



Title	Degeneration of Kunev surfaces I
Author(s)	Usui, Sampei
Citation	Proceedings of the Japan Academy, Series A, Mathematical Sciences. 1987, 63(4), p. 110-113
Version Type	VoR
URL	<a href="https://hdl.handle.net/11094/73386">https://hdl.handle.net/11094/73386</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## 31. Degeneration of Kunev Surfaces. I

By Sampei USUI\*)

Department of Mathematics, Faculty of Science,  
Kochi University

(Communicated by Kunihiko KODAIRA, M. J. A., April 13, 1987)

0. The purpose of this note is to outline our recent results on degeneration of Kunev surfaces. Details will be published elsewhere.

A Kunev surface is, by definition (see 1 below), a double cover of a K3 surface. We report descriptions of degenerations of Kunev surfaces over some fixed K3 surfaces (Theorems 1 and 2). These theorems have an interesting application: We can explain in a uniform way the failure of the Torelli theorem for Kunev surfaces and elliptic surfaces with  $p_g=1$  and  $q=0, 1$  (Corollary 3). We use the terminology a *homotopic K3 surface* and an *elliptic surface* as ones with  $\kappa=1$ .

1. A *Kunev surface* is defined as a minimal surface  $X$  of general type with  $p_g=c_1^2=1$  which has an involution  $\sigma$  such that  $Y':=X/\sigma$  is a K3 surface with rational double points (R.D.P. for short) and the bicanonical map of  $X$  is a Galois cover of  $\mathbf{P}^2$  factoring through  $Y'$ . Let  $X$  be a Kunev surface with ample  $K_X$ . Then it is known that the branch locus  $B \subset \mathbf{P}^2$  of the bicanonical map consists of two smooth cubics  $C_j$  ( $j=1, 2$ ) and of a line  $L$  such that  $B = \sum C_j + L$  has only nodes as singularities (see [1], [6]), and  $X$  is reconstructed as follows: (i) Take the double cover  $Y'$  of  $\mathbf{P}^2$  branched over  $\sum C_j$ . (ii) Take the minimal resolution  $Y$  of  $Y'$ . (iii) Take the double cover  $\tilde{X}$  of  $Y$  branched over  $L + \sum E_i$ , where  $E_i$  ( $1 \leq i \leq 9$ ) are  $(-2)$ -curves appeared in (ii). (iv) Contracting  $(-1)$ -curves on  $\tilde{X}$  induced from  $E_i$ , we recover the Kunev surface  $X$ .

2. Horikawa and Shah constructed a completion of the moduli space of K3 surfaces of degree 2 as a completion of  $\{\text{sextics in } \mathbf{P}^2\}$  by geometric invariant theory ([3], [5]), which contains our K3 surfaces  $Y$  appeared in 1. The latter form 10-dimensional submoduli  $\mathfrak{N}$  over which sits “a completion” of the moduli space  $\mathfrak{M}$  of Kunev surface. The first theorem is concerned with a completion of the fiber over a general point in  $\mathfrak{N}$ . Let  $C_1$  and  $C_2$  be general cubics in  $\mathbf{P}^2$ . Denote by  $\check{C}_j \subset \check{\mathbf{P}}^2$  the dual curve of  $C_j \subset \mathbf{P}^2$ , i.e., the image of the Gauss map. Then each  $\check{C}_j$  has nine cusps corresponding to nine inflexes on  $C_j$ ,  $\sum \check{C}_j$  has nine bitangents  $\check{D}_i$  with tangent points  $P_{i1}$  and  $P_{i2}$  ( $1 \leq i \leq 9$ ) subjected to nine nodes of  $\sum C_j$ , and we have two stratifications of  $\check{\mathbf{P}}^2$  determined by  $\sum \check{C}_j$  and  $\sum \check{D}_i$ :

\*) Partially supported by Grant-in-Aid for Encouragement of Young Scientist A-61740045 from the Ministry of Education, Science and Culture, Japan.

$$\begin{aligned}
\check{P}^2 &= (\check{P}^2 - \sum \check{C}_j) \cup (\sum \check{C}_j - (\sum \check{P}_{ji} + \text{Sing}(\sum \check{C}_j)) \cup (\sum \text{Sing}(\check{C}_j)) \\
&\quad \cup (\cap \check{C}_j) \cup (\sum \check{P}_{ji}) \\
&=: R_0 \cup R_1 \cup R'_1 \cup R_2 \cup R'_2, \\
\check{P}^2 &= (\check{P}^2 - \sum \check{D}_i) \cup (\sum \check{D}_i - \text{Sing}(\sum \check{D}_i)) \cup \text{Sing}(\sum \check{D}_i) \\
&=: S_0 \cup S_1 \cup S_2.
\end{aligned}$$

**Theorem 1.** *With the above notation, there exists a complete family  $f: \mathcal{X} \rightarrow \check{P}^2$  of degenerations of Kunev surfaces over the fixed general point  $[\sum C_j] \in \mathfrak{N}$ . This family has the following properties:*

(1.1) *The singularity of the total space  $\mathcal{X}$  consists of mutually disjoint compounds Veronese cone over  $\check{D}_i$  ( $1 \leq i \leq 9$ ), i.e., analytically isomorphic to the product of  $\check{D}_i$  and the cone over the Veronese embedding of  $P^2 \subset P^5$  by  $|\mathcal{O}_{P^2}(2)|$ . Hence a single blowing-up along the singular loci yields a resolution  $\check{f}: \tilde{\mathcal{X}} \rightarrow \check{P}^2$ . For each  $i$  ( $1 \leq i \leq 9$ ), the exceptional divisor  $\mathcal{W}_i$  is a family of  $P^2$  over  $\check{D}_i$ . The universal family  $\{L_t \mid t \in \check{P}^2\}$  of lines on  $P^2$  induces an irreducible divisor on  $\tilde{\mathcal{X}}$ . We denote by  $\mathcal{L}$  the divisor endowed with reduced structure. Then  $K_{\tilde{\mathcal{X}}} = \mathcal{L} + \sum \mathcal{W}_i$ .*

(1.2) *Besides the singularity of the total space  $\mathcal{X}$ , the fiber  $X_t$  has R.D.P. raised from the tangent points of  $L_t$  and  $\sum C_j$  on  $P^2$ . These singularities form two disjoint compounds  $A_1$  over  $R_1 \cup R_2$ , which degenerate to  $A_2$  over  $R'_1$ . Over  $R'_0$ , clash an  $A_1$  and a Veronese cone singularity. The effect is explained in (1.6) below.*

(1.3) *The fiber  $\tilde{X}_t = V_t + \sum W_{i,t}$ , where  $V_t$  is the main component and the summation runs over the indices  $i$  for which  $t \in \check{D}_i$ . Hence the canonical curve  $K_t$  of  $V_t$  coincides with  $\mathcal{L}|V_t$ .*

(1.4)  *$V_t$  is a (singular) Kunev surface, homotopic K3 surface, or K3 surface according to  $t \in S_0, S_1$ , or  $S_2$ .*

(1.5)  *$K_t$  is irreducible reduced and passes  $\text{Sing}(V_t)$ , if exists, and its geometric genus is  $2 - (m + n)$  for  $t \in (R_m \cup R'_m) \cap S_n$ .*

(1.6) *In case  $t \in S_1 - R'_0$ ,  $V_t \cap W_{i,t}$  is a smooth conic on  $W_{i,t} \simeq P^2$  and a rational curve with selfintersection  $-4$  on  $V_t$ , where  $t \in \check{D}_i$ . Whereas, in case  $t \in R'_0$ ,  $V_t \cap W_{i,t}$  decomposes into two distinct lines on  $W_{i,t}$  and two rational curves with selfintersection  $-3$  on  $V_t$ .*

**Remark.** (1) Since the isotropy group  $\text{Isot}[\sum C_j]$  of  $[\sum C_j]$  in  $PGL_2$  is trivial,  $\check{P}^2$  is actually a completion of the fiber of  $\mathfrak{M} \rightarrow \mathfrak{N}$  over  $[\sum C_j]$ . (2) We can compute easily the following numbers:  $\#(R_1) = 9 \cdot 2 = 18$ ,  $\#(R_2) = 6^2 = 36$ ,  $\#(R'_0) = 9 \cdot 2 = 18$ ,  $\#(R_1 \cap S_1) = 4 \cdot 9 = 36$ ,  $\#(S_2) = 9 \cdot 8/2 = 36$ . (3) We can describe easily a semi-stable reduction of a family induced over a disc.

3. Among the special cases with finite local monodromy in the pure second cohomology, we report here one of the most interesting cases. Let  $C_1$  (resp.  $C_2$ ) consists of three distinct lines  $\sum M_k$  (resp.  $\sum N_l$ ) passing through a common point  $T_1$  (resp.  $T_2$ ) such that  $C_1 \cap C_2$  are nine nodes  $D_i$  ( $1 \leq i \leq 9$ ). Denote by  $\check{M}_k$  and  $\check{N}_l$  (resp.  $\check{T}_j$  and  $\check{D}_i$ ) the dual points (resp. lines) on  $\check{P}^2$ . Then the three points  $\check{M}_k$  (resp.  $\check{N}_l$ ) are on the line  $\check{T}_1$  (resp.  $\check{T}_2$ ),  $\check{D}_i$  are the lines joining the points  $\check{M}_k$  and  $\check{N}_l$ , and these determine a

stratification on  $\check{P}^2$ :

$$\begin{aligned}\check{P}^2 &= (\check{P}^2 - (\sum \check{D}_i + \sum \check{T}_j)) \cup (\sum \check{D}_i - \text{Sing}(\sum \check{D}_i)) \\ &\quad \cup (\text{Sing}(\sum \check{D}_i) - (\sum \check{M}_k + \sum \check{N}_l)) \\ &\quad \cup (\sum \check{T}_j - (\check{T}_1 \cap \check{T}_2 + \sum \check{M}_k + \sum \check{N}_l)) \cup (\check{T}_1 \cap \check{T}_2) \cup (\sum \check{M}_k + \sum \check{N}_l) \\ &=: S_0 \cup S_1 \cup S_2 \cup S'_1 \cup S'_2 \cup S''_2.\end{aligned}$$

**Theorem 2.** *With the above notation, there exists a complete family  $f: \mathcal{X} \rightarrow \check{P}^2$  of degenerations of Kunev surfaces over the fixed  $[\sum C_j] \in \mathfrak{N}$  as above. This family has the following properties:*

(2.1) *Let  $\tilde{\mathcal{X}}$  be the blowing-up of  $\mathcal{X}$  along nine disjoint compounds Veronese cone over  $\check{D}_i$  ( $1 \leq i \leq 9$ ) raised from  $C_1 \cap C_2$ . Then the same statement as (1.1) holds, provided that  $\tilde{\mathcal{X}}$  still has singularity described in (2.2) below.*

(2.2) *Rising from two triple points  $T_1$  and  $T_2$ ,  $\mathcal{X}$  has four compounds R.D.P. of type  $D_4$  over  $\check{P}^2 - \sum \check{T}_j = S_0 \cup S_1 \cup S_2$ , each two of which clash to make up a compound elliptic singularity on  $f^{-1}(\check{T}_j - S'_2)$  ( $j=1, 2$ ) with a local equation*

$$z^2 + y(x^4 + y^2) = 0.$$

*In case  $t \in S''_2$ , say  $t = \check{M}_1$ , besides the two  $D_4$  raised from  $T_2$ , the main component  $V_t$  of the fiber  $\tilde{X}_t$  has the following singularity: We abuse the notation  $T_1$  for the point on  $V_t$  induced from  $T_1 \in \check{P}^2$ .  $V_t$  has ordinary double points along  $\mathcal{L}|V_t - T_1$  and a local equation at  $T_1 \in V_t$  is*

$$z^2 + y^2(x^2 + y^4) = 0.$$

*Hence  $T_1$  becomes an R.D.P. of type  $A_3$  on the normalization of  $V_t$ .*

(2.3) *The same statement as (1.3) holds.*

(2.4) *Analogously as (1.4),  $V_t$  is a singular Kunev surface, homotopic K3 surface, or K3 surface according to  $t \in S_0, S_1$ , or  $S_2$ . Whereas  $V_t$  becomes a singular elliptic surface with  $p_g = q = 1$ , abelian surface, or K3 surface according to  $t \in S'_1, S'_2$ , or  $S''_2$ .*

(2.5) *The canonical curve  $K_t$  on  $V_t$  is divided into two disjoint  $(-1)$ -curves in the case that  $t \in S_2$  and that  $t$  is a triple point of  $\sum \check{D}_i$ .  $K_t$  becomes a double rational curve in case  $t \in S''_2$ . In other cases,  $K_t$  is irreducible reduced, and its geometric genus is  $2-n$  for  $t \in S_n \cup S'_n$ .  $K_t$  passes the elliptic singular point or its degenerating point in case  $t \in \sum \check{T}_j = S'_1 \cup S'_2 \cup S''_2$ .*

(2.6) *An analogous statement as (1.6) holds according to  $t \in S_1 - S''_2$ , or  $S''_2$ . In case  $t \in S''_2$ ,  $V_t \cap W_{i,t}$  on  $V_t$  consists of two rational curves which cut the double curve  $\mathcal{L}|V_t$  transversely at a common point.*

**Remark.** We can give parallel remarks as those just after Theorem 1. We omit all but the version of (1).

(1')  *$\text{Isot}[\sum C_j]$  is a finite group and  $\check{P}^2 / \text{Isot}[\sum C_j]$  is a completion of the fiber of  $\mathfrak{M} \rightarrow \mathfrak{N}$  over  $[\sum C_j]$ .*

The proofs of Theorems 1 and 2 go on the same way as the construction of smooth Kunev surfaces with ample  $K$  over  $P^2$  explained in 1. In order to prove (1.4) and (2.4), we use the elliptic fibration on the minimal model of  $V_t$ , for  $t \in \check{D}_i$  or  $\check{T}_j$ , induced from the pencil of lines  $\{L_s \mid s \in \check{D}_i\}$  or

$\{L_s \mid s \in \dot{T}_j\}$  on  $P^2$ .

4. Combining the Clemens-Schumid exact sequence (see [2]), we can explain uniformly the failure of Torelli theorem for the period map  $\Phi_2$  of the pure second cohomology of Kunev surfaces and elliptic surfaces with  $p_g=1$  and  $q=0, 1$  (cf. [7], [8], [9], [4]).

**Corollary 3.**  *$S_0, S_1$  and  $S'_1$  in Theorems 1 and 2 appear as the fibers of the period map  $\Phi_2$  for Kunev surfaces, homotopic K3 surfaces and elliptic surfaces with  $p_g=q=1$  respectively.*

**Acknowledgement.** The author expresses his gratitude to Professors M. Miyanishi, A. Moriwaki, K. Nishiguchi, M.-H. Saito, and K. Ueno for stimulating discussions.

### References

- [1] Catanese, F.: Surfaces with  $K^2=p_g=1$  and their period mapping. Proc. Summer Meeting, Copenhagen 1978; Lect. Notes in Math., no. **732**, Springer-Verlag, 1–29 (1979).
- [2] Clemens, C. H.: Degenerations of Kähler manifolds. Duke Math. J., **44**, 215–290 (1977).
- [3] Horikawa, E.: Surjectivity of period map of K3 surfaces of degree 2. Math. Ann., **228**, 113–146 (1977).
- [4] Saito, M.-H.: On the infinitesimal Torelli problem of elliptic surfaces. J. Math. Kyoto Univ., **23-3**, 441–460 (1983).
- [5] Shah, J.: A complete moduli space for K3 surfaces of degree 2. Ann. Math., **112**, 485–510 (1980).
- [6] Saito, M.-H., Shimizu, Y. and Usui, S.: Variation of mixed Hodge structure and Torelli problem (to appear in Advanced Study of Pure Math., North-Holland Publ. Co. & Kinokuniya Co. Ltd.).
- [7] Todorov, A. N.: Surfaces of general type with  $p_g=1$  and  $(K, K)=1$ : I. Ann. scient. Éc. Norm. Sup., **4-13**, 1–21 (1980).
- [8] Usui, S.: Period map of surfaces with  $p_g=c_1^2=1$  and  $K$  ample. Mem. Fac. Sci. Kochi Univ. (Math.), **3**, 37–73 (1981).
- [9] ——: Effect of automorphisms on variation of Hodge structure. J. Math. Kyoto Univ., **21-4**, 645–672 (1981).