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A NUMERICAL CRITERION FOR ADMISSIBILITY OF SEMI-SIMPLE ELEMENTS

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Abstract. In this article, we shall generalize a theorem of Cattani and Kaplan on horizontal representations of $SL(2)$. Their theorem plays an important role in the construction of their partial compactifications of the classifying spaces D modulo an arithmetic subgroup of Hodge structures of weight 2.

Introduction. A horizontal SL_2-representation is a generalization of the notion of "(H_1)-homomorphism" of SL_2 in the case of the classical theory of Hermitian symmetric domains (cf., e.g., [Sa, III]). More precisely, let $G = G_R := \mathrm{Aut}(H_R, S)$ be the automorphism group of the classifying space D of Hodge structures of weight w (see §1). A representation $\rho: SL_2(R) \to G$ is said to be *horizontal* at $r \in D$ if the morphism $\rho_*: \mathfrak{sl}_2(R) \to \mathfrak{g}$ of the Lie algebras is a morphism of Hodge structures of type $(0, 0)$ with respect to the Hodge structures on $\mathfrak{sl}_2(C)$ and $\mathfrak{g}_C$ induced by $i \in U := (\text{upper-half plane})$ and $r \in D$ respectively (see Definition (2.1)). In this case, the pair (ρ, r) is uniquely determined by the pair $(Y, r) \in \mathfrak{g} \times D$ with

$$(0.1) \qquad\qquad Y := \rho_* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Conversely, a pair $(Y, r) \in \mathfrak{g} \times D$ is said to be *admissible* if there exists a representation $\rho: SL_2(R) \to G$ horizontal at r and satisfying (0.1). The main result in the present article is a numerical criterion for admissibility of a pair (Y, r) in the case of general weight.

Given a pair (ρ, r) as above, one can refine the Hodge decomposition $H_C = \bigoplus H_r^{p,q}$, corresponding to $r \in D$, under the horizontal action of $\mathfrak{sl}_2(C)$ at r, called a *Hodge-$(Z, X_\pm)$ decomposition* (see (2.7)). Our proof of the main result is based on an elementary but useful observation (Corollary (2.11), see also Remark (2.12)), which says that the transformation of the Hodge-$(Z, X_\pm)$ decomposition by the inverse c^{-1} of the Cayley element

$$c := \rho\left(\exp \frac{\pi i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

yields a split mixed Hodge structure, called a *mixed Hodge-*(Y, N_+) *decomposition*, which is nothing but the limiting mixed Hodge structure of the associated SL_2-orbit $\tilde{\rho}: U \to D$ defined by $\tilde{\rho}(gi) := \rho(g)r$ for $g \in SL_2(\mathbf{R})$ (cf. [Sc, Theorem (6.16)] and its proof). By virtue of this observation, we can view the relationship between the pairs (ρ, r) and (Y, r) from a better perspective, and generalize a numerical criterion [CK, Theorem (2.22)] for admissiblity of (Y, r) in the case of weight 2 to the case of general weight.

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1. Preliminaries. We recall first the definition of a (polarized) Hodge structure of weight w. Fix a free $\mathbf{Z}$-module $H_{\mathbf{Z}}$ of finite rank. Set $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$, $H = H_{\mathbf{R}} := \mathbf{R} \otimes H_{\mathbf{Z}}$ and $H_{\mathbf{C}} := \mathbf{C} \otimes H_{\mathbf{Z}}$, whose complex conjugation is denoted by σ. Let w be an integer. A *Hodge structure of weight* w on $H_{\mathbf{C}}$ is a decomposition

$$(1.1) \qquad H_{\mathbf{C}} = \bigoplus_{p+q=w} H^{p,q} \qquad \text{with} \quad \sigma H^{p,q} = H^{q,p}.$$

The integers

$$(1.2) \qquad h^{p,q} := \dim H^{p,q}$$

are called the Hodge numbers.

A polarization S for a Hodge structure (1.1) of weight w is a non-degenerate bilinear form on $H_{\mathbf{Q}}$, symmetric if w is even and skew-symmetric if w is odd, such that its $\mathbf{C}$-bilinear extension, denoted also by S, satisfies

$$(1.3) \qquad \begin{aligned} &S(H^{p,q}, \sigma H^{p',q'}) = 0 \quad \text{unless} \quad (p,q) = (p',q'), \\ &i^{p-q} S(v, \sigma v) > 0 \qquad \text{for all} \quad 0 \neq v \in H^{p,q}. \end{aligned}$$

REMARK (1.4). In the geometric case, i.e., the Hodge structure on the w-th cohomology group $H^w(X, \mathbf{Q})$ of a smooth projective variety $X \subset \mathbf{P}^N$ of dimension d over $\mathbf{C}$, we take as a polarization

$$S(u, v) := (-1)^{w(w-1)/2} \int_X u \wedge v \wedge \eta^{d-w}$$

for primitive classes $u, v \in H^w_{\mathrm{prim}}(X, \mathbf{C}) \simeq H^w_{\mathrm{prim}}(X, \Omega_X^{\bullet})$ where $\eta \in H^1(X, \Omega_X^1)$ is the cohomology class of a hyperplane section of X.

For fixed S and $\{h^{p,q}\}$, the classifying space D for Hodge structures and its "compact dual" $\check{D}$ are defined by

$$\check{D} := \{\{H^{p,q}\} \mid \text{Hodge structure on } H_C \text{ with } \dim H^{p,q} = h^{p,q},$$
$$\text{satisfying the first condition in (1.3)}\},$$

(1.5)

$$D := \{\{H^{p,q}\} \in \check{D} \mid \text{satisfying also the second condition in (1.3)}\}.$$

These are homogeneous spaces under the natural actions of the groups

(1.6) $$G_C := \text{Aut}(H_C, S), \qquad G = G_R := \{g \in G_C \mid g H_R = H_R\},$$

respectively. Taking a reference point $r \in D$, one obtains identifications

(1.7) $$\check{D} \simeq G_C/B_C, \qquad D \simeq G/V,$$

where B_C and V are the isotropy subgroups of G_C and of G at $r \in D$, respectively. It is a direct consequence of the definition that

(1.8) $$G \simeq \begin{cases} O(2h, k), \\ Sp(2h, R), \end{cases} \quad V \simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if } w = 2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w = 2t+1, \end{cases}$$

where $k := \sum_{|j| \le [t/2]} h^{t+2j,t-2j}$ and $h := (\dim H - k)/2$ if $w = 2t$, and $h := \dim H/2$ if $w = 2t + 1$. It is an important observation that V is compact, but not maximal compact in general. Hence D is a symmetric domain of Hermitian type if and only if

(1.9) $$h^{p,q} = 0 \quad \text{unless } (p, q) = \begin{cases} (t+1, t-1), & (t, t) \quad \text{or} \quad (t-1, t+1), \\ & \text{and } h^{t+1,t-1} = 1 \quad \text{if } w = 2t, \\ (t+1, t) & \text{or} \quad (t, t+1) \quad \text{if } w = 2t+1. \end{cases}$$

A reference Hodge structure $r = \{H_r^{p,q}\} \in D$ induces a Hodge structure of weight 0 on the Lie algebra $g_C := \text{Lie } G_C$ by

(1.10) $$g_C^{s,-s} := \{X \in g_C \mid X H_r^{p,q} \subset H_r^{p+s,q-s} \text{ for all } p, q\}.$$

One can define the associated Cartan involution θ_r on g_C by

(1.11) $$\theta_r(X) := \sum_s (-1)^s X^{s,-s} \quad \text{for} \quad X = \sum_s X^{s,-s} \in g_C = \bigoplus_s g_C^{s,-s}.$$

This can be interpreted in the following way: Set

(1.12)
$$H_r^+ := H_r^{w,0} \oplus H_r^{w-2,2} \oplus H_r^{w-4,4} \oplus \cdots,$$
$$H_r^- := H_r^{w-1,1} \oplus H_r^{w-3,3} \oplus H_r^{w-5,5} \oplus \cdots.$$

It is clear by definition that the isotropy subgroup of the decomposition $H_C = H_r^+ \oplus H_r^-$ induces the maximal compact subgroup

(1.13) $$K \simeq \begin{cases} O(2h) \times O(k) & \text{if } w = 2t, \\ U(h) & \text{if } w = 2t+1, \end{cases}$$

of G which contains V, and the Cartan involution θ_r in (1.11) is the one associated to

K. Define a C-linear automorphism

(1.14) $E_r: H_C \to H_C$ by $E_r := \begin{cases} 1 & \text{on } H_r^+, \\ -1 & \text{on } H_r^-. \end{cases}$

Then the Cartan involution θ_r in (1.11) can also be written as

(1.15) $\theta_r X = (\operatorname{Ad} E_r) X$ for $X \in \mathfrak{g}_C$.

We recall now well-known results on SL_2-representations. Let ξ, η be two variables, and write

(1.16) $\begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \begin{pmatrix} \xi^m \\ \xi^{m-1}\eta \\ \vdots \\ \eta^m \end{pmatrix}$ $(m = 0, 1, 2, \cdots)$.

A representation

(1.17) $\rho_m: SL_2(\mathbf{R}) \to SL_{m+1}(\mathbf{R})$ defined by $\rho_m(g)\begin{pmatrix} \xi \\ \eta \end{pmatrix}^{(m)} := \left(g\begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)^{(m)}$

is called a symmetric tensor representation of dimension $m+1$. It is known that the ρ_m $(m = 0, 1, 2, \cdots)$ are absolutely irreducible and constitute a full set of representatives for the equivalence classes of finite dimensional irreducible representations of $SL_2(\mathbf{R})$.

We take the standard generators for the Lie algebras $\mathfrak{sl}_2(\mathbf{R})$ and $\mathfrak{su}(1,1)$ which are related by the Cayley transformation $\operatorname{Ad} c_1$, where

(1.18) $c_1 := \exp\frac{\pi i}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$,

as follows:

$$\mathfrak{sl}_2(\mathbf{R}) \ni y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad n_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad n_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(1.19) $\operatorname{Ad} c_1 \downarrow \qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$

$$\mathfrak{su}(1,1) \ni z := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad x_+ := \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad x_- := \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.$$

The following lemma can be verified directly by using the monomial basis (1.16) and the definition (1.19), and so we omit the proof.

LEMMA (1.20).   (i)   *In the above notation,* $Y_m := \rho_{m*}(y)$ *and* $N_{m\pm} := \rho_{m*}(n_\pm)$ *satisfy*
$Y_m(\xi^{m-j}\eta^j) = (m-2j)\xi^{m-j}\eta^j$,
$N_{m+}(\xi^{m-j}\eta^j) = (m-j)\xi^{m-j-1}\eta^{j+1}$,
$N_{m-}(\xi^{m-j}\eta^j) = j\xi^{m-j+1}\eta^{j-1}$.

(ii) *For the Cayley element $c_m := \rho_m(c_1) \in SL_{m+1}(C)$,*
$\sigma c_m = c_m^{-1} \sigma$, *where σ is the complex conjugation.*
$$c_m^{\pm 2}(\xi^{m-j}\eta^j) = (\pm i)^m \eta^{m-j}\xi^j,$$
$$c_m^4(\xi^{m-j}\eta^j) = (-1)^m \xi^{m-j}\eta^j.$$

REMARK (1.21). The Hodge structure on $\mathfrak{g}_{1C} := \mathfrak{sl}_2(C)$ induced by $i \in U := $ (upper-half plane) $\simeq SL_2(R)/U(1)$ coincides with the canonical decomposition by the standard "H-element" $(n_+ - n_-)/2$ (cf., e.g., [Sa, II. §7]):

$$\mathfrak{g}_{1C} = \mathfrak{g}_{1C}^{1,-1} + \mathfrak{g}_{1C}^{0,0} + \mathfrak{g}_{1C}^{-1,1} = \mathfrak{p}_- + \mathfrak{k}_C + \mathfrak{p}_+ = \{x_-\}_C + \{z\}_C + \{x_+\}_C .$$

2. Horizontal SL_2-representations.

From now on, we assume that $w > 0$ and all Hodge structures of weight w satisfy $H^{p,q} = 0$ unless $p, q \geq 0$.

DEFINITION (2.1) (cf. [Sc, p. 258]). *An SL_2-representation $\rho: SL_2(R) \to G$ is said to be horizontal at $r = \{H_r^{p,q}\} \in D$ if $\rho_*(x_+) \in \mathfrak{g}_C^{-1,1} := \{X \in \mathfrak{g}_C \mid X H_r^{p,q} \subset H_r^{p-1,q+1} \text{ for all } p, q\}$.*

REMARK (2.2). It is clear that an SL_2-representation ρ is horizontal if and only if $\rho_*: \mathfrak{sl}_2(R) \to \mathfrak{g}$ is a morphism of Hodge structures of type $(0, 0)$ with respect to the Hodge structures induced by $i \in U$ and $r \in D$, respectively. A horizontal SL_2-representation ρ induces an equivariant horizontal map $\tilde{\rho}: P^1 \to \check{D}$ with $\tilde{\rho}(i) = r$:

$$
\begin{array}{ccc}
SL_2(C) & \xrightarrow{\rho} & G_C \\
\downarrow & & \downarrow \\
P^1 & \xrightarrow{\tilde{\rho}} & \check{D} .
\end{array}
$$

This is a generalization to the present context of the notion of '(H_1)-homomorphism' in the case of symmetric domains of Hermitian type (cf., e.g., [Sa, II. (8.5), III. §1]).

Let $\rho: SL_2(R) \to G$ be a representation horizontal at $r \in \{H_r^{p,q}\} \in D$, and set

(2.3) $\qquad Y := \rho_*(y), \quad N_\pm := \rho_*(n_\pm); \quad Z := \rho_*(z), \quad X_\pm := \rho_*(x_\pm).$

Notice that by (1.19) these are related under the Cayley transformation:

(2.4) $\qquad Z = (\operatorname{Ad} c)Y, \quad X_\pm = (\operatorname{Ad} c)N_\pm, \quad c := \rho(c_1).$

$(Y, N_\pm)$ and $(Z, X_\pm)$ define direct sum decompositions of H and H_C whose summands are

(2.5) $\qquad P_\lambda^{(\lambda+2k)} := N_-^k(H(Y; \lambda+2k) \cap \operatorname{Ker} N_+),$

(2.6) $\qquad Q_\lambda^{(\lambda+2k)} := X_-^k(H_C(Z; \lambda+2k) \cap \operatorname{Ker} X_+),$

for all eigenvalues $\lambda \in \{0, \pm 1, \pm 2, \ldots, \pm w\}$ of Y and Z and for $k \geq \max\{-\lambda, 0\}$, respectively. Here we denote by $H(Y; \lambda+2k)$ etc. the eigenspace of an endomorphism

Y of H with eigenvalue $\lambda + 2k$. Since ρ is horizontal at $r = \{H_r^{p,q}\}$, (2.6) is compatible with this Hodge structure and we set

$$(2.7) \qquad Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k} := Q_\lambda^{(\lambda + 2k)} \cap H_r^{a+k, b+\lambda+k} \qquad (a, b \geq 0).$$

These form a refined direct sum decomposition which we call the *Hodge-$(Z, X_\pm)$ decomposition* of (ρ, r) (cf. Remark (2.12) below). Transforming this by the inverse c^{-1} of the Cayley element, we define

$$(2.8) \qquad P_\lambda^{(\lambda + 2k)a + k, b + k} := c^{-1} Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k}.$$

LEMMA (2.9). (i) $\sigma Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k} = Q_{-\lambda}^{(-\lambda + 2(\lambda + k))b + \lambda + k, a + k}$.

(ii) $c Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k} = c^2 P_\lambda^{(\lambda + 2k)a + k, b + k} = P_{-\lambda}^{(-\lambda + 2(\lambda + k))a + \lambda + k, b + \lambda + k}$.

$c^{-1} P_\lambda^{(\lambda + 2k)a + k, b + k} = c^{-2} Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k} = Q_{-\lambda}^{(-\lambda + 2(\lambda + k))a + \lambda + k, b + k}$.

PROOF. It is easy to see, by definition, that $c P_\lambda^{(\lambda + 2k)} = Q_\lambda^{(\lambda + 2k)}$. Hence, by the first equality in (1.20.ii), we have

$$\sigma Q_\lambda^{(\lambda + 2k)} = \sigma c P_\lambda^{(\lambda + 2k)} = c^{-1} \sigma P_\lambda^{(\lambda + 2k)} = c^{-1} P_\lambda^{(\lambda + 2k)} = c^{-2} Q_\lambda^{(\lambda + 2k)}.$$

On the other hand, by the second equality in (1.20.ii), the third and the second equalities in (1.20.i), we see that on $P_\lambda^{(\lambda + 2k)}$

$$c^{-2} = \begin{cases} (-i)^{\lambda + 2k} \dfrac{k!}{(\lambda + k)!} N_-^\lambda & \text{if } \lambda \geq 0, \\[2ex] (-i)^{\lambda + 2k} \dfrac{(\lambda + k)!}{k!} N_+^{-\lambda} & \text{if } \lambda < 0. \end{cases}$$

Taking their Cayley transforms, we see that on $Q_\lambda^{(\lambda + 2k)}$

$$(2.10) \qquad c^{-2} = \begin{cases} (-i)^{\lambda + 2k} \dfrac{k!}{(\lambda + k)!} X_-^\lambda & \text{if } \lambda \geq 0, \\[2ex] (-i)^{\lambda + 2k} \dfrac{(\lambda + k)!}{k!} X_+^{-\lambda} & \text{if } \lambda < 0. \end{cases}$$

Thus, by the definition of the $Q_\lambda^{(\lambda + 2k)}$, we have in both cases that

$$\sigma Q_\lambda^{(\lambda + 2k)} = c^{-2} Q_\lambda^{(\lambda + 2k)} = X_\mp^{\pm\lambda} Q_\lambda^{(\lambda + 2k)} = Q_{-\lambda}^{(\lambda + 2k)} = Q_{-\lambda}^{(-\lambda + 2(\lambda + k))}.$$

This together with $\sigma H_r^{a+k, b+\lambda+k} = H_r^{b+\lambda+k, a+k}$ yields the assertion (i).

By horizontality, $X_- \in \mathfrak{g}_C^{1, -1}$ and $X_+ \in \mathfrak{g}_C^{-1, 1}$, hence $X_\mp^{\pm\lambda} \in \mathfrak{g}_C^{\lambda, -\lambda}$. This together with (2.10) shows that

$$c^{-2} Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k} = X_\mp^{\pm\lambda} Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k} = Q_{-\lambda}^{(-\lambda + 2(\lambda + k))a + \lambda + k, b + k}.$$

Thus we obtain the second equality in (ii). The first equality in (ii) follows from the second. ∎

COROLLARY (2.11). *Let (ρ, r) be as above. For each eigenvalue λ of Y and for $k \geq \max\{-\lambda, 0\}$, we see that*

$$C \otimes P_\lambda^{(\lambda + 2k)} = \bigoplus_{\substack{a + b + 2k = w - \lambda \\ a, b \geq 0}} P_\lambda^{(\lambda + 2k)a + k, b + k}$$

is a Hodge structure of weight $w - \lambda$. Moreover, in the case $\lambda = k = 0$, this is S-polarized.

PROOF. We should observe the behavior under the complex conjugation σ:

$$\sigma P_\lambda^{(\lambda + 2k)a + k, b + k} = \sigma c^{-1} Q_\lambda^{(\lambda + 2k)a + k, b + \lambda + k} = c Q_{-\lambda}^{(-\lambda + 2(\lambda + k))b + \lambda + k, a + k}$$

$$= c^2 P_{-\lambda}^{(-\lambda + 2(\lambda + k))b + \lambda + k, a + \lambda + k} = P_\lambda^{(\lambda + 2k)b + k, a + k}.$$

This shows the first assertion.

The representation ρ is trivial on $Q_0^{(0)}$, hence $P_0^{(0)a,b} = c^{-1} Q_0^{(0)a,b} = Q_0^{(0)a,b}$, and so the second assertion trivially holds. ■

We call a direct sum decomposition in (2.11) the *mixed Hodge-$(Y, N_\pm)$ decomposition of (ρ, r).*

REMARK (2.12). We remark here some observations which are verified easily by (1.20.i), their Cayley transforms and horizontality of ρ at r. A Hodge-$(Z, X_\pm)$ decomposition and a mixed Hodge-$(Y, N_\pm)$ decomposition form "nests of diamonds", respectively. For example, in the case of weight $w = 3$, these nests of diamonds are illustrated respectively as in Figures 1 and 2.
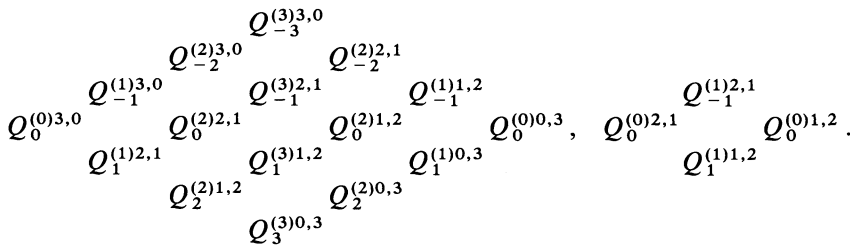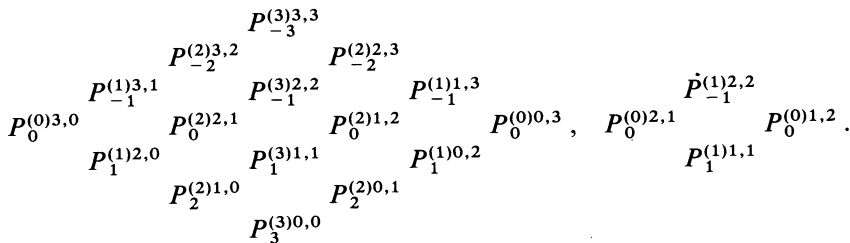
FIGURE 1.

FIGURE 2.

On these nests of diamonds, the complex conjugation by σ sends respectively a summand $Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}$ to a summand $Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k}$ which are symmetric with respect to the origin of the diamonds, and a summand $P_\lambda^{(\lambda+2k)a+k,b+k}$ to a summand $P_\lambda^{(\lambda+2k)b+k,a+k}$ which are symmetric with respect to the vertical axis. The operator X_+ (resp. X_-) sends a summand $Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}$ one step down (resp. up) to a summand $Q_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1,b+\lambda+2+k-1}$ (resp. $Q_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1,b+\lambda-2+k+1}$), and $X_\pm$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. Similarly, the operator N_+ (resp. N_-) sends a summand $P_\lambda^{(\lambda+2k)a+k,b+k}$ one step down (resp. up) to a summand $P_{\lambda+2}^{(\lambda+2+2(k-1))a+k-1,b+k-1}$ (resp. $P_{\lambda-2}^{(\lambda-2+2(k+1))a+k+1,b+k+1}$), and $N_\pm$ are inverse to each other up to non-zero constant between these summands whenever both summands actually appear in the nest of diamonds. The Cayley element c transforms the second nest of diamonds together with the action of the operators $Y, N_\pm$ to the first nest of diamonds together with the action of the operators $Z, X_\pm$: $cP_\lambda^{(\lambda+2k)a+k,b+k} = Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k}$.

By using these operators, we can explain why the summands outside the nests of diamonds vanish in the following way. We claim first that $Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = 0$ for $\lambda > 0$ and $b < 0$. Indeed, $X_-^{\lambda+k}$ is injective on this summand by the Cayley transform of the third equality in (1.20.i). On the other hand, looking at the Hodge type, we see that $X_-^{\lambda+k}Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} \subset Q_{-\lambda-2k}^{(-\lambda-2k+2(\lambda+2k))a+\lambda+2k,b} = 0$ by horizontality. Thus we get our claim. It follows by symmetry under the complex conjugation σ that $Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = 0$ for $\lambda < 0$ and $a < 0$. Finally, by the inverse of the Cayley transformation, we have $P_\lambda^{(\lambda+2k)a+k,b+k} = 0$ for $\lambda > 0$ and $b < 0$, and for $\lambda < 0$ and $a < 0$.

We call the length of the side of the biggest diamond in a nest the *size* of the nest of diamonds.

Another remark is that a mixed Hodge-$(Y, N_\pm)$ decomposition is nothing but the limiting split mixed Hodge structure of the associated SL_2-orbit $\tilde\rho : U \to D$, $\tilde\rho(gi) := \rho(g)r$ ($g \in SL_2(\mathbf{R})$), and the monodromy weight filtration L is described as $L_i = \bigoplus_{\lambda \le i} \bigoplus_k P_{-\lambda}^{(\lambda+2k)}$ (cf. [Sc, (6.16)] and its proof, [CK, pp. 13–14]).

In the above notation, for all λ, a and b, put

(2.13)
$$n_\lambda := \dim_{\mathbf{R}} H(Y; \lambda) = \dim_{\mathbf{C}} H_C(Z; \lambda) \, ,$$

$$p_\lambda^{a,b} := \dim_{\mathbf{C}} P_{\lambda-2k}^{(\lambda)a+k,b+k} = \dim_{\mathbf{C}} Q_{\lambda-2k}^{(\lambda)a+k,b+\lambda-k} \, .$$

Notice that, by construction, the middle terms and the terms on the extreme right hand side of the second equality in (2.13) are independent of k (cf. Remark (2.12)).

LEMMA (2.14).   *For (ρ, r) as above, the following hold*:
(i) $\sum_{a+b=w-\lambda} p_\lambda^{a,b} = n_\lambda - n_{\lambda+2}$ *for all* $0 \le \lambda \le w$.
(ii) $p_\lambda^{b,a} = p_\lambda^{a,b}$ *for all* λ, a, b *with* $0 \le \lambda \le w$, $a \ge 0$, $b \ge 0$ *and* $a+b = w - \lambda$.
(iii) $h^{a,b} = h^{a+1,b-1} - (p_0^{a+1,b-1} + p_1^{a+1,b-2} + \cdots + p_{b-1}^{a+1,0}) + (p_0^{a,b} + p_1^{a-1,b} + \cdots +$

$p_a^{0,b}$) *for all* a, b *with* $a \geq 0$, $b \geq 0$ *and* $a + b = w$.

PROOF. We first observe that there is an exact sequence

$$0 \longrightarrow P_\lambda^{(\lambda)} \longrightarrow H(Y; \lambda) \xrightarrow{N_+} H(Y; \lambda+2) \longrightarrow 0$$

for every $\lambda \geq 0$ (and N_- yields a right splitting). (i) and (ii) follow from this and (2.11).

In order to show (iii), we look at the morphism $X_+ : H^{a+1,b-1} \rightarrow H^{a,b}$ and its kernel and cokernel:

$$\mathrm{Ker} = Q_0^{(0)a+1,b-1} \oplus Q_1^{(1)a+1,b-1} \oplus \cdots \oplus Q_{b-1}^{(b-1)a+1,b-1}$$

$$\xleftarrow{c} P_0^{(0)a+1,b-1} \oplus P_1^{(1)a+1,b-2} \oplus \cdots \oplus P_{b-1}^{(b-1)a+1,0},$$

$$\mathrm{Coker} \simeq Q_0^{(0)a,b} \oplus Q_{-1}^{(1)a,b} \oplus \cdots \oplus Q_{-a}^{(a)a,b}$$

$$\simeq Q_0^{(0)a,b} \oplus Q_1^{(1)a-1,b+1} \oplus \cdots \oplus Q_a^{(a)0,b+a}$$

$$\xleftarrow{c} P_0^{(0)a,b} \oplus P_1^{(1)a-1,b} \oplus \cdots \oplus P_a^{(a)0,b}.$$

Looking at the dimension, we get (iii). ∎

DEFINITION (2.15). *We call a set of integers* $\{p_\lambda^{a,b}\}$, *which satisfies the conditions* (i), (ii) *and* (iii) *of* (2.14), *a set of primitive Hodge numbers belonging to* $\{h^{p,q}, n_\lambda\}$.

3. Admissible R-semi-simple elements.

We continue to use the notation in the previous sections.

PROPOSITION (3.1). *Given a pair* $(Y, r) \in \mathfrak{g} \times D$, *there exists at most one representation* $\rho : SL_2(\mathbf{R}) \rightarrow G$ *which is horizontal at* r *and* $\rho_*(y) = Y$.

PROOF. Since y and z generate $\mathfrak{sl}_2(C)$, it is enough to show that if such a representation ρ exists then the eigenspaces of Z, and hence Z itself, are determined by the pair (Y, r). Actually, we shall show by induction on the size w of the nest of diamonds of the Hodge-$(Z, X_\pm)$ decomposition (2.7) (cf. Remark (2.12)) that this nest of diamonds is completely determined by (Y, r).

First notice that

(3.2) $$Y = i(X_+ - X_-).$$

For a subspace M of H_C, we put, throughout this proof,

$$M^\perp := \{v \in H_C \mid S(v, \sigma u) = 0 \text{ for all } u \in M\},$$

$$\text{projection } \{M \rightarrow H_r^{p,q}\} := \mathrm{Im}\left\{M \subset H_C = \bigoplus_{p'+q'=w} H_r^{p',q'} \rightarrow H_r^{p,q}\right\}.$$

Then we see that

$$Q_w^{(w)0,w} = \text{projection} \{ Y^w H_r^{w,0} \to H_r^{0,w} \},$$

$$Q_{w-2k}^{(w)k,w-k} = \text{projection} \{ Y^k Q_w^{(w)0,w} \to H_r^{k,w-k} \} \qquad (0 \le k \le w),$$

$$\bigoplus_{0 \le \lambda \le w-1} Q_{-\lambda}^{(\lambda)w,0} = H_r^{w,0} \cap (Q_{-w}^{(w)w,0})^\perp,$$

$$Q_{w-1}^{(w-1)1,w-1} = \text{projection} \left\{ Y^{w-1} \left(\bigoplus_{0 \le \lambda \le w-1} Q_{-\lambda}^{(\lambda)w,0} \right) \to H_r^{1,w-1} \right\},$$

$$Q_{w-1-2k}^{(w-1)1+k,w-1-k} = \text{projection} \{ Y^k Q_{w-1}^{(w-1)1,w-1} \to H_r^{1+k,w-1-k} \} \quad (0 \le k \le w-1),$$

$$\bigoplus_{0 \le \lambda \le w-2} Q_{-\lambda}^{(\lambda)w,0} = H_r^{w,0} \cap \left(\bigoplus_{w-1 \le \lambda \le w} Q_{-\lambda}^{(\lambda)w,0} \right)^\perp,$$

$$Q_{w-2}^{(w-2)2,w-2} = \text{projection} \left\{ Y^{w-2} \left(\bigoplus_{0 \le \lambda \le w-2} Q_{-\lambda}^{(\lambda)w,0} \right) \to H_r^{2,w-2} \right\},$$

$$Q_{w-2-2k}^{(w-2)2+k,w-2-k} = \text{projection} \{ Y^k Q_{w-2}^{(w-2)2,w-2} \to H_r^{2+k,w-2-k} \} \quad (0 \le k \le w-2),$$

$$\cdots \cdots .$$

Thus $Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k}$ $(0 \le \lambda \le w, 0 \le k \le \lambda)$ are determined. Taking the complex conjugation by σ of these, we get $Q_{-\lambda+2k}^{(\lambda)\lambda-k,w-\lambda+k} = \sigma Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k}$ $(0 \le \lambda \le w, 0 \le k \le \lambda)$. Applying the induction hypothesis to the nest of diamonds of size $\le w-2$ in

$$\left(\bigoplus_{\substack{0 \le \lambda \le w \\ 0 \le k \le \lambda}} (Q_{\lambda-2k}^{(\lambda)w-\lambda+k,\lambda-k} \oplus Q_{-\lambda+2k}^{(\lambda)\lambda-k,w-\lambda+k}) \right)^\perp$$

(cf. Remark (2.12)), we get our assertion.   ∎

DEFINITION (3.3). A pair $(Y, r) \in \mathfrak{g} \times D$ is admissible if there exists a representation $\rho : SL_2(\mathbf{R}) \to G$ which is horizontal at r and $\rho_*(y) = Y$.

The set of primitive Hodge numbers $\{ p_\lambda^{a,b} \}$ belonging to $\{ h^{p,q}, n_\lambda \}$ is called the type of an admissible pair (Y, r).

$Y \in \mathfrak{g}$ is said to be admissible if (Y, r) is an admissible pair for some $r \in D$.

Now we prove the following numerical criterion for admissibility:

THEOREM (3.4). $Y \in \mathfrak{g}$ is admissible if and only if Y is semi-simple over $\mathbf{R}$ whose eigenvalues are contained in $\{ 0, \pm 1, \pm 2, \dots, \pm w \}$ and there exists a set of primitive Hodge numbers $\{ p_\lambda^{a,b} \}$ belonging to $\{ h^{p,q}, n_\lambda \}$, where $n_\lambda := \dim H(Y; \lambda)$ (cf. Definition (2.15)).

PROOF. Since Y is semi-simple over $\mathbf{R}$, the eigenspaces $H(Y; \lambda)$ are defined over $\mathbf{R}$ and $H(Y; \lambda)$ and $H(Y; \mu)$ are S-orthogonal unless $\lambda + \mu = 0$. Therefore $H(Y; \lambda)$ and $H(Y; -\lambda)$ are S-dual.

Since $n_{\lambda'} - n_{\lambda'+2} \ge 0$ for $\lambda' \ge 0$ by the condition (2.14.i), we can take a direct sum

decomposition

$$(3.5) \qquad H(Y; \lambda) = P_\lambda^{(\lambda)} \oplus P_\lambda^{(\lambda+2)} \oplus P_\lambda^{(\lambda+4)} \oplus \cdots \qquad \text{for} \quad \lambda \geq 0,$$

with $\dim P_\lambda^{(\lambda+2k)} = n_{\lambda+2k} - n_{\lambda+2k+2}$. Moreover, in the case $\lambda = 0$, the decomposition (3.5) can be taken to be S-orthogonal. We denote the S-dual decomposition by

$$(3.6) \qquad H(Y; -\lambda) = P_{-\lambda}^{(\lambda)} \oplus P_{-\lambda}^{(\lambda+2)} \oplus P_{-\lambda}^{(\lambda+4)} \oplus \cdots \qquad (\lambda \geq 0),$$

i.e., $P_{-\lambda}^{(\lambda+2k)}$ and $P_\lambda^{(\lambda+2m)}$ are S-orthogonal unless $k = m$.

By the conditions (i) and (ii) of (2.14), we can choose a Hodge decomposition

$$(3.7) \qquad C \otimes P_\lambda^{(\lambda+2k)} = \bigoplus_{\substack{a+b+2k=w-\lambda \\ a,b \geq 0}} P_\lambda^{(\lambda+2k)a+k,b+k} \qquad \text{for} \quad \lambda \geq 0, \quad k \geq 0,$$

with $\dim P_\lambda^{(\lambda+2k)a+k,b+k} = p_{\lambda+2k}^{a,b}$. Moreover, in the case $\lambda = k = 0$, the Hodge structure (3.7) can be chosen to be S-polarized. We denote the $S(\cdot, \sigma \cdot)$-orthogonal decomposition by

$$(3.8) \qquad C \otimes P_{-\lambda}^{(-\lambda+2(\lambda+k))} = \bigoplus_{\substack{a+b+2\lambda+2k=w+\lambda \\ a,b \geq 0}} P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k} \qquad (\lambda \geq 0, k \geq 0),$$

i.e., $S(P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}, \sigma P_\lambda^{(\lambda+2k)a'+k,b'+k}) = 0$ unless $(a,b) = (a',b')$. Notice that $P_{-\lambda}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k} = P_{-\lambda}^{(\lambda+2k)a+\lambda+k,b+\lambda+k}$.

Now we consider the cases $\lambda \geq 0$ and $\lambda < 0$ altogether. For $k \geq \max\{-\lambda, 0\}$ and $a \geq b$, let

$$(3.9) \qquad \{v_{\lambda,j}^{(\lambda+2k)a+k,b+k} \mid 1 \leq j \leq p_{\lambda+2k}^{a,b}\}$$

be a C-basis of $P_\lambda^{(\lambda+2k)a+k,b+k}$ such that

$$(3.10) \qquad S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}, \sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k}) = \delta_{jj'} (-1)^a i^{w-\lambda} \Bigg/ \binom{\lambda+2k}{k}.$$

In the case $a = b$, we can moreover take the above basis (3.9) to consist of real elements. Put

$$(3.11) \qquad v_{\lambda,j}^{(\lambda+2k)b+k,a+k} = \sigma v_{\lambda,j}^{(\lambda+2k)a+k,b+k} \qquad (a \geq b).$$

Define now C-linear endomorphisms $N_\pm$ of H_C by

$$(3.12) \qquad \begin{aligned} N_+ v_{\lambda,j}^{(\lambda+2k)a+k,b+k} &:= k v_{\lambda+2,j}^{((\lambda+2)+2(k-1))a+k-1,b+k-1}, \\ N_- v_{\lambda,j}^{(\lambda+2k)a+k,b+k} &:= (\lambda+k) v_{\lambda-2,j}^{((\lambda-2)+2(k+1))a+k+1,b+k+1}, \end{aligned}$$

for all λ, non-negative a, b and $k \geq \max\{-\lambda, 0\}$. By construction, it is easy to see that $N_\pm$ commute with the complex conjugation σ and satisfy the commutation relations: $[N_+, N_-] = Y$, and $[Y, N_\pm] = \pm 2 N_\pm$, respectively. It is also easy to verify that $S(N_\pm \cdot, \cdot) + S(\cdot, N_\pm \cdot) = 0$, respectively. Indeed, for example, one can compute as

$$S(N_+ v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}, \sigma v_{\lambda-2,j'}^{((\lambda-2)+2(k+1))a+k+1,b+k+1})$$

$$+ S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}, N_+\sigma v_{\lambda-2,j'}^{((\lambda-2)+2(k+1))a+k+1,b+k+1})$$

$$= \delta_{jj'}(-1)^a i^{w-\lambda+2} \frac{(\lambda+k)(\lambda+k-1)!(k+1)!}{(\lambda+2k)!} + \delta_{jj'}(-1)^a i^{w-\lambda}\frac{(k+1)k!(\lambda+k)!}{(\lambda+2k)!} = 0.$$

Thus we see that $N_\pm \in \mathfrak{g}$ and hence there exists a unique representation

(3.13) $\rho: SL_2(\mathbf{R}) \to G$ such that $\rho_*(y) = Y$ and $\rho_*(n_\pm) = N_\pm$,   respectively.

By using the Cayley element $c := \rho(c_1) \in G_{\mathbf{C}}$, we define

$$(3.14) \quad Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} := cP_\lambda^{(\lambda+2k)a+k,b+k}, \quad H^{p,q} := \bigoplus_{\substack{a+k=p \\ b+\lambda+k=q}} Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k},$$

where, on the right hand side of the second equality, the summation is taken over all the eigenvalues λ of Y, all integers $k \geq \max\{-\lambda, 0\}$ and all non-negative integers a, b with $a+b+\lambda+2k = w$. This defines a Hodge structure. Indeed, by using (1.20.ii), one sees that

$$\sigma Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = \sigma c P_\lambda^{(\lambda+2k)a+k,b+k} = c^{-1}\sigma P_\lambda^{(\lambda+2k)a+k,b+k}$$

$$= c^{-1}P_\lambda^{(\lambda+2k)b+k,a+k} = cP_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+\lambda+k} = Q_{-\lambda}^{(-\lambda+2(\lambda+k))b+\lambda+k,a+k},$$

and hence $\sigma H^{p,q} = H^{q,p}$. One can moreover verify that (3.14) is S-polarized. Indeed, the direct sum in (3.14) is S-orthogonal by construction and, for

$$cv_{\lambda,j}^{(\lambda+2k)a+k,b+k}, \quad cv_{\lambda,j'}^{(\lambda+2k)a+k,b+k} \in Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} \subset H^{p,q},$$

one can compute as

$$i^{p-q}S(cv_{\lambda,j}^{(\lambda+2k)a+k,b+k}, \sigma cv_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$

$$= i^{a-b-\lambda}S(cv_{\lambda,j}^{(\lambda+2k)a+k,b+k}, c^{-1}\sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$

$$= i^{a-b-\lambda}S(c^2 v_{\lambda,j}^{(\lambda+2k)a+k,b+k}, \sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$

$$= i^{a-b-\lambda+\lambda+2k}S(v_{-\lambda,j}^{(-\lambda+2(\lambda+k))a+\lambda+k,b+\lambda+k}, \sigma v_{\lambda,j'}^{(\lambda+2k)a+k,b+k})$$

$$= \delta_{jj'}i^{a-b+2k+2a+w-\lambda} \left/ \binom{\lambda+2k}{k}\right. = \delta_{jj'}\left/ \binom{\lambda+2k}{k}\right..$$

Thus we have $\{H^{p,q}\} \in D$.

Finally, we claim that the representation ρ in (3.13) is horizontal at $\{H^{p,q}\} \in D$. Indeed, since $Z = (\mathrm{Ad}\, c)Y$, $X_\pm = (\mathrm{Ad}\, c)N_\pm$, one can compute, by (1.20), as

$$ZQ_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = cYP_\lambda^{(\lambda+2k)a+k,b+k} = Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k},$$

$$X_\pm Q_\lambda^{(\lambda+2k)a+k,b+\lambda+k} = cN_\pm P_\lambda^{(\lambda+2k)a+k,b+k}$$

$$= cP_{\lambda \pm 2}^{((\lambda \pm 2) + 2(k \mp 1))a + k \mp 1, b + k \mp 1} = Q_{\lambda \pm 2}^{((\lambda \pm 2) + 2(k \mp 1))a + k \mp 1, b + \lambda + k \pm 1} .$$

This completes the proof of the theorem.   ∎

We remark that the condition on $\{n_\lambda\}$ in Theorem (3.4) coincides with the one in [CK, (2.20)] in the case of weight 2.

Fix identifications $D \simeq G/V$ and $R \simeq G/K$, where K is a maximal compact subgroup of G containing V and R is the associated Riemannian symmetric domain, and let θ_K be the associated Cartan involution. We denote the projection by

(3.15) $\pi : D \simeq G/V \to G/K \simeq R$.

PROPOSITION (3.16).   *We use the notation in Theorem (3.4). Let $Y \in \mathfrak{g}$ be an admissible element.*

(i)   *If there exists $r \in \pi^{-1}([K])$ such that (Y, r) is an admissible pair, then $\theta_r Y = -Y$, where θ_r is the Cartan involution on $\mathfrak{g}$ induced from (1.11).*

(ii)   *If $\theta_K Y = -Y$, then there exists $r \in \pi^{-1}([K])$ such that (Y, r) is an admissible pair.*

(iii)   *For each set of primitive Hodge numbers $\{p_\lambda^{a,b}\}$ belonging to $\{h^{p,q}, n_\lambda\}$, $G_Y := \{g \in G \mid (\mathrm{Ad}\, g) Y = Y\}$ acts transitively on the set $\{r \in D \mid (Y, r)$ is an admissible pair of type $\{p_\lambda^{a,b}\}\}$.*

PROOF.   (i) follows from (3.2) and (1.11).

(ii):   Assume $\theta_K Y = -Y$. Take a point $r' \in D$ at which Y is admissible and let K' be the maximal compact subgroup of G associated to the Cartan involution $\theta_{r'}$. By the result in (i) for (Y, r') and the assumption, Y can be viewed as a tangent vector to R at $[K']$ as well as at $[K]$: $Y \in T_R([K'])$, $Y \in T_R([K])$. By the transitivity of tangent spaces of a Riemannian symmetric domain, there exists $g \in G$ such that $(\mathrm{Int}\, g) K' = K$ and $(\mathrm{Ad}\, g) Y = Y$. Hence the admissibility of (Y, r') implies that of $((\mathrm{Ad}\, g) Y, gr') = (Y, gr')$, where $gr' \in \pi^{-1}([K])$.

(iii):   Suppose that $r, r' \in D$ are points at which Y is admissible of the same type $\{p_\lambda^{a,b}\}$. Let $\rho, \rho' : SL_2(\mathbf{R}) \to G$ be the corresponding representations. It is enough to show that there exists $g \in G$ such that $\rho' = (\mathrm{Int}\, g)\rho$. Indeed, if this is the case, then $(\mathrm{Ad}\, g) Y = (\mathrm{Ad}\, g)(\rho_*(y)) = \rho'_*(y) = Y$ and $gr = g\tilde{\rho}(i) = \tilde{\rho}'(i) = r'$.

We can construct such a $g \in G$ elementarily by using bases of $H_\mathbf{C}$ according to the S-polarized Hodge-$(Z, X_\pm)$ decompositions, where $(Z, X_\pm) = (\rho_*(z), \rho_*(x_\pm))$, $(\rho'_*(z), \rho'_*(x_\pm))$. Thus we get our assertion.   ∎

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