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TORELLI PROBLEM FOR SURFACES OF GENERAL TYPE (*)

Summary: *This article is a report of the present situation of the Torelli type problem for surfaces of general type. This article also contains new results on the surfaces with* $p_g = 1$, $c_1^2 = 2$ and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

In his articles [G.1], [G.2] and [G.3], Griffiths gave a fundamental set-up for period mappings as a theory of moduli. Among others, he showed that, for a polarized family $f:\mathscr{X} \longrightarrow S$ of projective manifolds,

(1.1) *the Hodge filtration* F *on* $R^n_{\mathcal{O}}(f) = R^n f_* \mathbb{Z} \otimes \mathcal{O}_S$ *consists of holomorphic subbundles,*

(1.2) *the Gauss-Manin connection* ∇ *of* $R^n_{\mathcal{O}}(f)$ *satisfies*

$$\nabla F^p(R^n_{\mathcal{O}}(f)) \subset \Omega^1_S \otimes F^{p-1}(R^n_{\mathcal{O}}(f)), \quad \textit{and}$$

(1.3) *thete exists a locally constant bilinear form* Q *defined over* $\mathbb{Q}$ *on the primitive part* $P^n_{\mathcal{O}}(f)$ *of* $R^n_{\mathcal{O}}(f)$ *which satisfies the Hodge-Riemann bilinear relations.*

Assume, for simplicity, that S is smooth. Fix a base point $s_0 \in S$. Let $\widetilde{S} \longrightarrow S$ be the covering defined by the representation $\rho: \pi_1(S, s_0) \longrightarrow \mathrm{Aut}(P^n(X_{s_0}, \mathbb{Z}), Q)$ and let $\widetilde{f}: \widetilde{\mathscr{X}} \longrightarrow \widetilde{S}$ be the family induced from $f: \mathscr{X} \longrightarrow S$.

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Then, we can define a map

$$\widetilde{\Phi} : \widetilde{S} \longrightarrow Fl = \text{flag manifold}$$

by $\widetilde{\Phi}(\tilde{s}) = \alpha[F^1(s) \supset \ldots \supset F^n(s)]$ for $\tilde{s} = (s, \alpha) \in \widetilde{S}$.

Using these notations, we can translate (1.1), (1.2) and (1.3) in the language of period map to

(2.1) *the period map* $\widetilde{\Psi}$ *is holomorphic,*

(2.2) *the differential* $d\widetilde{\Phi}$ *of* $\widetilde{\Phi}$ *is given (up to* $\underset{p}{\oplus}(-1)^p$*) by the composition*

$$T_{\widetilde{S}} \xrightarrow[\text{map}]{\text{Kodaira-Spencer}} R^1\widetilde{f}_* T_{\widetilde{f}} \xrightarrow{\substack{\text{via} \\ \text{contraction}}} \underset{p}{\oplus} \mathrm{Hom}^{Q}_{\mathcal{O}_{\widetilde{S}}}(\mathrm{Gr}^p_F(R^n_{\mathcal{O}}(\widetilde{f})), \mathrm{Gr}^{p-1}_F(R^n_{\mathcal{O}}(\widetilde{f}))) \longrightarrow T_{Fl} ,$$

and

(2.3) *the image of* $\widetilde{\Phi}$ *lies in a locally closed submanifold* D *of* Fl *restricted by the Hodge-Riemann bilinear relations.* $\Gamma = \mathrm{Im}\,\rho$ *acts properly discontinuously on* D*. In particular,* D/Γ *is a complex space and* $\widetilde{\Phi}$ *induces a holomorphic map* $\Phi : S \longrightarrow D/\Gamma$.

In the above frame-work, Griffiths proposed the problems. For a suitable family $f : \mathscr{X} \longrightarrow S$,

(3.1) $d\widetilde{\Phi}(\widetilde{S}) : T_{\widetilde{S}}(\widetilde{S}) \longrightarrow T_{Fl}(\widetilde{\Phi}(\widetilde{S}))$ *is injective or not (infinitesimal Torelli problem), and*

(3.2) Φ *or* $\widetilde{\Phi}$ *is injective or not (Torelli problem).*

(Note that there are various type of Torelli problems according to various type of formulations of period map.)

These are essential problems in the theory of moduli via period map.

Torelli theorem holds for the family of curves, which is one of the most remarkable classical results and also recently refined by Oort-Steenbrink ([O.S.]). Another famous examples are $K3$ surfaces and Enriques surfaces ([H.1], [H.2]). As for $K3$ surfaces, not only the injectivity ([P.S.], [B.R.], [Sh.], [L.P.]) but also the surjectivity ([H.3], [T.1], [Si.]) of the period map are verified.

Griffiths conjectured in [G.4] the following:

(4) *For a simply connected surface* X *of general type,*

$$H^1(X, T_X) \longrightarrow \mathrm{Hom}(H^0(X, \Omega^2_X), H^1(X, \Omega^1_X))$$

should be injective !?

(4) is affirmatively verified in the following cases:

- Smooth hypersurfaces in $\mathbb{P}^N$ ([G.2]).
- Smooth complete intersections in $\mathbb{P}^N$ ([U.1]).
- Cyclic branched coverings of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ ([P.]).
- Smooth weighted complete intersections satisfying some additional assumptions ([U.2]).
- Hypersurfaces in $\mathbb{P}^3$ with ordinary singularities whose singular loci are complete intersections in $\mathbb{P}^3$ ([U.3]).

But, unfortunately (?), Kinev gave a counter-example to the conjecture (4) ([Ki.]), which is a certain surface with $p_g = c_1^2 = 1$. After this, the surfaces with $p_g = c_1^2 = 1$ have attracted our attention.

Catanese proved in [C.1] that the canonical models of these surfaces are represented as weighted complete intersections of type (6,6) in $\mathbb{P}(1,2,2,3,3)$, which are the excluded case in [U.2], and that the period map is generically finite. Afterward, he proved that the degree of the period map is $\geqslant 2$ ([C.2]).

Todorov ([T.2]) and the author ([U.4]) showed independently and by different methods that the period map in question has some positive dimensional fibres. The former restricted himself only to those surfaces whose bicanonical maps are Galois coverings of $\mathbb{P}^2$. The latter pointed out also another surfaces, of course within the limit of $p_g = c_1^2 = 1$, at which the period map has positive dimensional fibres.

The author continued to study this curious phenomenon, and succeded in explaining it as an effect of automorphisms of the surfaces in question. The principle is very simple, that is, in the general set-up, an automorphism σ of X causes the decompositions of $H^n(X,\mathbb{C})$ and each $F^p(H^n(X,\mathbb{C}))$ into their eigen subspaces, and these decompositions must be compatible. This gives restrictions to the image of the period map. On this principle, he explained the above phenomenon after complete classification of the automorphisms of the surfaces in question ([U.5]).

When we restrict the period map to the surfaces with $p_g = c_1^2 = 1$

and with an automorphism of order 3 acting trivially on $H^0(X, \Omega_X^2)$, we get the global Torelli theorem ([U.6]). The author proved this by using the Torelli theorem for algebraic $K3$ surfaces.

The author generalized in [U.7] the mixed Hodge structures of Deligne [D.2] to the relative case arising from a family of logarithmic deformations of Kawamata [Ka.]. In order to state this more precisely, let $\mathscr{F} = (\mathscr{X}, \mathscr{Y}, \mathring{\mathscr{X}}, f, S, s_0, \varphi)$ be a family of logarithmic deformations of a non-singular triple $(X, Y, \mathring{X})$ (terminology is that in [Ka.]). We can consider the relative logarithmic De Rham complex $\Omega_f^{\cdot}(\log \mathscr{Y})$ together with its weight filtration W and its Hodge filtration F. For the spectral sequences

$$ {}_WE_1^{p,q} = \mathbb{R}^{p+q} f_*(\mathrm{Gr}_{-p}^W(\Omega_f^{\cdot}(\log \mathscr{Y}))) \Rightarrow E^{p+q} = \mathbb{R}^{p+q} f_*(\Omega_f^{\cdot}(\log \mathscr{Y})) \quad \text{and} $$
$$ {}_FE_1^{p,q} = \mathbb{R}^{p+q} f_*(\mathrm{Gr}_F^p(\Omega_f^{\cdot}(\log \mathscr{Y}))) \Rightarrow E^{p+q} = \mathbb{R}^{p+q} f_*(\Omega_f^{\cdot}(\log \mathscr{Y})) $$

of hypercohomologies defined respectively by $(\Omega_f^{\cdot}(\log \mathscr{Y}), W, f_*)$ and $(\Omega_f^{\cdot}(\log \mathscr{Y}), F, f_*)$, we can prove:

$$ (5.1) \qquad E^n = R^n \mathring{f}_*(\mathring{f}^{\cdot} \mathcal{O}_S) = R^n \mathring{f}_* \mathbb{Z} \otimes \mathcal{O}_S = R^n_{\mathcal{O}}(\mathring{f}), $$

where $\mathring{f} = f|\mathring{\mathscr{X}} : \mathring{\mathscr{X}} \longrightarrow S$. *The filtration* W *on* $R^n_{\mathcal{O}}(\mathring{f})$ *is induced from a filtration on* $R^n_{\mathbb{Q}}(\mathring{f})$ *consisting of its sub-local systems. The filtration* F *on* $R^n_{\mathcal{O}}(\mathring{f})$ *consists of holomorphic sub-bundles of* $R^n_{\mathcal{O}}(\mathring{f})$.

(5.2) *The triple* $(R^n_{\mathbb{Z}}(\mathring{f}), W[n], F)$ *is a variation of mixed Hodge structure,* i.e. *they satisfy*

$$ (\mathrm{M.H.1}) \qquad \nabla F^p(R^n_{\mathcal{O}}(\mathring{f})) \subset \Omega_S^1 \otimes F^{p-1}(R^n_{\mathcal{O}}(\mathring{f})), \quad \text{and} $$
$$ (\mathrm{M.H.2}) \qquad (R^n_{\mathbb{Z}}(\mathring{f}), W[n], F)(s) \simeq (H^n(X_s, \mathbb{Z}), W[n], F) $$

for each point $s \in S$, *where the left-hand-side is the restriction to the fibre over* s *and the right is the Deligne's mixed Hodge structure.*

$$ (5.3) \qquad {}_WE_2 = {}_WE_\infty \quad , \quad {}_FE_1 = {}_FE_\infty \ . $$

These are a generalization of (1.1) and (1.2), and hence we have the corresponding generalization of (2.1) and (2.2), that is,

(6.1) *The period map* $\widetilde{\Phi} : S \longrightarrow Fl$ (*= flag manifold*) *is holomorphic, and*

(6.2) *the differential* $d\widetilde{\Phi}$ *of* $\widetilde{\Phi}$ *is given (up to* $\underset{p}{\oplus} (-1)^p$) *by*

$$T_S \xrightarrow[\text{map}]{\text{Kodaira-Spencer}} R^1 f_* T_f(-\log \mathscr{Y}) \xrightarrow[\text{contraction}]{\text{via}} \bigoplus_p \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Gr}_F^p(R_{\mathcal{O}}^n(\mathring{f})), \mathrm{Gr}_F^{p-1}(R_{\mathcal{O}}^n(\mathring{f}))) \hookrightarrow T_{Fl}$$

Back to the surfaces with $p_g = c_1^2 = 1$, we can prove the following type of local Torelli theorem:

(7) *Let* X *be a surface with* $p_g = c_1^2 = 1$ *and let* C *be its unique canonical curve. Then, at the surface* X *with ample and smooth* C, *the local Torelli theorem holds in the sense of the variation of mixed Hodge structures,* i.e. *the map*

$$H^1(X, T_X(-\log C)) \longrightarrow \mathrm{Hom}(H^0(X, \Omega_X^2(\log C)), H^1(X, \Omega_X^1(\log C)))$$

induced from the contraction is injective.

Note that at least the surfaces, whose bicanonical maps are Galois coverings of $\mathbf{P}^2$, satisfy the conditions in (7).

Note. Todorov constructed further some surfaces of general type with $p_g = 1$ and $2 \leqslant c_1^2 \leqslant 8$ as double covers of quartic Kummer surfaces with rational double points. And he showed that the period map for these surfaces has positive dimensional fibres.

Catanese and Debarre descrived all the surfaces with $p_g = 1$ and $c_1^2 = 2$. According to the image of the bicanonical map

$$f_{|2K_S|} : S \longrightarrow \Sigma \subset \mathbf{P}^3 \quad , \quad \deg f_{|2K_S|} \cdot \deg \Sigma = 8$$

and the degree of the canonical map of the unique canonical curve C

$$f_{|\omega_C|} : C \longrightarrow \mathbf{P}^2 ,$$

they devided the following 6 cases:

(0) Σ is a quadric cone.

(1) Σ is a smooth quadric.

(2h) Σ is an irreducible quartic and $\deg f_{|\omega_C|} = 2$.

(2n) Σ is an irreducible quartic and $\deg f_{|\omega_C|} = 1$.

(3h) Σ is of degree 8 and $\deg f_{|\omega_C|} = 2$.

(3n) Σ is of degree 8 and $\deg f_{|\omega_C|} = 1$.

In case (0) $\pi_1(S) = Z/2Z$ and in the other cases $\pi_1(S) = 1$. For each case, they gave a structure theorem and descrived the canonical ring.

For the period map of these surfaces, the following things are known:

- In case (0), every fibre of the period map is positive dimensional (Oliverio).
- In case (2n) (this is the case investigated by Todorov), the period map has 3-dimensional fibres (Todorov).
- In the case when S is simply connected, the infinitesimal Torelli theorem holds on a Zariski open subset whose complement is nowhere dense (Catanese).

Additional Note. After writing out this manuscript I found a mistake in the preprint of Oliverio [O]. After a correction, what he proved is the following: *The infinitesimal Torelli theorem holds for general surfaces with* $p_g = 1$, $c_1^2 = 2$ *and* $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. In this case I also proved the following: *Assume furthermore that the canonical curve* C *of our surface* X *is smooth and ample, then the infinitesimal Torelli theorem by means of the mixed Hodge structure on* $X - C$ *holds, i.e. the map*

$$H^1(X, T_X(-\log C)) \longrightarrow \mathrm{Hom}\,(H^0(X, \Omega_X^2(\log x)), H^1(X, \Omega_X^1(\log C)))$$

is injective (cf. [U.8]).

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