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31. Degeneration of Kunev Surfaces. I

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0. The purpose of this note is to outline our recent results on degeneration of Kunev surfaces. Details will be published elsewhere.

A Kunev surface is, by definition (see 1 below), a double cover of a K3 surface. We report descriptions of degenerations of Kunev surfaces over some fixed K3 surfaces (Theorems 1 and 2). These theorems have an interesting application: We can explain in a uniform way the failure of the Torelli theorem for Kunev surfaces and elliptic surfaces with $p_g=1$ and $q=0, 1$ (Corollary 3). We use the terminology a *homotopic K3 surface* and an *elliptic surface* as ones with $\kappa=1$.

1. A *Kunev surface* is defined as a minimal surface X of general type with $p_g=c_1^2=1$ which has an involution σ such that $Y':=X/\sigma$ is a K3 surface with rational double points (R.D.P. for short) and the bicanonical map of X is a Galois cover of P^2 factoring through Y' . Let X be a Kunev surface with ample K_X . Then it is known that the branch locus $B \subset P^2$ of the bicanonical map consists of two smooth cubics C_j ($j=1, 2$) and of a line L such that $B=\sum C_j+L$ has only nodes as singularities (see [1], [6]), and X is reconstructed as follows: (i) Take the double cover Y' of P^2 branched over $\sum C_j$. (ii) Take the minimal resolution Y of Y' . (iii) Take the double cover \tilde{X} of Y branched over $L+\sum E_j$, where E_j ($1 \leq j \leq 9$) are (-2) -curves appeared in (ii). (iv) Contracting (-1) -curves on \tilde{X} induced from E_i , we recover the Kunev surface X .

2. Horikawa and Shah constructed a completion of the moduli space of K3 surfaces of degree 2 as a completion of {sextics in P^2 } by geometric invariant theory ([3], [5]), which contains our K3 surfaces Y appeared in 1. The latter form 10-dimensional submoduli \mathfrak{N} over which sits "a completion" of the moduli space \mathfrak{M} of Kunev surface. The first theorem is concerned with a completion of the fiber over a general point in \mathfrak{N} . Let C_1 and C_2 be general cubics in P^2 . Denote by $\check{C}_j \subset \check{P}^2$ the dual curve of $C_j \subset P^2$, i.e., the image of the Gauss map. Then each \check{C}_j has nine cusps corresponding to nine inflexes on C_j , $\sum \check{C}_j$ has nine bitangents \check{D}_i with tangent points P_{i1} and P_{i2} ($1 \leq i \leq 9$) subjected to nine nodes of $\sum C_j$, and we have two stratifications of \check{P}^2 determined by $\sum \check{C}_j$ and $\sum \check{D}_i$:

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$$\begin{aligned}
\check{P}^2 &= (\check{P}^2 - \sum \check{C}_j) \cup (\sum \check{C}_j - (\sum \check{P}_{j_i} + \text{Sing}(\sum \check{C}_j)) \cup (\sum \text{Sing}(\check{C}_j)) \\
&\quad \cup (\cap \check{C}_j) \cup (\sum \check{P}_{j_i}) \\
&= : R_0 \cup R_1 \cup R'_1 \cup R_2 \cup R'_0. \\
\check{P}^2 &= (\check{P}^2 - \sum \check{D}_i) \cup (\sum \check{D}_i - \text{Sing}(\sum \check{D}_i)) \cup \text{Sing}(\sum \check{D}_i) \\
&= : S_0 \cup S_1 \cup S_2.
\end{aligned}$$

Theorem 1. *With the above notation, there exists a complete family $f: \mathcal{X} \rightarrow \check{P}^2$ of degenerations of Kunev surfaces over the fixed general point $[\sum C_j] \in \mathfrak{N}$. This family has the following properties:*

(1.1) *The singularity of the total space \mathcal{X} consists of mutually disjoint compounds Veronese cone over \check{D}_i ($1 \leq i \leq 9$), i.e., analytically isomorphic to the product of \check{D}_i and the cone over the Veronese embedding of $P^2 \subset P^5$ by $|\mathcal{O}_{P^2}(2)|$. Hence a single blowing-up along the singular loci yields a resolution $\hat{f}: \hat{\mathcal{X}} \rightarrow \check{P}^2$. For each i ($1 \leq i \leq 9$), the exceptional divisor \mathcal{W}_i is a family of P^2 over \check{D}_i . The universal family $\{L_t | t \in \check{P}^2\}$ of lines on P^2 induces an irreducible divisor on $\hat{\mathcal{X}}$. We denote by \mathcal{L} the divisor endowed with reduced structure. Then $K_{\hat{\mathcal{X}}} = \mathcal{L} + \sum \mathcal{W}_i$.*

(1.2) *Besides the singularity of the total space \mathcal{X} , the fiber X_t has R.D.P. raised from the tangent points of L_t and $\sum C_j$ on P^2 . These singularities form two disjoint compounds A_1 over $R_1 \cup R_2$, which degenerate to A_2 over R'_1 . Over R'_0 , clash an A_1 and a Veronese cone singularity. The effect is explained in (1.6) below.*

(1.3) *The fiber $\tilde{X}_t = V_t + \sum W_{i,t}$, where V_t is the main component and the summation runs over the indices i for which $t \in \check{D}_i$. Hence the canonical curve K_t of V_t coincides with $\mathcal{L}|_{V_t}$.*

(1.4) *V_t is a (singular) Kunev surface, homotopic K3 surface, or K3 surface according to $t \in S_0, S_1$, or S_2 .*

(1.5) *K_t is irreducible reduced and passes $\text{Sing}(V_t)$, if exists, and its geometric genus is $2 - (m+n)$ for $t \in (R_m \cup R'_m) \cap S_n$.*

(1.6) *In case $t \in S_1 - R'_0$, $V_t \cap W_{i,t}$ is a smooth conic on $W_{i,t} \simeq P^2$ and a rational curve with selfintersection -4 on V_t , where $t \in \check{D}_i$. Whereas, in case $t \in R'_0$, $V_t \cap W_{i,t}$ decomposes into two distinct lines on $W_{i,t}$ and two rational curves with selfintersection -3 on V_t .*

Remark. (1) Since the isotropy group $\text{Isot}[\sum C_j]$ of $[\sum C_j]$ in PGL_2 is trivial, \check{P}^2 is actually a completion of the fiber of $\mathfrak{M} \rightarrow \mathfrak{N}$ over $[\sum C_j]$. (2) We can compute easily the following numbers: $\#(R_1) = 9 \cdot 2 = 18$, $\#(R_2) = 6^2 = 36$, $\#(R'_0) = 9 \cdot 2 = 18$, $\#(R_1 \cap S_1) = 4 \cdot 9 = 36$, $\#(S_2) = 9 \cdot 8/2 = 36$. (3) We can describe easily a semi-stable reduction of a family induced over a disc.

3. Among the special cases with finite local monodromy in the pure second cohomology, we report here one of the most interesting cases. Let C_1 (resp. C_2) consists of three distinct lines $\sum M_k$ (resp. $\sum N_l$) passing through a common point T_1 (resp. T_2) such that $C_1 \cap C_2$ are nine nodes D_i ($1 \leq i \leq 9$). Denote by \check{M}_k and \check{N}_l (resp. \check{T}_j and \check{D}_i) the dual points (resp. lines) on \check{P}^2 . Then the three points \check{M}_k (resp. \check{N}_l) are on the line \check{T}_1 (resp. \check{T}_2), \check{D}_i are the lines joining the points \check{M}_k and \check{N}_l , and these determine a

stratification on \check{P}^2 ;

$$\begin{aligned}\check{P}^2 &= (\check{P}^2 - (\sum \check{D}_i + \sum \check{T}_j)) \cup (\sum \check{D}_i - \text{Sing}(\sum \check{D}_i)) \\ &\quad \cup (\text{Sing}(\sum \check{D}_i) - (\sum \check{M}_k + \sum \check{N}_l)) \\ &\quad \cup (\sum \check{T}_j - (\check{T}_1 \cap \check{T}_2 + \sum \check{M}_k + \sum \check{N}_l)) \cup (\check{T}_1 \cap \check{T}_2) \cup (\sum \check{M}_k + \sum \check{N}_l) \\ &=: S_0 \cup S_1 \cup S_2 \cup S'_1 \cup S'_2 \cup S''_2.\end{aligned}$$

Theorem 2. *With the above notation, there exists a complete family $f: \mathcal{X} \rightarrow \check{P}^2$ of degenerations of Kunev surfaces over the fixed $[\sum C_j] \in \mathfrak{R}$ as above. This family has the following properties:*

(2.1) *Let $\tilde{\mathcal{X}}$ be the blowing-up of \mathcal{X} along nine disjoint compounds Veronese cone over \check{D}_i ($1 \leq i \leq 9$) raised from $C_1 \cap C_2$. Then the same statement as (1.1) holds, provided that $\tilde{\mathcal{X}}$ still has singularity described in (2.2) below.*

(2.2) *Rising from two triple points T_1 and T_2 , \mathcal{X} has four compounds R.D.P. of type D_4 over $\check{P}^2 - \sum \check{T}_j = S_0 \cup S_1 \cup S_2$, each two of which clash to make up a compound elliptic singularity on $f^{-1}(\check{T}_j - S'_2)$ ($j=1, 2$) with a local equation*

$$z^2 + y(x^4 + y^2) = 0.$$

In case $t \in S'_2$, say $t = \check{M}_1$, besides the two D_4 raised from T_2 , the main component V_t of the fiber \tilde{X}_t has the following singularity: We abuse the notation T_1 for the point on V_t induced from $T_1 \in \check{P}^2$. V_t has ordinary double points along $\mathcal{L}|V_t - T_1$ and a local equation at $T_1 \in V_t$ is

$$z^2 + y^2(x^2 + y^4) = 0.$$

Hence T_1 becomes an R.D.P. of type A_3 on the normalization of V_t .

(2.3) *The same statement as (1.3) holds.*

(2.4) *Analogously as (1.4), V_t is a singular Kunev surface, homotopic K3 surface, or K3 surface according to $t \in S_0, S_1$, or S_2 . Whereas V_t becomes a singular elliptic surface with $p_g = q = 1$, abelian surface, or K3 surface according to $t \in S'_1, S'_2$, or S''_2 .*

(2.5) *The canonical curve K_t on V_t is divided into two disjoint (-1) -curves in the case that $t \in S_2$ and that t is a triple point of $\sum \check{D}_i$. K_t becomes a double rational curve in case $t \in S'_2$. In other cases, K_t is irreducible reduced, and its geometric genus is $2-n$ for $t \in S_n \cup S'_n$. K_t passes the elliptic singular point or its degenerating point in case $t \in \sum \check{T}_j = S'_1 \cup S'_2 \cup S''_2$.*

(2.6) *An analogous statement as (1.6) holds according to $t \in S_1 - S'_2$, or S'_2 . In case $t \in S'_2$, $V_t \cap W_{i,t}$ on V_t consists of two rational curves which cut the double curve $\mathcal{L}|V_t$ transversely at a common point.*

Remark. We can give parallel remarks as those just after Theorem 1. We omit all but the version of (1).

(1') $\text{Isot}[\sum C_j]$ is a finite group and $\check{P}^2/\text{Isot}[\sum C_j]$ is a completion of the fiber of $\mathfrak{M} \rightarrow \mathfrak{R}$ over $[\sum C_j]$.

The proofs of Theorems 1 and 2 go on the same way as the construction of smooth Kunev surfaces with ample K over P^2 explained in 1. In order to prove (1.4) and (2.4), we use the elliptic fibration on the minimal model of V_t , for $t \in \check{D}_i$ or \check{T}_j , induced from the pencil of lines $\{L_s | s \in \check{D}_i\}$ or

$\{L_s | s \in \check{T}_j\}$ on P^2 .

4. Combining the Clemens-Schmid exact sequence (see [2]), we can explain uniformly the failure of Torelli theorem for the period map Φ_2 of the pure second cohomology of Kunev surfaces and elliptic surfaces with $p_g=1$ and $q=0, 1$ (cf. [7], [8], [9], [4]).

Corollary 3. *S_0, S_1 and S'_1 in Theorems 1 and 2 appear as the fibers of the period map Φ_2 for Kunev surfaces, homotopic K3 surfaces and elliptic surfaces with $p_g=q=1$ respectively.*

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