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# RECOVERY OF VANISHING CYCLES BY LOG GEOMETRY

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**Abstract.** We first construct compatible actions of the product of the unit interval and the unit circle as a monoid on a semi-stable degeneration of pairs and on the associated log topological spaces. Then we show that the log topological family is locally trivial in piecewise smooth category over the base, i.e., the associated log topological family recovers the vanishing cycles of the original semi-stable degeneration in the most naive sense. Using this result together with the log Riemann-Hilbert correspondence, we introduce two types of integral structure of the variation of mixed Hodge structure associated to a semi-stable degeneration of pairs.

**Introduction.** Kato and Nakayama [KN] constructed a ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ over a given fs log analytic space $(X, \mathcal{M}_X)$ and proved a log version of the Riemann-Hilbert correspondence on them (cf. (1.2), (1.3), (1.4)).

In the case where the fs log analytic space $(X, \mathcal{M}_X)$ is the one corresponding to a divisor $D$ with normal crossings on a complex manifold $X$, i.e., $\mathcal{M}_X := \{f \in \mathcal{O}_X \mid f$ is invertible outside $D_{\mathrm{red}}\}$ (cf. (1.1.4)), the projection $\tau_X : X^{\log} \to X$ is nothing but the real oriented blowing-up of $X$ along $D_{\mathrm{red}}$ (cf. (1.2.1)).

Let us consider a relative case. Let $f : X \to \Delta$ be a proper surjective flat morphism of a complex manifold onto an open disc such that $f$ is smooth over the punctured disc $\Delta^* := \Delta - \{0\}$ and that the central fiber $X_0 := f^{-1}(0)$ is a reduced divisor with simple normal crossings. Let $Y$ be a divisor on $X$, flat with respect to $f$. We assume that $X_0 + Y$ is also a divisor with simple normal crossings. Then, by [KN], we can construct a map $f^{\log} : X^{\log} \to \Delta^{\log}$ and a subspace $Y^{\log}$ of $X^{\log}$ over the given ones and we have a commutative diagram:

$$\begin{array}{ccc} (X^{\log}, Y^{\log}) & \xrightarrow{\tau_X} & (X, Y) \\ {\scriptstyle f^{\log}}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^{\log} & \xrightarrow{\tau_\Delta} & \Delta\,. \end{array} \tag{0.1}$$

The main result in the present paper is that the family

$$\overset{\circ}{f}{}^{\log} : (X^{\log} - Y^{\log}) \to \Delta^{\log} \tag{0.2}$$

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of open spaces is locally piecewise $C^\infty$ trivial over the base $\Delta^{\log}$ (Theorem (5.4)). As a consequence, we see that the family (0.2) is the one which recovers the vanishing cycles, in the most naive sense, of the degenerating family $\overset{\circ}{f}\colon (X - Y) \to \Delta$.

This implies, in particular, that $L_{\mathbf{Z}} := R^q(\overset{\circ}{f}{}^{\log})_*\mathbf{Z}$ is a locally constant sheaf of $\mathbf{Z}$-modules on $\Delta^{\log}$. On the other hand, Steenbrink and Zucker [SZ] showed that $\mathcal{V} := R^q f_* \Omega^\bullet_{X/\Delta}(\log(X_0 + Y))$ is a free $\mathcal{O}_\Delta$-module with the Gauss-Manin connection $\nabla$, filtered by the $W(Y)$-relative monodromy weight filtration $M$. We thus have

$$\mathcal{V} \simeq (\tau_\Delta)_*(\mathcal{O}_\Delta^{\log} \otimes_{\mathbf{C}} L_{\mathbf{C}}) \quad \text{on } \Delta$$

(Theorem (6.2)), under the log version of the Riemann-Hilbert correspondence established in [KN]. As a corollary, we have two types of integral structure of the degenerate variation of mixed Hodge structure on $\mathcal{V}$ (Theorem (6.4)).

We prove the main theorem (5.4) in a manner analogous to that in Clemens [C]. We first construct a suitable family of multi-valued $C^\infty$ global equations of the components of the divisor $X_0 + Y$ (Propositions (3.2), (4.3)) and with its aid we introduce compatible actions of the monoid $S := [0, 1] \times \mathbf{C}_1$ on the diagram (0.1), so that [0, 1] acts as shrinking and $\mathbf{C}_1$ acts as rounding (Theorem (5.2)).

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*Convention.* In this paper, for every partition of unity $\{\rho_W\}_{W\in\mathcal{W}}$ subordinate to a covering $\mathcal{W}$ of a manifold, the closure of $\operatorname{supp}\rho_W$ is assumed to be contained in $W$ for every $W \in \mathcal{W}$.

**1. Preliminary: Log geometry.** We summarize here the definitions of the notions and results in log geometry introduced and proved by Kato and Nakayama [KN], for our later use.

(1.1) A commutative semigroup with unity is called a *monoid.* A *homomorphism* of monoids is assumed to preserve the unity.

A monoid $P$ is called an *fs monoid* if the following three conditions are satisfied:

(1.1.1) $P$ is finitely generated.

(1.1.2) If $a, b, c \in P$ and $ab = ac$, then $b = c$.

(1.1.3) If $a \in P^{\mathrm{gp}}$ and $a^n \in P$ for some positive integer $n$, then $a \in P$. Here $P^{\mathrm{gp}}$ is the abelian group associated to $P$.

Let $X$ be a ringed space and $\mathcal{O}_X$ its sheaf of rings. A *pre-log structure* on $X$ is a sheaf of monoids $\mathcal{M}$ on $X$ endowed with a homomorphism of sheaves of monoids

$$\alpha : \mathcal{M} \to \mathcal{O}_X ,$$

where $\mathcal{O}_X$ is regarded as a sheaf of monoids with respect to multiplication. A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of ringed spaces with pre-log structures is a pair $(f, \varphi)$ consisting of a morphism $f$ of ringed spaces and a homomorphism $\varphi$ of pre-log structures which make the following diagram commutative:

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{\varphi} & \mathcal{M}_X \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f} & \mathcal{O}_X . \end{array}$$

A pre-log structure $\mathcal{M}$ is called a *log structure* if $\alpha$ induces an isomorphism

$$\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times ,$$

where $\mathcal{O}_X^\times$ is the subsheaf of $\mathcal{O}_X$ consisting of invertible elements. In this case, we regard $\mathcal{O}_X^\times$ as a subsheaf of $\mathcal{M}$ via the above isomorphism.

If $M$ is a pre-log structure on $X$, the *log structure* $\mathcal{M}^{\mathrm{a}}$ *associated to* $\mathcal{M}$ is defined to be the push out of the diagram

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^\times) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^\times & & \end{array}$$

in the category of sheaves of monoids, endowed with the induced homomorphism $\alpha^{\mathrm{a}} : \mathcal{M}^{\mathrm{a}} \to \mathcal{O}_X$.

Note that, in this case, we have $\mathcal{M}^{\mathrm{a}}/\mathcal{O}_X^\times \simeq \mathcal{M}/\alpha^{-1}(\mathcal{O}_X^\times)$.

A log structure $\mathcal{M} = (\mathcal{M}, \alpha)$ on $X$ is called an *fs log structure* if locally on $X$ there exists an fs monoid $P$ and a homomorphism $\beta : P \to \mathcal{O}_X$, where $P$ is regarded as a constant sheaf on $X$, such that $(\mathcal{M}, \alpha)$ is isomorphic to the associated log structure $(P^{\mathrm{a}}, \beta^{\mathrm{a}})$. In this case, $(P, \beta)$ with an isomorphism $P^{\mathrm{a}} \simeq \mathcal{M}$ is called a *chart* of $\mathcal{M}$. Charts exist locally on $X$.

Note that if $\mathcal{M}$ is an fs log structure, the stalk $(\mathcal{M}/\mathcal{O}_X^\times)_x$ is a torsion free fs monoid.

(1.1.4) EXAMPLE. If $X$ is a complex manifold and $D$ is a reduced divisor on $X$ with normal crossings, then

$$\mathcal{M} := \{f \in \mathcal{O}_X \mid f \text{ is invertible outside } D\} \overset{\alpha}{\hookrightarrow} \mathcal{O}_X$$

is an fs log structure, which is called the *log structure corresponding to D*. In fact, locally on $X$, $D$ is defined by $\prod_{1\le i\le r} z_i = 0$, and $\mathcal{M}$ is associated to

$$\alpha : \boldsymbol{N}^r \to \mathcal{O}_X , \quad \alpha(n) := \prod_{1\le i\le r} z_i^{n(i)} , \quad \text{where } n = (n(1), \dots , n(r)) .$$

Here $z_1, \dots , z_r$ is regarded as a part of local coordinates on $X$.

(1.2) Let $\boldsymbol{C}_1 := \{u \in \boldsymbol{C}^\times \mid |u| = 1\}$ be the unit circle and let $\boldsymbol{R}_{\ge 0}$ be the set of all non-negative real numbers. We consider $\boldsymbol{R}_{\ge 0} \times \boldsymbol{C}_1$ as a monoid by multiplication. The monoid homomorphism $\boldsymbol{R}_{\geq 0} \times \boldsymbol{C}_1 \ni (r, u) \mapsto ru \in \boldsymbol{C}$ makes

$$T := (\operatorname{Spec} \boldsymbol{C}, \boldsymbol{R}_{\geq 0} \times \boldsymbol{C}_1)$$

a log point. Let $X$ be an fs log analytic space. Then the *associated log topological space* $X^{\log}$ is defined to be the $T$-valued points

$$X^{\log} := \operatorname{Hom}(T, X)$$

as a set. This can be identified with the set

$$\{(x, h) \in X \times \operatorname{Hom}(\mathcal{M}^{\mathrm{gp}}_{X,x}, \boldsymbol{C}_1) \mid h \text{ extending } f \mapsto f/|f|, \ (f \in \mathcal{O}^{\times}_X)\}.$$

The topology on $X^{\log}$ is introduced as follows. Working locally on $X$, let $\alpha : P \to \mathcal{O}_X$ be a chart of $\mathcal{M}_X$. Then, by using the homomorphism $P^{\mathrm{gp}} \to \mathcal{M}^{\mathrm{gp}}_X$, $X^{\log}$ is identified with a closed subset of $X \times \operatorname{Hom}(P^{\mathrm{gp}}, \boldsymbol{C}_1)$. The topology of $X^{\log}$ is given by this identification. It is independent of the choice of a chart of $\mathcal{M}_X$, and defined globally. The projection $\tau : X^{\log} \to X$, $(x, h) \mapsto x$, is surjective, continuous and proper.

For $x \in X$, $\tau^{-1}(x)$ is isomorphic to $(\boldsymbol{C}_1)^r$ where $r = \operatorname{rank}(\mathcal{M}^{\mathrm{gp}}/\mathcal{O}^{\times}_X)_x$. In fact, since $(\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^{\times}_X)_x$ is a free abelian group, the exact sequence

$$1 \to \mathcal{O}^{\times}_{X,x} \xrightarrow{\iota} \mathcal{M}^{\mathrm{gp}}_{X,x} \xrightarrow{\pi} (\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^{\times}_X)_x \to 1$$

has a splitting $\sigma$ with $\pi \circ \sigma = \mathrm{id}$. Hence we have an isomorphism

$$\begin{aligned} \operatorname{Hom}(\mathcal{M}^{\mathrm{gp}}_{X,x}, \boldsymbol{C}_1) &\xrightarrow{\sim} \operatorname{Hom}(\mathcal{O}^{\times}_{X,x}, \boldsymbol{C}_1) \times \operatorname{Hom}((\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^{\times}_X)_x, \boldsymbol{C}_1). \\ h &\mapsto (h \circ \iota, h \circ \sigma). \end{aligned}$$

A morphism $f = (f, \varphi) : X \to Y$ of fs log complex spaces induces a continuous map $f^{\log} : X^{\log} \to Y^{\log}$, $(x, h) \mapsto (f(x), h \circ \varphi)$.

(1.2.1) EXAMPLE. Let $f : X \to \Delta$ be a proper surjective holomorphic map of a $d$-dimensional complex manifold $X$ to an open unit disc $\Delta$ such that $f$ is smooth over the punctured disc $\Delta^*$ and that $X_0 := f^{-1}(0)$ is a divisor with normal crossings. Then, as in Example (1.1.4), $X$ and $\Delta$ carry the fs log structures corresponding to the reduced divisor $(X_0)_{\mathrm{red}}$ and the origin $\{0\}$, respectively. The map $f$ can be regarded as a morphism of log complex manifolds which is described, in terms of local charts, as follows:

$$\begin{array}{ccccccc} \boldsymbol{N}^r & \xrightarrow{\alpha_X} & \mathcal{O}_X & \quad & m & \longmapsto & \prod_{1 \leq i \leq r} z_i^{m(i)} \\ {\scriptstyle\varphi}\uparrow & & {\scriptstyle f^*}\uparrow & & \uparrow & & \uparrow \\ \boldsymbol{N} & \xrightarrow{\alpha_\Delta} & \mathcal{O}_\Delta & & 1 & \longmapsto & t \end{array}$$

where $\prod_{1\le i\le r} z_i^{m(i)} = 0$ and $t = 0$ are local equations of $X_0$ in $X$ and $\{0\}$ in $\Delta$, respectively. This induces $f^{\log} : X^{\log} \to \Delta^{\log}$. These are locally described as follows:

$$
\begin{aligned}
X^{\log} &\overset{\text{locally}}{\simeq} \{(z_i, u_i)_{1\le i\le r} \in \boldsymbol{C}^r \times (\boldsymbol{C}_1)^r \mid z_i = |z_i| u_i \text{ for all } i\} \times \boldsymbol{C}^{d-r} \\
&\quad \overset{\sim}{\to} (\boldsymbol{R}_{\ge 0})^r \times (\boldsymbol{C}_1)^r \times \boldsymbol{C}^{d-r}, \\
&\quad ((z_i, u_i)_{1\le i\le r}, (z_j)_{r+1\le j\le d}) \mapsto ((|z_i|, u_i)_{1\le i\le r}, (z_j)_{r+1\le j\le d}). \\
\Delta^{\log} &\simeq \{(t, u) \in \Delta \times \boldsymbol{C}_1 \mid t = |t| u\} \overset{\sim}{\to} [0, 1) \times \boldsymbol{C}_1, \\
&\quad (t, u) \mapsto (|t|, u). \\
f^{\log} &: X^{\log} \to \Delta^{\log}, \\
&\quad ((z_i, u_i)_{1\le i\le r}, (z_j)_{r+1\le j\le d}) \mapsto (t, u) = \left( \prod_{1\le i\le r} z_i^{m(i)}, \prod_{1\le i\le r} u_i^{m(i)} \right).
\end{aligned}
$$

Note that the first local identifications of $X^{\log}$ and $\Delta^{\log}$ show that $\tau_X : X^{\log} \to X$ and $\tau_\Delta : \Delta^{\log} \to \Delta$ can be regarded as real oriented blowing-ups along $X_0$ and $\{0\}$, respectively (cf. [Mj]), and the second local identifications show that $X^{\log}$ and $\Delta^{\log}$ can be regarded as products of manifolds with corners, compact tori and complex manifolds (cf. [AMRT]).

(1.2.2) EXAMPLE. Let

$$\varpi : \mathfrak{h} \to \Delta^*, \quad z \mapsto t = e^{2\pi\sqrt{-1}z},$$

be the universal covering of the unit open disc. We add a point at infinity $\infty$ to $\boldsymbol{R}$ and extend the topology of $\boldsymbol{R}$ to $\hat{\boldsymbol{R}} := \boldsymbol{R} \sqcup \{\infty\}$ so that a fundamental system of open neighborhoods in $\hat{\boldsymbol{R}}$ of the point $\infty$ is given by

$$U_\eta := \{y \in \boldsymbol{R} \mid y > \eta\}, \quad \eta \text{ running over all positive real numbers}.$$

We introduce the product topology on $\hat{\mathfrak{h}} := \boldsymbol{R} + \sqrt{-1}\hat{\boldsymbol{R}}$. Then, by addition, $\boldsymbol{Z}$ acts on $\hat{\mathfrak{h}}$ continuously and freely and we have

$$\hat{\varpi} : \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}/\boldsymbol{Z} \simeq \Delta^{\log}, \quad x + \sqrt{-1}y \mapsto (e^{-2\pi y}, e^{2\pi\sqrt{-1}x}),$$

which can be regarded as the universal covering of $\Delta^{\log}$ in (1.2.1).

(1.3) Let $X$ be an fs log analytic space. Then $X$ is endowed with a sheaf of rings $\mathcal{O}_X^{\log}$ which is an enlargement of $\tau^{-1}\mathcal{O}_X$ by adding the 'logarithms' of local sections of $\tau^{-1}\mathcal{M}_X^{\mathrm{gp}}$.

The precise definition of $\mathcal{O}_X^{\log}$ is as follows. First, define a sheaf $\mathcal{L}$ of *logarithms of local sections of* $\tau^{-1}\mathcal{M}_X^{\mathrm{gp}}$ on $X^{\log}$ as the fiber product of

$$
\begin{array}{ccc}
 & & \tau^{-1}\mathcal{M}_X^{\mathrm{gp}} \\
 & & \downarrow \\
\mathrm{Map_c}(\ , \boldsymbol{R}\sqrt{-1}) & \xrightarrow{\exp} & \mathrm{Map_c}(\ , \boldsymbol{C}_1),
\end{array}
$$

where $\mathrm{Map_c}(\ , Y)$, for a topological space $Y$, denotes the sheaf of continuous maps into $Y$, and the vertical arrow comes from the definition of $X^{\log}$. We denote the projection $\mathcal{L} \to \tau^{-1}\mathcal{M}_X^{\mathrm{gp}}$

by exp. Then we have a commutative diagram of sheaves on $X^{\log}$ in which the horizontal rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}(1) \arrow[r] \arrow[d, equal] & \tau^{-1}\mathcal{O}_X \arrow[r, "\exp"] \arrow[d, "\theta"'] & \tau^{-1}\mathcal{O}_X^\times \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbf{Z}(1) \arrow[r] \arrow[d, equal] & \mathcal{L} \arrow[r, "\exp"] \arrow[d] & \tau^{-1}\mathcal{M}_X^{\mathrm{gp}} \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbf{Z}(1) \arrow[r] & \mathrm{Map}_{\mathrm{c}}(\ ,\boldsymbol{R}\sqrt{-1}) \arrow[r, "\exp"] & \mathrm{Map}_{\mathrm{c}}(\ ,\boldsymbol{C}_1) \arrow[r] & 1\,.
\end{tikzcd}$$

Here $\mathbf{Z}(1) := \mathbf{Z}\cdot 2\pi\sqrt{-1}$, and $\theta$ is induced from the map $\tau^{-1}\mathcal{O}_X \to \mathrm{Map}_{\mathrm{c}}(\ ,\boldsymbol{R}\sqrt{-1})$, $f \mapsto (f-\bar{f})/2$, and from the composite map $\tau^{-1}\mathcal{O}_X \xrightarrow{\exp} \tau^{-1}\mathcal{O}_X^\times \to \tau^{-1}\mathcal{M}_X^{\mathrm{gp}}$.

We define

$$\mathcal{O}_X^{\log} := (\tau^{-1}\mathcal{O}_X \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}\mathcal{L})/\mathcal{I}\,,$$

where $\mathrm{Sym}_{\mathbf{Z}}\mathcal{L}$ denotes the symmetric algebra of $\mathcal{L}$ over $\mathbf{Z}$ and $\mathcal{I}$ is the ideal of $\tau^{-1}\mathcal{O}_X \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}\mathcal{L}$ generated by local sections of the form $f\otimes 1 - 1\otimes\theta(f)$, $f\in\mathcal{O}_X$.

For $y\in X^{\log}$ and $x=\tau(y)$, the stalk $(\mathcal{O}_X^{\log})_y$ is described as follows. Let $r := \mathrm{rank}_{\mathbf{Z}}(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$ and let $(l_i)_{1\le i\le r}$ be a family of elements of $\mathcal{L}_y$ whose image under the composite map

$$\mathcal{L}_y \xrightarrow{\exp} \tau^{-1}(\mathcal{M}_X^{\mathrm{gp}})_y = \mathcal{M}_{X,x}^{\mathrm{gp}} \to (\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$$

is a $\mathbf{Z}$-basis of $(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$. Then, $(l_i)_{1\le i\le r}$ are algebraically independent over $\mathcal{O}_{X,x}$ and

$$(\mathcal{O}_X^{\log})_y = \tau^{-1}(\mathcal{O}_{X,x})[l_1,\cdots,l_r]\,.$$

Note that this is not a local ring.

For an fs log analytic space $X$, let $\omega_X^1$ be the *sheaf of differential forms on $X$ with log poles* defined by

$$\omega_X^1 := (\Omega_X^1 \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}_X^{\mathrm{gp}}))/N\,, \tag{1.3.1}$$

where $\Omega_X^1$ is the usual sheaf of Kähler differential forms on $X$, and $N$ is the $\mathcal{O}_X$-submodule of the direct sum generated by local sections of the form $(d\alpha(f),0)-(0,\alpha(f)\otimes f)$, $f\in\mathcal{M}_X$. For a local section of $\mathcal{M}_X^{\mathrm{gp}}$, the class of $(0,1\otimes f)$ in $\omega_X^1$ is denoted by $d\log(f)$.

Let $\omega_X^q$ be the $q$-th exterior power of $\omega_X^1$ over $\mathcal{O}_X$, and let

$$\omega_X^{q,\log} := \tau^*\omega_X^q = \mathcal{O}_X^{\log}\otimes_{\tau^{-1}\mathcal{O}_X}\tau^{-1}\omega_X^q\,.$$

We have derivations $d:\mathcal{O}_X\to\omega_X^1$, $f\mapsto df$, as well as

$$d:\mathcal{O}_X^{\log}\to\omega_X^{1,\log}\,,\quad df := d\log\exp(f)\quad\text{for } f\in\mathcal{L}\,.$$

These derivations are extended to de Rham complexes in a natural way:

$$\mathcal{O}_X \xrightarrow{d} \omega_X^1 \xrightarrow{d} \omega_X^2 \xrightarrow{d} \cdots\,,$$

$$\mathcal{O}_X^{\log} \xrightarrow{d} \omega_X^{1,\log} \xrightarrow{d} \omega_X^{2,\log} \xrightarrow{d} \cdots\,.$$

(1.4) We consider the following condition on a log analytic space $X$.

(1.4.1) $X$ is covered by open sets of type $(\mathrm{Spec}\,\boldsymbol{C}[P]/(\Sigma))_{\mathrm{an}}$, where $P$ is an fs monoid, $\Sigma$ is an ideal of $P$, i.e., $a \in P$ and $x \in \Sigma$ imply $ax \in \Sigma$, $(\Sigma)$ denotes the ideal of $\boldsymbol{C}[P]$ generated by $\Sigma$, and $(\mathrm{Spec}\,\boldsymbol{C}[P]/(\Sigma))_{\mathrm{an}}$ is endowed with the log structure associated to $P \to \mathcal{O}_X$.

If (1.4.1) is satisfied, then $\omega_X^1$ is a locally free $\mathcal{O}_X$-module of finite rank and the de Rham complex $\omega_X^{\bullet,\log}$ on $X^{\log}$ is a resolution of the constant sheaf $\boldsymbol{C}$ (cf. [KN]).

It is proved in [KN] that if a log analytic space $X$ satisfies the condition (1.4.1), then the following two categories $L_{\mathrm{unip}}(X^{\log})$ and $D_{\mathrm{nilp}}(X)$ are equivalent.

(1.4.2) $L_{\mathrm{unip}}(X^{\log})$: The category of locally constant sheaves $L$ of finite dimensional $\boldsymbol{C}$-vector spaces on $X^{\log}$, which have, locally on $X^{\log}$, a finite filtration $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ consisting of locally constant $\boldsymbol{C}$-subsheaves of $L$ such that each $L_i/L_{i-1}$ is the inverse image of a locally constant sheaf of $\boldsymbol{C}$-vector spaces on $X$.

(1.4.3) $D_{\mathrm{nilp}}(X)$: The category of locally free $\mathcal{O}_X$-modules $\mathcal{V}$ of finite rank on $X$ endowed with an integrable connection $\nabla : \mathcal{V} \to \omega_X^1 \otimes \mathcal{V}$, which have, locally on $X$, a finite filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$ consisting of $\mathcal{O}_X$-submodules of $\mathcal{V}$ such that $\nabla\mathcal{V}_i \subset \omega_X^1 \otimes \mathcal{V}_i$, $\mathcal{V}_i/\mathcal{V}_{i-1}$ is locally free and $\nabla$ on $\mathcal{V}_i/\mathcal{V}_{i-1}$ has no pole for all $i$.

The equivalence $L_{\mathrm{unip}}(X^{\log}) \to D_{\mathrm{nilp}}(X)$, $L \mapsto \mathcal{V}$ and its inverse $\mathcal{V} \mapsto L$ are defined as follows:

$$\begin{aligned} \mathcal{V} &:= \tau_*(\mathcal{O}_X^{\log} \otimes_{\boldsymbol{C}} L)\,, \\ L &:= \mathrm{Ker}(\tau^*\mathcal{V} \xrightarrow{\nabla} \omega_X^{1,\log} \otimes_{\mathcal{O}_X^{\log}} \tau^*\mathcal{V})\,, \end{aligned}$$

where $\tau^*(\ ) := \mathcal{O}_X^{\log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}(\ )$.

**2. Some lemmas.** In this section, we prove some elementary but non-trivial lemmas, which will be used in the proof of Proposition (3.2) in the next section.

(2.1) LEMMA. *Let $\varphi$ be a $C^\infty$ function in $\xi := (\xi_1, \dots, \xi_n)$ defined on a convex subset $U$ of $\boldsymbol{R}^n$. Then there exist $C^\infty$ functions $\psi_i$ in $\xi, \xi'$ defined on $U \times U$ $(i = 1, \dots, n)$ which satisfy the following conditions*:

(i) $$\varphi(\xi) - \varphi(\xi') = \sum_{1 \le i \le n} (\xi_i - \xi_i')\psi_i(\xi, \xi')\,,$$

(ii) $$\psi_i(\xi, \xi) = \frac{\partial\varphi}{\partial x_i}(\xi)\,.$$

PROOF. Since

$$\begin{aligned} \varphi(\xi) - \varphi(\xi') &= \int_0^1 \frac{d\varphi(t(\xi - \xi') + \xi')}{dt} dt \\ &= \int_0^1 \sum_{1 \le i \le n} \frac{\partial\varphi}{\partial x_i}(t(\xi - \xi') + \xi')(\xi_i - \xi_i') dt\,, \end{aligned}$$

we are done by taking

$$\psi_i(\xi, \xi') := \int_0^1 \frac{\partial \varphi}{\partial x_i}(t(\xi - \xi') + \xi')dt . \qquad \square$$

(2.2) Let $X$ be a complex manifold, and let $D$ be a compact complex submanifold of $X$. Let $\mathcal{V}$ be a finite family of open sets of $X$, which covers $D$. We assume that each $V \in \mathcal{V}$ has coordinates

$$\begin{bmatrix} (z_{i,V})_{1 \le i \le r} \\ (w_{j,V})_{1 \le j \le s} \end{bmatrix} , \tag{2.2.1}$$

arranged vertically, such that $(z_{i,v})_{1 \le i \le r}$ are local equations of $D$. Here $d := \dim X$ and $s := d - r$.

For $V \in \mathcal{V}$, by using local coordinates (2.2.1) on $V$, we define a column vector of functions

$$A_V(y, x) := (w_{j,V}(y) - w_{j,V}(x))_{1 \le j \le s} \quad (y \in V, x \in D \cap V) . \tag{2.2.2}$$

If we fix $x$, then these functions of $y$ induce a regular system of parameters of the local ring $\mathcal{O}_{D,x}$.

For $x \in V \cap V' \cap D$, we denote the Jacobian matrix for the change of parameters

$$\alpha' := A_{V'}(y, x)|_{y \in V' \cap D} \mapsto \alpha := A_V(y, x)|_{y \in V \cap D}$$

at $y = x$ by

$$J_{VV'}(x) := \left. \begin{bmatrix} \partial\alpha_1/\partial\alpha'_1 & \cdots & \partial\alpha_1/\partial\alpha'_s \\ \vdots & \ddots & \vdots \\ \partial\alpha_s/\partial\alpha'_1 & \cdots & \partial\alpha_s/\partial\alpha'_s \end{bmatrix} \right|_{y=x} . \tag{2.2.3}$$

Let

$$\{\rho_{V \cap D}\}_{V \in \mathcal{V}} \tag{2.2.4}$$

be a $C^\infty$ partition of unity on $D$, which is subordinate to the covering $\{V \cap D\}_{V \in \mathcal{V}}$. We modify the $A_V(y, x)$ as

$$B_V(y, z) := \sum_{V' \in \mathcal{V}} \rho_{V' \cap D}(x) J_{VV'}(x) A_{V'}(y, x) \quad (y \in V, x \in V \cap D) . \tag{2.2.5}$$

By the definition (2.2.5), we see easily that, for each $x \in V \cap V'' \cap D$ and $y$ in a neighborhood of $x$,

$$B_{V''}(y, x) = J_{V''V}(x) B_V(y, x) \quad \text{in } (\mathcal{O}_{X,x})^s . \tag{2.2.6}$$

(2.3) LEMMA. *For each fixed* $x \in V \cap D$, $(z_{i,V}(y))_{1 \le i \le r}$ *and* $B_V(y, x)$ *forms a regular system of parameters of* $\mathcal{O}_{X,x}$.

PROOF. For this, it is enough to show that $B_V(y,x)|_{y\in V\cap D}$ is a regular system of parameters of $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/((z_{i,V}(y))_{1\le i\le r})$. This follows from

$$\left.\frac{\partial(B_V(y,x)|_{y\in V\cap D})}{\partial(A_V(y,x)|_{y\in V\cap D})}\right|_{y=x} = \sum_{V'\in\mathcal{V}} \rho_{V'\cap D}(x) J_{VV'}(x) J_{V'V}(x) = 1_s\,. \qquad \square$$

(2.4) LEMMA. *In the situation of* (2.2), *there exists an open neighborhood $U$ of $D$ such that the following is a well-defined map*:

$$\pi : U \to D\,, \quad \textit{defined by } \pi(y) = x \textit{ if and only if } B_V(y,x) = 0 \tag{2.4.1}$$

*for some* (*hence, any*) $V \in \mathcal{V}$ *containing $x$ and $y$.*

PROOF. Denote $z_V := (z_{i,V})_{1\le i\le r}$. For each $x \in D$, choose $V \in \mathcal{V}$ containing $x$ and set

$$F_V(x) := \{y \in V \mid B_V(y,x) = 0\}\,. \tag{2.4.2}$$

*Claim* 1. For each $x \in D$, we can find a neighborhood $V_x$ of $x$ in $X$ contained in the intersection of all those $V \in \mathcal{V}$ with $V \ni x$ and a small number $\varepsilon > 0$, so that $F_V(x') \cap \{y \in V_x \mid z_V(y) = c\}$ contains only one point or is empty for every $x' \in V_x \cap D$ and for every constant vector $c \in \boldsymbol{C}^r$ with norm $|c| \le \varepsilon$.

We prove Claim 1. Define a function on $D$ by

$$\mu(x) := \max_{x\in V\in\mathcal{V}} \mu_V(x) \quad (x \in D)\,, \quad \text{where}$$
$$\mu_V(x) := \sup_{x\in W\subset V} (\text{distance from } x \text{ to the complement of } W)\,.$$

Here, in the second equation, $W$ runs over those neighborhoods of $x$ on which $z_V$ and $B_V(y,x)$ form local coordinates.

By Lemma (2.3) and the compactness of $D$, we see that the function $\mu(x)$, $x \in D$, is bounded away from 0. Suppose that there exist sequences of points $\{x_n\}_n$ and $\{x'_n\}_n$ on $D$ with $x_n \ne x'_n$, so that $x'_n \in F_V(x_n)$ for some $V \in \mathcal{V}$ for each $n$, and that the points $x_n$ and $x'_n$ approach each other as $n \to \infty$. Then $\mu(x_n) \to 0$ as $n \to \infty$, which contradicts the boundedness of $\mu$. Hence, for each $x \in D$, we can find a neighborhood $V'_x$ in $X$ contained in some $V \in \mathcal{V}$ such that $F_V(x') \cap V'_x \cap D = \{x'\}$ for every $x' \in V'_x \cap D$. Let $V_x \Subset V'_x$ be a neighborhood of $x$ in $X$ with the closure $\bar{V}_x$ compact and with $\bar{V}_x \subset V'_x$. Then it is easy to see that there exists a small number $\varepsilon > 0$ so that, for every $x' \in D \cap V_x$ and every constant vector $c \in \boldsymbol{C}^r$ with norm $|c| \le \varepsilon$, $F_V(x') \cap \{y \in V_x \mid z_V(y) = c\}$ consists of one point or empty. This proves Claim 1.

Let $\mathcal{V}^\flat$ be a finite open covering of a neighborhood of $D$ consisting of those open sets $V_x$ in Claim 1. For each $x \in D$, we define a slice by

$$F(x) := \bigcup_{x\in V^\flat\in\mathcal{V}^\flat} \{y \in V^\flat \mid B_V(y,x) = 0\}\,. \tag{2.4.3}$$

Here, on the right-hand-side, $V \in \mathcal{V}$ is chosen so as to contain $V^\flat$. Note that it is independent of the choice of $V$ by (2.2.6).

We choose an open neighborhood $U$ of $D$ so that $U$ is covered by $\mathcal{V}^\flat$.

*Claim* 2. If $U$ is small enough, then for each $y \in U$, there exists at most one slice $F(x)$ passing through $y$.

We prove Claim 2. For simplicity of notation, we denote $\rho_V := \rho_{V\cap D}$ in the proof of this claim. In order to prove Claim 2, it is enough to derive a contradiction from the assumption that there exist sequences of points $\{x_n\}_n$ and $\{x'_n\}_n$ on $D$ with $x_n \neq x'_n$ which approach each other as $n \to \infty$ and the corresponding slices $F(x_n)$ and $F(x'_n)$ have a common point $y_n$ which approaches $D$ as $n \to \infty$. Taking subsequences, by the compactness of $D$ and Claim 1, we may assume that the sequences $\{x_n\}_n$ and $\{x'_n\}_n$ converge to a common point $x_0 \in D$. By Claim 1, the sequence $\{y_n\}_n$ also converges to the unique point $x_0$ of $D \cap F(x_0)$. Take and fix a pair of open sets $V^\flat \subset V$ with $V^\flat \in \mathcal{V}^\flat$, $V \in \mathcal{V}$ and $x_0 \in V^\flat$. We denote $z := z_V$, $w := w_V$ and $B := B_V$ in this proof for simplicity of notation, and write the change of local coordinate as

$$w_{V'} = a_{V'} + b_{V'}z + c_{V'}w + h_{V'} \quad (V' \in \mathcal{V}), \tag{2.4.4}$$

where $a_{V'}$ is a constant vector, $b_{V'}$ and $c_{V'}$ are constant matrices and $h_{V'}$ is a vector whose entries are holomorphic functions of order $\geq 2$ in $z$, $w$. Then we have

$$\begin{aligned} J_{VV'}(x) &= \left.\frac{\partial w(y)}{\partial w_{V'}(y)}\right|_{y=x} = \left(\left.\frac{\partial w_{V'}(y)}{\partial w(y)}\right|_{y=x}\right)^{-1} \\ &= (c_{V'} + h'_{V'}(x))^{-1} = g_{V'}(x)c_{V'}^{-1}, \quad \text{where} \\ h'_{V'}(x) &:= \left.\frac{\partial h_{V'}(y)}{\partial w(y)}\right|_{y=x}, \quad g_{V'}(x) := \sum_{m\geq 0}(-c_{V'}^{-1}h'_{V'}(x))^m . \end{aligned} \tag{2.4.5}$$

Note that $g_{V'}(x)$ converges to a matrix consisting of holomorphic functions in $w(x)$ for $x$ near $x_0$. Hence, by (2.2.6), we have

$$\begin{aligned} 0 &= B(y_n, x_n) - B(y_n, x'_n) \\ &= \sum_{V'\in\mathcal{V}} [\rho_{V'}(x_n)J_{VV'}(x_n)A_{V'}(y_n, x_n) - \rho_{V'}(x'_n)J_{VV'}(x'_n)A_{V'}(y_n, x'_n)] \\ &= \sum_{V'\in\mathcal{V}} [\rho_{V'}(x_n)g_{V'}(x_n)c_{V'}^{-1}\{b_{V'}z(y_n)c_{V'}(w(y_n) - w(x_n)) + (h_{V'}(y_n) - h_{V'}(x_n))\} \\ &\quad - \rho_{V'}(x'_n)g_{V'}(x'_n)c_{V'}^{-1}\{b_{V'}z(y_n) + c_{V'}(w(y_n) - w(x'_n)) + (h_{V'}(y_n) - h_{V'}(x'_n))\}] \\ &= (w(x'_n) - w(x_n)) \\ &\quad + \sum_{V'\in\mathcal{V}} [\{\rho_{V'}(x_n)(-c_{V'}^{-1}h'_{V'}(x_n))g_{V'}(x_n)(w(y_n) - w(x_n)) \\ &\quad - \rho_{V'}(x'_n)(-c_{V'}^{-1}h'_{V'}(x'_n))g_{V'}(x'_n)(w(y_n) - w(x'_n))\} \\ &\quad + \{\rho_{V'}(x_n)g_{V'}(x_n)c_{V'}^{-1}(b_{V'}z(y_n) + (h_{V'}(y_n) - h_{V'}(x_n))) \\ &\quad - \rho_{V'}(x'_n)g_{V'}(x'_n)c_{V'}^{-1}(b_{V'}z(y_n) + (h_{V'}(y_n) - h_{V'}(x'_n)))\}] \end{aligned}$$

$$
\begin{aligned}
&= (w(x_n') - w(x_n)) \\
&\quad + \sum_{V' \in \mathcal{V}} [-\rho_{V'}(x_n) c_{V'}^{-1} \{h_{V'}'(x_n) g_{V'}(x_n) - h_{V'}'(x_n') g_{V'}(x_n')\} w(y_n) \\
&\quad - \{\rho_{V'}(x_n) - \rho_{V'}(x_n')\} c_{V'}^{-1} h_{V'}'(x_n') g_{V'}(x_n') w(y_n) \\
&\quad + \rho_{V'}(x_n) c_{V'}^{-1} \{h_{V'}'(x_n) g_{V'}(x_n) w(x_n) - h_{V'}'(x_n') g_{V'}(x_n') w(x_n')\} \\
&\quad + \{\rho_{V'}(x_n) - \rho_{V'}(x_n')\} c_{V'}^{-1} h_{V'}'(x_n') g_{V'}(x_n') w(x_n') \\
&\quad + \rho_{V'}(x_n) \{g_{V'}(x_n) - g_{V'}(x_n')\} c_{V'}^{-1} \{b_{V'} z(y_n) + h_{V'}(y_n)) \\
&\quad + \{\rho_{V'}(x_n) - \rho_{V'}(x_n')\} g_{V'}(x_n') c_{V'}^{-1} \{b_{V'} z(y_n) + h_{V'}(y_n)\} \\
&\quad - \rho_{V'}(x_n) \{g_{V'}(x_n) c_{V'}^{-1} h_{V'}(x_n) - g_{V'}(x_n') c_{V'}^{-1} h_{V'}(x_n')\} \\
&\quad - \{\rho_{V'}(x_n) - \rho_{V'}(x_n')\} g_{V'}(x_n') c_{V'}^{-1} h_{V'}(x_n')] .
\end{aligned}
$$

Since $\mathcal{V}$ is a finite covering, the extreme right-hand-side of the above equation contains only a finite number of terms. Moreover, by Lemma (2.1) applied to the $C^\infty$ function $\rho_{V'}(x_n) - \rho_{V'}(x_n')$, we can find $C^\infty$ functions $\psi_{V',j}'(x_n, x_n')$ and $\psi_{V',j}''(x_n, x_n')$ in $w(x_n)$, $\bar{w}(x_n)$, $w(x_n')$ and $\bar{w}(x_n')$ $(1 \leq j \leq s)$ so that

$$
\begin{aligned}
&\rho_{V'}(x_n) - \rho_{V'}(x_n') \\
&\quad = \sum_j [\{w_j(x_n) - w_j(x_n')\} \psi_{V',j}'(x_n, x_n') + \{\bar{w}_j(x_n) - \bar{w}_j(x_n')\} \psi_{V',j}''(x_n, x_n')] .
\end{aligned}
$$

Substituting this to the previous equation and dividing it by the norm $|w(x_n') - w(x_n)|$, we can observe that the first term does not converge to 0, whereas the other terms converge to 0. This contradiction finishes the proof of Claim 2.

*Claim* 3. If $U$ is small enough, then for each $y \in U$, there exists a slice $F(x)$ passing through $y$.

We prove Claim 3. By Claim 1 and the result of the 'only one' part, there exist open neighborhoods $U \Subset U'$ of $D$ in $\bigcup_{V^\flat \in \mathcal{V}^\flat} V^\flat$ such that the boundary of $F(x) \cap U'$ is contained in the boundary of $U'$ for every $x \in D$ and different slices do not intersect in $U'$. Here $U \Subset U'$ means that the closure of $U$ is contained in $U'$. Let $\bar{U}$ and $\bar{F}(x)$ be the closures of $U$ and of $F(x) \cap U$ in $U'$, respectively. Then $\bar{U}^* := \bar{U} - (\bigcup_{x \in D} \bar{F}(x))$ is open in $\bar{U}$. In fact, let $y \in \bar{U}^*$ and consider the function $\mu_y(x) :=$ distance$(y, \bar{F}(x))$ in $x \in D$. Since $D$ is compact, $\mu_y(x)$ attains its minimal value which should be positive because of the choice of $y$. Hence $\bar{U}^*$ is open in $\bar{U}$. Take a slice $F(x_0)$ which passes through a boundary point $y_0$ of the set $\bar{U}^*$, take $V^\flat \in \mathcal{V}^\flat$ and $V \in \mathcal{V}$ with $\bar{F}(x_0) \subset V^\flat \subset V$, take a small neighborhood $W$ of $x_0$ contained in $V^\flat$, take a constant vector $c$ with small enough norm and consider the map $\varphi : W \cap D \to \{y \in V^\flat \mid z(y) = c\}$ defined by $\varphi(x) := F(x) \cap \{z = c\}$, where $z := z_V$ as before. By construction, $\varphi$ is a $C^\infty$ injective map and hence its Jacobian determinant vanishes at at most isolated points of $W \cap D$. It follows that $\varphi$ is an open map and that its image intersects $\bar{U}^* \cap \{y \in V^\flat \mid z(y) = c\}$, a contradiction. This proves Claim 3.

If the neighborhood $U$ is small enough, then $z_V(y)$ is defined for $y$ near $F(x)$ in $U$ with $x \in \operatorname{supp} \rho_{V\cap D}$. In fact, since $\mathcal{V}$ is a finite covering and, for each $V \in \mathcal{V}$, the distance from $x$ to $F(x) \cap$ (complement of $V$), considered as a function in $x \in \operatorname{supp} \rho_{V\cap D}$, is bounded away from 0, our condition is fulfilled for a small enough neighborhood $U$. This proves Lemma (2.4). □

**3. A family of normal projections.** In the notation of (0.1), we construct in this section a family of 'normal projections' onto the strata of the divisor $X_0 + Y$ with normal crossings on the complex manifold $X$. Our method is analogous to that of Clemens [C, §5], but since we have to modify the argument in [C] to fit our situation and since it seems to the author that there are some points which are not clear in the proof of [C, Theorem 5.7] (cf. Remarks (3.4), (4.4)), we give here a complete proof for the readers' convenience.

(3.1) Let

$$f : X \to \Delta \tag{3.1.1}$$

be a proper, surjective, flat, holomorphic morphism from a $d$-dimensional complex manifold $X$ onto an open disc in the complex plane $\boldsymbol{C}$ with center 0. Let

$$Y = \sum_{1 \le j \le b} Y_j \tag{3.1.2}$$

be a reduced divisor on $X$ with simple normal crossings, where each $Y_j$ is a prime divisor. We assume that $Y$ is flat with respect to $f$. Let $t$ be a coordinate of $\Delta$. Let

$$f^{-1}(0) =: X_0 = \sum_{1 \le i \le a} m(i) X_i \tag{3.1.3}$$

be the central fiber of $f$. We assume that $f$ is smooth over the punctured disc $\Delta^*$ and that the sum

$$\sum_{1 \le i \le a} m(i) X_i + \sum_{1 \le j \le b} Y_j$$

of the central fiber $X_0$ and $Y$ is a divisor with simple normal crossings.

We denote

$$\begin{aligned} D_i &:= \begin{cases} X_i & \text{if } 1 \le i \le a\,, \\ Y_{i-a} & \text{if } a+1 \le i \le a+b\,, \end{cases} \\ D_I &:= \bigcap_{i \in I} D_i \quad \text{for } I \subset \{1, \dots, a+b\}\,, \end{aligned} \tag{3.1.4}$$

and

$$I(a) := I \cap \{1, \dots, a\}\,, \qquad I(b) := I \cap \{a+1, \dots, a+b\}\,. \tag{3.1.5}$$

Let $V$ be an open neighborhood in $X$ of a point of $D_I$, carrying coordinates

$$\begin{bmatrix} (z_{i,V})_{i \in I} \\ (w_{j,V})_{1 \le j \le d-|I|} \end{bmatrix} \tag{3.1.6}$$

arranged vertically, such that

$$\prod_{i\in I(a)} (z_{i,V})^{m(i)} = (t\circ f)|V \quad\text{and}\quad \left\{\prod_{i\in I(b)} z_{i,V} = 0\right\} = Y\cap V\,. \tag{3.1.7}$$

We call such local coordinates (3.1.6) *standard local coordinates* and such local equations $(z_{i,V})_{i\in I}$ *standard local equations*, respectively. We call such open set $V$ endowed with standard local coordinates a *standard local coordinate open set.*

For open subsets $U$ and $U'$ of a topological space $Z$, we denote

$$U \Supset U'$$

if $U$ contains the closure of $U'$ in $Z$.

(3.2) PROPOSITION. *In the above notation, shrinking $\Delta$, if necessary, we have a family $\{U_I\}$ of open tubular neighborhoods $U_I$ of $D_I$ in $X$ and a family $\{\pi_I\}$ of $C^\infty$ normal projections $\pi_I : U_I \to D_I$ with holomorphic fibers, where $I$ runs over all subsets of $\{1,\dots,a+b\}$ with $D_I \neq \emptyset$, which have the following properties*:

(i) $U_I \cap U_J = U_{I\cup J}$;

(ii) $\pi_I \circ \pi_J \,|\, U_I = \pi_I$ *for* $I \supset J$;

(iii) *If* $I \subset \{1,\dots,a\}$, *then* $\pi_I^{-1}(D_I\cap Y_j) = U_I \cap Y_j$ *for* $j = 1,\dots,b$.

PROOF. We prove the assertion by descending induction on the cardinality $|I|$ of a subset $I \subset \{1,\dots,a+b\}$. Since the proof is long, we divide it into five steps.

*Step* 1. Let

$$l := \max\{|I| \mid D_I \neq \emptyset\}\,, \tag{3.2.1}$$

and let $I \subset \{1,\dots,a+b\}$ with $|I| = l$ and $D_I \neq \emptyset$.

Choose first a finite covering $\mathcal{V}_I$ of an open neighborhood of $D_I$, consisting of standard coordinate open sets. Applying Lemmas (2.3) and (2.4) to $D = D_I$ and $\mathcal{V} = \mathcal{V}_I$, we have $B^I_V(y,x) := B_V(y,x)$, open neighborhoods $U_i^{(2l)}$ of $D_i$ ($1 \le i \le a+b$), and a $C^\infty$ projection

$$\pi_I : U_I^{(2l)} \to D_I\,, \quad \pi_I(y) = x \Leftrightarrow B^I_V(y,x) = 0\,. \tag{3.2.2}$$

Here $U_I^{(2l)} := \bigcap_{i\in I} U_i^{(2l)}$. Shrinking the $U_i^{(2l)}$, if necessary, we may assume that $U_I^{(2l)}$ is covered by $\mathcal{V}_I$. Put

$$\mathcal{W}_I := \{V \cap U_I^{(2|I|)}\}_{V\in\mathcal{V}_I}\,, \tag{3.2.3}$$

$$\rho_{W\cap D_I} := \rho_{V(W)\cap D_I} \quad \text{for } W \in \mathcal{W}_I\,, \tag{3.2.4}$$

$$B^I_W(y,x) := B^I_{V(W)}(y,x)|W \tag{3.2.5}$$

$$\text{for } W\in\mathcal{W}_I\,,\ x\in W\cap D_I\,, \text{ and } y \in W \text{ near } \pi_I^{-1}(x)\,.$$

Here $V(W)$ is the element of $\mathcal{V}_I$ defining $W = V(W)\cap U_I^{(2|I|)}$. We add a remark here. If there are different $V, V' \in \mathcal{V}_I$ which define the same $W = V\cap U_I^{(2|I|)} = V'\cap U_I^{(2|I|)}$, then we throw away one of them. Going on with this process and shrinking $U_I^{(2|I|)}$, if necessary, we may assume that $\mathcal{W}_I \ni W \mapsto V(W) \in \mathcal{V}$ is a well-defined map.

*Step* 2. By the induction hypothesis, we assume that, for a positive integer $m \le l$ and a subset $K \subset \{1, \dots, a+b\}$ with $|K| \ge m$ and $D_K \ne \emptyset$, the following data (3.2.6)–(3.2.11) have been constructed:

(3.2.6) The sequence of open neighborhoods $U_i^{(2l)} \Supset U_i^{(2l-1)} \Supset \cdots \Supset U_i^{(2m)}$ of $D_i$ $(1 \le i \le a+b)$;

(3.2.7) The finite family $\mathcal{V}_K$ of standard coordinate open sets which satisfies the conditions that

$$\begin{cases} \mathcal{V}_K \text{ covers } U_K^{(2|K|)} - \bigcup\limits_{\substack{M \supset K \\ |M|=|K|+1}} U_M^{(2|M|)}, \text{ and that} \\ \text{any element of } \mathcal{V}_K \text{ does not intersect } \bigcup\limits_{\substack{M \supset K \\ |M|=|K|+1}} U_M^{(2|M|-1)}; \end{cases}$$

(3.2.8) The finite, open covering $\mathcal{W}_K$ of $U_K^{(2|K|)}$, whose member $W \in \mathcal{W}_K$ carries the standard local equation $z_{k,W}$ of $D_k$ in $W$ $(k \in K)$;

(3.2.9) The $C^\infty$ partition $\{\rho_{W \cap D_K}\}_{W \in \mathcal{W}_K}$ of unity on $D_K$ subordinate to the covering $\mathcal{W}_K$;

(3.2.10) For $W \in \mathcal{W}_K$ and $x \in W \cap D_K$, the column vector $B_W^K(y, x)$ of holomorphic functions of $y$ near each fixed $x \in X$, which induce a regular system of parameters of $\mathcal{O}_{D_K,x}$;

(3.2.11) The $C^\infty$ projection $\pi_K : U_K^{(2|K|)} \to D_K$ such that $\pi_K(y) = x$ is equivalent to $B_W^K(y, x) = 0$ for some $W \in \mathcal{W}_K$. They satisfy $\pi_L \circ \pi_K = \pi_L$ on $U_L^{(2|L|)}$ $(L \supset K,$ $|L| = |K| + 1)$.

Now fix $I \subset \{1, \dots, a+b\}$ so that $|I| = m-1$ and $D_I \ne \emptyset$. If $D_I \cap D_k = \emptyset$ for any $k \in \{1, \dots, a+b\} - I$, then we apply for this $I$ the construction in Step 1. Otherwise, choose $k \in \{1, \dots, a+b\} - I$ so that $D_I \cap D_k \ne \emptyset$. Put $K := I \sqcup \{k\}$.

We define a multi-valued function $\zeta_k^K(y, \tilde{x})$ in $y$ near $\tilde{x} \in D_K$ by

$$\zeta_k^K(y, \tilde{x}) := \prod_{\tilde{W} \in \mathcal{W}_K} z_{k,\tilde{W}}(y)^{\rho_{\tilde{W} \cap D_K}(\tilde{x})}. \tag{3.2.12}$$

This definition is justified in the following way. Let $g_{k,\tilde{W}\tilde{W}'}$ be the transition function of the standard local equations of $D_k$:

$$z_{k,\tilde{W}} = g_{k,\tilde{W}\tilde{W}'} z_{k,\tilde{W}'} \quad (\tilde{W}, \tilde{W}' \in \mathcal{W}_K).$$

Then, for a fixed $\tilde{W}' \in \mathcal{W}_K$, we have

$$\begin{aligned} \zeta_k^K(y, \tilde{x}) &= \prod_{\tilde{W} \in \mathcal{W}_K} z_{k,\tilde{W}}(y)^{\rho_{\tilde{W} \cap D_K}(\tilde{x})} \\ &= \exp\Bigg( \sum_{\tilde{W} \in \mathcal{W}_K} \rho_{\tilde{W} \cap D_K}(\tilde{x}) \log(z_{k,\tilde{W}}(y)) \Bigg) \end{aligned}$$

$$= \exp\Bigg(\sum_{\tilde W \in \mathcal{W}_K} \rho_{\tilde W \cap D_K}(\tilde x)(\log(g_{k,\tilde W \tilde W'}(y)) + \log(z_{k,\tilde W'}(y)))\Bigg)$$

$$= \exp\Bigg(\sum_{\tilde W \in \mathcal{W}_K} \rho_{\tilde W \cap D_K}(\tilde x) \log(g_{k,\tilde W \tilde W'}(y))\Bigg) z_{k,\tilde W'}(y)\,.$$

The last equation is well-defined if we choose a branch of $\log(g_{k,\tilde W \tilde W'}(y))$. The ambiguity of the choice of a branch is the choice of a multiple

$$\prod_{\tilde W \in \mathcal{W}_K} \exp(2\pi n_{\tilde W} \rho_{\tilde W \cap D_K}(\tilde x)\sqrt{-1}) \quad (n_{\tilde W} \in \mathbf{Z})\,. \tag{3.2.13}$$

*Claim* 1. Let $\tilde W \in \mathcal{W}_K$. For each fixed $\tilde x \in \tilde W \cap D_K$, $(z_{i,\tilde W}(y))_{i \in I}$, $\zeta_k^K(y, \tilde x)$ and $B_{\tilde W}^K(y, \tilde x)$ form a regular system of parameters of $\mathcal{O}_{X,\tilde x}$.

In order to prove this claim, it is enough to show that $(z_{i,\tilde W}(y))_{i \in I}$ and $\zeta_k^K(y, \tilde x)$ induce a regular system of parameters of $\mathcal{O}_{X,\tilde x}/(B_{\tilde W}^K(y, \tilde x)) \simeq \mathcal{O}_{\tilde F,\tilde x}$. For simplicity of notation, we assume $I = \{1, \dots , m-1\}$ and $k = m$. By the definition (3.2.12), we have

$$\left(\partial\left(\begin{bmatrix}(z_{i,\tilde W}(y))_{i\in I} \\ \zeta_m^K(y, \tilde x)\end{bmatrix}\Bigg|_{y \in \tilde F}\right)\Bigg/ \partial((z_{j,\tilde W}(y))_{j \in K}|_{y \in \tilde F})\right)\Bigg|_{y = \tilde x} = \begin{bmatrix} 1_{m-1} & 0 \\ 0 & g^K_{m,\tilde W}(\tilde x, \tilde x)\end{bmatrix},$$

where $g^K_{m,\tilde W}(\tilde x, \tilde x) := \prod_{\tilde W' \in \mathcal{W}_K} g_{k,\tilde W',\tilde W}(\tilde x)^{\rho_{\tilde W' \cap D_K}(\tilde x)}$. Since $g^K_{m,\tilde W}(\tilde x, \tilde x) \neq 0$, Claim 1 is proved.

Let $I$, $k$ and $K$ be as above. For $\tilde W \in \mathcal{W}_K$, $\tilde x \in D_K \cap \tilde W$, $x \in \pi_K^{-1}(\tilde x) \cap D_I \cap \tilde W$ near $\tilde x$, and $y \in \tilde W$ near $\tilde x$, we define

$$A^I_{\tilde W}(y, x) := \begin{bmatrix}\zeta_k^K(y, \tilde x) - \zeta_k^K(x, \tilde x) \\ B^K_{\tilde W}(y, \tilde x)\end{bmatrix}. \tag{3.2.14}$$

Choose a family of small enough open neighborhoods $U_i^{(2m-1)}$ of $D_i$ such that

$$U_i^{(2m)} \Supset U_i^{(2m-1)} \quad (1 \le i \le a+b)\,. \tag{3.2.15}$$

We define a $C^\infty$ projection

$$\varpi_I^K : U_K^{(2|K|-1)} \to U_K^{(2|K|-1)} \cap D_I \tag{3.2.16}$$

by $\varpi_I^K(y) = x$ if and only if $B^K_{\tilde W}(y, \tilde x) = 0$ and $\zeta_k^K(y, \tilde x) - \zeta_k^K(x, \tilde x) = 0$ for some (hence, any) $\tilde W \in \mathcal{W}_K$ containing $x$ and $y$. Here $\tilde x := \pi_K(x)$. Obviously, we have

$$\pi_K \circ \varpi_I^K = \pi_K \quad \text{on } U_K^{(2|K|-1)}\,. \tag{3.2.17}$$

*Step* 3. Let $m \le l$ and $I \subset \{1, \dots , a+b\}$ be as in Step 2. Choose a family of open neighborhoods $U_i^{(2m-2)}$ of $D_i$ such that

$$U_i^{(2m-1)} \Supset U_i^{(2m-2)} \quad (1 \le i \le a+b) \tag{3.2.18}$$

and choose a finite family $\mathcal{V}_I$ of standard coordinate open sets which satisfy the following conditions:

$$\begin{cases} \mathcal{V}_I \text{ covers } U_I^{(2|I|)} - \bigcup_{\substack{K \supset I \\ |K|=|I|+1}} U_K^{(2|K|)}; \\ \text{Any element of } \mathcal{V}_I \text{ does not intersect } \bigcup_{\substack{K \supset I \\ |K|=|I|+1}} U_K^{(2|K|-1)}. \end{cases} \tag{3.2.19}$$

Let

$$\mathcal{W}_I^\sharp := \{V \cap U_I^{(2|I|)}\}_{V \in \mathcal{V}_I} \cup \bigcup_{\substack{K \supset I \\ |K|=|I|+1}} \{\tilde{W} \cap U_I^{(2|I|)}\}_{\tilde{W} \in \mathcal{W}_K}. \tag{3.2.20}$$

Choose a $C^\infty$ partition of unity

$$\{\rho_{W^\sharp \cap D_I}\}_{W^\sharp \in \mathcal{W}_I^\sharp} \tag{3.2.21}$$

on $D_I$ which is subordinate to the covering $\{W^\sharp \cap D_I\}_{W^\sharp \in \mathcal{W}_I^\sharp}$ and satisfies the following compatibility condition: If $L \supset K \supset I$, $|K| = |I| + 1$ and $V \in \mathcal{V}_L$, then

$$\sum_{\substack{W^\sharp \in \mathcal{W}_I^\sharp \\ V(W^\sharp)=V}} \rho_{W^\sharp \cap D_I} = \sum_{\substack{\tilde{W} \in \mathcal{W}_K \\ V(\tilde{W})=V}} \rho_{\tilde{W} \cap D_K} \circ \pi_K \quad \text{on } U_K^{(2|K|-1)} \cap D_I, \tag{3.2.22}$$

where $V(W^\sharp)$ (resp. $V(\tilde{W})$) is the unique element of $\bigcup_{M \supset I} \mathcal{V}_M$ containing $W^\sharp$ (resp. $\tilde{W}$). By the construction of the $\mathcal{V}_M$, $M \supset I$, and the same reasoning as that just after (3.2.5), $V(W^\sharp)$ and $V(\tilde{W})$ are indeed determined uniquely by $W^\sharp$ and $\tilde{W}$, respectively. As in [C, (5.14)], such a partition of unity (3.2.21) can be constructed descending inductively by using the properties (3.2.6), (3.2.15), (3.2.18), (3.2.7) and (3.2.19).

Divide each $W = \tilde{W} \cap U_I^{(2|I|)}$ ($\tilde{W} \in \mathcal{W}_K$) into two parts

$$\begin{cases} W_1(\tilde{W}) := \tilde{W} \cap \left( \bigcup_{\substack{M \supset I \\ |M|=|I|+1}} U_M^{(2|M|-1)} \right) \cap U_I^{(2|I|)}, \\ W_2(\tilde{W}) := \tilde{W} \cap \left( \bigcup_{\substack{M \supset I \\ |M|=|I|+1}} \left( U_M^{(2|M|)} - \overline{U_M^{(2|M|-2)}} \right) \right) \cap U_I^{(2|I|)}. \end{cases} \tag{3.2.23}$$

Here the overline denotes the closure. We define a covering

$$\mathcal{W}_I := \{V \cap U_I^{(2|I|)}\}_{V \in \mathcal{V}_I} \cup \bigcup_{\substack{K \supset I \\ |K|=|I|+1}} \{W_1(\tilde{W}), W_2(\tilde{W})\}_{\tilde{W} \in \mathcal{W}_K}. \tag{3.2.24}$$

We denote

$$\mathcal{W}_I^\flat := \{V \cap U_I^{(2|I|)}\}_{V \in \mathcal{V}_I} \cup \bigcup_{\substack{K \supset I \\ |K|=|I|+1}} \{W_2(\tilde{W})\}_{\tilde{W} \in \mathcal{W}_K}, \tag{3.2.25}$$

which is obtained from $\mathcal{W}_I$ by throwing away the open sets of 'type $W_1(\ )$'.

Let $W \in \mathcal{W}_I$ and $x \in W \cap D_I$. As a regular system of parameters of $\mathcal{O}_{X,x}$, we take (3.2.26), (3.2.27) and (3.2.28) below, according to the following types:

$$\begin{cases} \text{Type } 1: & W = W_1(\tilde{W}) \text{ for } \tilde{W} \in \mathcal{W}_K \text{ with } K \supset I \text{ and } |K| = |I|+1\,. \\ \text{Type } 2: & W = W_2(\tilde{W}) \text{ for } \tilde{W} \in \mathcal{W}_K \text{ with } K \supset I \text{ and } |K| = |I|+1\,. \\ \text{Type } 3: & W = V \cap U_I^{(2|I|)} \text{ for } V \in \mathcal{V}_I\,. \end{cases}$$

*Type* 1. When $W = W_1(\tilde{W})$ for $\tilde{W} \in \mathcal{W}_K$ with $K \supset I$ and $|K| = |I| + 1$, we put $K = I \sqcup \{k\}$ and define

$$\begin{bmatrix} (z_{i,W}(y))_{i\in I} \\ A^I_W(y,x) \end{bmatrix} := \begin{bmatrix} (z_{i,\tilde{W}}(y))_{i\in I} \\ A^I_{\tilde{W}}(y,x) \end{bmatrix} \quad (y \in W)\,. \tag{3.2.26}$$

Here $A^I_{\tilde{W}}(y,x)$ is the column vector of functions in (3.2.14).

*Type* 2. When $W = W_2(\tilde{W})$ for $\tilde{W} \in \mathcal{W}_K$ with $K$ as above, starting from $\tilde{W}$, we follow backward the construction process of $W_\nu(\ )$ ($\nu = 1, 2$) by descending induction, and take $W^* := W^*(\tilde{W}) \in \mathcal{W}_{I^*}$ which is just the previous open set belonging to $\mathcal{W}^\flat_{I^*}$ (cf. (3.2.25)). More precisely, let $W^{(0)} := \tilde{W}$. If $W^{(0)} \in \mathcal{W}^\flat_K$, we define $I^* := K$ and $W^*(\tilde{W}) := W^{(0)}$. Otherwise, there exist $W^{(j)} \in \mathcal{W}_{K^{(j)}}$ ($1 \le j \le n$) such that $W^{(j-1)} = W_1(W^{(j)})$ ($1 \le j \le n$) and $W^{(n)} \in \mathcal{W}^\flat_{K^{(n)}}$. Then we define $I^* := K^{(n)}$ and $W^*(\tilde{W}) := W^{(n)}$. Putting $K = I \sqcup \{k\}$, we define

$$\begin{bmatrix} (z_{i,W}(y))_{i\in I(a)} \\ (z_{i,W}(y))_{i\in I(b)} \\ A^I_W(y,x) \end{bmatrix} := \begin{bmatrix} \left(z_{i,\tilde{W}}(y)\left(\prod_{j\in I^*(a)-I(a)} (z_{j,\tilde{W}}(y))^{m(j)}\right)^{1/(|I(a)|m(i))}\right)_{i\in I(a)} \\ (z_{i,\tilde{W}}(y))_{i\in I(b)} \\ A^I_{\tilde{W}}(y,x) \end{bmatrix} \quad (y \in W)\,. \tag{3.2.27}$$

Here $I(a)$, $I(b)$, $I^*(a)$ are as in (3.1.5). Note that the precise definition of $z_{i,W}(y)$ ($i \in I(a)$) is as follows:

$$z_{i,W}(y) := z_{i,\tilde{W}}(y) \exp\left(\frac{1}{|I(a)|m(i)} \sum_{j\in I^*(a)-I(a)} m(j) \log z_{j,\tilde{W}}(y)\right) \quad (y \in W)\,.$$

Although, globally on $W_2(\tilde{W})$, these are multi-valued functions, they make sense as elements of $\mathcal{O}_{X,x}$ up to the choice of roots of unity.

*Type* 3. When $W = V \cap U_I^{(2|I|)}$ for $V \in \mathcal{V}_I$, we define

$$\begin{bmatrix} (z_{i,W}(y))_{i\in I} \\ A^I_W(y,x) \end{bmatrix} := \begin{bmatrix} (z_{i,V}(y))_{i\in I} \\ (w_{j,V}(y) - w_{j,V}(x))_{1\le j\le d-|I|} \end{bmatrix} \quad (y \in W)\,. \tag{3.2.28}$$

Choose a $C^\infty$ function $\chi_I$ on $U_I^{(2|I|)}$ which is constant on each fiber of $\varpi_I^K$, for every $K \supset I$ with $|K| = |I| + 1$, and takes values in [0, 1] so that

$$(3.2.29) \qquad \chi_I = \begin{cases} 1 \text{ on } \displaystyle\bigcup_{\substack{K \supset I \\ |K|=|I|+1}} U_K^{(2|K|-2)}, \\ 0 \text{ outside } \left( \displaystyle\bigcup_{\substack{K \supset I \\ |K|=|I|+1}} U_K^{(2|K|-1)} \right) \cap U_I^{(2|I|)}. \end{cases}$$

We define a $C^\infty$ partition of unity on $D_I$, subordinate to the covering $\{W \cap D_I\}_{W \in \mathcal{W}_I}$, by taking a refinement of (3.2.21) in the following way:

$$(3.2.30) \qquad \rho_{W \cap D_I} := \begin{cases} \chi_I \cdot \rho_{W^\sharp \cap D_I} & \text{if } W = W_1(\tilde{W}) \text{ for some } \tilde{W} \in \mathcal{W}_K, \\ (1 - \chi_I) \cdot \rho_{W^\sharp \cap D_I} & \text{if } W = W_2(\tilde{W}) \text{ for some } \tilde{W} \in \mathcal{W}_K, \\ \rho_{W \cap D_I} & \text{if } W = V \cap U_I^{(2|I|)} \text{ for some } V \in \mathcal{V}_I. \end{cases}$$

Here we denote $W^\sharp := \tilde{W} \cap U_I^{(2|I|)} \in \mathcal{W}_I^\sharp$.

For $W, W' \in \mathcal{W}_I$ and $x \in W \cap W' \cap D_I$, let

$$(3.2.31) \qquad J^I_{WW'}(x)$$

be the Jacobian matrix at $y = x$ of the change of the parameters of $\mathcal{O}_{D_I,x}$ induced by the $A^I_W(y, x)$ in (3.2.26), (3.2.27) and (3.2.28). Modifying the $A^I_W(y, x)$ as before, we define

$$(3.2.32) \qquad B^I_W(y, x) := \sum_{W' \in \mathcal{W}_I} \rho_{W' \cap D_I}(x) J^I_{WW'}(x) A^I_{W'}(y, x).$$

As in Step 1, shrinking the open neighborhoods $U_j^{(2|I|)}$ $(1 \le j \le a + b)$ and the disc $\Delta$, if necessary, we define a $C^\infty$ projection

$$(3.2.33) \qquad \pi_I : U_I^{(2|I|)} \to D_I, \quad \pi_I(y) = x \Leftrightarrow B^I_W(y, x) = 0.$$

*Step* 4. In this step, we prove that $\pi_I$ in (3.2.33) is well-defined. In fact, in the proof of Lemma (2.4), the arguments in the proofs of Claim 1 and Claim 3 work well for $D_I$, $\mathcal{W}_I$, $(z_{i,W}(y))_{i \in I}$ and $B^I_W(y, x)$. In order to show Claim 2 in the proof of Lemma (2.4) in the present situation, we divide its proof into two cases by (3.2.19): Let $x \in D_I$.

$$\begin{cases} \text{Case 1.} & \text{If } W \in \mathcal{W}_I \text{ contains } x, \text{ then } W \text{ is of Type 1 or Type 2}. \\ \text{Case 2.} & \text{There exists } W \in \mathcal{W}_I \text{ of Type 3 which contains } x. \end{cases}$$

*Case* 1. If $W \in \mathcal{W}_I$ contains $x$, then we can take $\tilde{W} \in \mathcal{W}_K$, $K = I \sqcup \{k\}$, so that $W = W_\nu(\tilde{W})$ $(\nu = 1, 2)$. Let $y \in U_K^{(2|I|)}$ and $\tilde{x} := \pi_K(x) \in D_K$. Then, by using (3.2.33), (3.2.32), (3.2.30), (3.2.22), (3.2.26) and (3.2.27), we have

$$\begin{aligned} 0 = B^I_W(y, x) &= \sum_{W' \in \mathcal{W}_I} \rho_{W' \cap D_I} J^I_{WW'}(x) A^I_{W'}(y, x) \\ &= \sum_{\tilde{W}' \in \mathcal{W}_K} \rho_{\tilde{W}' \cap D_K}(\tilde{x}) J^I_{WW'}(x) A^I_{\tilde{W}'}(y, x) = M(x) A^I_{\tilde{W}}(y, x), \end{aligned}$$

where

$$M(x) := \sum_{\tilde W' \in \mathcal{W}_K} \rho_{\tilde W' \cap D_K}(\tilde x) \begin{bmatrix} 1 & 0 \\ * & J^K_{\tilde W \tilde W'}(\tilde x) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & J^K_{\tilde W' \tilde W}(\tilde x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & 1_{d-|K|} \end{bmatrix}.$$

Here we use (3.2.14). Since $\det M(x) = 1 \neq 0$, $B^I_W(y,x) = 0$ is equivalent to $A^I_{\tilde W}(y,x) = 0$, and hence equivalent to $B^K_{\tilde W}(y,\tilde x) = 0$ and $\zeta^K_k(y,\tilde x) - \zeta^K_k(x,\tilde x) = 0$. Thus our assertion is verified in Case 1.

*Case* 2. As in the proof of Claim 2 in the proof of Lemma (2.4), let $\{x_n\}_n$ and $\{x'_n\}_n$ be the sequences on $D_I$ which converge to the common point $x_0$, and let $y_n$ be the common point of the slices $F(x_n)$ and $F(x'_n)$ so that $y_n \to x_0$ as $n \to \infty$. Take $W \in \mathcal{W}_I$ of Type 3 which contains $x_0$, and take $V \in \mathcal{V}$ with $W = V \cap U_I^{(2|I|)}$. Let $z := (z_{i,V})_{i \in I}$, $w := (w_{j,V})_{1 \le j \le d-|I|}$ and $B := B^I_W$. Then the change of local coordinates (2.4.4) becomes

$$B^I_{W'}(y,x) = a_{W'}(x) + b_{W'}(x)z(y) + c_{W'}(x)w(y) + h_{W'}(y,x) \tag{3.2.34}$$

for $W' \in \mathcal{W}_I$. Here $a_{W'}(x)$, $b_{W'}(x)$, $c_{W'}(x)$ and $h_{W'}(y,x)$ are $C^\infty$ in $x$. In $z$ and $w$, $a_{W'}(x)$ is a constant vector, $b_{W'}(x)$ and $c_{W'}(x)$ are constant matrices, and $h_{W'}(y,x)$ is a vector whose entries are functions of order $\ge 2$. As in (2.4.5), the Jacobian matrix $J_{WW'}(x) := J^I_{WW'}(x)$ becomes

$$\begin{aligned} J_{WW'}(x) &= \left.\frac{\partial w(y)}{\partial B^I_{W'}(y,x)}\right|_{y=x} = \left( \left.\frac{\partial B^I_{W'}(y,x)}{\partial w(y)}\right|_{y=x} \right)^{-1} \\ &= (c_{W'}(x) + h'_{W'}(x))^{-1} = g_{W'}(x) c_{W'}(x)^{-1}, \quad \text{where} \\ h'_{W'}(x) &:= \left.\frac{\partial h_{W'}(y,x)}{\partial w(y)}\right|_{y=x}, \quad g_{W'}(x) := \sum_{m \ge 0} (-c_{W'}^{-1} h'_{W'}(x))^m. \end{aligned} \tag{3.2.35}$$

Put $\rho_{W'} := \rho_{W' \cap D_I}$ and $A_{W'} := A^I_{W'}$. Since $A_{W'}(x,x) = 0$, the argument proceeds as

$$\begin{aligned} 0 &= B(y_n, x_n) - B(y_n, x'_n) \\ &= \sum_{W' \in \mathcal{W}_I} [\rho_{W'}(x_n) J_{WW'}(x_n) \{A_{W'}(y_n, x_n) - A_{W'}(x_n, x_n)\} \\ &\qquad - \rho_{W'}(x'_n) J_{WW'}(x'_n) \{A_{W'}(y_n, x'_n) - A_{W'}(x'_n, x'_n)\}] \\ &= \sum_{W' \in \mathcal{W}_I} [\rho_{W'}(x_n) g_{W'}(x_n) c_{W'}^{-1} \{b_{W'} z(y_n) \\ &\qquad + c_{W'}(w(y_n) - w(x_n)) + (h_{W'}(y_n) - h_{W'}(x_n))\} \\ &\quad - \rho_{W'}(x'_n) g_{W'}(x'_n) c_{W'}^{-1} \{b_{W'} z(y_n) \\ &\qquad + c_{W'}(w(y_n) - w(x'_n)) + (h_{W'}(y_n) - h_{W'}(x'_n))\}]. \end{aligned}$$

The rest of the argument works well and we get our assertion in Case 2.

*Step* 5. The descending induction is now completed. Finally we take

$$U_i := U_i^{(2)} \quad (1 \le i \le a+b). \tag{3.2.36}$$

Thus we obtain a desired family of $C^\infty$ projections

$$\pi_I : U_I \to D_I \quad (I \subset \{1, \dots, a+b\}) \tag{3.2.37}$$

with holomorphic fibers, which have the properties (i), (ii) and (iii). □

(3.3) REMARK. We use the notation in (3.1) and in Proposition (3.2) and its proof. If $x \in D_I \cap U_K^{(2|I|)}$ and if $W \in \mathcal{W}_I$ contains $x$, then $W$ is of Type 1 or Type 2, because of (3.2.19) and (3.2.23). Hence the proof in Case 1 in Step 4 in the proof of Proposition (3.2) shows the following:

Let $I \subset K := I \sqcup \{k\} \subset \{1, \dots, a+b\}$ and assume $D_k \neq \emptyset$. Let $x \in D_I$ and let $\tilde{x} := \pi_K(x) \in D_K$. Then, the fiber $\pi_I^{-1}(x)$ is a submanifold of the fiber $\pi_K^{-1}(\tilde{x})$ defined by $\zeta_k^K(y, \tilde{x}) - \zeta_k^K(x, \tilde{x}) = 0$ $(y \in \pi_K^{-1}(\tilde{x}))$.

(3.4) REMARK. In the proof of [C, Theorem 5.7], the argument to show the well-definedness of the projections $\pi_I$ is missing. We prove this in Step 4 in the proof of Proposition (3.2). Section 2 and the constructions in Step 3 in the proof of Proposition (3.2) is needed for this proof.

**4. A family of global equations.** By using the results in the previous section, we construct in Proposition (4.3) below a $C^\infty$ family of holomorphic coordinates of the fibers of the family of normal projections in Proposition (3.2), which fits to the original morphism $f : X \to \Delta$ in (3.1.1). The argument in this section is a refined version of the proof of the latter half of [C, Theorem 5.7] (cf. Remark (4.4)).

(4.1) Let $f : X \to \Delta$, $X_0$ and $Y$ be as in (3.1). We freely use the notation in Section 3, especially the notation in Proposition (3.2) and in its proof.

Before stating the main result in this section, we refine the compatibility condition (3.2.22) of the families of $C^\infty$ partition of unity $\{\rho_{W^\sharp \cap D_I}\}_{W^\sharp \in \mathcal{W}_I^\sharp}$ introduced in (3.2.21) into a form suitable for the proof of Proposition (4.3) below.

In Step 3 of the proof of Proposition (3.2), we have first constructed the covering $\mathcal{W}_I^\sharp$, in (3.2.20), of the tubular neighborhood $U_I^{(|I|)}$ of $D_I$, and then refined it into $\mathcal{W}_I$ in (3.2.24). The family $\mathcal{W}_I^\flat$ in (3.2.25) is obtained from $\mathcal{W}_I$ by throwing away the open sets of Type 1. Let $W \in \mathcal{W}_I$. We recall the notation $W^*(W)$ introduced just before (3.2.27), i.e., when we follow backward the construction process of $W \in \mathcal{W}_I$ by descending induction, $W^*(W)$ is just the previous open set belonging to $\mathcal{W}_M^\flat$ for some $M \subset \{1, \dots, a+b\}$ containing $I$. We understand here that $W^*(W) = W$ if $W \in \mathcal{W}_I^\flat$.

For $I \subset M \subset \{1, \dots, a+b\}$, we denote

$$\mathcal{W}_I(M) := \{W \in \mathcal{W}_I \mid W^*(W) \in \mathcal{W}_M^\flat\}. \tag{4.1.1}$$

Then, these form a division of $\mathcal{W}_I$:

$$\mathcal{W}_I = \bigsqcup_{M \supset I} \mathcal{W}_I(M). \tag{4.1.2}$$

By the definition (3.2.30) of the partition of unity $\{\rho_{W\cap D_I}\}_{W\in\mathcal{W}_I}$, it satisfies the following refined compatibility condition induced from the one in (3.2.22) for the partition of unity $\{\rho_{W^\sharp\cap D_I}\}_{W^\sharp\in\mathcal{W}_I^\sharp}$: If $M\supset K\supset I$, $|K|=|I|+1$ and $W^*\in\mathcal{W}_M^\flat$, then every $W\in\mathcal{W}_I(M)$ is of type $W_1(\ )$ and, on $U_K^{(2|K|-1)}\cap D_I$, we have

$$(4.1.3)\qquad \sum_{\substack{W\in\mathcal{W}_I(M)\\ W^*(W)=W^*}}\rho_{W\cap D_I}=\sum_{\substack{\tilde W\in\mathcal{W}_K(M)\\ W^*(\tilde W)=W^*}}\chi_I\cdot\rho_{\tilde W\cap D_K}\circ\pi_K\,.$$

Here the $\chi_I$ is the functions in (3.2.29) and the $\pi_K$ is the projections in Proposition (3.2).

We prove a lemma, which will be used in the proof of Proposition (4.3) below.

(4.2) LEMMA. *We use the above notation and also the notation in the proof of Proposition* (3.2). *Let $J\subset K=J\sqcup\{k\}\subset M\subset\{1,\dots,a+b\}$ with $k\le a$. Let $x\in D_J\cap U_K^{(2|K|-1)}$, $F:=\pi_J^{-1}(x)$ and $\tilde x:=\pi_K(x)$. Then, for $y\in F$, we have*

$$(4.2.1)\qquad \prod_{W\in\mathcal{W}_K(M)}z_{k,W}(y)^{\rho_{W\cap D_K}(\tilde x)}=\prod_{W\in\mathcal{W}_K(M)}z_{k,W}(x)^{\rho_{W\cap D_K}(\tilde x)}$$

*Here we regard $z_{k,W}(y)^{\rho_{W\cap D_K}(\tilde x)}:=\exp(\rho_{W\cap D_K}(\tilde x)\log z_{k,W}(y))$ etc. as usual.*

PROOF. Take a chain $K=:K_1\subset K_2\subset\cdots\subset K_n:=M$ with $|K_j|+1=|K_{j+1}|$. We have $\chi_{K_j}(x)=\chi_{K_j}(y)$ $(1\le j\le n)$ for $x$ and $y$ in the Lemma, since the function $\chi_{K_j}$ on $U_{K_j}^{(2|K_j|)}$ in (3.2.29) is constant on each fiber of the projection $\varpi_{K_j}^{K_{j+1}}$ in (3.2.16) by definition, since the projections $\varpi_{K_j}^{K_{j+1}}$ and $\pi_{K_j}$ coincide on $U_{K_{j+1}}^{(2|K_j|)}$ by Remark (3.3), and since the projections $\pi_{K_j}$ and $\pi_{K_{j+1}}$ are compatible on $U_{K_{j+1}}^{(2|K_j|)}$. Hence the assertion (4.2.1) follows from the following assertions (4.2.2) and (4.2.3).

For $K$, $M$, $k$, $y$ and $\tilde x$ in the Lemma,

$$(4.2.2)\qquad \prod_{W\in\mathcal{W}_K(M)}z_{k,W}(y)^{\rho_{W\cap D_K}(\tilde x)}=\zeta_k^M(y,\pi_M(y))^{(1-\chi_{K_n}(y))\chi_{K_{n-1}}(y)\cdots\chi_{K_2}(y)\chi_{K_1}(y)}\,.$$

For $M$, $k$, $y$ and $x$ in the Lemma,

$$(4.2.3)\qquad \zeta_k^M(y,\pi_M(y))=\zeta_k^M(x,\pi_M(x))\,.$$

By taking product over $W^*\in\mathcal{W}_M^\flat$ and using the definition (3.2.30) of the partition of unity $\{\rho_{W^*\cap D_M}\}_{W^*\in\mathcal{W}_M}$, (4.2.2) follows from the following assertion (4.2.4).

$$(4.2.4)\qquad \prod_{\substack{W\in\mathcal{W}_K(M)\\ W^*(W)=W^*}}z_{k,W}(y)^{\rho_{W\cap D_K}(\tilde x)}=(z_{k,W^*}(y)^{\rho_{W^*\cap D_M}(\pi_M(y))})^{\chi_{K_{n-1}}(y)\cdots\chi_{K_2}(y)\chi_{K_1}(y)}\,.$$

We prove (4.2.4) by induction on the length of the chain $n$. When $n=1$, i.e., $K=M$, (4.2.4) is obvious. When $n>1$, by the induction hypothesis, we have

$$(4.2.5)\qquad \prod_{\substack{\tilde W\in\mathcal{W}_{K_2}(M)\\ W^*(\tilde W)=W^*}}z_{k,\tilde W}(y)^{\rho_{\tilde W\cap D_{K_2}}(\tilde x)}=(z_{k,W^*}(y)^{\rho_{W^*\cap D_M}(\pi_M(y))})^{\chi_{K_{n-1}}(y)\cdots\chi_{K_3}(y)\chi_{K_2}(y)}\,.$$

Then, using (4.2.5), the definition of the standard local equations (3.2.26) and the compatibility (4.1.3), we have

$$\begin{aligned}
\prod_{\substack{W\in\mathcal{W}_{K_1}(M)\\ W^*(W)=W^*}} z_{k,W}(y)^{\rho_{W\cap D_{K_1}}(\tilde{x})} &= \prod_{\substack{W\in\mathcal{W}_{K_1}(M)\\ W^*(W)=W^*}} z_{k,W^*}(y)^{\rho_{W\cap D_{K_1}}(\tilde{x})} \\
&= z_{k,W^*}(y)^{\sum_{W\in\mathcal{W}_{K_1}(M),W^*(W)=W^*}\rho_{W\cap D_{K_1}}(\tilde{x})} \\
&= z_{k,W^*}(y)^{\sum_{\tilde{W}\in\mathcal{W}_{K_2}(M),W^*(\tilde{W})=W^*}\chi_{K_1}(y)\rho_{\tilde{W}\cap D_{K_2}}(\tilde{x})} \\
&= \left(\prod_{\substack{\tilde{W}\in\mathcal{W}_{K_2}(M)\\ W^*(\tilde{W})=W^*}} z_{k,\tilde{W}}(y)^{\rho_{\tilde{W}\cap D_{K_2}}(\tilde{x})}\right)^{\chi_{K_1}(y)} \\
&= (z_{k,W^*}(y)^{\rho_{W^*\cap D_M}(\pi_M(y))})^{\chi_{K_{n-1}}(y)\cdots\chi_{K_2}(y)\chi_{K_1}(y)}\,.
\end{aligned}$$

Thus, (4.2.4) and hence (4.2.2) is proved.

We prove (4.2.3). For any set $L$ with $K \subset L \subset M$ and $M = L \sqcup \{k\}$, we have, by the compatibility of the projections $\pi_\bullet$ and the result in Remark (3.3), that

$$\begin{aligned}
F &\subset \pi_L^{-1}(\pi_L(x)) \\
&= \pi_M^{-1}(\pi_M(x)) \cap \{y \mid \zeta_k^M(y, \pi_M(x)) = \zeta_k^M(\pi_L(x), \pi_M(x))\}\,.
\end{aligned}$$

This, together with $x \in \pi_L^{-1}(\pi_L(x))$, implies

$$\zeta_k^M(y, \pi_M(x)) = \zeta_k^M(\pi_L(x), \pi_M(x)) = \zeta_k^M(x, \pi_M(x))\,.$$

Thus, (4.2.3) is proved. □

We now prove the main result of this section:

(4.3) PROPOSITION. *We use the notation in* (4.1), *in Lemma* (4.2) *and in Section* 3. *Shrinking the open neighborhoods $U_i$ of $D_i$ $(1 \le i \le a+b)$ and the disc $\Delta$, and restricting the projections $\pi_J$ to the shrinked $U_J$, we have multi-valued $C^\infty$ functions*

$$z_i(y) \quad on\ X \quad (1 \le i \le a+b)\,,$$

*which satisfy the following conditions*:

(i) *For each $1 \le i \le a+b$, $z_i$ is a global equation of $D_i$ in $X$, which is multi-valued as a global function on $X$ with branches determined by the choice of multiples*

$$\prod_{I\ni i,\,W\in\mathcal{W}_I} \exp(2\pi n_{I,W}\psi_I(y)\rho_{W\cap D_I}(\pi_I(y))\sqrt{-1}) \quad (n_{I,W}\in\mathbf{Z},\ y\in X)\,.$$

(ii) *If $J \subset \{1,\ldots,a\}$ and $F$ is the fiber $\pi_J^{-1}(x)$ over $x \in D_J$, then the restricted functions $z_j|F$, $j \in J$, which are now single-valued after choosing branches, form a system of holomorphic coordinates on $F$.*

(iii) *Let $J$, $x$ and $F$ be as in* (ii). *For $k \in \{1, \dots, a+b\} - J$, the functions $z_k$ is constant on the fiber $F$, if it is defined.*

(iv) *Let $x \in X_0$. Let $J := \{j \mid 1 \le j \le a,\ D_j \ni x\}$ and let $F := \pi_J^{-1}(x)$. Then*

$$\prod_{j \in J} z_j^{m(j)} = (\text{constant}) t \circ f \quad \text{on } F\,,$$

*where the* (constant) *depends on the choice of $x$, the choice of $z_i$ $(1 \le i \le a)$ and the choice of their branches.*

PROOF. We continue to use the notation in the proof of Proposition (3.2). Let $U_i^{(2)}$ be the open neighborhoods $U_i$ of $D_i$ in Proposition (3.2) $(1 \le i \le a+b)$. We choose smaller open neighborhoods $U_i^{(1)}$ of $D_i$ $(1 \le i \le a+b)$ so that

$$U_i^{(1)} \Subset U_i^{(2)}\,. \tag{4.3.1}$$

For each $I \subset \{1, \dots, a+b\}$, we denote

$$U_I' := U_I^{(2)} - \left( \bigcup_{k \notin I} \overline{U_k^{(1)}} \right). \tag{4.3.2}$$

Put $U^{(2)} := \bigcup_{1 \le i \le a+b} U_i$. Note that these open sets $U_I'$ form a covering of $U^{(2)}$. Let

$$\{\psi_I\}_{I \subset \{1, \dots, a+b\}} \tag{4.3.3}$$

be a $C^\infty$ partition of unity on $U^{(2)}$ which is subordinate to the above covering and has the following property:

$$\begin{gathered}\text{For all } J \subset I \subset \{1, \dots, a+b\} \text{ and all } x \in D_J, \\ \psi_I \text{ is constant along the set } \pi_J^{-1}(x) \cap U_J^{(1)}\,. \end{gathered} \tag{4.3.4}$$

Such a partition of unity can be constructed easily by descending induction on $|I|$.

We restrict the projections $\pi_I$ to

$$\pi_I : U_I^{(1)} \to D_I \quad \text{for } I \subset \{1, \dots, a+b\}\,. \tag{4.3.5}$$

We shrink the disc $\Delta$, so that $X$ is covered by $U_i^{(1)}$ $(1 \le i \le a)$.

By using the $\zeta_i^I$ in (3.2.12), we define for each $1 \le i \le a+b$, a multi-valued $C^\infty$ function $z_i$ on $X$ by

$$z_i(y) = \prod_{I \ni i} \zeta_i^I(y, \pi_I(y))^{\psi_I(y)}\,. \tag{4.3.6}$$

Like (3.2.12), this definition is justified in the following way. For $I \subset \{1, \dots, a+b\}$, put $x := \pi_I(y)$, if defined, and take $W_I \in \mathcal{W}_I$ satisfying $\rho_{W \cap D_I}(x_I) \neq 0$. Let $g_{i, W W_I}$ be the transition function $z_{i,W} = g_{i,WW_I} z_{i,W_I}$ $(W \in \mathcal{W}_I,\ W \cap W_I \neq \emptyset)$. Put $W_i := W_{\{i\}}$. Let

$g_{i,W_I W_i}$ be the transition function $z_{i,W_I} = g_{i,W_I W_i} z_{i,W_i}$. Then we have

$$
\begin{aligned}
z_i(y) &= \prod_{I \ni i} \zeta_i^I(y, \pi_I(y))^{\psi_I(y)} = \prod_{I \ni i} \left( \prod_{W \in \mathcal{W}_I} z_{i,W}(y)^{\rho_{W \cap D_I}(x_I)} \right)^{\psi_I(y)} \\
&= \prod_{I \ni i} \left( z_{i,W_I}(y) \prod_{W \in \mathcal{W}_I} g_{i,WW_I}(y)^{\rho_{W \cap D_I}(x_I)} \right)^{\psi_I(y)} \\
&= z_{i,W_i}(y) \prod_{I \ni i} \left( \left( \prod_{W \in \mathcal{W}_I} g_{i,WW_I}(y)^{\rho_{W \cap D_I}(x_I)} \right) g_{i,W_I W_i}(y) \right)^{\psi_I(y)} \\
&= z_{i,W_i}(y) \exp \left( \sum_{I \ni i} \sum_{W \in \mathcal{W}_I} \psi_I(y) \rho_{W \cap D_I}(x_I) \log(g_{i,WW_I}(y) g_{i,W_I W_i}(y)) \right).
\end{aligned}
$$

The last equation is well-defined if we choose a branch of $\log(g_{i,WW_I}(y) g_{i,W_I W_i}(y))$. The ambiguity of the choice of a branch is the choice of a multiple

$$
\prod_{I \ni i, W \in \mathcal{W}_I} \exp(2\pi n_{I,W} \psi_I(y) \rho_{W \cap D_I}(\pi_I(y)) \sqrt{-1}) \quad (n_{I,W} \in \mathbf{Z}). \tag{4.3.7}
$$

Thus, (i) is proved.

We prove (ii). Let $J \subset \{1, \dots, a\}$, $x \in D_J$ and $F := \pi_J^{-1}(x)$, as in (ii). First, note that

$$
\text{if } I \subset \{1, \dots, a+b\} \text{ and } \operatorname{supp} \psi_I \cap F \neq \emptyset, \text{ then } I \supset J. \tag{4.3.8}
$$

This follows from $F \subset U_J^{(1)}$, $\operatorname{supp} \psi_I \subset U_I'$ and the definition (4.3.2) of $U_I'$. Hence, for $y \in F \cap U_I'$, we have

$$
\pi_I(y) = \pi_I \circ \pi_J(y) = \pi_I(x) \tag{4.3.9}
$$

by the compatibility in Proposition (3.2) (ii). This, together with the property (4.3.4) of the $\psi_I$ and the definition (4.3.6) of the $z_j$, implies that the restricted functions $z_j|F$, $j \in J$, which can be regarded as single-valued functions, form a system of holomorphic coordinates on $F$. (ii) is proved.

We prove (iii). Let $J, x, F$ and $k$ be as in (iii). Put $K := J \sqcup \{k\}$. If $D_K = \emptyset$, then $z_k$ is not defined on the fiber $F$. So, we assume $D_K \neq \emptyset$. For any $M$ with $K \subset M \subset \{1, \dots, a+b\}$ and $D_M \neq \emptyset$, the function $\zeta_k^M(y, \pi_M(y))$ is constant in $y \in F$ by (4.2.3) in the proof of Lemma (4.2). Hence, by the definition (4.3.6) of the function $z_k$ and by the definition (4.3.3) of the $\psi_K$, $z_k$ is constant on the fiber $F$. (iii) is proved.

We prove (iv). Let $x$, $J$ and $F$ be as in (iv). By (4.3.6), (iv) follows if we prove the following assertion:

For $I \supset J$ and $y \in F - X_0$, we have

$$
\left( \prod_{j \in J} \zeta_{j,W}^I(y, \pi_I(y))^{m(j)} \right)^{\psi_I(y)} = (\text{constant})(t \circ f(y))^{\psi_I(y)}, \tag{4.3.10}
$$

where both sides are considered as single-valued functions on $F$ after choosing their branches. In fact, taking the products, over all $I$ containing $J$, of the both sides of (4.3.10), we have

$$\prod_{I\supset J}(\text{constant})(t\circ f(y))^{\psi_I(y)} = \prod_{I\supset J}\left(\prod_{j\in J}\zeta^I_{j,W}(y,\pi_I(y))^{m(j)}\right)^{\psi_I(y)}$$
$$= \prod_{j\in J}\left(\prod_{I\supset J}\zeta^I_{j,W}(y,\pi_I(y))^{\psi_I(y)}\right)^{m(j)}$$

for $y\in F-X_0$. Applying (4.3.8) to both sides, we have (iv) on $F-X_0$, and hence on $F$ by the identity theorem.

In order to prove (4.3.10), we divide the problem into two cases according as $I=J$ or $I\supset J$.

*Case* 1. $I=J$: In the present case, by using the division (4.1.2) of $\mathcal{W}_J$, and the definition (3.2.12) of the $\zeta^J_j$ the assertion (4.3.10) follows from Claim 1 below.

*Claim* 1.

$$\prod_{j\in J}\prod_{W\in\mathcal{W}_J(M)}(z_{j,W}(y))^{m(j)\rho_{W\cap D_J}(x)}$$
$$= (\text{constant})(t\circ f(y))^{\sum_{W\in\mathcal{W}_J(M)}\rho_{W\cap D_J}(x)} \quad \text{for } y\in F-X_0\,.$$

We prove this by induction on $|M-J|$.

When $|M-J|=0$, we have $\mathcal{W}_J(M)=\mathcal{W}^\flat_J$ (see the definition (4.1.1)). Then, by the descending-inductive constructions (3.2.27) and (3.2.28) of the standard local equations on $W\in\mathcal{W}_J(M)=\mathcal{W}^\flat_J$, we have

$$\prod_{i\in I}(z_{i,W}(y))^{m(i)} = t\circ f(y) \quad \text{for } y\in F\,. \tag{4.3.11}$$

This implies Claim 1 in this case.

If $|M-J|>0$, we choose $k\in M-J$ and set $K:=J\sqcup\{k\}$. By (3.2.23), we may assume that such $k$ is chosen so that $U_K^{(2|K|-1)}$ contains $x$. The induction hypothesis yields

$$\begin{aligned}\prod_{j\in K(a)}\prod_{\tilde W\in\mathcal{W}_K(M)}&(z_{j,\tilde W}(y))^{m(j)\rho_{\tilde W\cap D_K}(\tilde x)}\\ &= (\text{constant})(t\circ f(y))^{\sum_{\tilde W\in\mathcal{W}_K(M)}\rho_{\tilde W\cap D_K}(\tilde x)} \quad \text{for } y\in\tilde F-X_0\,,\end{aligned}\tag{4.3.12}$$

where $\tilde x:=\pi_K(x)$ and $\tilde F:=\pi_K^{-1}(\tilde x)$.

If $k>a$, Claim 1 follows directly from (4.3.12) and the compatibility (4.1.3) of the partition of unity $\rho_\bullet$.

Now we assume $k\le a$. Taking the $\chi_J(x)$-th power of the right-hand-side of (4.3.12), and using the compatibility (4.1.3), we have

$$\begin{aligned}&((\text{constant})(t\circ f(y))^{\sum_{\tilde W\in\mathcal{W}_K(M)}\rho_{\tilde W\cap D_K}(\tilde x)})^{\chi_J(x)}\\ &= (\text{constant})^{\chi_J(x)}(t\circ f(y))^{\sum_{W\in\mathcal{W}_J(M)}\rho_{W\cap D_J}(x)} \quad \text{for } y\in F-X_0\,.\end{aligned}\tag{4.3.13}$$

Put $C := \prod_{\tilde{W}\in\mathcal{W}_K(M)}(z_{k,\tilde{W}}(y))^{m(j)\rho_{\tilde{W}\cap D_K}(\tilde{x})\chi_J(x)}$. Then, Lemma (4.2) shows that $C$ is constant in $y \in F$. Taking the $\chi_J(x)$-th power of the left-hand-side of (4.3.12), using the fact that the elements of $\mathcal{W}_K(M)$ and of $\mathcal{W}_J(M)$ are of Type 1 (see (4.1)), and using the compatibility (4.1.3), we have

$$\begin{aligned}
&\prod_{j\in K(a)}\ \prod_{\tilde{W}\in\mathcal{W}_K(M)} (z_{j,\tilde{W}}(y))^{m(j)\rho_{\tilde{W}\cap D_K}(\tilde{x})\chi_J(x)} \\
&\qquad = C\prod_{j\in J}\ \prod_{W^*\in\mathcal{W}_M^\flat}\ \prod_{\substack{\tilde{W}\in\mathcal{W}_K(M)\\ W^*(\tilde{W})=W^*}} (z_{j,W^*}(y))^{m(j)\rho_{\tilde{W}\cap D_K}(\tilde{x})\chi_J(x)} \\
(4.3.14)\qquad &\qquad = C\prod_{j\in J}\ \prod_{W^*\in\mathcal{W}_M^\flat}\ \prod_{\substack{W\in\mathcal{W}_J(M)\\ W^*(W)=W^*}} (z_{j,W}(y))^{m(j)\rho_{W\cap D_J}(x)} \\
&\qquad = C\prod_{j\in J}\ \prod_{W\in\mathcal{W}_J(M)} (z_{j,W}(y))^{m(j)\rho_{W\cap D_J}(x)} \quad \text{for } y\in F-X_0\,.
\end{aligned}$$

Combining (4.3.13) and (4.3.14), and denoting the new constant also by (constant), we have Claim 1 in this case.

*Case* 2. $I \supset J$: In this case, we prove (4.3.10) by induction on $|I - J|$.

If $|I - J| = 0$, our assertion is the result in Case 1.

Now we assume that (4.3.10) holds for every $I \supset J$ with $|I - J| \le m$. Let $J =: J' \sqcup \{j\}$, $x' \in D_{J'}$, $F' := \pi_{J'}^{-1}(x')$ and $\tilde{x} := \pi_I(x')$. Since $\operatorname{supp}\psi_I \subset U_I'$, the assertion (4.3.10) for $J'$ follows from:

*Claim* 2. $\zeta_j^I(y, \tilde{x})$ is constant for $y \in F' \cap U_I'$, where we choose and fix a branch of the function.

Note that the projections $\pi_M$, $M \subset I$, are compatible on $U_I'$. Let $I' := I - \{j\}$. Then $I \supset I' \supset J'$ and

$$F' \subset \pi_{I'}^{-1}(\tilde{x}')\,, \quad \text{where } \tilde{x}' := \pi_{I'}(x')\,.$$

Hence, Claim 2 follows from Remark (3.3), that is, $\pi_{I'}^{-1}(\tilde{x}')$ is the submanifold of $\pi_I^{-1}(\tilde{x})$ defined by

$$\zeta_j^I(y, \tilde{x}) - \zeta_j^I(\tilde{x}', \tilde{x}) = 0\,.$$

This completes the proof of (4.3.10) and the assertion (iv) is verified. □

(4.4) REMARK. The argument of Clemens to prove [C, Theorem 5.7 v)] (= our Proposition (4.3) (iv)) breaks down. This point is rescued by our subdivisions (3.2.23) and their resulting constructions in Step 4 in the proof of Proposition (3.2).

**5. Monoid actions and recovery of vanishing cycles.** In this section, by using the multi-valued $C^\infty$ global equations $z_j$ of the components of $X_0$ in Proposition (4.3), we lift the natural action of the monoid $S = [0, 1] \times \boldsymbol{C}_1$ on the disc $\Delta$ of the base compatibly to all spaces of the diagram (0.1) (Theorem (5.2)). Our argument here is a variation of the one in [C, Section 6]. As an application of this, we prove our main theorem (Theorem (5.4)).

(5.1) Let $f : X \to \Delta$, $X_0$ and $Y$ be as in (3.1).

As in Example (1.2.1), let

$$f^{\log} : X^{\log} \to \Delta^{\log} \tag{5.1.1}$$

be the log family induced by the flat family $f : X \to \Delta$ endowed with the fs log structures $\mathcal{M}_X$, $\mathcal{M}_\Delta$ asociated to $(X_0)_{\text{red}}$ and $\{0\}$, respectively. Let $Y^{\log}$ be the log topological space, defined by the fiber product

$$\begin{array}{ccc} Y^{\log} & \longrightarrow & X^{\log} \\ {\scriptstyle \tau_Y}\downarrow & & {\scriptstyle \tau_X}\downarrow \\ Y & \longrightarrow & X\,. \end{array} \tag{5.12}$$

Let

$$S := [0, 1] \times \boldsymbol{C}_1 \tag{5.1.3}$$

be the product of the closed unit interval $[0, 1]$ and the unit circle $\boldsymbol{C}_1$ with center 0 in the complex plane. $S$ is regarded as a monoid by multiplication. We consider the following actions of the monoid $S$ on the disc $\Delta$ and on the associated space $\Delta^{\log}$:

$$\begin{aligned} &(s, v) \cdot t = svt \quad \text{on } \Delta\,, \\ &(s, v) \cdot (t, u) = (svt, vu) \quad \text{on } \Delta^{\log}\,, \quad \text{where } t = |t|u\,. \end{aligned} \tag{5.1.4}$$

(5.2) THEOREM. *We use the notation in* (5.1) *in Section* 3 *and in Section* 4. *Shrinking the disc $\Delta$, if necessary, the actions* (5.1.4) *of the monoid $S$ lift to piecewise $C^\infty$ actions on $X$, on $Y$, on $X^{\log}$ and on $Y^{\log}$ with the following three properties.*

(i) *The actions of $S$ are compatible with the inclusions $X \supset Y$ and $X^{\log} \supset Y^{\log}$, and with the diagram*

$$\begin{array}{ccc} X^{\log} & \xrightarrow{\tau_X} & X \\ {\scriptstyle f^{\log}}\downarrow & & {\scriptstyle f}\downarrow \\ \Delta^{\log} & \xrightarrow{\tau_\Delta} & \Delta\,. \end{array}$$

(ii) *The action on $X_0$ of each element $(s, v) \in S$ is homotopic to the identity.*

(iii) *The action of $S$ on $X$ is compatible with the projections*

$$\pi_I : \left(U_I - \bigcup_{k \notin I, k \le a} U_{I \sqcup \{k\}}\right) \to \left(D_I - \bigcup_{k \notin I, k \le a} U_{I \sqcup \{k\}}\right)$$

*for any $I \subset \{1, \dots, a\}$, where $a$ is the number of irreducible components of the central fiber $X_0$.*

PROOF. We divide the proof into two steps.

*Step* 1. First we introduce 'hyperbolic polar coordinates' and an action of the monoid $[0, 1]$ on them. Recall the notation $X_0 = \sum_{1 \le i \le a} m(i) D_i$ for the central fiber of $f$.

Let

$$
\begin{aligned}
& C := [0,1]^a \quad \text{the unit cube in } \boldsymbol{R}^a\,, \\
(5.2.1)\qquad & C_\delta := \left\{ (r_i) \in C \;\middle|\; \sum_{1 \le i \le a} r_i^{m(i)} = \delta \right\} \quad \text{for } 0 \le \delta < 1\,, \\
& E_\delta := \bigcup_{\delta' \in [0,\delta]} C_{\delta'} = \left\{ (r_i) \in C \;\middle|\; \prod_{1 \le i \le a} r_i^{m(i)} \le \delta \right\}.
\end{aligned}
$$

Let $A := \{1, \dots, a\}$. We choose a real number

$$(5.2.2) \qquad 0 < \varepsilon < 1\,.$$

In the following, we assume that all the cuboids contained in $C$ are parallel to the cube $C$. Let $G$ be the cuboid in $C$ with the two points $B := (\varepsilon, \dots, \varepsilon)$ and $(1, \dots, 1)$ as the extreme vertices. We construct a family of projections from each face of $G$ containing the vertex $B$ to the union of the faces of $C$ containing the origin $O$ as follows:

For $I \subset A$, we denote by $B(I)$ the vertex of the cuboid $G$ whose $i$-th coordinate is 1 for $i \in I$ and the other coordinates are $\varepsilon$. Let $G(I)$ be the face of $G$ with the two points $B$ and $B(I)$ as the extreme vertices, and let $C(I)$ be the face of $C$ passing through $O$, parallel to $G(I)$ and of the same dimension as $G(I)$. For each point $Q \in G(I)$, let $G(I)^\perp + Q$ be the affine subspace which is the orthogonal complement of $G(I)$ passing through $Q$, and let $p_Q$ be the projection in $G(I)^\perp + Q$ from the point $Q$ whose rays are in the cuboid in $G(I)^\perp + Q$ with the two points $Q$ and $(G(I)^\perp + Q) \cap C(I)$ as the extreme vertices. We denote by $p_I$ the collection of the projections $p_Q$ $(Q \in G(I))$. We thus have a family $\{p_I\}_{I \subset A}$ of projections.

Choose a positive number $\delta_0 < 1$ so small that

$$(5.2.3) \qquad (r_i)_{i \in A} \in E_{\delta_0} \text{ implies } r_i < \varepsilon/2 \text{ for some } i \in A\,.$$

Then, for a fixed non-negative number $\delta \le \delta_0$ and any fixed point $(r_i) \in C_{\delta_0}$, the hypersurface $C_\delta$ and the unique ray of the family of projections $\{p_I\}_{I \subset A}$ passing through the point $(r_i)$ intersect at one point and, moreover, the intersections are transversal except at the points of the singular locus of $C_0$. Denote this intersection point by

$$(5.2.4) \qquad \langle r, (r_i) \rangle\,, \quad \text{where } r := \delta/\delta_0 \text{ and } (r_i) \in C_{\delta_0}\,,$$

and call this *hyperbolic polar coordinates* of the point in $E_{\delta_0}$.

We define a continuous action of the monoid $[0, 1]$ on $E_{\delta_0}$

$$(5.2.5) \qquad R : [0,1] \times E_{\delta_0} \to E_{\delta_0} \quad \text{by } R(s, \langle r, (r_i) \rangle) := \langle sr, (r_i) \rangle\,.$$

Then, this action has the following properties:

(5.2.5.1) $\quad R$ is piecewise $C^\infty$.

(5.2.5.2) $\quad R(s, C_\delta) = C_{s\delta}$ for $\delta \in [0, \delta_0]$.

(5.2.5.3) $\quad R(1, \ ) = \mathrm{id}$.

(5.2.5.4) $\quad R(s, \ )|_{C_0} = \mathrm{id}$ for any $s \in [0, 1]$.

Since $\delta_0$ is chosen to have the property (5.2.3), we see that

$$\{(r_i) \in C_{\delta_0} \mid r_j < \varepsilon/2\}_{j=1,\dots,a} \tag{5.2.6}$$

is an open covering of $C_{\delta_0}$. Take a $C^\infty$ partition of unity

$$\{\lambda_j\}_{j=1,\dots,a} \tag{5.2.7}$$

on $C_{\delta_0}$ subordinate to the covering (5.2.6), and extend this over $E_{\delta_0}$ by

$$\lambda_j(\langle r, (r_j)\rangle) := \lambda_j((r_i)) \quad \text{for all } r \in [0, 1]\,. \tag{5.2.8}$$

*Step* 2. We define here actions of the monoid $S$ on $X$ and on $X^{\log}$.

Let

$$r_i(y) := |z_i(y)| \quad \text{and} \quad r_i(y)u_i(y) := z_i(y) \quad (1 \le i \le a)\,, \tag{5.2.9}$$

where $z_i(y)$ is the equation of $D_i$ on $X$, constructed in Proposition (4.3). Note that the $r_i(y)$ are single-valued functions, whereas the $u_i(y)$ are multi-valued. We may assume that the positive number $\varepsilon$ in (5.2.2) are chosen so small that

$$\{y \in X \mid r_i(y) \le \varepsilon\} \subset U_i \quad (1 \le i \le a)\,, \tag{5.2.10}$$

where $U_i$ is the open neighborhood of $D_i$ in Proposition (4.3). We shrink $\Delta$ so that the $U_i$ $(1 \le i \le a)$ cover $X$, that $r_i(y) \le 1$ for all $y \in X$ and all $i \in \{1, \dots, a\}$, and that the radius of $\Delta$ is not greater than $\delta_0$ which is chosen in Step 1.

For $y \in X$, let

$$\begin{aligned} &I := \{i \mid 1 \le i \le a, U_i \ni y\}\,, \quad x := \pi_I(y)\,, \quad F := \pi_I^{-1}(x)\,, \\ &F^{\log} : \ \text{the closure of } \tau_X^{-1}(F - F \cap X_0) \ \text{in } X^{\log}\,. \end{aligned} \tag{5.2.11}$$

Note that $F \cap X_0$ is a divisor with normal crossings on $F$, and hence $F$ has the fs log structure $\mathcal{M}_F$ induced by $(F \cap X_0)_{\mathrm{red}}$ (see, (1.1.4)). $F^{\log} \to F$ in (5.2.11) is nothing but the one defined by $\mathcal{M}_F$ as in (1.2). Now for each $u_i$ $(i \in I)$, we choose a branch and regard $u_i$ as a single-valued function on $F^{\log}$. We thus have coordinates $(r_i(\ ), u_i(\ ))_{i \in I}$ on $F^{\log}$. Now we define an action of the monoid $S = [0, 1] \times \boldsymbol{C}_1$ on $F^{\log}$

$$S \times F^{\log} \to F^{\log} \quad \text{by} \quad \begin{cases} r_i((s, v) \cdot \eta) := R(s, y)_i \\ u_i((s, v) \cdot \eta) := v^{\lambda_i(y)/m(i)} u_i(\eta) \end{cases} \quad (i \in I)\,, \tag{5.2.12}$$

where

$$\begin{aligned} &\eta \in F^{\log}\,, \quad \tau_X(\eta) = y \in F\,, \\ &R(s, y) = (R(s, y)_j)_{1 \le j \le a} := R(s, (r_j(y))_{1 \le j \le a})\,, \\ &\lambda_i(y) := \lambda_i((r_j(y))_{1 \le j \le a})\,. \end{aligned} \tag{5.2.13}$$

*Claim* 1. The action (5.2.12) is compatible with the restricted morphism $f^{\log} : F^{\log} \to \Delta^{\log}$.

In fact, by (5.2.12) and Proposition (4.3) (ii), we have, for $(s, v) \in S$ and $\eta \in F^{\log}$, that

$$r \circ f^{\log}((s, v) \cdot \eta) = (\text{constant}) \left( s \prod_{i \in I} r_i(\eta)^{m(i)} \right) = s \cdot r \circ f^{\log}(\eta)\,,$$

$$u \circ f^{\log}((s, v) \cdot \eta) = (\text{constant}) \left( v \prod_{i \in I} u_i(\eta)^{m(i)} \right) = v \cdot u \circ f^{\log}(\eta)\,.$$

*Claim* 2. The monoid actions (5.2.12) on the fibers $F^{\log}$ fit together to give a continuous action on $X^{\log}$.

In fact, in the notation in Section 4, it follows from (5.2.10) that

$$r_i(y) > \varepsilon \quad \text{for } y \in U_i^{(2)} - U_i \quad (1 \le i \le a)\,.$$

Therefore, for each pair $I \subset M \subset \{1, \dots, a\}$, by construction, especially by the property of hyperbolic coordinates, we have

$$R(s, y)_j = r_j(y) \quad \text{and} \quad \lambda_j(y) = 0$$
$$\text{for all } s \in [0, 1]\,, \text{ all } y \in U_I \cap (U_M^{(2)} - U_M) \text{ and all } j \in M - I\,.$$

From this together with (5.2.12), we get Claim 2.

By Proposition (4.3) (iii), we see that the $S$-action on $X^{\log}$ preserves the subspace $Y^{\log}$. Now, it is obvious that $S$-actions on $X^{\log}$ and on $Y^{\log}$ drop down to induce $S$-actions on $X$ and on $Y$ and that these $S$-actions satisfy the other conditions in the theorem. □

(5.3) We assume that the family $f : X \to \Delta$, in Section 3, is reduced. Using the action of the monoid $S$ in Theorem (5.2), we introduce a horizontal projection of the family of log topological spaces $f^{\log} : X^{\log} \to \Delta^{\log}$ in (5.1) in the following way. We denote

$$0^{\log} := \tau_\Delta^{-1}(0) \simeq \boldsymbol{C}_1\,. \tag{5.3.1}$$

For $(0, 1), (s, 1) \in S$, we define a continuous map

$$\tilde{\pi} : X^{\log} \to X^{\log}_{0^{\log}} \quad \text{by } \tilde{\pi}(\eta) := (0, 1) \cdot \eta = \lim_{s \to 0} (s, 1) \cdot \eta\,. \tag{5.3.2}$$

Note that, by Proposition (4.3) (iii), $\tilde{\pi}$ is compatible with the inclusion $Y^{\log} \subset X^{\log}$.

(5.4) THEOREM. *We use the notation in* (3.1) *and in* (5.1). *We assume that the family $f : X \to \Delta$ is reduced. Then, the family of pairs of log topological spaces*

$$f^{\log} : (X^{\log}, Y^{\log}) \to \Delta^{\log}$$

*is locally piecewise $C^\infty$ trivial over the base. In particular, the family of open spaces*

$$\mathring{f}^{\log} : (X^{\log} - Y^{\log}) \to \Delta^{\log}$$

*is also locally piecewise $C^\infty$ trivial over the base. This means that the above family $\mathring{f}^{\log}$ recovers the vanishing cycles of the given degenerating family $\mathring{f}: (X - Y) \to \Delta$ in the most naive sense.*

PROOF. Let $\tilde{t} = (t, u) \in \Delta^{\log}$ and $\tilde{t}_0 = (0, 1) \cdot \tilde{t} = (0, u) \in 0^{\log}$. The theorem follows from the remark after (5.3.2) and the following:

*Claim.* The restricted map $\tilde{\pi} : X_{\tilde{t}}^{\log} \to X_{\tilde{t}_0}^{\log}$ is bijective.

First, note that

$$U_{\mathrm{I}}'' := U_I - \left( \bigcup_{k \notin I, k \leq a} U_k \right) \quad (I \subset \{1, \dots, a\}) \tag{5.4.1}$$

form a subdivision of $X = \bigcup_{1 \leq j \leq a} U_j$. Hence, by Theorem (5.2) (iii), it is enough to examine the above claim on each $U_I''$. Let $x \in D_I \cap U_I''$, $F := \pi_I^{-1}(x)$, and let $F^{\log}$ be the proper transform of $F$ by $\tau_X$ as in (5.2.11). By Claim 2 in the proof of Theorem (5.2), it is enough to examine the claim even on each $F^{\log}$. Here, by Definition (5.2.12), our restricted map becomes

$$\tilde{\pi} : F_{\tilde{t}}^{\log} \to F_{\tilde{t}_0}^{\log}, \quad \begin{cases} r_i(\eta) \mapsto r_i((0, 1) \cdot \eta) = R(0, y)_i \\ u_i(\eta) \mapsto u_i((0, 1) \cdot \eta) = u_i(\eta) \end{cases} \quad (i \in I).$$

Here $y := \pi_X(\eta)$. This is obviously bijective by construction. □

(5.5) COROLLARY. *In the situation of Theorem* (5.4), *we have the surjective homomorphism of fundamental groups*

$$\pi_1(X_t - Y_t) \twoheadrightarrow \pi_1(X_0 - Y_0),$$

*induced by the restriction of the shrinking map from the general fiber* $X_t - Y_t = (\overset{\circ}{f})^{-1}(t)$ *to the central fiber* $X_0 - Y_0 = (\overset{\circ}{f})^{-1}(0)$.

PROOF. Let $t \in \Delta$, $\tilde{t} \in \tau_\Delta^{-1}(t)$ and $\tilde{t}_0 := (0, 1) \cdot \tilde{t} \in \tau_\Delta^{-1}(0)$. The assertion follows immediately from the observation that the composite map $\tau_X \circ \tilde{\pi}$ of the shrinking map (5.3.2) and the projection induces a continuous surjective map

$$X_t - Y_t = X_t^{\log} - Y_t^{\log} \xrightarrow{\sim} X_{t_0}^{\log} - Y_{t_0}^{\log} \twoheadrightarrow X_0 - Y_0,$$

whose fibers are products of circles, in particular connected. □

**6. Integral structure of limit of variation of mixed Hodge structure and its local monodromy.** In this section, we introduce two types of integral structure and local monodromy on the variation of mixed Hodge structure associated to a semi-stable degeneration of pairs, as an application of the local topological triviality of the family $\overset{\circ}{f}{}^{\log} : (X^{\log} - Y^{\log}) \to \Delta^{\log}$ in Theorem (5.4) and the log version of the Riemann-Hilbert correspondence by Kato-Nakayama.

(6.1) Let $f : X \to \Delta$, $X_0$ and $Y$ be as in (3.1). In this section, we assume moreover that $X \subset \boldsymbol{P}^N \times \Delta$, for some $N$, and $f$ is the restriction of the second projection, and that $X_0$ is reduced.

Let

$$\mathcal{M}_X(X_0 + Y) \hookrightarrow \mathcal{O}_X \tag{6.1.1}$$

be the fs log structure on $X$ corresponding to the divisor with normal crossings $X_0 + Y$ (cf. Example (1.1.4)), and let

$$\omega^1_X\langle Y\rangle$$

be the sheaf of differential forms on $X$ with log poles associated to the fs log structure $\mathcal{M}_X(X_0+Y)$ (cf. (1.3.1)). We denote by

$$\omega^1_{X/\Delta}\langle Y\rangle := \omega^1_X\langle Y\rangle / f^*\omega^1_\Delta \tag{6.1.2}$$

the sheaf of relative differential forms on $X$ with log poles along $X_0 + Y$, and by $\omega^\bullet_{X/\Delta}\langle Y\rangle$ its de Rham complex. In the present case, $\omega^\bullet_{X/\Delta}\langle Y\rangle = \Omega^\bullet_{X/\Delta}(\log(X_0+Y))$ in the classical notation.

(6.2) THEOREM. *In the notation in* (6.1), (5.4) *and* (1.4), *let* $\mathcal{V} := R^q f_*\omega^\bullet_{X/\Delta}\langle Y\rangle$ *and* $L_{\boldsymbol{C}} := R^q(\overset{\circ}{f}{}^{\log})_*\boldsymbol{C}$ *for any integer q. Then we have*

(i) $\quad L_{\boldsymbol{C}} \simeq \mathrm{Ker}(\nabla : (\tau_\Delta)^*\mathcal{V} \to \omega^{1,\log}_\Delta \otimes_{\mathcal{O}^{\log}_\Delta} (\tau_\Delta)^*\mathcal{V}) \quad$ *on* $\Delta^{\log}$,

*or, equivalently,*

(ii) $\quad \mathcal{V} \simeq (\tau_\Delta)_*(\mathcal{O}^{\log}_\Delta \otimes_{\boldsymbol{C}} L_{\boldsymbol{C}}) \quad$ *on* $\Delta$.

PROOF. Let $N := \log\gamma$ be the monodromy logarithm of the locally free sheaf $\mathcal{V}|\Delta^*$ on $\Delta^* := \Delta - \{0\}$ with the Gauss-Manin connection $\nabla$. Choose a multi-valued, flat frame $\{e_1, \dots, e_r\}$ of $\mathcal{V}|\Delta^*$. Modifying

$$\tilde{e}_j := \exp(-zN)\cdot e_j, \quad z := (2\pi\sqrt{-1})^{-1}\log t, \tag{6.2.1}$$

we get an invariant frame $\{\tilde{e}_1, \dots, \tilde{e}_r\}$ which extends over $\Delta$ and induces a frame of the canonical extension $\mathcal{V}$ of $\mathcal{V}|\Delta^*$.

Let $W$ be the weight filtration corresponding to $Y$. [SZ, Section 5] showed that there exists a $W$-relative monodromy weight filtration $M$ of the central fiber $\mathcal{V}(0)$, which is characterized by the properties

$$NM_k \subset M_{k-2}, \quad N^k : \mathrm{gr}^M_{j+k}\mathrm{gr}^W_j \xrightarrow{\sim} \mathrm{gr}^M_{j-k}\mathrm{gr}^W_j. \tag{6.2.2}$$

We may assume that the basis of $\mathcal{V}(0)$ induced by $\{\tilde{e}_1, \dots, \tilde{e}_r\}$ respects the filtration $M$. Then, by using the frame $\{\tilde{e}_1, \dots, \tilde{e}_r\}$, we extend $M$ over $\mathcal{V}$. Here we add some comments on (6.2.2) for the readers' convenience. For $\tilde{e}_j \in M_k$, we have

$$\begin{aligned}\nabla\tilde{e}_j &= -dz\otimes N\cdot\exp(-zN)\cdot e_j = -dz\otimes\exp(-zN)\cdot Ne_j\\ &= -dz\otimes\exp(-zN)\cdot\left(\sum_i a_ie_i\right) = -(2\pi\sqrt{-1})^{-1}d\log t\otimes\left(\sum_i a_i\tilde{e}_i\right),\end{aligned}$$

where $\sum_i a_ie_i := Ne_j \in M_{k-2}$, $a_i \in \boldsymbol{C}$. Hence

$$\nabla(h(t)\tilde{e}_j) = dh(t)\otimes\tilde{e}_j - (2\pi\sqrt{-1})^{-1}h(t)d\log t\otimes\left(\sum_i a_i\tilde{e}_i\right) \in \omega^1_\Delta\otimes M_k.$$

Note also that $\exp(-zN)$ and $N$ commute. Hence we have

$$N = -2\pi\sqrt{-1}\,\mathrm{Res}_0(\nabla)$$

under the identification

$$\varpi^*(\mathcal{V}|\Delta^*)(u) \xrightarrow{\sim} \mathcal{V}(0)\,, \quad \tilde{e}_j(u) \mapsto \tilde{e}_j(0)\,,$$

where $\varpi : \mathfrak{h} \to \Delta^*$ is the universal covering and $u \in \mathfrak{h}$.

Thus we see that $\mathcal{V}$ endowed with $\nabla$ and $M$ is an object of $D_{\mathrm{nilp}}(\Delta)$ in (1.4.3). Applying the log Riemann-Hilbert correspondence (1.4), we get a locally constant sheaf

$$\mathrm{Ker}(\nabla : (\tau_\Delta)^*\mathcal{V} \to \omega_\Delta^{1,\log} \otimes_{\mathcal{O}_X^{\log}} \tau_\Delta^*\mathcal{V})$$

on $\Delta^{\log}$ of $\boldsymbol{C}$-vector spaces.

On the other hand, since by Theorem (5.4) the family

$$\mathring{f}^{\log} : (X^{\log} - Y^{\log}) \to \Delta^{\log}$$

is locally piecewise $C^\infty$ trivial over the base, $L_{\boldsymbol{C}} := R^q(\mathring{f}^{\log})_*\boldsymbol{C}$ is also a locally constant sheaf on $\Delta^{\log}$ of $\boldsymbol{C}$-vector spaces. These two locally constant sheaves coincide over $(\tau_\Delta)^{-1}(\Delta^*) \xrightarrow{\sim} \Delta^*$ by construction, and hence they coincide over all $\Delta^{\log}$.

The second isomorphism follows from the first by the inverse correspondence. □

(6.3) We use the notation in (6.1) and (6.2). Choose now a multi-valued, flat frame

$$\{e_1, \dots, e_r\} \tag{6.3.1}$$

of $\mathcal{V}|\Delta^*$ from the image of

$$(R^q(\mathring{f}^{\log})_*\boldsymbol{Z}) \mid (\tau_\Delta)^{-1}(\Delta^*) \to ((\tau_\Delta)^*\mathcal{V}) \mid (\tau_\Delta)^{-1}(\Delta^*) \xrightarrow{\sim} \mathcal{V} \mid \Delta^*\,.$$

We regard (6.3.1) also as a multi-valued, flat frame of $(\tau_\Delta)^*\mathcal{V}$, by abuse of notation. Putting

$$\tilde{e}_j := \exp(-zN)\cdot e_j\,, \quad z := (2\pi\sqrt{-1})^{-1}\log t\,, \tag{6.3.2}$$

as before in the proof of Theorem (6.2), we have an invariant frame

$$\{\tilde{e}_1, \dots, \tilde{e}_r\} \tag{6.3.3}$$

of $\mathcal{V}|\Delta^*$, which extends over $\Delta$ and induces a frame of the canonical extension $\mathcal{V}$ of $\mathcal{V}|\Delta^*$. We use the same letters for the induced frame of $\mathcal{V}$, by abuse of notation.

(6.4) THEOREM. *In the notation of Theorem* (6.2) *and of* (6.3), *we have two types of integral structure on* $\mathcal{V} = R^q f_*\omega^\bullet_{X/\Delta}\langle Y\rangle$:

(i) *The integral structure determined by the multi-valued, flat frame* (6.3.1) *of*

$$(\tau_\Delta)^*\mathcal{V} \simeq \mathcal{O}_\Delta^{\log} \otimes_{\boldsymbol{Z}} R^q(\mathring{f}^{\log})_*\boldsymbol{Z} \quad \textit{on } \Delta^{\log}\,.$$

*Here the local monodromy is induced by the* $\boldsymbol{C}_1$*-action on* $\Delta^{\log}$.

(ii) *The integral structure determined by the invariant frame* (6.3.3) *of*

$$\mathcal{V} \simeq \mathcal{O}_\Delta \otimes_{\boldsymbol{Z}} (\tau_\Delta)_* R^q(\mathring{f}^{\log})_*(\mathring{f}^{\log})^{-1}\boldsymbol{Z}[z] \quad \textit{on } \Delta\,.$$

*Here* $z = (2\pi\sqrt{-1})^{-1}\log t$ *as before, and the monodromy logarithm is given by* $-2\pi\sqrt{-1}\,\mathrm{Res}_0(\nabla)$.

PROOF. We prove (i). By Theorem (6.2) and the universal coefficient theorem, we see that

$$(\tau_\Delta)^*\mathcal{V} \simeq \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{C}} L_{\mathbf{C}} \simeq \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{C}} \mathbf{C} \otimes_{\mathbf{Z}} R^q(\mathring{f}^{\log})_*\mathbf{Z} \simeq \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} R^q(\mathring{f}^{\log})_*\mathbf{Z}.$$

We prove (ii). We regard $\mathbf{Z}[z]$ as a sheaf of algebras on $\Delta^{\log}$.

*Claim.* $R^q(\mathring{f}^{\log})_*((\mathring{f}^{\log})^{-1}\mathbf{Z}[z]) \simeq \mathbf{Z}[z] \otimes_{\mathbf{Z}} R^q(\mathring{f}^{\log})_*\mathbf{Z}$.

Set $\varphi = \mathring{f}^{\log}$. For a small open set $U \subset \Delta^{\log}$, we will prove, by the Čech method, that

$$H^q(\varphi^{-1}U, \varphi^{-1}\mathbf{Z}[z]) \simeq \mathbf{Z}[z](U) \otimes_{\mathbf{Z}} H^q(\varphi^{-1}U, \mathbf{Z}). \tag{6.4.1}$$

In fact, let $\mathcal{U}$ be a suitable open covering of $\varphi^{-1}U$. Then, by the property of the sheaf $\mathbf{Z}[z]$, we have

$$\Gamma(U_0 \cap \cdots \cap U_q, \varphi^{-1}\mathbf{Z}[z]) \simeq \mathbf{Z}[z](U) \otimes_{\mathbf{Z}} \Gamma(U_0 \cap \cdots \cap U_q, \mathbf{Z}).$$

Hence

$$\mathcal{C}^q(\mathcal{U}, \varphi^{-1}\mathbf{Z}[z]) \simeq \mathbf{Z}[z](U) \otimes_{\mathbf{Z}} \mathcal{C}^q(\mathcal{U}, \mathbf{Z}).$$

This implies (6.4.1), and the Claim is proved.

By the Claim, we see that

$$\begin{aligned}
\mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} R^q\varphi_*\mathbf{Z} &\simeq \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}[z]} \mathbf{Z}[z] \otimes_{\mathbf{Z}} R^q\varphi_*\mathbf{Z} \\
&\simeq \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}[z]} R^q\varphi_*\varphi^{-1}\mathbf{Z}[z] \\
&\simeq (\tau_\Delta)^{-1}\mathcal{O}_\Delta \otimes_{\mathbf{Z}} \mathbf{Z}[z] \otimes_{\mathbf{Z}[z]} R^q\varphi_*\varphi^{-1}\mathbf{Z}[z] \\
&\simeq (\tau_\Delta)^{-1}\mathcal{O}_\Delta \otimes_{\mathbf{Z}} R^q\varphi_*\varphi^{-1}\mathbf{Z}[z].
\end{aligned}$$

This together with (i) yields

$$(\tau_\Delta)^{-1}\mathcal{O}_\Delta \otimes_{\mathbf{Z}} R^q\varphi_*\varphi^{-1}\mathbf{Z}[z] \simeq (\tau_\Delta)^*\mathcal{V}.$$

Taking $(\tau_\Delta)_*$, we obtain (ii), by the projection formula.

The other assertions are obvious by construction. □

Note that the integral structures (i) and (ii) in Theorem (6.4) are independent of the choice of a multi-valued, flat frame (6.3.1). However, the integral structure (ii) depends on the choice of a coordinate $t$ on $\Delta$. Note also, in the case $Y = \emptyset$, that the integral structure (i) in Theorem (6.4) is the one in the limiting mixed Hodge structure of Schmid [Sc], whereas the integral structure (ii) is the one in the limiting mixed Hodge structure of Steenbrink [St1].

(6.5) REMARK. (i) In this paper, we restrict ourselves to the case of one-dimensional base throughout, and we use the log version of the Riemann-Hilbert correspondence of Kato-Nakayama [KN] in the proof of Theorem (6.2). However, we note that our argument can be generalized to the case of higher-dimensional base. We note also that Section 6 can be rewritten by using the theory of canonical extensions of P. Deligne instead of the above correspondence.

(ii) The author was informed by Morihiko Saito, on May 24, 1996, that there is a correction of [St, (5.9)] in [Sa, 4.2].

(iii) When the base is $0^{\log}$ in (5.3.1) and $Y = \emptyset$, the assertion of type (ii) in Theorem (6.4) is stated in [KN]. The proof of the Claim in the proof of Theorem (6.4) is due to C. Nakayama.

(iv) Steenbrink [St2] introduced an 'integral structure' of the limiting mixed Hodge structure by using the log structure associated to the pair $X_0 \subset X$ (cf. (1.1.4)). However, he used fractions there and consequently neglected torsions. In this sense, his structure can be regarded as a $\boldsymbol{Q}$-structure. In contrast, in our formulation in Theorem (6.4) (i), we can consider, for example, $R^q(\overset{\circ}{f}{}^{\log})_*(\boldsymbol{Z}/(l))$ in the notation there, and hence $l$-adic cohomologies.

(v) After writing up this manuscript, Kazuya Kato and the author, we introduced in [KU] a notion of *polarized logarithmic variation of Hodge structure* and a notion of *polarized logarithmic Hodge structure*. The latter is the weaker notion obtained from the former by forgetting the Griffiths transversality. Consider the case $Y = \emptyset$. Let $H_{\boldsymbol{Z}}$ be one of the integral structures in Theorem (6.4), $F$ be the Hodge filtration on $\mathcal{V} = R^q f_* \omega^\bullet_{X/\Delta}$, and $\langle\ ,\ \rangle$ be the cup product modified by the Lefschetz decomposition. Then the result in this section shows that $(H_{\boldsymbol{Z}}, \langle\ ,\ \rangle, \tau_\Delta^* F)$ is a polarized logarithmic variation of Hodge structure.

(vi) After writing up this manuscript, the author was informed by T. Matsubara and by F. Kato, independently, that they obtained in [Mt] and in [K] the integral structure of type Theorem (6.4) (i) in the case where $Y = \emptyset$ in our notation. Their method is different from ours. They first proved a log version of the relative Poincaré lemma and then used it to obtain the integral structure.

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