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SUMMARY OF STUDIES OF CLOSED/OPEN MIRROR SYMMETRY FOR QUINTIC THREEFOLDS THROUGH LOG MIXED HODGE THEORY

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0. Introduction and Statements

This is a summary of [U14p].

We correct the definitions and descriptions of the integral structures in our previous paper [U14]. We use $\hat{\Gamma}$-integral structure of Iritani in [I11] for A-model. Using the corrected version, we study open mirror symmetry for quintic threefolds through log mixed Hodge theory, especially the recent result on Néron models for admissible normal functions with non-torsion extensions in the joint work [KNU14] with K. Kato and C. Nakayama. We positively use integral structures of local systems with graded polarizations over the boundary points.

In a series of joint works with Kato and Nakayama, we are constructing a fundamental diagram which consists of various kind of partial compactifications of classifying space

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of mixed Hodge structures and their relations. We try to understand Hodge theoretic aspects of mirror symmetry in this framework of the fundamental diagram.

Fundamental Diagram

For a classifying space D of Hodge structures of specified type, we have

$$\begin{array}{ccccccc} & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \longrightarrow & D_{\mathrm{BS},\mathrm{val}} \\ & & & & \downarrow & & \downarrow \\ \Gamma\backslash D_{\Sigma,\mathrm{val}} & \longleftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \longrightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\ \downarrow & & \downarrow & & & & \\ \Gamma\backslash D_{\Sigma} & \longleftarrow & D^{\sharp}_{\Sigma} & & & & \end{array}$$

in pure case: [KU99], [KU02], [KU09]. For mixed case, we should extend to an amplified diagram: [KNU08], [KNU09], [KNU11], [KNU13], continuing.

Mirror symmetry for quintic threefolds

Mirror symmetry for the A-model of quintic threefold V and the B-model of its mirror V° was predicted in the famous paper [CDGP91]. We recall two styles of the theorem (1) and (2) below. Every statement in the present paper is near the large radius point q_0 of the complexified Kähler moduli $\mathcal{KM}(V)$ and the maximally unipotent monodromy point p_0 of the complex moduli $\mathcal{M}(V^{\circ})$.

Let $t := y_1/y_0$, $u := t/2\pi i$ be the canonical parameters and $q := e^t = e^{2\pi i u}$ be the canonical coordinate from 2.2 below and the respective ones in 2.3 below.

The following theorem is due to Lian-Liu-Yau [LLuY97].

(1) (*Potential*). The potentials of the two models coincide: $\Phi^{V}_{\mathrm{GW}}(t) = \Phi^{V^{\circ}}_{\mathrm{GM}}(t)$.

The following theorem is formulated by Morrison [M97] and proved by Iritani [I11].
(2) (*Variation of Hodge structure*). The isomorphism $(q_0 \in \overline{\mathcal{KM}}(V)) \overset{\sim}{\leftarrow} (p_0 \in \overline{\mathcal{M}}(V^{\circ}))$ of neighborhoods of the compactifications, by the canonical coordinate $q = \exp(2\pi i u)$, lifts to an isomorphism, over the punctured neighborhoods $\mathcal{KM}(V) \overset{\sim}{\leftarrow} \mathcal{M}(V^{\circ})$, of polarized $\mathbf{Z}$-variations of Hodge structure with a specified section

$$(\mathcal{H}^V, S, \nabla^{\mathrm{even}}, \mathcal{H}^V_{\mathbf{Z}}, F; 1) \overset{\sim}{\leftarrow} (\mathcal{H}^{V^{\circ}}, Q, \nabla^{\mathrm{GM}}, \mathcal{H}^{V^{\circ}}_{\mathbf{Z}}, F; \tilde{\Omega}).$$

Our (3) below is equivalent to (1) and (2) by a log version [KU09, 2.5.14] of the nilpotent orbit theorem of Schmid [S73] (this part of [U14] is valid).

(3) (*Log Hodge structure, Log period map*). The isomorphism $(q_0 \in \overline{\mathcal{KM}}(V)) \overset{\sim}{\leftarrow} (p_0 \in \overline{\mathcal{M}}(V^{\circ}))$ of neighborhoods of the compactifications uniquely lifts to an isomorphism of B-model log variation of polarized Hodge structure with a specified section $\tilde{\Omega}$ for V° and A-model log variation of polarized Hodge structure with a specified section

1 for V, whose restriction over the punctured $\mathcal{KM}(V) \stackrel{\sim}{\leftarrow} \mathcal{M}(V^\circ)$ coincides with the isomorphism of variations of polarized Hodge structure with specified sections in (2).

This rephrases as follows. Let σ be the common monodromy cone, transformed by a level structure into End of a reference fiber of the local system, for the A-model and for the B-model. Then, we have a commutative diagram of horizontal log period maps

$$\begin{array}{ccc} (q_0 \in \overline{\mathcal{KM}}(V)) & \stackrel{\sim}{\leftarrow} & (p_0 \in \overline{\mathcal{M}}(V^\circ)) \\ & \searrow \quad \swarrow & \\ & ([\sigma, \exp(\sigma_{\mathbf{C}})F_0] \in \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma) & \end{array}$$

with extensions of specified sections in (2), where $(\sigma, \exp(\sigma_{\mathbf{C}})F_0)$ is the nilpotent orbit, regarded as a boundary point, and $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is the fine moduli of log Hodge structures of specified type. (For fine moduli $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$, or more generally $\Gamma \backslash D_\Sigma$, see [KU09].)

Open mirror symmetry for quintic threefolds

The following theorem is due to Walcher [W07] and Morrison-Walcher [MW09].

(4) (*Inhomogenous solutions*).
Let $\mathcal{L}$ be the Picard-Fuchs differential operator for quintic mirror (cf. 2.2). Let

$$\mathcal{T}_A = \frac{u}{2} \pm \Big(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} n_d q^{d/2}\Big)$$

be the A-model domainwall tension in [MW09], and

$$\mathcal{T}_B = \int_{C_-}^{C_+} \Omega$$

be the B-model domainwall tension, where $C_\pm \subset V^\circ$ are the disjoint smooth curves coming from the two conics in $\{x_1 + x_2 = x_3 + x_4 = 0\} \cap V_\psi \subset \mathbf{P}^4(\mathbf{C})$ [ibid].
Then

$$\mathcal{L}(y_0(z)\mathcal{T}_A(z)) = \mathcal{L}(\mathcal{T}_B(z))\Big(= \frac{15}{16\pi^2}\sqrt{z}\Big) \quad \Big(z = \frac{1}{(5\psi)^5}\Big).$$

Concerning this, we have the following observations.

(5) (*Log mixed Hodge structure, Log normal function*). We describe for B-model. The same holds for A-model by (1)–(3) and the correspondence table in 2.5 below.

Put $\mathcal{H} := \mathcal{H}^{V^\circ}$ and $\mathcal{T} := \mathcal{T}_B$. We use $e^0 \in I^{0,0}$, $e^1 \in I^{1,1}$ which are a part of a basis of $\mathcal{H}_{\mathcal{O}^{\log}}$ respecting the Deligne decomposition at p_0 (see 2.5 (3B)) and a flat sections $s^0 = e^0$, $s^1 = e^1 - ue^0$ (see 2.5 (5B)). To make the local monodromy of $\mathcal{T}$ unipotent, we take a double cover $z^{1/2} \mapsto z$. Let $L_{\mathbf{Q}}$ be the translated local system from the trivial extension $\mathbf{Q} \oplus \mathcal{H}_{\mathbf{Q}}$ by $-(\mathcal{T}/y_0)s^0$ in $\mathcal{E}xt^1(\mathbf{Q}, \mathcal{H}_{\mathbf{Q}})$. Let $J_{L_{\mathbf{Q}}}$ be the Néron model on a neighborhood S of p_0 in the $z^{1/2}$-plane which lies over $L_{\mathbf{Q}}$ in [KNU14]. Then,

$J_{L_{\mathbf{Q}}} = \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, \mathcal{H})$ (extension group of log mixed Hodge structures over S) in the present case ([KNU13, III, Corollary 6.1.6], cf. 1.4 below), and we have the following (5.1)–(5.3).

(5.1) The normalized tension $\mathcal{T}/y_0$ is understood as a truncated normal function by $(\mathcal{T}/y_0)s^0$. This extends as a truncated log normal function over the puncture. Then it lifts uniquely to a log normal function $S \to J_{L_{\mathbf{Q}}}$ so that the corresponding exact sequence $0 \to \mathcal{H} \to H \to \mathbf{Z} \to 0$ of log mixed Hodge structures over S is given by the liftings $1_{\mathbf{Z}}$ and 1_F in H of $1 \in \mathbf{Z} \simeq (\mathrm{gr}^W)_{\mathbf{Z}}$ respecting the lattice and the Hodge filtration, respectively, which are defined as follows: $1_{\mathbf{Z}} := 1 - (\mathcal{T}/y_0)s^0$ with $(\mathcal{T}/y_0)s^0 \in \mathcal{H}_{\mathcal{O}^{\log}} = (\mathrm{gr}^W_3)_{\mathcal{O}^{\log}}$, and $1_F - 1_{\mathbf{Z}} := -(\theta(\mathcal{T}/y_0))e^1 + (\mathcal{T}/y_0)e^0$.

(5.2) A splitting of the weight filtration W of the local system $H_{\mathbf{Z}}$, i.e., a splitting compatible with the monodromy of the local system $H_{\mathbf{Z}}$, is given by $1^{\mathrm{spl}}_{\mathbf{Z}} = 1_{\mathbf{Z}} + s^1/2$, and the log normal function over it is given by $1^{\mathrm{spl}}_F - 1^{\mathrm{spl}}_{\mathbf{Z}} = -(\theta(\mathcal{T}/y_0))e^1 + (\mathcal{T}/y_0)e^0$.

(5.3) (4) says that the inverse of the truncated normal function in (5.1) from its image is given by $16\pi^2/15$ times the Picard-Fuchs differential operator $\mathcal{L}$.

Some geometric backgrounds of (5) are explained in Section 3.

We treat Tate twists case by case in this article.

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1. Log mixed Hodge theory

In this section, we recall some notions and results of log mixed Hodge theory from [KU09], [KNU13] and [KNU14] adapting to the present context.

1.1. Category $\mathcal{B}(\log)$

Let S be a subset of an analytic space Z. The *strong topology of S in Z* is the strongest one among those topologies on S in which, for any analytic space A and any morphism $f : A \to Z$ with $f(A) \subset S$ as sets, $f : A \to S$ is continuous. S is regarded as a local ringed space by the pullback sheaf of $\mathcal{O}_Z$.

Let $\mathcal{B}$ be the category of local ringed spaces S over $\mathbf{C}$ which have an open covering $(U_\lambda)_\lambda$ satisfying the following condition: For each λ, there exist an analytic space Z_λ, and a subset S_λ of Z_λ such that, as local ringed space over $\mathbf{C}$, U_λ is isomorphic to S_λ which is endowed with the strong topology in Z_λ and the inverse image of $\mathcal{O}_{Z_\lambda}$.

A *log structure* on a local ringed space S is a sheaf of monoids M on S together with a homomorphisim $\alpha : M \to \mathcal{O}_S$ such that $\alpha^{-1}\mathcal{O}_S^\times \xrightarrow{\sim} \mathcal{O}_S^\times$. *fs* log structure means, locally on the underlying space, the log structure has a chart which is finitely generated, integral and saturated.

Let $\mathcal{B}(\log)$ be the category of objects of $\mathcal{B}$ endowed with an fs log structure (more precisely, cf. [KU09]).

1.2. Ringed space $(S^{\log}, \mathcal{O}_S^{\log})$

Let $S \in \mathcal{B}(\log)$. As a set define
$S^{\log} := \{(s,h) \mid s \in S,\ h : M_s^{\mathrm{gp}} \to \mathbf{S}^1 \text{ homomorphism s.t. } h(u) = u/|u| \ (u \in \mathcal{O}_{S,s}^\times)\}$.
Endow $S^{\log}$ with the weakest topology such that the following two maps are continuous.
(1) $\tau : S^{\log} \to S, (s,h) \mapsto s$.
(2) For any open set $U \subset S$ and any $f \in \Gamma(U, M^{\mathrm{gp}})$, $\tau^{-1}(U) \to \mathbf{S}^1$, $(s,h) \mapsto h(f_s)$.

Then, τ is proper and surjective with fiber $\tau^{-1}(s) = (\mathbf{S}^1)^{r(s)}$, where $r(s)$ is the rank of $(M^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ which varies with $s \in S$.

For $s \in S$ and $t \in S^{\log}$ lying over s, let $q_j \in M_s^{\mathrm{gp}}$ $(1 \le j \le r(s))$ be elements such that their images in $(M^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ form a basis. Let $t_j := \log(q_j)$ and define $\mathcal{O}_{S,t}^{\log}$ to be a polynomial ring $\mathcal{O}_{S,s}[t_j \ (1 \le j \le r(s)]$ over $\mathcal{O}_{S,s}$. Thus $\tau : (S^{\log}, \mathcal{O}_S^{\log}) \to (S, \mathcal{O}_S)$ is a morphism of ringed spaces over $\mathbf{C}$ (more precisely, cf. [KU09]).

1.3. Graded polarized log mixed Hodge structure

Let $S \in \mathcal{B}(\log)$. A *pre-graded polarized log mixed Hodge structure on S* is a tuple $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ consisting of a local system of $\mathbf{Z}$-free modules $H_{\mathbf{Z}}$ of finite rank on $S^{\log}$, an increasing filtration W of $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$, a nondegenerate $(-1)^w$-symmetric $\mathbf{Q}$-bilinear form $\langle\ ,\ \rangle_w$ on gr_w^W, a locally free $\mathcal{O}_S$-module $H_{\mathcal{O}}$ on S, a specified isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ (*log Riemann-Hilbert correspondence*), and a specified decreasing filtration $FH_{\mathcal{O}}$ of $H_{\mathcal{O}}$ such that $F^pH_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^pH_{\mathcal{O}}$ are locally free. Put $F^p := \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^pH_{\mathcal{O}}$. Then $\tau_*F^p = F^pH_{\mathcal{O}}$. For each integer w, the orthogonality condition $\langle F^p(\mathrm{gr}_w^W), F^q(\mathrm{gr}_w^W)\rangle_w = 0$ $(p+q > w)$ is imposed.

A pre-graded polarized log mixed Hodge structure on S is a *graded polarized log mixed Hodge structure on S* if its pullback to each $s \in S$ is a graded polarized log mixed Hodge structure on s in the following sense.

Let $(H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ be a pre-graded polarized log mixed Hodge structure on a log point s. It is a *graded polarized log mixed Hodge structure* if it satisfies the following three conditions.

(1) (Admissibility). For each logarithm N of the local monodromy of the local system $(H_{\mathbf{R}}, W, (\langle\ ,\ \rangle_w)_w)$, there exists a W-relative N-filtration $M(N,W)$.

(2) (Griffiths transversality). For any integer p, $\nabla F^p \subset \omega_s^{1,\log} \otimes F^{p-1}$ is satisfied, where $\omega_s^{1,\log}$ is the sheaf of $\mathcal{O}^{\log}$-module of log differential 1-forms on $(s^{\log}, \mathcal{O}_s^{\log})$, and $\nabla = d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_s^{\log} \otimes H_{\mathbf{Z}} \to \omega_s^{1,\log} \otimes H_{\mathbf{Z}}$ is the log Gauss-Manin connection.

(3) (Positivity). For a point $t \in s^{\log}$ and a $\mathbf{C}$-algebra homomorphism $a : \mathcal{O}_{s,t}^{\log} \to \mathbf{C}$, define a filtration $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ on $H_{\mathbf{C},t}$. Then, $(H_{\mathbf{Z},t}(\mathrm{gr}_w^W), \langle\ ,\ \rangle_w, F(a))$ is a polarized Hodge structure of weight w in the usual sense if a is sufficiently twisted, i.e., for $(q_j)_{1 \le j \le n} \subset M_s$ inducing generators of $M_s/\mathcal{O}_s^\times$, $|\exp(a(\log q_j))| \ll 1$ for any j.

1.4. Néron model for admissible normal function

We review some results from [KNU14, Theorem 1.3], [KNU13, III, Section 6.1] and [KNU10, Section 8] adapted to the situation (5) in Introduction.

For a pure case $h^{p,q} = 1$ ($p+q=3$, $p,q \geq 0$) and $h^{p,q} = 0$ otherwise, a complete fan is constructed in [KU09, Section 12.3]. For a mixed case $h^{p,q} = 1$ (the above (p,q), plus $(p,q) = (2,2)$) and $h^{p,q} = 0$ otherwise, over the above fan, a weak fan of Néron model for given admissible normal function is constructed in [KNU14, Theorem 3.1], and we have a Néron model in the following sense.

Let $S \in \mathcal{B}(\log)$, $U := S_{\mathrm{triv}} \subset S$ (consisting of those points with trivial log structure), $H_{(-1)}$ be a polarized variation of Hodge structure of weight -1 (Tate-twisted by 2 from $\mathcal{H}$ in Introduction (5)) on U and $L_{\mathbf{Q}}$ be a local system of $\mathbf{Q}$-vector spaces which is an extension of $\mathbf{Q}$ by $H_{(-1),\mathbf{Q}}$. An admissible normal function over U for $H_{(-1)}$ underlain by the local system $L_{\mathbf{Q}}$ can be regarded as an admissible variation of mixed Hodge structure which is an extension of $\mathbf{Z}$ by $H_{(-1)}$ and lies over local system $L_{\mathbf{Q}}$.

For any given unipotent admissible normal function over U as above, $H_{(-1)}$ and $L_{\mathbf{Q}}$ extend to a polarized log mixed Hodge structure on S and a local system on $S^{\log}$, respectively, denoted by the same symbols, and there is a relative log manifold $J_{L_{\mathbf{Q}}}$ over S (cf. [KU09]) which is strict over S (i.e., endowed with the pullback log structure from S) and which represents the following functor on $\mathcal{B}/S^{\circ}$ ($S^{\circ} \in \mathcal{B}$ is the underlying space of S):

$S' \mapsto \{$LMH H on S' satisfying $H(\mathrm{gr}^W_w) = H_{(w)}|_{S'}$ ($w = -1, 0$) and $(*)$ below$\}/$isom.

$(*)$ Locally on S', there is an isomorphism $H_{\mathbf{Q}} \simeq L_{\mathbf{Q}}$ on $(S')^{\log}$ preserving W.

Here $H_{(w)}|_{S'}$ is the pullback of $H_{(w)}$ by the structure morphism $S' \to S^{\circ}$, and S' is endowed with the pullback log structure from S.

Put $H' := H_{(-1)}$. In the present case, we have $J_{L_{\mathbf{Q}}} = \mathcal{E}\mathrm{xt}^1_{\mathrm{LMH}/S}(\mathbf{Z}, H')$ by [KNU13, Corollary 6.1.6]. This is the subgroup of $\tau_*(H'_{\mathcal{O}^{\log}}/(F^0+H'_{\mathbf{Z}}))$ restricted by admissibility condition and log-point-wise Griffiths transversality condition ([KNU10, Section 8], cf. 1.3). Define $\bar{J}_{L_{\mathbf{Q}}}$ as the image of the composite map $J_{L_{\mathbf{Q}}} \to \tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}})) \to \tau_*(H'_{\mathcal{O}^{\log}}/(F^{-1} + \mathcal{H}_{\mathbf{Z}})$. By using the polarization, we have a commutative diagram:

$$\begin{array}{ccccccc} J_{L_{\mathbf{Q}}} & = & \mathcal{E}\mathrm{xt}^1_{\mathrm{LMH}/S}(\mathbf{Z}, H') & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}})) & \xrightarrow[\sim]{\mathrm{pol}} & \tau_*((F^0)^*/H'_{\mathbf{Z}}) \\ \downarrow & & & & \downarrow & & \downarrow \\ \bar{J}_{L_{\mathbf{Q}}} & & & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/(F^{-1} + \mathcal{H}_{\mathbf{Z}}) & \xrightarrow[\sim]{\mathrm{pol}} & \tau_*((F^1)^*/H'_{\mathbf{Z}}). \end{array}$$

2. Quintic threefolds

In this section, we give a correspondence table of A-model for quintic threefold and B-model for its mirror. This is a correction of our previous [U14, 3] by using $\hat{\Gamma}$-integral structure of Iritani [I11].

2.1. Quintic threefold and its mirror

Let V be a general quintic threefold in $\mathbf{P}^4$.

Let $V_\psi : f := \frac{1}{5}\sum_{j=1}^5 x_j^5 - \psi\prod_{j=1}^5 x_j = 0$ ($\psi \in \mathbf{P}^1$) be a pencil of quintics in $\mathbf{P}^4$. Let μ_5 be the group consisting of the fifth roots of the unity in $\mathbf{C}$. Then the group $G := \{(a_j) \in (\mu_5)^5 \,|\, a_1 \dots a_5 = 1\}$ acts on V_ψ by $x_j \mapsto a_j x_j$. Let V_ψ° be a crepant resolution of quotient singularity of V_ψ/G (cf. [MW09]). Divide further by the action $(x_1, \dots, x_5) \mapsto (a^{-1}x_1, x_2, \dots, x_5)$ ($a \in \mu_5$).

2.2. Picard-Fuchs equation on the mirror V°

Let Ω be a 3-form on V_ψ° with a log pole over $\psi = \infty$ induced from

$$\left(\frac{5}{2\pi i}\right)^3 \mathrm{Res}_{V_\psi}\left(\frac{\psi}{f}\sum_{j=1}^{5}(-1)^{j-1}x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \wedge \cdots \wedge dx_5\right).$$

Let $z := 1/(5\psi)^5$ and $\theta := zd/dz$. Let

$$\mathcal{L} := \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)$$

be the Picard-Fuchs differential operator for Ω, i.e., $\mathcal{L}\Omega = 0$ via the Gauss-Manin connection ∇.

At $z = 0$, the Picard-Fuchs differential equation $\mathcal{L}y = 0$ has the indicial equation $\rho^4 = 0$ (ρ is indeterminate), i.e., maximally unipotent. By the Frobenius method, we have a basis of solutions $y_j(z)$ $(0 \le j \le 3)$ as follows. Let

$$\tilde{y}(-z;\rho) := \sum_{n=0}^{\infty}\frac{\prod_{m=1}^{5n}(5\rho+m)}{\prod_{m=1}^{n}(\rho+m)^5}(-z)^{n+\rho}$$

be a solution of $\mathcal{L}(\tilde{y}(-z;\rho)) = \rho^4(-z)^\rho$, and let

$$\tilde{y}(-z;\rho) = y_0(z) + y_1(z)\rho + y_2(z)\rho^2 + y_3(z)\rho^3 + \cdots, \quad y_j(z) := \frac{1}{j!}\frac{\partial^i \tilde{y}(-z;\rho)}{\partial\rho^j}|_{\rho=0}$$

be the Taylor expansion at $\rho = 0$. Then, y_j $(0 \le j \le 3)$ form a basis of solutions for the equation $\mathcal{L}y = 0$. We have

$$y_0 = \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n,$$

$$y_1 = y_0\log z + 5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\Big(\sum_{j=n+1}^{5n}\frac{1}{j}\Big)z^n.$$

Define the canonical parameters by $t := y_1/y_0$, $u := t/2\pi i$, and the canonical coordinate by $q := e^t = e^{2\pi i u}$ which is a specific chart of the log structure given by the divisor $(z = 0)$ of $\mathbf{P}^1$ and gives a mirror map.

y_0 is holomorphic in z and invertible at $z = 0$. Write $z = z(q)$ which is holomorphic in q. Then we have

$$\log z = 2\pi i u - \frac{5}{y_0(z(q))} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \Big(\sum_{j=n+1}^{5n} \frac{1}{j} \Big) z(q)^n.$$

The Gauss-Manin potential of V_z° is

$$\Phi_{\mathrm{GM}}^{V^\circ} = \frac{5}{2} \Big(\frac{y_1}{y_0} \frac{y_2}{y_0} - \frac{y_3}{y_0} \Big).$$

Let $\tilde{\Omega} := \Omega / y_0$. Then, the Yukawa coupling at $z = 0$ is

$$Y := - \int_{V^\circ} \tilde{\Omega} \wedge \nabla_\theta \nabla_\theta \nabla_\theta \tilde{\Omega} = \frac{5}{(1 + 5^5 z) y_0(z)^2}.$$

2.3. A-model of quintic V

Let V be a general quintic hypersurface in $\mathbf{P}^4$. Let $T^2 = H$ be the cohomology class of a hyperplane section of V in $\mathbf{P}^4$, $K(V) = \mathbf{R}_{>0} T^2$ be the Kähler cone of V, and u be the coordinate of $\mathbf{C}T^2$. Put $t := 2\pi i u$. A complexified Kähler moduli is defined as

$$\mathcal{KM}(V) := (H^2(V, \mathbf{R}) + i K(V)) / H^2(V, \mathbf{Z}) \xrightarrow{\sim} \Delta^*, \quad uT^2 \mapsto q := e^{2\pi i u}.$$

Let $C \in H_2(V, \mathbf{Z})$ be the homology class of a line on V, and $T^1 \in H^4(V, \mathbf{Z})$ be the cohomology class Poincaré duality isomorphic to C.

For $\beta = dC \in H_2(V, \mathbf{Z})$, define $q^\beta := q^{\int_\beta T^1} = q^d$. The Gromov-Witten potential of V is defined as

$$\Phi_{\mathrm{GW}}^V := \frac{1}{6} \int_V (tT^2)^3 + \sum_{0 \neq \beta \in H_2(V, \mathbf{Z})} N_d q^\beta = \frac{5t^3}{6} + \sum_{d>0} N_d q^d.$$

Here the Gromov-Witten invariant N_d is

$$\overline{M}_{0,0}(\mathbf{P}^4, d) \xleftarrow{\pi_1} \overline{M}_{0,1}(\mathbf{P}^4, d) \xrightarrow{e_1} \mathbf{P}^4,$$

$$N_d := \int_{\overline{M}_{0,0}(\mathbf{P}^4, d)} c_{5d+1}(\pi_{1*} e_1^* \mathcal{O}_{\mathbf{P}^4}(5)).$$

Note that $N_d = 0$ if $d \leq 0$. Let $N_d = \sum_{k|d} n_{d/k} k^{-3}$. Then $n_{d/k}$ is the instanton number.

2.4. Integral structure

Let S^* be $\mathcal{KM}(V)$ for A-model of V and $\mathcal{M}(V^\circ)$ for B-model for V°, and let S be $\overline{\mathcal{KM}}(V)$ for A-model and $\overline{\mathcal{M}}(V^\circ)$ for B-model (see 2.2, 2.3). Endow S with the log structure associated to the divisor $S \setminus S^*$.

The B-model variation of Hodge structure $\mathcal{H}^{V^\circ}$ is the usual variation of Hodge structure arising from the smooth projective family $f : X \to S^*$ of the quintic mirrors over a punctured neighborhood of the maximally unipotent monodromy point p_0. Its integral structure is the usual one $\mathcal{H}^{V^\circ}_{\mathbf{Z}} = R^3 f_* \mathbf{Z}$. This is compatible with the monodromy weight filtration M around p_0. Define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}^{V^\circ}_{\mathbf{Z}}$ for all k.

For the A-model $\mathcal{H}^V$ on S^*, the locally free sheaf on S^*, the Hodge filtration, and the monodromy weight filtration M around the large radius point q_0 are given by $\mathcal{H}^V_{\mathcal{O}} := \mathcal{O}_{S^*} \otimes (\bigoplus_{0\le p\le 3} H^{2p}(V))$, $F^p := \mathcal{O}_{S^*} \otimes H^{\le 2(3-p)}(V)$, and $M_{2p} := H^{\ge 2(3-p)}(V)$, respectively. Iritani defined $\hat{\Gamma}$-integral structure in more general setting in [I11, Definition 3.6]. In the present case, it is characterized as follows. Let H and C be a hyperplane section and a line on V, respectively. Then, in the present case, a basis of the $\hat{\Gamma}$-integral structure is given by $\{s(\mathcal{E}) \,|\, \mathcal{E} \text{ is } \mathcal{O}_V, \mathcal{O}_H, \mathcal{O}_C, \mathcal{O}_{\rm pt}\}$ [ibid, Example 6.18], where $s(\mathcal{E})$ is a unique $\nabla^{\rm even}$-flat section satisfying

$$s(\mathcal{E}) \sim (2\pi i)^{-3} e^{-2\pi i u H} \cdot \hat{\Gamma}(T_V) \cdot (2\pi i)^{\deg/2} \operatorname{ch}(\mathcal{E})$$

at the large radius point q_0. Here, for the Chern roots $c(T_V) = \prod_{j=1}^3 (1+\delta_j)$, the Gamma class $\hat{\Gamma}(T_V)$ is defined by

$$\begin{aligned}\hat{\Gamma}(T_V) := \prod_{j=1}^{3} \Gamma(1+\delta_j) &= \exp(-\gamma c_1(V) + \sum_{k\ge 2} (-1)^k (k-1)! \zeta(k) \operatorname{ch}_k(T_V) \\ &= \exp(\zeta(2) \operatorname{ch}_2(T_V) - 2\zeta(3) \operatorname{ch}_3(T_V))\end{aligned}$$

where γ is the Euler constant, and $\deg|_{H^{2p}(V)} := 2p$. The important point is that this class $\hat{\Gamma}(T_V)$ plays the role of a "square root" of the Todd class in Hirzebruch-Riemann-Roch ([I09, 1], [I11, 1, (13)]). Denote this $\hat{\Gamma}$-integral structure by $\mathcal{H}^V_{\mathbf{Z}}$. This is compatible with the monodromy weight filtration M and we define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}^V_{\mathbf{Z}}$ for all k. For a direct definition of $\hat{\Gamma}$-integral structure, see [I11, Definition 3.6].

In both A-model case and B-model case, the integral structures $\mathcal{H}^V_{\mathbf{Z}}$ and $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$ on S^* extend to the local systems of $\mathbf{Z}$-modules over $S^{\log}$ ([O03], [KU09, Proposition 2.3.5]), still denoted $\mathcal{H}^V_{\mathbf{Z}}$ and $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$, respectively.

Consider a diagram:

$$\begin{array}{ccc} \tilde{S}^{\log} := (\mathbf{R} \times i(0,\infty])^r & \supset & \tilde{S}^* := (\mathbf{R} \times i(0,\infty))^r \\ \downarrow & & \downarrow \\ S^{\log} & \supset & S^* \\ \tau\downarrow & & \\ S & & \end{array}$$

The coordinate u of $\tilde{S}^*$ extends over $\tilde{S}^{\log}$. Fix base points as $u_0 = 0 + i\infty \in \tilde{S}^{\log} \mapsto b := \bar{0} + i\infty \in S^{\log} \mapsto q = 0 \in S$, where $q = 0$ corresponds to q_0 for A-model and p_0

for B-model. Note that fixing a base point $u = u_0$ on $\tilde{S}^{\log}$ is equivalent to fixing a base point b on $S^{\log}$ and also a branch of $(2\pi i)^{-1} \log q$.

Let $B := \mathcal{H}^V_{\mathbf{Z}}(u_0) = \mathcal{H}^V_{\mathbf{Z}}(b)$ for A-model and $B := \mathcal{H}^{V^\circ}_{\mathbf{Z}}(u_0) = \mathcal{H}^{V^\circ}_{\mathbf{Z}}(b)$ for B-model.

2.5. Correspondence table

In this section, we complete the approximation in the previous paper [U14]. These results will be used in Section 3.

We use (1) and (2) in Introduction. Put $\Phi := \Phi^V_{\mathrm{GW}} = \Phi^{V^\circ}_{\mathrm{GM}}$.

(1A) *Polarization of A-model of V.*

$$S(\alpha, \beta) := (-1)^p \int_V \alpha \cup \beta \quad (\alpha \in H^{p,p}(V), \beta \in H^{3-p,3-p}(V)).$$

(1B) *Polarization of B-model of V°.*

$$Q(\alpha, \beta) := (-1)^{3(3-1)/2} \int_{V^\circ} \alpha \cup \beta = - \int_{V^\circ} \alpha \cup \beta \quad (\alpha, \beta \in H^3(V^\circ)).$$

(2A) **Z**-*basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}^V_{\mathbf{Z}}(u_0) = \mathcal{H}^V_{\mathbf{Z}}(b)$. Then we have a basis b^0, b^1, b^2, b^3 of B compatible with the monodromy weight filtration M [I11, Example 6.18].

(2B) **Z**-*basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}^{V^\circ}_{\mathbf{Z}}(u_0) = \mathcal{H}^{V^\circ}_{\mathbf{Z}}(b)$. Then we have a basis b^0, b^1, b^2, b^3 of B compatible with the monodromy weight filtration M [ibid].

For both cases (2A) and (2B), we regard B as a constant sheaf endowed with M on $S^{\log}$ and also on S.

(3A) *Specified sections inducing* **Z**-*basis of* gr^M *for A-model of V.*

$$T^3 := 1 \in H^0(V, \mathbf{Z}), \quad T^2 := H \in H^2(V, \mathbf{Z}),$$
$$T^1 := C \in H^4(V, \mathbf{Z}), \quad T^0 := [\mathrm{pt}] \in H^6(V, \mathbf{Z}),$$

where H is a hyperplane section of V and C is a line on V. Then $S(T^3, T^0) = 1$ and $S(T^2, T^1) = -1$. Hence T^3, T^2, $-T^0$, T^1 form a symplectic base for S in (1A).

(3B) *Specified sections inducing* **Z**-*basis of* gr^M *for B-model of V°.*

We use Deligne decomposition [D97]. We consider B in (2B) as a constant sheaf on $S^{\log}$. We have locally free $\mathcal{O}_S$-submodules $\mathcal{M}_{2p} := \tau_*(\mathcal{O}^{\log}_S \otimes_{\mathbf{Z}} M_{2p}B)$ and $\mathcal{F}^p$ of $\tau_*(\mathcal{O}^{\log}_S \otimes_{\mathbf{Z}} B) = \mathcal{O}_S \otimes_{\mathbf{Z}} B$. The mixed Hodge structure of Hodge-Tate type $(\mathcal{M}, \mathcal{F})$ has decomposition:

$$\mathcal{O}_S \otimes_{\mathbf{Z}} B = \bigoplus_p I^{p,p}, \qquad I^{p,p} := \mathcal{M}_{2p} \cap \mathcal{F}^p \xrightarrow{\sim} \mathrm{gr}^{\mathcal{M}}_{2p}.$$

Transporting the basis b^p $(0 \le p \le 3)$ of B in (2B), regarded as sections of the constant sheaf B on $S^{\log}$, via isomorphism

$$I^{p,p} \xrightarrow{\sim} \mathcal{O}_S \otimes_{\mathbf{Z}} \mathrm{gr}^M_{2p} B$$

we define sections $e^p \in I^{p,p}$ $(0 \le p \le 3)$. Then e^3, e^2,$-e^0$, e^1 form a symplectic basis for Q in (1B).

Note that $e^3 = \tilde{\Omega}$.

(4A) *A-model connection* $\nabla = \nabla^{\mathrm{even}}$ *of* V.

$$\nabla_\theta T^0 := 0, \quad \nabla_\theta T^1 := T^0,$$
$$\nabla_\theta T^2 := \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}T^1 = \Big(5 + \frac{1}{(2\pi i)^3}\frac{d^3\Phi_{\mathrm{hol}}}{du^3}\Big)T^1,$$
$$\nabla_\theta T^3 := T^2.$$

∇ is flat, i.e., $\nabla^2 = 0$.

(4B) *B-model connection* $\nabla = \nabla^{\mathrm{GM}}$ *of* V°.

$$\nabla_\theta e^0 = 0, \quad \nabla_\theta e^1 = e^0,$$
$$\nabla_\theta e^2 = \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}e^1 = Ye^1 = \frac{5}{(1+5^5)y_0(z)^2}\Big(\frac{q}{z}\frac{dz}{dq}\Big)^3 e^1,$$
$$\nabla_\theta e^3 = e^2.$$

(5A) ∇*-flat* $\mathbf{Z}$*-basis for* $\mathcal{H}^V_{\mathbf{Z}}$.

$$s^0 := T^0,$$
$$s^1 := T^1 - uT^0,$$
$$s^2 := T^2 - \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)T^1 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u - \frac{25}{12}\Big)T^0,$$
$$s^3 := T^3 - uT^2 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) - \frac{25}{12}\Big)T^1$$
$$- \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) - \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)T^0.$$

Then s^3, s^2,$-s^0$, s^1 form a symplectic basis for S in (1A).

(5B) ∇*-flat* $\mathbf{Z}$*-basis for* $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$.

$$s^0 := e^0,$$
$$s^1 := e^1 - ue^0,$$
$$s^2 := e^2 - \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)e^1 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u - \frac{25}{12}\Big)e^0,$$
$$s^3 := e^3 - ue^2 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) - \frac{25}{12}\Big)e^1$$
$$- \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) - \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)e^0.$$

Then s^3, s^2,$-s^0$, s^1 form a symplectic basis for Q in (1B).

(6A) *Expression of the T^p by the s^p.*
It is computed that T^p are written by the ∇-flat $\mathbf{Z}$-basis s^p of $\mathcal{H}_{\mathbf{Z}}^V$ as follows.

$$\begin{aligned}
T^0 &= s^0,\\
T^1 &= s^1 + us^0,\\
T^2 &:= s^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) + \frac{25}{12}\Big)s^0,\\
T^3 &= s^3 + us^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u + \frac{25}{12}\Big)s^1\\
&\quad + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) + \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)s^0.
\end{aligned}$$

Note that the section $1 = T^3$ varies with respect to the the lattice $\mathcal{H}_{\mathbf{Z}}^V$ as above while the section $[\mathrm{pt}] = T^0 = s^0$ does not.

(6B) *Expression of the e^p by the s^p.*
It is computed that e^p are written by the ∇-flat $\mathbf{Z}$-basis s^p of $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ as follows.

$$\begin{aligned}
e^0 &= s^0,\\
e^1 &= s^1 + us^0,\\
e^2 &:= s^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) + \frac{25}{12}\Big)s^0,\\
e^3 &= s^3 + us^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u + \frac{25}{12}\Big)s^1\\
&\quad + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) + \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)s^0.
\end{aligned}$$

Note that the normalized holomorphic 3-form $\tilde{\Omega} = \Omega/y_0 = e^3$ varies with respect to the lattice $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ as above, while the section $e^0 = s^0$ does not.

Idea of proof of (4A) and (4B). We prove (4B). (4A) follows by mirror symmetry theorems (1) and (2) in Introduction.

We improve the proof of [CoK99, Prop. 5.6.1] carefully by a log Hodge theoretic understanding of the relation among a constant sheaf and a local system on $S^{\log}$, of the canonical extension of Deligne on S, and of the Deligne decomposition.

Idea of proofs of (5A), (5B), (6A) and (6B). In [I11, Introduction] (cf. 2.4), the asymptotic condition in the large radius limit is stated for the flat integral section corresponding to $\mathcal{E} = \mathcal{O}_V \in K(V)$ in the situation (5A). Up to Tate twists, this condition coincides with the one in [CDGP91, (5.5)] stated in the situation (6A). By the mirror symmetry in [I11] (cf. (2) in Introduction), this condition is interpreted in the situation (6B). Our previous results in [U14, Sections 3.5–3.6] are insufficient (see Remark below). In order to complete them, we compute here higher approximations in the situation (6B). The result in the situation (5B) is a linear algebraic solution of this.

Remark. The author was pointed out by Hiroshi Iritani that the definitions and the descriptions of integral structures in [U14, 3.5, 3.6] are insufficient. Actually, they were the first approximations of integral structures by means of gr^M, and the second proof in [ibid, 3.9] works well even in this approximation.

3. Discussions on geometries for (5) in Introduction

We discuss here the relation with geometries and local systems considered in [W07] and [MW09]. Forgetting Hodge structures, we consider only local systems corresponding to the monodromy of integral periods and tensions.

Let V_ψ and V_ψ° be a quintic threefold and its mirror from 2.1. Let S be a small neighborhood in the z-plane (z in 2.2) of the maximal unipotent monodromy point p_0 endowed with the log structure associated to the divisor p_0.

We first consider B-model. Let the setting be as in [MW09, 4]. For $z \neq 0$ near 0, i.e., near p_0, let V_z° be the mirror quintic and $C_{+,z} \cup C_{-,z}$ be the disjoint union of smooth rational curves on V_z° coming from the two conics contained in $V_\psi \cap \{x_1 + x_2 = x_3 + x_4 = 0\} \subset \mathbf{P}^4(\mathbf{C})$. From the relative homology sequence for $(V_z^\circ, (C_{+,z} \cup C_{-,z}))$, we have

$$(1) \qquad 0 \to H_3(V_z^\circ; \mathbf{Z}) \to H_3(V_z^\circ, (C_{+,z} \cup C_{-,z}); \mathbf{Z}) \xrightarrow{\partial} \mathbf{Z}([C_{+,z}] - [C_{-,z}]) \to 0,$$

where $\mathbf{Z}([C_{+,z}] - [C_{-,z}])$ is $\mathrm{Ker}(H_2(C_{+,z} \cup C_{-,z}); \mathbf{Z}) \to H_2(V_z^\circ); \mathbf{Z})$. The monodromy T_∞ around p_0 interchanges $C_{+,z}$ and $C_{-,z}$.

Respecting the sequence (1), we take a family of cycles Poincaré duality isomorphic to the flat integral basis s^p ($0 \leq p \leq 3$) in 2.5 (5B) and a family of chains joining from $C_{-,z}$ to $C_{+,z}$ (a choice up to integral cycles and up to half twists), and over them integrate the family of 3-forms $\Omega(z)$ with log pole over $z = 0$ (z in the punctured disc in the z-plan) in 2.2, then we have a family of vectors $(\eta_0, \eta_1, \eta_2, \eta_3, \mathcal{T})$ consisting of periods and a tension. This corresponds to the data in [W07], [MW09]. Since $T_\infty(\mathcal{T}) = -(\mathcal{T} + \eta_1 + \eta_0)$ by [W07, (3.14)], we find $\mathcal{T} + \frac{1}{2}\eta_1 + \frac{1}{4}\eta_0 = \frac{15}{\pi^2}\tau$ is an eigenvector of the monodromy T_∞ with eigenvalue -1.

The family of sequences (1) ($z \neq 0$) forms an exact sequence of local systems of $\mathbf{Z}$-modules. To make the monodromy of this system unipotent, we take a double cover $z^{1/2} \mapsto z$. Let S be a neighborhood disc of p_0 in the $z^{1/2}$-plane endowed with log structure associated to the divisor p_0 in S, and let $S^{\log}$ be as in 1.2. Let S^* be the punctured disc $S \smallsetminus \{p_0\}$. Pull back the above local system to S^* and then extend it over $S^{\log}$.

Applying Tate twist (-3) and Poincaré duality isomorphism to the left and the right ends of this exact sequence, we have a local system L' over $S^{\log}$ which is an extension of $\mathbf{Z}(-2)$ by $\mathcal{H}_\mathbf{Z}$:

$$(2) \qquad 0 \to \mathcal{H}_\mathbf{Z} \to L' \to \mathbf{Z}(-2) \to 0.$$

Let $1 \in \mathbf{Z} \simeq \mathrm{gr}_4^W \mathbf{Z}(-2)$, take a lifting $1_{\mathbf{Z}} := 1 - (\mathcal{T}/\eta_0)s^0$ in L' of 1, and extend ∇ on $\mathcal{H}_{\mathbf{Z}}$ over L' by $\nabla(1_{\mathbf{Z}}) = 0$. We look for a T_∞^2-invariant ∇-flat element associated to $1_{\mathbf{Z}}$. This is computed as $1_{\mathbf{Z}}^{\mathrm{spl}} := 1_{\mathbf{Z}} - (s^1/2)$, and we know that L' coincides with $H_{\mathbf{Z}}$ in (5) in Introduction.

For the relative monodromy weight filtration $M = M(N, W)$, we see that $1_{\mathbf{Z}} \in M_4$ and $s^1 \in M_2$ are the smallest filters containing the elements in question. Taking the graded quotients by M of the sequence (2), we have

$$\begin{aligned}
&\mathrm{gr}_6^M \mathcal{H}_{\mathbf{Z}} \xrightarrow{\sim} \mathrm{gr}_6^M L', \\
0 \to &\mathrm{gr}_4^M \mathcal{H}_{\mathbf{Z}} \to \mathrm{gr}_4^M L' \to \mathbf{Z}(-2) \to 0, \\
0 \to &\mathrm{gr}_2^M \mathcal{H}_{\mathbf{Z}} \to \mathrm{gr}_2^M L' \to (2\text{-torsion}) \to 0, \\
&\mathrm{gr}_0^M \mathcal{H}_{\mathbf{Z}} \xrightarrow{\sim} \mathrm{gr}_0^M L'.
\end{aligned} \tag{3}$$

The 2-torsion in the third sequence of (3) corresponds to a half twist of chains from C_- to C_+. Standing on a half integral point and looking at the integral points nearby, we have two orientations. These correspond to the two orientations of a half twist of the chains, and also correspond to $\mathcal{T}_\pm := \pm(\frac{15}{\pi^2}\tau - \frac{\eta_0}{4}) - \frac{\eta_1}{2}$ in [W07]. $\mathcal{T}_-$ is different from $-\mathcal{T}_+$ by the complementary half twist, i.e., $\mathcal{T}_+ + \mathcal{T}_- = -\eta_1$.

For A-model, we consider the setting in [W07, 2.1]. Let $V = V_\psi$ with $\psi = 0$ from 2.1 be a Fermat quintic threefold in $\mathbf{P}^4(\mathbf{C})$ and $Lg := V \cap \mathbf{P}^4(\mathbf{R})$ be a Lagrangian submanifold of its real locus. From the exact sequence of relative homology for (V, Lg), we have

$$\begin{aligned}
&H_6(V;\mathbf{Z}) \xrightarrow{\sim} H_6(V, Lg;\mathbf{Z}), \\
0 \to &H_4(V;\mathbf{Z}) \to H_4(V, Lg;\mathbf{Z}) \to H_3(Lg;\mathbf{Z}) \to 0, \\
0 \to &H_2(V;\mathbf{Z}) \to H_2(V, Lg;\mathbf{Z}) \to H_1(Lg;\mathbf{Z}) \to 0, \\
&H_0(V;\mathbf{Z}) \xrightarrow{\sim} H_0(V, Lg;\mathbf{Z}).
\end{aligned} \tag{4}$$

Let $H' = H_\bullet(V)$, $H = H_\bullet(V, Lg)$ and $H'' = H_\bullet(Lg)$, and let

$$H_{\mathrm{even}}(V) := \bigoplus_{0 \le p \le 3} (H')_{2p}, \quad H_{\mathrm{even}}(V, Lg) := \bigoplus_{0 \le p \le 3} H_{2p}, \quad H_{\mathrm{odd}}(Lg) := \bigoplus_{0 \le p \le 1} (H'')_{2p+1}.$$

Then we have an exact sequence

$$0 \to H_{\mathrm{even}}(V) \to H_{\mathrm{even}}(V, Lg) \to H_{\mathrm{odd}}(Lg) \to 0. \tag{5}$$

The weight filtration W is given by $W_3 H_{\mathrm{even}}(V, Lg) := H_{\mathrm{even}}(V)$, $W_4 H_{\mathrm{even}}(V, Lg) := H_{\mathrm{even}}(V, Lg)$, and the relative monodromy weight filtration $M = M(N, W)$ is given by $M_{2p} H_{\mathrm{even}}(V, Lg) = H_{\le 2p}(V, Lg)$ $(0 \le p \le 3)$.

In the above setting, the projection from $\mathbf{P}^4(\mathbf{R})$ to the real hyperplane $\{x_5 = 0\} = \mathbf{P}^3(\mathbf{R})$ with center $(0,0,0,0,1)$ induces a homeomorphism $Lg \simeq \mathbf{P}^3(\mathbf{R})$. Therefore there are two choices of flat $U(1)$ connections on Lg. Denote Lg endowed with these structures by $Lg_\pm$. Morrison-Walcher [MW09, 3] explain the relation between $Lg_\pm$ for A-model of V and $C_\pm$ for B-model of V°.

After pulling back to the double cover $z^{1/2} \mapsto z$ ($z \neq 0$) and extending over $S^{\log}$, the sequence for A-model (5) and the sequence for B-model (2), and the set of sequences for A-model (4) and the set of sequences for B-model (3), respectively, seem to correspond in mirror symmetry. By Poincaré duality isomorphisms, $H^{\text{even}}(V) = H_{\text{even}}(V)(-3)$ and $H^{\text{even}}(Lg) \simeq H_{\text{odd}}(Lg)$.

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