31. *Degeneration of Kunev Surfaces. I*

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0. The purpose of this note is to outline our recent results on degeneration of Kunev surfaces. Details will be published elsewhere.

A Kunev surface is, by definition (see 1 below), a double cover of a K3 surface. We report descriptions of degenerations of Kunev surfaces over some fixed K3 surfaces (Theorems 1 and 2). These theorems have an interesting application: We can explain in a uniform way the failure of the Torelli theorem for Kunev surfaces and elliptic surfaces with $p_g=1$ and $q=0, 1$ (Corollary 3). We use the terminology a *homotopic K3 surface* and an *elliptic surface* as ones with $\kappa=1$.

1. A *Kunev surface* is defined as a minimal surface X of general type with $p_g=c_1^2=1$ which has an involution σ such that $Y':=X/\sigma$ is a K3 surface with rational double points (R.D.P. for short) and the bicanonical map of X is a Galois cover of $\boldsymbol{P}^2$ factoring through Y'. Let X be a Kunev surface with ample K_X. Then it is known that the branch locus $B\subset \boldsymbol{P}^2$ of the bicanonical map consists of two smooth cubics C_j $(j=1, 2)$ and of a line L such that $B=\sum C_j+L$ has only nodes as singularities (see [1], [6]), and X is reconstructed as follows: (i) Take the double cover Y' of $\boldsymbol{P}^2$ branched over $\sum C_j$. (ii) Take the minimal resolution Y of Y'. (iii) Take the double cover $\tilde{X}$ of Y branched over $L+\sum E_j$, where E_j $(1\leq i\leq 9)$ are (-2)-curves appeared in (ii). (iv) Contracting (-1)-curves on $\tilde{X}$ induced from E_i, we recover the Kunev surface X.

2. Horikawa and Shah constructed a completion of the moduli space of K3 surfaces of degree 2 as a completion of {sextics in $\boldsymbol{P}^2$} by geometric invariant theory ([3], [5]), which contains our K3 surfaces Y appeared in 1. The latter form 10-dimensional submoduli $\mathfrak{N}$ over which sits "a completion" of the moduli space $\mathfrak{M}$ of Kunev surface . The first theorem is concerned with a completion of the fiber over a general point in $\mathfrak{N}$. Let C_1 and C_2 be general cubics in $\boldsymbol{P}^2$. Denote by $\check{C}_j\subset\check{\boldsymbol{P}}^2$ the dual curve of $C_j\subset\boldsymbol{P}^2$, i.e., the image of the Gauss map. Then each $\check{C}_j$ has nine cusps corresponding to nine inflexes on C_j, $\sum\check{C}_j$ has nine bitangents $\check{D}_i$ with tangent points P_{i1} and P_{i2} $(1\leq i\leq 9)$ subjected to nine nodes of $\sum C_j$, and we have two stratifications of $\check{\boldsymbol{P}}^2$ determined by $\sum\check{C}_j$ and $\sum\check{D}_i$:

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$$\check{P}^2=(\check{P}^2-\sum \check{C}_j)\cup(\sum \check{C}_j-(\sum \check{P}_{ji}+\mathrm{Sing}(\sum \check{C}_j))\cup(\sum \mathrm{Sing}(\check{C}_j))$$
$$\cup(\cap \check{C}_j)\cup(\sum \check{P}_{ji})$$
$$=: R_0\cup R_1\cup R_1'\cup R_2\cup R_0'.$$
$$\check{P}^2=(\check{P}^2-\sum \check{D}_i)\cup(\sum \check{D}_i-\mathrm{Sing}(\sum \check{D}_i))\cup \mathrm{Sing}(\sum \check{D}_i)$$
$$=: S_0\cup S_1\cup S_2.$$

Theorem 1. *With the above notation, there exists a complete family $f:\mathcal{X}\to\check{P}^2$ of degenerations of Kunev surfaces over the fixed general point $[\sum C_j]\in\mathfrak{N}$. This family has the following properties:*

(1.1) *The singularity of the total space $\mathcal{X}$ consists of mutually disjoint compounds Veronese cone over $\check{D}_i$ ($1\leq i\leq 9$), i.e., analytically isomorphic to the product of $\check{D}_i$ and the cone over the Veronese embedding of $P^2\subset P^5$ by $|\mathcal{O}_{P^2}(2)|$. Hence a single blowing-up along the singular loci yields a resolution $\tilde{f}:\tilde{\mathcal{X}}\to\check{P}^2$. For each i ($1\leq i\leq 9$), the exceptional divisor $\mathcal{W}_i$ is a family of P^2 over $\check{D}_i$. The universal family $\{L_t \mid t\in\check{P}^2\}$ of lines on P^2 induces an irreducible divisor on $\tilde{\mathcal{X}}$. We denote by $\mathcal{L}$ the divisor endowed with reduced structure. Then $K_{\tilde{\mathcal{X}}}=\mathcal{L}+\sum \mathcal{W}_i$.*

(1.2) *Besides the singularity of the total space $\mathcal{X}$, the fiber X_t has R.D.P. raised from the tangent points of L_t and $\sum C_j$ on P^2. These singularities form two disjoint compounds A_1 over $R_1\cup R_2$, which degenerate to A_2 over R_1'. Over R_0', clash an A_1 and a Veronese cone singularity. The effect is explained in (1.6) below.*

(1.3) *The fiber $\tilde{X}_t=V_t+\sum W_{i,t}$, where V_t is the main component and the summation runs over the indices i for which $t\in\check{D}_i$. Hence the canonical curve K_t of V_t coincides with $\mathcal{L}\mid V_t$.*

(1.4) *V_t is a (singular) Kunev surface, homotopic K3 surface, or K3 surface according to $t\in S_0$, S_1, or S_2.*

(1.5) *K_t is irreducible reduced and passes* Sing(V_t), *if exists, and its geometric genus is $2-(m+n)$ for $t\in(R_m\cup R_m')\cap S_n$.*

(1.6) *In case $t\in S_1-R_0'$, $V_t\cap W_{i,t}$ is a smooth conic on $W_{i,t}\simeq P^2$ and a rational curve with selfintersection -4 on V_t, where $t\in\check{D}_i$. Whereas, in case $t\in R_0'$, $V_t\cap W_{i,t}$ decomposes into two distinct lines on $W_{i,t}$ and two rational curves with selfintersection -3 on V_t.*

Remark. (1) Since the isotropy group Isot$[\sum C_j]$ of $[\sum C_j]$ in PGL_2 is trivial, $\check{P}^2$ is actually a completion of the fiber of $\mathfrak{M}\to\mathfrak{N}$ over $[\sum C_j]$. (2) We can compute easily the following numbers: $\sharp(R_1)=9\cdot 2=18$, $\sharp(R_2)=6^2=36$, $\sharp(R_0')=9\cdot 2=18$, $\sharp(R_1\cap S_1)=4\cdot 9=36$, $\sharp(S_2)=9\cdot 8/2=36$. (3) We can describe easily a semi-stable reduction of a family induced over a disc.

3. Among the special cases with finite local monodromy in the pure second cohomology, we report here one of the most interesting cases. Let C_1 (resp. C_2) consists of three distinct lines $\sum M_k$ (resp. $\sum N_l$) passing through a common point T_1 (resp. T_2) such that $C_1\cap C_2$ are nine nodes D_i ($1\leq i\leq 9$). Denote by $\check{M}_k$ and $\check{N}_l$ (resp. $\check{T}_j$ and $\check{D}_i$) the dual points (resp. lines) on $\check{P}^2$. Then the three points $\check{M}_k$ (resp. $\check{N}_l$) are on the line $\check{T}_1$ (resp. $\check{T}_2$), $\check{D}_i$ are the lines joining the points $\check{M}_k$ and $\check{N}_l$, and these determine a stratification on $\check{\boldsymbol{P}}^2$;

$$\begin{aligned}\check{\boldsymbol{P}}^2 &= (\check{\boldsymbol{P}}^2-(\sum \check{D}_i+\sum \check{T}_j))\cup(\sum \check{D}_i-\mathrm{Sing}(\sum \check{D}_i))\\ &\quad\cup(\mathrm{Sing}(\sum \check{D}_i)-(\sum \check{M}_k+\sum \check{N}_l))\\ &\quad\cup(\sum \check{T}_j-(\check{T}_1\cap \check{T}_2+\sum \check{M}_k+\sum \check{N}_l))\cup(\check{T}_1\cap\check{T}_2)\cup(\sum \check{M}_k+\sum \check{N}_l)\\ &=: S_0\cup S_1\cup S_2\cup S_1'\cup S_2'\cup S_2''.\end{aligned}$$

Theorem 2. *With the above notation, there exists a complete family $f:\mathfrak{X}\to\check{\boldsymbol{P}}^2$ of degenerations of Kunev surfaces over the fixed $[\sum C_j]\in\mathfrak{N}$ as above. This family has the following properties:*

(2.1) *Let $\tilde{\mathfrak{X}}$ be the blowing-up of $\mathfrak{X}$ along nine disjoint compounds Veronese cone over $\check{D}_i$ $(1\leq i\leq 9)$ raised from $C_1\cap C_2$. Then the same statement as* (1.1) *holds, provided that $\tilde{\mathfrak{X}}$ still has singularity described in* (2.2) *below.*

(2.2) *Rising from two triple points T_1 and T_2, $\mathfrak{X}$ has four compounds R.D.P. of type D_4 over $\check{\boldsymbol{P}}^2-\sum \check{T}_j=S_0\cup S_1\cup S_2$, each two of which clash to make up a compound elliptic singularity on $f^{-1}(\check{T}_j-S_2'')$ $(j=1,2)$ with a local equation*

$$z^2+y(x^4+y^2)=0.$$

In case $t\in S_2''$, say $t=\check{M}_1$, besides the two D_4 raised from T_2, the main component V_t of the fiber $\tilde{X}_t$ has the following singularity: We abuse the notation T_1 for the point on V_t induced from $T_1\in\boldsymbol{P}^2$. V_t has ordinary double points along $\mathcal{L}|V_t-T_1$ and a local equation at $T_1\in V_t$ is

$$z^2+y^2(x^2+y^4)=0.$$

Hence T_1 becomes an R.D.P. of type A_3 on the normalization of V_t.

(2.3) *The same statement as* (1.3) *holds.*

(2.4) *Analogously as* (1.4), *V_t is a singular Kunev surface, homotopic K3 surface, or K3 surface according to $t\in S_0, S_1$, or S_2. Whereas V_t becomes a singular elliptic surface with $p_g=q=1$, abelian surface, or K3 surface according to $t\in S_1', S_2'$, or S_2''.*

(2.5) *The canonical curve K_t on V_t is divided into two disjoint (-1)-curves in the case that $t\in S_2$ and that t is a triple point of $\sum \check{D}_i$. K_t becomes a double rational curve in case $t\in S_2''$. In other cases, K_t is irreducible reduced, and its geometric genus is $2-n$ for $t\in S_n\cup S_n'$. K_t passes the elliptic singular point or its degenerating point in case $t\in\sum \check{T}_j=S_1'\cup S_2'\cup S_2''$.*

(2.6) *An analogous statement as* (1.6) *holds according to $t\in S_1-S_2''$, or S_2''. In case $t\in S_2''$, $V_t\cap W_{i,t}$ on V_t consists of two rational curves which cut the double curve $\mathcal{L}|V_t$ transversely at a common point.*

Remark. We can give parallel remarks as those just after Theorem 1. We omit all but the version of (1).

(1′) $\mathrm{Isot}[\sum C_j]$ is a finite group and $\check{\boldsymbol{P}}^2/\mathrm{Isot}[\sum C_j]$ is a completion of the fiber of $\mathfrak{M}\to\mathfrak{N}$ over $[\sum C_j]$.

The proofs of Theorems 1 and 2 go on the same way as the construction of smooth Kunev surfaces with ample K over $\boldsymbol{P}^2$ explained in **1.** In order to prove (1.4) and (2.4), we use the elliptic fibration on the minimal model of V_t, for $t\in\check{D}_i$ or $\check{T}_j$, induced from the pencil of lines $\{L_s|s\in\check{D}_i\}$ or

$\{L_s \mid s \in \check{T}_j\}$ on $\boldsymbol{P}^2$.

4. Combining the Clemens-Schumid exact sequence (see [2]), we can explain uniformly the failure of Torelli theorem for the period map Φ_2 of the pure second cohomology of Kunev surfaces and elliptic surfaces with $p_g=1$ and $q=0, 1$ (cf. [7], [8], [9], [4]).

Corollary 3. *S_0, S_1 and S_1' in Theorems* 1 *and* 2 *appear as the fibers of the period map Φ_2 for Kunev surfaces, homotopic* $K3$ *surfaces and elliptic surfaces with $p_g=q=1$ respectively.*

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