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<tr>
<td>Author(s)</td>
<td>Usui, Sampei</td>
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<tr>
<td>Citation</td>
<td>Memoirs of the Faculty of Science, Kochi University. Ser. A, Mathematics. 1 P.1-P.3</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1980</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/11094/73379">http://hdl.handle.net/11094/73379</a></td>
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<td>DOI</td>
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NOTE ON THE EXAMPLE OF KINEF

Sampei Usui
(Received June 12, 1979)

1. Statement

In [2], Kinef constructed an interesting example of a surface of general type with $p_g = K^2 = 1$, for which the local Torelli theorem does not hold. In this short paper, the author makes a remark on the Kuranishi family of the deformations of the Kinef's example.

Let $X$ be the surface constructed by Kinef [2], which has the following properties.

(1.1) $X$ is a simply connected non-singular projective surface of general type over $\mathbb{C}$.

(1.2) $p_g = K^2 = 1$. $q = 0$.

(1.3) $H^0(X, \Theta_X) = H^2(X, \Theta_X) = 0$. $h^1(X, \Theta_X) = 18$. $h^1(X, \Omega_X^1) = 19$.

(1.4) $C \in |K|$ is a smooth curve of genus 2 and $h^0(C, \Omega_X^1 \otimes \mathcal{O}_C) = 2$.

Let $\mathcal{E} \longrightarrow S$ be the Kuranishi family of the deformations of $X = f^{-1}(0) (0 \in S)$. From (1.3), $S$ is smooth and $\dim S = 18$. Since $p_g = 1$, the infinitesimal period map $\varphi$ at $0 \in S$ is nothing but

$$H^1(X, \Theta_X) \longrightarrow H^1(X, \Omega_X^1)$$

obtained from the exact sequence

(1.5)

$$0 \longrightarrow \Theta_X \otimes \omega \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1 \otimes \mathcal{O}_C \longrightarrow 0,$$

where $\omega$ is the global equation of the canonical divisor $C$. From (1.5), we get

$$H^0(X, \Omega_X^1) \longrightarrow H^0(X, \Omega_X^1 \otimes \mathcal{O}_C) \longrightarrow H^1(X, \Theta_X) \longrightarrow H^1(X, \Omega_X^1)$$

and hence, from (1.2) and (1.4), $\dim \ker \varphi = h^0(X, \Omega_X^1 \otimes \mathcal{O}_C) = 2$.

Our result in this paper is the following.

**Proposition (1.6).** There is a subspace $S'$ of $S$ with codimension $\leq 4$, such that, at each point $s \in S'$, the infinitesimal period map $\varphi_s$ has the 2-dimensional kernel.
We will prove this in the next section.

2. Proof of (1.6)

**Lemma (2.1).** There exists $\omega \in H^0(\mathcal{X}, \Omega^2_X)$ which induces a global equation $\omega$ of the canonical divisor $C$ on $X = f^{-1}(0)$.

**Proof.** From (1.2), we get $H^1(\mathcal{X}, \mathcal{O}_X) = 0$ by the Serre duality. Hence $\Omega^2_X$ is cohomologically flat in dimension 0 over $S$ by the base change theorem. Therefore we can get the required section $\omega$. QED.

We may assume $\omega$ is not zero on each fibre of $f$ and hence we get a flat family $g = \text{res}(f): \mathcal{C} \longrightarrow S$ of the deformations of the canonical divisor $C$ on $X$. Actually $g$ is smooth because of (1.4).

There is a natural exact sequence

\begin{equation}
0 \longrightarrow \tilde{N}_{\mathcal{C}/\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{O}/\mathcal{S}} \longrightarrow 0
\end{equation}

which induces, on each fibre $C_s = g^{-1}(s)$, the exact sequence

\begin{equation}
0 \longrightarrow \tilde{N}_{C_s/\mathcal{X}_s} \longrightarrow \Omega^1_{\mathcal{C}_s/\mathcal{O}_{C_s}} \longrightarrow \Omega^1_{\mathcal{O}_s/\mathcal{S}_s} \longrightarrow 0.
\end{equation}

**Lemma (2.4).** $(\Omega^1_{\mathcal{O}/\mathcal{S}})^\vee \otimes \tilde{N}_{\mathcal{C}/\mathcal{X}}$ is cohomologically flat in dimension 1 over $S$ and $R^1 g^*((\Omega^1_{\mathcal{O}/\mathcal{S}})^\vee \otimes \tilde{N}_{\mathcal{C}/\mathcal{X}})$ is of rank 4.

**Proof.** By the base change theorem, it is enough to show that $h^1((\Omega^1_{\mathcal{O}/\mathcal{S}})^\vee \otimes \tilde{N}_{C_s/\mathcal{X}_s}) = 4$ for all $s \in S$. Since $\text{deg}((\Omega^1_{\mathcal{O}/\mathcal{S}})^\vee \otimes \tilde{N}_{C_s/\mathcal{X}_s}) = -3K_X^2 = -3$, we have the required result by the Riemann-Roch theorem on $C_s$. QED.

Let $e_i (i = 1, 2, 3, 4)$ be the basis of $R^1 g^*((\Omega^1_{\mathcal{O}/\mathcal{S}})^\vee \otimes \tilde{N}_{\mathcal{C}/\mathcal{X}})$ and let $e = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$ be the global section of $R^1 g^*((\Omega^1_{\mathcal{O}/\mathcal{S}})^\vee \otimes \mathcal{N}_{\mathcal{C}/\mathcal{X}})$ corresponding to the extension (2.2). Denote by $S'$ the subspace of $S$ defined by the $\mathcal{O}_S$-ideal generated by $a_i (i = 1, 2, 3, 4)$, we get the following result.

**Lemma (2.5).** There is a subspace $S'$ of $S$ with codimension $\leq 4$, such that, at each point $s \in S'$, the exact sequence (2.3) splits.

It is easy to see that (1.6) follows from (2.5). In fact, (2.5) is a little finer result than (1.6).

References

Example of Kinef


Kochi University
Department of Mathematics
Faculty of Science
Kochi, 780 Japan