Mem. Fac. Sci.
Kochi Univ. (Math.)
1 (1980), 1–3

NOTE ON THE EXAMPLE OF KĬNEF

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(Received June 12, 1979)

1. Statement

In [2], Kĭnef constructed an interesting example of a surface of general type with $p_g = K^2 = 1$, for which the local Torelli theorem does not hold. In this short paper, the author makes a remark on the Kuranishi family of the deformations of the Kĭnef's example.

Let X be the surface constructed by Kĭnef [2], which has the following properties.

(1.1) X is a simply connected non-singular projective surface of general type over $\mathbf{C}$.

(1.2) $p_g = K^2 = 1$. $q = 0$.

(1.3) $H^0(X, \Theta_X) = H^2(X, \Theta_X) = 0$. $h^1(X, \Theta_X) = 18$. $h^1(X, \Omega_X^1) = 19$.

(1.4) $C \in |K|$ is a smooth curve of genus 2 and $h^0(C, \Omega_X^1 \otimes \mathcal{O}_C) = 2$.

Let $\mathfrak{X} \xrightarrow{f} S$ be the Kuranishi family of the deformations of $X = f^{-1}(0)$ $(0 \in S)$. From (1.3), S is smooth and $\dim S = 18$. Since $p_g = 1$, the infinitesimal period map φ at $0 \in S$ is nothing but

$$H^1(X, \Theta_X) \longrightarrow H^1(X, \Omega_X^1)$$

obtained from the exact sequence

$$0 \longrightarrow \Theta_X \xrightarrow{\otimes \omega} \Omega_X^1 \longrightarrow \Omega_X^1 \otimes \mathcal{O}_C \longrightarrow 0, \tag{1.5}$$

where ω is the global equation of the canonical divisor C. From (1.5), we get

$$H^0(X, \Omega_X^1) \longrightarrow H^0(X, \Omega_X^1 \otimes \mathcal{O}_C) \longrightarrow H^1(X, \Theta_X) \longrightarrow H^1(X, \Omega_X^1)$$

and hence, from (1.2) and (1.4), $\dim \operatorname{Ker} \varphi = h^0(X, \Omega_X^1 \otimes \mathcal{O}_C) = 2$.

Our result in this paper is the following.

PROPOSITION (1.6). *There is a subspace S' of S with codimension $\leqq 4$, such that, at each point $s \in S'$, the infinitesimal period map φ_s has the 2-dimensional kernel.*

We will prove this in the next section.

2. Proof of (1.6)

LEMMA (2.1). *There exists $\tilde{\omega} \in H^0(\mathfrak{X}, \Omega^2_f)$ which induces a global equation ω of the canonical divisor C on $X = f^{-1}(0)$.*

PROOF. From (1.2), we get $H^1(X, \Omega^2_X) = 0$ by the Serre duality. Hence Ω^2_f is cohomologically flat in dimension 0 over S by the base change theorem. Therefore we can get the required section $\tilde{\omega}$. QED.

We may assume $\tilde{\omega}$ is not zero on each fibre of f and hence we get a flat family

$$g = \mathrm{res}\,(f)\colon \mathfrak{C} \longrightarrow S$$

of the deformations of the canonical divisor C on X. Actually g is smooth because of (1.4).

There is a natural exact sequence

$$0 \longrightarrow \check{N}_{\mathfrak{C}/\mathfrak{X}} \longrightarrow \Omega^1_f \otimes \mathcal{O}_{\mathfrak{C}} \longrightarrow \Omega^1_g \longrightarrow 0 \tag{2.2}$$

which induces, on each fibre $C_s = g^{-1}(s)$, the exact sequence

$$0 \longrightarrow \check{N}_{C_s/X_s} \longrightarrow \Omega^1_{X_s} \otimes \mathcal{O}_{C_s} \longrightarrow \Omega^1_{C_s} \longrightarrow 0. \tag{2.3}$$

LEMMA (2.4). *$(\Omega^1_g)^\vee \otimes \check{N}_{\mathfrak{C}/\mathfrak{X}}$ is cohomologically flat in dimension* 1 *over S and $R^1 g^*((\Omega^1_g)^\vee \otimes \check{N}_{\mathfrak{C}/\mathfrak{X}})$ is of rank* 4.

PROOF. By the base change theorem, it is enough to show that $h^1((\Omega^1_{C_s})^\vee \otimes \check{N}_{C_s/X_s}) = 4$ for all $s \in S$. Since $\deg((\Omega^1_{C_s})^\vee \otimes \check{N}_{C_s/X_s}) = -3K^2 = -3$, we have the required result by the Riemann-Roch theorem on C_s. QED.

Let e_i $(i = 1, 2, 3, 4)$ be the basis of $R^1 g^*((\Omega^1_g)^\vee \otimes \check{N}_{\mathfrak{C}/\mathfrak{X}})$ and let $e = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$ be the global section of $R^1 g^*((\Omega^1_g)^\vee \otimes N_{\mathfrak{C}/\mathfrak{X}})$ corresponding to the extension (2.2). Denote by S' the subspace of S defined by the $\mathcal{O}_S$-ideal generated by a_i $(i = 1, 2, 3, 4)$, we get the following result.

LEMMA (2.5). *There is a subspace S' of S with codimension ≤ 4, such that, at each point $s \in S'$, the exact sequence* (2.3) *splits.*

It is easy to see that (1.6) follows from (2.5). In fact, (2.5) is a little finer result than (1.6).

References

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