition and inducing* **Z**-*basis of* gr^M *for B-model of* V°.

We use Deligne decomposition [4]. We consider B in (2B) as a constant sheaf on $S^{\log}$. We have locally free $\mathcal{O}_S$-submodules $\mathcal{M}_{2p} := \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} M_{2p} B)$ and $\mathcal{F}^p$ in $\tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} B) = \mathcal{O}_S \otimes_{\mathbf{Z}} B$ (canonical extension of Deligne in Proposition 1.4.2). The mixed Hodge structure of Hodge–Tate type $(\mathcal{M}, \mathcal{F})$

has decomposition:

$$\mathcal{O}_S \otimes_{\mathbf{Z}} B = \bigoplus_p I^{p,p}, \qquad I^{p,p} := \mathcal{M}_{2p} \cap \mathcal{F}^p \xrightarrow{\sim} \mathrm{gr}^{\mathcal{M}}_{2p}.$$

Transporting the basis b^p $(0 \le p \le 3)$ of B in (2B), regarded as sections of the constant sheaf B on $S^{\log}$, via isomorphism

$$I^{p,p} \xrightarrow{\sim} \mathcal{O}_S \otimes_{\mathbf{Z}} \mathrm{gr}^M_{2p} B$$

we define sections $e^p \in I^{p,p}$ $(0 \le p \le 3)$ over S. Then, e^3, e^2,$-e^0$, e^1 form a symplectic basis for Q in (1B), and $e^3 = \tilde{\Omega} = \Omega/y_0$ over S.

The asymptotic relation of the T^p in (3A) (resp. the e^p in (3B)) can be computed, via the s^p, from (7A) (resp. (7B)) below.

(4A) *A-model connection* $\nabla = \nabla^{even}$ *of* V.

Let $\delta = d/du = 2\pi i q d/dq$. The Dubrovin connection ∇ (cf. [1], Sect. 8.4) is characterized by

$$\nabla_\delta T^0 = 0, \quad \nabla_\delta T^1 = T^0, \quad \nabla_\delta T^2 = \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}T^1 = \Big(5 + \frac{1}{(2\pi i)^3}\frac{d^3\Phi_{\mathrm{hol}}}{du^3}\Big)T^1,$$
$$\nabla_\delta T^3 = T^2.$$

∇ is flat, i.e., $\nabla^2 = 0$, and extends to a log connection over $S^{\log}$.

(4B) *B-model connection* $\nabla = \nabla^{\mathrm{GM}}$ *of* V°.

Let $\delta = d/du = 2\pi i q d/dq$. The Gauss-Manin connection ∇ is computed as

$$\nabla_\delta e^0 = 0, \quad \nabla_\delta e^1 = e^0, \quad \nabla_\delta e^2 = \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}e^1 = \frac{5}{(1+5^5)y_0(z)^2}\Big(\frac{q}{z}\frac{dz}{dq}\Big)^3 e^1,$$
$$\nabla_\delta e^3 l = e^2.$$

∇ is flat, i.e., $\nabla^2 = 0$, and extends to a log connection over $S^{\log}$.

(5A) ∇*-flat basis of* $\mathcal{H}^V_{\mathbf{C}}$ *inducing* $\mathbf{Z}$*-basis of* gr^M.

$$\tilde{s}^0 := T^0, \quad \tilde{s}^1 := T^1 - uT^0, \quad \tilde{s}^2 := T^2 - \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}T^1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}T^0,$$
$$\tilde{s}^3 := T^3 - uT^2 + \frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{dlu}\Big)T^1 - \frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big)T^0.$$

Then $\tilde{s}^3$, $\tilde{s}^2$,$-\tilde{s}^0$, $\tilde{s}^1$ form a symplectic basis for S in (1A).

(5B) ∇*-flat basis for* $\mathcal{H}^{V^\circ}_{\mathbf{C}}$ *inducing* $\mathbf{Z}$*-basis of* gr^M.

$$\tilde{s}^0 := e^0, \quad \tilde{s}^1 := e^1 - ue^0, \quad \tilde{s}^2 := e^2 - \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}e^1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}e^0,$$
$$\tilde{s}^3 := e^3 - ue^2 + \frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\Big)e^1 - \frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big)e^0.$$

Then $\tilde{s}^3$, $\tilde{s}^2$,$-\tilde{s}^0$, $\tilde{s}^1$ form a symplectic basis for Q in (1B).

For both cases (5A) and (5B), by using (4A) and (4B), the ∇-flat bases $\tilde{s}^p$ are determined inductively on $0 \le p \le 3$ from the T^p in (3A) and the e^p in (3B). These ∇-flat bases $\tilde{s}^p$ are characterized by the Frobenius solutions y_j $(0 \le j \le 3)$ in 2.2 such that $y_0 T^3$ in A-model and $y_0 e^3 = \Omega$ in B-model coincide with

$$y_0 \tilde{s}^3 + (2\pi i)^{-1} y_1 \tilde{s}^2 + 5(2\pi i)^{-2} y_2 \tilde{s}^1 + 5(2\pi i)^{-3} y_3 \tilde{s}^0.$$

(6A), (6B) *Relations of* ∇*-flat* **Z***-basis* s^p *and the* ∇*-flat basis* $\tilde{s}^p$.

$$s^0 = \tilde{s}^0, \quad s^1 = \tilde{s}^1 + \tilde{s}^0, \quad s^2 = \tilde{s}^2 - \frac{5}{2}\tilde{s}^1 + \frac{35}{12}\tilde{s}^0, \quad s^3 = \tilde{s}^3 + \frac{25}{12}\tilde{s}^1 + \frac{25i\zeta(3)}{\pi^3}\tilde{s}^0.$$

(7A) *Expression of the* T^p *by the* s^p *over* $S^{\log}$.

It is computed that T^p are written by the ∇-flat **Z**-basis s^p of $\mathcal{H}^V_{\mathbf{Z}}$ as follows.

$$T^0 = s^0, \quad T^1 = s^1 + (u-1)s^0,$$

$$T^2 = s^2 + \Big(\frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2} + \frac{5}{2}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\Big) - 5u - \frac{65}{12}\Big)s^0,$$

$$T^3 = s^3 + us^2 + \Big(\frac{1}{(2\pi i)^3}\frac{d\Phi}{du} + \frac{5}{2}u - \frac{25}{12}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big) - \frac{65}{12}u + \frac{25}{12} - \frac{25i}{\pi^3}\zeta(3)\Big)s^0.$$

(7B) *Expression of the* e^p *by the* s^p *over* $S^{\log}$.

It is computed that e^p are written by the ∇-flat **Z**-basis s^p of $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$ as follows.

$$e^0 = s^0, \quad e^1 = s^1 + (u-1)s^0,$$

$$e^2 = s^2 + \Big(\frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2} + \frac{5}{2}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\Big) - 5u - \frac{65}{12}\Big)s^0,$$

$$e^3 = s^3 + us^2 + \Big(\frac{1}{(2\pi i)^3}\frac{d\Phi}{du} + \frac{5}{2}u - \frac{25}{12}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big) - \frac{65}{12}u + \frac{25}{12} - \frac{25i}{\pi^3}\zeta(3)\Big)s^0.$$

(8A), (8B) *Relations of integral periods and Frobenius solutions.*

Let η_j $(0 \le j \le 3)$ be the integral periods defined by the condition that $y_0 T^3$ in A-model and $y_0 e^3 = \Omega$ in B-model coincide with $\eta_0 s^3 + \eta_1 s^2 + \eta_2 s^1 + \eta_3 s^0$. Then the relations in (6A), (6B) are interpreted as

$$\eta_0 = y_0, \quad \eta_1 = (2\pi i)^{-1} y_1, \quad \eta_2 = 5(2\pi i)^{-2} y_2 + \frac{5}{2}(2\pi i)^{-1} y_1 - \frac{25}{12} y_0,$$

$$\eta_3 = 5(2\pi i)^{-3} y_3 - 5(2\pi i)^{-2} y_2 + \frac{65}{12}(2\pi i)^{-1} y_1 + \Big(\frac{25}{12} - \frac{25i\zeta(3)}{\pi^3}\Big) y_0.$$

Remark. The η_j coincide with the corresponding coefficients of the expression of y_0T^3 in (7A) and of $y_0e^3 = \Omega$ in (7B), and yield the same integral structure for periods given in [2] and [18]. Indeed, for the notation ϖ_j in [18], (8), they are related as $\varpi_0 = \eta_0$, $\varpi_1 = \eta_1$, $\varpi_2 = \eta_2 - 5\eta_1$, and $\varpi_3 = -\eta_3 - \eta_2 - 5\eta_1$.

2.6. *Proofs of results in 2.5*

Proofs of (4A) and (4B) in 2.5. We prove (4B). (4A) follows by mirror symmetry theorems in Introduction 0.2.

We improve the proof of [1], Prop. 5.6.1 carefully by a log Hodge theoretic understanding in 1.4 of the relation among a constant sheaf and the local system on $S^{\log}$, the canonical extension of Deligne on S, and the Deligne decomposition.

We investigate the Gauss-Manin connection ∇, corresponding to the local system $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$, contracted with $\delta = 2\pi iqd/dq$. Since e^p maps to a $\nabla(\mathrm{gr}^{\mathcal{M}}_{2p})$-flat element of $\mathrm{gr}^{\mathcal{M}}_{2p}$, $\nabla_\delta(e^p)$ lies in $\mathcal{M}_{2p-1} = \mathcal{M}_{2p-2}$. But e^p is also an element of $\mathcal{F}^p$, so that $\nabla_\delta(e^p)$ lies in $\mathcal{F}^{p-1}$ by Griffiths transversality. This shows that $\nabla_\delta(e^p)$ is an element of $I^{p-1,p-1}$, and it follows that

$$\nabla_\delta(e^3) = Y_3e^2, \quad \nabla_\delta(e^2) = Y_2e^1, \quad \nabla_\delta(e^1) = Y_1e^0, \quad \nabla_\delta(e^0) = 0$$

for some $Y_1, Y_2, Y_3 \in \mathcal{O}_S^{\log}$. However, since $Q(e^3, e^1) = 0$ by orthogonality of Hodge filtration, we have

$$\begin{aligned} 0 &= \delta Q(e^3,e^1) = Q(\nabla_\delta(e^3), e^1) + Q(e^3, \nabla_\delta(e^1)) = Y_3Q(e^2,e^1) + Y_1Q(e^3,e^0) \\ &= -Y_3 + Y_1, \end{aligned}$$

where the last equality follows from 2.5 (3B).

Since ∇ has a regular singular point and $\delta = 2\pi iqd/dq$, Y_1 is holomorphic over S. Considering over the log point p_0, we claim $Y_1(0) = \pm 1$. Since e^1 is taken to be the canonical extension over p_0, we have $\nabla_\delta(e^1) = N(e^1) = -e^0$ by [17], Prop. 2.3.4 (ii) (cf. Proposition 1.4.3). Replacing e^1, e^2 by $-e^1, -e^2$, we have $Y_1(0) = 1$.

Since we use the canonical coordinate q in 2.2, the arguments in [1], Sect. 5.6.4, Sect. 2.3, yield

$$q = \exp\Big(\int Y_1(q)\frac{dq}{q}\Big).$$

Taking logarithm of both sides and differentiating them by $d/d\log q$, we have $Y_1(q) = 1$, hence $\nabla_\delta e^1 = e^0$ and $\nabla_\delta e^3 = e^2$. Thus, relative to the basis

e^0, e^1, e^2, e^3 and using the canonical coordinate q, ∇_δ has the connection matrix

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & Y & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

where $Y = Y_2$.

The proofs in [ibid, Proof of Prop. 5.6.1] for the following assertions work well: e^3 is the normalized 3-form $\tilde{\Omega}$; the Picard–Fuchs equation for $\tilde{\Omega}$ is $\nabla_\delta^2(\nabla_\delta^2\tilde{\Omega}/Y) = 0$; Y is the Yukawa coupling. The notation e^0, e^1, e^2, e^3 (resp. T^0, T^1, T^2, T^3) in the present paper corresponds to e_3, e_2, e_1, e_0 in [ibid, p.105] (resp. T^0, T^1, T_1, T_0 in [ibid, Sect. 8.5.3]). □

Proofs of (5A), (5B), (6A), (6B), (7A), and (7B) in 2.5.

We use mirror symmetry theorems in Introduction 0.2.

From e^p in (3B) and ∇_δ in (4B), we produce $\tilde{s}^p$ inductively on $0 \le p \le 3$ as in (5B). These are transported as (5A) in A-model. For the last assertion in (5B) on the relation of ∇-flat basis $\tilde{s}^p$ and the basis of the Frobenius solutions y_j, since $y_0 T^3$, $y_0 e^3 = \Omega$, and the last expression in (5B) are killed by the operator $\mathcal{L}$, it is enough to show the equality on the fiber $B_{\mathbf{C}} := \mathbf{C} \otimes B$, i.e., the coincidence of the initial conditions. We work in A-model. By the asymptotics of the reverse relation of (5A) and of the expressions of y_j in 2.2, we have

$$\begin{aligned} y_0 T^3 &= y_0 \tilde{s}^3 + y_0 u \tilde{s}^2 + y_0 (2\pi i)^{-3} \frac{d\Phi}{du} \tilde{s}^1 + y_0 (2\pi i)^{-3} \Big(u \frac{d\Phi}{du} - 2\Phi\Big) \tilde{s}^0 \\ &\sim y_0 \tilde{s}^3 + y_0 u \tilde{s}^2 + \frac{5}{2} y_0 u^2 \tilde{s}^1 + \frac{5}{6} y_0 u^3 \tilde{s}^0 \\ &\sim \tilde{s}^3 + ((2\pi i)^{-1} \log z) \tilde{s}^2 + \frac{5}{2}((2\pi i)^{-1} \log z)^2 \tilde{s}^1 + \frac{5}{6}((2\pi i)^{-1} \log z)^3 \tilde{s}^0 \\ &\sim y_0 \tilde{s}^3 + (2\pi i)^{-1} y_1 \tilde{s}^2 + 5(2\pi i)^{-2} y_2 \tilde{s}^1 + 5(2\pi i)^{-3} y_3 \tilde{s}^0. \end{aligned}$$

To prove (6A), (6B), (7A), and (7B), we want to find $c^{10}, c^{21}, c^{20}, c^{32}, c^{31}, c^{30} \in \mathbf{C}$ such that, on the fiber B,

$$\tilde{s}^0(u_0) = b^0, \quad \tilde{s}^1(u_0) = b^1 + c^{10}\tilde{s}^0(u_0), \quad \tilde{s}^2(u_0) = b^2 + c^{21}\tilde{s}^1(u_0) + c^{20}\tilde{s}^0(u_0),$$

$$\tilde{s}^3(u_0) = b^3 + c^{32}\tilde{s}^2(u_0) + c^{31}\tilde{s}^1(u_0) + c^{30}\tilde{s}^0(u_0).$$

Then, since $\tilde{s}^p$ and s^p are ∇-flat, we have

$$s^0 = \tilde{s}^0, \; s^1 = \tilde{s}^1 - c^{10}\tilde{s}^0, \; s^2 = \tilde{s}^2 - c^{21}\tilde{s}^1 - c^{20}\tilde{s}^0, \; s^3 = \tilde{s}^3 - c^{32}\tilde{s}^2 - c^{31}\tilde{s}^1 - c^{30}\tilde{s}^0.$$

Express the e^p by the s^p by using the inverse expressions of the above and of (5B). Transporting these into A-model, we get expression of the T^p by the s^p

and the c^{jk}. Using Iritani asymptotics for the s^p in A-model in 2.4, we get

$$
\begin{aligned}
T^0 &= \frac{1}{5}H^3, \quad T^1 = (2\pi i)^{-1}\frac{1}{5}H^2 + \frac{c^{10}}{5}H^3,\\
T^2 &\sim (2\pi i)^{-2}H + (2\pi i)^{-1}\left(-\frac{1}{2} + \frac{c^{21}}{5}\right)H^2 + \left(\left(\frac{3}{2} - \frac{c^{21}}{5} + c^{10}\right)u\right.\\
&\quad \left. + \left(\frac{7}{12} + \frac{c^{21}}{5} + \frac{c^{21}c^{10}}{5} + \frac{c^{20}}{5}\right)\right)H^3,\\
T^3 &\sim (2\pi i)^{-3} + (2\pi i)^{-2}c^{32}H + (2\pi i)^{-1}\left(\left(-\frac{1}{2} + \frac{c^{21}}{5} - c^{32}\right)u\right.\\
&\quad \left. + \left(\frac{5}{12} - \frac{c^{32}}{2} + \frac{c^{32}c^{21}}{5} + \frac{c^{31}}{5}\right)\right)H^2\\
&\quad + \left(\left(\frac{1}{6} - \frac{c^{31}}{5} + \frac{c^{20}}{5}\right)u + \left(\frac{5i\zeta(3)}{\pi^3} + \frac{c^{30}}{5}\right)\right)H^3.
\end{aligned}
$$

Since $T^p \in \mathcal{F}^p$ by construction ((3A), (3B) in 2.5), we see that all coefficients of H^j in the above expressions of T^p are zero for $j+p > 3$. Thus we get

$$c^{10} = -1,\ c^{21} = 5/2,\ c^{20} = -35/12,\ c^{32} = 0,\ c^{31} = -25/12,\ c^{30} = -25i\zeta(3)/\pi^3.$$

(6A), (6B), (7A), and (7B) follow from this and (5A), (5B).

We prove (8A), (8B). The same argument goes for both cases. We use the notation in B-model. By the definition of the η_j and (5B), $\Omega = \eta_0 s^3 + \eta_1 s^2 + \eta_2 s^1 + \eta_3 s^0 = y_0 \tilde{s}^3 + (2\pi i)^{-1} y_1 \tilde{s}^2 + 5(2\pi i)^{-2} y_2 \tilde{s}^1 + 5(2\pi i)^{-3} y_3 \tilde{s}^0$. Substituting (6B) and comparing the coefficients of $\tilde{s}^p$, we have expressions of the y_j by the η_j. Solving these for the η_j, we get (8B). □

Remark. It was pointed out by Hiroshi Iritani that the definitions and the descriptions of integral structures 3.5, 3.6 in [30] are insufficient. Actually, they were the first approximations of integral structures by means of gr^M, which are characterized by the Frobenius solutions as in the last statement of (5A) and (5B) in 2.5. The second proof in [ibid, 3.9] works well even in this approximation.

3 Proof of Theorem 0.4.1

In this section, we prove Theorem 0.4.1 in Introduction for open mirror symmetry of quintic threefolds. We prove it by constructing a normal function in log mixed Hodge theory for B-model in 3.1–3.2 below. This argument is applicable to the case of A-model by the theorems in Introduction 0.2 and 0.3 and the correspondence table in 2.5. We give some discussions on geometries and local systems in 3.3.

3.1. Proof of Theorem 0.4.1 over log disc S

We consider B-model. To make the monodromy of $\mathcal{T}_B$ unipotent, we take a double cover $z^{1/2} \mapsto z$. Let S be a neighborhood disc of p_0 in the $z^{1/2}$-plane endowed with log structure associated to the divisor p_0 in S. Denote by $\mathcal{H}$ and $\mathcal{T}$ the pullbacks of the log Hodge structure $\mathcal{H}^{V^\circ}$ and the tension $\mathcal{T}_B$ by the double covering, respectively.

We are looking for an extension $\tilde{\mathcal{H}}$:

$$0 \to \mathcal{H} \to \tilde{\mathcal{H}} \to \mathbf{Z}(-2) \to 0$$

of log mixed Hodge structures with $\mathrm{gr}_4^W \tilde{\mathcal{H}} = \mathbf{Z}(-2)$ and $\mathrm{gr}_3^W \tilde{\mathcal{H}} = \mathcal{H}$, which has liftings $1_{\mathbf{Z}}$ and 1_F of $(2\pi i)^{-2} \cdot 1 \in \mathbf{Z}(-2)$ in $\tilde{\mathcal{H}}$ respecting the lattice and the Hodge filtration, respectively, such that the tension $\mathcal{T}$ is described as

$$(1) \qquad Q(1_F - 1_{\mathbf{Z}}, \Omega) = \int_{C_-}^{C_+} \Omega = \mathcal{T},$$

where Q is the polarization of $\mathcal{H}$ coming from 2.5 (1B) and Ω is the 3-form from 2.2.

To find such a log mixed Hodge structure, we use the basis e^p $(0 \le p \le 3)$ respecting the Deligne decomposition of $(\mathcal{H}, \mathcal{M}, \mathcal{F})$ from 2.5 (3B), and the ∇-flat integral basis s^p $(0 \le p \le 3)$ from 2.5 (5B). We also use the integral periods η_j $(0 \le j \le 3)$ in 2.5 (8B). Note that these players are already extended and live together over $S^{\log}$.

Let the local system $L_{\mathbf{Q}}$ and the Néron model $J_{L_{\mathbf{Q}}}$ be as in 0.4 in Introduction (see also 1.8). Then $J_{L_{\mathbf{Q}}} = \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}(-2), \mathcal{H})$, and let $1_{\mathbf{Z}} := ((2\pi i)^{-2} \cdot 1, -(\mathcal{T}/\eta_0)s^0) \in \tilde{\mathcal{H}}_{\mathbf{Z}}$ be a lifting of $(2\pi i)^{-2} \cdot 1 \in \mathbf{Z}(-2) = (\mathrm{gr}_4^W)_{\mathbf{Z}}$, where $(\mathcal{T}/\eta_0)s^0 \in \mathcal{H}_{\mathcal{O}^{\log}} = (\mathrm{gr}_3^W)_{\mathcal{O}^{\log}}$. In particular, the connection $\nabla = \nabla^{\mathrm{GM}}$ on $\mathcal{H}$ is extended over $\tilde{\mathcal{H}}$ by $\nabla(1_{\mathbf{Z}}) = 0$.

To find 1_F, we write $1_F - 1_{\mathbf{Z}} = ae^3 + be^2 + ce^1 - (\mathcal{T}/\eta_0)e^0$ with $a, b, c \in \mathcal{O}_S^{\log}$ by using (1). The Griffiths transversality condition on $1_F - 1_{\mathbf{Z}}$ is understood as vanishing of the coefficient of e^0 in $\nabla_\delta(1_F - 1_{\mathbf{Z}})$. Using 2.5 (4B), we have

$$\nabla_\delta(1_F - 1_{\mathbf{Z}}) = (\delta a)e^3 + (a + \delta b)e^2 + \left(b\frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3} + \delta c\right)e^1 + (c - \delta(\mathcal{T}/\eta_0))e^0.$$

Hence, the above condition is equivalent to $c = \delta(\mathcal{T}/\eta_0)$ and a, b arbitrary. Using the relation "modulo F^2," we can take $a = b = 0$. Thus

$$1_F = 1_{\mathbf{Z}} + (\delta(\mathcal{T}/\eta_0))e^1 - (\mathcal{T}/\eta_0)e^0.$$

The pair $1_{\mathbf{Z}}$ and 1_F yields the desired element of $\mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}(-2), \mathcal{H})$, hence $1_F - 1_{\mathbf{Z}}$ yields the desired log normal function. Theorem 0.4.1 (1) is proved.

Next, we will find a splitting of the weight filtration W of the local system $L_{\mathbf{Q}}$, i.e., a splitting of W which is compatible with the local monodromy of the local system $L_{\mathbf{Q}}$. We use the monodromy table (3.14) in [31]. This is computed for A-model but applicable also for B-model by the theorems in Introduction 0.2–0.3 and in Section 2.5. Let T_∞^2 be the monodromy around $p_0 : z^{1/2} = 0$ and put $N := \log(T_\infty^2)$. By [ibid], $N(\mathcal{T}/\eta_0) = -1$ hence $N(1_{\mathbf{Z}}) = s^0$. On the other hand, we have $N(s^1) = -2s^0$. (Here we use the rotation of the monodromy as $\log z \mapsto \log z + 2\pi i$.) Define

$$1_{\mathbf{Z}}^{\mathrm{spl}} := 1_{\mathbf{Z}} + \frac{1}{2}s^1 = \left((2\pi i)^{-2}\cdot 1, \frac{1}{2}s^1 - \frac{\mathcal{T}}{\eta_0}s^0\right) \in \tilde{\mathcal{H}}_{\mathbf{Q}}.$$

Then $N(1_{\mathbf{Z}}^{\mathrm{spl}}) = 0$, and this gives the desired splitting of W of the local system $L_{\mathbf{Q}}$.

A lifting 1_F^{spl} for $1_{\mathbf{Z}}^{\mathrm{spl}}$, respecting the Hodge filtration, is computed as before and we get

(2) $$1_F^{\mathrm{spl}} = 1_{\mathbf{Z}}^{\mathrm{spl}} + \left(\delta\left(\frac{\mathcal{T}}{\eta_0}\right)\right)e^1 - \frac{\mathcal{T}}{\eta_0}e^0.$$

The pair $1_{\mathbf{Z}}^{\mathrm{spl}}$ and 1_F^{spl} yields the desired element of $\mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}(-2), \mathcal{H})$ which splits the weight filtration W of the local system $L_{\mathbf{Q}}$. Note that $1_F^{\mathrm{spl}} - 1_{\mathbf{Z}}^{\mathrm{spl}} = 1_F - 1_{\mathbf{Z}} = (\delta(\mathcal{T}/\eta_0))e^1 - (\mathcal{T}/\eta_0)e^0$. Theorem 0.4.1 (2) is proved.

Theorem 0.4.1 (3) follows immediately from the above results.

We add a remark that the W-relative N-filtration $M = M(N, W)$ on $H_{\mathbf{R}}$ in the admissibility condition 1.5 (1) is given by

$$M_{-1} = 0 \subset M_0 = M_1 = \mathbf{R}s^0 \subset M_2 = M_3 = M_1 + \mathbf{R}s^1$$

$$\subset M_4 = M_5 = M_3 + \mathbf{R}s^2 + \mathbf{R}1_{\mathbf{Z}} \subset M_6 = \tilde{\mathcal{H}}_{\mathbf{R}} = M_5 + \mathbf{R}s^3.$$

3.2. *Proofs of (1) and (2) in Theorem 0.4.1 over log point p_0*

We still consider B-model. We show here that (1) and (2) in Theorem 0.4.1 have meanings just over the log point p_0 and that the computations in their proofs become simpler.

Recall that

(1) $$\mathcal{T} = -\frac{\eta_1}{2} - \frac{\eta_0}{4} + a_0\tau \qquad \left(a_0 := \frac{15}{\pi^2},\ \tau : \text{tau function}\right)$$

from [31]. We substitute $z^{1/2} = 0$ to $\mathcal{T}$ carefully as follows. Recall $\eta_1 = \eta_0 u$ from 2.2 and $u = x + iy$ from 2.4. Write $v := x + i\infty$ and define

$$\mathcal{T}(0) := -\frac{v}{2} - \frac{1}{4} + a_0 \qquad \text{in } \mathcal{O}_{p_0}^{\log} = \mathbf{C}[v].$$

We abuse the notation e^p and s^p also for their restrictions over the log point p_0, and so they live together over $p_0^{\log} = (\mathbf{S}^1, \mathbf{C}[v])$.

Similarly as in 0.4 in Introduction, but using now $\mathcal{T}(0)s^0$ instead of $(\mathcal{T}/\eta^0)s^0$ because $\eta^0(0) = 1$, we define a local system $L_{\mathbf{Q}}$ and a Néron model $J_{L_{\mathbf{Q}}}$ lying over $L_{\mathbf{Q}}$. Let $\tilde{\mathcal{H}}$ be an extension of log mixed Hodge structures over the log point p_0, we are looking for, like in 3.1, and let $1_{\mathbf{Z}} := ((2\pi i)^{-2} \cdot 1, -\mathcal{T}(0)s^0)$ be a lifting of $(2\pi i)^{-2} \cdot 1 \in \mathbf{Z}(-2) = (\mathrm{gr}_4^W)_{\mathbf{Z}}$ in $\tilde{\mathcal{H}}_{\mathbf{Z}}$. Hence the connection ∇ on $\mathcal{H}$ is extended over $\tilde{\mathcal{H}}$ by $\nabla(1_{\mathbf{Z}}) = 0$. Note that both $2\pi iq\frac{d}{dq}$ and $2\pi iz\frac{d}{dz}$ coincide with $\frac{d}{dv}$ now, which is denoted by δ. To find 1_F, write $1_F - 1_{\mathbf{Z}} = ae^3 + be^2 + ce^1 - \mathcal{T}(0)e^0$ $(a,b,c \in \mathbf{C}[v], \eta_0(0) = 1)$ and compute $\nabla_\delta(1_F - 1_{\mathbf{Z}})$ as in 3.1. Then, by the Griffiths transversality, we have $c = -1/2$, a and b arbitrary. By the relation "modulo F^2," a and b can be reduced to 0. Thus, we have

$$1_F = 1_{\mathbf{Z}} + (\delta\mathcal{T}(0))e^1 - \mathcal{T}(0)e^0 = 1_{\mathbf{Z}} - \frac{1}{2}e^1 + \left(\frac{v}{2} + \frac{1}{4} - a_0\right)e^0.$$

The pair $1_{\mathbf{Z}}$ and 1_F yields the desired element of $\mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}(-2), \mathcal{H})$. Theorem 0.4.1 (1) is proved.

The splitting of the weight filtration W of the local system $L_{\mathbf{Q}}$ is computed as in 3.1 but more simply, and we define

$$1_{\mathbf{Z}}^{\mathrm{spl}} := 1_{\mathbf{Z}} + \frac{1}{2}s^1 = \left((2\pi i)^{-2} \cdot 1, \frac{1}{2}s^1 - \mathcal{T}(0)s^0\right) \in \tilde{\mathcal{H}}_{\mathbf{Q}}.$$

Similarly, a lifting 1_F^{spl} for $1_{\mathbf{Z}}^{\mathrm{spl}}$ is computed simply, and we get

$$1_F^{\mathrm{spl}} = 1_{\mathbf{Z}}^{\log} + (\delta\mathcal{T}(0))e^1 - \mathcal{T}(0)e^0.$$

Theorem 0.4.1 (2) is proved. □

Remark. Note that 0.4.1 (3) does not have meaning in the present context. This is because tau function disappears except its constant term when $z^{1/2} = 0$ is substituted. That is, in this step, we lose the transcendental data of the tension $\mathcal{T}$, contained as the extension of its underlying local system, from which we can recover the position of the quintic mirror in its complex moduli space.

3.3. Discussions on geometries and local systems

We discuss here the relation with geometries and local systems considered in [31] and [23]. Forgetting Hodge structures, we consider only local systems corresponding to the monodromy of integral periods and tensions.

Let V_ψ and V_ψ° be a quintic threefold and its mirror from 2.1. Let S be a small neighborhood in the z-plane (z in 2.2) of the maximal unipotent monodromy point p_0 endowed with the log structure associated to the divisor p_0.

We first consider B-model. Let the setting be as in [23], Sect. 4. For $z \neq 0$ near 0, i.e., near p_0, let V_z° be the mirror quintic and $C_{+,z} \cup C_{-,z}$ be the disjoint union of smooth rational curves on V_z° coming from the two conics contained in $V_\psi \cap \{x_1+x_2=x_3+x_4=0\} \subset \mathbf{P}^4(\mathbf{C})$. From the relative homology sequence for $(V_z^\circ,(C_{+,z}\cup C_{-,z}))$, we have

(1) $\quad 0 \to H_3(V_z^\circ;\mathbf{Z}) \to H_3(V_z^\circ,(C_{+,z}\cup C_{-,z});\mathbf{Z}) \stackrel{\partial}{\to} \mathbf{Z}([C_{+,z}]-[C_{-,z}]) \to 0,$

where $\mathbf{Z}([C_{+,z}]-[C_{-,z}])$ is $\mathrm{Ker}(H_2(C_{+,z}\cup C_{-,z};\mathbf{Z}) \to H_2(V_z^\circ;\mathbf{Z}))$. The monodromy T_∞ around p_0 interchanges $C_{+,z}$ and $C_{-,z}$.

Respecting the sequence (1), we take a family of cycles Poincaré duality isomorphic to the flat integral basis s^p ($0 \le p \le 3$) in 2.4 and a family of chains joining from $C_{-,z}$ to $C_{+,z}$ (a choice up to integral cycles and up to half twists), and over them integrate the family of 3-forms $\Omega(z)$ with log pole over $z=0$ (z in the punctured disc in the z-plane) in 2.2, then we have a family of vectors $(\eta_0,\eta_1,\eta_2,\eta_3,\mathcal{T})$ consisting of periods (2.5 (8B)) and a tension. This corresponds to the data in [31], [23] (cf. Remark in 2.5). Since $T_\infty(\mathcal{T})=-(\mathcal{T}+\eta_1+\eta_0)$ by [31], (3.14), we find $\mathcal{T}+\frac{1}{2}\eta_1+\frac{1}{4}\eta_0=\frac{15}{\pi^2}\tau$ (see 3.2 (1)) is an eigenvector of the monodromy T_∞ with eigenvalue -1.

The family of sequences (1) ($z\neq 0$) forms an exact sequence of local systems of $\mathbf{Z}$-modules. Pulling this back to S^* in 3.1 by the double cover $z^{1/2}\mapsto z$, we have a sequence with unipotent local monodromy and its extension over $S^{\log}$. Applying Tate twist (-3) and Poincaré duality isomorphism to the left and the right ends of this exact sequence, we have a local system L' over $S^{\log}$ which is an extension of $\mathbf{Z}(-2)$ by $\mathcal{H}_{\mathbf{Z}}$:

(2) $$0 \to \mathcal{H}_{\mathbf{Z}} \to L' \to \mathbf{Z}(-2) \to 0.$$

Take a lifting $1_{\mathbf{Z}} := ((2\pi i)^{-2}\cdot 1, (\mathcal{T}/\eta_0)s^0)$ in L' of $(2\pi i)^{-2}\cdot 1 \in \mathbf{Z}(-2)$, and extend ∇ on $\mathcal{H}_{\mathbf{Z}}$ over L' by $\nabla(1_{\mathbf{Z}})=0$. We look for a ∇-flat T_∞^2-invariant element associated to $1_{\mathbf{Z}}$. This is done as in 3.1, and we get $1_{\mathbf{Z}}^{\mathrm{spl}} := 1_{\mathbf{Z}}-(s^1/2)$. Thus we know that L' coincides with $\tilde{\mathcal{H}}_{\mathbf{Z}}$ in 3.1.

For the relative monodromy weight filtration $M=M(N,W)$ of L', we see that $1_{\mathbf{Z}}\in M_4$ and $s^1\in M_2$ are the smallest filters containing each element in question. Taking the graded quotients by M of the sequence (2), we have

(3) $$\mathrm{gr}_6^M\mathcal{H}_{\mathbf{Z}} \stackrel{\sim}{\to} \mathrm{gr}_6^M L',$$
$$0 \to \mathrm{gr}_4^M\mathcal{H}_{\mathbf{Z}} \to \mathrm{gr}_4^M L' \to \mathbf{Z}(-2) \to 0,$$
$$0 \to \mathrm{gr}_2^M\mathcal{H}_{\mathbf{Z}} \to \mathrm{gr}_2^M L' \to (\text{2-torsion}) \to 0,$$
$$\mathrm{gr}_0^M\mathcal{H}_{\mathbf{Z}} \stackrel{\sim}{\to} \mathrm{gr}_0^M L'.$$

Here we abuse the notation M also for the monodromy filtration on $\mathcal{H}_{\mathbf{Z}}$, because it coincides with the restriction of $M=M(N,W)$ to $\mathcal{H}_{\mathbf{Z}}$. The 2-torsion in the third sequence of (3) corresponds to a half twist of chains from C_- to C_+. Standing on a half integral point and looking at the integral points nearby, we have two orientations. These correspond to the two orientations of a half twist of the chains, and also correspond to $\mathcal{T}_{\pm} := \pm(\frac{15}{\pi^2}\tau - \frac{\eta_0}{4}) - \frac{\eta_1}{2}$ in [31]. $\mathcal{T}_-$ is different from $-\mathcal{T}_+$ by the complementary half twist, i.e., $\mathcal{T}_+ + \mathcal{T}_- = -\eta_1$.

By using mirror symmetries in 0.2–0.4, or more precisely, by the results in Section 2.5 and Section 3.1, $\mathcal{H}_{\mathbf{Z}} = \mathcal{H}_{\mathbf{Z}}^{V^\circ}$, $\mathcal{T} = \mathcal{T}_B$, $1_{\mathbf{Z}}$, $\nabla = \nabla^{\mathrm{GM}}$, $1_{\mathbf{Z}}^{\mathrm{spl}}$, and $M = M(N, W)$ of B-model are transformed to the corresponding $\mathcal{H}_{\mathbf{Z}} = \mathcal{H}_{\mathbf{Z}}^{V}$, $\mathcal{T} = y_0\mathcal{T}_A$, $1_{\mathbf{Z}}$, $\nabla = \nabla^{\mathrm{even}}$, $1_{\mathbf{Z}}^{\mathrm{spl}}$, and $M = M(N, W)$ of A-model, and the exact sequences (2) and (3) of B-model are transformed to the corresponding exact sequences of A-model.

It is interesting to study the relations of these exact sequences with the geometries of Fermat quintic $V = V_\psi$ with $\psi = 0$ and its Lagrangian submanifold $Lg := V \cap \mathbf{P}^4(\mathbf{R})$ in [31], 2.1; [23], 3.

Remark.

(1) The argument in 3.3 can be performed even over the log point p_0.
(2) [26] and [19] are related with the topics in this subsection.

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