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A NUMERICAL CRITERION FOR ADMISSIBILITY OF SEMI-SIMPLE ELEMENTS

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Abstract. In this article, we shall generalize a theorem of Cattani and Kaplan on horizontal representations of $SL(2)$. Their theorem plays an important role in the construction of their partial compactifications of the classifying spaces D modulo an arithmetic subgroup of Hodge structures of weight 2.

Introduction. A horizontal SL_2-representation is a generalization of the notion of "(H_1)-homomorphism" of SL_2 in the case of the classical theory of Hermitian symmetric domains (cf., e.g., [Sa, III]). More precisely, let $G=G_{\boldsymbol{R}}:=\mathrm{Aut}(H_{\boldsymbol{R}}, S)$ be the automorphism group of the classifying space D of Hodge structures of weight w (see §1). A representation $\rho: SL_2(\boldsymbol{R})\to G$ is said to be *horizontal* at $r\in D$ if the morphism $\rho_*: \mathfrak{sl}_2(\boldsymbol{R})\to\mathfrak{g}$ of the Lie algebras is a morphism of Hodge structures of type (0, 0) with respect to the Hodge structures on $\mathfrak{sl}_2(\boldsymbol{C})$ and $\mathfrak{g}_{\boldsymbol{C}}$ induced by $i\in U:=$(upper-half plane) and $r\in D$ respectively (see Definition (2.1)). In this case, the pair (ρ, r) is uniquely determined by the pair $(Y, r)\in\mathfrak{g}\times D$ with

$$Y:=\rho_*\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{0.1}$$

Conversely, a pair $(Y, r)\in\mathfrak{g}\times D$ is said to be *admissible* if there exists a representation $\rho: SL_2(\boldsymbol{R})\to G$ horizontal at r and satisfying (0.1). The main result in the present article is a numerical criterion for admissibility of a pair (Y, r) in the case of general weight.

Given a pair (ρ, r) as above, one can refine the Hodge decomposition $H_{\boldsymbol{C}}=\bigoplus H_r^{p,q}$, corresponding to $r\in D$, under the horizontal action of $\mathfrak{sl}_2(\boldsymbol{C})$ at r, called a *Hodge-$(Z, X_\pm)$ decomposition* (see (2.7)). Our proof of the main result is based on an elementary but useful observation (Corollary (2.11), see also Remark (2.12)), which says that the transformation of the Hodge-$(Z, X_\pm)$ decomposition by the inverse c^{-1} of the Cayley element

$$c:=\rho\left(\exp\frac{\pi i}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$$

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yields a split mixed Hodge structure, called a *mixed Hodge-*$(Y, N_{\pm})$ *decomposition*, which is nothing but the limiting mixed Hodge structure of the associated SL_2-orbit $\tilde{\rho}: U \to D$ defined by $\tilde{\rho}(gi) := \rho(g)r$ for $g \in SL_2(\boldsymbol{R})$ (cf. [Sc, Theorem (6.16)] and its proof). By virtue of this observation, we can view the relationship between the pairs (ρ, r) and (Y, r) from a better perspective, and generalize a numerical criterion [CK, Theorem (2.22)] for admissiblity of (Y, r) in the case of weight 2 to the case of general weight.

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1. Preliminaries. We recall first the definition of a (polarized) Hodge structure of weight w. Fix a free $\boldsymbol{Z}$-module $H_{\boldsymbol{Z}}$ of finite rank. Set $H_{\boldsymbol{Q}} := \boldsymbol{Q} \otimes H_{\boldsymbol{Z}}$, $H = H_{\boldsymbol{R}} := \boldsymbol{R} \otimes H_{\boldsymbol{Z}}$ and $H_{\boldsymbol{C}} := \boldsymbol{C} \otimes H_{\boldsymbol{Z}}$, whose complex conjugation is denoted by σ. Let w be an integer. A *Hodge structure of weight* w on $H_{\boldsymbol{C}}$ is a decomposition

$$(1.1) \qquad H_{\boldsymbol{C}} = \bigoplus_{p+q=w} H^{p,q} \quad \text{with} \quad \sigma H^{p,q} = H^{q,p} .$$

The integers

$$(1.2) \qquad h^{p,q} := \dim H^{p,q}$$

are called the Hodge numbers.

A polarization S for a Hodge structure (1.1) of weight w is a non-degenerate bilinear form on $H_{\boldsymbol{Q}}$, symmetric if w is even and skew-symmetric if w is odd, such that its $\boldsymbol{C}$-bilinear extension, denoted also by S, satisfies

$$(1.3) \qquad \begin{aligned} &S(H^{p,q}, \sigma H^{p',q'}) = 0 \quad \text{unless} \quad (p, q) = (p', q') , \\ &i^{p-q} S(v, \sigma v) > 0 \qquad \text{for all} \quad 0 \neq v \in H^{p,q} . \end{aligned}$$

REMARK (1.4). In the geometric case, i.e., the Hodge structure on the w-th cohomology group $H^w(X, \boldsymbol{Q})$ of a smooth projective variety $X \subset \boldsymbol{P}^N$ of dimension d over $\boldsymbol{C}$, we take as a polarization

$$S(u, v) := (-1)^{w(w-1)/2} \int_X u \wedge v \wedge \eta^{d-w}$$

for primitive classes $u, v \in H^w_{\text{prim}}(X, \boldsymbol{C}) \simeq H^w_{\text{prim}}(X, \Omega^{\bullet}_X)$ where $\eta \in H^1(X, \Omega^1_X)$ is the cohomology class of a hyperplane section of X.

For fixed S and $\{h^{p,q}\}$, the classifying space D for Hodge structures and its "compact dual" $\check{D}$ are defined by

$$\begin{aligned}(1.5)\quad &\check{D} := \{\{H^{p,q}\} \mid \text{Hodge structure on } H_{\boldsymbol{C}} \text{ with } \dim H^{p,q} = h^{p,q}, \\ &\qquad\qquad \text{satisfying the first condition in (1.3)}\}, \\ &D := \{\{H^{p,q}\} \in \check{D} \mid \text{satisfying also the second condition in (1.3)}\}.\end{aligned}$$

These are homogeneous spaces under the natural actions of the groups

$$(1.6)\qquad G_{\boldsymbol{C}} := \mathrm{Aut}(H_{\boldsymbol{C}}, S), \qquad G = G_{\boldsymbol{R}} := \{g \in G_{\boldsymbol{C}} \mid gH_{\boldsymbol{R}} = H_{\boldsymbol{R}}\},$$

respectively. Taking a reference point $r \in D$, one obtains identifications

$$(1.7)\qquad \check{D} \simeq G_{\boldsymbol{C}}/B_{\boldsymbol{C}}, \qquad D \simeq G/V,$$

where $B_{\boldsymbol{C}}$ and V are the isotropy subgroups of $G_{\boldsymbol{C}}$ and of G at $r \in D$, respectively. It is a direct consequence of the definition that

$$(1.8)\quad G \simeq \begin{cases} O(2h, k), \\ \mathrm{Sp}(2h, \boldsymbol{R}), \end{cases} \quad V \simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if } w = 2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w = 2t+1, \end{cases}$$

where $k := \sum_{|j| \le [t/2]} h^{t+2j,t-2j}$ and $h := (\dim H - k)/2$ if $w = 2t$, and $h := \dim H/2$ if $w = 2t+1$. It is an important observation that V is compact, but not maximal compact in general. Hence D is a symmetric domain of Hermitian type if and only if

$$(1.9)\qquad h^{p,q} = 0 \quad \text{unless } (p, q) = \begin{cases} (t+1, t-1), \ (t, t) \ \text{ or } \ (t-1, t+1), \\ \qquad\qquad \text{and } \ h^{t+1,t-1} = 1 \quad \text{if } w = 2t, \\ (t+1, t) \ \text{ or } \ (t, t+1) \quad \text{if } w = 2t+1. \end{cases}$$

A reference Hodge structure $r = \{H_r^{p,q}\} \in D$ induces a Hodge structure of weight 0 on the Lie algebra $\mathfrak{g}_{\boldsymbol{C}} := \mathrm{Lie}\, G_{\boldsymbol{C}}$ by

$$(1.10)\qquad \mathfrak{g}_{\boldsymbol{C}}^{s,-s} := \{X \in \mathfrak{g}_{\boldsymbol{C}} \mid XH_r^{p,q} \subset H_r^{p+s,q-s} \text{ for all } p, q\}.$$

One can define the associated Cartan involution θ_r on $\mathfrak{g}_{\boldsymbol{C}}$ by

$$(1.11)\qquad \theta_r(X) := \sum_s (-1)^s X^{s,-s} \qquad \text{for } \ X = \sum_s X^{s,-s} \in \mathfrak{g}_{\boldsymbol{C}} = \bigoplus_s \mathfrak{g}_{\boldsymbol{C}}^{s,-s}.$$

This can be interpreted in the following way: Set

$$(1.12)\qquad \begin{aligned} H_r^+ &:= H_r^{w,0} \oplus H_r^{w-2,2} \oplus H_r^{w-4,4} \oplus \cdots, \\ H_r^- &:= H_r^{w-1,1} \oplus H_r^{w-3,3} \oplus H_r^{w-5,5} \oplus \cdots. \end{aligned}$$

It is clear by definition that the isotropy subgroup of the decomposition $H_{\boldsymbol{C}} = H_r^+ \oplus H_r^-$ induces the maximal compact subgroup

$$(1.13)\qquad K \simeq \begin{cases} O(2h) \times O(k) & \text{if } w = 2t, \\ U(h) & \text{if } w = 2t+1, \end{cases}$$

of G which contains V, and the Cartan involution θ_r in (1.11) is the one associated to

K. Define a C-linear automorphism

$$E_r\colon H_C \to H_C \quad \text{by} \quad E_r := \begin{cases} 1 & \text{on } H_r^+, \\ -1 & \text{on } H_r^-. \end{cases} \tag{1.14}$$

Then the Cartan involution θ_r in (1.11) can also be written as

$$\theta_r X = (\mathrm{Ad}\, E_r) X \qquad \text{for} \quad X \in \mathfrak{g}_C. \tag{1.15}$$

We recall now well-known results on SL_2-representations. Let ξ, η be two variables, and write

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \begin{pmatrix} \xi^m \\ \xi^{m-1}\eta \\ \vdots \\ \eta^m \end{pmatrix} \qquad (m = 0, 1, 2, \cdots). \tag{1.16}$$

A representation

$$\rho_m\colon SL_2(\boldsymbol{R}) \to SL_{m+1}(\boldsymbol{R}) \quad \text{defined by} \quad \rho_m(g)\begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \left(g\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)^{(m)} \tag{1.17}$$

is called a symmetric tensor representation of dimension $m+1$. It is known that the ρ_m $(m=0, 1, 2, \cdots)$ are absolutely irreducible and constitute a full set of representatives for the equivalence classes of finite dimensional irreducible representations of $SL_2(\boldsymbol{R})$.

We take the standard generators for the Lie algebras $\mathfrak{sl}_2(\boldsymbol{R})$ and $\mathfrak{su}(1, 1)$ which are related by the Cayley transformation $\mathrm{Ad}\, c_1$, where

$$c_1 := \exp \frac{\pi i}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{1.18}$$

as follows:

$$\begin{array}{llll} \mathfrak{sl}_2(\boldsymbol{R}) & \ni y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & n_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & n_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \mathrm{Ad}\, c_1 \downarrow & \quad\downarrow & \quad\downarrow & \quad\downarrow \\ \mathfrak{su}(1, 1) & \ni z := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & x_+ := \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, & x_- := \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \end{array} \tag{1.19}$$

The following lemma can be verified directly by using the monomial basis (1.16) and the definition (1.19), and so we omit the proof.

LEMMA (1.20). (i) *In the above notation,* $Y_m := \rho_{m*}(y)$ *and* $N_{m\pm} := \rho_{m*}(n_\pm)$ *satisfy*
$Y_m(\xi^{m-j}\eta^j) = (m-2j)\xi^{m-j}\eta^j$,
$N_{m+}(\xi^{m-j}\eta^j) = (m-j)\xi^{m-j-1}\eta^{j+1}$,
$N_{m-}(\xi^{m-j}\eta^j) = j\xi^{m-j+1}\eta^{j-1}$.

(ii) *For the Cayley element* $c_m := \rho_m(c_1) \in SL_{m+1}(\boldsymbol{C})$,
$\sigma c_m = c_m^{-1}\sigma$, *where* σ *is the complex conjugation.*
$c_m^{\pm 2}(\xi^{m-j}\eta^j) = (\pm i)^m \eta^{m-j}\xi^j$,
$c_m^4(\xi^{m-j}\eta^j) = (-1)^m \xi^{m-j}\eta^j$.

REMARK (1.21). The Hodge structure on $\mathfrak{g}_{1C} := \mathfrak{sl}_2(\boldsymbol{C})$ induced by $i \in U :=$ (upper-half plane) $\simeq SL_2(\boldsymbol{R})/U(1)$ coincides with the canonical decomposition by the standard "H-element" $(n_+ - n_-)/2$ (cf., e.g., [Sa, II. §7]):

$$\mathfrak{g}_{1C} = \mathfrak{g}_{1C}^{1,-1} + \mathfrak{g}_{1C}^{0,0} + \mathfrak{g}_{1C}^{-1,1} = \mathfrak{p}_- + \mathfrak{k}_C + \mathfrak{p}_+ = \{x_-\}_C + \{z\}_C + \{x_+\}_C\,.$$

2. Horizontal SL_2-representations. From now on, we assume that $w > 0$ and all Hodge structures of weight w satisfy $H^{p,q} = 0$ unless $p, q \geq 0$.

DEFINITION (2.1) (cf. [Sc, p. 258]). *An SL_2-representation $\rho\colon SL_2(\boldsymbol{R}) \to G$ is said to be horizontal at $r = \{H_r^{p,q}\} \in D$ if $\rho_*(x_+) \in \mathfrak{g}_C^{-1,1} := \{X \in \mathfrak{g}_C \mid XH_r^{p,q} \subset H_r^{p-1,q+1}$ for all $p, q\}$.*

REMARK (2.2). It is clear that an SL_2-representation ρ is horizontal if and only if $\rho_*\colon \mathfrak{sl}_2(\boldsymbol{R}) \to \mathfrak{g}$ is a morphism of Hodge structures of type $(0, 0)$ with respect to the Hodge structures induced by $i \in U$ and $r \in D$, respectively. A horizontal SL_2-representation ρ induces an equivariant horizontal map $\tilde{\rho}\colon \boldsymbol{P}^1 \to \check{D}$ with $\tilde{\rho}(i) = r$:

$$\begin{array}{ccc} SL_2(\boldsymbol{C}) & \xrightarrow{\ \rho\ } & G_C \\ \downarrow & & \downarrow \\ \boldsymbol{P}^1 & \xrightarrow{\ \tilde{\rho}\ } & \check{D}\,. \end{array}$$

This is a generalization to the present context of the notion of '(H_1)-homomorphism' in the case of symmetric domains of Hermitian type (cf., e.g., [Sa, II. (8.5), III. §1]).

Let $\rho\colon SL_2(\boldsymbol{R}) \to G$ be a representation horizontal at $r \in \{H_r^{p,q}\} \in D$, and set

(2.3) $$Y := \rho_*(y)\,, \quad N_\pm := \rho_*(n_\pm)\,; \quad Z := \rho_*(z)\,, \quad X_\pm := \rho_*(x_\pm)\,.$$

Notice that by (1.19) these are related under the Cayley transformation:

(2.4) $$Z = (\mathrm{Ad}\, c)Y\,, \quad X_\pm = (\mathrm{Ad}\, c)N_\pm\,, \quad c := \rho(c_1)\,.$$

$(Y, N_\pm)$ and $(Z, X_\pm)$ define direct sum decompositions of H and H_C whose summands are

(2.5) $$P_\lambda^{(\lambda+2k)} := N_-^k(H(Y; \lambda+2k) \cap \operatorname{Ker} N_+)\,,$$

(2.6) $$Q_\lambda^{(\lambda+2k)} := X_-^k(H_C(Z; \lambda+2k) \cap \operatorname{Ker} X_+)\,,$$

for all eigenvalues $\lambda \in \{0, \pm 1, \pm 2, \ldots, \pm w\}$ of Y and Z and for $k \geq \max\{-\lambda, 0\}$, respectively. Here we denote by $H(Y; \lambda+2k)$ etc. the eigenspace of an endomorphism

Y of H with eigenvalue $\lambda+2k$. Since ρ is horizontal at $r=\{H_r^{p,q}\}$, (2.6) is compatible with this Hodge structure and we set

$$Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}:=Q_\lambda^{(\lambda+2k)}\cap H_r^{a+k,b+\lambda+k} \qquad (a,b\geq 0)\,. \tag{2.7}$$

These form a refined direct sum decomposition which we call the *Hodge-*$(Z, X_\pm)$ *decomposition* of (ρ, r) (cf. Remark (2.12) below). Transforming this by the inverse c^{-1} of the Cayley element, we define

$$P_\lambda^{(\lambda+2k)a+k,b+k}:=c^{-1}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}\,. \tag{2.8}$$

LEMMA (2.9). (i) $\sigma Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k}$.
(ii) $cQ_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=c^2P_\lambda^{(\lambda+2k)a+k,b+k}=P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}$.
$c^{-1}P_\lambda^{(\lambda+2k)a+k,b+k}=c^{-2}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=Q_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+k}$.

PROOF. It is easy to see, by definition, that $cP_\lambda^{(\lambda+2k)}=Q_\lambda^{(\lambda+2k)}$. Hence, by the first equality in (1.20.ii), we have

$$\sigma Q_\lambda^{(\lambda+2k)}=\sigma cP_\lambda^{(\lambda+2k)}=c^{-1}\sigma P_\lambda^{(\lambda+2k)}=c^{-1}P_\lambda^{(\lambda+2k)}=c^{-2}Q_\lambda^{(\lambda+2k)}\,.$$

On the other hand, by the second equality in (1.20.ii), the third and the second equalities in (1.20.i), we see that on $P_\lambda^{(\lambda+2k)}$

$$c^{-2}=\begin{cases}(-i)^{\lambda+2k}\dfrac{k!}{(\lambda+k)!}N_-^\lambda & \text{if } \lambda\geq 0\,,\\[2ex] (-i)^{\lambda+2k}\dfrac{(\lambda+k)!}{k!}N_+^{-\lambda} & \text{if } \lambda<0\,.\end{cases}$$

Taking their Cayley transforms, we see that on $Q_\lambda^{(\lambda+2k)}$

$$c^{-2}=\begin{cases}(-i)^{\lambda+2k}\dfrac{k!}{(\lambda+k)!}X_-^\lambda & \text{if } \lambda\geq 0\,,\\[2ex] (-i)^{\lambda+2k}\dfrac{(\lambda+k)!}{k!}X_+^{-\lambda} & \text{if } \lambda<0\,.\end{cases} \tag{2.10}$$

Thus, by the definition of the $Q_\lambda^{(\lambda+2k)}$, we have in both cases that

$$\sigma Q_\lambda^{(\lambda+2k)}=c^{-2}Q_\lambda^{(\lambda+2k)}=X_\mp^{\pm\lambda}Q_\lambda^{(\lambda+2k)}=Q_{-\lambda}^{(\lambda+2k)}=Q_{-\lambda}^{(-\lambda+2(\lambda+k))}\,.$$

This together with $\sigma H_r^{a+k,b+\lambda+k}=H_r^{b+\lambda+k,a+k}$ yields the assertion (i).

By horizontality, $X_-\in\mathfrak{g}_C^{1,-1}$ and $X_+\in\mathfrak{g}_C^{-1,1}$, hence $X_\mp^{\pm\lambda}\in\mathfrak{g}_C^{\lambda,-\lambda}$. This together with (2.10) shows that

$$c^{-2}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=X_\mp^{\pm\lambda}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=Q_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+k}\,.$$

Thus we obtain the second equality in (ii). The first equality in (ii) follows from the second. ∎

COROLLARY (2.11). *Let (ρ, r) be as above. For each eigenvalue λ of Y and for $k \geq \max\{-\lambda, 0\}$, we see that*

$$C \otimes P_\lambda^{(\lambda+2k)} = \bigoplus_{\substack{a+b+2k=w-\lambda \\ a,b\geq 0}} P_\lambda^{(\lambda+2k)a+k,b+k}$$

is a Hodge structure of weight $w-\lambda$. Moreover, in the case $\lambda=k=0$, this is S-polarized.

PROOF. We should observe the behavior under the complex conjugation σ:

$$\begin{aligned}\sigma P_\lambda^{(\lambda+2k)a+k,b+k} &= \sigma c^{-1} Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = c Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k} \\ &= c^2 P_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+\lambda+k} = P_\lambda^{(\lambda+2k)b+k,a+k}.\end{aligned}$$

This shows the first assertion.

The representation ρ is trivial on $Q_0^{(0)}$, hence $P_0^{(0)a,b} = c^{-1} Q_0^{(0)a,b} = Q_0^{(0)a,b}$, and so the second assertion trivially holds. ∎

We call a direct sum decomposition in (2.11) the *mixed Hodge-$(Y, N_\pm)$ decomposition of (ρ, r)*.

REMARK (2.12). We remark here some observations which are verified easily by (1.20.i), their Cayley transforms and horizontality of ρ at r. A Hodge-$(Z, X_\pm)$ decomposition and a mixed Hodge-$(Y, N_\pm)$ decomposition form "nests of diamonds", respectively. For example, in the case of weight $w=3$, these nests of diamonds are illustrated respectively as in Figures 1 and 2.

$$\begin{array}{ccccccccccc}
 & & & Q_{-3}^{(3)3,0} & & & & & & & \\
 & & Q_{-2}^{(2)3,0} & & Q_{-2}^{(2)2,1} & & & & & & \\
 & Q_{-1}^{(1)3,0} & & Q_{-1}^{(3)2,1} & & Q_{-1}^{(1)1,2} & & & & Q_{-1}^{(1)2,1} & \\
Q_0^{(0)3,0} & & Q_0^{(2)2,1} & & Q_0^{(2)1,2} & & Q_0^{(0)0,3}, & & Q_0^{(0)2,1} & & Q_0^{(0)1,2}. \\
 & Q_1^{(1)2,1} & & Q_1^{(3)1,2} & & Q_1^{(1)0,3} & & & & Q_1^{(1)1,2} & \\
 & & Q_2^{(2)1,2} & & Q_2^{(2)0,3} & & & & & & \\
 & & & Q_3^{(3)0,3} & & & & & & &
\end{array}$$

FIGURE 1.

$$\begin{array}{ccccccccccc}
 & & & P_{-3}^{(3)3,3} & & & & & & & \\
 & & P_{-2}^{(2)3,2} & & P_{-2}^{(2)2,3} & & & & & & \\
 & P_{-1}^{(1)3,1} & & P_{-1}^{(3)2,2} & & P_{-1}^{(1)1,3} & & & & P_{-1}^{(1)2,2} & \\
P_0^{(0)3,0} & & P_0^{(2)2,1} & & P_0^{(2)1,2} & & P_0^{(0)0,3}, & & P_0^{(0)2,1} & & P_0^{(0)1,2}. \\
 & P_1^{(1)2,0} & & P_1^{(3)1,1} & & P_1^{(1)0,2} & & & & P_1^{(1)1,1} & \\
 & & P_2^{(2)1,0} & & P_2^{(2)0,1} & & & & & & \\
 & & & P_3^{(3)0,0} & & & & & & &
\end{array}$$

FIGURE 2.

On these nests of diamonds, the complex conjugation by σ sends respectively a summand $Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}$ to a summand $Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k}$ which are symmetric with respect to the origin of the diamonds, and a summand $P_{\lambda}^{(\lambda+2k)a+k,b+k}$ to a summand $P_{\lambda}^{(\lambda+2k)b+k,a+k}$ which are symmetric with respect to the vertical axis. The operator X_+ (resp. X_-) sends a summand $Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}$ one step down (resp. up) to a summand $Q_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1,b+\lambda+2+k-1}$ (resp. $Q_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1,b+\lambda-2+k+1}$), and $X_{\pm}$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. Similarly, the operator N_+ (resp. N_-) sends a summand $P_{\lambda}^{(\lambda+2k)a+k,b+k}$ one step down (resp. up) to a summand $P_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1,b+k-1}$ (resp. $P_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1,b+k+1}$), and $N_{\pm}$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. The Cayley element c transforms the second nest of diamonds together with the action of the operators $Y, N_{\pm}$ to the first nest of diamonds together with the action of the operators $Z, X_{\pm}$: $cP_{\lambda}^{(\lambda+2k)a+k,b+k}=Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}$.

By using these operators, we can explain why the summands outside the nests of diamonds vanish in the following way. We claim first that $Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}=0$ for $\lambda>0$ and $b<0$. Indeed, $X_-^{\lambda+k}$ is injective on this summand by the Cayley transform of the third equality in (1.20.i). On the other hand, looking at the Hodge type, we see that $X_-^{\lambda+k}Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}\subset Q_{-\lambda-2k}^{(-\lambda-2k+2(\lambda+2k))a+\lambda+2k,b}=0$ by horizontality. Thus we get our claim. It follows by symmetry under the complex conjugation σ that $Q_{\lambda}^{(\lambda+2k)a+k,b+\lambda+k}=0$ for $\lambda<0$ and $a<0$. Finally, by the inverse of the Cayley transformation, we have $P_{\lambda}^{(\lambda+2k)a+k,b+k}=0$ for $\lambda>0$ and $b<0$, and for $\lambda<0$ and $a<0$.

We call the length of the side of the biggest diamond in a nest the *size* of the nest of diamonds.

Another remark is that a mixed Hodge-$(Y, N_{\pm})$ decomposition is nothing but the limiting split mixed Hodge structure of the associated SL_2-orbit $\tilde{\rho}: U\to D$, $\tilde{\rho}(gi):=\rho(g)r$ $(g\in SL_2(\boldsymbol{R}))$, and the monodromy weight filtration L is described as $L_i=\bigoplus_{\lambda\le i}\bigoplus_k P_{-\lambda}^{(\lambda+2k)}$ (cf. [Sc, (6.16)] and its proof, [CK, pp. 13–14]).

In the above notation, for all λ, a and b, put

$$n_{\lambda}:=\dim_{\boldsymbol{R}}H(Y;\lambda)=\dim_{\boldsymbol{C}}H_{\boldsymbol{C}}(Z;\lambda)\,,\tag{2.13}$$
$$p_{\lambda}^{a,b}:=\dim_{\boldsymbol{C}}P_{\lambda-2k}^{(\lambda)a+k,b+k}=\dim_{\boldsymbol{C}}Q_{\lambda-2k}^{(\lambda)a+k,b+\lambda-k}\,.$$

Notice that, by construction, the middle terms and the terms on the extreme right hand side of the second equality in (2.13) are independent of k (cf. Remark (2.12)).

LEMMA (2.14). *For (ρ, r) as above, the following hold:*
(i) $\sum_{a+b=w-\lambda}p_{\lambda}^{a,b}=n_{\lambda}-n_{\lambda+2}$ *for all* $0\le\lambda\le w$.
(ii) $p_{\lambda}^{b,a}=p_{\lambda}^{a,b}$ *for all* λ, a, b *with* $0\le\lambda\le w$, $a\ge0$, $b\ge0$ *and* $a+b=w-\lambda$.
(iii) $h^{a,b}=h^{a+1,b-1}-(p_0^{a+1,b-1}+p_1^{a+1,b-2}+\cdots+p_{b-1}^{a+1,0})+(p_0^{a,b}+p_1^{a-1,b}+\cdots+$

$p_a^{0,b}$) *for all* a, b *with* $a \geq 0$, $b \geq 0$ *and* $a+b=w$.

PROOF. We first observe that there is an exact sequence

$$0 \longrightarrow P_\lambda^{(\lambda)} \longrightarrow H(Y;\lambda) \xrightarrow{N_+} H(Y;\lambda+2) \longrightarrow 0$$

for every $\lambda \geq 0$ (and N_- yields a right splitting). (i) and (ii) follow from this and (2.11).

In order to show (iii), we look at the morphism $X_+ : H^{a+1,b-1} \to H^{a,b}$ and its kernel and cokernel:

$$\begin{aligned}\mathrm{Ker} &= Q_0^{(0)a+1,b-1} \oplus Q_1^{(1)a+1,b-1} \oplus \cdots \oplus Q_{b-1}^{(b-1)a+1,b-1}\\ &\overset{C}{\leftarrow} P_0^{(0)a+1,b-1} \oplus P_1^{(1)a+1,b-2} \oplus \cdots \oplus P_{b-1}^{(b-1)a+1,0}\,,\\ \mathrm{Coker} &\simeq Q_0^{(0)a,b} \oplus Q_{-1}^{(1)a,b} \oplus \cdots \oplus Q_{-a}^{(a)a,b}\\ &\simeq Q_0^{(0)a,b} \oplus Q_1^{(1)a-1,b+1} \oplus \cdots \oplus Q_a^{(a)0,b+a}\\ &\overset{C}{\leftarrow} P_0^{(0)a,b} \oplus P_1^{(1)a-1,b} \oplus \cdots \oplus P_a^{(a)0,b}\,.\end{aligned}$$

Looking at the dimension, we get (iii). ■

DEFINITION (2.15). *We call a set of integers* $\{p_\lambda^{a,b}\}$, *which satisfies the conditions* (i), (ii) *and* (iii) *of* (2.14), *a set of primitive Hodge numbers belonging to* $\{h^{p,q}, n_\lambda\}$.

3. Admissible *R*-semi-simple elements. We continue to use the notation in the previous sections.

PROPOSITION (3.1). *Given a pair* $(Y, r) \in \mathfrak{g} \times D$, *there exists at most one representation* $\rho: SL_2(\boldsymbol{R}) \to G$ *which is horizontal at* r *and* $\rho_*(y) = Y$.

PROOF. Since y and z generate $\mathfrak{sl}_2(\boldsymbol{C})$, it is enough to show that if such a representation ρ exists then the eigenspaces of Z, and hence Z itself, are determined by the pair (Y, r). Actually, we shall show by induction on the size w of the nest of diamonds of the Hodge-$(Z, X_\pm)$ decomposition (2.7) (cf. Remark (2.12)) that this nest of diamonds is completely determined by (Y, r).

First notice that

$$Y = i(X_+ - X_-)\,. \tag{3.2}$$

For a subspace M of H_C, we put, throughout this proof,

$$M^\perp := \{v \in H_C \mid S(v, \sigma u) = 0 \text{ for all } u \in M\}\,,$$

$$\text{projection } \{M \to H_r^{p,q}\} := \mathrm{Im}\left\{M \subset H_C = \bigoplus_{p'+q'=w} H_r^{p',q'} \to H_r^{p,q}\right\}.$$

Then we see that

$$Q_{w}^{(w)0,w}=\text{projection}\{Y^{w}H_{r}^{w,0}\to H_{r}^{0,w}\},$$

$$Q_{w-2k}^{(w)k,w-k}=\text{projection}\{Y^{k}Q_{w}^{(w)0,w}\to H_{r}^{k,w-k}\}\qquad(0\leq k\leq w),$$

$$\bigoplus_{0\leq\lambda\leq w-1}Q_{-\lambda}^{(\lambda)w,0}=H_{r}^{w,0}\cap(Q_{-w}^{(w)w,0})^{\perp},$$

$$Q_{w-1}^{(w-1)1,w-1}=\text{projection}\left\{Y^{w-1}\left(\bigoplus_{0\leq\lambda\leq w-1}Q_{-\lambda}^{(\lambda)w,0}\right)\to H_{r}^{1,w-1}\right\},$$

$$Q_{w-1-2k}^{(w-1)1+k,w-1-k}=\text{projection}\{Y^{k}Q_{w-1}^{(w-1)1,w-1}\to H_{r}^{1+k,w-1-k}\}\quad(0\leq k\leq w-1),$$

$$\bigoplus_{0\leq\lambda\leq w-2}Q_{-\lambda}^{(\lambda)w,0}=H_{r}^{w,0}\cap\left(\bigoplus_{w-1\leq\lambda\leq w}Q_{-\lambda}^{(\lambda)w,0}\right)^{\perp},$$

$$Q_{w-2}^{(w-2)2,w-2}=\text{projection}\left\{Y^{w-2}\left(\bigoplus_{0\leq\lambda\leq w-2}Q_{-\lambda}^{(\lambda)w,0}\right)\to H_{r}^{2,w-2}\right\},$$

$$Q_{w-2-2k}^{(w-2)2+k,w-2-k}=\text{projection}\{Y^{k}Q_{w-2}^{(w-2)2,w-2}\to H_{r}^{2+k,w-2-k}\}\quad(0\leq k\leq w-2),$$

$$\cdots\cdots.$$

Thus $Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k}$ $(0\leq\lambda\leq w, 0\leq k\leq\lambda)$ are determined. Taking the complex conjugation by σ of these, we get $Q_{-\lambda+2k}^{(\lambda)\lambda-k,w-\lambda+k}=\sigma Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k}$ $(0\leq\lambda\leq w, 0\leq k\leq\lambda)$. Applying the induction hypothesis to the nest of diamonds of size $\leq w-2$ in

$$\left(\bigoplus_{\substack{0\leq\lambda\leq w\\0\leq k\leq\lambda}}(Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k}\oplus Q_{-\lambda+2k}^{(\lambda)\lambda-k,w-\lambda+k})\right)^{\perp}$$

(cf. Remark (2.12)), we get our assertion. ■

DEFINITION (3.3). A pair $(Y, r)\in\mathfrak{g}\times D$ is admissible if there exists a representation $\rho\colon SL_2(\boldsymbol{R})\to G$ which is horizontal at r and $\rho_*(y)=Y$.

The set of primitive Hodge numbers $\{p_{\lambda}^{a,b}\}$ belonging to $\{h^{p,q}, n_\lambda\}$ is called the type of an admissible pair (Y, r).

$Y\in\mathfrak{g}$ is said to be admissible if (Y, r) is an admissible pair for some $r\in D$.

Now we prove the following numerical criterion for admissibility:

THEOREM (3.4). *$Y\in\mathfrak{g}$ is admissible if and only if Y is semi-simple over $\boldsymbol{R}$ whose eigenvalues are contained in $\{0, \pm1, \pm2, \ldots, \pm w\}$ and there exists a set of primitive Hodge numbers $\{p_{\lambda}^{a,b}\}$ belonging to $\{h^{p,q}, n_\lambda\}$, where $n_\lambda:=\dim H(Y;\lambda)$ (cf. Definition (2.15)).*

PROOF. Since Y is semi-simple over $\boldsymbol{R}$, the eigenspaces $H(Y;\lambda)$ are defined over $\boldsymbol{R}$ and $H(Y;\lambda)$ and $H(Y;\mu)$ are S-orthogonal unless $\lambda+\mu=0$. Therefore $H(Y;\lambda)$ and $H(Y;-\lambda)$ are S-dual.

Since $n_{\lambda'}-n_{\lambda'+2}\geq0$ for $\lambda'\geq0$ by the condition (2.14.i), we can take a direct sum decomposition

$$H(Y;\lambda)=P_\lambda^{(\lambda)}\oplus P_\lambda^{(\lambda+2)}\oplus P_\lambda^{(\lambda+4)}\oplus\cdots\quad\text{for}\quad \lambda\ge 0\,, \tag{3.5}$$

with $\dim P_\lambda^{(\lambda+2k)}=n_{\lambda+2k}-n_{\lambda+2k+2}$. Moreover, in the case $\lambda=0$, the decomposition (3.5) can be taken to be S-orthogonal. We denote the S-dual decomposition by

$$H(Y;-\lambda)=P_{-\lambda}^{(\lambda)}\oplus P_{-\lambda}^{(\lambda+2)}\oplus P_{-\lambda}^{(\lambda+4)}\oplus\cdots\quad(\lambda\ge 0)\,, \tag{3.6}$$

i.e., $P_{-\lambda}^{(\lambda+2k)}$ and $P_\lambda^{(\lambda+2m)}$ are S-orthogonal unless $k=m$.

By the conditions (i) and (ii) of (2.14), we can choose a Hodge decomposition

$$\boldsymbol{C}\otimes P_\lambda^{(\lambda+2k)}=\bigoplus_{\substack{a+b+2k=w-\lambda\\ a,b\ge 0}} P_\lambda^{(\lambda+2k)a+k,b+k}\quad\text{for}\quad\lambda\ge 0\,,\quad k\ge 0\,, \tag{3.7}$$

with $\dim P_\lambda^{(\lambda+2k)a+k,b+k}=p_{\lambda+2k}^{a,b}$. Moreover, in the case $\lambda=k=0$, the Hodge structure (3.7) can be chosen to be S-polarized. We denote the $S(\cdot,\sigma\cdot)$-orthogonal decomposition by

$$\boldsymbol{C}\otimes P_{-\lambda}^{(-\lambda+2(\lambda+k))}=\bigoplus_{\substack{a+b+2\lambda+2k=w+\lambda\\ a,b\ge 0}} P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}\qquad(\lambda\ge 0,k\ge 0)\,, \tag{3.8}$$

i.e., $S(P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k},\sigma P_\lambda^{(\lambda+2k)a'+k,b'+k})=0$ unless $(a,b)=(a',b')$. Notice that $P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}=P_{-\lambda}^{(\lambda+2k)a+\lambda+k,b+\lambda+k}$.

Now we consider the cases $\lambda\ge 0$ and $\lambda<0$ altogether. For $k\ge\max\{-\lambda,0\}$ and $a\ge b$, let

$$\{v_{\lambda,j}^{(\lambda+2k)a+k,b+k}\mid 1\le j\le p_{\lambda+2k}^{a,b}\} \tag{3.9}$$

be a $\boldsymbol{C}$-basis of $P_\lambda^{(\lambda+2k)a+k,b+k}$ such that

$$S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k},\sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k})=\delta_{jj'}(-1)^a i^{w-\lambda}\Big/\binom{\lambda+2k}{k}. \tag{3.10}$$

In the case $a=b$, we can moreover take the above basis (3.9) to consist of real elements. Put

$$v_{\lambda,j}^{(\lambda+2k)b+k,a+k}=\sigma v_{\lambda,j}^{(\lambda+2k)a+k,b+k}\qquad(a\ge b)\,. \tag{3.11}$$

Define now $\boldsymbol{C}$-linear endomorphisms $N_\pm$ of $H_{\boldsymbol{C}}$ by

$$\begin{aligned} N_+v_{\lambda,j}^{(\lambda+2k)a+k,b+k}&:=kv_{\lambda+2,j}^{((\lambda+2)+2(k-1))a+k-1,b+k-1}\,,\\ N_-v_{\lambda,j}^{(\lambda+2k)a+k,b+k}&:=(\lambda+k)v_{\lambda-2,j}^{((\lambda-2)+2(k+1))a+k+1,b+k+1}\,, \end{aligned} \tag{3.12}$$

for all λ, non-negative a, b and $k\ge\max\{-\lambda,0\}$. By construction, it is easy to see that $N_\pm$ commute with the complex conjugation σ and satisfy the commutation relations: $[N_+,N_-]=Y$, and $[Y,N_\pm]=\pm 2N_\pm$, respectively. It is also easy to verify that $S(N_\pm\cdot,\cdot)+S(\cdot,N_\pm\cdot)=0$, respectively. Indeed, for example, one can compute as

$$S(N_+v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k},\sigma v_{\lambda-2,j'}^{((\lambda-2)+2(k+1))a+k+1,b+k+1})$$
$$+S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k},N_+\sigma v_{\lambda-2,j'}^{((\lambda-2)+2(k+1))a+k+1,b+k+1})$$
$$=\delta_{jj'}(-1)^a i^{w-\lambda+2}\frac{(\lambda+k)(\lambda+k-1)!(k+1)!}{(\lambda+2k)!}+\delta_{jj'}(-1)^a i^{w-\lambda}\frac{(k+1)k!(\lambda+k)!}{(\lambda+2k)!}=0\,.$$

Thus we see that $N_\pm\in\mathfrak{g}$ and hence there exists a unique representation

(3.13) $\rho: SL_2(\boldsymbol{R})\to G$ such that $\rho_*(y)=Y$ and $\rho_*(n_\pm)=N_\pm$, respectively.

By using the Cayley element $c:=\rho(c_1)\in G_{\boldsymbol{C}}$, we define

$$\text{(3.14)}\qquad Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}:=cP_\lambda^{(\lambda+2k)a+k,b+k},\quad H^{p,q}:=\bigoplus_{\substack{a+k=p\\ b+\lambda+k=q}}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k},$$

where, on the right hand side of the second equality, the summation is taken over all the eigenvalues λ of Y, all integers $k\ge\max\{-\lambda,0\}$ and all non-negative integers a, b with $a+b+\lambda+2k=w$. This defines a Hodge structure. Indeed, by using (1.20.ii), one sees that

$$\sigma Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=\sigma cP_\lambda^{(\lambda+2k)a+k,b+k}=c^{-1}\sigma P_\lambda^{(\lambda+2k)a+k,b+k}$$
$$=c^{-1}P_\lambda^{(\lambda+2k)b+k,a+k}=cP_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+\lambda+k}=Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k},$$

and hence $\sigma H^{p,q}=H^{q,p}$. One can moreover verify that (3.14) is S-polarized. Indeed, the direct sum in (3.14) is S-orthogonal by construction and, for

$$cv_{\lambda,j}^{(\lambda+2k)a+k,b+k},\quad cv_{\lambda,j'}^{(\lambda+2k)a+k,b+k}\in Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}\subset H^{p,q},$$

one can compute as

$$i^{p-q}S(cv_{\lambda,j}^{(\lambda+2k)a+k,b+k},\sigma cv_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$
$$=i^{a-b-\lambda}S(cv_{\lambda,j}^{(\lambda+2k)a+k,b+k},c^{-1}\sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$
$$=i^{a-b-\lambda}S(c^2v_{\lambda,j}^{(\lambda+2k)a+k,b+k},\sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$
$$=i^{a-b-\lambda+\lambda+2k}S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k},\sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$
$$=\delta_{jj'}i^{a-b+2k+2a+w-\lambda}\Big/\binom{\lambda+2k}{k}=\delta_{jj'}\Big/\binom{\lambda+2k}{k}.$$

Thus we have $\{H^{p,q}\}\in D$.

Finally, we claim that the representation ρ in (3.13) is horizontal at $\{H^{p,q}\}\in D$. Indeed, since $Z=(\operatorname{Ad}c)Y$, $X_\pm=(\operatorname{Ad}c)N_\pm$, one can compute, by (1.20), as

$$ZQ_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=cYP_\lambda^{(\lambda+2k)a+k,b+k}=Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k},$$
$$X_\pm Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}=cN_\pm P_\lambda^{(\lambda+2k)a+k,b+k}$$

$$=cP_{\lambda\pm 2}^{((\lambda\pm 2)+2(k\mp 1))a+k\mp 1,b+k\mp 1}=Q_{\lambda\pm 2}^{((\lambda\pm 2)+2(k\mp 1))a+k\mp 1,b+\lambda+k\pm 1}\,.$$

This completes the proof of the theorem. ∎

We remark that the condition on $\{n_\lambda\}$ in Theorem (3.4) coincides with the one in [CK, (2.20)] in the case of weight 2.

Fix identifications $D\simeq G/V$ and $R\simeq G/K$, where K is a maximal compact subgroup of G containing V and R is the associated Riemannian symmetric domain, and let θ_K be the associated Cartan involution. We denote the projection by

(3.15) $$\pi: D\simeq G/V\rightarrow G/K\simeq R\,.$$

PROPOSITION (3.16). *We use the notation in Theorem* (3.4). *Let $Y\in\mathfrak{g}$ be an admissible element.*

(i) *If there exists $r\in\pi^{-1}([K])$ such that (Y, r) is an admissible pair, then $\theta_r Y=-Y$, where θ_r is the Cartan involution on $\mathfrak{g}$ induced from* (1.11).

(ii) *If $\theta_K Y=-Y$, then there exists $r\in\pi^{-1}([K])$ such that (Y, r) is an admissible pair.*

(iii) *For each set of primitive Hodge numbers $\{p_\lambda^{a,b}\}$ belonging to $\{h^{p,q}, n_\lambda\}$, $G_Y:=\{g\in G\,|\,(\mathrm{Ad}\,g)Y=Y\}$ acts transitively on the set $\{r\in D\,|\,(Y, r)$ is an admissible pair of type $\{p_\lambda^{a,b}\}\}$.*

PROOF. (i) follows from (3.2) and (1.11).

(ii): Assume $\theta_K Y=-Y$. Take a point $r'\in D$ at which Y is admissible and let K' be the maximal compact subgroup of G associated to the Cartan involution $\theta_{r'}$. By the result in (i) for (Y, r') and the assumption, Y can be viewed as a tangent vector to R at $[K']$ as well as at $[K]$: $Y\in T_R([K'])$, $Y\in T_R([K])$. By the transitivity of tangent spaces of a Riemannian symmetric domain, there exists $g\in G$ such that $(\mathrm{Int}\,g)K'=K$ and $(\mathrm{Ad}\,g)Y=Y$. Hence the admissibility of (Y, r') implies that of $((\mathrm{Ad}\,g)Y, gr')=(Y, gr')$, where $gr'\in\pi^{-1}([K])$.

(iii): Suppose that $r, r'\in D$ are points at which Y is admissible of the same type $\{p_\lambda^{a,b}\}$. Let $\rho, \rho': SL_2(\boldsymbol{R})\rightarrow G$ be the corresponding representations. It is enough to show that there exists $g\in G$ such that $\rho'=(\mathrm{Int}\,g)\rho$. Indeed, if this is the case, then $(\mathrm{Ad}\,g)Y=(\mathrm{Ad}\,g)(\rho_*(y))=\rho'_*(y)=Y$ and $gr=g\tilde{\rho}(i)=\tilde{\rho}'(i)=r'$.

We can construct such a $g\in G$ elementarily by using bases of H_C according to the S-polarized Hodge-$(Z, X_\pm)$ decompositions, where $(Z, X_\pm)=(\rho_*(z), \rho_*(x_\pm))$, $(\rho'_*(z), \rho'_*(x_\pm))$. Thus we get our assertion. ∎

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