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PERIOD MAP OF SURFACES WITH $p_g=c_1^2=1$ AND K AMPLE

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Introduction

Let X be a smooth, complete surface with $p_g=c_1^2=1$ and K ample. Let $\pi: \mathscr{X}\to S$ be the Kuranishi family of the deformations of $X_{s_0}=X(s_0\in S)$ and let $\phi: S\to D$ be the period map in the second cohomology (cf. 2.1). In this paper, we investigate what kind of fibre ϕ has. Our main results are Theorem (5.1) and Theorem (6.1).

In Theorem (5.1), we show that the period map ϕ actually has 0-dimensional, 1-dimensional and 2-dimensional fibers. The method here is an elementary caluculus by using the Jacobian of the period map ϕ (cf. Lemma (4.10)) and the kernel of the differential map $d\phi(s)$ at $s\in S$ of the period map ϕ (cf. Lemma (4.3) and Lemma (4.10)). Unfortunately, our result is imcomplete (cf. Remark (5.9)).

In Theorem (6.1), we get the relationship among some properties on $\{s\in S|$ $\dim \operatorname{Ker} d\phi(s)=2\}$ (cf. also Lemma (3.18)).

In § 1, we summarize some results on the surfaces X with $p_g=c_1^2=1$: Representation of such surfaces as weighted complete intersections of type (6, 6) in $\mathbf{P}(1, 2, 2, 3, 3)$ (the result of Catanese [2]). Some results on the cohomology groups of these surfaces X.

In § 2, we calculate the Jacobian of the period map ϕ by means of the coefficients of the defining equations of the surface X.

In § 3, we study the ramification divisor Δ of the period map ϕ: The singular locus of Δ. The subset $\{s\in\Delta\,|\dim \operatorname{Ker} d\phi(s)=2\}$.

In § 4, we calculate explicitly the kernel of the differential $d\phi(s)$ by means of the coefficients of the defining equations of the corresponding surface X.

In § 5 and § 6, we prove Theorem (5.1) and Theorem (6.1) mentioned above.

In this paper, the author owes much to the result of Catanese [2].

The present paper is deeply concerned with the author's forthcoming paper [7].

Notation and Convention

Every variety in this paper is defined over the field $\mathbf{C}$ of complex numbers.

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For a complex manifold X, we denote

Ω_X^1 = the sheaf of holomorphic differential 1-forms,

$K_X = \det \Omega_X^1$,

T_X = the dual of Ω_X^1.

More generally, for a smooth morphism $f: X \to Y$, we use the notation

Ω_f^1 = the sheaf of relative holomorphic differential 1-forms i.e. $\Omega_X^1/f^*\Omega_Y^1$,

$K_f = \det \Omega_f^1$,

T_f = the dual of Ω_f^1.

For a submanifold Z of a manifold X, we denote

$N_{Z/X}$ = the sheaf associated to the normal bundle to Z in X,

$\check{N}_{Z/X}$ = the dual of $N_{Z/X}$.

1. Surfaces with $p_g = c_1^2 = 1$

1.1. F. Catanese ([2]) showed that the canonical model of the surfaces of general type with $p_g = c_1^2 = 1$ are represented as weighted complete intersections of type (6, 6) in $\mathbf{P}(1, 2, 2, 3, 3)$. (For the concept of weighted complete intersection, see [5].) If we assume, furthermore, that their canonical invertible sheaves are ample, they contain no rational curves with self-intersection number -2 and hence their canonical models are smooth.

1.2. We summarize here some cohomological properties of the surface in question, which will be used later.

Let X be a smooth weighted complete intersection of type (6, 6) in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$. Denote by C the canonical divisor of X.

By using the facts

$$N_{X/\mathbf{P}} \simeq \mathcal{O}_X(6)^{\oplus 2} \quad \text{and} \quad K_X \simeq \mathcal{O}_X(1)$$

and the well-known exact sequences

$$0 \longrightarrow T_X \longrightarrow T_{\mathbf{P}} \otimes \mathcal{O}_X \longrightarrow N_{X/\mathbf{P}} \longrightarrow 0, \tag{1.1}$$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \bigoplus_{0 \leqq i \leqq 4} \mathcal{O}_X(e_i) \longrightarrow T_{\mathbf{P}} \otimes \mathcal{O}_X \longrightarrow 0 \tag{1.2}$$

(where $e_0 = 1$, $e_1 = e_2 = 2$ and $e_3 = e_4 = 3$) and

$$0 \longrightarrow \check{N}_{C/X} \longrightarrow \Omega_X^1 \otimes \mathcal{O}_C \longrightarrow \Omega_C^1 \longrightarrow 0, \tag{1.3}$$

we can calculate easily the following data on cohomology groups (cf. [5]):

$$H^0(X, T_X) = H^2(X, T_X) = 0, \quad \dim H^1(X, T_X) = 18. \tag{1.4}$$

$$H^0(X, \Omega_X^1) = 0, \quad \dim H^1(X, \Omega_X^1) = 19. \tag{1.5}$$

$$\dim H^0(C, \Omega_X^1 \otimes \mathcal{O}_C) \leqq 2. \tag{1.6}$$

(1.7) $\dim H^0(X, T_{\mathbf{P}}\otimes\mathcal{O}_X)=16, \quad H^1(X, T_{\mathbf{P}}\otimes\mathcal{O}_X)=H^2(X, T_{\mathbf{P}}\otimes\mathcal{O}_X)=0.$

(1.8) $\dim H^0(X, T_{\mathbf{P}}\otimes K_X)=28, \quad \dim H^1(X, T_{\mathbf{P}}\otimes K_X)=1.$

(1.9) $\dim H^0(C, T_{\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C)=12, \quad \dim H^1(C, T_{\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C)=1.$

(1.10) $\dim H^0(X, N_{X/\mathbf{P}})=34, \quad H^1(X, N_{X/\mathbf{P}})=H^2(X, N_{X/\mathbf{P}})=0.$

(1.11) $\dim H^0(X, N_{X/\mathbf{P}}\otimes K_X)=46, \quad H^1(X, N_{X/\mathbf{P}}\otimes K_X)=0.$

(1.12) $\dim H^0(C, N_{X/\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C)=12, \quad H^1(C, N_{X/\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C)=0.$

Let ω be the fundamental (1, 1)-form corresponding to the canonial polarization of X and let

$$H^1(X, T_X\otimes K_X)\xrightarrow{\omega} H^2(X, K_X)$$

be the map defined as the contraction with ω. Tensoring K_X to the exact sequence (1.1) and taking the cohomology sequence, we have

$$H^0(X, N_{X/\mathbf{P}}\otimes K_X)\xrightarrow{\delta} H^1(X, T_X\otimes K_X)\longrightarrow H^1(X, T_{\mathbf{P}}\otimes K_X).$$

LEMMA (1.13).

$H^0(X, N_{X/\mathbf{P}}\otimes K_X)\xrightarrow{\delta} H^1(X, T_X\otimes K_X)\xrightarrow{\omega} H^2(X, K_X)$ *is exact.*

PROOF. $\omega\in H^1(X, \Omega^1_X)$ comes from some $\tilde\omega\in H^1(X, \Omega^1_{\mathbf{P}}\otimes\mathcal{O}_X)$ and we have a canonical factorization

$$\begin{tikzcd} H^1(X, T_X\otimes K_X) \arrow[r, "\omega"] \arrow[d] & H^2(X, K_X). \\ H^1(X, T_{\mathbf{P}}\otimes K_X) \arrow[ur, "\tilde\omega"'] & \end{tikzcd}$$

Since ω is surjective and $\dim H^2(X, K_X)=\dim H^1(X, T_{\mathbf{P}}\otimes K_X)=1$ (1.8), we get our assertion. Q.E.D.

1.3. Let H be the Hilbert scheme parametrizing smooth weighted complete intersections of type (6, 6) in $\mathbf{P}=\mathbf{P}(1, 2, 2, 3, 3)$ and let

$$\begin{tikzcd} \mathscr{X}^{\cdot} \arrow[r] \arrow[d, "\pi^{\cdot}"'] & \mathbf{P}\times H \arrow[dl, "\text{projection}"] \\ H & \end{tikzcd}$$

be the universal family.

Let $R=\mathbf{C}[x_0, y_1, y_2, z_3, z_4]$ be the weighted polynomial ring with $\deg x_0=1$, $\deg y_i=2$ $(i=1, 2)$ and $\deg z_i=3$ $(i=3, 4)$ and let Aut (R) be the group of automorphisms of the graded $\mathbf{C}$-algebra R. Denote by $G^{\cdot}$ the image of the antiautomorphism Aut $(R) \to$ Aut $(\mathbf{P})$ of groups, i.e. $G^{\cdot}=$the group of linear automorphisms of $\mathbf{P}$. Since $K_{X_t} \simeq \mathcal{O}_{X_t}(1)$ for $X_t = \pi^{\cdot -1}(t)$ $(t \in H)$, $H/G^{\cdot}$ is the coarse moduli scheme of complete, smooth, minimal surfaces with $p_g = c_1^2 = 1$ and K ample.

The defining equations of a smooth weighted complete intersection of type (6, 6) in $\mathbf{P}(1, 2, 2, 3, 3)$ can be normalized as follows (see [2]):

$$\text{(1.14)} \qquad \begin{cases} f = z_3^2 + f^{(1)} z_4 x_0 + f^{(3)}, \\ g = z_4^2 + g^{(1)} z_3 x_0 + g^{(3)}, \end{cases}$$

where $f^{(1)}$ and $g^{(1)}$ are linear and $f^{(3)}$ and $g^{(3)}$ are cubic forms in x_0^2, y_1 and y_2, i.e., by using the notation $y_0 = x_0^2$,

$$f^{(1)} = \sum_{0 \le i \le 2} f_i y_i, \quad f^{(3)} = \sum_{0 \le i \le j \le k \le 2} f_{ijk} y_i y_j y_k,$$

$$g^{(1)} = \sum_{0 \le i \le 2} g_i y_i, \quad g^{(3)} = \sum_{0 \le i \le j \le k \le 2} g_{ijk} y_i y_j y_k.$$

These coefficients form a Zariski open set U in 26-dimensional affine space, that is,

$$U = \left\{ u \in \mathbf{A}^{26} \;\middle|\; \begin{array}{l} \textit{the corresponding surface is a} \\ \textit{smooth weighted complete intersection} \\ \textit{of type } (6, 6) \textit{ in } \mathbf{P}(1, 2, 2, 3, 3) \end{array} \right\}.$$

Note that U can be considered as a closed subscheme of the Hilbert scheme H. Denote by $\pi' : \mathscr{X}' \to U$ the pull-back of the universal family $\pi^{\cdot} : \mathscr{X}^{\cdot} \to H$ by $U \hookrightarrow H$.

Set

$$G = \{ \sigma \in G^{\cdot} \mid \sigma(U) \subset U \}.$$

Then an element $\sigma \in G$ can be represented by a non-singular matrix

$$\text{(1.15)} \qquad \left(\begin{array}{c|cc|cc} d_0 & & & & \\ \hline d_{10} & d_{11} & d_{12} & & \\ d_{20} & d_{21} & d_{22} & \multicolumn{2}{c}{0} \\ \hline & 0 & & d_3 & 0 \\ & & & 0 & d_4 \end{array}\right)$$

or

$$
(1.16)\qquad
\begin{array}{|c|cc|cc|}
\hline
d_0 & & & & \\
\cline{1-3}
d_{10} & d_{11} & d_{12} & & 0 \\
d_{20} & d_{21} & d_{22} & & \\
\hline
 & & & 0 & d_3 \\
 & 0 & & d_4 & 0 \\
\hline
\end{array}
$$

with the action

$$
\begin{cases}
\sigma x_0 = d_0 x_0, \\
\sigma y_i = \sum\limits_{0 \leqq j \leqq 2} d_{ij} y_j \qquad (i=1,2), \\
\sigma z_i = d_i z_i \qquad (i=3,4)
\end{cases}
$$

in case (1.15), and

$$
\begin{cases}
\sigma x_0 = d_0 x_0, \\
\sigma y_i = \sum\limits_{0 \leqq j \leqq 2} d_{ij} y_j \qquad (i=1,2), \\
\sigma z_3 = d_3 z_4, \\
\sigma z_4 = d_4 z_3,
\end{cases}
$$

in case (1.16).

The induced action of G on U is, for $u=(f, g) \in U$,

$$u=(f, g) \longmapsto \sigma u=(\sigma f/d_3^2, \sigma g/d_4^2)$$

in case (1.15), and

$$u=(f, g) \longmapsto \sigma u=(\sigma g/d_4^2, \sigma f/d_3^2)$$

in case (1.16), and we have

$$U/G \xrightarrow{\sim} H/G^{\cdot} = \text{the coarse moduli scheme of complete, smooth surfaces with } p_g = x_1^2 = 1 \text{ and } K \text{ ample.}$$

2. Period map and its Jacobian

2.1. Let X be a complete, smooth surface with $p_g = c_1^2 = 1$ and K_X ample. By (1.7), we see that the Kuranishi family $\pi\colon \mathscr{X} \to S$ of the deformations of $X = X_{s_0} = \pi^{-1}(s_0)$ $(s_0 \in S)$ is a universal family with the smooth parameter space S of dimmension 18.

Let

$$\phi: S \longrightarrow D \tag{2.1}$$

be the period map arising from the Kuranishi family $\pi: \mathscr{X} \to S$. Recall that (2.1) is constructed in the following way (cf. [3]): Fixing a C^∞-trivialization of the family $\pi: \mathscr{X} \to S$, we get the isomorphisms $\alpha_s: P^2(X_s, \mathbf{C}) \to P^2(X, \mathbf{C})$ $(s \in S)$ of the primitive cohomology groups preserving the Hodge-Riemann bilinear form Q. Then the map

$$\phi: S \longrightarrow \mathbf{P}^{19} = \{\text{lines in } P^2(X, \mathbf{C}) \text{ through the origin}\}$$

defined by

$$\phi(s) = \text{the line } \alpha_s(P^{2,0}(X_s)) \text{ in } P^2(X, \mathbf{C})$$

is holomorphic and factorizes

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{P}^{19} \\ & \searrow & \cup \\ & & D \subset \check{D} \end{array}$$

where

$$\check{D} = \{\xi \in \mathbf{P}^{19} \mid Q(\xi, \xi) = 0\} \quad \text{and}$$

$$D = \{\xi \in \check{D} \mid Q(\xi, \bar{\xi}) > 0\}.$$

This map $\phi: S \to D$ is the period map (2.1).

2.2. We continue to use the notation in 2.1.

By the universality of the Kuranishi family, we have uniquely determined morphism $p^\cdot: H \to S$ (in the sense of germs) from the Hilbert scheme H to S. Since the composite morphism $p': U \hookrightarrow H \to S$ (in the sense of germs) is smooth[1], we can take a section $S \hookrightarrow U$ and hence we can consider that the Kuranishi family $\pi: \mathscr{X} \to S$ is a subfamily of $\pi': \mathscr{X}' \to U$ in 1.3, that is, we have the following diagram:

$$\begin{array}{ccccccc} \mathscr{X} & \subset & \mathscr{X}' & \subset & \mathscr{X}^\cdot & \subset & \mathbf{P} \times H \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi^\cdot} & \swarrow & \\ S & \subset & U & \subset & H & & \end{array}$$

[1] In the paper [7], we constructs the fine moduli $\tilde{\pi}: \tilde{\mathscr{X}} \to \tilde{M}$ of marked surfaces with $p_g = c_1^2 = 1$ and K ample, whose parameter space $\tilde{M}$ is finite over the course moduli space M of surfaces with $p_g = c_1^2 = 1$ and K ample. By the universality of the fine moduli, we get the morphism $\tilde{p}: U \to \tilde{M}$ whose germ at the point in question coincides with the morphism $p': U \to S$, and we can prove that the morphism $\tilde{p}: U \to \tilde{M}$ is smooth by its construction.

Denote by q' the first projection $\mathbf{P}\times U\to\mathbf{P}$.

Recall also that the canonical divisors C_u on $X_u=\pi'^{-1}(u)$ $(u\in U)$ form a flat family $\mathscr{C}'\to U$ satisfying the commutative diagram (cf. [6]):

$$\begin{tikzcd} \mathscr{C}' \arrow[d] & \subset \mathscr{X} \arrow[dl, "\pi'"] \\ U. & \end{tikzcd}$$

Taking the direct image sheaves of the commutative exact diagram

(2.2)

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & T_{\pi'} \arrow[r] \arrow[d] & q'^*T_{\mathbf{P}}\otimes\mathcal{O}_{\mathscr{X}'} \arrow[r] \arrow[d] & N_{\mathscr{X}'/\mathbf{P}\times U} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_{\pi'}\otimes K_{\pi'} \arrow[r] \arrow[d] & q'^*T_{\mathbf{P}}\otimes K_{\pi'} \arrow[r] \arrow[d] & N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_{\pi'}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'} \arrow[r] \arrow[d] & q'^*T_{\mathbf{P}}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'} \arrow[r] \arrow[d] & N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

we get, by using (1.4), (1.5), (1.7), (1.10), (1.11) and (1.12), the diagram (2.3) in the next page.

LEMMA (2.4) *The following sheaves are cohomologically flat in dimension* 0 *with respect to the morphism $\pi'\colon\mathscr{X}'\to U$ and their direct images $\pi'_*(\ )$ have the ranks indicated just after them respectively*:

$q'^*T_{\mathbf{P}}\otimes\mathcal{O}_{\mathscr{X}'}$, 16. $q'^*T_{\mathbf{P}}\otimes K_{\pi'}$, 28.

$q'^*T_{\mathbf{P}}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}$, 12. $N_{\mathscr{X}'/\mathbf{P}\times U}$, 34.

$N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'}$, 46. $N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}$, 12.

The following are cohomologically flat in dimension 1 *and their first direct images $R^1\pi'_*(\ )$ have the ranks indicating just after them respectively*:

$T_{\pi'}$, 18. $T_{\pi'}\otimes K_{\pi'}$, 19.

$q'^*T_{\mathbf{P}}\otimes K_{\pi'}$, 1. $q'^*T_{\mathbf{P}}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}$, 1.

PROOF. By Continuity Theorem (cf. [1]), we get our assertion immediately from (1.4), (1.5), (1.7), (1.8), (1.9), (1.10), (1.11) and (1.12). Q. E. D.

(2.3)

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & {} \arrow[d] & \\
 & & 0 \arrow[d] & 0 \arrow[r] \arrow[d] & \pi'_*(T_{\pi'}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}) \arrow[r] \arrow[d] & \pi'_*(q'^*T_{\mathbf{P}}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}) \arrow[r] \arrow[d] & {} \\
 & 0 \arrow[r] & \pi'_*(q'^*T_{\mathbf{P}}\otimes\mathcal{O}_{\mathscr{X}'}) \arrow[r, "j"] \arrow[d, "i_2"] & \pi'_*N_{\mathscr{X}'/\mathbf{P}\times U} \arrow[r, "s"] \arrow[d, "i_1"] & R^1\pi'_*T_{\pi'} \arrow[r] \arrow[d, "i"] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \pi'_*(q'^*T_{\mathbf{P}}\otimes K_{\pi'}) \arrow[r, "j_1"] \arrow[d, "r_2"] & \pi'_*(N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'}) \arrow[r, "s_1"] \arrow[d, "r_1"] & R^1\pi'_*(T_{\pi'}\otimes K_{\pi'}) \arrow[r] \arrow[d, "r"] & R^1\pi'_*(q'^*T_{\mathbf{P}}\otimes K_{\pi'}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi'_*(T_{\pi'}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}) \arrow[r] \arrow[d] & \pi'_*(q'^*T_{\mathbf{P}}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}) \arrow[r, "j_2"] \arrow[d] & \pi'_*(N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}) \arrow[r, "s_2"] \arrow[d] & R^1\pi'_*(T_{\pi'}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}) \arrow[r] \arrow[d] & R^1\pi'_*(q'^*T_{\mathbf{P}}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'}) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & R^1\pi'_*T_{\pi'} \arrow[r] \arrow[d, "i"] & 0 & 0 & 0 & 0 & \\
{} \arrow[r] & R^1\pi'_*(T_{\pi'}\otimes K_{\pi'}) \arrow[r] \arrow[d] & {} & & & & \\
 & {} & & & & &
\end{tikzcd}
$$

2.3. For the later use, we will represent the morphism $j_1: \pi'_*(q'^*T_{\pi'}\otimes K_{\pi'}) \to \pi'_*(N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'})$ in (2.3) by choosing suitable frames.

We consider, as in 2.2, the family $\pi': \mathscr{X}' \to U$ to be that in the sense of germ at some fixed point $u \in U$.

Let f and g be the normalized forms in (1.14) of the defining equations of $\mathscr{X}'$ in $\mathbf{P}\times U$, where their coefficients f_i, f_{ijk}, g_i and g_{ijk} are considered to be parameters on U. Let $\mathscr{U}$ be the open subset of $\mathscr{X}'$ defined by $x_0 \neq 0$ and $\partial(f/x_0^6, g/x_0^6)/\partial(z_3/x_0^3, z_4/x_0^3) \neq 0$. For a sheaf F on $\mathscr{X}'$, we denote by res π'_*F the sheaf on U defined by

$\Gamma(V$, res $\pi'_*F)=$ the restriction of $\Gamma(\pi'^{-1}(V), F)$ to $\pi'^{-1}(V)\cap\mathscr{U}$ for an open subset V of U.

Note that, by Poincaré residue formura, we can take

$$(2.5)\qquad (x_0/x_0)\left(\frac{\partial(f/x_0^6, g/x_0^6)}{\partial(z_3/x_0^3, z_4/x_0^3)}\right)^{-1} d(y_1/x_0^2)\wedge d(y_2/x_0^2)$$

as a frame of res $\pi'_*K_{\pi'}$. Set

$$(2.6)\qquad \psi' = \left(\frac{\partial(f/x_0^6, g/x_0^6)}{\partial(z_3/x_0^3, z_4/x_0^3)}\right)^{-1} d(y_1/x_0^2)\wedge d(y_2/x_0^2).$$

Note also that res $\pi'_*N_{\mathscr{X}'/\mathbf{P}\times U}$ (resp. res $\pi'_*(N_{\mathscr{X}'/\mathbf{P}\times U}\otimes K_{\pi'})$) has a frame

$$(2.7)\qquad \begin{aligned} &\left\{(a/x_0^6)\frac{\partial}{\partial(f/x_0^6)}\,\middle|\, a \text{ is a monomial in } R \text{ of degree } 6\right\}\\ &\cup\left\{(a/x_0^6)\frac{\partial}{\partial(g/x_0^6)}\,\middle|\, a \text{ is a monomial in } R \text{ of degree } 6\right\}\end{aligned}$$

modulo

$$(2.8)\qquad \begin{aligned} &(f/x_0^6)\frac{\partial}{\partial(f/x_0^6)},\quad (g/x_0^6)\frac{\partial}{\partial(f/x_0^6)},\\ &(f/x_0^6)\frac{\partial}{\partial(g/x_0^6)} \quad\text{and}\quad (g/x_0^6)\frac{\partial}{\partial(g/x_0^6)}\end{aligned}$$

$$(2.9)\qquad \begin{aligned} &\Bigg(\text{resp. } \left\{(a/x_0^7)\frac{\partial}{\partial(f/x_0^6)}\otimes\psi'\,\middle|\, a \text{ is a monomial in } R \text{ of degree } 7\right\}\\ &\cup\left\{(a/x_0^7)\frac{\partial}{\partial(g/x_0^6)}\otimes\psi'\,\middle|\, a \text{ is a monomial in } R \text{ of degree } 7\right\}\end{aligned}$$

modulo

$$(2.10)\qquad \begin{aligned} &(fx_0/x_0^7)\frac{\partial}{\partial(f/x_0^6)}\otimes\psi',\quad (gx_0/x_0^7)\frac{\partial}{\partial(f/x_0^6)}\otimes\psi',\\ &(fx_0/x_0^7)\frac{\partial}{\partial(g/x_0^6)}\otimes\psi' \quad\text{and}\quad (gx_0/x_0^7)\frac{\partial}{\partial(g/x_0^6)}\otimes\psi'\Bigg)\end{aligned}$$

Let N (resp. I, N', I') be the free $\mathcal{O}_U$-mdule of rank 38 (resp. 4, 50, 4) with (2.7) (resp. (2.8), (2.9), (2.10)) as its free frame. Then we have natural exact sequences:

$$0 \longrightarrow I \longrightarrow N \longrightarrow \pi'_* N_{\mathfrak{X}'/\mathbf{P}\times U} \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow I' \longrightarrow N' \longrightarrow \pi'_*(N_{\mathfrak{X}'/\mathbf{P}\times U} \otimes K_{\pi'}) \longrightarrow 0.$$

Let

$$\tilde{j}\colon \operatorname{res} \pi'_*(q'^* T_{\mathbf{P}} \otimes \mathcal{O}_{\mathfrak{X}'}) \longrightarrow N$$

$$(\text{resp. } \tilde{j}_1\colon \operatorname{res} \pi'_*(q'^* T_{\mathbf{P}} \otimes K_{\pi'}) \longrightarrow N')$$

be a lifting of the morphism j (resp. j_1) in (2.3). Set

$$T = \operatorname{res} \pi'_*(q'^* T_{\mathbf{P}} \otimes \mathcal{O}_{\mathfrak{X}'}) \oplus I \quad \text{and}$$

$$T' = \operatorname{res} \pi'_*(q'^* T_{\mathbf{P}} \otimes K_{\pi'}) \oplus I'.$$

Let

$$j'\colon T \longrightarrow N \qquad (\text{resp. } j'_1\colon T' \longrightarrow N')$$

be the morphism defined by $\tilde{j}$(resp. $\tilde{j}_1$) and natural inclusion $I \hookrightarrow N$ (resp. $I' \hookrightarrow N'$).

Then we can replace the part

$$\begin{array}{ccc} \pi'_*(q'^* T_{\mathbf{P}} \otimes \mathcal{O}_{\mathfrak{X}'}) & \xrightarrow{j} & \pi'_* N_{\mathfrak{X}'/\mathbf{P}\times U} \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} \\ \pi'_*(q'^* T_{\mathbf{P}} \otimes K_{\pi'}) & \xrightarrow{j_1} & \pi'_*(N_{\mathfrak{X}'/\mathbf{P}\times U} \otimes K_{\pi'}) \end{array}$$

of the diagram (2.3) by

$$(2.11) \qquad \begin{array}{ccc} T & \xrightarrow{j'} & N \\ \downarrow{\scriptstyle i'_2} & & \downarrow{\scriptstyle i'_1} \\ T' & \xrightarrow{j'_1} & N' \end{array}$$

without losing commutativity and exactness, where i'_1 and i'_2 are the morphisms obtained by tensoring (2.5).

Now we represent the morphism j'_1 in (2.11) by using the following frames.

(2.12) *Frame of* T': We can divide the frame of T' into two parts (2.12.1) and (2.12.2) bellow, so that the last part (2.12.2) is the frame of the image of i'_2 in (2.11).

$$(2.12.1)\quad \left\{\begin{array}{l}
(y_1/x_0^2)\theta, \quad (y_2/x_0^2)\theta, \quad (z_4/x_0^3)\dfrac{\partial}{\partial(y_1/x_0^2)}\otimes\psi', \\[2ex]
(z_4/x_0^3)\dfrac{\partial}{\partial(y_2/x_0^2)}\otimes\psi', \quad (z_3/x_0^3)\dfrac{\partial}{\partial(y_1/x_0^2)}\otimes\psi', \\[2ex]
(z_3/x_0^3)\dfrac{\partial}{\partial(y_2/x_0^2)}\otimes\psi', \\[2ex]
(y_1^2/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \quad (y_1y_2/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \\[2ex]
(y_2^2/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \quad (y_1^2/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes x\psi', \\[2ex]
(y_1y_2/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes\psi', \quad (y_2^2/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes\psi';
\end{array}\right.$$

$$(2.12.2)\quad \left\{\begin{array}{l}
(z_3x_0/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \quad (z_4x_0/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \\[2ex]
(y_1x_0^2/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \quad (y_2x_0^2/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \\[2ex]
(x_0^4/x_0^4)\dfrac{\partial}{\partial(z_3/x_0^3)}\otimes\psi', \quad (z_3x_0/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes\psi', \\[2ex]
(z_4x_0/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes\psi', \quad (y_1x_0^2/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes\psi', \\[2ex]
(y_2x_0^2/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes\psi', \quad (x_0^4/x_0^4)\dfrac{\partial}{\partial(z_4/x_0^3)}\otimes\psi', \\[2ex]
(y_1x_0/x_0^3)\dfrac{\partial}{\partial(y_1/x_0^2)}\otimes\psi', \quad (y_2x_0/x_0^3)\dfrac{\partial}{\partial(y_1/x_0^2)}\otimes\psi', \\[2ex]
(x_0^3/x_0^3)\dfrac{\partial}{\partial(y_1/x_0^2)}\otimes\psi', \quad (y_1x_0/x_0^3)\dfrac{\partial}{\partial(y_2/x_0^2)}\otimes\psi', \\[2ex]
(y_2x_0/x_0^3)\dfrac{\partial}{\partial(y_2/x_0^2)}\otimes\psi', \quad (x_0^3/x_0^3)\dfrac{\partial}{\partial(y_2/x_0^2)}\otimes\psi', \\[2ex]
(fx_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', \quad (gx_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', \\[2ex]
(fx_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', \quad (gx_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi',
\end{array}\right.$$

where ψ' is (2.6) and

$$\theta = -\Big(\sum_{1\leq i\leq 2} 2(y_i/x_0^2)\frac{\partial}{\partial(y_i/x_0^2)} + \sum_{3\leq i\leq 4} 3(z_i/x_0^3)\frac{\partial}{\partial(z_i/x_0^3)}\otimes\psi'.$$

(2.13) *Frame of* N': The frame of N' can be divided into three parts (2.13.1), (2.13.2) and (2.13.3) bellow in the following way. The image of (2.13.1) in $\pi'_*(N_{\mathfrak{X}'/\mathbf{P}\times U}\otimes K_{\pi'}\otimes\mathcal{O}_{\mathscr{C}'})$ forms its frame (cf. (2.3)). The image of (2.13.3) in $\pi'_*(N_{\mathfrak{X}'/\mathbf{P}\times U}\otimes K_{\pi'})$ forms a frame of $i_1(T_U)$, where the sheaf T_U of the tangent bundle to U can be considered naturally a subsheaf of $T_H\otimes\mathcal{O}_U\simeq\pi'_*N_{\mathfrak{X}'/\mathbf{P}\times U}$ (cf. also (2.3)).

$$
(2.13.1)\quad \begin{cases}
(z_4y_1^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_4y_1y_2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi',\\
(z_4y_2^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_3y_1^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi',\\
(z_3y_1y_2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (z_3y_2^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi',\\
(z_3y_1^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_3y_1y_2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi',\\
(z_3y_2^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_4y_1^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi',\\
(z_4y_1y_2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (z_4y_2^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi';
\end{cases}
$$

$$
(2.13.2)\quad \begin{cases}
(z_3^2x_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_3z_4x_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi',\\
(z_4^2x_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_3y_1x_0^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi',\\
(z_3y_2x_0^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_3x_0^4/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi',\\
(z_3^2x_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (z_3z_4x_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi''\\
(z_4^2x_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (z_4y_1x_0^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi',\\
(z_4y_2x_0^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (z_4x_0^4/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi';
\end{cases}
$$

$$
\begin{cases}
(z_4y_1x_0^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_4y_2x_0^2/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi',\\
(z_4x_0^4/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (z_3y_1x_0^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi',
\end{cases}
$$

$$
(2.13.3)\qquad
\left\{
\begin{array}{ll}
(z_3 y_2 x_0^2/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (z_3 x_0^4/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', \\[2ex]
(y_1^3 x_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (y_1^2 y_2 x_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', \\[2ex]
(y_1 y_2^2 x_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (y_2^3 x_0/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', \\[2ex]
(y_1^2 x_0^3/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (y_1 y_2 x_0^3/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', \\[2ex]
(y_2^2 x_0^3/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (y_1 x_0^5/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', \\[2ex]
(y_2 x_0^5/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', & (x_0^7/x_0^7)\dfrac{\partial}{\partial(f/x_0^6)}\otimes\psi', \\[2ex]
(y_1^3 x_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (y_1^2 y_2 x_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', \\[2ex]
(y_1 y_2^2 x_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (y_2^3 x_0/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', \\[2ex]
(y_1^2 x_0^3/x_0^7)\dfrac{\partial}{\partial(g/x_0^7)}\otimes\psi', & (y_1 y_2 x_0^3/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', \\[2ex]
(y_2^2 x_0^3/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (y_1 x_0^5/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', \\[2ex]
(y_2 x_0^5/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi', & (x_0^7/x_0^7)\dfrac{\partial}{\partial(g/x_0^6)}\otimes\psi'.
\end{array}
\right.
$$

Let J be the matrix corresponding to the morphism j_1' in (2.11) with respect to the frames (2.12) and (2.13). J is a 50×32 matrix with functions on U as its entries and J is divided into 6 blocks according to the division of the frames. Set

$$
(2.14)\qquad J=\begin{array}{c}
\left[\begin{array}{c|c}
J_{11} & 0 \\ \hline
J_{21} & J_{22} \\ \hline
J_{31} & J_{32}
\end{array}\right]
\begin{array}{l}
\}12 \\ \}12 \\ \}26
\end{array} \\
\begin{array}{cc}
\underbrace{\qquad}_{12} & \underbrace{\qquad}_{20}
\end{array}
\end{array}
$$

Then, we have

(2.15)

$$J_{11}=\begin{array}{|cc|cc|cc|cccccc|c}
\hline
f_1 & 0 & 3f_{111} & f_{112} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
f_2 & f_1 & 2f_{112} & 2f_{122} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & f_2 & f_{122} & 3f_{222} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\cline{1-6}
g_1 & 0 & 0 & 0 & 3g_{111} & g_{112} & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
g_2 & g_1 & 0 & 0 & 2g_{112} & 2g_{122} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & g_2 & 0 & 0 & g_{122} & 3g_{222} & 0 & 0 & 0 & 0 & 0 & 0 & \\
\hline
0 & 0 & 0 & 0 & 3f_{111} & f_{112} & 2 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 2f_{112} & 2f_{122} & 0 & 2 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & f_{122} & 3f_{222} & 0 & 0 & 2 & 0 & 0 & 0 & \\
\cline{3-6}
0 & 0 & 3g_{111} & g_{112} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 6 \\
0 & 0 & 2g_{112} & 2g_{122} & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \\
0 & 0 & g_{122} & 3g_{222} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \\
\hline
\multicolumn{6}{c}{\underbrace{\hphantom{f_1\;0\;3f_{111}\;f_{112}\;3g_{111}\;g_{112}}}_{6}} & \multicolumn{6}{c}{\underbrace{\hphantom{2\;2\;2\;2\;2\;2}}_{6}} &
\end{array}$$

The other matrices $J_{ij}(i=2, 3; j=1, 2)$ can be also calculate in a trivial way. Since the space is limitted, we omit to write down these matrices.

2.4. We continue to use the notation above.

Recall that, by the result of Griffiths ([3]), the differential

$$d\phi\colon T_S \longrightarrow \phi^* T_D$$

of the period map ϕ in (2.1) can be identified with the morphism

$$R^1\pi_* T_\pi \longrightarrow \mathcal{H}om\,(\pi_* K_\pi,\ P^1\pi_*\Omega^1_\pi)$$

induced from the pairing

$$T_\pi \otimes K_\pi \longrightarrow \Omega^1_\pi,$$

where

$$P^1\pi_*\Omega^1_\pi = \text{the primitive part of } R^1\pi_*\Omega^1_\pi.$$

Hence, by the functoriality, the pull-back of $d\phi$ by the morphism $p'\colon U \to S$ (in the sense of germs as before) can be identified with

$$R^1\pi'_* T_{\pi'} \longrightarrow \mathcal{H}om\,(\pi'_* K_{\pi'}\ P^1\pi'_*\Omega^1_{\pi'}).$$

Let ψ be a frame of $\pi'_* K_{\pi'}$. Then, we have a commutative diagram

$$(2.16)\qquad \begin{tikzcd} p'^* T_S \arrow[r, "p'^*(d\phi)"] \arrow[d, phantom, "\wr"] & p'^* \phi^* T_D \arrow[d, phantom, "\wr"] \\ R^1 \pi'_* T_{\pi'} \arrow[r] \arrow[d, equal] & \mathcal{H}om(\pi'_* K_{\pi'}, P^1 \pi'_* \Omega^1_{\pi'}) \arrow[d, "\otimes\psi"] \\ R^1 \pi'_* T_{\pi'} \arrow[r, "\otimes\psi"] & P^1 \pi'_* \Omega^1_{\pi'}. \end{tikzcd}$$

From (1.13), (2.3), (2.11) and (2.16), we get a commutative diagram

$$(2.17)\qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & T \arrow[r, "j'"] \arrow[d, "i'_2"] & N \arrow[r, "s'"] \arrow[d, "i'_1"] & R^1 \pi'_* T_{\pi'} \arrow[r] \arrow[d, "i'"] & 0 \\ 0 \arrow[r] & T' \arrow[r, "j'_1"] \arrow[d, "r'_2"] & N' \arrow[r, "s'_1"] \arrow[d, "r'_1"] & P^1 \pi'_* \Omega^1_{\pi'} \arrow[r] & 0 \\ & \pi'_*(q'^* T_{\mathbf{P}} \otimes K_{\pi'} \otimes \mathcal{O}_{\mathscr{C}'}) \arrow[r, "j_2"] \arrow[d] & \pi'_*(N_{\mathscr{X}'/\mathbf{P}\times U} \otimes K_{\pi'} \otimes \mathcal{O}_{\mathscr{C}'}) \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

We denote by A the 6×6 minor matrix of J_{11} in (2.15) consisting of the first 6 rows and the first 6 columns, i.e.

$$(2.18)\qquad J_{11} = \left[\begin{array}{c|c} A & 0 \\ \hline A' & 2I_6 \end{array}\right], \quad \text{where } I_6 \text{ is the } 6\times 6 \text{ unit matrix.}$$

LEMMA (2.19) *For suitable coordinates of S and D, we have*

$$\textit{the Jacobian of } \phi = \det A,$$

where A is the matrix in (2.18).

PROOF. Set

$$(2.20)\qquad J' = \left[\begin{array}{c} J_{11} \\ \hline J_{21} \\ \hline J_{31} \end{array}\right] \quad \text{and} \quad J'' = \left[\begin{array}{c} J_{22} \\ \hline J_{32} \end{array}\right].$$

Note that the matrix J'' (resp. J_{11}) represents the morphism j' (resp. j_2) in (2.17) for suitable frames. Since j' and j_1' in (2.17) are injective, the matrices J'' and J in (1.14) are of maximal rank and hence we get that rank $J'=12$, rank $J''=20$ and rank $J=32$.

Let E be a subset of $\{1, 2,\dots, 50\}$ such that det $J(E)$ does not vanish at the point $u\in U$ in question, where we denote by $J(E)$ the 32×32 minor matrix of J consisting of the e-th rows $(e\in E)$. We can divide $E=E'\amalg E''$ so that det $J'(E')$ and det $J''(E''-12)$ do not vanish at the point $u\in U$, where we use the notation

$$E''-12=\{e-12\mid e\in E''\}$$

and also the notation $J'(E')$ and $J''(E''-12)$ as above.

Denote by $v_1', v_2',\dots, v_{50}'$ the frame of N' in (2.13). Then, from (2.17),

$$\{s'\circ i_1'^{-1}(v_e')\mid e\in\{13,\dots, 50\}\setminus E''\}$$

$$(\text{resp. } \{s_1'(v_e')\mid e\in\{1,\dots, 50\}\setminus E\})$$

form a fram of $R^1\pi'_*T_{\pi'}$ (resp. $P^1\pi'_*\Omega^1_{\pi'}$). By using these frames, the morphism i' in (2.17) can be represented by the matrix

$$\left|\begin{array}{c|c} -J_{11}J'(E')^{-1} & 0 \\ \hline * & I_6 \end{array}\right|.$$

Since

$$\det\left|\begin{array}{c|c} -J_{11}J'(E')^{-1} & 0 \\ \hline * & I_6 \end{array}\right| = (\det A)\cdot 2^6\cdot\det J'(E')^{-1}$$

and det $J'(E')$ is non-vanishing at $u\in U$, we get our assertion. Q. E. D.

3. Ramification locus of period map

We continue to use the notation in the previous section.

3.1. As in the beginning of 2.2, $\pi\colon\mathscr{X}\to S$ is the Kuranishi family of the deformations of $X=X_{u_0}=X_{s_0}$ with $u_0\in U$, $s_0\in S$ and $p'(u_0)=s_0$. Let $d\phi(s_0)$: $T_S(s_0)\to T_D(\phi(s_0))$ be the differential of the period map ϕ in (2.1) at $s_0\in S$ and let $A(u_0)$ be the matrix obtained by evaluating A in (2.18) at $u_0\in U$. Let C be the canonical divisor on X as before.

LEMMA (3.1) *We have*

$$\dim\operatorname{Ker} d\phi(s_0)=\operatorname{corank} A(u_0)=\dim H^0(C, \Omega^1_X\otimes\mathcal{O}_C)\leqq 2.$$

In particular, the fibre of the period map ϕ in (2.1) *through s_0 is at most* 2-*dimensional.*

PROOF. Restricting (2.2) to $X=X_{u_0}$ and taking the cohomology diagram, we get an exact diagram

(3.2)

$$\begin{array}{ccccc} & 0 & & & \\ & \downarrow & & & \\ 0\to H^0(C, T_X\otimes K_X\otimes\mathcal{O}_C) & \to & H^0(C, T_{\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C) & \xrightarrow{j_2(u_0)} & H^0(C, N_{X/\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C). \\ & \downarrow{\scriptstyle i(u_0)} & & & \\ & H^1(X, T_X) & & & \\ & \downarrow & & & \\ & H^1(X, T_X\otimes K_X) & & & \end{array}$$

Since $i(u_0)$ in (3.2) can be identified with $d\phi(s_0)$ in the same sense as in the beginning of 2.4 and since $j_2(u_0)$ in (3.2) is represented by the matrix $J_{11}(u_0)$ obtained from J_{11} in (2.15) by evaluating at u_0, we have

$$\dim \operatorname{Ker} d\phi(s_0)=\operatorname{corank} A(u_0)=\dim H^0(C, \Omega_X^1\otimes\mathcal{O}_C),$$

where we used the natural identification $T_X\otimes K_X\simeq\Omega_X^1$. Now the assertion follows from (1.6). Q. E. D.

Set

$$(3.3)\qquad \begin{aligned} &U_i=\{u\in U \mid \operatorname{corank} A(u)=i\} && \text{and} \\ &S_i=\{s\in S \mid \dim \operatorname{Ker} d\phi(s)=i\} && (i=0, 1, 2). \end{aligned}$$

Then, U_i $(i=0, 1, 2)$ (resp. S_i $(i=0, 1, 2)$) form a stratification of U (resp. S).

We also use the following notation:

$$\Delta'=U_1\cup U_2.$$

$$\Delta_0'=\{u\in\Delta' \mid f_1=f_2=g_1=g_2=0 \quad \text{at} \quad u\}.$$

$$\Delta_1'=\left\{u\in\Delta' \;\middle|\; \begin{array}{l} \text{not } f_1=f_2=0 \text{ but } g_1=g_2=0, \text{ or} \\ f_1=f_2=0 \text{ but not } g_1=g_2=0 \text{ at } u \end{array}\right\}.$$

$$(3.4)\qquad \Delta_{11}'=\left\{u\in\Delta' \;\middle|\; \begin{array}{l} \operatorname{rank}\begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix}=1, \text{ and neither} \\ f_1=f_2=0 \text{ nor } g_1=g_2=0 \text{ at } u \end{array}\right\}.$$

$$\Delta'_2=\left\{u\in\Delta' \,\middle|\, \operatorname{rank}\begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix}=2 \quad \text{at} \quad u\right\}.$$

$$\Delta=S_1\cup S_2.$$

$$\Delta_i=p'(\Delta'_i) \qquad (i=0,\ 1,\ 11,\ 2).$$

LEMMA (3.5) Δ'_0, Δ'_1, Δ'_{11} and Δ'_2 are G-stable.

PROOF. The assertion follows immediately from the obserbation of the induced action of G on f_1, f_2, g_1 and g_2. Q.E.D.

Note that Δ'_i (resp. Δ_i) ($i=0$, 1, 11, 2) form a stratification of Δ' (resp. Δ).

We normalize, further, the defining equations of X_u according to $u\in\Delta'_i$ ($i=0$, 1, 11, 2) as follows:

$$(3.5)\qquad \begin{cases} f=z_3^2+f_0z_4x_0^3+f^{(3)} \\ g=z_4^2+g_0z_3x_0^3+g^{(3)}, \end{cases} \qquad \text{for} \quad u\in\Delta'_0.$$

$$(3.6)\qquad \begin{cases} f=z_3^2+z_4x_0y_1+f^{(3)}, \\ g=z_4^2+g_0z_3x_0^3+g^{(3)}, \end{cases} \qquad \text{for} \quad u\in\Delta'_1.$$

$$(3.7)\qquad \begin{cases} f=z_3^2+z_4x_0y_1+f^{(3)}, \\ g=z_4^2+z_3x_0y_1+g_0z_3x_0^3+g^{(3)}, \end{cases} \qquad \text{for} \quad u\in\Delta'_{11}.$$

$$(3.8)\qquad \begin{cases} f=z_3^2+z_4x_0y_1+f^{(3)}, \\ g=z_4^2+z_3x_0y_2+g^{(3)}, \end{cases} \qquad \text{for} \quad u\in\Delta'_2.$$

3.2. *For simplicity of notation, we use the following*:

$$(3.9)\qquad (f)\ \ (\text{resp.}\,(g))=\textit{the matrix}\begin{pmatrix} 3f_{111} & f_{112} \\ 2f_{112} & 2f_{122} \\ f_{122} & 3f_{222} \end{pmatrix} \quad \left(\text{resp.}\begin{pmatrix} 3g_{111} & g_{112} \\ 2g_{112} & 2g_{122} \\ g_{122} & 3g_{222} \end{pmatrix}\right).$$

$|f(i, j)|$ (resp. $|g(i, j)|$) $=$ *the* 2×2 *minor determinant of the matrix* (f) (resp. (g)) *with the i-th and the j-th rows.*

LEMMA (3.10) *Set*

$$\Delta''=\{u\in\Delta'_{11}\cup\Delta'_2 \mid \operatorname{rank}(f)=\operatorname{rank}(g)=1 \quad at \quad u\}.$$

Then, we have:

(i) $U_2=\Delta_0'\cup\Delta''$ *(disjoint union).*
Δ_0' and Δ'' are both G-stable and of codimension 4 *in U.*
(ii) $S_2=\Delta_0\cup p'(\Delta'')$ *(disjoint union).*
Δ_0 and $p'(\Delta'')$ are both of codimension 4 *in S.*

PROOF. It is enough to prove (i), since (ii) follows immediately from (i).

The inclusion $U_2\supset\Delta_0'\cup\Delta''$ is obvious. We will prove the other inclusion by dividing cases according to $u\in\Delta_i'$ ($i=0$, 1, 11, 2).

In case $u\in\Delta_1'$, we use the normalized form (3.6) of the defining equations of X_u. Then the matrix A in (2.18) is evaluated at u as

$$A(u)=\left|\begin{array}{cc|cc|cc} 1 & 0 & 3f_{111} & f_{112} & & \\ 0 & 1 & 2f_{112} & 2f_{122} & \multicolumn{2}{c}{0} \\ 0 & 0 & f_{122} & 3f_{222} & & \\ \hline & & & & 3g_{111} & g_{112} \\ \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 2g_{112} & 2g_{122} \\ & & & & g_{122} & 3g_{222} \end{array}\right| . \tag{3.11}$$

If $f_{122}=f_{222}=0$, the points on X_u with $x_0=y_1=z_3=0$ become singular points of X_u. Hence

$$f_{122}\neq 0 \quad\text{or}\quad f_{222}\neq 0. \tag{3.12}$$

If rank $(g)\neq 2$, $g^{(3)}(0, y_1, y_2)=0$ has a triple root, say $y_i=a_i$ ($i=1$, 2), and hence the points on X_u with $x_0=z_4=0$ and $y_i=a_i$ ($i=1$, 2) become singular points of X_u. Therefore we have

$$\operatorname{rank}(g)=2. \tag{3.13}$$

By (3.11), (3.12) and (3.13), we see that corank $A=1$.

In case $u\in\Delta_{11}'$, we use the normalized form (3.7) of the defining equations of X_u. We assume

$$\text{not}\quad \operatorname{rank}(f)=\operatorname{rank}(g)=1 \quad\text{at}\quad u.$$

This is equivalent to assume

$$\text{not}\quad |f(1,3)|=|f(2,3)|=|g(1,3)|=|g(2,3)|=0 \quad\text{at}\quad u. \tag{3.14}$$

The matrix A is evaluated at u as

$$(3.15) \qquad A(u)=\left|\begin{array}{cc|cc|cc} 1 & 0 & 3f_{111} & f_{112} & & \\ 0 & 1 & 2f_{112} & 2f_{122} & & 0 \\ 0 & 0 & f_{122} & 3f_{222} & & \\ \hline 1 & 0 & & & 3g_{111} & g_{112} \\ 0 & 1 & & 0 & 2g_{112} & 2g_{122} \\ 0 & 0 & & & g_{122} & 3g_{222} \end{array}\right| .$$

As in the above case, we see that

$$(3.16) \qquad f_{122} \neq 0 \quad \text{or} \quad f_{222} \neq 0, \quad \text{and}$$

$$(3.17) \qquad g_{122} \neq 0 \quad \text{or} \quad g_{222} \neq 0.$$

By (3.14), (3.16) and (3.17), we can observe that at least one of the following 5×5 minor determinants of $A(u)$ does not vanish:

$$|2;6| = \pm g_{122}|f(1,3)|, \qquad |1;6| = \pm g_{122}|f(2,3)|,$$

$$|2;5| = \pm 3g_{222}|f(1,3)|, \qquad |1;5| = \pm 3g_{222}|f(2,3)|,$$

$$|5;4| = \pm f_{122}|g(1,3)|, \qquad |4;4| = \pm f_{122}|g(2,3)|,$$

$$|5;3| = \pm 3f_{222}|g(1,3)|, \qquad |4;3| = \pm 3f_{222}|g(2,3)|,$$

where

$|i;j|$ = the 5×5 minor determinant of $A(u)$ omtting the i-th row and the j-th column.

In case $u \in \Delta_2'$, we use the normalized form (3.8). The matrix A becomes

$$A(u)=\left|\begin{array}{cc|cc|cc} 1 & 0 & 3f_{111} & f_{112} & & \\ 0 & 1 & 2f_{112} & 2f_{222} & & 0 \\ 0 & 0 & f_{122} & 3f_{222} & & \\ \hline 0 & 0 & & & 3g_{111} & g_{112} \\ 1 & 0 & & 0 & 2g_{112} & 2g_{122} \\ 0 & 1 & & & g_{122} & 3g_{222} \end{array}\right| .$$

Observing the following 5×5 minor determinants of $A(u)$

$$|2;6|=\pm 3g_{111}|f(1,3)|, \quad |6;6|=\pm 3g_{111}|f(2,3)|,$$
$$|2;5|=\pm g_{112}|f(1,3)|, \quad |1;5|=\pm g_{112}|f(2,3)|,$$
$$|5;3|=\pm 3f_{222}|g(1,3)|, \quad |6;3|=\pm 3f_{222}|g(1,2)|,$$
$$|5;4|=\pm f_{122}|g(1,3)|, \quad |6;4|=\pm f_{122}|g(1,2)|,$$

we can find a non-vanishing one in the same manner as in the case $u\in\Delta'_{11}$.

Thus we have proven $U_2=\Delta'_0\cup\Delta''$.

It is easy to see that the condition rank $(f)=1$ (resp. rank $(g)=1$) is equivalent to the condition that, in $\mathbf{P}^2$ with homogeneous coordinates y_0, y_1 and y_2, the cubic curve $f^{(3)}=0$ (resp. $g^{(3)}=0$) and the line $y_0=0$ intersects at only one point. From this, it follows that Δ'' is G-stable.

The other assertions are easy to verify and we omit their proof.

Q. E. D.

3.3. We investigate in the next lemma the singular loci of Δ' and Δ in (3.4).

LEMMA (3.18) *Set*

$$\begin{aligned}\Delta'''=\{u\in\Delta'_1\,|\,&(|f(1,3)|\,|f(1,3)|-|f(1,2)|\,|f(2,3)|)\\ &\times(|g(1,3)|\,|g(1,3)|-|g(1,2)|\,|g(2,3)|)=0 \quad \textit{and}\\ &(f_1(|f(2,3)|\,|g(1,3)|-|f(1,3)|\,|g(1,3)|)\\ &-f_2(|f(1,3)|\,|g(1,3)|-|f(1,2)|\,|g(2,3)|))\\ &\times(g_1(|g(2,3)|\,|f(1,3)|-|g(1,3)|\,|f(1,3)|)\\ &-g_2(|g(1,3)|\,|f(1,3)|-|g(1,2)|\,|f(2,3)|))=0 \quad \textit{at} \quad u\}.\end{aligned}$$

Then we have:

(i) $Sing(\Delta')=U_2\cup\Delta'''$.

(ii) $Sing(\Delta)=S_2\cup p'(\Delta''')$.

PROOF. Since Δ' is smooth over Δ, (ii) follows from (i). Hence it is enough to prove (i).

We use the new affine coordinates ξ_i, ξ_{ijk}, η_i and η_{ijk} of U obtained from the old ones f_i, f_{ijk}, g_i and g_{ijk} by translating the origin to u, i.e.

$$(3.19)\qquad \begin{aligned}&\xi_i=f_i-f_i(u), \quad \eta_i=g_i-g_i(u) \qquad (0\leqq i\leqq 2),\\ &\eta_{ijk}=f_{ijk}-f_{ijk}(u), \quad \eta_{ijk}=g_{ijk}(u) \qquad (0\leqq i\leqq j\leqq k\leqq 2).\end{aligned}$$

With these new coordinates, the determinant

$$
\det \left(\begin{array}{cc|cc|cc}
\xi_1+f_1(u) & 0 & 3(\xi_{111}+f_{111}(u)) & \xi_{112}+f_{112}(u) & & \\
\xi_2+f_2(u) & \xi_1+f_1(u) & 2(\xi_{112}+f_{112}(u)) & 2(\xi_{122}+f_{122}(u)) & \multicolumn{2}{c}{0} \\
0 & \xi_2+f_2(u) & \xi_{122}+f_{122}(u) & 3(\xi_{222}+f_{222}(u)) & & \\
\hline
\eta_1+g_1(u) & 0 & & & 3(\eta_{111}+g_{111}(u)) & \eta_{112}+g_{112}(u) \\
\eta_2+g_2(u) & \eta_1+g_1(u) & \multicolumn{2}{c|}{0} & 2(\eta_{112}+g_{112}(u)) & 2(\eta_{122}+g_{122}(u)) \\
0 & \eta_2+g_2(u) & & & \eta_{122}+g_{122}(u) & 3(\eta_{222}+g_{222}(u))
\end{array} \right) \tag{3.20}
$$

gives the Taylor expansion of det A at u.

We will prove (i) in dividing cases with respect to the stratification Δ'_i ($i=0, 1, 11, 2$) of Δ'.

In case $u \in \Delta'_1$. We observe first the subcase

$$not \quad f_1=f_2=0 \quad but \quad g_1=g_2=0 \quad at \quad u.$$

In this subcase, the linear term of (3.20) is

$$\begin{aligned}&\eta_1(f_1(|f(2,3)|\,|g(1,3)|-|f(1,3)|\,|g(2,3)|)\\ &\quad -f_2(|f(1,3)|\,|g(1,3)|-|f(1,2)|\,|g(2,3)|))(u)\\ &\quad -\eta_2(f_1(|f(2,3)|\,|g(1,2)|-|f(1,3)|\,|g(1,3)|)\\ &\quad -f_2(|f(1,3)|\,|g(1,2)|-|f(1,2)|\,|g(1,3)|))(u)\end{aligned}$$

Since f_1 or $f_2 \neq 0$ at u, $u \in Sing(\Delta')$ is equivalent to

(3.21)

$$\begin{aligned}&\det\begin{bmatrix}|f(2,3)|\,|g(1,3)|-|f(1,3)|\,|g(2,3)| & |f(1,3)|\,|g(1,3)|-|f(1,2)|\,|g(2,3)|\\ |f(2,3)|\,|g(1,2)|-|f(1,3)|\,|g(1,3)| & |f(1,3)|\,|g(1,2)|-|f(1,2)|\,|g(1,3)|\end{bmatrix}\\ &=(|f(1,3)|\,|f(1,3)|-|f(1,2)|\,|f(2,3)|)(|g(1,3)|\,|g(1,3)|-|g(1,2)|\,|g(2,3)|)\\ &=0 \quad \text{at} \quad u\end{aligned}$$

and

$$\begin{aligned}(3.22)\qquad &f_1(|f(2,3)|\,|g(1,3)|-|f(1,3)|\,|g(2,3)|)\\ &\quad -f_2(|f(1,3)|\,|g(1,3)|-|f(1,2)|\,|g(2,3)|)=0 \quad \text{at} \quad u.\end{aligned}$$

Similarly, we see, in the subcase $f_1=f_2=0$ but not $g_1=g_2=0$ at u, that $u \in Sing(\Delta')$ is equivalent to (3.21) and

$$\begin{aligned}(3.23)\qquad &g_1(|g(2,3)|\,|f(1,3)|-|g(1,3)|\,|f(2,3)|)\\ &\quad -g_2(|g(1,3)|\,|f(1,3)|-|g(1,2)|\,|f(2,3)|)=0 \quad \text{at} \quad u.\end{aligned}$$

Combining (3.21), (3.22) and (3.23), we see that

$$Sing(\Delta') \cap \Delta'_1 = \Delta'''.$$

In case $u \in \Delta'_{11}$, we will first prove the assertion for such u that the defining equations of X_u is normalized as (3.7). At such point u, the linear term of (3.20) becomes

$$
\begin{aligned}
(3.24)\qquad & \xi_1(-|f(2,3)|\,|g(1,3)|+|f(1,3)|\,|g(2,3)|)(u)\\
&+\xi_2(|f(1,3)|\,|g(1,3)|-|f(1,2)|\,|g(2,3)|)(u)\\
&+\eta_1(|f(1,3)|\,|g(2,3)|-|f(2,3)|\,|g(1,3)|)(u)\\
&+\eta_2(-|f(1,3)|\,|g(1,3)|+|f(2,3)|\,|g(1,2)|)(u)\\
&+\xi_{111}(3\cdot 3f_{222}|g(2,3)|)(u)\\
&+\xi_{112}(-f_{122}|g(2,3)|-2\cdot 3f_{222}|g(1,3)|)(u)\\
&+\xi_{122}(-f_{112}|g(2,3)|+2\cdot 2f_{122}|g(1,3)|)(u)\\
&+\xi_{222}(3\cdot 3f_{111}|g(2,3)|-3\cdot 2f_{112}|g(1,3)|)(u)\\
&+\eta_{111}(-3\cdot 3g_{222}|f(2,3)|)(u)\\
&+\eta_{112}(g_{122}|f(2,3)|+2\cdot 3g_{222}|f(1,3)|)(u)\\
&+\eta_{122}(g_{112}|f(2,3)|-2\cdot 2g_{122}|f(1,3)|)(u)\\
&+\eta_{222}(-3\cdot 3g_{111}|f(2,3)|+3\cdot 2g_{112}|f(1,3)|)(u).
\end{aligned}
$$

We claim that every coefficient of (3.24) is zero if and only if

$$(3.25)\qquad |f(1,3)|=|f(2,3)|=|g(1,3)|=|g(2,3)|=0 \quad \text{at} \quad u.$$

It is evident that (3.25) is a sufficient condition. Assume now that $|g(2,3)|\neq 0$ at u. This implies that $f_{222}=f_{122}=f_{112}=f_{111}=0$ at u. But if $f_{222}=f_{122}=0$ at u, the points with $x_0=y_1=z_3=0$ become singular points of X_u. Hence $|g(2,3)|=0$ and also $|g(1,3)|=0$ at u. The same reasoning assures that $|f(2,3)|=|f(1,3)|=0$ at u.

By using the fact $(f_1, f_2)\neq(0,0)$ and $(g_1, g_2)\neq(0,0)$ at u once more, we see that (3.25) is equivalent to

$$(3.26)\qquad \operatorname{rank}(f)=\operatorname{rank}(g)=1.$$

As we have shown at the final stage of the proof of Lemma (3.10), the condition (3.26) is invariant under the action of G. Thus the assertion in case $u\in\Delta'_{11}$ is proven.

In case $u\in\Delta'_2$, by using the normalized form (3.8), we can get the condition (3.26) with a similar argument as in the previous case $u\in\Delta'_{11}$. Q.E.D.

REMARK (3.27) We remark here that Δ'' in (3.10) and Δ''' in (3.18) are not empty sets.

Δ'' contain such point u with

$$X_u : \begin{cases} f = z_3^2 + z_4 x_0 y_1 + y_2^3, \\ g = z_4^2 + y_1(y_1^2 - y_2^2) + x_0^6. \end{cases}$$

$\Delta''' \cap \Delta'_{11}$ contain such point u with

$$X_u : \begin{cases} f = z_3^2 + z_4 x_0 (y_1 - y_2) + y_1^3, \\ g = z_4^2 + z_3 x_0 (y_1 - y_2) + y_2^3 + x_0^6. \end{cases}$$

$\Delta''' \cap \Delta'_2$ contain such point u with

$$X_u : \begin{cases} f = z_3^2 + z_4 x_0 y_1 + y_2^3, \\ g = z_4^2 + z_3 x_0 y_2 + y_1^3 + x_0^6. \end{cases}$$

4. Explicit calculation of Ker $d\phi(s)$

Let $u_0 \in U$. Let $\pi: \mathscr{X} \to S$ be the Kuranishi family of the deformations of $X_{u_0} = X_{s_0}$ ($s_0 \in S$) and let $p': U \to S$ be the morphism (in the sense of germs) sending u_0 to s_0 as in the beginning of 2.2. Take $u \in U$ and set $s = p'(u) \in S$. In this section, we will calculate explicitly a lifting $\in T_U(u)$ of the kernel of the differential $d\phi(s)$: $T_S(s) \to T_D(\phi(s))$ of the period map ϕ at s in terms of the coordinates of $u \in U$.

We use the notation in the previous sections.

4.1. Taking the fibers at $u \in U$ of the diagram (2.17), we have a commutative, exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T(u) \arrow[r, "j'(u)"] \arrow[d, "i'_2(u)"] & N(u) \arrow[r, "s'(u)"] \arrow[d, "i'_1(u)"] & H^1(X, T_X) \arrow[r] \arrow[d, "i'(u)"] & 0 \\
0 \arrow[r] & T'(u) \arrow[r, "j'_1(u)"] \arrow[d, "r'_2(u)"] & N'(u) \arrow[r, "s'_1(u)"] \arrow[d, "r'_1(u)"] & P^{1,1}(X) \arrow[r] & 0 \\
 & H^0(C, T_{\mathbf{P}} \otimes K_X \otimes \mathcal{O}_C) \arrow[r, "j_2(u)"] \arrow[d] & H^0(C, N_{X/\mathbf{P}} \otimes K_X \otimes \mathcal{O}_C) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{4.1}$$

where $X=X_u$ and $C=C_u$.

LEMMA (4.2) *By using the notation in* (4.1), *we see that*

$$\operatorname{Ker} j_2(u) \xrightarrow{\sim} \operatorname{Ker} i'(u)=\operatorname{Ker} d\phi(s)$$

via $s'(u)\circ i'_1(u)^{-1}\circ j'_1(u)\circ r'_2(u)^{-1}$.

PROOF. This follows immediately from the diagram (4.1) and (2.16). (cf. also the proof of Lemma (3.1).) Q.E.D.

4.2. Let

$$w'_1,\dots, w'_{12};\ w'_{13},\dots, w'_{32}$$

$$(\text{resp. } v'_1,\dots, v'_{12};\ v'_{13},\dots, v'_{24};\ v'_{25},\dots, v'_{50})$$

be the basis of $T'(u)$ (resp. $N'(u)$) induced from the frame (2.12) (resp. (2.13)). Let

$$w_{13},\dots, w_{32}$$

$$(\text{resp. } v_{13},\dots, v_{24};\ v_{25},\dots, v_{50})$$

be the basis of $T(u)$ (resp. $N(u)$) defined by

$$i'_2(u)(w_e)=w'_e \qquad (13\leqq e\leqq 32)$$

$$(\text{resp. } i'_1(u)(v_e)=v'_e \qquad (13\leqq e\leqq 50)).$$

Then

$$r'_2(u)(w'_e) \qquad (1\leqq e\leqq 12)$$

$$(\text{resp. } r'_1(u)(v'_e) \qquad (1\leqq e\leqq 12))$$

form a basis of $H^0(C, T_{\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C)$ (resp. $H^0(C, N_{X/\mathbf{P}}\otimes K_X\otimes\mathcal{O}_C)$).

By using these bases the morphisms

$$j'(u),\ j'_1(u) \quad \text{and} \quad j'_2(u)$$

are represented by the evaluation of the matrices

J'' in (2.20), J in (2.14) and J_{11} in (2.15) at u respectively.

4.3. Using the affine coordinates (3.19) of U, $T_U(u)$ is identified with a subspace of $N(u)$ in the sense that

$$\frac{\partial}{\partial\xi_i}=(z_4x_0y_i/x_0^6)\frac{\partial}{\partial(f(u)/x_0^6)},$$

$$\frac{\partial}{\partial \eta_i} = (z_3 x_0 y_i / x_0^6) \frac{\partial}{\partial (g(u)/x_0^6)},$$

$$\frac{\partial}{\partial \xi_{ijk}} = (y_i y_j y_k / x_0^6) \frac{\partial}{\partial (f(u)/x_0^6)},$$

$$\frac{\partial}{\partial \eta_{ijk}} = (y_i y_j y_k / x_0^6) \frac{\partial}{\partial (g(u)/x_0^6)}.$$

This means that $T_U(u)$ is considered to be the subspace of $N(u)$ spanned by v_e $(25 \leqq e \leqq 50)$.

4.4. Now we caluculate a lifting of $\mathrm{Ker}\, d\phi(s)$ to $T_U(u)$ by tracing each step of the morphism in (4.2). We use the notation above.

LEMMA (4.3) *A lifting of every vector in* $\mathrm{Ker}\, d\phi(s)$ *to* $T_U(u)$ *is obtained by*

$$(4.4) \qquad (v_{25}, \ldots, v_{50}) B \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix},$$

where a_e $(1 \leqq e \leqq 6)$ *is a solution of the linear equation*

$$(4.5) \qquad A \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

(A is the matrix in (2.18)) *and B is the* 26×6 *matrix* (4.6) *in the next pages.*

The converse is also true, i.e. every vector in $T_U(u)$ *of the form* (4.4) *is a lifting of some vector in* $\mathrm{Ker}\, d\phi(s)$.

PROOF. Take $\gamma = \sum_{1 \leqq i \leqq 12} a_i r_2'(w_i') \in \mathrm{Ker}\, j_2(u)$. By the remark in 4.2, this is equivalent to finding coefficients $a_e (1 \leqq e \leqq 12)$ satisfying

$$J_{11} \begin{pmatrix} a_1 \\ \vdots \\ a_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

By the form of the matrix J_{11}, this is also equivalent to finding $a_e (1 \leqq e \leqq 6)$ with

$$A \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

since a_e $(7 \leqq e \leqq 12)$ are given by

$$(4.7) \qquad \begin{pmatrix} a_7 \\ \vdots \\ a_{12} \end{pmatrix} = -\frac{1}{2} A' \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix},$$

(4.6)

$$B=\begin{array}{|cc|c|c|}
\hline
3f_0 & 0 & 2f_{011} & f_{012} \\
0 & 3f_0 & f_{012} & 2f_{022} \\
0 & 0 & f_{001} & f_{002} \\
\hline
\multicolumn{2}{|c|}{(*)} & \multicolumn{2}{c|}{0} \\
\hline
2f_{011} & 0 & -\frac{5}{2}f_1g_{111} & -\frac{1}{2}f_1g_{112}-f_2g_{111} \\
2f_{012} & 2f_{011} & -2f_1g_{112}-\frac{3}{2}f_2g_{111} & -f_1g_{122}-\frac{3}{2}f_2g_{112} \\
2f_{022} & 2f_{012} & -\frac{3}{2}f_1g_{122}-f_2g_{112} & -\frac{3}{2}f_1g_{222}-2f_2g_{122} \\
0 & 2f_{022} & -f_1g_{222}-\frac{1}{2}f_2g_{122} & -\frac{5}{2}f_2g_{222} \\
4f_{001} & 0 & -\frac{3}{2}f_0g_{111}-2f_1g_{011} & -\frac{1}{2}f_0g_{112}-\frac{1}{2}f_1g_{012}-f_2g_{011} \\
4f_{002} & 4f_{001} & -f_0g_{112}-\frac{3}{2}f_1g_{012}-f_2g_{011} & -f_0g_{122}-f_1g_{022}-\frac{3}{2}f_2g_{012} \\
0 & 4f_{002} & -\frac{1}{2}f_0g_{122}-f_1g_{022}-\frac{1}{2}f_2g_{012} & -\frac{3}{2}f_0g_{222}-2f_2g_{022} \\
6f_{000} & 0 & -f_0g_{011}-\frac{3}{2}f_1g_{001} & -\frac{1}{2}f_0g_{012}-\frac{1}{2}f_1g_{002}-f_2g_{001} \\
0 & 6f_{000} & -\frac{1}{2}f_0g_{012}-f_1g_{002}-\frac{1}{2}f_2g_{001} & -f_0g_{022}-\frac{3}{2}f_2g_{002} \\
0 & 0 & -\frac{1}{2}f_0g_{001}-f_1g_{000} & -\frac{1}{2}f_0g_{002}-f_2g_{000} \\
\hline
\multicolumn{2}{|c|}{(**)} & \multicolumn{2}{c|}{\text{obtained from (3, 3)-block by replacing } f \text{ and } g \text{ respectively}} \\
\hline
\end{array}$$

(*) (resp. (**)) is obtained from (1, 1)-block (resp. (1, 3)-block) by repracing f and g

0	
obtained from (1,2)-block by repracing f and g respectively	
0	0
$-\frac{3}{4}f_1^2g_1$	$-\frac{1}{4}f_1f_2g_1-\frac{1}{2}f_1^2g_2$
$-\frac{5}{4}f_1f_2g_1-\frac{1}{4}f_1^2g_2$	$-\frac{1}{4}f_2^2g_1 \quad -\frac{5}{4}f_1f_2g_2$
$-\frac{1}{2}f_2^2g_1 \quad -\frac{1}{4}f_1f_2g_2$	$-\frac{3}{4}f_2^2g_2$
$-\frac{7}{4}f_0f_1g_1 \qquad -\frac{1}{4}f_1^2g_0$	$-\frac{1}{4}f_0f_2g_1-\frac{3}{2}f_0f_1g_2-\frac{1}{4}f_1f_2g_0$
$-\frac{1}{4}f_0f_1g_2-\frac{3}{2}f_0f_2g_1-\frac{1}{4}f_1f_2g_0$	$-\frac{7}{4}f_0f_2g_2 \qquad -\frac{1}{4}f_2^2g_0$
$-\frac{1}{4}f_0f_1g_0-\frac{1}{2}f_0^2g_1$	$-\frac{1}{4}f_0f_2g_0-\frac{1}{2}f_0^2g_2$
obtained from (3, 2)-block by replacing f and g respectively	

respectively.

where A' is the 6×6 matrix in (2.18).

Set

$$\gamma_1=\sum_{1\leqq i\leqq 12} a_i w_i' \in r_2'(u)^{-1}(\gamma)\subset T'(u) \quad \text{and}$$

$$\gamma_2=j_1'(u)(\gamma_1)\in N'(u).$$

Then, by the remark in 4.2 again, we see that

$$\gamma_2=(v_1',\cdots,v_{50}')J\begin{pmatrix} a_1\\ \vdots\\ a_{12}\\ 0\\ \vdots\\ 0\end{pmatrix}=(v_{13}',\dots,v_{50}')\begin{pmatrix} J_{21}\\ \\ J_{31}\end{pmatrix}\begin{pmatrix} a_1\\ \vdots\\ a_{12}\end{pmatrix}.$$

Set

$$\gamma_3=(v_{13},\dots,v_{50})\begin{pmatrix} J_{21}\\ \\ J_{31}\end{pmatrix}\begin{pmatrix} a_1\\ \vdots\\ a_{12}\end{pmatrix}.$$

By the form of the matrix J_{22}, we can find a vector

$$\gamma'=(v_{13},\dots,v_{50})J''\begin{pmatrix} a_{13}\\ \vdots\\ a_{32}\end{pmatrix}\in \operatorname{Im} j'(u)$$

so that

$$a_{13}=a_{19}=a_{23}=a_{24}=a_{25}=a_{26}=a_{27}=a_{28}=a_{29}=a_{32}=0 \tag{4.8}$$

and

$$\gamma_3+\gamma'=(v_{13},\dots,v_{50})\begin{pmatrix} J_{21} & J_{22}\\ \\ J_{31} & J_{32}\end{pmatrix}\begin{pmatrix} a_1\\ \vdots\\ a_{12}\\ a_{13}\\ \vdots\\ a_{32}\end{pmatrix}=(v_{25},\dots,v_{50})(J_{31}J_{32})\begin{pmatrix} a_1\\ \vdots\\ a_{32}\end{pmatrix} \tag{4.9}$$

i.e. $\gamma_3+\gamma'\in T_U(u)$.

Set $\gamma_4=\gamma_3+\gamma'$. Then, by the construction, γ_4 is a lifting to $T_U(u)$ of the vector in $\operatorname{Ker} d\phi(s)$ corresponding to $\gamma\in\operatorname{Ker} j_2(u)$ under the isomorphism in (4.2).

From (4.7), (4.8) and (4.9), a_e $(7\leqq e\leqq 32)$ can be expressed as linear combinations of a_e $(1\leqq e\leqq 6)$. Thus we can calculate the matrix B so that γ_4 is the form (4.4). Q.E.D.

The following lemma can be obtained elementarily and we omit the proof.

LEMMA (4.10) Using the normalized forms (3.5), (3.6), (3.7) and (3.8) of the defining equations of X_u with $u \in \Delta'_i$ ($i=0, 1, 11, 2$), we get the folling solutions of the linear equation (4.5):

		a_1	a_2	a_3	a_4	a_5	a_6
Δ_0		t_1	t_2	0	0	0	0
Δ_1		$\|f(1,3)\|$	$\|f(2,3)\|$	$-3f_{222}$	f_{122}	0	0
Δ_{11}	not $\|f(1,3)\| = \|g(1,3)\| = 0$	$\|f(1,3)\| \times \|g(1,3)\|$	$\|f(1,3)\| \times \|g(2,3)\|$	$-3f_{222} \times \|g(1,3)\|$	$f_{122} \times \|g(1,3)\|$	$-\|f(1,3)\| \times 3g_{222}$	$\|f(1,3)\| \times g_{122}$
	not $\|f(2,3)\| = \|g(2,3)\| = 0$	$\|f(2,3)\| \times \|g(1,3)$	$\|f(2,3)\| \times \|g(2,3)\|$	$-3f_{222} \times \|g(2,3)\|$	$f_{122} \times \|g(2,3)\|$	$-\|f(2,3)\| \times 3g_{222}$	$\|f(2,3)\| \times g_{122}$
	rank $(f)=$ rank $(g)=1$	0	0	$-3f_{222}t_1$	$f_{122}t_1$	$-3g_{222}t_2$	$g_{122}t_2$
Δ_2	not $\|f(1,3)\| = \|g(1,2)\| = 0$	$\|f(1,3)\| \times \|g(1,2)\|$	$\|f(2,3)\| \times \|g(1,2)\|$	$-3f_{222} \times \|g(1,2)\|$	$f_{122} \times \|g(1,2)\|$	$\|f(1,3)\| \times g_{112}$	$-\|f(1,3)\| \times 3g_{111}$
	not $\|f(2,3)\| = \|g(1,3)\| = 0$	$\|f(1,3)\| \times \|g(1,3)\|$	$\|f(2,3)\| \times \|g(1,3)\|$	$-3f_{222} \times \|g(1,3)\|$	$f_{122} \times \|g(1,3)\|$	$\|f(2,3)\| \times g_{112}$	$-\|f(2,3)\| \times 3g_{111}$
	rank $(f)=$ rank $(g)=1$	0	0	$-3f_{222}t_1$	$f_{122}t_1$	$g_{112}t_2$	$-3g_{111}t_2$

where t_1 and t_2 are independent parameters, and we have relations $|f(1, 3)|\,|g(2, 3)|-|f(2, 3)|\,|g(1, 3)|=0$ in case $u\in\Delta_{11}$ and $|f(1, 3)|\,|g(1, 3)|-|f(2, 3)|\,|g(1, 2)|=0$ in case $u\in\Delta_2$.

5. Fibres of the period map

Let $u_0\in U$, let $\pi: \mathscr{X}\to S$ be the Kuranishi family of the deformations of $X_{u_0}\simeq X_{s_0}$ ($s_0\in S$) and let $p': U\to S$ be the morphism (in the sense of germs) sending u_0 to s_0. For $u\in U$, we set $s=p'(u)\in S$.

We will investigate, in this section, the fibres of the period map $\phi: S\to D$ in (2.1).

We use the notation in the previous sections.

THEOREM (5.1) *With the above notation, we get the following assertions for the fibres of the period map ϕ.*

(5.1.1) *In case $u_0\in\Delta'_0$, using the normalized form* (3.5), *we have*:

$f_0=g_0=0$ *at u_0 if and only if the period map ϕ has 2-dimensional fibre through s_0.*

If $f_0=0$ and $g_0\neq 0$ at u_0, the period map ϕ has 1-dimensional fibre through s_0.

If $f_0\neq 0$ and $g_0=0$ at u_0, the period map ϕ has 1-dimensional fibre through s_0.

If $f_0g_0\neq 0$ at u_0 and if the period map ϕ has positive dimensional fibre through s_0, it must be 1-diemnsional and the point $s\in S$, which starts from s_0 and moves along this fibre, will go into Δ_{11}.

(5.1.2) *In case $u_0\in\Delta'_1$, using the normalized form* (3.6), *we have*:

If $g_0=0$ at u_0, the period map ϕ has 1-dimensional fibre through s_0.

If $g_0\neq 0$ at u_0 and if the period map ϕ has positive dimensional fibre through s_0, the point $s\in S$, which starts from s_0 and moves along this fibre, will go into Δ_2.

(5.1.3) *In case $u_0\in\Delta'_{11}$, using the normalized form* (3.7), *we get the following*:

If $f_{112}=f_{122}=f_{012}=f_{022}=f_{002}=g_{112}=g_{122}=g_{012}=g_{022}=g_{002}=0$ at u_0, the period map ϕ has 1-dimensional fibre through s_0.

If rank $(f)=$ rank $(g)=1$ *at u_0, i.e. $u_0\in\Delta''$ (see* (3.10)), *and if the period map ϕ has positive dimensional fibre through s_0, it must be 1-dimensional and the point $s\in S$, which starts from s_0 and moves along this fibre, will leave from $p'(\Delta'')$.*

(5.1.4) *In case $u_0\in\Delta'_2$, using the normalized form* (3.8), *we have the following*:

If rank $(f)=$ rank $(g)=1$ *at u_0, i.e. $u_0\in\Delta''$, and if the period map ϕ has* *positive dimensional fibre through s_0, the point $s \in S$, which stars from s_0 and moves along this fibre, will leave from $p'(\Delta'')$.*

PROOF. Set

$$Z_1' = \{u \in U \mid g_1 = g_2 = g_0 = 0 \quad \text{at} \quad u\},$$

$$Z_2' = \{u \in U \mid f_1 = f_2 = f_0 = g_1 = g_2 = g_0 = 0 \quad \text{at} \quad u\} \quad \text{and}$$

$$Z_3' = \left\{ u \in U \;\middle|\; \begin{array}{l} f_2 = f_{112} = f_{122} = f_{012} = f_{022} = f_{002} = g_2 \\ = g_{112} = g_{122} = g_{012} = g_{022} = g_{002} = 0 \text{ at } u \end{array} \right\},$$

and set also $Z_i = p'(Z_i')$ $(i = 1, 2, 3)$. Then Z_i' $(i = 1, 2, 3)$ are "linear" subvarieties of U and Z_i $(i = 1, 2, 3)$ are analytic submanifolds of S (see, for detail, [7]).

It is easy to see that $Z_3' \cap \Delta'' = \phi$, where we use the notation Δ'' in (3.10). Hence we see from (3.10), that

$$\text{(5.2)} \qquad \begin{array}{lll} \dim \operatorname{Ker} d\phi(s) = 2 & \text{for} \quad s \in Z_2 \quad \text{and} & \\ \dim \operatorname{Ker} d\phi(s) = 1 & \text{for} \quad s \in Z_i \setminus Z_2 & (i = 1, 3). \end{array}$$

In case $u \in Z_1'$, we know, from (4.3) and (4.10), that the coefficients of the liftings $\in T_U(u)$ of $\operatorname{Ker} d\phi(s)$ corresponding to the part $v_{28} = \frac{\partial}{\partial \eta_1}$, $v_{29} = \frac{\partial}{\partial \eta_2}$ and $v_{30} = \frac{\partial}{\partial \eta_0}$ of the basis v_e $(25 \leqq e \leqq 50)$ of $T_U(u)$ are all zero. This implies that

$$\text{(5.3)} \qquad \operatorname{Ker} d\phi(s) \subset T_{Z_1}(s) \qquad \text{at any point} \qquad s \in Z_1$$

Combining (5.2) and (5.3), we can conclude that, in case $u_0 \in Z_1' \setminus Z_2'$, the period map ϕ has smooth, 1-dimensional fibre through s_0.

In a similar manner, we get also that the period map ϕ has smooth, 2-dimensional (resp. 1-dimensional) fibre through s_0 in case $u_0 \in Z_2'$ (resp. $u_0 \in Z_3' \setminus Z_2'$).

For the other assertion, we will prove them case by case.

First of all, we identify $T_U(u_0)$ with the ambiant space $\mathbf{A}^{26}$ of U by means of the affine coordinates ξ_i, η_i, ξ_{ijk} and η_{ijk} in (3.19).

Case $f_1 = f_2 = g_1 = g_2 = 0$ and $f_0 g_0 \neq 0$ at u_0: From (4.3) and (4.10), we see that the coefficients of the liftings $T_U(u_0)$ of $\operatorname{Ker} d\phi(s_0)$ corresponding to the part $v_{25} = \frac{\partial}{\partial \xi_1}$, $v_{26} = \frac{\partial}{\partial \xi_2}$, $v_{27} = \frac{\partial}{\partial \xi_0}$, $v_{28} = \frac{\partial}{\partial \eta_1}$, $v_{29} = \frac{\partial}{\partial \eta_2}$ and $v_{30} = \frac{\partial}{\partial \eta_0}$ of the basis v_e $(25 \leqq e \leqq 50)$ of $T_U(u_0)$ are as follows.

$$\text{(5.4)} \qquad \begin{array}{l} \xi_1 = 3 f_0 t_1, \quad \xi_2 = 3 f_0 t_2, \quad \xi_0 = 0, \\ \eta_1 = 3 g_0 t_1, \quad \eta_2 = 3 g_0 t_2, \quad \eta_0 = 0, \end{array}$$

where t_i $(i=1, 2)$ are independent parameters.

If the period map ϕ has positive dimensional fibre through s_0, starting from $s=s_0$ and moving along this fibre, the point s, by (5.4), will go into $\Delta_{11}\setminus p'(\Delta'')$. In particular, the fibre of the period map ϕ through s_0 is at most 1-dimensional.

In a similar way, we get the last half of (5.1.2).

Case $u_0 \in \Delta'_{11} \cap \Delta''$: We use the normalized form (3.7). It is easy to see that $f_{222}g_{222} \neq 0$ at u_0, unless otherwise X_{u_0} has singular points. Hence the condition

$$\operatorname{rank}(f)=\operatorname{rank}(g)=1$$

is equivalent to

$$|f(1, 3)|=|f(2, 3)|=|g(1, 3)|=|g(2, 3)|=0$$

and also equivalent to

$$(5.5)\qquad \begin{aligned} &f_{111}=a^3 f_{222}, \quad f_{112}=3a^2 f_{222}, \quad f_{122}=3a f_{222}, \\ &g_{111}=b^3 g_{222}, \quad g_{112}=3b^2 g_{222}, \quad g_{122}=3b g_{222}, \end{aligned}$$

for some $a, b \in \mathbf{C}$. Note that (5.5) means

$$\begin{aligned} &f^{(3)}(0, y_1, y_2)=f_{222}(ay_1+y_2)^3 \quad \text{and} \\ &g^{(3)}(0, y_1, y_2)=g_{222}(by_1+y_2)^3, \end{aligned}$$

and hence we see that $a \neq b$, unless otherwise X_{u_0} is not contained in $\mathbf{P}(1, 2, 2, 3, 3)$.

From (4.3), (4.10) and (5.5), we see that the coefficients of the liftings $T_U(u_0)$ of $\operatorname{Ker} d\phi(s_0)$ corresponding to the part

$$v_{32}=\frac{\partial}{\partial\xi_{112}}, \quad v_{33}=\frac{\partial}{\partial\xi_{122}}, \quad v_{34}=\frac{\partial}{\partial\xi_{222}}, \quad v_{42}=\frac{\partial}{\partial\eta_{112}}, \quad v_{43}=\frac{\partial}{\partial\eta_{122}}$$

and $v_{44}=\dfrac{\partial}{\partial\eta_{222}}$ of the basis v_e $(25 \leqq e \leqq 50)$ of $T_U(u_0)$ are as follows.

$$(5.6)\qquad \begin{aligned} &\xi_{112}=3^2 f_{222}g_{222}t_1(2b^2-ab), \quad \xi_{122}=\frac{3^2}{2} f_{222}g_{222}t_1(3b-a), \\ &\xi_{222}=3f_{222}g_{222}t_1, \quad \eta_{112}=3^2 f_{222}g_{222}t_2(2a^2-ab), \\ &\eta_{122}=\frac{3^2}{2} f_{222}g_{222}t_2(3a-b), \quad \eta_{222}=3f_{222}g_{222}t_2, \end{aligned}$$

where t_i $(i=1, 2)$ are independent parameters.

The equations of the tangent spaces $|f(2, 3)|=0$ and $|g(2, 3)|=0$ in $\mathbf{A}^{26}$ are

$$(5.7)\qquad \begin{vmatrix} 2\xi_{112} & 2f_{122} \\ \xi_{122} & 3f_{222} \end{vmatrix} + \begin{vmatrix} 2f_{112} & 2\xi_{122} \\ f_{122} & 3\xi_{222} \end{vmatrix} \quad \text{and}$$

$$\begin{vmatrix} 2\eta_{112} & 2g_{122} \\ \eta_{122} & 3g_{222} \end{vmatrix} + \begin{vmatrix} 2g_{112} & 2\eta_{122} \\ g_{112} & 3\eta_{222} \end{vmatrix} \quad \text{respectively.}$$

Substituting (5.5) and (5.6) into (5.7), we get

$$(5.8)\qquad 2^2 \cdot 3^3 f_{222}^2 g_{222}(a-b)^2 t_1 \quad \text{and}$$

$$2^2 \cdot 3^3 f_{222} g_{222}^2 (a-b)^2 t_2 \quad \text{respectively.}$$

If the period map ϕ has positive dimensional fibre through s_0, we see from (5.8) that, starting from $s=s_0$ and moving along this fibre, the point s will leave from $p'(\Delta'')$. Thus, we get the desired assertion in this case.

In a similar way, we get (5.1.4). Q.E.D.

REMARK (5.9) *In the cases* (5.1.3) *and* (5.1.4), *our results in Theorem* (5.1) *are imperfect.*

In these cases, we know a rough idea of the algorithm to determine what kind of fibre the period map ϕ *has through a given point* $s_0 \in S$:

1) *Given a point* $u_0 \in U$. *Set*

$$h_1 = \textit{the pull-back of the Jacobian } \det A \textit{ of } d\phi$$
$$\textit{by the morphism } p' \colon U \to S.$$

In case $h_1(u_0) \neq 0$, *we stop the process at this stage.*

2) *In case* $h_1(u_0)=0$: *For* $u \in U$ *with* $h_1(u)=0$, *we can calculate explicitly, by* (4.3) *and* (4.10), *the liftings* $\tau(u) \in T_U(u)$ *of* $\operatorname{Ker} d\phi(s)$. *Calculate also the equation* $h_1'(u)$ *of the tangent cone to* $\{u \in U \mid h_1(u)=0\}$ *at* u *and set*

$$h_2(u) = h_1'(\tau(u)).$$

In case $h_2(u_0) \neq 0$, *we stop the process at this stage.*

3) *In case* $h_2(u_0)=0$: *Calculate the equation* $h_2'(u)$ *of the tangent cone to* $\{u \in U \mid h_2(u)=0\}$ *at* u *and set*

$$h_3(u) = h_2'(\tau(u)).$$

In case $h_3(u_0) \neq 0$, *we stop the process at this stage. In case* $h_3(u_0)=0$, *continue the process and get the equations* $h_4, h_5, \dots$.

4) *We can conclude the period map* ϕ *has positive dimensional fibre through* s_0 *if and only if*

$$\dim_{u_0} \{u \in U \mid h_i(u)=0 \quad (i=1, 2, \dots)\} > 0.$$

Unfortunately, the above algorithm is too complicated to perform at present.

6. Some properties on $S_2=\{s\in S|\ \dim \operatorname{Ker} d\phi(u)=2\}$

We will investigate, in this section, some properties on S_2 in (3.3) and their relations. We use the notation in the previous sections.

THEOREM (6.1) *Let $u_0\in U$ and set $X=X_{u_0}$ Consider the following properties*:

(a) *The bi-canonical map $\Phi_{|2K|}: X\to\mathbf{P}^2$ is a Galois covering with the group $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$.*

(a′) $f^{(1)}=g^{(1)}=0$ at u_0.

(a″) *The period map ϕ has 2-dimensional fibre through $s_0=p'(u_0)$.*

(b) *The canonical divisor C of X is smooth and the exact sequence*

$$0\longrightarrow \check{N}_{C/X}\longrightarrow \Omega_X^1\otimes\mathcal{O}_C\longrightarrow\Omega_C^1\longrightarrow 0$$

splits.

(b′) $f_1=f_2=g_1=g_2=0$ at u_0.

(c) $\dim\operatorname{Ker} d\phi(s_0)=2$.

(c′) $f_1=f_2=g_1=g_2=0$ *or* $\operatorname{rank}(f)=\operatorname{rank}(g)=1$ at u_0.

Then, among the above properties, we have the following equivalences and implications:

$$(a)\Longleftrightarrow(a')\Longleftrightarrow(a'')\Longrightarrow(b)\Longleftrightarrow(b')\Longrightarrow(c)\Longleftrightarrow(c'').$$

We kow also that the subvarieties of S satisfying the property (a) (*resp.* (b), (c)) *is of codimension* 6 (*resp.* 4, 4) *in S and irreducible* (*resp. irreducible, of disjoint union of two irreduclble components*).

PROOF. The implication (a′)⇒(a) is immediatly verified by observing that the involusions $z_3\mapsto -z_3$ and $z_4\mapsto -z_4$ form a system of generators of the covering transformation group of $\Phi_{|2K|}: X\to\mathbf{P}^2$. The converse is an exercise in the Galois theory (for detail, see [2]).

The equivalence between (a′) and (a″) is the first part in (5.1.1) and we have already proven.

The implication (a)⇒(b) follows from the observation of the induced action of the involution $x_0\mapsto -x_0$ (for detail, see [2]).

The implication (b)⇒(c) is easy (see [4] and also [2]).

The equivalence between (c) and (c′) has already stated in Lemma (3.10).

We have known that the subvariety of S satisfying the property (b) is of codimension $\leqq 4$ in S (see [6]). It is easy to see that the subvariety of S satisfying the property (c′) is of codimension 4 in S. Note also that the canonical divisor of X_{u_0} has singularity if rank $(f)=$ rank $(g)=1$ at u_0. Combining the above three facts and the implication (b)$\Rightarrow$(c′), which has been already proven, we get the equivalence between (b) and (b′).

The last assertion is an immediate consequence of the characterizations (a′), (b′) and (c′) of (a), (b) and (c) respectively. Q. E. D.

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