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Néron models in log mixed Hodge theory by weak fans

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Abstract: We construct Néron models of intermediate Jacobians in degenerations over higher dimensional bases by means of log geometry.

Key words: Hodge theory; log geometry; Néron model; weak fan; intermediate Jacobian; admissible normal function.

Introduction. The construction of Néron models of intermediate Jacobians in degenerations is now a prevalent subject in progress. In this paper, we give a method to construct such models over higher dimensional bases by means of log geometry. Over one dimensional bases, our construction is reduced to that in our previous paper [KNU10a] 8.2, where we used the method of fans. But, this method does not work over higher dimensional bases (cf. 4.13). To overcome the difficulty, we generalize fans to weak fans, which is the key idea of this paper.

Our model represents the moduli functor of log mixed Hodge structures with the given graded quotients at each stalk (4.8), but the constructions are not compatible with localization (see 4.16).

We omit the details of proofs, which are to be published in our project [KNU.p1], [KNU.p2], ...

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1. Weak fans. Here we introduce the concept of weak fans, which is the main subject of this paper.

1.1. Let D be the classifying space of mixed Hodge structures with polarized graded quotients, introduced in [U84], for a fixed data $\Lambda := (H_0, W, (\langle\ ,\ \rangle_k)_{k\in\mathbf{Z}}, ((h_k^{p,q})_{p,q})_k)$. This is a natural generalization of Griffiths' classifying space of polarized Hodge structures [G68]. Here the notation is compatible with that in [KNU09] §1. In particular, H_0 is a finitely generated free $\mathbf{Z}$-module, W is a rational increasing filtration on $H_{0,\mathbf{R}}$, $\langle\ ,\ \rangle_k$ is a rational, non-degenerate, $(-1)^k$-symmetric $\mathbf{R}$-bilinear form on gr_k^W, and $h_k^{p,q}$ $(p,q,k\in\mathbf{Z})$ is a non-negative integer such that $h_k^{p,q}=0$ unless $p+q=k$, that $h_k^{p,q}=h_k^{q,p}$ for all p,q,k, and that $\mathrm{rank}_{\mathbf{Z}}(H_0)=\sum_{p,q,k}h_k^{p,q}$, $\dim_{\mathbf{R}}(\mathrm{gr}_k^W)=\sum_{p,q}h_k^{p,q}$ for all k.

1.2. We continue to use the same notation as in [KNU09] §1. Let G_A $(A=\mathbf{Z},\mathbf{Q},\mathbf{R},\mathbf{C})$ be the group of A-automorphisms of $H_{0,A}$ which are compatible with the weight filtration W and the polarizations on each graded piece. Let $\mathfrak{g}_A$ $(A=\mathbf{Q},\mathbf{R},\mathbf{C})$ be its Lie algebra.

1.3. Admissible nilpotent cone. A *nilpotent cone* is a sharp cone in $\mathfrak{g}_{\mathbf{R}}$ generated by finitely many, mutually commuting nilpotent elements.

A nilpotent cone is *admissible* [SZ85, K86] if for any $N\in\sigma$, the relative monodromy filtration $M(N,W)$ exists and depends only on the face of σ spanned by N.

1.4. Nilpotent orbit. Let σ be an admissible nilpotent cone.

A *σ-nilpotent orbit* is a subset Z of $\check{D}$ satisfying the following (1)–(3) for some (and hence for all) $F\in Z$, where $\check{D}\supset D$ is the compact dual of D.

(1) $Z=\exp(\sigma_{\mathbf{C}})F$. Here $\sigma_{\mathbf{C}}$ is the vector space over $\mathbf{C}$ generated by σ in $\mathfrak{g}_{\mathbf{C}}$.

(2) $N(F^p)\subset F^{p-1}$ for all $N\in\sigma$ and $p\in\mathbf{Z}$.

(3) Let $N_1,\dots,N_n$ be a set of generators of σ. Then, $\exp(\sum_{j=1}^n iy_jN_j)F\in D$ for all $y_j\gg 0$ $(1\le j\le n)$.

1.5. Fan in $\mathfrak{g}_{\mathbf{Q}}$. We review the notion of fan introduced in [KNU10a, KNU10b]. This is the mixed version of that in [KU09].

A *fan Σ in $\mathfrak{g}_{\mathbf{Q}}$* is a set of nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ satisfying the following (1)–(3).

(1) All $\sigma\in\Sigma$ are rational and admissible.

(2) If $\sigma\in\Sigma$, all faces of σ belong to Σ.

(3) If $\sigma,\sigma'\in\Sigma$, $\sigma\cap\sigma'$ is a face of σ.

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1.6. Weak fan in $\mathfrak{g}_{\mathbf{Q}}$. Now we define weak fans by relaxing the last condition in 1.5.

A *weak fan* Σ *in* $\mathfrak{g}_{\mathbf{Q}}$ is a set of nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ satisfying the following (1)–(3).

(1) All $\sigma \in \Sigma$ are rational and admissible.

(2) If $\sigma \in \Sigma$, all faces of σ belong to Σ.

(3) Let $\sigma, \sigma' \in \Sigma$, and assume that σ and σ' have a common interior point. Assume that there is an $F \in \check{D}$ such that (σ, F) and (σ', F) generate nilpotent orbits. Then $\sigma = \sigma'$.

1.7. We remark that in the definition in 1.6, under the conditions (1) and (2), the condition (3) is equivalent to (3)$'$ below and also is equivalent to (3)$''$ below.

(3)$'$ If $\sigma, \sigma' \in \Sigma$ and if there is an $F \in \check{D}$ such that (σ, F), (σ', F), and $(\sigma \cap \sigma', F)$ generate nilpotent orbits, then $\sigma \cap \sigma'$ is a face of σ.

(3)$''$ Let τ be an admissible nilpotent cone. Let (τ, F) $(F \in \check{D})$ generate a nilpotent orbit. If the set $A := \{\sigma \in \Sigma \mid \tau \subset \sigma, (\sigma, F)$ generates a nilpotent orbit$\}$ is not empty, A has a smallest element, and it is a face of any element of A.

1.8. We will explain in §3 that the results of [KNU10b] can be generalized by replacing fans by weak fans. For this, we give a few necessary definitions in the rest of this section. Let D_Σ be the set of all pairs (σ, Z), where $\sigma \in \Sigma$ and Z is a σ-nilpotent orbit. We have embeddings $D \subset D_\Sigma, F \mapsto (\{0\}, \{F\})$.

1.9. Compatibility with Γ. Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$. Let Γ be a subgroup of $G_{\mathbf{Z}}$. We say Σ and Γ are *compatible*, if for any $\gamma \in \Gamma$ and $\sigma \in \Sigma$, we have $\mathrm{Ad}(\gamma)\sigma \in \Sigma$. Further, we say Σ and Γ are *strongly compatible* if they are compatible and if any element of any $\sigma \in \Sigma$ can be written as a finite sum of elements of the form aN, where $a \in \mathbf{R}_{\geq 0}$ and $N \in \sigma$ satisfies $\exp(N) \in \Gamma$.

2. Basic example. In this section, for a given weak fan in $\mathfrak{g}_{\mathbf{Q}}(\mathrm{gr}^W)$ and a given subgroup of $G_{\mathbf{Q},u} := \mathrm{Ker}(G_{\mathbf{Q}} \to G_{\mathbf{Q}}(\mathrm{gr}^W))$, we construct a weak fan in $\mathfrak{g}_{\mathbf{Q}}$. This result will be used in the construction of Néron models in §4.

Fix a subgroup Υ of $G_{\mathbf{Q},u}$, and fix a splitting $H_0(\mathrm{gr}^W) := \bigoplus_w H_0(\mathrm{gr}^W_w) \simeq H_0$ of W over $\mathbf{Z}$. Here $H_0(\mathrm{gr}^W_w) := (H_0 \cap W_w)/(H_0 \cap W_{w-1})$.

Theorem 2.1. *For each $w \in \mathbf{Z}$, let Σ'_w be a weak fan on gr^W_w. Let $\Sigma' := \prod_w \Sigma'_w$. Then, $\Sigma :=$ (Υ-translations of the trivial extension of Σ') is a weak fan. Here the trivial extension of Σ' is the set of cones on $H_{0,\mathbf{R}}$ which come from the cones in Σ' under the fixed splitting of W.*

Proof. The conditions (1) and (2) in 1.6 are trivially satisfied by construction. We examine the condition (3) in 1.6.

Let $\sigma', \tau' \in \Sigma'$ and let σ, τ be their trivial extensions, respectively. Let $\upsilon \in \Upsilon$ and let $\tau_\upsilon = \mathrm{Ad}(\upsilon)(\tau)$. Assume that N is in the interior of σ and of τ_υ. Let $F \in \check{D}$. Assume that (σ, F) and (τ_υ, F) generate nilpotent orbits.

We first claim that $N\upsilon = \upsilon N$. In fact, $\mathrm{gr}^W(N) = \mathrm{gr}^W(\mathrm{Ad}(\upsilon)^{-1}N)$ because $\upsilon \in \Upsilon \subset G_{\mathbf{R},u}$. On the other hand, $N \in \sigma$ and $\mathrm{Ad}(\upsilon)^{-1}N \in \tau$ are pure of weight 0 with respect to W under the fixed splitting of W. Hence $N = \mathrm{Ad}(\upsilon)^{-1}N$, that is, $N\upsilon = \upsilon N$.

Since $\mathrm{gr}^W(N)$ is in the interior of σ' and of τ' and since $(\sigma', F(\mathrm{gr}^W))$ and $(\tau', F(\mathrm{gr}^W))$ generate nilpotent orbits, we have $\sigma' = \tau'$ because Σ' is a weak fan.

Since σ' is admissible on gr^W with respect to the canonical weight filtration by [CK82], so is σ on $H_{0,\mathbf{R}}$ with respect to W. It follows that the adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}$ is admissible with respect to the weight filtration $W\mathfrak{g}_{\mathbf{R}}$ on $\mathfrak{g}_{\mathbf{R}}$ induced by W.

Let $M = M(\mathrm{Ad}(\sigma), W\mathfrak{g}_{\mathbf{R}})$ be the relative monodromy filtration on $\mathfrak{g}_{\mathbf{R}}$ defined by $\mathrm{Ad}(N_0)$ and $W\mathfrak{g}_{\mathbf{R}}$, where N_0 is any interior point of σ. Since υ commutes with N as shown as above, υ preserves M. We see, moreover, $\upsilon - 1 \in M_{-1}$. This is a special case of the general fact Lemma 2.2 below, applied to $\upsilon - 1$ which is in $W\mathfrak{g}_{\mathbf{R},-1}$ and is in the kernel of $\mathrm{Ad}(N)$.

Let $h \in \sigma$. Then $h \in M_{-2}$. Hence, by the above result, $h\upsilon - \upsilon h = h(\upsilon - 1) - (\upsilon - 1)h \in M_{-3}$. Applying υ^{-1} from the right, we have $h - \upsilon h \upsilon^{-1} \in M_{-3}$. Since (σ, F) and (σ_υ, F) generate nilpotent orbits, we have $h \in F^{-1}\mathfrak{g}_{\mathbf{C}}$ and $\upsilon h \upsilon^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$, respectively (Griffiths transversality). Hence $h - \upsilon h \upsilon^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$, and hence $h - \upsilon h \upsilon^{-1} \in M_{-3} \cap F^{-1}\mathfrak{g}_{\mathbf{C}} \cap \bar{F}^{-1}\mathfrak{g}_{\mathbf{C}} = 0$. Here we use the fact that $(M, F\mathfrak{g}_{\mathbf{C}})$ is an $\mathbf{R}$-mixed Hodge structure. Thus we have $\sigma = \sigma_\upsilon = \tau_\upsilon$. The condition (3) in 1.6 is verified. □

For the relative monodromy filtration $M(N, W)$, the following is easily proved.

Lemma 2.2. *If $x \in W_w$ and if $N(x) = 0$, then $x \in M(N, W)_w$.*

3. Partial toroidal compactifications of classifying spaces of mixed Hodge structures. The results in [KNU10b] can be generalized by replacing fans by weak fans.

We define the moduli functor of log mixed Hodge structures with polarized graded quotients.

3.1. Fix $\Phi := (\Lambda, \Sigma, \Gamma)$. Here Λ is as in 1.1, Σ is a weak fan, and Γ is a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Γ.

3.2. Let S be an object of $\mathcal{B}(\log)$. By a *log mixed Hodge structure of type* Φ over S, we mean an LMH with polarized graded quotients $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_k)_k, F)$ endowed with a global section μ of the sheaf $\Gamma\backslash \operatorname{Isom}((H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_k)_k), (H_0, W, (\langle\ ,\ \rangle_k)_k))$ on $S^{\log}$ which satisfies the following conditions (1) and (2).

(1) $\operatorname{rank}_{\mathbf{Z}}(H_{\mathbf{Z}}) = \sum_{p,q,k} h_k^{p,q}$, $\operatorname{rank}_{\mathcal{O}_S^{\log}}(F^p) = \sum_{k\in\mathbf{Z}, r\geq p} h_k^{r,k-r}$ for all p.

(2) For any $s \in S$ and $t \in S^{\log}$ lying over s, if $\tilde{\mu}_t : (H_{\mathbf{Z},t}, W, (\langle\ ,\ \rangle_k)_k) \xrightarrow{\simeq} (H_0, W, (\langle\ ,\ \rangle_k)_k)$ is a representative of the germ of μ at t, then there exists $\sigma \in \Sigma$ such that the image of the composite map $\operatorname{Hom}(M_{S,s}/\mathcal{O}_{S,s}^{\times}, \mathbf{N}) \hookrightarrow \pi_1(\tau^{-1}(s)) \to \operatorname{Aut}(H_{\mathbf{Z},t}, W, (\langle\ ,\ \rangle_k)_k) \xrightarrow{\text{by } \tilde{\mu}_t} \operatorname{Aut}(H_0, W, (\langle\ ,\ \rangle_k)_k)$ is contained in $\exp(\sigma)$ and such that the $\exp(\sigma_{\mathbf{C}})$-orbit Z including $\tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}_{S,t}^{\log}} F_t)$, which is independent of the choice of a $\mathbf{C}$-algebra homomorphism $\mathcal{O}_{S,t}^{\log} \to \mathbf{C}$, is a σ-nilpotent orbit (cf. [KU09] 0.4.24, 2.5.1, 2.5.5).

3.3. Let $\mathrm{LMH}_{\Phi} : \mathcal{B}(\log) \to (\mathrm{set})$ be the contravariant functor defined as follows: $\mathrm{LMH}_{\Phi}(S)$ for an object S of $\mathcal{B}(\log)$ is the set of isomorphism classes of log mixed Hodge structures of type Φ over S.

3.4. We define the sheaf of complex analytic functions on $\Gamma\backslash D_{\Sigma}$ and its log structure in the same way as our previous paper [KNU10b]. Everything is parallel to the pure case [KU09]. We have the same results as in [KNU10b] by replacing fans by weak fans. In particular, we have the following

Theorem 3.5. (i) *$\Gamma\backslash D_{\Sigma}$ is Hausdorff.*

(ii) *Assume that Γ is neat. Then $\Gamma\backslash D_{\Sigma}$ is a log manifold. In particular, it belongs to $\mathcal{B}(\log)$. Further, it represents the functor LMH_{Φ} in 3.3.*

The period map $\mathrm{LMH}_{\Phi} \to \operatorname{Mor}(\bullet, \Gamma\backslash D_{\Sigma})$ which is the isomorphism in this Theorem is as follows: Let S and H be as in 3.2. Let $s \in S$. The associated point of $\Gamma\backslash D_{\Sigma}$ by this period map is the image of $(\sigma, Z) \in D_{\Sigma}$ in $\Gamma\backslash D_{\Sigma}$. Here σ is the smallest cone of Σ satisfying 3.2 (2), which exists by 1.7 (3)$''$, and Z is the associated $\exp(\sigma_{\mathbf{C}})$-orbit as in 3.2 (2).

The proof of 3.5 is similar to that in [KU09].

4. Néron models.

4.1. Assume that $\Sigma' := \prod_w \Sigma'_w$ as in 2.1 is given and $\Gamma' = \prod \Gamma'_w$ is a subgroup of $G_{\mathbf{Z}}(\mathrm{gr}^W)$ such that Γ'_w is strongly compatible with Σ'_w for each w. Let Γ be the inverse image of Γ' in $G_{\mathbf{Z}}$. So Γ is a semi-direct product of Γ' and $G_{\mathbf{Z},u}$.

Take a subgroup Υ of $G_{\mathbf{Q},u}$. Assume

(1) $G_{\mathbf{Z},u} \subset \Upsilon$.

(2) $\gamma\Upsilon\gamma^{-1} = \Upsilon$ for any $\gamma \in \Gamma$.

An example of Υ which satisfies (1) and (2) is $\Upsilon = G_{\mathbf{Z},u}$.

The other extreme example is $\Upsilon = G_{\mathbf{Q},u}$.

Fix a splitting $H_0(\mathrm{gr}^W) \simeq H_0$. Let Σ be the Υ-translations of the trivial extension of Σ'. By 2.1, Σ is a weak fan.

Lemma 4.2. *Σ is strongly compatible with Γ.*

This follows from 4.1 (1) and (2), and the following lemma.

Lemma 4.3. *Let $\sigma' \in \Sigma'$ and let $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ be the trivial extension of σ' to $H_{0,\mathbf{R}}$. Then for any $\upsilon \in G_{\mathbf{Q},u}$, the canonical map $\Gamma(\operatorname{Ad}(\upsilon)(\sigma)) \to \Gamma'(\sigma')$ is of kummer type, that is, it is injective and some power of any element of $\Gamma'(\sigma')$ belongs to its image. Here $\Gamma(\operatorname{Ad}(\upsilon)(\sigma)) := \Gamma \cap \exp(\operatorname{Ad}(\upsilon)(\sigma))$ and $\Gamma'(\sigma') := \Gamma' \cap \exp(\sigma')$.*

4.4. Let $D' := D(\mathrm{gr}^W)$. Assume that we have a morphism $S \to \Gamma'\backslash D'_{\Sigma'}$ from an object S of $\mathcal{B}(\log)$. Let J_{Σ} be the fiber product of

$$S \to \Gamma'\backslash D'_{\Sigma'} \leftarrow \Gamma\backslash D_{\Sigma}$$

in the category $\mathcal{B}(\log)$.

In the case $\Upsilon = G_{\mathbf{Z},u}$, we will call J_{Σ} the *connected Néron model.* (Here "connected" refers to the fact that the fibers of $J_{\Sigma} \to S$ are connected.)

4.5. Now assume $\Sigma' = \operatorname{face}(\sigma')$ for some $\sigma' \in \Sigma'$. Let $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ be the trivial extension of σ' to $H_{0,\mathbf{R}}$. Let Υ_1 be the subset of $G_{\mathbf{Q},u}$ consisting of all elements υ such that $\Gamma(\operatorname{Ad}(\upsilon)(\sigma)) \to \Gamma'(\sigma')$ is an isomorphism.

A morphism $S \to T$ in $\mathcal{B}(\log)$ is called *strict* if the log structure of S is the pullback of the log structure of T.

Proposition 4.6. *In 4.5, we assume further that there are $w, w' \in \mathbf{Z}$ such that $\mathrm{gr}_k^W = 0$ if $k \neq w, w'$.*

(i) *Υ_1 is a subgroup of $G_{\mathbf{Q},u}$ and satisfies (1) and (2) in 4.1.*

(ii) *For any subgroup Υ of $G_{\mathbf{Q},u}$ satisfying (1) and (2) in 4.1, if Σ denotes the weak fan in $\mathfrak{g}_{\mathbf{Q}}$ corresponding to Υ, the following conditions (a) and (b) are equivalent.*

(*a*) *$\Upsilon \subset \Upsilon_1$.*

(*b*) *$\Gamma\backslash D_{\Sigma} \to \Gamma'\backslash D'_{\Sigma'}$ is strict.*

(iii) *If the equivalent conditions in* (ii) *are satisfied, then,* $X := \Gamma\backslash D_\Sigma \to X' := \Gamma'\backslash D'_{\Sigma'}$ *is a relative manifold with slits, that is, locally on* X *and on* X'*, there are an* $n \geq 0$*, an open set* U *of* $X' \times \mathbf{C}^n$*, a finite subset* I *of* $\Gamma(U, \omega^1_U)$*, and a strict* X'*-open immersion* $X \to U' := \{u \in U \mid$ *the image of* I *in* ω^1_u *is zero*$\}$*, where* U' *is endowed with the strong topology in* U *and with the inverse images of* $\mathcal{O}_{X'\times\mathbf{C}^n}$ *and* $M_{X'}$*. Here* ω^1_U *denotes the sheaf of log differential forms on* U*, and* ω^1_u *denotes the space of log differential forms on the log point* u*. The strong topology of* U' *means the strongest topology on* U' *such that for any complex analytic space* A *and any morphism* $f : A \to U$ *of* $\mathcal{B}$ *such that* $f(A) \subset U'$*,* $f : A \to U'$ *is continuous.*

Note that relative manifold with slits in 4.6 (iii) is a relative notion of log manifold in [KU09] 3.5.7. Note also that n in (iii) must coincide with the relative dimension of $D \to D(\mathrm{gr}^W)$ in this case.

4.6 (i) is easily checked by using the assumption that there are only two non-trivial gr^W_k. (ii) is clear. (iii) is, roughly speaking, deduced from the fact that $\mathrm{toric}_\sigma \times \check{D} \to \mathrm{toric}_{\sigma'} \times \check{D}'$ is smooth by taking into account slits coming from Griffiths transversality. Here toric_σ is the toric variety associated to σ.

4.7. Assume that we are as in the beginning of 4.6. Assume that $S \to \Gamma'\backslash D'_{\Sigma'}$ is strict. This means in this case that for any $s \in S$, the map $\pi_1^+(s^{\log}) := \mathrm{Hom}(M_{S,s}/\mathcal{O}^\times_{S,s}, \mathbf{N}) \to \Gamma'(\sigma')$ is injective and the image is a face of $\Gamma'(\sigma')$.

Let Σ_1 be the weak fan corresponding to Υ_1. We call J_{Σ_1} the *Néron model*. By 4.6 (iii), $J_{\Sigma_1} \to S$ is strict and a relative manifold with slits.

Proposition 4.8. *In* 4.6*, assume that* $S \to \Gamma'\backslash D'_{\Sigma'}$ *is strict. We identify the category* $\mathcal{B}/S^\circ$ *(*S° *denotes the underlying analytic space of* S *forgetting the log structure) with the category of strict objects of* $\mathcal{B}(\log)$ *over* S*.*

(i) *For an object* S' *of* $\mathcal{B}/S^\circ$*, there is a natural functorial injection* $\iota(S') : \mathrm{Mor}_{S^\circ}(S', J_{\Sigma_1}) \to \{H$*; LMH on* S' *with the given* gr^W *such that* W *on* $H_{\mathbf{Q}}$ *splits locally on* $S'\}$*.*

(ii) *If* $\dim(\sigma') = 1$*, then* ι *is bijective.*

(iii) *Assume that* $s \in S$ *is given such that* $\pi_1^+(s^{\log}) \to \Gamma'(\sigma')$ *is bijective. Let* $S' \to S$ *be a strict morphism, and let* $s' \in S'$ *be a point lying over* s*. Then, the stalk of* ι *at* s' *is bijective.*

Remark. The above ι is not bijective in general. See 4.16.

In 4.8, ι is given by the diagram in 4.4. The problem is that an LMH with the given gr^W does not necessarily satisfy the condition 3.2 (2). We can show that 3.2 (2) is satisfied either on a base of log rank ≤ 1 or after localization, which implies (ii) and (iii) respectively.

4.9. Assume that we are as in 4.8. From now on, assume further $\mathrm{gr}^W_0 = \mathbf{Z}$ with the standard polarization, and there is $w < 0$ such that $\mathrm{gr}^W_k = 0$ for $k \neq 0, w$.

Let $H'_0 = \mathrm{gr}^W_w(H_0)$. So $H_0(\mathrm{gr}^W) = H'_0 \oplus \mathbf{Z}$.

Hence,

$$G_{\mathbf{Z},u} \simeq H'_0, \quad G_{\mathbf{Q},u} \simeq H'_{0,\mathbf{Q}},$$

and Υ satisfying (1) and (2) in 4.1 corresponds to a subgroup Q of $H'_{0,\mathbf{Q}}$ containing H'_0 which is stable under the action of $\Gamma' = \Gamma'_w$.

The next is straightforward.

Proposition 4.10. *In the situation* 4.9*,*

(i) $\Upsilon = G_{\mathbf{Z},u}$ *corresponds to* $Q = H'_0$*.*

(ii) $\Upsilon = G_{\mathbf{Q},u}$ *corresponds to* $Q = H'_{0,\mathbf{Q}}$*.*

(iii) *Assume* $\Sigma' = \mathrm{face}(\sigma')$ *and define* Υ_1 *as in* 4.5*. Then* Υ_1 *corresponds to*

$$Q_1 := \{x \in H'_{0,\mathbf{Q}} \mid \gamma x - x \in H'_0 \text{ for all } \gamma \in \Gamma'(\sigma')\}.$$

The next variant of 4.8 is proved easily (cf. [KNU10a] 8.1 $(*)$).

Proposition 4.11. *In the situation* 4.9*, assume* $\Upsilon \subset \Upsilon_1$*. Then, on* $\mathcal{B}/S^\circ$*,* J_Σ *represents the functor* $\{$*LMH with the given* gr^W *whose image in* $R^1\tau_*H'_{\mathbf{Z}}$ *belongs to the kernel of* $R^1\tau_*H'_{\mathbf{Z}} \to R^1\tau_*Q\}$*. Here we abuse notation and denote by* Q *as the sub local system of* $H'_{\mathbf{Q}}$ *corresponding to the subspace* $Q \subset H'_{0,\mathbf{Q}}$*.*

By 4.10 (i) and by 4.11, we have

Proposition 4.12. *In the situation* 4.9*, the connected Néron model* (4.4) *represents the horizontal part of* $\tau_*H_{\mathbf{Z}}\backslash H_{\mathcal{O}}/F^0$*, i.e., the part consisting of sections corresponding to pre-log mixed Hodge structures satisfying the small Griffiths transversality* (*see* [KNU10a] 8.1)*.*

4.13. Proposition 4.12 means that in the situation of 4.9, the connected Néron model is the Zucker model [Z76] with slits (cf. [KNU10a] §5–§6). Note that we can not necessarily get the slit Zucker model by using a fan. That is, we need a weak fan.

For example, assume that σ' is generated by N'_1 and N'_2. Let $\Sigma' = \mathrm{face}(\sigma')$. Let $\Gamma' = \exp(\sigma'^{\mathrm{gp}}) \cap G_{\mathbf{Z}}(\mathrm{gr}^W_w)$. Let Σ correspond to $G_{\mathbf{Z},u}$. Then, if Σ is a fan, then $\mathrm{Ker}(N'_1 + N'_2)$ must be contained in

$\mathrm{Ker}(N'_1)$ (which does not hold in general). In fact, let $q \in \mathrm{Ker}(N'_1 + N'_2) \cap H'_0$. Let $\upsilon \in G_{\mathbf{Z},u}$ be the corresponding element. Let N_j be the trivial extension of N'_j ($j = 1, 2$). Let σ be the trivial extension of σ'. Then, the sum of the elements $\mathrm{Ad}(\upsilon)N_1$ and $\mathrm{Ad}(\upsilon)N_2$ of $\mathrm{Ad}(\upsilon)(\sigma)$ belongs to σ. So, if $\sigma \cap \mathrm{Ad}(\upsilon)(\sigma)$ is a face of $\mathrm{Ad}(\upsilon)(\sigma)$, then $\mathrm{Ad}(\upsilon)N_1$ belongs to σ, which means $q \in \mathrm{Ker}(N'_1)$.

4.14. Let S be an object of $\mathcal{B}(\log)$ whose log rank is ≤ 1. We continue to assume that we are in the situation 4.9, and assume that we are given an LMH on gr^W over S. Then, we can always take, locally on S, Σ' and Γ' such that we have a strict morphism $S \to \Gamma' \backslash D'_{\Sigma'}$ (cf. [KU09] 4.3).

The next is another variant of 4.8.

Proposition 4.15. *Assume that we are in the situation* 4.9. *Assume that* $S \to \Gamma' \backslash D'_{\Sigma'}$ *is strict. Let* $S' \to S$ *be a strict morphism in* $\mathcal{B}(\log)$. *Assume that* S' *is a smooth manifold endowed with the log structure by a normal crossing divisor.*

(i) *There is a natural functorial injection* $\iota(S') : \mathrm{Mor}_S(S', J_{\Sigma_1}) \to \{$*admissible normal functions on* S' *whose image in* $R^1\tau_* H'_{\mathbf{Q}}$ *is trivial*$\}$.

(ii) *Assume that* $\dim(\sigma') = 1$. *Then,* $\iota(S')$ *is bijective.*

(iii) *The same statement as* 4.8 (iii) *holds.*

On such an S' as in 4.15 (i), an LMH with the given gr^W is nothing but an admissible normal function in the classical sense. From this, (i) follows. (ii) and (iii) are special cases of 4.8.

4.16. In 4.15, ι is not bijective in general. In fact, in 4.9, let $w = -1$, $H'_0 = \mathbf{Z}^4 = \mathbf{Z}e_1 + \cdots + \mathbf{Z}e_4$, $\langle e_j, e_k \rangle_{-1}$ is equal to 1 if $j = k+2$, -1 if $j = k-2$, and 0 if otherwise. Let $h^{0,-1}_{-1} = h^{-1,0}_{-1} = 2$, i.e., we consider a degeneration of abelian surfaces. Define $N'_1 \in \mathfrak{g}'_{\mathbf{Q}}$ by $N'_1(e_3) = e_1, N'_1(e_4) = 2e_2$, and $N'_1(e_1) = N'_1(e_2) = 0$, and $N'_2 \in \mathfrak{g}'_{\mathbf{Q}}$ by $N'_2(e_4) = e_2$, and $N'_2(e_j) = 0$ unless $j = 4$. Let σ' be the cone spanned by N'_1 and N'_2. Let $\Sigma' = \mathrm{face}(\sigma')$. Let $\Gamma' = \exp(\sigma'^{\mathrm{gp}}) \cap G'_{\mathbf{Z}}$. Let $S = \Gamma' \backslash D'_{\Sigma'}$. Let S' be a smooth curve endowed with the log structure by a point. Consider an admissible normal function on S' whose local monodromy coincides with N'_1 on gr^W_{-1} and sends the generator 1 of $\mathrm{gr}^W_0 = \mathbf{Z}$ to e_2. Then, this does not come from $\mathrm{Mor}_S(S', J_{\Sigma_1})$. In fact, in this case, any element x of Q_1 in 4.10 (iii) is of the form $ae_1 + be_2 + ce_3 + de_4$ $(a, b \in \mathbf{Q}, c, d \in \mathbf{Z})$ so that, for the $\upsilon \in \Upsilon_1$ corresponding to x, $\mathrm{Ad}(\upsilon)(N_1)$ sends $1 \in \mathrm{gr}^W_0$ to $-ce_1 - 2de_2$, which cannot coincide with e_2. Here N_1 is the trivial extension of N'_1. Hence the former local monodromy cannot appear from the admissible normal functions belonging to $\mathrm{Mor}_S(S', J_{\Sigma_1})$.

4.17. Discussion about the usual Néron model. In 4.14, in the case where S is a smooth curve over $\mathbf{C}$ with log structure given by a finite subset P, $w = -1$, and $F^1 H'_{\mathcal{O}} = 0$ and $F^{-1} H'_{\mathcal{O}} = H'_{\mathcal{O}}$, then J_{Σ_1} is the Néron model in the usual sense of its restriction to $S_{\mathrm{triv}} = S - P$ which is an abelian variety J over S_{triv}.

4.18. Relations with other works. In case where the base is of one dimension, Green-Griffiths-Kerr [GGK10] constructed a Néron model. In this case, our construction is in [KNU10a] 8.2, where we do not need weak fans. Recently, T. Hayama [H.p] studied the relation of their model and our model and proved that there is a homeomorphism between them.

In [BPS.p], Brosnan-Pearlstein-Saito constructed a generalization of the Néron model of Green-Griffiths-Kerr to the case of a higher dimensional base. In [S.p], C. Schnell also constructed a connected Néron model over a higher dimensional base. The relationship between these constructions and the Néron model in this paper should be investigated.

We remark that our constructions of models in this section obviously imply some special cases (cf. 4.4, 4.15) of analyticity shown by Brosnan-Pearlstein ([BP.p]; cf. also [S.p] by Schnell).

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