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# Generic Torelli theorem for quintic-mirror family

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**Abstract:** This article is a geometric application of polarized logarithmic Hodge theory of Kazuya Kato and Sampei Usui. We prove generic Torelli theorem for the well-known quintic-mirror family in two ways by using different logarithmic points at the boundary of the fine moduli of polarized logarithmic Hodge structures.

**Key words:** Quintic-mirror family; logarithmic Hodge theory; moduli; Torelli theorem.

**1. Review of quintic-mirror family.** We review the construction of the mirror family of the pencil joining the quintic hypersurface of Fermat type and the union of the coordinate hyperplanes in a 4-dimensional complex projective space after [M1].

Let  $\psi \in \mathbf{P}^1$ , and let

$$(1) \quad Q_\psi = \left\{ \sum_{j=1}^5 x_j^5 - 5\psi \prod_{j=1}^5 x_j = 0 \right\} \subset \mathbf{P}^4.$$

Let  $\mu_5 = \{\alpha \in \mathbf{C} \mid \alpha^5 = 1\}$ .

The singular members of the pencil (1) are listed as follows:

(2) Over each point  $\psi \in \mu_5 \subset \mathbf{C} \subset \mathbf{P}^1$ ,  $Q_\psi$  contains  $5^3 = 125$  ordinary double points  $(\alpha_1, \dots, \alpha_5) \in (\mu_5)^5 / \mu_5 \subset \mathbf{P}^4$  with  $\psi \alpha_1 \cdots \alpha_5 = 1$ .

(3) Over  $\infty \in \mathbf{P}^1$ ,  $Q_\infty = \bigcup_j \{x_j = 0\}$ . The union of coordinate hyperplanes.

Let  $G = \{\alpha \in (\mu_5)^5 \mid \alpha_1 \cdots \alpha_5 = 1\} / \mu_5$ . By multiplication, this becomes a group which acts on  $\mathbf{P}^4$  coordinate-wise. This action of  $G$  preserves  $Q_\psi$ . Taking the quotient  $Q_\psi / G$ , the following canonical singularities appear:

(4) For each pair of distinct indices  $j, k$ , a compound du Val singularity  $cA_4$  appears as the quotient of the curve  $Q_\psi \cap \{x_j = x_k = 0\} \setminus (\bigcup_{m \neq j,k} \{x_m = 0\})$ .

(5) For each triple of distinct indices  $j, k, l$ , the point which is the quotient of the five points  $Q_\psi \cap \{x_j = x_k = x_l = 0\}$  belongs to the closure of three curves in (4).

Moreover, we see that holomorphic 3-forms on  $Q_\psi$  are  $G$ -invariant for every  $\psi \in \mathbf{C}$  by adjunc-

tion formula.

For  $\psi \in \mathbf{C}$ , it is known that there is a simultaneous minimal desingularization  $W_\psi$  of these quotient singularities, and that holomorphic 3-forms extend to nowhere vanishing forms on  $W_\psi$ .

We thus have a pencil  $(W_\psi)_{\psi \in \mathbf{P}^1}$  whose singular fibers are listed as follows:

(6) Over each point  $\psi \in \mu_5 \subset \mathbf{C} \subset \mathbf{P}^1$ ,  $W_\psi$  has one ordinary double point.

(7) Over  $\psi = \infty \in \mathbf{P}^1$ ,  $W_\psi$  is a normal crossing divisor in the total space, whose components are all rational.

The other members  $W_\psi$  are smooth with Hodge numbers  $h^{p,q} = 1$  for  $p + q = 3$ .

By the action of  $\alpha \in \mu_5$ ,  $(x_1, \dots, x_5) \mapsto (\alpha^{-1}x_1, x_2, \dots, x_5)$ , we have  $W_{\alpha\psi} \simeq W_\psi$ . Let  $\lambda = \psi^5$ , and let

$$\begin{array}{ccc} (W_\lambda)_\lambda & \xlongequal{\quad} & ((W_\psi)_\psi) / \mu_5 \\ \downarrow & & \downarrow \\ (\lambda\text{-plan}) & \xlongequal{\quad} & (\psi\text{-plan}) / \mu_5. \end{array}$$

This pencil  $(W_\lambda)_{\lambda \in \mathbf{P}^1}$  is the mirror of the original pencil (1). (For more details of the above construction, see e.g. [M1].)

**2. Review of fine moduli space  $\Gamma \backslash D_\Xi$ .** We review some facts in [KU] that are necessary in the present article.

Let  $w = 3$ , and  $h^{p,q} = 1$  ( $p + q = 3, p, q \geq 0$ ). Let  $H_0 = \bigoplus_{j=1}^4 \mathbf{Z}e_j$ , and  $\langle e_3, e_1 \rangle_0 = \langle e_4, e_2 \rangle_0 = 1$ . Let  $D$  be the corresponding classifying space of polarized Hodge structures, and  $\check{D}$  the compact dual.

Let  $S$  = (square free positive integers), and let  $m \in S$ . Define  $N_\alpha, N_\beta, N_m \in \text{End}(H_0, \langle, \rangle_0)$  as follows:

$$N_\alpha(e_3) = e_1, \quad N_\alpha(e_j) = 0 \quad (j \neq 3);$$

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$$\begin{aligned}
N_\beta(e_4) &= e_3, & N_\beta(e_3) &= -e_1, \\
N_\beta(e_1) &= -e_2, & N_\beta(e_2) &= 0; \\
N_m(e_1) &= e_3, & N_m(e_4) &= -me_2, \\
N_m(e_2) &= N_m(e_3) &= 0.
\end{aligned}$$

Then the respective Hodge diamonds are

$$\begin{array}{c}
0 : \quad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ (3,0) & (2,1) & (1,2) & (0,3) \end{array} \\
\\
N_\alpha : \quad \begin{array}{ccc} & \bullet & \\ & (2,2) & \\ (3,0) & \downarrow & (0,3) \\ & \bullet & \\ & (1,1) & \end{array} \\
\\
N_\beta : \quad \begin{array}{c} \bullet \\ (3,3) \\ \downarrow \\ \bullet \\ (2,2) \\ \downarrow \\ \bullet \\ (1,1) \\ \downarrow \\ \bullet \\ (0,0) \end{array} \\
\\
N_m : \quad \begin{array}{cc} \bullet & \bullet \\ (3,1) & (1,3) \\ \downarrow & \downarrow \\ \bullet & \bullet \\ (2,0) & (0,2) \end{array}
\end{array}$$

Let  $\sigma_\alpha = \mathbf{R}_{\geq 0}N_\alpha$ ,  $\sigma_\beta = \mathbf{R}_{\geq 0}N_\beta$ , and  $\sigma_m = \mathbf{R}_{\geq 0}N_m$  ( $m \in S$ ).

**Proposition** [KU, §12.3]. *Let  $\Xi = (\text{rational nilpotent cones in } \mathfrak{g}_{\mathbf{R}} \text{ of rank } \leq 1)$ . Then  $\Xi = \{\text{Ad}(g)\sigma \mid \sigma = \{0\}, \sigma_\alpha, \sigma_\beta, \sigma_m \ (m \in S), g \in G_{\mathbf{Q}}\}$ . This is a complete fan, i.e., if there exists  $Z \subset \check{D}$  such that  $(\sigma, Z)$  is a nilpotent orbit, then  $\sigma \in \Xi$ .*

Here a pair  $(\sigma, Z)$  is a *nilpotent orbit* if  $Z = \exp(\mathbf{C}N)F$ ,  $NF^p \subset F^{p-1}$  ( $\forall p$ ) and  $\exp(iyN)F \in D$  ( $y \gg 0$ ) hold for  $\mathbf{R}_{\geq 0}N = \sigma$  and  $F \in Z$ .

In [KU], the fine moduli space  $\Gamma \backslash D_\Xi$  of polarized logarithmic Hodge structures is constructed. We briefly explain this according to the present case. Let  $\Gamma$  be a neat subgroup of  $G_{\mathbf{Z}} := \text{Aut}(H_0, \langle, \rangle_0)$  of finite index. As a set,  $D_\Xi = \{(\sigma, Z) : \text{nilpotent orbit} \mid \sigma \in \Xi, Z \subset \check{D}\}$ . Let  $\sigma \in \Xi$ ,

and let  $\Gamma(\sigma) = \Gamma \cap \exp(\sigma)$ . If  $\sigma = \{0\}$ , then  $D \simeq \{(\{0\}, F) \mid F \in D\} \subset D_\Xi$ . If  $\sigma \neq \{0\}$ , then  $\Gamma(\sigma) \simeq \mathbf{N}$ . Let  $\gamma$  be its generator and let  $N = \log \gamma$ . Define

$$E_\sigma = \left\{ (q, F) \in \mathbf{C} \times \check{D} \left| \begin{array}{l} \exp((2\pi i)^{-1} \log(q)N)F \in D \text{ if } q \neq 0, \text{ and} \\ \exp(\mathbf{C}N)F \text{ is a } \sigma\text{-nilpotent orbit if } q = 0 \end{array} \right. \right\},$$

and the map

$$(1) \ E_\sigma \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma, \quad (q, F) \mapsto \begin{cases} \exp((2\pi i)^{-1} \log(q)N)F \bmod \Gamma(\sigma)^{\text{gp}} & \text{if } q \neq 0, \\ (\sigma, \exp(\mathbf{C}N)F) \bmod \Gamma(\sigma)^{\text{gp}} & \text{if } q = 0. \end{cases}$$

Here  $\Gamma(\sigma)^{\text{gp}}$  is the subgroup of  $\Gamma$  generated by  $\Gamma(\sigma)$ .  $\mathbf{C} \times \check{D}$  is obviously an analytic manifold. We endow this with the logarithmic structure  $M$  associated to the divisor  $\{0\} \times \check{D}$ . The strong topology of  $E_\sigma$  in  $\mathbf{C} \times \check{D}$  is defined as follows: A subset  $U$  of  $E_\sigma$  is open if, for any analytic space  $Y$  and any analytic morphism  $f : Y \rightarrow \mathbf{C} \times \check{D}$  such that  $f(Y) \subset E_\sigma$ ,  $f^{-1}(U)$  is open in  $Y$ . By (1), the quotient topology, the sheaf  $\mathcal{O}$  of local rings over  $\mathbf{C}$ , and the logarithmic structure  $M$  are introduced on  $\Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ . Introduce the corresponding structures  $\mathcal{O}$ ,  $M$  on  $\Gamma \backslash D_\Xi$  so that  $\Gamma(\sigma)^{\text{gp}} \backslash D_\sigma \rightarrow \Gamma \backslash D_\Xi$  ( $\sigma \in \Xi$ ) are local isomorphisms and form an open covering. Then, the resulting  $\Gamma \backslash D_\Xi$  is a “logarithmic manifold”, which is nearly a logarithmic analytic space but has “slits” at the boundaries. In fact, in the present case, we have in [KU, §12.3] that  $\dim_{\mathbf{C}} D = 4$ , but  $\dim_{\mathbf{C}} \Gamma(\sigma_\alpha)^{\text{gp}} \backslash (D_{\sigma_\alpha} - D) = 2$ ,  $\dim_{\mathbf{C}} \Gamma(\sigma_\beta)^{\text{gp}} \backslash (D_{\sigma_\beta} - D) = 1$  and  $\dim_{\mathbf{C}} \Gamma(\sigma_m)^{\text{gp}} \backslash (D_{\sigma_m} - D) = 1$ .

**3. Period map.** Let  $\Delta$  be a unit disc,  $\Delta^* = \Delta - \{0\}$ , and  $\mathfrak{h} \rightarrow \Delta^*$ ,  $z \mapsto q = e^{2\pi iz}$ , the universal covering. Let  $\varphi : \Delta^* \rightarrow \langle \gamma \rangle \backslash D$  be a period map, where  $\langle \gamma \rangle$  is the monodromy group generated by a unipotent element  $\gamma$ , and  $\tilde{\varphi} : \mathfrak{h} \rightarrow D$  a lifting. Let  $N = \log \gamma$ . The map  $\exp(-zN)\tilde{\varphi}(z)$  from  $\mathfrak{h}$  to  $D$  drops down to the map  $\psi : \Delta^* \rightarrow D$ .

Then the nilpotent orbit theorem of Schmid asserts that there exists the limit  $\psi(0)$ , denoted by  $F$ , and that  $\tilde{\varphi}(z)$  and  $\exp(zN)F$  are closing as  $\text{Im } z \rightarrow \infty$ .

Let  $\sigma = \mathbf{R}_{\geq 0}N$ . In our space  $\langle \gamma \rangle \backslash D_\sigma$  we have, moreover,  $\exp(zN)F \rightarrow ((\sigma, \exp(\sigma_{\mathbf{C}})F) \bmod \langle \gamma \rangle)$  as  $\text{Im } z \rightarrow \infty$ . Hence  $\varphi(q) \rightarrow ((\sigma, \exp(\sigma_{\mathbf{C}})F) \bmod \langle \gamma \rangle)$  as  $q \rightarrow 0$  in  $\langle \gamma \rangle \backslash D_\sigma$ . (For details, see [KU].)

Fix a point  $b \in \mathbf{P}^1 - \{0, 1, \infty\}$  on the  $\lambda$ -plane, identify  $H^3(W_b, \mathbf{Z}) = H_0$ , and let

$$(1) \quad \Gamma = \text{Image}(\pi_1(\mathbf{P}^1 - \{0, 1, \infty\}) \rightarrow G_{\mathbf{Z}}).$$

This  $\Gamma$  is not neat. In fact, the local monodromy around 0 is of order 5. Let

$$(2) \quad \mathbf{P}^1 - \{0, 1, \infty\} \rightarrow \Gamma \backslash D$$

be the period map. Since  $K_{W_\lambda}$  is trivial, the differential of (2) is injective everywhere. Endow  $\mathbf{P}^1$  with the logarithmic structure associated to the divisor  $\{1, \infty\}$ . Then, by §1, §2 and [KU], (2) extends to a morphism

$$(3) \quad \begin{aligned} \varphi : \mathbf{P}^1 &\rightarrow \Gamma \backslash D_\Xi, \\ 0 &\mapsto (\text{point}) \bmod \Gamma \in \Gamma \backslash D, \\ 1 &\mapsto (\text{Ad}(g)(\sigma_\alpha)\text{-nilpotent orbit} \\ &\quad \text{for some } g \in G_{\mathbf{Q}}) \bmod \Gamma, \\ \infty &\mapsto (\text{Ad}(g)(\sigma_\beta)\text{-nilpotent orbit} \\ &\quad \text{for some } g \in G_{\mathbf{Q}}) \bmod \Gamma. \end{aligned}$$

of logarithmic ringed spaces (cf. [KU, 4.3.1 (i)]). The image of the extended period map  $\varphi$  is an analytic curve, which is not affected by the slits of the space  $\Gamma \backslash D_\Xi$ .

Let  $X = \Gamma \backslash D_\Xi$ . Let  $P_1 = 1, P_\infty = \infty \in \mathbf{P}^1$ , and let  $Q_1 = \varphi(P_1), Q_\infty = \varphi(P_\infty) \in X$ . Then, by the observation of local monodromy and holomorphic 3-form basing on the descriptions in §1 and §2, we have

$$(4) \quad \varphi^{-1}(Q_\lambda) = \{P_\lambda\} \quad \text{for } \lambda = 1, \infty.$$

**4. Generic Torelli theorem.** We use the notation in the previous sections.

**Theorem.** *The period map  $\varphi$  in §3 (3) is the normalization of analytic spaces over its image.*

*Proof.* We use the fs logarithmic point  $P_1$  and its image  $Q_1$  at the boundaries (§3). Since  $\varphi^{-1}(Q_1) = \{P_1\}$  (§3 (4)), it is enough to show that the local ramification index at  $Q_1$  is one, i.e.,

**Claim.**  $(M_X/\mathcal{O}_X^\times)_{Q_1} \rightarrow (M_{\mathbf{P}^1}/\mathcal{O}_{\mathbf{P}^1}^\times)_{P_1}$  is surjective.

Let  $N := N_\alpha$  be the nilpotent endomorphism introduced in §2.

Let  $\tilde{q}$  be a local coordinate on a neighborhood  $U$  of  $P_1 = 1$  in  $\mathbf{P}^1$ , and let  $z = (2\pi i)^{-1} \log \tilde{q}$  be a branch over  $U - \{P_1\}$ . Then  $\exp(-zN)e_3 = e_3 - ze_1$  is single-valued. Let  $\omega(\tilde{q})$  be a local frame of the locally free  $\mathcal{O}_{\mathbf{P}^1}$ -module  $F^3$ . Write  $\omega(\tilde{q}) = \sum_{j=1}^4 a_j(\tilde{q})e_j$ , and define  $t = -a_1(\tilde{q})/a_3(\tilde{q})$ . Then

$$t = \frac{\langle e_3, \omega(\tilde{q}) \rangle_0}{\langle e_1, \omega(\tilde{q}) \rangle_0} = \frac{\langle \exp(-zN)e_3, \omega(\tilde{q}) \rangle_0 + z\langle e_1, \omega(\tilde{q}) \rangle_0}{\langle e_1, \omega(\tilde{q}) \rangle_0}$$

$$= z + (\text{single-valued holomorphic function in } \tilde{q}).$$

Let  $q = e^{2\pi i t}$ . Then  $q = u\tilde{q}$  for some  $u \in \mathcal{O}_{\mathbf{P}^1, P_1}^\times$ . Let  $V$  be a neighborhood of  $Q_1$  in  $X = \Gamma \backslash D_\Xi$ . We have a composite morphism of fs logarithmic local ringed spaces

$$U \rightarrow V \rightarrow \mathbf{C}, \quad \tilde{q} \mapsto q = e^{2\pi i(-a_1/a_3)} (= u\tilde{q}).$$

Hence the composite morphism  $(M_{\mathbf{P}^1}/\mathcal{O}_{\mathbf{P}^1}^\times)_{P_1} \leftarrow (M_X/\mathcal{O}_X^\times)_{Q_1} \leftarrow (M_{\mathbf{C}}/\mathcal{O}_{\mathbf{C}}^\times)_0$  of reduced logarithmic structures is an isomorphism. The claim follows. In fact, that is an isomorphism since the rank of  $(M_X/\mathcal{O}_X^\times)_{Q_1}$  is one in the present case.  $\square$

**5. The second proof, and logarithmic generic Torelli theorem.** In this section, we give another proof of the generic Torelli theorem in §4 by using the fs logarithmic points  $P_\infty$  and  $Q_\infty$  at the boundaries.

Since  $\varphi^{-1}(Q_\infty) = \{P_\infty\}$  (§3 (4)), it is enough to show the following

**Claim.**  $(M_X/\mathcal{O}_X^\times)_{Q_\infty} \rightarrow (M_{\mathbf{P}^1}/\mathcal{O}_{\mathbf{P}^1}^\times)_{P_\infty}$  is surjective.

*Proof.* Let  $N$  be the logarithm of the local monodromy at  $\lambda = \infty$ . Let  $\beta^1, \beta^2, \alpha_1, \alpha_2$  be the integral symplectic basis of  $H_0$  given in [M1, Appendix C], and let

$$g_0 = \alpha_2, \quad g_1 = 2\alpha_1 + \beta^1, \quad g_2 = \beta^2, \quad g_3 = \alpha_1 + \beta^1.$$

Then  $g_3, g_2, g_1, g_0$  is another integral symplectic basis such that  $g_0, g_1$  is a good integral basis of  $\text{Image}(N^2)$  in the sense of [M1, 2, Appendix C]. Using the result there, we have

$$(N(g_3), N(g_2), N(g_1), N(g_0)) = (g_3, g_2, g_1, g_0) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & -9/2 & 0 & 0 \\ -9/2 & 25/6 & 1 & 0 \end{pmatrix}.$$

Let  $\tilde{q}$  be a local coordinate on a neighborhood  $U$  of  $P_\infty = \infty$  in  $\mathbf{P}^1$ , and let  $z = (2\pi i)^{-1} \log \tilde{q}$  be a branch over  $U - \{P_\infty\}$ . Then  $\exp(-zN)g_1 = g_1 - zg_0$  is single-valued. Let  $\omega(\tilde{q})$  be a local frame of the locally free  $\mathcal{O}_{\mathbf{P}^1}$ -module  $F^3$ . Write  $\omega(\tilde{q}) = \sum_{j=0}^3 b_j(\tilde{q})g_j$ , and define  $t = b_3(\tilde{q})/b_2(\tilde{q})$  and  $q = e^{2\pi i t}$ . Then, as in §4, we see  $q = u\tilde{q}$  for some  $u \in \mathcal{O}_{\mathbf{P}^1, P_\infty}^\times$ , and the claim follows.  $\square$

Combining the results in §4 and §5, we have the following refinement.

**Theorem.** *The period map  $\varphi$  in §3 (3) is the normalization of fs logarithmic analytic spaces over its image. Here the normalization of the image of  $\varphi$  is endowed with the pull-back logarithmic structure.*

**Note.** Canonical coordinates in [M1,M2], like  $q$  in the proofs of Claims in §4 and §5, can be understood more naturally in the context of PLH.

**Problem 1.** Eliminate the possibility that the image  $\varphi(\mathbf{P}^1)$  would have singularities in Theorems in §4 and §5.

**Problem 2.** Describe the global monodromy group  $\Gamma$  explicitly. Generators of  $\Gamma$  are computed [COGP], [M1, Appendix C], and  $\Gamma$  is known to be Zariski dense in  $G_{\mathbf{Z}} = \mathrm{Sp}(4, \mathbf{Z})$  [COGP], [D, 13].

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