Classifying Spaces of Degenerating Polarized Hodge Structures

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PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD
2009

Published by Princeton University Press
41 William Street, Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press
6 Oxford Street, Woodstock, Oxfordshire 0X20 1TW

Library of Congress Cataloging-in-Publication Data

Kato, K. (Kazuya)
Classifying spaces of degenerating polarized Hodge structures / Kazuya Kato and Sampei Usui.
p. cm. — (Annals of mathematics studies ; no. 169)
Includes bibliographical references and index.
ISBN 978-0-691-13821-3 (cloth : acid-free paper) — ISBN 978-0-691-13822-0 (pbk. : acid-free paper)
1. Hodge theory. 2. Logarithms. I. Usui, Sampei. II. Title.
QA564.K364 2009
514′.74—dc22 2008039091

British Library Cataloging-in-Publication Data is available

This book has been composed in LaTeX

Printed on acid-free paper. ∞

The publisher would like to thank the authors of this volume for
providing the camera-ready copy from which this book was printed

press.princeton.edu

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Chapter Zero

Overview

In this chapter, we introduce the main ideas and results of this book.

In Section 0.1, we review the basic idea of Hodge theory. In Section 0.2, we introduce the basic idea of logarithmic Hodge theory. In Section 0.3, we review classifying spaces D of Griffiths (i.e., Griffiths domains) as the moduli spaces of polarized Hodge structures. In Section 0.4, we describe our toroidal partial compactifications of the classifying spaces of Griffiths and our result that they are the fine moduli spaces of polarized logarithmic Hodge structures. In Section 0.5, we describe the other seven enlargements of D in the fundamental diagram (3) in Introduction and state our results on these spaces.

In this chapter, we explain the above subjects by presenting examples. Hodge theory (Section 0.1) and logarithmic Hodge theory (Section 0.2) are explained by using the example of the Hodge structure on $H^1(E, \mathbf{Z})$ of an elliptic curve E and its degeneration arising from the degeneration of E. This example appears first in 0.1.3 and then continues to appear as an example of each subject. The classifying space D (Section 0.3) and its various enlargements (Sections 0.4 and 0.5) are explained by using the following three examples: (i) $D = \mathfrak{h}$, the upper half plane; (ii) $D = \mathfrak{h}_g$, Siegel's upper half space ((i) is a special case of (ii). In the case (ii), we mainly consider the case $g = 2$.); (iii) an example of weight 2 for which D is not a symmetric Hermitian domain. These examples appear first in 0.3.2 and then continue to appear as examples of each subject.

In this chapter, we do not generally give proofs.

0.1 HODGE THEORY

0.1.1

First we recall the basic idea of Hodge theory.

For a topological space X, the homology groups $H_m(X, \mathbf{Z})$ and the cohomology groups $H^m(X, \mathbf{Z})$ are important invariants of X.

If X is a projective complex analytic manifold, the cohomology groups $H^m(X, \mathbf{Z})$ have finer structures: $\mathbf{C} \otimes_{\mathbf{Z}} H^m(X, \mathbf{Z}) = H^m(X, \mathbf{C})$ is endowed with a decreasing filtration $F = (F^p)_{p \in \mathbf{Z}}$, called Hodge filtration.

The cohomology group $H^m(X, \mathbf{Z})$ remembers X merely as a topological space, but, with this Hodge filtration, the pair $(H^m(X, \mathbf{Z}), F)$ becomes a finer invariant of X which remembers the analytic structure of X (not just the topological structure of X) often very well.

0.1.2

For example, in the case $m = 1$, the Hodge filtration F on $H^1(X, \mathbf{C})$ is given by $F^p = H^1(X, \mathbf{C})$ for $p \leq 0$, $F^p = 0$ for $p \geq 2$, and F^1 is the image of the injective map

$$H^0(X, \Omega_X^1) \to H^1(X, \mathbf{C}). \qquad (1)$$

Here $H^0(X, \Omega_X^1)$ is the space of holomorphic differential forms on X, and (1) is the map that sends a differential form $\omega \in H^0(X, \Omega_X^1)$ to its cohomology class in $H^1(X, \mathbf{C})$: Under the identification $H^1(X, \mathbf{C}) = \mathrm{Hom}(H_1(X, \mathbf{Z}), \mathbf{C})$, the cohomology class of ω is given by $\gamma \mapsto \int_\gamma \omega \quad (\gamma \in H_1(X, \mathbf{Z}))$.

For the definition of F^p of $H^m(X, \mathbf{C})$ for general p, m, see 0.1.7.

0.1.3

Elliptic curves. An elliptic curve X over $\mathbf{C}$ is isomorphic to $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ for some $\tau \in \mathfrak{h}$, where $\mathfrak{h}$ is the upper half plane. For $X = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ with $\tau \in \mathfrak{h}$, $H_1(X, \mathbf{Z})$ is identified with $\mathbf{Z}\tau + \mathbf{Z}$, $H^1(X, \mathbf{Z})$ is identified with $\mathrm{Hom}(\mathbf{Z}\tau + \mathbf{Z}, \mathbf{Z})$, and the Hodge filtration on $H^1(X, \mathbf{C}) = \mathrm{Hom}(\mathbf{Z}\tau + \mathbf{Z}, \mathbf{C})$ is described as follows. The space $H^0(X, \Omega_X^1)$ is a one-dimensional $\mathbf{C}$-vector space with the basis dz, where z is the coordinate function of $\mathbf{C}$, and where we regard dz as a differential form on the quotient space $X = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ of $\mathbf{C}$. Let $(\gamma_j)_{j=1,2}$ be the $\mathbf{Z}$-basis of $H_1(X, \mathbf{Z})$ that is identified with the $\mathbf{Z}$-basis $(\tau, 1)$ of $\mathbf{Z}\tau + \mathbf{Z}$, and let $(e_j)_{j=1,2}$ be the dual $\mathbf{Z}$-basis of $H^1(X, \mathbf{Z})$. Since $\int_{\gamma_1} dz = \tau$ and $\int_{\gamma_2} dz = 1$, the cohomology class of dz coincides with $\tau e_1 + e_2$, and hence $F^1 H^1(X, \mathbf{C})$ is the $\mathbf{C}$-subspace of $H^1(X, \mathbf{C})$ generated by $\tau e_1 + e_2$.

0.1.4

Elliptic curves (continued). If X is an elliptic curve, we cannot recover X merely from $H^1(X, \mathbf{Z})$. In fact, $H^1(X, \mathbf{Z}) \simeq \mathbf{Z}^2$ for any elliptic curve X over $\mathbf{C}$, and we cannot distinguish different elliptic curves from this information. However, if we consider the Hodge filtration, we can recover X from $(H^1(X, \mathbf{Z}), F)$ as

$$X \simeq \mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})/H_1(X, \mathbf{Z}) = \mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})/\mathrm{Hom}(H^1(X, \mathbf{Z}), \mathbf{Z}). \qquad (1)$$

Here $H_1(X, \mathbf{Z})$ is embedded in $\mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})$ via the map $\gamma \mapsto (\omega \mapsto \int_\gamma \omega)$ ($\gamma \in H_1(X, \mathbf{Z})$, $\omega \in \Gamma(X, \Omega_X^1)$), and the isomorphism $X \simeq \mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})/H_1(X, \mathbf{Z})$ sends $x \in X$ to the class of the homomorphism $F^1 \to \mathbf{C}$, $\omega \mapsto \int_\gamma \omega$, where γ is a path in X from the origin 0 of X to x (the choice of γ is not unique, but the class of the map $\omega \mapsto \int_\gamma \omega$ modulo $H_1(X, \mathbf{Z})$ is independent of the choice of γ). In (1), the middle group is identified with the right one in which $\mathrm{Hom}(H^1(X, \mathbf{Z}), \mathbf{Z})$ is embedded in $\mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})$ via the composition $\mathrm{Hom}(H^1(X, \mathbf{Z}), \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{C}}(H^1(X, \mathbf{C}), \mathbf{C}) \to \mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})$, which is injective.

If $X = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ with $\tau \in \mathfrak{h}$, this isomorphism $X \simeq \mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})/H_1(X, \mathbf{Z})$ is nothing but the original presentation $X = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ where $\mathrm{Hom}_{\mathbf{C}}(F^1, \mathbf{C})$ is identified with $\mathbf{C}$ by the evaluation at dz.

0.1.5

Now we discuss Hodge structures.

A *Hodge structure* of weight w is a pair $(H_{\mathbf{Z}}, F)$ consisting of a free $\mathbf{Z}$-module $H_{\mathbf{Z}}$ of finite rank and of a decreasing filtration F on $H_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ (that is, a family $(F^p)_{p \in \mathbf{Z}}$ of $\mathbf{C}$-subspaces of $H_{\mathbf{C}}$ such that $F^p \supset F^{p+1}$ for all p), which satisfies the following condition (1):

$$H_{\mathbf{C}} = \bigoplus_{p+q=w} H^{p,q}, \quad \text{where } H^{p,q} = F^p \cap \bar{F}^q. \tag{1}$$

Here $\bar{F}^q$ denotes the image of F^q under the complex conjugation $H_{\mathbf{C}} \to H_{\mathbf{C}}$, $a \otimes x \mapsto \bar{a} \otimes x$ ($a \in \mathbf{C}, x \in H_{\mathbf{Z}}$).

We have

$$F^p = \bigoplus_{p' \geq p} H^{p', w-p'}, \quad F^p/F^{p+1} \simeq H^{p,w-p}. \tag{2}$$

We say $(H_{\mathbf{Z}}, F)$ is of Hodge type $(h^{p,q})_{p,q \in \mathbf{Z}}$, where $h^{p,q} = \dim_{\mathbf{C}} H^{p,q}$ if $p + q = w$, and $h^{p,q} = 0$ otherwise (these numbers $h^{p,q}$ are called the *Hodge numbers*).

0.1.6

For a projective analytic manifold X and for $m \in \mathbf{Z}$, the pair $(H_{\mathbf{Z}}, F)$ with $H_{\mathbf{Z}} = H^m(X, \mathbf{Z})/(\text{torsion})$ and F the Hodge filtration becomes a Hodge structure of weight m.

For example, if X is the elliptic curve $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ with $\tau \in \mathfrak{h}$, then $H^{1,0} = \mathbf{C}(\tau e_1 + e_2)$, $H^{0,1} = \mathbf{C}(\bar{\tau} e_1 + e_2)$, and $H^1(X, \mathbf{C}) = H^{1,0} \oplus H^{0,1}$ since $\tau \neq \bar{\tau}$.

The theory of homology groups and cohomology groups is important in the study of topological spaces. Similarly, Hodge theory (the theory of Hodge structures) is important for the study of analytic spaces.

0.1.7

For a projective analytic manifold X, the Hodge filtration F^p on $H^m(X, \mathbf{C})$ is defined as follows. Let

$$\Omega_X^\bullet = (\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots)$$

be the de Rham complex of X where $\Omega_X^p = \bigwedge_{\mathcal{O}_X}^p \Omega_X^1$ is the sheaf of holomorphic p-forms on X ($\mathcal{O}_X$ is set in degree 0). Let $\Omega_X^{\geq p}$ be the degree $\geq p$ part of $\Omega_X^\bullet$. Then F^p is defined as

$$F^p := H^m(X, \Omega_X^{\geq p}) \hookrightarrow H^m(X, \Omega_X^\bullet) \simeq H^m(X, \mathbf{C}).$$

Here $H^m(X, \Omega_X^{\geq p})$ and $H^m(X, \Omega_X^\bullet)$ denote the mth hypercohomology groups of complexes of sheaves. The canonical homomorphism $H^m(X, \Omega_X^{\geq p}) \to H^m(X, \Omega_X^\bullet)$ is known to be injective. The isomorphism $H^m(X, \Omega_X^\bullet) \simeq H^m(X, \mathbf{C})$ comes from the exact sequence of Dolbeault

$$0 \to \mathbf{C} \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots.$$

We have isomorphisms

$$H^{p,m-p} \simeq F^p/F^{p+1} \simeq H^{m-p}(X, \Omega_X^p),$$

where the second isomorphism is obtained by applying $H^m(X, \ )$ to the exact sequence of complexes of sheaves $0 \to \Omega_X^{\geq p+1} \to \Omega_X^{\geq p} \to \Omega_X^p[-p] \to 0$.

0.1.8

It is often important to consider a *polarized Hodge structure*, that is, a Hodge structure endowed with a polarization.

A *polarization on a Hodge structure* $(H_{\mathbf{Z}}, F)$ *of weight* w is a nondegenerate bilinear form $\langle\ ,\ \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$ ($H_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$) which is symmetric if w is even and is antisymmetric if w is odd, satisfying the following conditions (1) and (2).

(1) $\langle F^p, F^q \rangle = 0$ for $p+q > w$.
(2) Let $C_F : H_{\mathbf{C}} \to H_{\mathbf{C}}$ be the $\mathbf{C}$-linear map defined by $C_F(x) = i^{p-q}x$ for $x \in H^{p,q}$. Then the Hermitian form $(\ ,\ )_F : H_{\mathbf{C}} \times H_{\mathbf{C}} \to \mathbf{C}$, defined by $(x, y)_F = \langle C_F(x), \bar{y} \rangle$, is positive definite.

Here, in (2), $\langle\ ,\ \rangle$ is regarded as the natural extension to the $\mathbf{C}$-bilinear form. The Hermitian form $(\ ,\ )_F$ in (2) is called the *Hodge metric* associated with F. The condition (1) (resp. (2)) is called the *Riemann-Hodge first* (resp. *second*) *bilinear relation*.

0.1.9

For a projective analytic manifold X, we have the intersection form

$$\langle\ ,\ \rangle : H^m(X, \mathbf{Q}) \times H^m(X, \mathbf{Q}) \to \mathbf{Q}$$

induced by an ample line bundle on X (see [G1], [GH]). The triple $(H^m(X, \mathbf{Z}), \langle\ ,\ \rangle, F)$ becomes a polarized Hodge structure.

Elliptic curves (continued)

For the elliptic curve $X = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ ($\tau \in \mathfrak{h}$), the standard polarization of X gives the antisymmetric pairing $\langle\ ,\ \rangle : H^1(X, \mathbf{Q}) \times H^1(X, \mathbf{Q}) \to \mathbf{Q}$ characterized by $\langle e_2, e_1 \rangle = 1$. This pairing is nothing but the cup product $H^1(X, \mathbf{Q}) \times H^1(X, \mathbf{Q}) \to H^2(X, \mathbf{Q}) \simeq \mathbf{Q}$. It satisfies

$$\langle \tau e_1 + e_2, \tau e_1 + e_2 \rangle = 0,$$
$$(\tau e_1 + e_2, \tau e_1 + e_2)_F = i^{1-0}\langle \tau e_1 + e_2, \bar{\tau} e_1 + e_2 \rangle = i(\bar{\tau} - \tau) = 2\,\mathrm{Im}(\tau) > 0.$$

Hence $(H^1(X, \mathbf{Z}), \langle\ ,\ \rangle, F)$ is indeed a polarized Hodge structure.

0.1.10

It is often very useful to consider analytic families of Hodge structures and of polarized Hodge structures.

Let X be an analytic manifold.

After Griffiths ([G3], also [D2], [Sc]), a *variation of Hodge structure* (VH) *on X of weight w* is a pair $(H_{\mathbf{Z}}, F)$ consisting of a locally constant sheaf $H_{\mathbf{Z}}$ of free $\mathbf{Z}$-modules of finite rank on X and of a decreasing filtration F of $H_{\mathcal{O}} := \mathcal{O}_X \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ by $\mathcal{O}_X$-submodules which satisfy the following three conditions:

(1) $F^p = H_{\mathcal{O}}$ for $p \ll 0$, $F^p = 0$ for $p \gg 0$, and $\mathrm{gr}^p_F = F^p/F^{p+1}$ is a locally free $\mathcal{O}_X$-module for any p.
(2) For any $x \in X$, the fiber $(H_{\mathbf{Z},x}, F(x))$ is a Hodge structure of weight w.
(3) $(d \otimes 1_{H_{\mathbf{Z}}})(F^p) \subset \Omega^1_X \otimes_{\mathcal{O}_X} F^{p-1}$ for all p.

Here (3) is called the *Griffiths transversality*.

A *polarization of a variation of Hodge structure* $(H_{\mathbf{Z}}, F)$ *of weight w* on X is a bilinear form $\langle\ ,\ \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$ which yields for each $x \in X$ a polarization $\langle\ ,\ \rangle_x$ on the fiber $(H_{\mathbf{Z},x}, F(x))$. In this case, the triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ is called a *variation of polarized Hodge structure* (VPH).

0.1.11

Let X and Y be analytic manifolds and $f : Y \to X$ be a projective, smooth morphism. Then for each $m \in \mathbf{Z}$, we obtain a VH (variation of Hodge structure) of weight m on X:

$$H_{\mathbf{Z}} = R^m f_* \mathbf{Z}/(\text{torsion}),$$
$$F^p = R^m f_*(\Omega^{\geq p}_{Y/X}) \hookrightarrow R^m f_*(\Omega^{\bullet}_{Y/X}) \simeq \mathcal{O}_X \otimes_{\mathbf{Z}} H_{\mathbf{Z}}.$$

Indeed, on each fiber $Y_x := f^{-1}(x)$ $(x \in X)$, $H_{\mathbf{Z},x} = H^m(Y_x, \mathbf{Z})/(\text{torsion})$ and $F^p(x) = H^m(Y_x, \Omega^{\geq p}_{Y_x})$ form the Hodge structure.

If we fix a polarization of Y over X, this VH becomes a VPH (variation of polarized Hodge structure) on X ([G3]; cf. also [Sc], [GH]).

0.2 LOGARITHMIC HODGE THEORY

From now on, we discuss degeneration of Hodge structures by the method of logarithmic Hodge theory. The logarithmic Hodge theory uses the magic of the theory of logarithmic structures introduced by Fontaine-Illusie. It has a strong connection with the theory of nilpotent orbits as discussed in Section 0.4.

In the story of Beauty and the Beast, the Beast becomes a nice man because of the love of the heroine. Similarly, a degenerate object becomes a nice object because of the magic of LOG.

$$(\text{Beast}) \xrightarrow{\text{Love Of Girl}} (\text{a nice man}),$$
$$(\text{degenerate object}) \xrightarrow{\text{LOG}} (\text{a nice object}).$$

The authors learned this mysterious coincidence of letters from Takeshi Saito.

0.2.1

Elliptic curves (continued). We observe what happens for the Hodge structure on H^1 of an elliptic curve when the elliptic curve degenerates. In this section, the idea of logarithmic Hodge theory is explained by use of this example.

Let $\Delta = \{q \in \mathbf{C} \mid |q| < 1\}$ be the unit disc. Then we have a standard family of degenerating elliptic curve

$$f : E \to \Delta,$$

which is a morphism of analytic manifolds, having the following property.

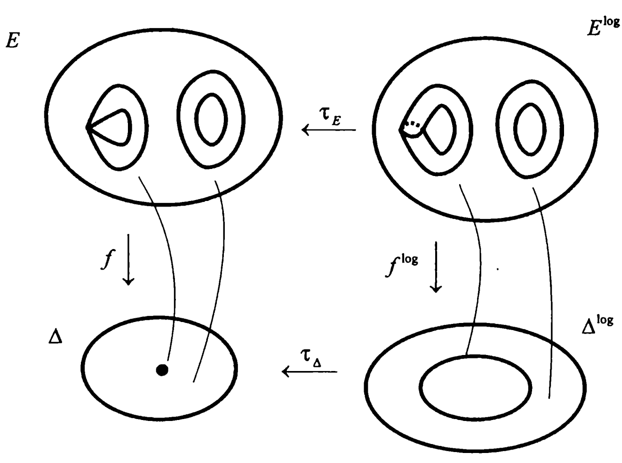
(1) For $q \in \Delta$ with $q \neq 0$, $f^{-1}(q) = \mathbf{C}^\times/q^{\mathbf{Z}}$. This is an elliptic curve. In fact, taking $\tau \in \mathbf{C}$ with $q = \exp(2\pi i \tau)$, we have $\tau \in \mathfrak{h}$ (since $|q| < 1$), and

$$\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}) \xrightarrow{\sim} \mathbf{C}^\times/q^{\mathbf{Z}}, \quad (z \bmod (\mathbf{Z}\tau + \mathbf{Z})) \mapsto (\exp(2\pi i z) \bmod q^{\mathbf{Z}}).$$

(2) $f^{-1}(0) = \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$.

The definition of E will be given in 0.2.10 below. E is a two-dimensional analytic manifold and $f^{-1}(0)$ is a divisor with normal crossings on E. These look like the left-hand side of Figure 1. This family degenerates at $q = 0$ as is described on the left-hand side of Figure 1. All the fibers $f^{-1}(q)$ for $q \in \Delta^*$ are homeomorphic to the surface of a doughnut, whereas the central fiber $f^{-1}(0)$ has a degenerate shape.

However, as we will see below, the central fiber recovers its lost body as in the right-hand side of Figure 1 by the magic of its logarithmic structure. We will explain this magical process in the following.



$$(z_1, z_2) = (r_1u_1, r_2u_2) \leftarrow\!\shortmid (r_1, r_2, u_1, u_2)$$

$$q = z_1z_2 = sv \leftarrow\!\shortmid (s, v) = (r_1r_2, u_1u_2)$$

Figure 1

0.2.2

Elliptic curves (continued). Let $\Delta^* = \Delta - \{0\}$, let $E^* = f^{-1}(\Delta^*) \subset E$, and let $f' : E^* \to \Delta^*$ be the restriction of f to E^*. Since $f' : E^* \to \Delta^*$ is projective and smooth, the polarized Hodge structures on H^1 of the elliptic curves $\mathbf{C}^\times/q^{\mathbf{Z}}$ $(q \in \Delta^*)$ form a variation of polarized Hodge structure $(H'_{\mathbf{Z}}, F')$. Here, $H'_{\mathbf{Z}} = R^1 f'_* \mathbf{Z}$ is a locally constant sheaf on Δ^* of $\mathbf{Z}$-modules of rank 2, and the filtration F' on $\mathcal{O}_{\Delta^*} \otimes_{\mathbf{Z}} H'_{\mathbf{Z}} = R^1 f'_*(\Omega^\bullet_{E^*/\Delta^*}) = R^1 f'_*(\mathcal{O}_{E^*} \xrightarrow{d} \Omega^1_{E^*/\Delta^*})$ is given by $(F')^p = \mathcal{O}_{\Delta^*} \otimes_{\mathbf{Z}} H'_{\mathbf{Z}}$ for $p \leq 0$, $(F')^p = 0$ for $p \geq 2$, and $(F')^1 = f'_*(\Omega^1_{E^*/\Delta^*}) \subset \mathcal{O}_{\Delta^*} \otimes_{\mathbf{Z}} H'_{\mathbf{Z}}$.

0.2.3

Elliptic curves (continued). This variation of Hodge structure on Δ^* does not extend to a VH on Δ. First of all, the local system $H'_{\mathbf{Z}}$ on Δ^* does not extend to a local system on Δ. $H'_{\mathbf{Z}}$ extends to the sheaf $R^1 f_* \mathbf{Z}$ on Δ, but this sheaf is not locally constant. The stalk $(R^1 f_* \mathbf{Z})_0 = H^1(f^{-1}(0), \mathbf{Z})$ of this sheaf at $0 \in \Delta$ is of rank 1, not 2. In fact, $e_1 \in H^1(\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}), \mathbf{Z}) \simeq H^1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z})$ for $q \in \Delta^*$ (0.1.3) extends to a global section of $R^1 f_* \mathbf{Z}$ on Δ, but e_2 is defined only locally on Δ^*, depending on the choice of τ with $q = \exp(2\pi i \tau)$. There is no element of $(R^1 f_* \mathbf{Z})_0$ that gives e_2 in $H^1(\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}), \mathbf{Z}) \simeq H^1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z})$ for $q \in \Delta^*$ near to 0.

We show that by a magic of the theory of logarithmic structure, $H'_{\mathbf{Z}}$ does extend over the origin as a local system in the logarithmic world (0.2.4–0.2.10), and the variation of polarized Hodge structure $(H'_{\mathbf{Z}}, F')$ also extends over the origin as a logarithmic variation of polarized Hodge structure (0.2.15–0.2.20).

0.2.4

By a monoid, we mean a commutative semigroup with a neutral element 1. A homomorphism of monoids is assumed to preserve 1.

A *logarithmic structure* on a local ringed space $(X, \mathcal{O}_X)$ is a sheaf of monoids M_X on X endowed with a homomorphism $\alpha : M_X \to \mathcal{O}_X$, where $\mathcal{O}_X$ is regarded as a sheaf of monoids with respect to the multiplication, such that $\alpha : \alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism.

We regard $\mathcal{O}_X^\times$ as a subsheaf of M_X via α^{-1}.

0.2.5

Example. A standard example of a logarithmic structure is given as follows. Let X be an analytic manifold, let D be a divisor on X with normal crossings, and let $U = X - D$. (That is, locally, $X = \Delta^n$ a polydisc with coordinates $q_1, \ldots, q_n$, $D = \{q_1 \cdots q_r = 0\}$ $(0 \leq r \leq n)$, and $U = (\Delta^*)^r \times \Delta^{n-r}$.) Then

$$M_X = \{f \in \mathcal{O}_X \mid f \text{ is invertible on } U\} \overset{\alpha}{\hookrightarrow} \mathcal{O}_X$$

is a logarithmic structure on X. This is called the logarithmic structure on X associated with D.

0.2.6

Elliptic curves (continued). Let $f : E \to \Delta$ be as in 0.2.1. Define the logarithmic structure M_Δ on Δ by taking $X = \Delta$ and $D = \{0\}$ in 0.2.5, and define a logarithmic structure M_E on E by taking $X = E$ and $D = f^{-1}(0)$.

Then the stalk of M_Δ is given by $M_{\Delta,q} = \mathcal{O}^\times_{\Delta,q}$ if $q \in \Delta^*$, and $M_{\Delta,0} = \bigsqcup_{n\geq 0} \mathcal{O}^\times_{\Delta,0} \cdot q^n$ where the last q denotes the coordinate function of Δ.

0.2.7

For a complex analytic space X endowed with a logarithmic structure M_X, let

$$X^{\log} = \{(x, h) \mid x \in X;\ h \text{ is a homomorphism } M_{X,x} \to \mathbf{S}^1 \text{ satisfying (1) below}\}.$$

Here

$$\mathbf{S}^1 = \{z \in \mathbf{C}^\times \mid |z| = 1\}$$

is regarded as a multiplicative group.

$$h(u) = \frac{u(x)}{|u(x)|} \quad \text{for any } u \in \mathcal{O}^\times_{X,x}. \tag{1}$$

We have a canonical map

$$\tau : X^{\log} \to X, \quad (x, h) \mapsto x.$$

The space $X^{\log}$ has a natural topology, the weakest topology for which the map τ and the maps $\tau^{-1}(U) \to \mathbf{S}^1$, $(x, h) \mapsto h(f)$, given for each open set U of X and for each $f \in \Gamma(U, M_X)$, are continuous.

0.2.8

Elliptic curves (continued). For Δ with the logarithmic structure M_Δ, the shape of $\Delta^{\log}$ is as in Figure 2. Here, in $\Delta^{\log}$, $\{0\} \subset \Delta$ is replaced by $\mathbf{S}^1$ since $h : M_{\Delta,0} \to \mathbf{S}^1$ can send $q \in M_{\Delta,0}$ to any element of $\mathbf{S}^1$. Thus, roughly speaking, $\Delta^{\log}$ has a shape like Δ^* ($\Delta^{\log}$ is an extension of Δ^* over the origin without a change of shape).

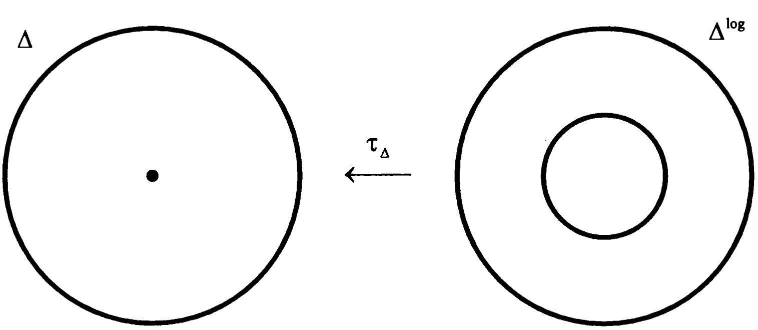


Figure 2

Precisely speaking, the inclusion map $\Delta^* \to \Delta^{\log}$ is a homotopy equivalence. From this, we see that $H'_{\mathbf{Z}}$ on Δ^* extends to a local system on $\Delta^{\log}$ of rank 2.

We will see in 0.2.10 below, for the continuous map $f^{\log} : E^{\log} \to \Delta^{\log}$ induced by $f : E \to \Delta$, all the fibers $(f^{\log})^{-1}(p)$ of any $p \in \Delta^{\log}$ are homeomorphic to the surface of a doughnut even if p lies over $0 \in \Delta$, as described on the right-hand side of Figure 1. The above local system of rank 2 on $\Delta^{\log}$ is in fact the sheaf $R^1 f^{\log}_*(\mathbf{Z})$.

0.2.9

Example. In 0.2.5, consider the case $X = \Delta^n$, $D = \{q_1 \cdots q_r = 0\}$. Then $X^{\log} = (|\Delta| \times \mathbf{S}^1)^r \times \Delta^{n-r}$, where $|\Delta| = \{t \in \mathbf{R} \mid 0 \le t < 1\}$, which has the natural topology. The map $\tau : X^{\log} \to X$ is given by $((r_j, u_j)_{1 \le j \le r}, (x_j)_{r<j\le n}) \mapsto (x_j)_{1\le j \le n}$ where $x_j = r_j u_j$ for $1 \le j \le r$. For $x \in X$, the inverse image $\tau^{-1}(x)$ is isomorphic to $(\mathbf{S}^1)^m$ where m is the number of j such that $1 \le j \le r$ and $x_j = 0$.

0.2.10

Elliptic curves (continued). Let $f : E \to \Delta$ be the family of degenerating elliptic curves in 0.2.1. Explicitly, this E is defined as $X/\sim$, where $X = \{(t_1, t_2) \in \mathbf{C}^2 \mid |t_1 t_2| < 1\}$ and $\sim$ is the following equivalence relation. Let $g : X \to \Delta$, $(t_1, t_2) \mapsto t_1 t_2$. For $a, b \in X$, if $a \sim b$, then $g(a) = g(b)$. The restriction of $\sim$ to $g^{-1}(q)$ for each $q \in \Delta$ is defined as follows. Assume first $q \neq 0$. Consider the map $g^{-1}(q) \simeq \mathbf{C}^\times \to \mathbf{C}^\times/q^{\mathbf{Z}}$ where the first isomorphism is $(t_1, t_2) \mapsto t_1$. For $a, b \in g^{-1}(q)$, $a \sim b$ if and only if the images of a, b in $\mathbf{C}^\times/q^{\mathbf{Z}}$ coincide, i.e., $(b_1, b_2) = (q^n a_1, q^{-n} a_2)$ for some $n \in \mathbf{Z}$. Next assume $q = 0$. Consider $g^{-1}(0) = \{(t_1, t_2) \mid t_1 t_2 = 0\} \to \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$, where the arrow sends $(t_1, 0)$ to t_1 and $(0, t_2)$ to t_2^{-1}. Then, for $a, b \in g^{-1}(0)$, $a \sim b$ if and only if the images of a, b in $\mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$ coincide, i.e., $a = b$ or $\{a, b\} = \{(t, 0), (0, t^{-1})\}$ for some $t \in \mathbf{C}^\times$.

The projection $X \to E$ is a local homeomorphism. The analytic structure and the logarithmic structure of E are the unique ones for which this projection is locally an isomorphism of analytic spaces with logarithmic structures.

We have

$$X^{\log} = \{(r_1, r_2, u_1, u_2) \in (\mathbf{R}_{\ge 0}) \times (\mathbf{R}_{\ge 0}) \times \mathbf{S}^1 \times \mathbf{S}^1 \mid r_1 r_2 < 1\},$$
$$\Delta^{\log} = |\Delta| \times \mathbf{S}^1$$

($|\Delta|$ is as in 0.2.9) and the projection $X^{\log} \to \Delta^{\log}$ is $(r_1, r_2, u_1, u_2) \mapsto (r_1 r_2, u_1 u_2)$. The projection $X^{\log} \to E^{\log}$ is a local homeomorphism. For $a = (r_1, r_2, u_1, u_2)$, $a' = (r'_1, r'_2, u'_1, u'_2) \in X^{\log}$, in the case $r_1 r_2 \neq 0$, the images of a and a' in $E^{\log}$ coincide if and only if there exists $n \in \mathbf{Z}$ such that

$$r'_1 = r_1(r_1 r_2)^n, \quad r'_2 = r_2(r_1 r_2)^{-n}, \quad u'_1 = u_1(u_1 u_2)^n, \quad u'_2 = u_2(u_1 u_2)^{-n}.$$

In the case $r_1 r_2 = 0$, the images of a and a' in $E^{\log}$ coincide if and only if either $a = a'$ or

$$\{a, a'\} = \{(c, 0, u_1, u_2), (0, c^{-1}, u_1(u_1 u_2), u_2(u_1 u_2)^{-1})\} \quad \text{for some } c \in \mathbf{R}_{>0}.$$

For example, if $p = (0, 1) \in |\Delta| \times \mathbf{S}^1 = \Delta^{\log}$, we have a homeomorphism $(f^{\log})^{-1}(p) \simeq [0, \infty]/(0 \sim \infty) \times \mathbf{S}^1$ which sends the image of $(c, 0, u_1, u_2) \in X^{\log}$ in $E^{\log}$ to (c, u_1), and the image of $(0, c, u_1, u_2)$ to (c^{-1}, u_1). Hence we have a homeomorphism $(f^{\log})^{-1}(p) \simeq \mathbf{S}^1 \times \mathbf{S}^1$.

We show that, for each $p \in \Delta^{\log}$, there are an open neighborhood V of p and a homeomorphism

$$(f^{\log})^{-1}(V) \simeq V \times \mathbf{S}^1 \times \mathbf{S}^1$$

over V. This shows that any fiber of $f^{\log} : E^{\log} \to \Delta^{\log}$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$.

Let $B :=$ (the complement of $(1, 1)$ in $[0, 1] \times [0, 1]) \subset (\mathbf{R}_{\geq 0}) \times (\mathbf{R}_{\geq 0})$, and let $A := B \times \mathbf{S}^1 \times \mathbf{S}^1 \subset X^{\log}$. Then the projection $A \to E^{\log}$ is surjective, and we have a homeomorphism

$$A/\sim \xrightarrow{\sim} E^{\log},$$

where for $a, a' \in A$, $a \sim a'$ if and only if either $a = a'$ or

$$\{a, a'\} = \{(1, r, u_1, u_2), (r, 1, u_1(u_1u_2), u_2(u_1u_2)^{-1})\}$$

for some $r \in |\Delta|$ and for some $u_1, u_2 \in \mathbf{S}^1$. Note that this equivalence relation $\sim$ in A comes from the equivalence relation associated with the local homeomorphism $X^{\log} \to E^{\log}$.

Take a continuous map $s : B \to [0, 1]$ such that we have a homeomorphism

$$B \xrightarrow{\sim} |\Delta| \times [0, 1], \quad (r_1, r_2) \mapsto (r_1r_2, s(r_1, r_2)),$$

and such that

$$s(r_1, r_2) = 0 \Leftrightarrow r_2 = 1, \quad s(r_1, r_2) = 1 \Leftrightarrow r_1 = 1.$$

For example, the function $s(r_1, r_2) = (1 - r_2)/((1 - r_1) + (1 - r_2))$ has this property.

Let $p = (r_0, u_0) \in \Delta^{\log}$ $(r_0 \in |\Delta|, u_0 \in \mathbf{S}^1)$ and take an open neighborhood V' of u_0 in $\mathbf{S}^1$ and a continuous map $k : V' \times [0, 1] \to \mathbf{S}^1$ such that $k(u, 1) = k(u, 0)u$ for any $u \in V'$. For example, if $u_0 = e^{i\theta}$ with $\theta \in \mathbf{R}$, we can take $V' = \{e^{i\lambda} \mid a < \lambda < b\}$ for any fixed $a, b \in \mathbf{R}$ such that $a < \theta < b$ and $b - a < 2\pi$ and $k(e^{i\lambda}, t) = e^{i\lambda t}$ $(a < \lambda < b, 0 \leq t \leq 1)$. Let

$$V = [0, 1) \times V' \subset \Delta^{\log}, \quad V'' = \{(u_1, u_2) \mid u_1u_2 \in V'\} \subset \mathbf{S}^1 \times \mathbf{S}^1.$$

Then we have a homeomorphism

$$B \times V'' \xrightarrow{\sim} V \times [0, 1] \times \mathbf{S}^1,$$
$$(r_1, r_2, u_1, u_2) \mapsto (r_1r_2, u_1u_2, s(r_1, r_2), u_1k(u_1u_2, s(r_1, r_2))).$$

The subset $B \times V''$ of A is stable for the relation $\sim$, and this homeomorphism induces a homeomorphism

$$(B \times V'')/\sim \xrightarrow{\sim} V \times ([0, 1]/(0 \sim 1)) \times \mathbf{S}^1.$$

Since $(B \times V'')/\sim \xrightarrow{\sim} (f^{\log})^{-1}(V)$, we have a homeomorphism

$$(f^{\log})^{-1}(V) \xrightarrow{\sim} V \times ([0,1]/(0 \sim 1)) \times \mathbf{S}^1$$

over V. Note that $[0,1]/(0 \sim 1) \simeq \mathbf{S}^1$.

See [U3] for a generalization of this local topological triviality of the family $E^{\log} \to \Delta^{\log}$ over the base.

Thus Figure 1 is completely explained.

0.2.11

The above magic for $f : E \to \Delta$ is generalized in logarithmic complex analytic geometry as follows. In logarithmic complex analytic geometry, we consider mainly logarithmic structures called fs logarithmic structures.

We say a monoid $\mathcal{S}$ is *integral* if $ab = ac$ implies $b = c$ in $\mathcal{S}$. An integral monoid $\mathcal{S}$ is embedded in the group $\mathcal{S}^{\mathrm{gp}} = \{ab^{-1} \mid a, b \in \mathcal{S}\}$. We say a monoid $\mathcal{S}$ is *saturated* if it is integral and if $a \in \mathcal{S}^{\mathrm{gp}}$ and $a^n \in \mathcal{S}$ for some integer $n \geq 1$ imply $a \in \mathcal{S}$. We say a monoid is *fs* if it is finitely generated and saturated.

For an fs monoid $\mathcal{S}$, $\mathcal{S}^{\mathrm{gp}}$ is a finitely generated abelian group, and $\mathcal{S}^{\mathrm{gp}}$ is torsion free if $\mathcal{S}$ is torsion free.

Let X be a local ringed space.

Let $\mathcal{S}$ be an fs monoid which is considered as a constant sheaf on X, and let $h : \mathcal{S} \to \mathcal{O}_X$ be a homomorphism of sheaves of monoids. The *associated logarithmic structure* on X is defined as the push-out $\tilde{\mathcal{S}}$ of the diagram

$$\begin{array}{ccc} h^{-1}(\mathcal{O}_X^\times) & \longrightarrow & \mathcal{S} \\ \downarrow & & \\ \mathcal{O}_X^\times & & \end{array}$$

in the category of sheaves of monoids, which is endowed with the induced homomorphism $\alpha : \tilde{\mathcal{S}} \to \mathcal{O}_X$ (for an explicit description of the push-out, see 2.1.1).

A logarithmic structure on X is *fs* if it is locally isomorphic to the one above.

The logarithmic structure in 0.2.5 associated with a divisor with normal crossings on an analytic manifold is an fs logarithmic structure. This can be checked locally by the fact that the logarithmic structure on $X = \Delta^n$ associated with the divisor $q_1 \cdots q_r = 0$ $(0 \leq r \leq n)$ is induced from the homomorphism $\mathbf{N}^r \to \mathcal{O}_X$, $(n_j)_{1 \leq j \leq r} \mapsto \prod_{j=1}^r q_j^{n_j}$. Note that $\mathbf{N}^r$ (the semigroup law is the addition) is an fs monoid.

An fs logarithmic structure M_X on X is integral, and hence M_X is embedded in the sheaf of commutative groups M_X^{gp}. For an fs logarithmic structure M_X on X, the stalk $(M_X/\mathcal{O}_X^\times)_x$ at $x \in X$ is a sharp fs monoid, and, in particular, torsion-free. Here we say that a monoid $\mathcal{S}$ is sharp if $\mathcal{S}^\times = \{1\}$, where $\mathcal{S}^\times$ denotes the set of all invertible elements of $\mathcal{S}$. Hence $((M_X/\mathcal{O}_X^\times)_x)^{\mathrm{gp}} = (M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$ is a free $\mathbf{Z}$-module of finite rank.

For instance, in the above example $X = \Delta^n$, if $x = (x_j)_{1 \leq j \leq n} \in X$ and if the number of those j satisfying $1 \leq j \leq r$ and $x_j = 0$ is m, then $(M_X/\mathcal{O}_X^\times)_x \simeq \mathbf{N}^m$

and this monoid is generated by (q_j mod $\mathcal{O}_{X,x}^\times$) for such j. (Note that $q_j \in \mathcal{O}_{X,x}^\times$ for the other j's.) We have $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x \simeq \mathbf{Z}^m$.

An analytic space with an fs logarithmic structure is called an *fs logarithmic analytic space*.

For an fs logarithmic analytic space X, the canonical map $\tau : X^{\log} \to X$ is proper, and $\tau^{-1}(x) \simeq (\mathbf{S}^1)^m$ for $x \in X$, where m is the rank of $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$.

Here "proper" means "proper in the sense of Bourbaki [Bn] and separated" (see 0.7.5). We keep this terminology throughout this book.

0.2.12

Example: *Toric varieties.* Let $\mathcal{S}$ be an fs monoid, and $X := \mathrm{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}} = \mathrm{Hom}(\mathcal{S}, \mathbf{C}^{\mathrm{mult}})$ (here $\mathbf{C}^{\mathrm{mult}}$ denotes the set $\mathbf{C}$ regarded as a multiplicative monoid) be the analytic toric variety. Then $\mathcal{S} \subset \mathbf{C}[\mathcal{S}] \to \mathcal{O}_X$ induces a canonical fs logarithmic structure. We have

$$X^{\log} = \mathrm{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\mathrm{mult}} \times \mathbf{S}^1)$$

(here $\mathbf{R}_{\geq 0}^{\mathrm{mult}}$ denotes the set $\mathbf{R}_{\geq 0}$ regarded as a multiplicative monoid).

Using this, we have a local presentation of $X^{\log}$ for any fs logarithmic analytic space X. Let X be an analytic space, let $\mathcal{S}$ be an fs monoid, let $\mathcal{S} \to \mathcal{O}_X$ be a homomorphism, and endow X with the induced logarithmic structure. Then

$$X^{\log} = X \times_{\mathrm{Hom}(\mathcal{S}, \mathbf{C}^{\mathrm{mult}})} \mathrm{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\mathrm{mult}} \times \mathbf{S}^1) \qquad (1)$$

(the fiber product as a topological space).

For a morphism $f : Y \to X$ of local ringed spaces and for a logarithmic structure M on X, the inverse image f^*M of M, which is a logarithmic structure on Y, is defined as in 2.1.3. If M is an fs logarithmic structure associated with a homomorphism $\mathcal{S} \to \mathcal{O}_X$ with $\mathcal{S}$ an fs monoid, the inverse image f^*M is the fs logarithmic structure associated with the homomorphism $\mathcal{S} \to \mathcal{O}_Y$ induced by f. Hence the inverse image of an fs logarithmic structure is an fs logarithmic structure. If $f : Y \to X$ is a morphism of analytic spaces, for an fs logarithmic structure M on X and for f^*M on Y, we have $Y^{\log} = Y \times_X X^{\log}$ as a topological space. The description of $X^{\log}$ in (1) is explained by this since, in that case, the logarithmic structure of X is the inverse image of the canonical logarithmic structure of $\mathrm{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$.

Example. Let $x = 0 \in \Delta$, and define the logarithmic structure M_x of x as the inverse image of M_Δ. This logarithmic structure is induced from the homomorphism $\mathcal{S} = \mathbf{N} \to \mathcal{O}_x = \mathbf{C}$, $n \mapsto$ (the image of q^n in $\mathcal{O}_x$) $= 0^n$ (note $0^0 = 1$). Hence $M_x = \bigsqcup_{n \geq 0}(\mathbf{C}^\times \cdot q^n) \simeq \mathbf{C}^\times \times \mathbf{N}$ with $\alpha : M_x \to \mathcal{O}_x = \mathbf{C}$, $c \cdot q^n \mapsto c \cdot 0^n$ ($c \in \mathbf{C}^\times$, $n \in \mathbf{N}$). Thus a one-point set can have a nontrivial logarithmic structure. We have $x^{\log} = \mathbf{S}^1$ for this logarithmic structure M_x.

0.2.13

The morphism $f : E \to \Delta$ in 0.2.1 is an easiest nontrivial example of logarithmically smooth morphisms (see 2.1.11 for the definition) of fs logarithmic analytic spaces. A logarithmically smooth morphism can have degeneration in the sense of classical complex analytic geometry, but, with the magic of the logarithmic structure, it can behave in logarithmic complex analytic geometry like a smooth morphism in classical complex analytic geometry.

A wider example of a logarithmically smooth morphism is a morphism with semistable degeneration $f : Y \to \Delta$ (Δ is endowed with M_Δ), that is, a morphism which, locally on Y, has the form $\Delta^n \to \Delta$, $(q_j)_{1\le j\le n} \mapsto \prod_{j=1}^r q_j (1 \le r \le n)$ with the logarithmic structure of Δ^n in 0.2.9. Indeed, it is shown in [U3] that, if f is proper (0.7.5), the associated continuous map $f^{\log} : Y^{\log} \to \Delta^{\log}$ is topologically trivial locally over the base.

Kajiwara and Nakayama [KjNc] proved the following:

Let $f : Y \to X$ be a proper logarithmically smooth morphism of fs logarithmic analytic spaces. Then, for any $m \ge 0$, the higher direct image functor $R^m f_*^{\log}$ sends locally constant sheaves of abelian groups on $Y^{\log}$ to locally constant sheaves on $X^{\log}$.

An fs logarithmic analytic space X is said to be logarithmically smooth if the structural morphism $X \to \mathrm{Spec}(\mathbf{C})$ is logarithmically smooth. Here $\mathrm{Spec}(\mathbf{C})$ is endowed with the trivial logarithmic structure $\mathbf{C}^\times$. An fs logarithmic analytic space X is logarithmically smooth if and only if, locally on X, there is an open immersion of analytic spaces $i : X \hookrightarrow Z = \mathrm{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ with $\mathcal{S}$ an fs monoid such that the logarithmic structure of X is the inverse image of the canonical logarithmic structure of Z (i.e., the logarithmic structure of X induced by the homomorphism $\mathcal{S} \to \mathcal{O}_X$ defined by i).

0.2.14

Logarithmic differential forms. The name "logarithmic structure" comes from its relation to differential forms with logarithmic poles.

For an fs logarithmic analytic space X, the sheaves of logarithmic differential q-forms ω_X^q on X ($q \in \mathbf{N}$) are defined as in 2.1.7. If X is an analytic manifold and D is a divisor on X with normal crossings, and if X is endowed with the logarithmic structure associated with D, ω_X^q coincides with the sheaf $\Omega_X^q(\log(D))$, the sheaf of differential q-forms on X which may have logarithmic poles along D. In general, ω_X^q is an $\mathcal{O}_X$-module, there is a canonical homomorphism of $\mathcal{O}_X$-modules $\Omega_X^q \to \omega_X^q$, ω_X^q is the qth exterior power of ω_X^1, and ω_X^1 is generated over $\mathcal{O}_X$ by Ω_X^1 and the image of a homomorphism $d\log : M_X^{\mathrm{gp}} \to \omega_X^1$. For a morphism $Y \to X$ of fs logarithmic analytic spaces, the logarithmic version $\omega_{Y/X}^q$ of $\Omega_{Y/X}^q$ is also defined (2.1.7).

Example. Let $X = \Delta^n$ with the logarithmic structure as in 0.2.9. Then for $x = (x_j)_{1\le j\le n} \in X$, the stalk of ω_X^1 at x is a free $\mathcal{O}_{X,x}$-module with basis $(\omega_j)_{1\le j\le n}$ where $\omega_j = d\log(q_j)$ if $1 \le j \le r$ and $x_j = 0$, and $\omega_j = dq_j$ otherwise.

For a logarithmically smooth morphism $Y \to X$ of fs logarithmic analytic spaces, the sheaf $\omega_{Y/X}^1$ is locally free although $\Omega_{Y/X}^1$ may not be locally free if degeneration occurs in $Y \to X$. Consider the case $n = r = 2$ of the last example, and consider the logarithmically smooth morphism $X \to \Delta$, $(x_1, x_2) \mapsto x_1x_2$. The sheaf $\Omega_{X/\Delta}^1$ is generated by dq_1 and dq_2 which satisfy the relation $q_1dq_2 + q_2dq_1 = d(q_1q_2) = 0$.

This indicates that $\Omega^1_{X/\Delta}$ is not locally free. However, $\omega^1_{X/\Delta}$ is generated by $d\log(q_1)$ and $d\log(q_2)$, which satisfy $d\log(q_1)+d\log(q_2) = d\log(q_1q_2) = 0$. This shows that $\omega^1_{X/\Delta}$ is a free $\mathcal{O}_X$-module of rank 1 generated by $d\log(q_1)$.

Example. Let $\mathcal{S}$ be an fs monoid and let $X = \operatorname{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ endowed with the canonical logarithmic structure. Then we have an isomorphism

$$\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{S}^{\mathrm{gp}} \xrightarrow{\sim} \omega^1_X,\ f \otimes g \mapsto fd\log(g).$$

Example. Let $x = 0 \in \Delta$ and endow x with the inverse image of M_Δ. Then ω^1_x is the one-dimensional $\mathbf{C}$-vector space generated by the image of $d\log(q) \in \Gamma(\Delta, \omega^1_\Delta)$. Thus a one point set can have a nontrivial logarithmic differential form.

Now we talk about Hodge filtration and logarithmic Hodge theory. The key point is that, for an fs logarithmic analytic space X, we define a sheaf of rings $\mathcal{O}^{\log}_X$ on $X^{\log}$. Roughly speaking, $\mathcal{O}^{\log}_X$ is the ring generated over $\mathcal{O}_X$ by $\log(M^{\mathrm{gp}})$. First, by considering the example $f : E \to \Delta$ of 0.2.1, we will see why such a ring is necessary.

0.2.15

Elliptic curves (continued). Let $f' : E^* \to \Delta^*$ be as in 0.2.2. Let $H'_{\mathbf{Z}} = R^1f'_*(\mathbf{Z})$ and let $\mathcal{M}' = R^1f'_*(\Omega^\bullet_{E^*/\Delta^*})$.

We have seen that $H'_{\mathbf{Z}}$ extends to the local system $H_{\mathbf{Z}} = R^1f^{\log}_*(\mathbf{Z})$ on $\Delta^{\log}$ (0.2.8).

On the other hand, the $\mathcal{O}_{\Delta^*}$-module $\mathcal{M}'$ with a decreasing filtration F' (0.2.2) extends to a locally free $\mathcal{O}_\Delta$-module $\mathcal{M} := R^1f_*(\omega^\bullet_{E/\Delta}) = R^1f_*(\mathcal{O}_E \xrightarrow{d} \omega^1_{E/\Delta})$ of rank 2 with the decreasing filtration defined by $\mathcal{M}^0 = \mathcal{M}, \mathcal{M}^1 = f_*(\omega^1_{E/\Delta}) \hookrightarrow \mathcal{M}$, $\mathcal{M}^2 = 0$. We have

$$\mathcal{M} = \mathcal{O}_\Delta e_1 \oplus \mathcal{O}_\Delta \omega \supset \mathcal{M}^1 = \mathcal{O}_\Delta \omega, \tag{1}$$

where we denote by the same letter e_1 the image of the global section e_1 of $R^1f_*\mathbf{Z}$ under $R^1f_*\mathbf{Z} \to R^1f_*(\omega^\bullet_{E/\Delta})$, and we denote by ω the differential form $(2\pi i)^{-1}dt_1/t_1$ on E with logarithmic poles along $f^{-1}(0)$ (where t_1 is the coordinate function of X in 0.2.10).

Although the relation between $H'_{\mathbf{Z}}$ and $\mathcal{M}'$ is simply $\mathcal{M}' = \mathcal{O}_{\Delta^*} \otimes_{\mathbf{Z}} H'_{\mathbf{Z}}$, the relation between $H_{\mathbf{Z}}$ and $\mathcal{M}$ is not so direct. They are related only after being tensored by a sheaf of rings $\mathcal{O}^{\log}_\Delta$ defined below.

Let q be the coordinate function of Δ. Then on Δ^*, via the identification $\mathcal{M}' = \mathcal{O}_{\Delta^*} \otimes_{\mathbf{Z}} H'_{\mathbf{Z}}$, we have

$$\omega = (2\pi i)^{-1}\log(q)e_1 + e_2 \tag{2}$$

where e_2 is taken by fixing a branch of the multivalued function $(2\pi i)^{-1}\log(q)$, and the same branch of this function is used in the formula (2). This formula (2) follows from the fact that the restriction of ω to each fiber $\mathbf{C}^\times/q^{\mathbf{Z}} \simeq \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ ($q \in \Delta^*$, $\tau = (2\pi i)^{-1}\log(q)$) is $dz = \tau e_1 + e_2$ on $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$.

Locally on $\Delta^{\log}$, e_2 extends to a local section of $H_{\mathbf{Z}}$, and we have

$$H_{\mathbf{Z}} = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2. \tag{3}$$

When we compare (1), (2), and (3), since $\log(q)$ in (2) does not extend over the origin in the classical sense, we do not find a simple relation between $H_{\mathbf{Z}}$ and $\mathcal{M}$. However, $\log(q)$ exists on $\Delta^{\log}$ locally as a local section of $j_*^{\log}(\mathcal{O}_{\Delta^*})$ where $j^{\log}$ is the inclusion map $\Delta^* \hookrightarrow \Delta^{\log}$. In fact, if $y = (0, e^{i\theta}) \in \Delta^{\log} = |\Delta| \times \mathbf{S}^1$ ($\theta \in \mathbf{R}$), where $|\Delta|$ is as in 0.2.9, if we take $a, b \in \mathbf{R}$ such that $a < \theta < b$ and $b - a < 2\pi$, and if we denote by U the open neighborhood $\{(r, e^{ix}) \mid r \in |\Delta|, a < x < b\}$ of y in $\Delta^{\log}$, the holomorphic map $re^{i\theta} \mapsto \log(r) + i\theta$ defined on $\Delta^* \cap U = \{(r, e^{i\theta}) \mid 0 < r < 1, a < \theta < b\}$ is an element of $\Gamma(\Delta^* \cap U, \mathcal{O}_{\Delta^*}) = \Gamma(U, j_*^{\log}(\mathcal{O}_{\Delta^*}))$, which is a branch of $\log(q)$. All branches of $\log(q)$ in $j_*^{\log}(\mathcal{O}_{\Delta^*})$ are congruent modulo $2\pi i\mathbf{Z}$. Let $\mathcal{O}_\Delta^{\log} \subset j_*^{\log}(\mathcal{O}_{\Delta^*})$ be the sheaf of subrings of $j_*^{\log}(\mathcal{O}_{\Delta^*})$ on $\Delta^{\log}$ generated over $\tau^{-1}(\mathcal{O}_\Delta)$ by $\log(q)$. Here $\tau^{-1}(\,)$ is the inverse image of a sheaf. Then from (1), (2), and (3), we have

$$\mathcal{O}_\Delta^{\log} \otimes_{\tau^{-1}(\mathcal{O}_\Delta)} \tau^{-1}(\mathcal{M}) = \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}.$$

0.2.16

The sheaf of rings $\mathcal{O}_X^{\log}$. For an fs logarithmic analytic space X, we have a sheaf of rings $\mathcal{O}_X^{\log}$, which generalizes the above $\mathcal{O}_\Delta^{\log}$.

First we consider the case $X = \operatorname{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ for an fs monoid $\mathcal{S}$. Let U be the open subspace $\operatorname{Spec}(\mathbf{C}[\mathcal{S}^{\mathrm{gp}}])_{\mathrm{an}}$ of X, and let $j^{\log} : U \to X^{\log}$ be the canonical map. Then we define $\mathcal{O}_X^{\log}$ as a sheaf of subrings of $j_*^{\log}(\mathcal{O}_U)$ generated over $\tau^{-1}(\mathcal{O}_X)$ by the logarithms of local sections of M_X^{gp} (these logarithms exist in $j_*^{\log}(\mathcal{O}_U)$ and are determined mod $2\pi i\mathbf{Z}$ just as in the case of Δ).

The definition of $\mathcal{O}_X^{\log}$ for a general fs logarithmic analytic space X is given in 2.2.4. If the logarithmic structure of X is induced from a homomorphism $\mathcal{S} \to \mathcal{O}_X$ with $\mathcal{S}$ an fs monoid, we have

$$\mathcal{O}_X^{\log} = \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Z^{\log} \quad \text{with} \quad Z = \operatorname{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$$

(note that $\mathcal{O}_Z^{\log}$ is explained just above), where we denote the inverse images on $X^{\log}$ of the sheaves $\mathcal{O}_X$, $\mathcal{O}_Z$, and $\mathcal{O}_Z^{\log}$, by $\mathcal{O}_X$, $\mathcal{O}_Z$, and $\mathcal{O}_Z^{\log}$, respectively, for simplicity. We have a homomorphism $\log : M_X^{\mathrm{gp}} \to \mathcal{O}_X^{\log}/2\pi i\mathbf{Z}$, and $\mathcal{O}_X^{\log}$ is generated over $\tau^{-1}(\mathcal{O}_X)$ by $\log(M_X^{\mathrm{gp}})$. For an fs logarithmic analytic space X and $x \in X$, if the free $\mathbf{Z}$-module $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$ is of rank r with basis $(f_j \bmod \mathcal{O}_{X,x}^\times)_{1 \le j \le r}$ ($f_j \in M_{X,x}^{\mathrm{gp}}$), then for any point y of $X^{\log}$ lying over x, the stalk $\mathcal{O}_{X,y}^{\log}$ of $\mathcal{O}_X^{\log}$ is isomorphic to the polynomial ring in r variables over $\mathcal{O}_{X,x}$ by

$$\mathcal{O}_{X,x}[T_1, \ldots, T_r] \xrightarrow{\sim} \mathcal{O}_{X,y}^{\log}, \quad T_j \mapsto \log(f_j)$$

($\log(f_j)$ is defined only modulo $2\pi i\mathbf{Z}$ but we choose a branch (a representative) for each j). Let

$$\omega_X^{q,\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_X^q).$$

Then we have the de Rham complex $\omega_X^{\bullet,\log}$ on $X^{\log}$ with the differential $d : \omega_X^{q,\log} \to \omega_X^{q+1,\log}$ defined as in 2.2.6.

Example. Let $X = \Delta^n$ endowed with the logarithmic structure in 0.2.9. Let $x \in \Delta^n$ and let y be a point of $X^{\log}$ lying over x. Let m be the number of j such that $1 \leq j \leq r$ and $x_j = 0$. Then the stalk $\mathcal{O}^{\log}_{X,y}$ is a polynomial ring over $\mathcal{O}_{X,x}$ in m variables $\log(q_j)$ for such j.

Example. Let $x = \mathrm{Spec}(\mathbf{C})$ endowed with an fs logarithmic structure (we will call such x an *fs logarithmic point*). Then $M_x = \mathbf{C}^\times \times \mathcal{S}$ for some fs monoid $\mathcal{S}$ with no invertible element other than 1, where $\alpha : M_x \to \mathbf{C}$ sends $(c, t) \in M_x$ ($c \in \mathbf{C}^\times$, $t \in \mathcal{S}$) to 0 if $t \neq 1$, and to c if $t = 1$. Let r be the rank of $\mathcal{S}^{\mathrm{gp}}$ which is a free $\mathbf{Z}$-module of finite rank. We have

$$x^{\log} \simeq (\mathbf{S}^1)^r, \quad \omega^1_x \simeq \mathbf{C}^r, \quad \mathcal{O}^{\log}_{x,y} \simeq \mathbf{C}[T_1, \ldots, T_r] \quad (y \in x^{\log}).$$

0.2.17

Let X be an fs logarithmic analytic space, let $x \in X$, and let y be a point of $X^{\log}$ lying over x. The stalk $\mathcal{O}^{\log}_{X,y}$ is not necessarily a local ring, and has a global ring-theoretic nature. Let $\mathrm{sp}(y)$ be the set of all ring homomorphisms $s : \mathcal{O}^{\log}_{X,y} \to \mathbf{C}$ such that $s(f) = f(x)$ for any $f \in \mathcal{O}_{X,x}$. If we fix $s_0 \in \mathrm{sp}(y)$, we have a bijection

$$\mathrm{sp}(y) \xrightarrow{\sim} \mathrm{Hom}((M^{\mathrm{gp}}_X/\mathcal{O}^\times_X)_x, \mathbf{C}^{\mathrm{add}}), \tag{1}$$
$$s \mapsto (f \mapsto s(\log(f)) - s_0(\log(f))) \quad \text{for} \quad f \in M^{\mathrm{gp}}_{X,x}.$$

Here $\mathbf{C}^{\mathrm{add}}$ is $\mathbf{C}$ regarded as an additive group.

Let $(H_{\mathbf{Z}}, F)$ be a pair of a local system $H_{\mathbf{Z}}$ of free $\mathbf{Z}$-modules of finite rank on $X^{\log}$ and of a decreasing filtration F on the $\mathcal{O}^{\log}_X$-module $\mathcal{O}^{\log}_X \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ such that F^p and $(\mathcal{O}^{\log}_X \otimes_{\mathbf{Z}} H_{\mathbf{Z}})/F^p$ are locally free as $\mathcal{O}^{\log}_X$-modules for all p. Then, for each $s \in \mathrm{sp}(y)$, we have a decreasing filtration $F(s)$ on $H_{\mathbf{C},y} = \mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z},y}$ (called the *specialization of F at s*) defined by $F^p(s) = \mathbf{C} \otimes_{\mathcal{O}^{\log}_{X,y}} F^p_y$. Here $\mathcal{O}^{\log}_{X,y} \to \mathbf{C}$ is s. We will see later that a nilpotent orbit can be regarded as the family $(F(s))_{s \in \mathrm{sp}(y)}$ associated with such $(H_{\mathbf{Z}}, F)$. The reason why an orbit of Hodge filtrations, called a nilpotent orbit (not a single Hodge filtration), appears in the degeneration of Hodge structures is, from the point of view of logarithmic Hodge theory, that the stalk $\mathcal{O}^{\log}_{X,y}$ is still a global ring and we have many specializations at a point y.

0.2.18

Elliptic curves (continued). Let $f : E \to \Delta$ be as in 0.2.1, and consider $H_{\mathbf{Z}} = R^1 f^{\log}_*(\mathbf{Z})$, $\mathcal{M} = R^1 f_*(\mathcal{O}_E \to \omega^1_{E/\Delta})$, and the filtration $(\mathcal{M}^p)_p$ as in 0.2.15. Define a filtration F on $\mathcal{O}^{\log}_\Delta \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ by

$$F^p = \mathcal{O}^{\log}_\Delta \otimes_{\tau^{-1}(\mathcal{O}_\Delta)} \tau^{-1}(\mathcal{M}^p) \subset \mathcal{O}^{\log}_\Delta \otimes_{\tau^{-1}(\mathcal{O}_\Delta)} \tau^{-1}(\mathcal{M}) = \mathcal{O}^{\log}_\Delta \otimes_{\mathbf{Z}} H_{\mathbf{Z}}.$$

Take $y \in \Delta^{\log}$ lying over $0 \in \Delta$. We consider the specializations $F(s)$ for $s \in \mathrm{sp}(y)$. Take a branch of $e_2 \in H_{\mathbf{Z}}$ at y and take the corresponding branch of $\log(q)$ at y. Then F^1_y is a free $\mathcal{O}^{\log}_{\Delta,y}$-module of rank 1 generated by $(2\pi i)^{-1}\log(q)e_1 + e_2$

(0.2.15 (2)). Since $\mathcal{O}^{\log}_{\Delta,y}$ is a polynomial ring in one variable $\log(q)$ over $\mathcal{O}_{\Delta,0}$, an element $s \in \operatorname{sp}(y)$ is determined by $s(\log(q)) \in \mathbf{C}$. The filtration $F(s)$ is described as: $F^0(s) = H_{\mathbf{C},y}$, $F^2(s) = 0$, and $F^1(s)$ is the one-dimensional $\mathbf{C}$-subspace of $H_{\mathbf{C},y}$ generated by $s((2\pi i)^{-1}\log(q))e_1 + e_2$.

If the imaginary part of $s((2\pi i)^{-1}\log(q))$ is > 0 (that is, if $|\exp(s(\log(q)))| < 1$), then $(H_{\mathbf{Z},y}, F(s))$ is a Hodge structure of weight 1, and for the antisymmetric pairing $\langle\ ,\ \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$ defined as $\langle e_2, e_1\rangle = 1$, $(H_{\mathbf{Z},y}, \langle\ ,\ \rangle_y, F(s))$ becomes a polarized Hodge structure.

0.2.19

The observation in 0.2.18 leads us to the notion of "logarithmic variation of polarized Hodge structure."

Let X be a logarithmically smooth fs logarithmic analytic space. A *logarithmic variation of polarized Hodge structure* (LVPH) *on* X *of weight* w is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a locally constant sheaf $H_{\mathbf{Z}}$ of free $\mathbf{Z}$-modules of finite rank on $X^{\log}$, a bilinear form $\langle\ ,\ \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$, and a decreasing filtration F of $\mathcal{O}^{\log}_X \otimes H_{\mathbf{Z}}$ by $\mathcal{O}^{\log}_X$-submodules which satisfy the following three conditions (1)–(3).

(1) There exist a locally free $\mathcal{O}_X$-module $\mathcal{M}$ and a decreasing filtration $(\mathcal{M}^p)_{p\in\mathbf{Z}}$ by $\mathcal{O}_X$-submodules of $\mathcal{M}$ such that $\mathcal{O}^{\log}_X \otimes_{\mathbf{Z}} H_{\mathbf{Z}} = \mathcal{O}^{\log}_X \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{M})$ and $F^p = \mathcal{O}^{\log}_X \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{M}^p)$ for all p, and such that $\mathcal{M}^p = \mathcal{M}$ for $p \ll 0$, $\mathcal{M}^p = 0$ for $p \gg 0$, and $\mathcal{M}^p/\mathcal{M}^{p+1}$ are locally free for all p.

(2) Let $x \in X$, and let $(f_j)_{1\le j\le n}$ be elements of $M_{X,x}$ that are not contained in $\mathcal{O}^\times_{X,x}$ such that $(f_j \bmod \mathcal{O}^\times_{X,x})_{1\le j\le n}$ generates the monoid $(M_X/\mathcal{O}^\times_X)_x$. Let $y \in \tau^{-1}(x) \subset X^{\log}$. Then if $s \in \operatorname{sp}(y)$ and if $\exp(s(\log(f_j)))$ are sufficiently near to 0 for all j, $(H_{\mathbf{Z},y}, \langle\ ,\ \rangle_y, F(s))$ is a polarized Hodge structure of weight w.

(3) $(d \otimes_{\mathbf{Z}} 1_{H_{\mathbf{Z}}})(F^p) \subset \omega^{1,\log}_X \otimes_{\mathcal{O}^{\log}_X} F^{p-1}$ for all p.

0.2.20

Elliptic curve (continued). The pair $(H_{\mathbf{Z}}, F)$ in 0.2.18 arising from $E \to \Delta$ is an LVPH on Δ. (The Griffiths transversality (3) in 0.2.19 is satisfied automatically.)

At a point of Δ^*, the fiber of $(H_{\mathbf{Z}}, F)$ is a polarized Hodge structure. However, at $0 \in \Delta$, the fiber of $(H_{\mathbf{Z}}, F)$ should be understood as a family $(F(s))_{s\in\operatorname{sp}(y)}$ for some fixed $y \in \tau^{-1}(0)$ and for varying s. As is explained in Section 0.4, this family is the so-called nilpotent orbit. This is expressed in the schema (2) in Introduction.

0.2.21

LVPH arising from geometry. By the weakly semistable reduction theorem of Abramovich and Karu [AK], any projective fiber space is modified, by alteration and birational modification, to a projective, toroidal morphism $f : Y \to X$ without horizontal divisor, which is equivalent to a projective, vertical, logarithmically smooth morphism (2.1.11; see also 0.2.13) with $\operatorname{Coker}\big((M^{\mathrm{gp}}_X/\mathcal{O}^\times_X)_{f(y)} \to (M^{\mathrm{gp}}_Y/\mathcal{O}^\times_Y)_y\big)$ being torsion-free at any $y \in Y$.

Then by Kato-Matsubara-Nakayama [KMN], for any $m \in \mathbf{Z}$, a variation of polarized logarithmic Hodge structure $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ of weight m on X is obtained in the following way.

$$H_{\mathbf{Z}} = R^m(f^{\log})_* \mathbf{Z}/(\text{torsion}),$$
$$\langle\ ,\ \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q} \ \text{ induced from an ample line bundle},$$
$$\mathcal{M} = R^m f_*(\omega^{\bullet}_{Y/X}),$$
$$\mathcal{M}^p = R^m f_*(\omega^{\geq p}_{Y/X}) \hookrightarrow \mathcal{M}$$
$$F^p = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{M}^p) \hookrightarrow \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{M}) = \mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}.$$

There are many other contributors: [F], [Kf2], [Ma1], [Ma2], [U3], etc.

This is a generalization of work of Steenbrink [St].

0.2.22

The nilpotent orbit theorem of Schmid [Sc] is interpreted as follows (see Theorem 2.5.14).

Let X be a logarithmically smooth, fs logarithmic analytic space, and let $U = \{x \in X \mid M_{X,x} = \mathcal{O}^{\times}_{X,x}\}$ be the open set of X consisting of all points at which the logarithmic structure is trivial. Let H be a VPH on U with unipotent local monodromy along $X - U$. Then H extends to a LVPH on X.

0.2.23

The theory of logarithmic structure was started in p-adic Hodge theory to construct the logarithmic crystalline cohomology theory for varieties with semistable reduction ([I1], [HK], etc.).

Usually, the theory over p-adic fields begins by following its analogue over $\mathbf{C}$. But in the theory of logarithmic structure, applications appeared first in p-adic Hodge theory. We hope that the theory of logarithmic structure will also be useful in Hodge theory.

0.3 GRIFFITHS DOMAINS AND MODULI OF PH

In [G1], Griffiths defined and studied classifying spaces D of polarized Hodge structures. We review the definition of D. We regard D as moduli of polarized Hodge structures by discarding the Griffiths transversality from VPH (0.3.5–0.3.7).

Fix

$$(w, (h^{p,q})_{p,q \in \mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$$

where w is an integer, $(h^{p,q})_{p,q \in \mathbf{Z}}$ is a family of non-negative integers such that $h^{p,q} = 0$ unless $p + q = w$, $h^{p,q} \neq 0$ for only finitely many (p, q), and such that $h^{p,q} = h^{q,p}$ for all p, q, H_0 is a free $\mathbf{Z}$-module of rank $\sum_{p,q} h^{p,q}$, and $\langle\ ,\ \rangle_0$ is a nondegenerate bilinear form $H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$, which is symmetric if w is even and antisymmetric if w is odd.

Definition 0.3.1 *The classifying space D of polarized Hodge structures of type $\Phi_0 = \big(w, (h^{p,q})_{p,q}, H_0, \langle\ ,\ \rangle_0\big)$ is the set of all decreasing filtrations F on $H_{0,\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} H_0$ such that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})_{p,q}$.*

The "compact dual" $\check{D}$ *of D is defined to be the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that* $\dim_{\mathbf{C}}(F^p/F^{p+1}) = h^{p,w-p}$ *for all p that satisfy the Riemann-Hodge first bilinear relation* 0.1.8 (1).

The spaces D and $\check{D}$ have natural structures of analytic manifolds, and D is an open submanifold of $\check{D}$.

The space D is called the *Griffiths domain* and also the *period domain*.

0.3.2

Examples. (i) *Upper half plane.* Consider the case $w = 1$, $h^{1,0} = h^{0,1} = 1$ and $h^{p,q} = 0$ for other (p, q). Let H_0 be a free $\mathbf{Z}$-module of rank 2 with basis e_1, e_2, and define an antisymmetric $\mathbf{Z}$-bilinear form $\langle\ ,\ \rangle_0 : H_0 \times H_0 \to \mathbf{Z}$ by $\langle e_2, e_1\rangle_0 = 1$. Then $D \simeq \mathfrak{h}$, the upper half plane, where we identify a point $\tau \in \mathfrak{h}$ with $F(\tau) \in D$ defined by

$$F^0(\tau) = H_{0,\mathbf{C}}, \quad F^1(\tau) = \mathbf{C}(\tau e_1 + e_2), \quad F^2(\tau) = \{0\}.$$

In this case, $\check{D}$ is identified with $\mathbf{P}^1(\mathbf{C})$.

(ii) *Upper half space* (a generalization of Example (i)). Let $g \geq 1$ and consider the case $w = 1$, $h^{1,0} = h^{0,1} = g$ and $h^{p,q} = 0$ for other (p, q). Let H_0 be a free $\mathbf{Z}$-module with basis $(e_j)_{1\leq j\leq 2g}$ and define a $\mathbf{Z}$-bilinear form $\langle\ ,\ \rangle_0 : H_0 \times H_0 \to \mathbf{Z}$ by

$$(\langle e_j, e_k\rangle_0)_{j,k} = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}.$$

Then $D \simeq \mathfrak{h}_g$, the Siegel upper half space of degree g. Recall that $\mathfrak{h}_g$ is the space of all symmetric matrices over $\mathbf{C}$ of degree g whose imaginary parts are positive definite. We identify a matrix $\tau \in \mathfrak{h}_g$ with $F(\tau) \in D$ as follows:

$$F^0(\tau) = H_{0,\mathbf{C}}, \quad F^1(\tau) = \left(\begin{matrix}\text{subspace of } H_{0,\mathbf{C}} \text{ spanned} \\ \text{by the column vectors of}\end{matrix} \begin{pmatrix} \tau \\ 1_g \end{pmatrix}\right), \quad F^2(\tau) = \{0\}.$$

(The symmetry of τ corresponds to the Riemann-Hodge first bilinear relation for $F(\tau)$ and the positivity of $\mathrm{Im}(\tau)$ corresponds to the second bilinear relation (cf. 0.1.8).)

(iii) *Example with* $w = 2$, $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ (a special case of the example investigated in Section 12.2 where the weight w is shifted to 0). Let H_0 be a free $\mathbf{Z}$-module of rank 5 with basis $(e_j)_{1\leq j\leq 5}$, and let $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$ be the bilinear form defined by

$$\left\langle \sum_{1\leq j\leq 5} c_j e_j, \sum_{1\leq j\leq 5} c'_j e_j \right\rangle_0 = -c_1c'_1 - c_2c'_2 - c_3c'_3 + c_4c'_5 + c_5c'_4 \quad (c_j, c'_j \in \mathbf{Q}).$$

Let

$$Q := \{(z_1 : z_2 : z_3) \in \mathbf{P}^2(\mathbf{C}) \mid z_1^2 + z_2^2 + z_3^2 = 0\},$$

$$X := \left\{ (z, a) \;\middle|\; \begin{array}{l} z = (z_1 : z_2 : z_3) \in Q, \ a \in (\sum_{j=1}^{3} \mathbf{C}e_j)/\mathbf{C}(z_1e_1 + z_2e_2 + z_3e_3), \\ a \notin \mathrm{Image}(\mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_3) \end{array} \right\}.$$

For $z \in Q$, define a decreasing filtration $F(z)$ of $H_{0.\mathbf{C}}$ by $F^p(z) = H_{0.\mathbf{C}}$ for $p \leq 0$, $F^p(z) = 0$ for $p \geq 3$; $F^2(z)$ is the two-dimensional $\mathbf{C}$-subspace generated by $z_1e_1 + z_2e_2 + z_3e_3$ and e_5, and $F^1(z)$ is the annihilator of F^2 with respect to $\langle\ ,\ \rangle_0$. For $a \in \sum_{j=1}^{3} \mathbf{C}e_j$, let $N_a : H_{0.\mathbf{C}} \to H_{0.\mathbf{C}}$ be the nilpotent $\mathbf{C}$-linear map defined by

$$N_a(e_5) = a, \quad N_a(b) = -\langle a, b\rangle_0 e_4 \text{ for } b \in \sum_{j=1}^{3} \mathbf{C}e_j, \quad N_a(e_4) = 0.$$

Then $a \mapsto N_a$ is $\mathbf{C}$-linear, and $N_aN_b = N_bN_a$ for any a, b. For $a, b \in \sum_{j=1}^{3} \mathbf{C}e_j$ and $z \in Q$, $\exp(N_a)F(z) = \exp(N_b)F(z)$ if and only if $a \equiv b \bmod \mathbf{C}(z_1e_1 + z_2e_2 + z_3e_3)$. We have

$$X \xrightarrow{\sim} D, \quad (z, a) \mapsto \exp(N_a)F(z).$$

The complex dimension of D is 3.

In this example, D is not a symmetric Hermitian domain.

0.3.3

Let $G_{\mathbf{Z}} = \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0)$, and for $R = \mathbf{Q}, \mathbf{R}, \mathbf{C}$ let $G_R = \mathrm{Aut}_R(H_{0.R}, \langle\ ,\ \rangle_0)$ and $\mathfrak{g}_R = \mathrm{Lie}\, G_R = \{A \in \mathrm{End}_R(H_{0.R}) \mid \langle A(x), y\rangle_0 + \langle x, A(y)\rangle_0 = 0\ (\forall x, y \in H_{0.R})\}$.

Then D is a homogeneous space under the natural action of $G_{\mathbf{R}}$. For $\mathbf{r} \in D$, let $K_{\mathbf{r}}$ be the maximal compact subgroup of $G_{\mathbf{R}}$ consisting of all elements that preserve the Hodge metric $(\ ,\ )_{\mathbf{r}}$ associated with $\mathbf{r}$ (0.1.8). The isotropy subgroup $K'_{\mathbf{r}}$ of $G_{\mathbf{R}}$ at $\mathbf{r} \in D$ is contained in $K_{\mathbf{r}}$, but they need not coincide for general D (cf. [G1], and also [Sc]). The following conditions (1) and (2) are equivalent for any $\mathbf{r} \in D$:

(1) D is a symmetric Hermitian domain.
(2) $\dim(K_{\mathbf{r}}) = \dim(K'_{\mathbf{r}})$.

0.3.4

Examples. Upper half space (continued). In this case, we have, for $R = \mathbf{Q}, \mathbf{R}, \mathbf{C}$,

$$G_R = \mathrm{Sp}(g, R) = \{h \in \mathrm{GL}(2g, R) \mid {}^t h J_g h = J_g\},$$

where $J_g = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}$. The matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbf{R})$ acts on D by

$$F(\tau) \mapsto F(\tau'), \quad \tau' = (A\tau + B)(C\tau + D)^{-1}.$$

Let $\mathbf{r} = F(i1_g) \in D$. Then $K_\mathbf{r} = K'_\mathbf{r}$, and this group is isomorphic to the unitary group $U(g)$ by $A+iB \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \in K_\mathbf{r} \subset \mathrm{Sp}(g, \mathbf{R})$.

Example with $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ (continued). As in 0.3.2 (iii), let $Q = \{(z_1 : z_2 : z_3) \in \mathbf{P}^2(\mathbf{C}) \mid z_1^2 + z_2^2 + z_3^2 = 0\}$. We have a homeomorphism

$$\theta : Q \xrightarrow{\sim} \mathbf{S}^2 = \left\{ \sum_{j=1}^{3} x_j e_j \in H_{0,\mathbf{R}} \;\middle|\; x_j \in \mathbf{R}, x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

characterized as follows. Let $z = (a_1 + ib_1 : a_2 + ib_2 : a_3 + ib_3) \in Q$ $(a_j, b_j \in \mathbf{R})$. Then $(a_j)_j$ and $(b_j)_j$ are orthogonal in $\mathbf{R}^3$ and have the same length. The characterization of θ is that for $\theta(z) = \sum_{j=1}^{3} c_j e_j \in \mathbf{S}^2$, $(c_j)_j$ is orthogonal to $(a_j)_j$ and $(b_j)_j$, and $\det((a_j)_j, (b_j)_j, (c_j)_j) > 0$.

For $v \in \mathbf{S}^2$, write

$$\mathbf{r}(v) = \exp(iN_v)F(\theta^{-1}(v)) \in D.$$

If we take a basis $(f_j)_{1\le j\le 5}$ of $H_{0,\mathbf{Q}}$ given by $f_j := e_j$ $(j = 1, 2, 3)$, $f_4 := e_5 - \frac{1}{2}e_4$, $f_5 := e_5 + \frac{1}{2}e_4$, then $(\langle f_j, f_k\rangle_0)_{j,k} = \left(\begin{smallmatrix} -1_4 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and hence $G_\mathbf{R} \simeq O(1, 4, \mathbf{R})$. Furthermore, for $v \in \mathbf{S}^2$, for the Hodge decomposition $H_{0,\mathbf{C}} = \bigoplus_{p,q} H^{p,q}_{\mathbf{r}(v)}$ corresponding to $\mathbf{r}(v)$, $(f_j)_{1\le j\le 4}$ is a $\mathbf{C}$-basis of $H^{2,0}_{\mathbf{r}(v)} \oplus H^{0,2}_{\mathbf{r}(v)}$, f_5 is a $\mathbf{C}$-basis of $H^{1,1}_{\mathbf{r}(v)}$, and $((f_j, f_k)_{\mathbf{r}(v)})_{j,k} = 1_5$. Hence the maximal compact subgroup $K_{\mathbf{r}(v)}$ is $O(4, \mathbf{R}) \times O(1, \mathbf{R})$ for the basis $(f_j)_{1\le j\le 5}$ and is independent of v. For this basis, the isotropy subgroup $K'_{\mathbf{r}(e_2)}$ is the image of

$$U(2) \times O(1, \mathbf{R}) \to O(4, \mathbf{R}) \times O(1, \mathbf{R}), \quad (A+iB) \times (\pm 1) \mapsto \begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

We have $\dim_\mathbf{R}(K_{\mathbf{r}(v)}) = 6 > \dim_\mathbf{R}(K'_{\mathbf{r}(v)}) = 4$. (This shows, following 0.3.3, that D is not a symmetric Hermitian domain.)

0.3.5

Now we consider the moduli of polarized Hodge structures. We consider what functors D and $\Gamma\backslash D$, for torsion-free subgroups Γ of $G_\mathbf{Z}$, represent as analytic spaces. That is, we ask what are $\mathrm{Mor}_\mathcal{A}(\ , D)$ and $\mathrm{Mor}_\mathcal{A}(\ , \Gamma\backslash D)$ for the category $\mathcal{A}$ of analytic spaces.

We consider $\mathrm{Mor}_\mathcal{A}(\ , D)$ first.

For an analytic space X, a morphism $X \to D$ is identified with a decreasing filtration $F = (F^p)_{p\in\mathbf{Z}}$ on the $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_\mathbf{Z} H_0$ having the following properties (1) and (2).

(1) $F^p = \mathcal{O}_X \otimes_\mathbf{Z} H_0$ for $p \ll 0$, $F^p = 0$ for $p \gg 0$, and the $\mathcal{O}_X$-modules F^p/F^{p+1} are locally free for all $p \in \mathbf{Z}$.

(2) For any $x \in X$, the fiber $(H_0, \langle\ ,\ \rangle_0, F(x))$ is a PH of weight w and of Hodge type $(h^{p,q})_{p,q}$.

Here the object $(\mathcal{O}_X \otimes_{\mathbf{Z}} H_0, F)$ need not satisfy the Griffiths transversality (3) in Section 0.1.10. Corresponding to the identity morphism $D \to D$, there is a universal Hodge filtration F on $\mathcal{O}_D \otimes_{\mathbf{Z}} H_0$, but this F need not satisfy the Griffiths transversality. Hence, for an analytic space X, we will consider an object $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ which satisfies all conditions of variation of polarized Hodge structure in 0.1.10 except the Griffiths transversality (3). The correct name of such an object might be "the analytic family of polarized Hodge structures parametrized by X without the assumption of Griffiths transversality," but in this book, for simplicity, we will call this object just a *polarized Hodge structure on* X, or, simply, a PH on X.

For an analytic space X, by a PH on X of type $\Phi_0 = \big(w, (h^{p,q})_{p,q}, H_0, \langle\ ,\ \rangle_0\big)$, we mean a PH on X of weight w and of Hodge type $(h^{p,q})_{p,q}$ endowed with an isomorphism $(H_{\mathbf{Z}}, \langle\ ,\ \rangle) \simeq (H_0, \langle\ ,\ \rangle_0)$ of local systems on X. Here $(H_0, \langle\ ,\ \rangle_0)$ is considered as a constant sheaf on X. Let $\underline{\mathrm{PH}}_{\Phi_0}(X)$ be the set of all isomorphism classes of PH on X of type Φ_0. The above interpretation of $\mathrm{Mor}_{\mathcal{A}}(\ , D)$ is rewritten as follows.

LEMMA 0.3.6 *We have an isomorphism* $\underline{\mathrm{PH}}_{\Phi_0} \simeq \mathrm{Mor}_{\mathcal{A}}(\ , D)$ *of functors from* $\mathcal{A}$ *to* (Sets).

If $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ is a PH on X of type Φ_0 and $\varphi : X \to D$ is the corresponding morphism, $\varphi(x) \in D$ for $x \in X$ is nothing but the fiber $F(x)$ of F at x regarded as a filtration of $H_{0,\mathbf{C}}$ via the endowed isomorphism $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle) \simeq (H_0, \langle\ ,\ \rangle_0)$.

Let Γ be a torsion-free subgroup of $G_{\mathbf{Z}}$. Then $\Gamma\backslash D$ is an analytic manifold. Let X be an analytic space and let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a PH on X. By a Γ*-level structure* of H, we mean a global section of the sheaf

$$\Gamma\backslash \underline{\mathrm{Isom}}((H_{\mathbf{Z}}, \langle\ ,\ \rangle), (H_0, \langle,\rangle_0)),$$

on X, where $(H_0, \langle\ ,\ \rangle_0)$ is considered as a constant sheaf on X.

A level structure appears as follows. Let X be a connected analytic space and let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a PH on X of weight w and of Hodge type $(h^{p,q})$, let $x \in X$, and define $(H_0, \langle\ ,\ \rangle_0)$ to be the stalk $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x)$. Assume Γ contains the image of $\pi_1(X, x) \to G_{\mathbf{Z}}$. Then H has a unique Γ-level structure μ such that the germ μ_x is the germ of the identity map of H_0 modulo Γ.

Let $\Phi_1 = \big(w, (h^{p,q})_{p,q}, H_0, \langle\ ,\ \rangle_0, \Gamma\big)$. For an analytic space X, by a PH on X of type Φ_1, we mean a PH on X of weight w and of Hodge type $(h^{p,q})_{p,q}$ endowed with a Γ-level structure. Let $\underline{\mathrm{PH}}_{\Phi_1}(X)$ be the set of all isomorphism classes of PH on X of type Φ_1.

LEMMA 0.3.7 *We have an isomorphism* $\underline{\mathrm{PH}}_{\Phi_1} \simeq \mathrm{Mor}_{\mathcal{A}}(\ , \Gamma\backslash D)$ *of functors from* $\mathcal{A}$ *to* (Sets).

This is deduced from Lemma 0.3.6 as follows. The isomorphism $\underline{\mathrm{PH}}_{\Phi_0} \simeq \mathrm{Mor}_{\mathcal{A}}(\ , D)$ in 0.3.6 preserves the actions of Γ. The functor $\mathrm{Mor}_{\mathcal{A}}(\ , \Gamma\backslash D) : \mathcal{A} \to$ (Sets) is identified with the quotient $\Gamma\backslash \mathrm{Mor}_{\mathcal{A}}(\ , D)$ in the category of sheaf functors, that is, for each object X of $\mathcal{A}$, $\mathrm{Mor}_{\mathcal{A}}(X, \Gamma\backslash D)$ coincides with the set of global sections of the sheaf on X associated with the presheaf $U \mapsto \Gamma\backslash \mathrm{Mor}_{\mathcal{A}}(U, D)$ on X (here U is an open set of X). Similarly, $\underline{\mathrm{PH}}_{\Phi_1}$ is identified with the quotient $\Gamma\backslash \underline{\mathrm{PH}}_{\Phi_0}$ in the category of sheaf functors. Hence $\underline{\mathrm{PH}}_{\Phi_1} \simeq \mathrm{Mor}_{\mathcal{A}}(\ , \Gamma\backslash D)$. □

There is a torsion-free subgroup of $G_{\mathbf{Z}}$ of finite index. More strongly, there is a neat subgroup of $G_{\mathbf{Z}}$ of finite index (see 0.4.1 below).

0.3.8

We emphasize again here that the moduli conditions for $\underline{\mathrm{PH}}_{\Phi_0}$ and $\underline{\mathrm{PH}}_{\Phi_1}$ do not contain the Griffiths transversality on X (0.1.10 (3)). Griffiths transversality should be understood as an important property of the period map, which is satisfied by period maps coming from geometry. Let X be an analytic manifold, let H be a PH on X of type Φ_0, and let $\varphi : X \to D$ be the morphism corrsponding to H (the period map of H). Then the following two conditions (1) and (2) are equivalent.

(1) H satisfies the Griffths transversality.
(2) The image of the morphism of tangent bundles $d\varphi : T_X \to T_D$ induced by φ is contained in the horizontal tangent bundle $T_D^h \subset T_D$.

Here the horizontal tangent bundle is defined by

$$\begin{aligned} T_D^h &= F^{-1}(\mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}}))/F^0(\mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}})) \\ &\subset T_D = \mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}})/F^0(\mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}})) \end{aligned}$$

where $\mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}}) = \{A \in \mathcal{E}nd_{\mathcal{O}}(H_{\mathcal{O}}) \mid \langle Ax, y\rangle + \langle x, Ay\rangle = 0 \ \ (\forall\, x, y \in H_{\mathcal{O}})\}$ and $F(\mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}}))$ is the filtration on $\mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}})$ induced by the universal Hodge filtration on $H_{\mathcal{O}} = \mathcal{O}_D \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ ([G2]; also [Sc]).

In particular, by applying this equivalence to the identity morphism of D, we see that the universal PH on D satisfies the Griffiths transversality if and only if the tangent bundle of D coincides with the horizontal tangent bundle. These equivalent conditions are satisfied by Example (ii) (and hence by Example (i)) but not by Example (iii) in 0.3.2. In Example (iii) in 0.3.2, T_D^h is a vector bundle of rank 2 whereas T_D is of rank 3.

0.3.9

Let $f : Y \to X$ be a projective, smooth morphism of analytic manifolds. Fix a polarization of Y over X. Let $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be the associated VPH (0.1.11). Assume that X is connected, fix a base point $x \in X$, and let $(H_0, \langle\ ,\ \rangle_0) := (H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x)$, and assume that $\Gamma := \mathrm{Image}(\pi_1(X, x) \to \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0)$ is torsion-free. Let $\varphi : X \to \Gamma\backslash D$ be the associated period map (0.3.7). For the differential $d\varphi$ of the period map φ, Griffiths obtained the following commutative diagram:

$$\begin{array}{ccc} T_X & \xrightarrow{\ d\varphi\ } & \varphi^* T^h_{\Gamma\backslash D} = \mathrm{gr}_F^{-1}\,\mathcal{E}nd_{\langle\,,\,\rangle}(H_{\mathcal{O}}) \\ {\scriptstyle \text{K-S}}\downarrow & & \cap\downarrow \\ R^1 f_* T_{Y/X} & \xrightarrow{\text{via coupling}} & \bigoplus_p \mathcal{H}om_{\mathcal{O}_X}(R^{m-p} f_*\Omega^p_{Y/X}, R^{m-p+1} f_*\Omega^{p-1}_{Y/X}) \end{array}$$

where $T^h_{\Gamma\backslash D}$ is the horizontal tangent bundle in the tangent bundle $T_{\Gamma\backslash D}$, K-S on the left vertical arrow means the Kodaira-Spencer map, and the right vertical arrow is the canonical map (for details, see [G2]). The bottom horizontal arrow is often more computable than the top horizontal arrow. This gives a geometric presentation of the differential of the period map.

0.4 TOROIDAL PARTIAL COMPACTIFICATIONS OF $\Gamma\backslash D$ AND MODULI OF PLH

We discuss how to add points at infinity to D to construct a kind of toroidal partial compactification $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$. Since "nilpotent orbits" appear in degenerations of Hodge structures, it is natural to add "nilpotent orbits" as points at infinity. We do this in 0.4.1–0.4.12. We describe the space $\Gamma\backslash D_\Sigma$ in 0.4.13–0.4.19. We then explain in 0.4.20–0.4.34 that this enlarged classifying space is a moduli space of "polarized logarithmic Hodge structures."

We fix $(w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ as in Section 0.3.

0.4.1

We say a subgroup Γ of $G_\mathbf{Z}$ is *neat* if, for each $\gamma \in \Gamma$, the subgroup of $\mathbf{C}^\times$ generated by all the eigenvalues of γ is torsion-free. It is known that there exists a neat subgroup of $G_\mathbf{Z}$ of finite index (cf. [B]). A neat subgroup of $G_\mathbf{Z}$ is, in particular, torsion-free.

Let Γ be a neat subgroup of $G_\mathbf{Z}$. We will construct some toroidal partial compactifications of $\Gamma\backslash D$ by adding "nilpotent orbits" as points at infinity.

Definition 0.4.2 *A subset σ of $\mathfrak{g}_\mathbf{R} = \mathrm{Lie}\, G_\mathbf{R}$ (0.3.3) is called a* nilpotent cone *if the following conditions* (1)–(3) *are satisfied.*

(1) $\sigma = (\mathbf{R}_{\geq 0})N_1 + \cdots + (\mathbf{R}_{\geq 0})N_n$ *for some* $n \geq 1$ *and for some* $N_1, \ldots, N_n \in \sigma$.

(2) *Any element of σ is nilpotent as an endomorphism of $H_{0,\mathbf{R}}$.*

(3) *$NN' = N'N$ for any $N, N' \in \sigma$ as endomorphisms of $H_{0,\mathbf{R}}$.*

A nilpotent cone is said to be rational *if we can take $N_1, \ldots, N_n \in \mathfrak{g}_\mathbf{Q}$ in* (1) *above.*

0.4.3

For a nilpotent cone σ, a *face* of σ is a nonempty subset τ of σ satisfying the following two conditions.

(1) If $x, y \in \tau$ and $a \in \mathbf{R}_{\geq 0}$, then $x + y, ax \in \tau$.

(2) If $x, y \in \sigma$ and $x + y \in \tau$, then $x, y \in \tau$.

One can show that a face of a nilpotent cone (resp. rational nilpotent cone) is a nilpotent cone (resp. rational nilpotent cone), and that a nilpotent cone has only finitely many faces.

For example, let $N_j \in \mathfrak{g}_\mathbf{R}$ $(1 \leq j \leq n)$ be mutually commutative nilpotent elements that are linearly independent over $\mathbf{R}$. Then $\sigma := \sum_{j=1}^n (\mathbf{R}_{\geq 0})N_j$ is a nilpotent cone, and the faces of σ are $\sum_{j\in J}(\mathbf{R}_{\geq 0})N_j$ for finite subsets J of $\{1, \ldots, n\}$.

Definition 0.4.4 *A* fan in $\mathfrak{g}_{\mathbf{Q}}$ *is a nonempty set* Σ *of rational nilpotent cones in* $\mathfrak{g}_{\mathbf{R}}$ *satisfying the following three conditions:*

(1) *If* $\sigma \in \Sigma$, *any face of* σ *belongs to* Σ.
(2) *If* $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ *is a face of* σ *and of* σ'.
(3) *Any* $\sigma \in \Sigma$ *is sharp. That is,* $\sigma \cap (-\sigma) = \{0\}$.

0.4.5

Examples. (i) Let

$$\Xi := \{(\mathbf{R}_{\geq 0})N \mid N \text{ is a nilpotent element of } \mathfrak{g}_{\mathbf{Q}}\}.$$

Then Ξ is a fan in $\mathfrak{g}_{\mathbf{Q}}$.

(ii) Let $\sigma \in \mathfrak{g}_{\mathbf{R}}$ be a sharp rational nilpotent cone. Then the set of all faces of σ is a fan in $\mathfrak{g}_{\mathbf{Q}}$.

0.4.6

Nilpotent orbits. Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}} = \operatorname{Lie} G_{\mathbf{R}}$. For $R = \mathbf{R}, \mathbf{C}$, we denote by σ_R the R-linear span of σ in $\mathfrak{g}_R$.

Definition 0.4.7 *Let* $\sigma = \sum_{1 \leq j \leq r} (\mathbf{R}_{\geq 0}) N_j$ *be a nilpotent cone. A subset* Z *of* $\check{D}$ *is called a* σ-nilpotent orbit *if there is* $F \in \check{D}$ *which satisfies* $Z = \exp(\sigma_{\mathbf{C}})F$ *and satisfies the following two conditions.*

(1) $NF^p \subset F^{p-1}$ $(\forall p, \forall N \in \sigma)$.
(2) $\exp(\sum_{1 \leq j \leq r} z_j N_j) F \in D$ *if* $z_j \in \mathbf{C}$ *and* $\operatorname{Im}(z_j) \gg 0$.

In this case, the pair (σ, Z) *is called a* nilpotent orbit.

Definition 0.4.8 *Let* Σ *be a fan in* $\mathfrak{g}_{\mathbf{Q}}$. *We define the* space D_Σ of nilpotent orbits in the directions in Σ *by*

$$D_\Sigma := \{(\sigma, Z) \mid \sigma \in \Sigma,\ Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit}\}.$$

Note that we have the inclusion map

$$D \hookrightarrow D_\Sigma, \quad F \mapsto (\{0\}, \{F\}).$$

For a sharp rational nilpotent cone σ in $\mathfrak{g}_{\mathbf{R}}$, we denote $D_{\{\text{face of } \sigma\}}$ by D_σ. Then, for a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$, we have $D_\Sigma = \cup_{\sigma \in \Sigma} D_\sigma$.

0.4.9

Upper half plane (continued). Let Ξ be as in 0.4.5. Then $D_\Xi = D \cup \mathbf{P}^1(\mathbf{Q})$. This is explained as follows. For $a \in \mathbf{P}^1(\mathbf{Q})$, let V_a be the one-dimensional $\mathbf{R}$-vector subspace of $H_{0,\mathbf{R}}$ corresponding to a, that is, $V_a = \mathbf{R}(ae_1 + e_2)$ if $a \in \mathbf{Q}$, and $V_\infty = \mathbf{R}e_1$.

For $a \in \mathbf{P}^1(\mathbf{Q})$, define a sharp rational nilpotent cone $\sigma_a \in \Xi$ by

$$\sigma_a = \{N \in \mathfrak{g}_{\mathbf{R}} \mid N(H_{0,\mathbf{R}}) \subset V_a,\ N(V_a) = \{0\},\\ \langle x, N(x)\rangle_0 \geq 0 \ \text{ for any } x \in H_{0,\mathbf{R}}\}.$$

We identify $a \in \mathbf{P}^1(\mathbf{Q})$ with the nilpotent orbit $(\sigma_a, Z_a) \in D_\Xi$ where $Z_a = \{F \in \check{D} \mid F^1 \neq V_{a,\mathbf{C}}\}$. For example,

$$\sigma_\infty = \begin{pmatrix} 0 & \mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix}, \qquad Z_\infty = \mathbf{C} \subset \mathbf{P}^1(\mathbf{C}) = \check{D}.$$

Definition 0.4.10 *Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$.*

(i) *We say Γ is* compatible with Σ *if the following condition* (1) *is satisfied.*

(1) *If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then* $\mathrm{Ad}(\gamma)(\sigma) \in \Sigma$. *Here,* $\mathrm{Ad}(\gamma)(\sigma) = \gamma\sigma\gamma^{-1}$. *Note that, if Γ is compatible with Σ, Γ acts on D_Σ by*

$$\gamma : (\sigma, Z) \mapsto (\mathrm{Ad}(\gamma)(\sigma), \gamma Z) \quad (\gamma \in \Gamma).$$

(ii) *We say Γ is* strongly compatible with Σ *if it is compatible with Σ and the following condition* (2) *is also satisfied. For $\sigma \in \Sigma$, define*

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma).$$

(2) *The cone σ is generated by* $\log \Gamma(\sigma)$, *that is, any element of σ can be written as a sum of a* $\log(\gamma)$ *($a \in \mathbf{R}_{\geq 0}$, $\gamma \in \Gamma(\sigma)$).*

Note that $\Gamma(\sigma)$ is a sharp fs monoid and $\Gamma(\sigma)^{\mathrm{gp}} = \Gamma \cap \exp(\sigma_{\mathbf{R}})$.

0.4.11

Example. If $\Sigma = \Xi$ in 0.4.5 and Γ is of finite index in $G_{\mathbf{Z}}$, then Γ is strongly compatible with Σ. If $\Sigma = \Xi$ and Γ is just a subgroup of $G_{\mathbf{Z}}$, it is compatible with Σ but is not necessarily strongly compatible with Σ.

0.4.12

Assume that Γ and Σ are strongly compatible.

In Chapter 3, we will define a topology of $\Gamma\backslash D_\Sigma$ for which $\Gamma\backslash D$ is a dense open subset of $\Gamma\backslash D_\Sigma$ and which has the following property. Let $(\sigma, Z) \in D_\Sigma$, $F \in D$, and $N_j \in \mathfrak{g}_{\mathbf{R}}$ $(1 \leq j \leq n)$, and assume $\sigma = \sum_{j=1}^{n} (\mathbf{R}_{\geq 0}) N_j$. Then

$$\left(\exp\left(\sum_{j=1}^{n} z_j N_j\right) F \bmod \Gamma\right)\\ \to ((\sigma, Z) \bmod \Gamma) \quad \text{if } z_j \in \mathbf{C} \text{ and } \mathrm{Im}(z_j) \to \infty\ (\forall j).$$

Furthermore, in Chapter 3, we will introduce on $\Gamma\backslash D_\Sigma$ a structure of a local ringed space over $\mathbf{C}$ and also a logarithmic structure. In 0.4.13 (resp. 0.4.18), we describe what this local ringed structure looks like in the cases of Examples (i) and (ii) (resp. Example (iii)) in 0.3.2. If Γ is neat, the logarithmic structure M of $\Gamma\backslash D_\Sigma$ is an fs logarithmic structure and has the form $M = \{f \in \mathcal{O} \mid f$ is invertible on $\Gamma\backslash D\} \overset{\alpha}{\hookrightarrow} \mathcal{O}$ for the structure sheaf of rings $\mathcal{O}$ of $\Gamma\backslash D_\Sigma$.

0.4.13

Examples. Upper half plane (continued). For

$$\Gamma = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix} \subset \mathrm{SL}(2, \mathbf{Z}), \quad \sigma = \sigma_\infty = \begin{pmatrix} 0 & \mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix},$$

we have a commutative diagram of analytic spaces

$$\begin{array}{ccc} \Delta^* & \simeq & \Gamma\backslash D \\ \cap & & \cap \\ \Delta & \simeq & \Gamma\backslash D_\sigma. \end{array}$$

Here the upper isomorphism sends $e^{2\pi i \tau} \in \Delta^*$ ($\tau \in \mathfrak{h} = D$) to ($\tau$ mod Γ); the lower isomorphism extends the upper isomorphism by sending $0 \in \Delta$ to the class of the nilpotent orbit ((σ_∞, $\mathbf{C}$) mod Γ).

Next we consider a subgroup Γ of $G_\mathbf{Z}$ of finite index. Let $\sigma = \sigma_\infty$. Let $n \geq 1$ and let $\Gamma = \Gamma(n)$ be the kernel of $\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/n\mathbf{Z})$. Then

$$\Gamma(\sigma) = \begin{pmatrix} 1 & n\mathbf{N} \\ 0 & 1 \end{pmatrix}, \quad \Gamma(\sigma)^{\mathrm{gp}} = \Gamma \cap \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n\mathbf{Z} \\ 0 & 1 \end{pmatrix}.$$

Γ is neat if and only if $n \geq 3$. For $n \geq 3$, as is well known in the theory of modular curves, we have the following local description of $\Gamma\backslash D_\Xi$ at the boundary point (∞ mod Γ) in $\Gamma\backslash D_\Xi = \Gamma\backslash(\mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q}))$. We have

$$\Delta \xrightarrow{\sim} \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \xrightarrow{\text{local isom}} \Gamma\backslash D_\Xi,$$

where the first arrow is an isomorphism of analytic spaces which sends $q \in \Delta^*$ to (τ mod $\Gamma(\sigma)^{\mathrm{gp}}$) with $q = e^{2\pi i \tau/n}$, and sends $0 \in \Delta$ to ((σ_∞, $\mathbf{C}$) mod $\Gamma(\sigma)^{\mathrm{gp}}$), and the second arrow is the canonical projection and is locally an isomorphism of analytic spaces.

Upper half space (continued). We consider the case $g = 2$, i.e., $D = \mathfrak{h}_2$. Let U be the open set of $\mathbf{C}^3$ defined by

$$U = \{(q_1, q_2, a) \in \Delta^2 \times \mathbf{C} \mid \text{if } q_1 q_2 \neq 0, \\ \text{then } \log(|q_1|)\log(|q_2|) > (2\pi\, \mathrm{Im}(a))^2\}.$$

Let $N_1, N_2 \in \mathfrak{g}_\mathbf{R} = \mathfrak{sp}(2, \mathbf{R})$ be the nilpotent elements defined by

$$N_1(e_3) = e_1, N_1(e_j) = 0 \text{ for } j \neq 3, \quad N_2(e_4) = e_2, N_2(e_j) = 0 \text{ for } j \neq 4.$$

Then

$$\exp(z_1 N_1 + z_2 N_2) F \begin{pmatrix} a & b \\ b & c \end{pmatrix} = F \begin{pmatrix} a + z_1 & b \\ b & c + z_2 \end{pmatrix}$$

for any $z_1, z_2, a, b, c \in \mathbf{C}$. Let

$$\sigma = (\mathbf{R}_{\geq 0})N_1 + (\mathbf{R}_{\geq 0})N_2 \subset \mathfrak{g}_\mathbf{R}.$$

For $\Gamma = \exp(\mathbf{Z}N_1 + \mathbf{Z}N_2) = 1 + \mathbf{Z}N_1 + \mathbf{Z}N_2$, Γ is strongly compatible with the fan $\{\{0\}, (\mathbf{R}_{\geq 0})N_1, (\mathbf{R}_{\geq 0})N_2, \sigma\}$ of all faces of σ, and we have the following isomorphism of analytic spaces. For $q_j = e^{2\pi i \tau_j}$,

$$\Delta^2 \times \mathbf{C} \supset U \simeq \Gamma \backslash D_\sigma,$$

$$(q_1, q_2, a) \mapsto F\begin{pmatrix} \tau_1 & a \\ a & \tau_2 \end{pmatrix} \bmod \Gamma \quad (q_1 q_2 \neq 0),$$

$$(0, q_2, a) \mapsto \left((\mathbf{R}_{\geq 0})N_1, F\begin{pmatrix} \mathbf{C} & a \\ a & \tau_2 \end{pmatrix}\right) \bmod \Gamma \quad (q_2 \neq 0),$$

$$(q_1, 0, a) \mapsto \left(\mathbf{R}_{\geq 0}N_2, F\begin{pmatrix} \tau_1 & a \\ a & \mathbf{C} \end{pmatrix}\right) \bmod \Gamma \quad (q_1 \neq 0),$$

$$(0, 0, a) \mapsto \left(\sigma, F\begin{pmatrix} \mathbf{C} & a \\ a & \mathbf{C} \end{pmatrix}\right) \bmod \Gamma.$$

More generally, for any strongly compatible pair (Γ, Σ) such that $\sigma \in \Sigma$ and Γ is neat and such that $\Gamma(\sigma)^{\text{gp}} = \exp(\mathbf{Z}N_1 + \mathbf{Z}N_2)$, the above isomorphism $U \simeq \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ induces a morphism of analytic spaces $U \to \Gamma \backslash D_\Sigma$ which is locally an isomorphism.

0.4.14

We say that we are in the classical situation if D is a symmetric Hermitian domain and the tangent bundle of D coincides with the horizontal tangent bundle. Example (ii) (and hence Example (i)) in 0.3.2 belongs to the classical situation, but Example (iii) does not.

In the classical situation, $\Gamma \backslash D_\Sigma$ is a toroidal partial compactification as constructed by Mumford et al. [AMRT]. Under the assumption $D \neq \emptyset$, the classical situation is listed as follows.

Case (1) $w = 2t + 1$, $h^{t+1,t} = h^{t,t+1} \geq 0$, $h^{p,q} = 0$ for other (p, q).
Case (2) $w = 2t$, $h^{t+1,t-1} = h^{t-1,t+1} \leq 1$, $h^{t,t} \geq 0$, $h^{p,q} = 0$ for other (p, q).

For general D, the space $\Gamma \backslash D_\Sigma$ is not necessarily a complex analytic space because it may have "slits" caused by "Griffiths transversality at the boundary". But still it has a kind of complex structure, period maps can be extended to $\Gamma \backslash D_\Sigma$, and infinitesimal calculus can be performed nicely. This was first observed by the simplest example in [U2].

In the terminology of this book, $\Gamma \backslash D_\Sigma$ is a "logarithmic manifold," as explained in 0.4.15–0.4.17 below.

0.4.15

Strong topology. The underlying local ringed space over $\mathbf{C}$ of $\Gamma \backslash D_\Sigma$ is not necessarily an analytic space in general. Sometimes, it can be something like

(1) $S := \{(x, y) \in \mathbf{C}^2 \mid x \neq 0\} \cup \{(0, 0)\} = \{(x, y) \in \mathbf{C}^2 \mid \text{if } x = 0, \text{ then } y = 0\}$

endowed with a topology that is stronger than the topology as a subspace of $\mathbf{C}^2$, called the "strong topology."

Let Z be an analytic space and S be a subset of Z. A subset U of S is open in the *strong topology of S in Z* if and only if, for any analytic space Y and any morphism $\lambda : Y \to Z$ such that $\lambda(Y) \subset S$, $\lambda^{-1}(U)$ is open on Y.

If S is a locally closed analytic subspace of Z, the strong topology coincides with the topology as a subspace of Z. However the strong topology of the set S in (1) is stronger than the topology as a subspace of $\mathbf{C}^2$. For example,

(2) Let $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a map such that for each integer $n \geq 1$, there exists $\varepsilon_n > 0$ for which $f(s) \leq s^n$ if $0 < s < \varepsilon_n$. (An example of $f(s)$ is $e^{-1/s}$.) Then, if $s > 0$ and $s \to 0$, $(f(s), s)$ converges to $(0, 0)$ for the topology of S as a subspace of $\mathbf{C}^2$, but it does not converge for the strong topology (see 3.1.3). (Roughly speaking, $(f(s), s)$ runs too near to the "bad line" $\{0\} \times \mathbf{C}$.)

0.4.16

Categories $\mathcal{A}$, $\mathcal{A}(\log)$, $\mathcal{B}$, $\mathcal{B}(\log)$. We define the categories

$$\mathcal{A} \subset \mathcal{B}, \quad \mathcal{A}(\log) \subset \mathcal{B}(\log)$$

as follows (cf. 3.2.4).

We denote by

$$\mathcal{A}, \quad \mathcal{A}(\log),$$

the category of analytic spaces and the category of fs logarithmic analytic spaces, respectively.

Let $\mathcal{B}$ be the category of local ringed spaces X over $\mathbf{C}$ (over $\mathbf{C}$ means that $\mathcal{O}_X$ is a $\mathbf{C}$-algebra) having the following property: X has an open covering $(U_\lambda)_\lambda$ such that, for each λ, there exists an isomorphism $U_\lambda \simeq S_\lambda$ of local ringed spaces over $\mathbf{C}$ for some subset S_λ of an analytic space Z_λ, where S_λ is endowed with the strong topology in Z_λ and with the inverse image of $\mathcal{O}_{Z_\lambda}$.

Let $\mathcal{B}(\log)$ be the category of objects of $\mathcal{B}$ endowed with an fs logarithmic structure.

0.4.17

Logarithmic manifolds. Our space $\Gamma\backslash D_\Sigma$ is a very special object in $\mathcal{B}(\log)$, called a "logarithmic manifold" (cf. Section 3.5).

We first describe the idea of the logarithmic manifold by using the example $S \subset \mathbf{C}^2$ in 0.4.15 (1). Let $Z = \mathbf{C}^2$ with coordinate functions x, y, and endow Z with the logarithmic structure M_Z associated with the divisor "$x = 0$." Then the sheaf ω^1_Z of logarithmic differential forms on Z (= the sheaf of differential forms with logarithmic poles along $x = 0$) is a free $\mathcal{O}_Z$-module with basis $(d\log(x), dy)$. For each $z \in Z$, let ω^1_z be the module of logarithmic differential forms on the point z which is regarded as an fs logarithmic analytic space endowed with the ring $\mathbf{C}$ and with the inverse image of M_Z. Then, if z does not belong to the part $x = 0$ of Z, z is just the usual point Spec(**C**) with the trivial logarithmic structure $\mathbf{C}^\times$, and $\omega_z^1 = 0$. If z is in the part $x = 0$, z is a point Spec(**C**) with the induced logarithmic structure $M_z = \bigsqcup_{n\geq 0} \mathbf{C}^\times x^n \simeq \mathbf{C}^\times \times \mathbf{N}$, and hence ω_z^1 is a one-dimensional **C**-vector space generated by $d\log(x)$. Thus ω_z^1 is not equal to the fiber of ω_Z^1 at z which is a two-dimensional **C**-vector space with basis $(d\log(x), dy)$. Now the above set S has a presentation

$$S = \{z \in Z \mid \text{the image of } y d\log(x) \text{ in } \omega_z^1 \text{ is zero}\}. \qquad (1)$$

Recall that zeros of a holomorphic function on Z form a closed analytic subset of Z. Here we discovered that S is the set of "zeros" of the differential form $yd\log(x)$ on Z, but the meaning of "zero" is not that the image of $yd\log(x)$ in the fiber of ω_Z^1 is zero (the latter "zeros" form the closed analytic subset $y = 0$ of Z). The set "zeros in the new sense" of a differential form with logarithmic poles is the idea of a "logarithmic manifold."

The precise definition is as follows (cf. 3.5.7). By a *logarithmic manifold*, we mean a local ringed space over **C** endowed with an fs logarithmic structure which has an open covering $(U_\lambda)_\lambda$ with the following property: For each λ, there exist a logarithmically smooth fs logarithmic analytic space Z_λ, a finite subset I_λ of $\Gamma(Z_\lambda, \omega_{Z_\lambda}^1)$, and an isomorphism of local ringed spaces over **C** with logarithmic structures between U_λ and an open subset of

$$S_\lambda = \{z \in Z_\lambda \mid \text{the image of } I_\lambda \text{ in } \omega_z^1 \text{ is zero}\}, \qquad (2)$$

where S_λ is endowed with the strong topology in Z_λ and with the inverse images of $\mathcal{O}_{Z_\lambda}$ and M_{Z_λ}.

0.4.18

Example with $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ (continued). Let Γ be a neat subgroup of $G_\mathbf{Z}$ of finite index. We give a local description of the space $\Gamma\backslash D_\Xi$ and observe that this space has a slit.

Fix $v \in \mathbf{S}^2 \cap (\sum_{j=1}^3 \mathbf{Q}e_j)$ and fix a nonzero element v' of $\sum_{j=1}^3 \mathbf{R}e_j$ which is linearly independent of v over **R**. Define $\ell \in \mathbf{Q}_{>0}$ by $\Gamma \cap \exp(\mathbf{Q}N_v) = \exp(\ell\mathbf{Z}N_v)$. Let

$$U = \{(q, a, z) \in \mathbf{C}^2 \times Q \mid (q, a, z) \text{ satisfies (1) and (2) below}\}.$$

(1) If $q \neq 0$ and $q = e^{2\pi i\tau/\ell}$, then $\langle \mathrm{Im}(\tau)v + \mathrm{Im}(a)v', \theta(z)\rangle_0 < 0$ (for θ, see 0.3.4).
(2) If $q = 0$, then $\theta(z) = v$.

Endow U with the strong topology in $Z := \mathbf{C}^2 \times Q$, with the sheaf of rings $\mathcal{O}_Z|_U$, and with the inverse image of the logarithmic structure of Z associated to the divisor $q = 0$. By the above condition (2), U has a slit and it is not an analytic space. But U is a logarithmic manifold.

Let $\sigma = (\mathbf{R}_{\geq 0})N_v$. We have morphisms of local ringed spaces over **C** with logarithmic structures

$$U \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Xi$$

which are locally isomorphisms, and the left of which sends $(q, a, z) \in U$ to the class of $\exp(\tau N_v + aN_{v'})F(z) = \exp(\tau N_v)\exp(aN_{v'})F(z)$ if $q \neq 0$ and $q = e^{2\pi i\tau/\ell}$, and to the class of $(\sigma, \exp(\sigma_{\mathbf{C}})\exp(aN_{v'})F(z))$ if $q = 0$ and $\theta(z) = v$.

The above local description of $\Gamma\backslash D_\Xi$ is obtained from Proposition 12.2.5. Note that e_1, e_2 in Section 12.2 are e_4, e_5 here, respectively. The slit in $\Gamma\backslash D_\Xi$ corresponding to the slit in U appears by "small Griffiths transversality," as explained in 0.4.29 below.

THEOREM 0.4.19 (cf. Theorem A in Section 4.1) *Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a neat subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ.*

(i) *Then $\Gamma\backslash D_\Sigma$ is a logarithmic manifold. It is a Hausdorff space.*
(ii) *For any $\sigma \in \Sigma$, the canonical projection $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ is locally an isomorphism of logarithmic manifolds.*

The Hausdorff property of $\Gamma\backslash D_\Sigma$ is by virtue of the strong topology. Proposition 12.3.6 gives an example such that $\Gamma\backslash D_\Sigma$ is not Hausdorff if we use a naive topology that is weaker than the strong topology.

Now we consider a polarized logarithmic Hodge structure and its moduli.

0.4.20

For an object X of $\mathcal{B}(\log)$, a ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ is defined just as in the case of fs logarithmic analytic spaces (see Section 2.2). It is described locally as follows. Assume that the logarithmic structure of X is induced from a homomorphism $\mathcal{S} \to \mathcal{O}_X$ with $\mathcal{S}$ an fs monoid. Let $Z = \mathrm{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$. Then

$$X^{\log} = X \times_Z Z^{\log} = X \times_{\mathrm{Hom}(\mathcal{S}, \mathbf{C}^{\mathrm{mult}})} \mathrm{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\mathrm{mult}} \times \mathbf{S}^1), \quad \mathcal{O}_X^{\log} = \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Z^{\log}.$$

0.4.21

Let X be an object of $\mathcal{B}(\log)$. A *prepolarized logarithmic Hodge structure (pre-PLH) on X of weight w* is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a locally constant sheaf $H_{\mathbf{Z}}$ of free $\mathbf{Z}$-modules of finite rank on $X^{\log}$, a bilinear form $\langle\ ,\ \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$, and a decreasing filtration F on $\mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ by $\mathcal{O}_X^{\log}$-submodules which satisfy the following condition (1).

(1) There exist a locally free $\mathcal{O}_X$-module $\mathcal{M}$ and a decreasing filtration $(\mathcal{M}^p)_{p\in\mathbf{Z}}$ by $\mathcal{O}_X$-submodules of $\mathcal{M}$ such that $\mathcal{M}^p = \mathcal{M}$ for $p \ll 0$, $\mathcal{M}^p = 0$ for $p \gg 0$, and $\mathcal{M}^p/\mathcal{M}^{p+1}$ are locally free for all p, and such that $\mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{M})$ and $F^p = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{M}^p)$ for all p. Furthermore, the annihilator of F^p with respect to $\langle\ ,\ \rangle$ coincides with F^{w+1-p} for any p.

We give two remarks concerning pre-PLH.

(i) If $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ is a pre-PLH on X, the $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{M}^p$ in (1) above are determined by H as

$$\mathcal{M} = \tau_*(\mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}), \quad \mathcal{M}^p = \tau_*(F^p).$$

This follows from Proposition 2.2.10.

(ii) (Proposition 2.3.3 (ii)) If $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ is a pre-PLH on X, then for any $x \in X$ and for any $y \in X^{\log}$ lying over x, the action of $\pi_1(x^{\log})$ on $H_{\mathbf{Z},y}$ is unipotent.

0.4.22

Let X be an object of $\mathcal{B}(\log)$ and let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a pre-PLH on X. We call H a *polarized logarithmic Hodge structure (PLH) on* X if for each $x \in X$, it satisfies the following two conditions.

Positivity on x. Let y be any element of $X^{\log}$ lying over x. Take a finite family f_j $(1 \leq j \leq n)$ of elements of $M_{X,x}$ which do not belong to $\mathcal{O}_{X,x}^\times$ such that the monoid $(M_X/\mathcal{O}_X^\times)_x$ is generated by the images of f_j. Then, if $s \in \mathrm{sp}(y)$ and if $\exp(s(\log(f_j)))$ are sufficiently near to 0, $(H_{\mathbf{Z},y}, \langle\ ,\ \rangle_y, F(s))$ is a polarized Hodge structure.

Griffiths transversality on x. $(d \otimes 1_{H_{\mathbf{Z}}})(F^p|_{x^{\log}}) \subset \omega_x^{1,\log} \otimes_{\mathcal{O}_x^{\log}} (F^{p-1}|_{x^{\log}})$ for any p.

Here $\mathcal{O}_x^{\log}$ and $\omega_x^{1,\log}$ are those of the point $x = \mathrm{Spec}(\mathbf{C})$ endowed with the inverse image of M_X, and $F|_{x^{\log}}$ denotes the module-theoretic inverse image of F under the morphism of ringed spaces $(x^{\log}, \mathcal{O}_x^{\log}) \to (X^{\log}, \mathcal{O}_X^{\log})$.

In other words, PLH is a pre-PLH whose pullback to the fs logarithmic point x for any $x \in X$ satisfies the conditions in 0.2.19 for an LVPH (we take x as X in 0.2.19 here). Although x is not logarithmically smooth unless the logarithmic structue of x is trivial, the conditions in 0.2.19 make sense when we replace X there by x.

In the case that X is a logarithmically smooth, fs logarithmic analytic space, the validity of the above Griffiths transversality on x for all $x \in X$ (we call this the *small Griffiths transversality*) is much weaker than the Griffiths transversality (3) in 0.2.19 (we call this the *big Griffiths transversality*). In fact if the logarithmic structure of X is trivial (that is, $M_X = \mathcal{O}_X^\times$), $\omega_x^{1,\log} = 0$ for any point $x \in X$, and hence the small Griffiths transversality is an empty condition. Hence an LVPH on X is a PLH on X, but a PLH on X is not necessarily an LVPH on X.

0.4.23

In 0.4.23–0.4.25, we will see that the notion of a "nilpotent orbit" is nothing but a "PLH on an fs logarithmic point".

Let x be an fs logarithmic point. Then $x^{\log} \simeq \mathrm{Hom}(M_x^{\mathrm{gp}}/\mathcal{O}_x^\times, \mathbf{S}^1)$ and hence $\pi_1(x^{\log}) \simeq \mathrm{Hom}(M_x^{\mathrm{gp}}/\mathcal{O}_x^\times, \mathbf{Z})$. Let $\pi_1^+(x^{\log}) \subset \pi_1(x^{\log})$ be the part corresponding to the part $\mathrm{Hom}(M_x/\mathcal{O}_x^\times, \mathbf{N}) \subset \mathrm{Hom}(M_x^{\mathrm{gp}}/\mathcal{O}_x^\times, \mathbf{Z})$. Then $\pi_1^+(x^{\log})$ is an fs monoid.

PROPOSITION 0.4.24 (cf. Propositions 2.5.1, and 2.5.5) *Let x be an fs logarithmic point and let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a pre-PLH on x. Let $y \in x^{\log}$.*

(i) *Let $h_j \in \mathrm{Hom}(M_x^{\mathrm{gp}}/\mathcal{O}_x^\times, \mathbf{Z})$ $(1 \le j \le n)$, let $\gamma_j \in \pi_1(x^{\log})$ be the element corresponding to h_j, and let $N_j : H_{\mathbf{Q},y} \to H_{\mathbf{Q},y}$ be the logarithm of the (unipotent) action of γ_j on $H_{\mathbf{Q},y}$. Let $z_j \in \mathbf{C}$ $(1 \le j \le n)$, $s_0 \in \mathrm{sp}(y)$, and let s be the element of $\mathrm{sp}(y)$ characterized by*

$$s((2\pi i)^{-1}\log(f)) - s_0((2\pi i)^{-1}\log(f)) = \sum_{j=1}^{n} z_j h_j(f) \quad \textit{for any} \quad f \in M_x^{\mathrm{gp}}$$

(see 0.2.17). Then

$$F(s) = \exp\left(\sum_{j=1}^{n} z_j N_j\right) F(s_0).$$

(ii) *Let $(\gamma_j)_{1\le j\le n}$ be a finite family of generators of the monoid $\pi_1^+(x^{\log})$ and let $N_j : H_{\mathbf{Q},y} \to H_{\mathbf{Q},y}$ be the logarithm of the action of γ_j on $H_{\mathbf{Q},y}$. Fix $s \in \mathrm{sp}(y)$. Then H satisfies the positivity on x in 0.4.22 if and only if the following condition is satisfied:*

$$\left(H_{\mathbf{Z},y}, \langle\ ,\ \rangle_y, \exp\left(\sum_{j=1}^{n} z_j N_j\right) F(s)\right) \textit{ is a PH if } \mathrm{Im}(z_j) \gg 0\ (\forall j).$$

(iii) *Let N_j $(1 \le j \le n)$ be as in (ii) and let $s \in \mathrm{sp}(y)$. Then H satisfies the Griffiths transversality on x in 0.4.22 if and only if*

$$N_j F^p(s) \subset F^{p-1}(s) \textit{ for any } j \textit{ and } p.$$

0.4.25

With the notation in (ii) in 0.4.24, let $\sigma = \sum_{j=1}^{n} (\mathbf{R}_{\ge 0}) N_j$. By (i) of 0.4.24, $\{F(s)\}_{s \in \mathrm{sp}(y)}$ is an $\exp(\sigma_{\mathbf{C}})$-orbit. By (ii) and (iii) of 0.4.24, this $\exp(\sigma_{\mathbf{C}})$-orbit is a σ-nilpotent orbit if and only if H is a PLH on x. In other words,

(a PLH on an fs logarithmic point) = (a nilpotent orbit).

Hence if H is a PLH on an object X of $\mathcal{B}(\log)$, for each $x \in X$, the pullback $H(x)$ of H to the fs logarithmic point x is regarded as a nilpotent orbit. This fact is presented in schema (2) in Introduction.

0.4.26

We generalize the functor $\underline{\mathrm{PH}}_{\Phi_1} : \mathcal{A} \to (\mathrm{Sets})$ in 0.3.7 to the logarithmic case.

Let Γ be a neat subgroup of $G_{\mathbf{Z}}$ and let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$. Assume that Γ and Σ are strongly compatible (0.4.10). Let $\Phi = \big(w, (h^{p,q})_{p,q}, H_0, \langle\ ,\ \rangle_0, \Gamma, \Sigma\big)$.

For a PLH $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ on X, by a Γ-level structure on H, we mean a global section of the sheaf $\Gamma\backslash \underline{\mathrm{Isom}}((H_{\mathbf{Z}}, \langle\ ,\ \rangle), (H_0, \langle\ ,\ \rangle_0))$ on $X^{\log}$.

We define a functor $\underline{\mathrm{PLH}}_\Phi : \mathcal{B}(\log) \to (\mathrm{Sets})$ as follows. For $X \in \mathcal{B}(\log)$, let $\underline{\mathrm{PLH}}_\Phi(X)$ be the set of all isomorphism classes of PLH on X of type $(w, (h^{p,q})_{p,q\in\mathbf{Z}})$ endowed with a Γ-level structure μ satisfying the following condition:

For any $x \in X$, any $y \in x^{\log}$ and a lifting $\tilde{\mu}_y : (H_{\mathbf{Z},y}, \langle\ ,\ \rangle_y) \xrightarrow{\sim} (H_0, \langle\ ,\ \rangle_0)$ of the germ of μ at y, we have the following (1) and (2).

(1) There exists $\sigma \in \Sigma$ such that

$$\mathrm{Image}\left(\pi_1^+(x^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},y}) \xrightarrow{\text{by } \tilde{\mu}_y} Aut(H_0)\right) \subset \exp(\sigma).$$

(2) For the smallest such $\sigma \in \Sigma$ and for $s \in \mathrm{sp}(y)$, $\exp(\sigma_{\mathbf{C}})\tilde{\mu}_y(F(s))$ is a σ-nilpotent orbit.

Note that the Griffiths transversality that is required for PLH is only the Griffiths transversality. But this definition fits well the moduli problem.

Now the precise form of Theorem for Subject I in Introduction is stated as follows.

Theorem 0.4.27 (cf. Theorem B in Section 4.2) *Let Γ be a neat subgroup of $G_{\mathbf{Z}}$ and let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$. Assume that Γ and Σ are strongly compatible.*

(i) *The logarithmic manifold $\Gamma\backslash D_\Sigma$ represents the functor $\underline{\mathrm{PLH}}_\Phi : \mathcal{B}(\log) \to$ (Sets), that is, there exists an isomorphism $\varphi : \underline{\mathrm{PLH}}_\Phi \xrightarrow{\sim} \mathrm{Mor}(\ ,\Gamma\backslash D_\Sigma)$.*

(ii) *For any local ringed space Z over $\mathbf{C}$ with a logarithmic structure (which need not be fs) and for any morphism of functors $h : \underline{\mathrm{PLH}}_\Phi|_{\mathcal{A}(\log)} \to \mathrm{Mor}(\ ,Z)|_{\mathcal{A}(\log)}$ (where $|_{\mathcal{A}(\log)}$ denotes the restrictions to $\mathcal{A}(\log)$), there exists a unique morphism $f : \Gamma\backslash D_\Sigma \to Z$ such that $h = (f \circ \varphi)|_{\mathcal{A}(\log)}$, where f is regarded as a morphism $\mathrm{Mor}(\ ,\Gamma\backslash D_\Sigma) \to \mathrm{Mor}(\ ,Z)$.*

0.4.28

For $X \in \mathcal{B}(\log)$ and for $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F, \mu) \in \underline{\mathrm{PLH}}_\Phi(X)$, the morphism $\varphi_H : X \to \Gamma\backslash D_\Sigma$ corresponding to H is called the associated *period map*, which is set-theoretically given by sending $x \in X$ to the Γ-equivalence class of the nilpotent orbit $(\sigma, \exp(\sigma_{\mathbf{C}})\tilde{\mu}_y(F(s)))$ at x (which is independent of the choices of y, $\tilde{\mu}_y$, and s) in 0.4.26 (2). Note that this map is an extension of the classical period map. If U is an open set on which the logarithmic structure of X is trivial (that is, $M_X|_U = \mathcal{O}_U^\times$), then the restriction $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F, \mu)|_U$ is a PH on U with a Γ-level structure, and the period map of H is an extension of the period map $U \to \Gamma\backslash D$.

Theorem 0.4.27 (ii) characterizes $\Gamma\backslash D_\Sigma$ as the universal object among the targets of period maps from objects of $\mathcal{A}(\log)$ into local ringed spaces over $\mathbf{C}$ with logarithmic structures. This indicates that the topology of $\Gamma\backslash D_\Sigma$, its ringed space structure, and the logarithmic structure, that we define in this book are in fact intrinsic structures (not artificial ones) determined by this universality.

0.4.29

The reason that the moduli space $\Gamma\backslash D_\Sigma$ of PLH is not necessarily an analytic space but a logarithmic manifold is as follows. We also explain why slits and the strong topology naturally appear.

Any PLH of type Φ on an fs logarithmic analytic space X (2.5.8) comes, locally on X, from a universal pre-PLH H_σ on some logarithmically smooth, fs logarithmic analytic space $\check{E}_\sigma$ (3.3.2) for some $\sigma \in \Sigma$ by pulling back via a morphism $X \to \check{E}_\sigma$ (cf. Sections 3.3 and 8.2). Let

$$\tilde{E}_\sigma = \{x \in \check{E}_\sigma \mid \text{the inverse image of } H_\sigma \text{ on } x \text{ satisfies Griffiths transversality}\},$$
$$\supset E_\sigma = \{x \in \check{E}_\sigma \mid \text{the inverse image of } H_\sigma \text{ on } x \text{ is a PLH}\}.$$

Locally on $\check{E}_\sigma$, $\tilde{E}_\sigma$ is the zeros in the new sense (0.4.17) of a finite family of differential forms on $\check{E}_\sigma$ (see Proposition 3.5.10). Hence $\tilde{E}_\sigma$ can have slits. (Note that the Griffiths transversality of the inverse image of H_σ on $x \in \tilde{E}_\sigma$ is the small Griffiths transversality (0.4.22).) Furthermore, as in Theorem A (i) stated in Section 4.1, we have

$$E_\sigma \text{ is open in } \tilde{E}_\sigma \text{ for the strong topology of } \tilde{E}_\sigma \text{ in } \check{E}_\sigma.$$

This openness is not true in general if we use the topology of $\tilde{E}_\sigma$ as a subset of $\check{E}_\sigma$ (12.3.10). Consequently, E_σ is a logarithmic manifold.

For a neat subgroup Γ of $G_{\mathbf{Z}}$ and a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ that are strongly compatible, the local shape of $\Gamma\backslash D_\Sigma$ is similar to that of E_σ. More precisely, $\Gamma\backslash D_\Sigma$ is covered by the images of morphisms $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ ($\sigma \in \Sigma$) which are locally isomorphisms, and E_σ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$, where $\sigma_{\mathbf{C}}$ is the $\mathbf{C}$-vector space spanned by σ (Theorem A (iii) and (iv) in Section 4.1). Thus, slits, the strong topology, and logarithmic manifolds naturally appear in the moduli of PLH.

In the nonlogarithmic case where Σ consists of one element $\{0\}$ and $\Gamma = \{1\}$, $\check{E}_{\{0\}} = \check{D}$ with the universal H, and we have $\tilde{E}_{\{0\}} = \check{D}$ and $E_{\{0\}} = D$.

Example with $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ (continued). In this example, the fact that the slit "if $q = 0$ then $v = \theta(z)$" appears from the small Griffiths transversality is explained as follows. Let $U \subset \mathbf{C}^2 \times Q$ be as in 0.4.18. The pullback on U of the universal PLH on $\Gamma\backslash D_\Xi$ extends to a pre-PLH $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ on the fs logarithmic analytic space $\mathbf{C}^2 \times Q$ whose logarithmic structure is defined by the divisor $\{(q, a, z) \in \mathbf{C}^2 \times Q \mid q = 0\}$. For $x = (0, a, z) \in \mathbf{C}^2 \times Q$, the inverse image of H on x satisfies Griffiths transversality if and only if $v = \pm\theta(z)$.

One of the motivations of the dream of Griffiths to enlarge D was the hope of extending the period map of VPH to the boundary. Concerning this, we have the following result.

Theorem 0.4.30 (Theorem 4.3.1) *Let X be a connected, logarithmically smooth, fs logarithmic analytic space and let $U = X_{\mathrm{triv}} = \{x \in X \mid M_{X,x} = \mathcal{O}_{X,x}^\times\}$ be the open subspace of X consisting of all points of X at which the logarithmic structure of X is trivial. Let H be a variation of polarized Hodge structure on U with unipotent local monodromy along $X - U$. Fix a base point $u \in U$ and let $(H_0, \langle\ ,\ \rangle_0) = (H_{\mathbf{Z},u}, \langle\ ,\ \rangle_u)$. Let Γ be a subgroup of $G_{\mathbf{Z}}$ which contains the global monodromy* *group* Image$(\pi_1(U, u) \to G_{\mathbf{Z}})$ *and assume* Γ *is neat. Let* $\varphi : U \to \Gamma\backslash D$ *be the associated period map.*

(i) *Assume that* $X - U$ *is a smooth divisor. Then the period map* φ *extends to a morphism* $X \to \Gamma\backslash D_\Sigma$ *of logarithmic manifolds for some fan* Σ *in* $\mathfrak{g}_{\mathbf{Q}}$ *that is strongly compatible with* Γ.

(ii) *For any* $x \in X$, *there exist an open neighborhood* W *of* x, *a logarithmic modification* W' *of* W, *a commutative subgroup* Γ' *of* Γ, *and a fan* Σ *in* $\mathfrak{g}_{\mathbf{Q}}$ *that is strongly compatible with* Γ' *such that the period map* $\varphi|_{U\cap W}$ *lifts to a morphism* $U \cap W \to \Gamma'\backslash D$ *and extends to a morphism* $W' \to \Gamma'\backslash D_\Sigma$ *of logarithmic manifolds.*

Here in (ii) a logarithmic modification is a special kind of proper morphism $W' \to W$ which is an isomorphism over $U \cap W$ (3.6.12). This theorem can be deduced from the nilpotent orbit theorem of Schmid and some results for fans.

Note that (i) can be applied to $X = \Delta$.

Elliptic curves (continued). In theorem 0.4.30, consider the case where $X = \Delta$, $U = \Delta^*$, and H is the LVPH on Δ in 0.2.18. Fix a branch of e_2 in $H_{\mathbf{Z},u} = H_0 = \mathbf{Z}e_1 + \mathbf{Z}e_2$. The image Γ of $\pi_1(U, u) \to G_{\mathbf{Z}}$ is isomorphic to $\mathbf{Z}$ and is generated by the element γ such that $\gamma(e_1) = e_1$, $\gamma(e_2) = e_1 + e_2$. The classical period map $\Delta^* \to \Gamma\backslash D = \Gamma\backslash\mathfrak{h}$ extends to the period map $\Delta \to \Gamma\backslash D_\sigma$ where $\sigma = \sigma_\infty$ (0.4.13), which coincides with the isomorphism $\Delta \simeq \Gamma\backslash D_\sigma$ in 0.4.13.

0.4.31

Infinitesimal period maps. Let $f : Y \to X$ be a projective, logarithmically smooth (2.1.11), vertical morphism of logarithmically smooth fs logarithmic analytic spaces with connected X. Assume, for any $y \in Y$, that Coker $\big((M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_{f(y)} \to (M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times)_y\big)$ is torsion-free. Let $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be the associated LVPH of weight m on X as in 0.2.21.

Let Γ and Σ be a strongly compatible pair (0.4.10). Assume that Γ is neat and contains Image$(\pi_1(X^{\log}) \to G_{\mathbf{Z}})$, and assume that we have the associated period map $\varphi : X \to \Gamma\backslash D_\Sigma$ (0.4.28). Note that, by Theorem 0.4.30 (ii), these assumptions will be fulfilled locally on X, if we allow a logarithmic modification of it.

Then as a generalization of 0.3.9, for the differential $d\varphi$ of the period map φ, we have the following commutative diagram:

$$\begin{array}{ccc}
\theta_X & \xrightarrow{d\varphi} & \varphi^*\theta^h_{\Gamma\backslash D_\Sigma} = \mathrm{gr}^{-1}\,\mathcal{E}nd_{\langle\,,\,\rangle}(\mathcal{M}) \\
\text{K-S}\downarrow & & \cap\downarrow \\
R^1 f_*\theta_{Y/X} & \xrightarrow{\text{via coupling}} & \bigoplus_p \mathcal{H}om_{\mathcal{O}_X}(R^{m-p}f_*\omega^p_{Y/X}, R^{m-p+1}f_*\omega^{p-1}_{Y/X})
\end{array}$$

where $\theta_{Y/X} := \mathcal{H}om_{\mathcal{O}_Y}(\omega^1_{Y/X}, \mathcal{O}_Y)$, and $\theta^h_{\Gamma\backslash D_\Sigma}$ is the horizontal logarithmic tangent bundle of the logarithmic tangent bundle $\theta_{\Gamma\backslash D_\Sigma}$, K-S on the left vertical arrow means the logarithmic version of the Kodaira-Spencer map, and the right vertical arrow is the canonical map (Section 4.4).

0.4.32

Note that the classifying space D for polarized Hodge structures on H^2 of surfaces of general type with $p_g \geq 2$, or on H^3 of Calabi-Yau threefolds, is not classical in the sense of 0.4.14. By the construction in the present book, we can now talk about the extended period maps and their differentials associated with degenerations of all complex projective manifolds.

0.4.33

Moduli of PLH with coefficients. We can generalize the above theorems of the moduli of PLH to the moduli of PLH with coefficients (see Chapter 11). Let A be a finite-dimensional semisimple $\mathbf{Q}$-algebra endowed with a map $A \to A, a \mapsto a^\circ$, satisfying

$$(a+b)^\circ = a^\circ + b^\circ, \quad (ab)^\circ = b^\circ a^\circ \quad (a, b \in A).$$

By a *polarized logarithmic Hodge structure with coefficients in A (A-PLH)* we mean a PLH $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ endowed with a ring homomorphism $A \to \mathrm{End}_{\mathbf{Q}}(H_{\mathbf{Q}})$ satisfying

$$\langle ax, y\rangle = \langle x, a^\circ y\rangle \quad (a \in A, x, y \in H_{\mathbf{Q}}).$$

The theorems 0.4.19 and 0.4.27 can be generalized to the moduli $\Gamma\backslash D_\Sigma^A$ of A-PLH (11.1.7, 11.3.1).

0.4.34

In the classical situation 0.4.14, in the work [AMRT], they constructed a fan Σ which is strongly compatible with $G_{\mathbf{Z}}$ such that $G_{\mathbf{Z}}\backslash D_\Sigma$ is compact. In our general situation, it can often happen that $\Gamma\backslash D_\Sigma$ is not locally compact for any Σ such that $D_\Sigma \neq D$. However, we can define the notion of a complete fan (a sufficiently big fan, roughly speaking) such that, in the classical situation, Σ is complete if and only if $G_{\mathbf{Z}}\backslash D_\Sigma$ is compact (see Section 12.6). If Σ is complete and is strongly compatible with Γ, the classical period map $U \to \Gamma\backslash D$ in 0.4.30 always extends globally to a morphism $X' \to \Gamma\backslash D_\Sigma$ of logarithmic manifolds for some logarithmic modification $X' \to X$ (Theorem 12.6.6).

One problem which we cannot solve in this book is that of finding a complete fan in general.*

In Example 0.3.2 (iii) (see also 0.3.4, 0.4.18, and 0.4.29), the fan Ξ in 0.4.5 is complete. But $\Gamma\backslash D_\Xi$ is not compact, not even locally compact, since it has slits.

0.5 FUNDAMENTAL DIAGRAM AND OTHER ENLARGEMENTS OF D

We fix $(w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ as in Section 0.3. Let D be the classifying space of polarized Hodge structures, i.e., a Griffiths domain, as in 0.3.1.

*See the end of section 12.7.

To prove the main theorems 0.4.19 and 0.4.27, as already mentioned in the Introduction, we need to construct the fundamental diagram (3) in Introduction and to study all the spaces and their relations there. Roughly speaking, in this fundamental diagram, the construction of the four spaces in the right-hand side is based on arithmetic theory of algebraic groups, and that of the four spaces in the left-hand side is based on Hodge theory. They are joined by the central continuous map $D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$, which is a geometric interpretation of the SL(2)-orbit theorem of Cattan-Kaplan-Schmid [CKS]. We give an overview of our results concerning these spaces.

The organization of Section 0.5 is as follows. In 0.5.1, we describe the rough ideas of all the enlargements of D in the fundamental diagram. In 0.5.2, in the case of Example (i) in 0.3.2 (the case of the upper half plane), we give the complete descriptions of all the enlargements of D, other than D_Σ which was already described in Section 0.4. After that we explain each of these enlargements (other than D_Σ) one by one in the general case; $D^{\sharp}_{\Sigma}$ in 0.5.3–0.5.6, D_{BS} in 0.5.7–0.5.10, $D_{\mathrm{SL}(2)}$ in 0.5.11–0.5.18, and the "valuative spaces" completing the fundamental diagram in 0.5.19–0.5.29. In particular, explicit descriptions of the central bridge $D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ in Examples (ii) and (iii) in 0.3.2 are given in 0.5.26 and 0.5.27, respectively. In 0.5.30, we overview ♭-spaces, related to the work of Cattani and Kaplan [CK1].

0.5.1

First we give some general observations.

Recall that D_Σ (Σ is a fan in $\mathfrak{g}_{\mathbf{Q}}$) is the set of nilpotent orbits (σ, Z), where $\sigma \in \Sigma$ and Z is an $\exp(\sigma_{\mathbf{C}})$-orbit in $\check{D}$ satisfying a certain condition (0.4.7).

(i) *Space* $D^{\sharp}_{\Sigma}$. The space $D^{\sharp}_{\Sigma}$ (Σ is a fan in $\mathfrak{g}_{\mathbf{Q}}$) is a set of pairs (σ, Z) where $\sigma \in \Sigma$ and Z is an $\exp(i\sigma_{\mathbf{R}})$-orbit in $\check{D}$ satisfying a certain condition (see 0.5.3 below). The space $D^{\sharp}_{\Sigma}$ has a natural topology, D is a dense open subset of $D^{\sharp}_{\Sigma}$, and, roughly speaking, the element $(\sigma, Z) \in D^{\sharp}_{\Sigma}$ is the limit point of elements of Z which "run in the direction of degeneration conducted by σ." The space $D^{\sharp}_{\Sigma}$ is covered by open subsets $D^{\sharp}_{\sigma} = \{(\sigma', Z) \in D^{\sharp}_{\Sigma} \mid \sigma' \subset \sigma\}$, where σ runs over elements of Σ.

(ii) *Space* D_{BS}. The space D_{BS} is a set of pairs (P, Z) where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and Z is a subset of D satisfying a certain condition (see 0.5.7 below). The set Z is a torus orbit in the following sense. For $(P, Z) \in D_{\mathrm{BS}}$, there is an associated homomorphism of algebraic groups $s : (\mathbf{R}^\times)^n \to P$ over $\mathbf{R}$ such that Z is an $s(\mathbf{R}^n_{>0})$-orbit in D. The space D_{BS} has a natural topology, D is a dense open subset of D_{BS}, and, roughly speaking, the element $(P, Z) \in D_{\mathrm{BS}}$ is the limit point of elements of Z which "run in the direction of degeneration conducted by P." The space D_{BS} is covered by open subsets $D_{\mathrm{BS}}(P) = \{(P', Z) \in D_{\mathrm{BS}} \mid P' \supset P\}$, where P runs over all $\mathbf{Q}$-parabolic subgroups of G.

(iii) *Space* $D_{\mathrm{SL}(2)}$. The space $D_{\mathrm{SL}(2)}$ is a set of pairs (W, Z) where W is a compatible family $(W^{(j)})_{1\le j\le r}$ (i.e., distributive families in [K]; see 5.2.12) of rational weight filtrations $W^{(j)} = (W^{(j)}_k)_{k\in\mathbf{Z}}$ on $H_{0,\mathbf{R}}$ and Z is a subset of D satisfying a certain condition (see 0.5.11–0.5.13 below). The set Z is a torus orbit in the following sense. For $(W, Z) \in D_{\mathrm{SL}(2)}$, there is an associated homomorphism of algebraic groups over $\mathbf{R}$

$$s : (\mathbf{R}^{\times})^r \to G_{W,\mathbf{R}} = \{g \in G_{\mathbf{R}} \mid gW_k^{(j)} = W_k^{(j)} \text{ for all } j, k\}$$

such that Z is an $s(\mathbf{R}_{>0})^n$-orbit in D. The space $D_{\mathrm{SL}(2)}$ has a natural topology, D is a dense open subset of $D_{\mathrm{SL}(2)}$, and, roughly speaking, the element $(W, Z) \in D_{\mathrm{SL}(2)}$ is the limit point of elements of Z which "run in the direction of degeneration conducted by W." The space $D_{\mathrm{SL}(2)}$ is covered by open subsets $D_{\mathrm{SL}(2)}(W) = \{(W', Z) \in D_{\mathrm{SL}(2)} \mid W' \text{ is a "subfamily" of } W\}$, where W runs over all compatible families of rational weight filtrations of $H_{0,\mathbf{R}}$.

(iv) The *other four spaces.* The other four spaces are spaces of "valuative" orbits which are located over D_{Σ}, $D_{\Sigma}^{\sharp}$, $D_{\mathrm{SL}(2)}$, and D_{BS}, respectively. These upper spaces in the fundamental diagram (3) in Introduction are obtained from the lower spaces as the limits when the directions of degenerations are divided into narrower and narrower directions. We can say also that the vertical arrows in that diagaram are projective limits of kinds of blow-ups.

0.5.2

Upper half plane (continued). In the easiest case $D = \mathfrak{h}$, the sets $D_{\Xi}^{\sharp}$, $D_{\mathrm{SL}(2)}$, and D_{BS} are described as follows.

First we describe $D_{\Xi}^{\sharp}$. Recall that $\Xi = \{\{0\}, \sigma_a\ (a \in \mathbf{P}^1(\mathbf{Q}))\}$ (0.4.9). Recall that

$$\sigma_{\infty} = \begin{pmatrix} 0 & \mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix}.$$

The space $D_{\Xi}^{\sharp}$ is covered by open sets $D_{\sigma}^{\sharp}$ for $\sigma \in \Xi$. The space $D_{\sigma}^{\sharp}$ for $\sigma = \{0\}$ is identified with D ($F \in D$ is identified with the pair (σ, Z) with $\sigma = \{0\}$ and $Z = \{F\}$). The complement $D_{\sigma_{\infty}}^{\sharp} - D$ is the set of all pairs (σ_{∞}, Z) where Z is a subset of $\mathbf{C} \subset \check{D} = \mathbf{P}^1(\mathbf{C})$ of the form $x + i\mathbf{R}$ for some $x \in \mathbf{R}$. This is a set of all $\exp(i\sigma_{\infty,\mathbf{R}})$-orbits in $\mathbf{C}$. We have a homeomorphism

$$D_{\sigma_{\infty}}^{\sharp} \simeq \{x + iy \mid x \in \mathbf{R},\ 0 < y \leq \infty\}, \quad (\sigma_{\infty}, x + i\mathbf{R}) \mapsto x + i\infty,$$

which extends the identity map of D. Hence $(\sigma_{\infty}, x + i\mathbf{R})$ is the limit of $x + iy \in D$ $(y > 0)$ for $y \to \infty$. Let $a \in \mathbf{P}^1(\mathbf{Q})$ and let g be any element of $\mathrm{SL}(2, \mathbf{Q})$ such that $a = g \cdot \infty$. Then $D_{\sigma_a}^{\sharp} - D$ is the set of all pairs (σ_a, Z) where Z is a subset of $\check{D} = \mathbf{P}^1(\mathbf{C})$ such that $(\sigma_{\infty}, g^{-1}(Z)) \in D_{\sigma_{\infty}}^{\sharp}$. We have a homeomorphism $D_{\sigma_{\infty}}^{\sharp} \xrightarrow{\sim} D_{\sigma_a}^{\sharp}$, $(\sigma_{\infty}, Z) \mapsto (\sigma_a, g(Z))$.

We describe D_{BS}. A $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}} = \mathrm{SL}(2, \mathbf{R})$ is either $G_{\mathbf{R}}$ itself or P_a $(a \in \mathbf{P}^1(\mathbf{Q}))$ defined by

$$P_a = \{g \in \mathrm{SL}(2, \mathbf{R}) \mid ga = a\} = \{g \in \mathrm{SL}(2, \mathbf{R}) \mid gV_a = V_a\}$$

where V_a is as in 0.4.9. For example,

$$P_{\infty} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \ a, b, d \in \mathbf{R},\ ad = 1 \right\}.$$

The space D_{BS} is covered by open sets $D_{\mathrm{BS}}(P)$ for $P = G_{\mathbf{R}}$, P_a ($a \in \mathbf{P}^1(\mathbf{Q})$). The space $D_{\mathrm{BS}}(G_{\mathbf{R}})$ is identified with D ($F \in D$ is identified with the pair $(G_{\mathbf{R}}, Z)$ with $Z = \{F\}$). The complement $D_{\mathrm{BS}}(P_\infty) - D$ is the set of all pairs (P_∞, Z) where Z is a subset of $\mathfrak{h} = D$ of the form $x + i\mathbf{R}_{>0}$ for some $x \in \mathbf{R}$. We have a homeomorphism

$$D_{\mathrm{BS}}(P_\infty) \simeq \{x + iy \mid x \in \mathbf{R}, 0 < y \le \infty\}, \quad (P_\infty, x + i\mathbf{R}_{>0}) \mapsto x + i\infty,$$

which extends the identity map of D. Hence $(P_\infty, x + i\mathbf{R}_{>0})$ is the limit of $x + iy \in D$ ($y > 0$) for $y \to \infty$. Let $a \in \mathbf{P}^1(\mathbf{Q})$ and let g be any element of $\mathrm{SL}(2, \mathbf{Q})$ such that $a = g \cdot \infty$. Then $D_{\mathrm{BS}}(P_a) - D$ is the set of all pairs (P_a, Z) where Z is a subset of D such that $(P_\infty, g^{-1}(Z)) \in D_{\mathrm{BS}}(P_\infty)$. We have a homeomorphism $D_{\mathrm{BS}}(P_\infty) \xrightarrow{\sim} D_{\mathrm{BS}}(P_a)$, $(P_\infty, Z) \mapsto (P_a, g(Z))$.

We describe $D_{\mathrm{SL}(2)}$. For $a \in \mathbf{P}^1(\mathbf{Q})$, let $W(a)$ be the increasing filtration on $H_{0,\mathbf{R}}$ defined by

$$W_1(a) = H_{0,\mathbf{R}}, \quad W_0(a) = W_{-1}(a) = V_a, \quad W_{-2}(a) = 0.$$

For example,

$$W_1(\infty) = H_{0,\mathbf{R}} \supset W_0(\infty) = W_{-1}(\infty) = \mathbf{R}e_1 \supset W_{-2}(\infty) = 0.$$

The space $D_{\mathrm{SL}(2)}$ is covered by the open subsets $D_{\mathrm{SL}(2)}(W(a))$ where $W(a)$ now denotes the family of weight filtrations consisting of the single member $W(a)$. The space $D_{\mathrm{SL}(2)}(\emptyset)$ for the empty family $\emptyset$ is identified with D ($F \in D$ is identified with the pair $(\emptyset, Z)$ with $Z = \{F\}$). The complement $D_{\mathrm{SL}(2)}(W(\infty)) - D$ is the set of all pairs $(W(\infty), Z)$ where Z is a subset of $\mathfrak{h} = D$ of the form $x + i\mathbf{R}_{>0}$ for some $x \in \mathbf{R}$. We have a homeomorphism

$$D_{\mathrm{SL}(2)}(W(\infty)) \simeq \{x + iy \mid x \in \mathbf{R}, 0 < y \le \infty\}, \quad (W(\infty), x + i\mathbf{R}_{>0}) \mapsto x + i\infty,$$

which extends the identity map of D. Hence $(W(\infty), x + i\mathbf{R}_{>0})$ is the limit of $x + iy \in D$ ($y > 0$) for $y \to \infty$. Let $a \in \mathbf{P}^1(\mathbf{Q})$ and let g be any element of $\mathrm{SL}(2, \mathbf{Q})$ such that $a = g \cdot \infty$. Then $D_{\mathrm{SL}(2)}(W(a)) - D$ is the set of all pairs $(W(a), Z)$ where Z is a subset of D such that $(W(\infty), g^{-1}(Z)) \in D_{\mathrm{SL}(2)}(W(\infty))$. We have a homeomorphism $D_{\mathrm{SL}(2)}(W(\infty)) \xrightarrow{\sim} D_{\mathrm{SL}(2)}(W(a))$, $(W(\infty), Z) \mapsto (W(a), g(Z))$.

The valuative spaces in this case are naturally identified with the spaces under them respectively in the fundamental diagram. That is, the canonical maps $D^{\sharp}_{\Xi,\mathrm{val}} \to D^{\sharp}_{\Xi}$, $D_{\mathrm{BS,val}} \to D_{\mathrm{BS}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ are homeomorphisms and the canonical map $D_{\Xi,\mathrm{val}} \to D_{\Xi}$ is bijective.

The identity map of D extends to $G_{\mathbf{Q}}$-equivariant homeomorphisms $D^{\sharp}_{\Xi} \simeq D_{\mathrm{SL}(2)} \simeq D_{\mathrm{BS}}$, which induce homeomorphisms $D^{\sharp}_{\sigma_a} \simeq D_{\mathrm{SL}(2)}(W(a)) \simeq D_{\mathrm{BS}}(P_a)$ for each $a \in \mathbf{P}^1(\mathbf{Q})$ described as

$$(\sigma_a, Z') \leftrightarrow (W(a), Z) \leftrightarrow (P(a), Z), \quad Z' = \exp(i\sigma_{a,\mathbf{R}})Z, \quad Z = Z' \cap D.$$

Thus the fundamental diagram in this case becomes like (4) in Introduction.

0.5.3

Space $D^\sharp_\Sigma$. In 0.5.3–0.5.6, we consider the space $D^\sharp_\Sigma$ which is on the left-hand side (the Hodge side) of the fundamental diagram, next to the space D_Σ of nilpotent orbits considered in Section 0.4.

A *nilpotent i-orbit* is a pair (σ, Z) consisting of a nilpotent cone $\sigma = \sum_{1\le j\le r}(\mathbf{R}_{\ge 0})N_j$ and a subset $Z \subset \check{D}$ which satisfy, for some $F \in Z$,

$$\begin{cases} Z = \exp(i\sigma_{\mathbf{R}})F, \\ NF^p \subset F^{p-1} \quad (\forall p,\ \forall N \in \sigma), \\ \exp\left(\sum_{1\le j\le r} iy_jN_j\right) F \in D \quad (\forall\ y_j \gg 0). \end{cases}$$

Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$. As a set, we define

$$D^\sharp_\Sigma := \{(\sigma, Z) \text{ nilpotent } i\text{-orbit} \mid \sigma \in \Sigma,\ Z \subset \check{D}\}.$$

Note that we have the inclusion map $D \hookrightarrow D^\sharp_\Sigma$, $F \mapsto (\{0\}, \{F\})$. There is a canonical surjection $D^\sharp_\Sigma \to D_\Sigma$, $(\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z)$. For a rational nilpotent cone σ in $\mathfrak{g}_{\mathbf{R}}$, we denote $D^\sharp_{\{\text{face of } \sigma\}}$ by $D^\sharp_\sigma$. Then, for a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$, we have $D^\sharp_\Sigma = \bigcup_{\sigma\in\Sigma} D^\sharp_\sigma$.

0.5.4

In Chapter 3, we will define a topology of $D^\sharp_\Sigma$ that has the following property. Let $(\sigma, Z) \in D^\sharp_\Sigma$, let $N_j \in \mathfrak{g}_{\mathbf{Q}}$ $(1 \le j \le n)$, $F \in Z$, and assume $\sigma = \sum_{j=1}^n (\mathbf{R}_{\ge 0})N_j$. Then

$$\exp\left(\sum_{j=1}^{n} iy_jN_j\right) F \to (\sigma, Z) \quad \text{if } y_j \in \mathbf{R} \text{ and } y_j \to \infty.$$

Theorem 0.5.5 (Theorem A in Section 4.1)

(i) *The space $D^\sharp_\Sigma$ is Hausdorff.*

(ii) *Assume that Γ is strongly compatible with Σ. Then $\Gamma\backslash D^\sharp_\Sigma$ is Hausdorff.*

(iii) *Assume that Γ is strongly compatible with Σ and is neat. Then the canonical projection $D^\sharp_\Sigma \to \Gamma\backslash D^\sharp_\Sigma$ is a local homeomorphism.*

(iv) *Assume that Γ is strongly compatible with Σ and is neat. Then we have a canonical homeomorphism*

$$\Gamma\backslash D^\sharp_\Sigma \simeq (\Gamma\backslash D_\Sigma)^{\log}$$

which is compatible with the projections to $\Gamma\backslash D_\Sigma$.

By (iv), for Γ as in (iv), the canonical map $\Gamma\backslash D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$ is proper, and the fibers are products of finite copies of $\mathbf{S}^1$.

0.5.6

Here we give local descriptions of $D_\Sigma^\sharp \to \Gamma\backslash D_\Sigma^\sharp \to \Gamma\backslash D_\Sigma$ for Example (i), Example (ii) ($g = 2$), and Example (iii) in 0.3.2 (for some choices of Σ and Γ).

Upper half plane (continued). Let $\Gamma = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}$, $\sigma = \sigma_\infty$. Then we have a commutative diagram of topological spaces:

$$\begin{array}{ccc} \{x+iy \mid x \in \mathbf{R},\ 0 < y \le \infty\} & \simeq & D_\sigma^\sharp \\ \downarrow & & \downarrow \\ \Delta^{\log} & \simeq & \Gamma\backslash D_\sigma^\sharp \\ \downarrow & & \downarrow \\ \Delta & \simeq & \Gamma\backslash D_\sigma. \end{array}$$

Here the lower horizontal isomorphism is that in 0.4.13, the upper horizontal isomorphism is the one described in 0.5.2, and the upper left vertical arrow sends $\tau = x + iy$ ($0 < y < \infty$) to $e^{2\pi i\tau} \in \Delta^* \subset \Delta^{\log}$, and $x + i\infty$ to $(0, e^{2\pi ix}) \in \Delta^{\log} = |\Delta| \times \mathbf{S}^1$.

Upper half space (continued). Let $g = 2$ and $D = \mathfrak{h}_2$. Let U be the open set of $\Delta^2 \times \mathbf{C}$ defined in 0.4.13, and let $\Gamma = \exp(\mathbf{Z}N_1 + \mathbf{Z}N_2) = 1 + \mathbf{Z}N_1 + \mathbf{Z}N_2$, $\sigma = (\mathbf{R}_{\ge 0})N_1 + (\mathbf{R}_{\ge 0})N_2$. We describe $D_\sigma^\sharp$ and $\Gamma\backslash D_\sigma^\sharp$. We have a commutative diagram of topological spaces:

$$\begin{array}{ccccc} (|\Delta| \times \mathbf{R})^2 \times \mathbf{C} & \supset & \tilde{U}^{\log} & \simeq & D_\sigma^\sharp \\ \downarrow & & \downarrow & & \downarrow \\ (|\Delta| \times \mathbf{S}^1)^2 \times \mathbf{C} & \supset & U^{\log} & \simeq & \Gamma\backslash D_\sigma^\sharp \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^2 \times \mathbf{C} & \supset & U & \simeq & \Gamma\backslash D_\sigma. \end{array}$$

Here the upper left vertical arrow is induced by $\mathbf{R} \to \mathbf{S}^1$, $x \mapsto e^{2\pi ix}$, the lower left vertical arrow is induced by $|\Delta| \times \mathbf{S}^1 \to \Delta$, $(r, u) \mapsto ru$, and $\tilde{U}^{\log}$ is the inverse image of U in $(|\Delta| \times \mathbf{R})^2 \times \mathbf{C}$. The space $U^{\log}$ is identified with the inverse image of U in $(|\Delta| \times \mathbf{S}^1)^2 \times \mathbf{C}$. The inclusions $\supset$ in this diagram are open immersions. Let $r_j = e^{-2\pi y_j}$. The upper horizontal isomorphism of this diagram sends $((r_1, x_1), (r_2, x_2), a) \in \tilde{U}^{\log}$ to

$$\begin{array}{ll} F\begin{pmatrix} x_1 + iy_1 & a \\ a & x_2 + iy_2 \end{pmatrix} & \text{if } r_1r_2 \ne 0, \\ \left((\mathbf{R}_{\ge 0})N_1,\ F\begin{pmatrix} x_1 + i\mathbf{R} & a \\ a & x_2 + iy_2 \end{pmatrix}\right) & \text{if } r_1 = 0 \text{ and } r_2 \ne 0, \\ \left((\mathbf{R}_{\ge 0})N_2,\ F\begin{pmatrix} x_1 + iy_1 & a \\ a & x_2 + i\mathbf{R} \end{pmatrix}\right) & \text{if } r_1 \ne 0 \text{ and } r_2 = 0, \\ \left(\sigma,\ F\begin{pmatrix} x_1 + i\mathbf{R} & a \\ a & x_2 + i\mathbf{R} \end{pmatrix}\right) & \text{if } r_1 = r_2 = 0. \end{array}$$

Example with $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ (continued). Let the notation be as in 0.4.18. Let Γ be a neat subgroup of $G_\mathbf{Z}$ of finite index. We have a commutative diagram of topological spaces

$$\begin{array}{ccccc}
((\mathbf{R}_{\geq 0}) \times \mathbf{R}) \times \mathbf{C} \times Q & \supset & \tilde{U}^{\log} & \to & D^{\sharp}_{\Xi} \\
\downarrow & & \downarrow & & \downarrow \\
((\mathbf{R}_{\geq 0}) \times \mathbf{S}^1) \times \mathbf{C} \times Q & \supset & U^{\log} & \to & \Gamma\backslash D^{\sharp}_{\Xi} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{C} \times \mathbf{C} \times Q & \supset & U & \to & \Gamma\backslash D_{\Xi}
\end{array}$$

in which the three horizontal arrows are local homeomorphisms. Here the upper left vertical arrow is induced by $\mathbf{R} \to \mathbf{S}^1$, $x \mapsto e^{2\pi i x}$, the lower left vertical arrow is induced from $(\mathbf{R}_{\geq 0}) \times \mathbf{S}^1 \to \mathbf{C}$, $(r, u) \mapsto ru$, and $\tilde{U}^{\log}$ is the inverse image of U in $((\mathbf{R}_{\geq 0}) \times \mathbf{R}) \times \mathbf{C} \times Q$. The space $U^{\log}$ is identified with the inverse image of U in $((\mathbf{R}_{\geq 0}) \times \mathbf{S}^1) \times \mathbf{C} \times Q$. Recall that U is endowed with the strong topology. The spaces $U^{\log}$ and $\tilde{U}^{\log}$ are endowed here with the topologies as fiber products by left squares. The upper horizontal arrow sends $(r, x, a, z) \in \tilde{U}^{\log}$ $(r \neq 0)$ to $\exp((x + iy)N_v + aN_{v'})F(z)$, where $r = e^{-2\pi y/\ell}$ with ℓ, v, and v' as in 0.4.18, and $(0, x, a, \theta^{-1}(v))$ to $((\mathbf{R}_{\geq 0})N_v, \exp(i\mathbf{R}N_v)\exp(xN_v + aN_{v'})F(\theta^{-1}(v)))$.

0.5.7

Space D_{BS}. In 0.5.7–0.5.10, we consider the space D_{BS} which is on the right-hand side (algebraic group side) of the fundamental diagram.

D_{BS} is a real manifold with corners.

The definition of D_{BS} will be reviewed in Section 5.1. Here we give an explicit presentation of the open set $D_{\mathrm{BS}}(P)$ of D_{BS}, for simplicity, under the assumptions that the $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$ is an $\mathbf{R}$-minimal parabolic subgroup of $G_{\mathbf{R}}$ and that the largest $\mathbf{R}$-split torus in the center of P/P_u (P_u is the unipotent radical of P) is $\mathbf{Q}$-split. In this case, $D_{\mathrm{BS}}(P)$ is described by using the Iwasawa decomposition of $G_{\mathbf{R}}$.

Upper half plane (continued). We first observe the easiest case $D = \mathfrak{h}$. In this case, we have a homeomorphism

$$\begin{pmatrix} 1 & \mathbf{R} \\ 0 & 1 \end{pmatrix} \times (\mathbf{R}_{>0}) \times \mathrm{SO}(2, \mathbf{R}) \simeq G_{\mathbf{R}} = \mathrm{SL}(2, \mathbf{R}),$$

$$(g, t, k) \mapsto gs(t)k, \quad \text{where } s(t) = \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix}.$$

This is an Iwasawa decomposition of $\mathrm{SL}(2, \mathbf{R})$. By $\mathrm{SL}(2, \mathbf{R})/\mathrm{SO}(2, \mathbf{R}) \xrightarrow{\sim} \mathfrak{h}$, $g \mapsto g \cdot i$, this Iwasawa decomposition induces a homeomorphism

$$\mathbf{R} \times \mathbf{R}_{>0} \xrightarrow{\sim} \mathfrak{h} = D, \quad (x, t) \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} s(t) \cdot i = x + t^{-2} i.$$

This homeomorphism extends to a homeomorphism

$$\mathbf{R} \times \mathbf{R}_{\geq 0} \simeq D_{\mathrm{BS}}(P_\infty), \quad (x, 0) \mapsto (P_\infty, \ x + i\mathbf{R}_{>0}).$$

In general, let K be a maximal compact subgroup of $G_{\mathbf{R}}$ and let P be an $\mathbf{R}$-minimal parabolic subgroup of $G_{\mathbf{R}}$. Denote by P_u the unipotent radical of P. Then we have a homeomorphism (Iwasawa decomposition)

$$P_u \times (\mathbf{R}^n_{>0}) \times K \simeq G_{\mathbf{R}}, \quad (g, t, k) \mapsto gs(t)k,$$

for a unique pair (n, s) where $n \geq 0$ is an integer and s is a homomorphism $(\mathbf{R}^\times)^n \to P$ of $\mathbf{R}$-algebraic groups satisfying the following conditions (1)–(3).

(1) The composition $(\mathbf{R}^\times)^n \xrightarrow{s} P \to P/P_u$ induces an isomorphism from $(\mathbf{R}^\times)^n$ onto the largest $\mathbf{R}$-split torus in the center of P/P_u.
(2) The Cartan involution $G_{\mathbf{R}} \xrightarrow{\sim} G_{\mathbf{R}}$ associated with K (see below) sends $s(t)$ to $s(t)^{-1}$.
(3) For $t = (t_j)_{1 \leq j \leq n} \in (\mathbf{R}^\times)^n$, $|t_j| < 1$ for any j if and only if all eigenvalues of $\mathrm{Ad}(s(t))$ on $\mathrm{Lie}(P_u)$ have absolute values > 1.

Here the Cartan involution associated with a maximal compact subgroup K of $G_{\mathbf{R}}$ is the unique homomorphism $\iota : G_{\mathbf{R}} \to G_{\mathbf{R}}$ of $\mathbf{R}$-algebraic groups such that $\iota^2 = \mathrm{id}$ and such that $K = \{g \in G_{\mathbf{R}} \mid \iota(g) = g\}$. For $\mathbf{r} \in D$, the Cartan involution of $G_{\mathbf{R}}$ associated with $K_{\mathbf{r}}$ coincides with the map $g \mapsto C_{\mathbf{r}} g C_{\mathbf{r}}^{-1}$ where $C_{\mathbf{r}}$ is the operator in 0.1.8 (2).

Now let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ which is an $\mathbf{R}$-minimal parabolic subgroup of $G_{\mathbf{R}}$ such that the largest $\mathbf{R}$-split torus in the center of P/P_u is $\mathbf{Q}$-split. Let $\mathbf{r} \in D$, let $K_{\mathbf{r}}$ be the maximal compact subgroup of $G_{\mathbf{R}}$ corresponding to $\mathbf{r}$ (0.3.3), and consider the Iwasawa decomposition $P_u \times (\mathbf{R}^n_{>0}) \times K_{\mathbf{r}} \simeq G_{\mathbf{R}}$ with respect to $(P, K_{\mathbf{r}})$ satisfying (1)–(3) as above. It induces a homeomorphism

$$P_u \times (\mathbf{R}^n_{>0}) \times (K_{\mathbf{r}}/K'_{\mathbf{r}}) \simeq D, \quad (g, t, k) \mapsto gs(t)k \cdot \mathbf{r}.$$

This extends to a homeomorphism

$$P_u \times (\mathbf{R}^n_{\geq 0}) \times (K_{\mathbf{r}}/K'_{\mathbf{r}}) \simeq D_{\mathrm{BS}}(P).$$

The element of $D_{\mathrm{BS}}(P)$ corresponding to $(g, 0, k) \in P_u \times (\mathbf{R}^n_{\geq 0}) \times (K_{\mathbf{r}}/K'_{\mathbf{r}})$ ($g \in P_u$, $k \in K_{\mathbf{r}}$) coincides with the pair (P, Z) where $Z = \{gs(t)k \cdot \mathbf{r} \mid t \in \mathbf{R}^n_{>0}\}$. More generally, the element of $D_{\mathrm{BS}}(P)$ corresponding to (g, t, k) ($g \in P_u, t \in \mathbf{R}^n_{\geq 0}$, $k \in K_{\mathbf{R}}$) coincides with the element (Q, Z) of $D_{\mathrm{BS}}(P)$ where Q is the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ containing P which corresponds to the subset $J = \{j \mid 1 \leq j \leq n,\ t_j \neq 0\}$ of the set $\{1, \ldots, n\}$ (there is a bijection between the set of all subsets of $\{1, \ldots, n\}$ and the set of all $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$ containing P), and $Z = \{gs(t')k \cdot \mathbf{r} \mid t' \in \mathbf{R}^n_{>0},\ t'_j = t_j \text{ if } j \in J\} \subset D$.

Thus D_{BS} is understood by the theory of algebraic groups, rather than by Hodge theory.

Theorem 0.5.8

(i) *D_{BS} is a real manifold with corners. For any $p \in D_{\mathrm{BS}}$, there are an open neighborhood U of p in D_{BS}, integers $m, n \geq 0$, and a homeomorphism*

$$U \simeq \mathbf{R}^m \times \mathbf{R}^n_{\geq 0}$$

which sends p to $(0, 0)$. The point p belongs to D if and only if $n = 0$.

(ii) *For any subgroup* Γ *of* $G_{\mathbf{Z}}$, $\Gamma\backslash D_{\mathrm{BS}}$ *is Hausdorff.*
(iii) *If* Γ *is of finite index in* $G_{\mathbf{Z}}$, $\Gamma\backslash D_{\mathrm{BS}}$ *is compact.*
(iv) *If* Γ *is a neat subgroup of* $G_{\mathbf{Z}}$, *the projection* $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$ *is a local homeomorphism.*

The definition of D_{BS} and the proof of this theorem were given in [KU2] and [BJ] independently. The definition of D_{BS} is a modification of the definition in [BS] of the original Borel-Serre space $\mathcal{X}_{\mathrm{BS}}$, which is an enlargement of the symmetric Hermitian space $\mathcal{X}$ of all maximal compact subgroups of $G_{\mathbf{R}}$. There is a canonical surjection $D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}$ which sends $F \in D$ to $K_F \in \mathcal{X}$. The proof of the above theorem is a reduction to the similar properties of the original Borel-Serre space proved in [BS].

In 0.5.9 and 0.5.10 below, we give local descriptions of D_{BS} for Example (ii) ($g = 2$) and Example (iii) in 0.3.2, respectively.

0.5.9

Upper half space (continued). Let $g = 2$ and $D = \mathfrak{h}_2$. Let P be the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ consisting of elements that preserve the $\mathbf{R}$-subspaces $\mathbf{R}e_1$, $\mathbf{R}e_1 + \mathbf{R}e_2$, and $\mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_4$ of $H_{0,\mathbf{R}}$. This is an $\mathbf{R}$-minimal parabolic subgroup of $G_{\mathbf{R}}$. There is a homeomorphism $\mathbf{R}^4 \simeq P_u$ (not an isomorphism of groups, for P_u is noncommutative). Let $s : (\mathbf{R}^\times)^2 \to P$ be the homomorphism of algebraic groups given by

$$s(t)e_1 = t_1^{-1}t_2^{-1}e_1, \quad s(t)e_2 = t_2^{-1}e_2, \quad s(t)e_4 = t_2e_4, \quad s(t)e_3 = t_1t_2e_3.$$

Put $\mathbf{r} = F\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$. Then we have $K_{\mathbf{r}} = K'_{\mathbf{r}} = \mathrm{Sp}(2, \mathbf{R}) \cap O(4, \mathbf{R}) \simeq U(2)$. We have a homeomorphism $P_u \times (\mathbf{R}^\times)^2 \simeq P$, $(g, t) \mapsto gs(t)$. We have a homeomorphism (Iwasawa decomposition)

$$P_u \times (\mathbf{R}^2_{>0}) \times K_{\mathbf{r}} \simeq G_{\mathbf{R}}, \quad (g, t, k) \mapsto gs(t)k,$$

which satisfies the conditions (1)–(3) in 0.5.7. This induces a homeomorphism

$$P_u \times \mathbf{R}^2_{>0} \simeq D, \quad (g, t) \mapsto gs(t) \cdot \mathbf{r},$$

which extends to a homeomorphism

$$P_u \times \mathbf{R}^2_{\geq 0} \simeq D_{\mathrm{BS}}(P).$$

0.5.10

Example with $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ (continued). We use the notation in 0.3.4. Let $G^\circ_{\mathbf{R}}$ be the kernel of the determinant map $G_{\mathbf{R}} \to \{\pm 1\}$ (for the action on $H_{0,\mathbf{R}}$), and let P be the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ consisting of all elements of $G^\circ_{\mathbf{R}}$ which preserve the subspaces $\mathbf{R}e_4$ and $\sum_{j=1}^4 \mathbf{R}e_j$ of $H_{0,\mathbf{R}}$. This is an $\mathbf{R}$-minimal parabolic subgroup of $G_{\mathbf{R}}$. We have an isomorphism of $\mathbf{R}$-algebraic groups $\mathbf{R}^3 \xrightarrow{\sim} P_u$, $a \mapsto \exp(N_a)$. Let $s : \mathbf{R}^\times \to P$ be the homomorphism of $\mathbf{R}$-algebraic groups defined by

$$s(t)e_j = e_j \quad (1 \le j \le 3), \quad s(t)e_4 = t^{-1}e_4, \quad s(t)e_5 = te_5.$$

Then we have a homeomorphism $P_u \times \mathbf{R}^\times \xrightarrow{\sim} P$, $(g, t) \mapsto gs(t)$.

Let $v \in \mathbf{S}^2$. We have a homeomorphism (Iwasawa decomposition)

$$P_u \times (\mathbf{R}_{>0}) \times K_{\mathbf{r}(v)} \xrightarrow{\sim} G_{\mathbf{R}}, \quad (g, t, k) \mapsto gs(t)k$$

which satisfies the conditions (1)–(3) in 0.5.7. We have a homeomorphism

$$\{\pm 1\} \times \mathbf{S}^2 \xrightarrow{\sim} K_{\mathbf{r}(v)} \cdot \mathbf{r}(v), \quad (\pm 1, v') \mapsto s(\pm 1) \cdot \mathbf{r}(v').$$

Hence this Iwasawa decomposition induces a homeomorphism

$$\mathbf{R}^3 \times (\mathbf{R}_{>0}) \times \{\pm 1\} \times \mathbf{S}^2 \xrightarrow{\sim} D, \quad (a, t, \pm 1, v) \mapsto \exp(N_a)s(\pm t) \cdot \mathbf{r}(v),$$

which extends to a homeomorphism

$$\mathbf{R}^3 \times (\mathbf{R}_{\ge 0}) \times \{\pm 1\} \times \mathbf{S}^2 \xrightarrow{\sim} D_{\mathrm{BS}}(P).$$

In this example, all $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$ other than $G_{\mathbf{R}}^\circ$ are conjugate to P under $G_{\mathbf{Q}}$, and hence D_{BS} is covered by open sets $D_{\mathrm{BS}}(gPg^{-1})$ $(g \in G_{\mathbf{Q}})$ each of which is homeomorphic to $D_{\mathrm{BS}}(P)$ via the homeomorphism that extends $g^{-1} : D \to D$.

0.5.11

The space $D_{\mathrm{SL}(2)}$. In general, D_Σ and D_{BS} are still far from each other in nature. We find an intermediate existence $D_{\mathrm{SL}(2)}$ to connect them. We consider this space $D_{\mathrm{SL}(2)}$ in 0.5.11–0.5.18. Hodge theory and algebraic group theory are unified on this space. This unification is based on a fundamental property of the SL(2)-action on the upper half plane $\mathfrak{h}$:

$$\exp\left(iy\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)(0) = \begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}(i).$$

When $y > 0$ varies, the left-hand side produces a nilpotent i-orbit, while the right-hand side produces a torus orbit in the Borel-Serre space.

We define $D_{\mathrm{SL}(2)}$ as follows.

A pair (ρ, φ), consisting of a homomorphism $\rho : \mathrm{SL}(2, \mathbf{C})^r \to G_{\mathbf{C}}$ of algebraic groups which is defined over $\mathbf{R}$ and a holomorphic map $\varphi : \mathbf{P}^1(\mathbf{C})^r \to \check{D}$, is called an SL(2)-*orbit of rank* r if it satisfies the following conditions (1)–(4) ([CKS, Chapter 4], [KU2, Chapter 3]):

(1) $\varphi(gz) = \rho(g)\varphi(z)$ for all $g \in \mathrm{SL}(2, \mathbf{C})^r$ and all $z \in \mathbf{P}^1(\mathbf{C})^r$.
(2) The Lie algebra homomorphism $\rho_* : \mathfrak{sl}(2, \mathbf{C})^{\oplus r} \to \mathfrak{g}_{\mathbf{C}}$ is injective.
(3) $\varphi(\mathfrak{h}^r) \subset D$.
(4) Let $z \in \mathfrak{h}^r$, let $F_z^\bullet(\mathfrak{sl}(2, \mathbf{C})^{\oplus r})$ be the Hodge filtration of $\mathfrak{sl}(2, \mathbf{C})^{\oplus r}$ induced by the Hodge filtration of $(\mathbf{C}^2)^{\oplus r}$ corresponding to z, and let $F_{\varphi(z)}^\bullet(\mathfrak{g}_{\mathbf{C}})$ be the Hodge filtration of $\mathfrak{g}_{\mathbf{C}}$ induced by the Hodge filtration $\varphi(z)$ of $H_{0,\mathbf{C}}$. Then $\rho_* : \mathfrak{sl}(2, \mathbf{C})^{\oplus r} \to \mathfrak{g}_{\mathbf{C}}$ sends $F_z^p(\mathfrak{sl}(2, \mathbf{C})^{\oplus r})$ into $F_{\varphi(z)}^p(\mathfrak{g}_{\mathbf{C}})$ for any p.

Here in (4), for $F \in D$, the Hodge filtration on $\mathfrak{g}_{\mathbf{C}}$ induced by F is defined as

$$F^p_F(\mathfrak{g}_{\mathbf{C}}) = \{h \in \mathfrak{g}_{\mathbf{C}} \mid h(F^s) \subset F^{s+p} \ (\forall s)\}.$$

The Hodge filtration of $(\mathbf{C}^2)^{\oplus r} = \bigoplus_{j=1}^r (\mathbf{C}e_{1j} \oplus \mathbf{C}e_{2j})$ corresponding to z is defined as $F^0(z) = (\mathbf{C}^2)^{\oplus r}$, $F^1(z) = \bigoplus_{j=1}^r \mathbf{C}(z_j e_{1j} + e_{2j})$, and $F^2(z) = 0$.

Let $\mathbf{i} = (i, \ldots, i) \in \mathfrak{h}^r$. Then, if the condition (1) is satisfied, (3) is satisfied if and only if $\varphi(\mathbf{i}) \in D$, and (4) is satisfied if and only if ρ_* sends $F^p_{\mathbf{i}}(\mathfrak{sl}(2, \mathbf{C})^{\oplus r})$ into $F^p_{\varphi(\mathbf{i})}(\mathfrak{g}_{\mathbf{C}})$ for any p.

0.5.12

For an SL(2)-orbit (ρ, φ) of rank r, let $N_j \in \mathfrak{g}_{\mathbf{R}}$ be the image under ρ_* of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2, \mathbf{R})$ in the jth factor. Let

$$W^{(j)} = W(N_1 + \cdots + N_j) \quad (1 \le j \le r),$$

where $W(N)$ for a nilpotent linear operator N denotes the monodromy weight filtration associated with N (Deligne [D5]; see 5.2.4). The family $W = (W^{(j)})_{1 \le j \le r}$ is called the *family of weight filtrations associated with* (ρ, φ).

Definition 0.5.13 ([KU2, 3.6], 5.2.6) *Two* SL(2)*-orbits* (ρ_1, φ_1) *and* (ρ_2, φ_2) *of rank* r *are equivalent if there exists* $(t_1, \ldots, t_r) \in \mathbf{R}^r_{>0}$ *such that*

$$\rho_2 = \mathrm{Int}\left(\rho_1\left(\begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix}, \ldots, \begin{pmatrix} t_r^{-1} & 0 \\ 0 & t_r \end{pmatrix}\right)\right) \circ \rho_1,$$

$$\varphi_2 = \rho_1\left(\begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix}, \ldots, \begin{pmatrix} t_r^{-1} & 0 \\ 0 & t_r \end{pmatrix}\right) \cdot \varphi_1.$$

Here $\mathrm{Int}(g)$ *means the inner automorphism by* g.

Define $D_{\mathrm{SL}(2),r}$ *to be the set of all equivalence classes of* SL(2)*-orbits* (ρ, φ) *of rank* r *whose associated family of weight filtrations is defined over* $\mathbf{Q}$.

Define $D_{\mathrm{SL}(2)} = \bigsqcup_{r \ge 0} D_{\mathrm{SL}(2),r}$ *where* $D_{\mathrm{SL}(2),0} = D$.

For an SL(2)-orbit (ρ, φ) of rank r, the family W of weight filtrations associated with (ρ, φ) and the set $Z = \{\varphi(iy_1, \ldots, iy_r) \mid y_j > 0 \ (1 \le j \le r)\}$ are determined by the class $[\rho, \varphi]$ of (ρ, φ) in $D_{\mathrm{SL}(2)}$. Conversely, $[\rho, \varphi]$ is determined by the pair (W, Z) (see [KU2, 3.10]). We will denote $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$ also by (W, Z).

0.5.14

If the condition (2) in 0.5.11 is omitted, a pair (ρ, φ) is called an SL(2)*-orbit in* r *variables*.

For an SL(2)-orbit (ρ, φ) in n variables, there exists a unique SL(2)-orbit (ρ', φ') such that for some $J \subset \{1, \ldots, n\}$, $(\rho, \varphi) = (\rho', \varphi') \circ \pi_J$, where $\pi_J : (\mathrm{SL}(2, \mathbf{C}) \times \mathbf{P}^1(\mathbf{C}))^n \to (\mathrm{SL}(2, \mathbf{C}) \times \mathbf{P}^1(\mathbf{C}))^r$ is the projection to the J-factor and such that (ρ', φ') is an SL(2)-orbit of rank $r := \sharp(J)$. We denote the point $[\rho', \varphi']$ of $D_{\mathrm{SL}(2)}$ also by $[\rho, \varphi]$.

The notion of the SL(2)-orbit generalizes the (H_1)-homomorphism in the context of equivariant holomorphic maps of symmetric domains (cf. [Sa2, II § 8]).

In the classical situation (0.4.14), the Satake-Baily-Borel compactification of $\Gamma\backslash D$ for a subgroup Γ of $G_{\mathbf{Z}}$ of finite index is a quotient of $\Gamma\backslash D_{\mathrm{SL}(2)}$, and is philosophically close to $\Gamma\backslash D_{\mathrm{SL}(2)}$.

0.5.15

In 5.2.13, we will review the definition of the topology of $D_{\mathrm{SL}(2)}$ given in [KU2]. In this topology, for $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$, we have

$$\varphi(iy_1, \ldots, iy_n) \to [\rho, \varphi] \quad \text{if } y_j \in \mathbf{R}_{>0} \text{ and } y_j/y_{j+1} \to \infty \ (y_{n+1} \text{ denotes } 1).$$

Theorem 0.5.16 (5.2.16, 5.2.15)

(i) *Let $p \in D_{\mathrm{SL}(2)}$ be an element of rank r. Then there are an open neighborhood U of p in $D_{\mathrm{SL}(2)}$, a finite dimensional vector space V over* $\mathbf{R}$, $\mathbf{R}$*-vector subspaces V_J of V given for each subset J of the set $\{1, \ldots, r\}$, satisfying $V_J \supset V_{J'}$ if $J \subset J' \subset \{1, \ldots, r\}$ and $V_\emptyset = V$, and a homeomorphism*

$$U \simeq \{(a, t) \in V \times \mathbf{R}^r_{\geq 0} \mid a \in V_J \text{ where } J = \{j \mid t_j = 0\}\}$$

which sends p to $(0, 0)$.

(ii) *For any subgroup Γ of $G_{\mathbf{Z}}$, $\Gamma\backslash D_{\mathrm{SL}(2)}$ is Hausdorff.*

(iii) *If Γ is a neat subgroup of $G_{\mathbf{Z}}$, the projection $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ is a local homeomorphism.*

0.5.17

We consider the relation between D_{BS} and $D_{\mathrm{SL}(2)}$.

In D_{BS}, the direction of degeneration is determined by a parabolic subgroup of $G_{\mathbf{R}}$. On the other hand, in $D_{\mathrm{SL}(2)}$, it is determined by a family of weight filtrations.

Let $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$ be an element of rank r and let $W = (W^{(j)})_{1\leq j\leq r}$ be the family of weight filtrations associated with (ρ, φ).

Let $G^\circ_{\mathbf{R}}$ be the kernel of the determinant map $G_{\mathbf{R}} \to \{\pm 1\}$, and let $G^\circ_{W,\mathbf{R}}$ be the subgroup of $G^\circ_{\mathbf{R}}$ consisting of all elements which preserve $W^{(j)}_k$ for any j, k. If $r = 1$, i.e., if W consists of one weight filtration, $G^\circ_{W,\mathbf{R}}$ is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. Let $D_{\mathrm{SL}(2),\leq 1}$ be the part of $D_{\mathrm{SL}(2)}$ consisting of all elements of rank ≤ 1. Then $D_{\mathrm{SL}(2),\leq 1}$ is an open set of $D_{\mathrm{SL}(2)}$. The identity map of D is extended to a continuous map $D_{\mathrm{SL}(2),\leq 1} \to D_{\mathrm{BS}}$ which has the form $(W, Z) \mapsto (P, Z')$, where $P = G^\circ_{W,\mathbf{R}}$ and Z' is a certain subset of D containing Z (5.1.5).

Even for $r \geq 2$, in the case $h^{p,q} = 0$ for any $(p, q) \neq (1, 0), (0, 1)$ (the case $D = \mathfrak{h}_g$), the associated family $W = (W^{(j)})_{1\leq j\leq r}$ of weight filtrations is so simple that all filters in this family are linearly ordered,

$$0 = W^{(j)}_{-2} \subset W^{(1)}_{-1} \cdots \subset W^{(r)}_{-1} \subset W^{(r)}_0 \subset \cdots \subset W^{(1)}_0 \subset W^{(j)}_1 = H_{\mathbf{Q}},$$

for any j, i.e., they form a single long filtration. With one exception below, the same is true for other classical situations in 0.4.14. Hence W and P are related directly by $P = G^{\circ}_{W,\mathbf{R}}$, and we have $D_{\mathrm{SL}(2)}(W) = D_{\mathrm{BS}}(G^{\circ}_{W,\mathbf{R}})$, $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ (which also coincides with the classical Borel-Serre space $\mathcal{X}_{\mathrm{BS}}$) as in (5) in Introduction (cf. 12.1.2, [KU2, 6.7]).

Exceptional Case ([KU2, 6.7]). The weight w is even, rank $H_0 = 4$, and there exists a $\mathbf{Q}$-basis $(e_j)_{1\le j\le 4}$ of $H_{0,\mathbf{Q}}$ such that $\langle e_j, e_k\rangle_0 = 1$ if $j+k=5$, and $=0$ otherwise.

In general, we have the following criterion.

Criterion ([KU2, 6.3]). *The following are equivalent.*

(i) *The identity map of D extends to a continuous map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$.*
(ii) *At any point of $D_{\mathrm{SL}(2)}$, the filters $W^{(j)}_k$ which appear in the associated family $W = (W^{(j)})_j$ of weight filtrations are linearly ordered by inclusion.*

For examples with no continuous extension $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ of the identity map of D, see [KU2, 6.10] and 12.4.7.

0.5.18

We have the following criterion for the local compactness of $D_{\mathrm{SL}(2)}$.

Criterion (Theorem 10.1.6). *Let $p = [\rho, \varphi] \in D_{\mathrm{SL}(2)}$. The following* (i) *and* (ii) *are equivalent.*

(i) *There exists a compact neighborhood of p in $D_{\mathrm{SL}(2)}$.*
(ii) *The following conditions* (1) *and* (2) *hold.*

(1) *All filters $W^{(j)}_k$ appeared in the associated compatible family $W = (W^{(j)})_j$ of weight filtrations at p are linearly ordered by inclusion.*
(2) $\mathrm{Lie}(K_{\mathbf{r}}) \subset \mathrm{Lie}(G_{W,\mathbf{R}}) + \mathrm{Lie}(K'_{\mathbf{r}})$, *where* $\mathbf{r} = \varphi(\mathbf{i})$ *and* $G_{W,\mathbf{R}} = \{g \in G_{\mathbf{R}} \mid gW^{(j)}_k = W^{(j)}_k\ (\forall j, \forall k)\}$.

By this criterion, $D_{\mathrm{SL}(2)}$ for the example with $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ in 0.3.2 (iii) is locally compact, and hence has no slit. We have $D_{\mathrm{SL}(2)} = D_{\mathrm{SL}(2),\le 1} \xrightarrow{\sim} D_{\mathrm{BS}}$ in this case. But even for the examples of similar kind in 12.2.10, $D_{\mathrm{SL}(2)}$ can have slits in general.

0.5.19

Valuative spaces. In the general case, the family W of weight filtrations associated with $p \in D_{\mathrm{SL}(2)}$ becomes more complicated and we do not have a direct relation between the W and the parabolic subgroups P (Criterion in 0.5.17). We have to introduce the valuative spaces $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\mathrm{BS},\mathrm{val}}$ to relate the spaces $D_{\mathrm{SL}(2)}$ and D_{BS}. These are the projective limits of certain kinds of blow-ups of the respective spaces.

To relate the spaces D_Σ and $D_{\mathrm{SL}(2)}$, we also have to introduce the valuative space $D^{\natural}_{\Sigma,\mathrm{val}}$ of $D^{\natural}_{\Sigma}$. This is the projective limit of a kind of blow-up of $D^{\natural}_{\Sigma}$ corresponding to rational subdivisions of the fan Σ. We have a continuous map $D^{\natural}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ which is a geometric interpretation of the SL(2)-orbit theorem [CKS] as in 0.5.24 below.

In all cases, the valuative spaces are projective limits over the corresponding original spaces so as to divide the directions of degenerations into narrower and narrower.

THEOREM 0.5.20 *Let X be one of $D^{\natural}_{\Sigma,\mathrm{val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{BS},\mathrm{val}}$. Then*

(i) *X is Hausdorff.*

(ii) *Let Γ be a subgroup of $G_{\mathbf{Z}}$. In the case X is $D^{\natural}_{\Sigma,\mathrm{val}}$, assume Γ is strongly compatible with Σ. Then $\Gamma\backslash X$ is Hausdorff. If furthermore Γ is neat, the canonical projection $X \to \Gamma\backslash X$ is a local homeomorphism.*

The definitions of the four valuative spaces in the fundamental diagram are given in Chapter 5. Here we just give in 0.5.22–0.5.23 explicit local descriptions of them in the case of Example (ii) with $g = 2$ in 0.3.2. (The cases of Examples (i) and (iii) are not interesting concerning valuative spaces, for, in these cases, the valuative spaces are identified with the spaces under them in the fundamental diagram.)

0.5.21

Example $(\mathbf{C}^2)_{\mathrm{val}}$ *and* $(\mathbf{R}^2_{\geq 0})_{\mathrm{val}}$. In 0.5.22 and 0.5.23, we give explicit descriptions of some valuative spaces in the case $D = \mathfrak{h}_2$. For this, we introduce here the spaces $(\mathbf{C}^2)_{\mathrm{val}}$ and $(\mathbf{R}^2_{\geq 0})_{\mathrm{val}}$ obtained as projective limits of blow-ups from $\mathbf{C}^2$ and $\mathbf{R}^2_{\geq 0}$, respectively. In general, for any object X of $\mathcal{B}(\log)$, we will define in Section 3.6 (see 3.6.18 and 3.6.23) a space X_{val} obtained from X by taking blow-ups along the logarithmic structure. The space $(\mathbf{C}^2)_{\mathrm{val}}$ is X_{val} for $X = \mathbf{C}^2$ which is endowed with the logarithmic structure associated with the normal crossing divisor $\mathbf{C}^2 - (\mathbf{C}^\times)^2$.

Let $X = X_0 = \mathbf{C}^2$, and let X_1 be the blow-up of X at the origin $(0, 0)$. Then $(\mathbf{C}^\times)^2 \subset X_1$, and the complement $X_1 - (\mathbf{C}^\times)^2$ is the union of three irreducible divisors C_0, C_1, C_∞ where C_0 is the closure of $\{0\} \times \mathbf{C}^\times$, C_∞ is the closure of $\mathbf{C}^\times \times \{0\}$, and C_1 is the inverse image of $(0, 0)$. Next let X_2 be the blow-up of X_1 at two points, the intersection of C_0 and C_1 and the intersection of C_1 and C_∞. Then $(\mathbf{C}^\times)^2 \subset X_2$, and the complement $X_2 - (\mathbf{C}^\times)^2$ is the union of five irreducible divisors C_0, $C_{1/2}$, C_1, C_2, and C_∞. Here $C_{1/2}$ is the inverse image of the intersection of C_0 and C_1 in X_1, C_2 is the inverse image of the intersection of C_1 and C_∞, and we denote the proper transformations of C_0, C_1, and C_∞ in X_1 simply by C_0, C_1, and C_∞, respectively. In this way, we have a sequence of blow-ups

$$\cdots \to X_3 \to X_2 \to X_1 \to X_0 = X,$$

where X_{n+1} is obtained from X_n by blow-up the intersections of different irreducible components of $X_n - (\mathbf{C}^\times)^2$. We define

$$(\mathbf{C}^2)_{\mathrm{val}} = \varprojlim_n X_n.$$

This $(\mathbf{C}^2)_{\text{val}}$ is obtained also as the inverse limit of the toric varieties [KKMS, Od] corresponding to finite rational subdivisions of the cone $\mathbf{R}^2_{\geq 0}$ in $\mathbf{R}^2$. For example, the above X_2 is the toric variety corresponding to the finite subdivision of $\mathbf{R}^2_{\geq 0}$ consisting of the subcones $\{0\}$, σ_0, $\sigma_{0,1/2}$, $\sigma_{1/2}$, $\sigma_{1/2,1}$, σ_1, $\sigma_{1,2}$, σ_2, $\sigma_{2,\infty}$, and σ_∞ of $\mathbf{R}^2_{\geq 0}$. Here σ_s ($s = 0,\ 1/2,\ 1,\ 2$) is the half line $\{(x, sx) \mid x \in \mathbf{R}_{\geq 0}\}$ of slope s, σ_∞ is the half line $\{0\} \times \mathbf{R}_{\geq 0}$, and $\sigma_{s,t}$ is the cone generated by σ_s and σ_t. The open subvariety of X_2 corresponding to the cone σ_s is $X_2 - \cup_{s' \neq s} C_{s'}$, and the open subvariety of X_2 corresponding to the cone $\sigma_{s,t}$ is $X_2 - \cup_{s' \neq s,t} C_{s'}$. Let q_1 and q_2 be the coordinate functions of $\mathbf{C}^2$. Then for a finite rational subdivision of $\mathbf{R}^2_{\geq 0}$, the corresponding toric variety is the union of the open subvarieties $\operatorname{Spec}(\mathbf{C}[P(\sigma)])_{\text{an}}$ for cones σ in this subdivision, where

$$P(\sigma) = \{q_1^m q_2^n \mid (m, n) \in \mathbf{Z},\ am + bn \geq 0\ \ \forall (a, b) \in \sigma\}.$$

The projection $f : (\mathbf{C}^2)_{\text{val}} \to \mathbf{C}^2$ is proper and surjective, and f induces a homeomorphism $f^{-1}(\mathbf{C}^2 - \{(0,0)\}) \xrightarrow{\sim} \mathbf{C}^2 - \{(0,0)\}$. We regard $\mathbf{C}^2 - \{(0,0)\}$ as an open subspace of $(\mathbf{C}^2)_{\text{val}}$ via f^{-1}. The complement $f^{-1}((0,0))$ is described as

$$\begin{aligned} f^{-1}((0,0)) = \{&(0,0)_s \mid s \in [0,\infty],\ s \notin \mathbf{Q}_{>0}\} \\ &\cup \{(0,0)_{s,z} \mid s \in \mathbf{Q}_{>0},\ z \in \mathbf{P}^1(\mathbf{C})\}. \end{aligned}$$

Here if $s \in \mathbf{Q}_{>0}$ and $z \in \mathbf{C}^\times$ and if s is expressed as m/n with $m, n \in \mathbf{Z}, m > 0$, $n > 0$ and $\operatorname{GCD}(m, n) = 1$, then $(0,0)_{s,z}$ is the unique point of $f^{-1}((0,0))$ at which both $q_1^m q_2^{-n}$ and $q_1^{-m} q_2^n$ are holomorphic and the value of $q_1^m q_2^{-n}$ is z. For any finite rational subdivision of $\mathbf{R}^2_{\geq 0}$ containing the half line $\sigma = \{(x, sx) \mid x \in \mathbf{R}_{\geq 0}\}$ of slope s, the map from $(\mathbf{C}^2)_{\text{val}}$ to the toric variety corresponding to this subdivision induces a bijection from $\{(s, z) \mid z \in \mathbf{C}^\times\}$ onto the fiber over $(0,0) \in \mathbf{C}^2$ of the open subvariety corresponding to σ. For $s \in \mathbf{Q}_{>0}$, $(0,0)_{s,0}$ (resp. $(0,0)_{s,\infty}$) is the limit of $(0,0)_{s,z}$ ($z \in \mathbf{C}^\times$) for $z \to 0$ (resp. $z \to \infty$). The point $(0,0)_0$ (resp. $(0,0)_\infty$) is the limit of $(0, z) \in \mathbf{C}^2$ (resp. $(z, 0) \in \mathbf{C}^2$) ($z \in \mathbf{C}^\times$) for $z \to 0$. Finally, $(0,0)_s$ for $s \in \mathbf{R}_{>0} - \mathbf{Q}_{>0}$ is the unique point of $f^{-1}((0,0))$ at which $q_1^m q_2^{-n}$ ($m, n \in \mathbf{Z}, m > 0$, $n > 0$) is holomorphic if and only if $m/n > s$ and $q_1^{-m} q_2^n$ is holomorphic if and only if $s > m/n$. A point of $f^{-1}((0,0))$ has the form $(0,0)_s$ ($s \in [0,\infty] - \mathbf{Q}_{>0}$) if and only if for any $n \geq 0$, its image in X_n is the intersection of two different irreducible components of the divisor $X_n - (\mathbf{C}^\times)^2$.

These points of $f^{-1}((0,0))$ are characterized as the limits of points of $(\mathbf{C}^\times)^2 \subset (\mathbf{C}^2)_{\text{val}}$ as follows. If $s \in [0,\infty] - \mathbf{Q}_{>0}$, $(q_1, q_2) \in (\mathbf{C}^\times)^2$ converges to $(0,0)_s$ if and only if $(q_1, q_2) \to (0,0)$ and $\log(|q_2|)/\log(|q_1|) \to s$. If $s \in \mathbf{Q}_{>0}$ and $z \in \mathbf{P}^1(\mathbf{C})$, and if s is expressed as m/n with $m, n \in \mathbf{Z}, m > 0, n > 0$ and $\operatorname{GCD}(m, n) = 1$, $(q_1, q_2) \in (\mathbf{C}^\times)^2$ converges to $(0,0)_{s,z}$ if and only if $(q_1, q_2) \to (0,0)$, $\log(|q_2|)/\log(|q_1|) \to s$, and $q_1^m / q_2^n \to z$.

Let $(\mathbf{R}^2_{\geq 0})_{\text{val}} \subset (\mathbf{C}^2)_{\text{val}}$ be the closure of the subset $\mathbf{R}^2_{\geq 0} - \{(0,0)\}$ (regarded as a subset of $\mathbf{C}^2 - \{(0,0)\} \subset (\mathbf{C}^2)_{\text{val}}$). That is, $(\mathbf{R}^2_{\geq 0})_{\text{val}}$ is the union of $\mathbf{R}^2_{\geq 0} - \{(0,0)\}$ and the part of $f^{-1}((0,0))$ consisting of elements $(0,0)_s$ for $s \in [0,\infty]$ such that $s \notin \mathbf{Q}_{>0}$, and elements $(0,0)_{s,z}$ with $s \in \mathbf{Q}_{>0}$ and with $z \in \mathbf{R}_{\geq 0} \cup \{\infty\} \subset \mathbf{P}^1(\mathbf{C})$. The canonical projection $(\mathbf{R}^2_{\geq 0})_{\text{val}} \to \mathbf{R}^2_{\geq 0}$ is proper and surjective. The inverse image of $(0,0)$ in $(\mathbf{R}^2_{\geq 0})_{\text{val}}$ is regarded as a very long totally ordered set by the following rule: $(0,0)_s < (0,0)_{s',z} < (0,0)_{s',z'} < (0,0)_{s''}$ if $0 \le s < s' < s'' \le \infty$, $s \notin \mathbf{Q}_{>0}$, $s' \in \mathbf{Q}_{>0}$, $s'' \notin \mathbf{Q}_{>0}$, $0 \le z < z' \le \infty$. Closed intervals form a base of closed sets in this totally ordered set.

0.5.22

Upper half space (continued). Let $g = 2$ and $D = \mathfrak{h}_2$. We have the following commutative diagrams of topological spaces in which all inclusions are open immersions.

$$\begin{array}{ccccc} (\Delta^2)_{\text{val}} \times \mathbf{C} & \supset & U_{\text{val}} & \simeq & \Gamma\backslash D_{\sigma,\text{val}} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^2 \times \mathbf{C} & \supset & U & \simeq & \Gamma\backslash D_\sigma . \end{array} \qquad (1)$$

$$\begin{array}{ccccc} (|\Delta|^2)_{\text{val}} \times \mathbf{R}^2 \times \mathbf{C} & \supset & \tilde{U}^{\log}_{\text{val}} & \simeq & D^{\sharp}_{\sigma,\text{val}} \\ \downarrow & & \downarrow & & \downarrow \\ |\Delta|^2 \times \mathbf{R}^2 \times \mathbf{C} & \supset & \tilde{U}^{\log} & \simeq & D^{\sharp}_{\sigma} . \end{array} \qquad (2)$$

$$\begin{array}{ccc} P_u \times (\mathbf{R}^2_{\ge 0})_{\text{val}} & \simeq & D_{\text{BS},\text{val}}(P) \\ \downarrow & & \downarrow \\ P_u \times \mathbf{R}^2_{\ge 0} & \simeq & D_{\text{BS}}(P). \end{array} \qquad (3)$$

Here in (1), $(\Delta^2)_{\text{val}} \subset (\mathbf{C}^2)_{\text{val}}$ is the inverse image of Δ^2 under $(\mathbf{C}^2)_{\text{val}} \to \mathbf{C}^2$, U and Γ are as in 0.4.13, and U_{val} denotes the inverse image of $U \subset \Delta^2 \times \mathbf{C}$ in $(\Delta^2)_{\text{val}} \times \mathbf{C}$. In (2), $(|\Delta|^2)_{\text{val}} \subset (\mathbf{R}^2_{\ge 0})_{\text{val}}$ is the inverse image of $|\Delta|^2$ under $(\mathbf{R}_{\ge 0})^2_{\text{val}} \to \mathbf{R}^2_{\ge 0}$, $\tilde{U}^{\log}$ is as in 0.5.6, and $\tilde{U}^{\log}_{\text{val}}$ denotes the inverse image of $\tilde{U}^{\log}$ in $(|\Delta|^2)_{\text{val}} \times \mathbf{R}^2 \times \mathbf{C}$. In (3), P is as in 0.5.9, and $D_{\text{BS},\text{val}}(P)$ denotes the inverse image of $D_{BS}(P)$ in $D_{\text{BS},\text{val}}$.

The lower rows of these diagrams are those obtained in 0.4.13, 0.5.6, and 0.5.9, respectively.

The first diagram is obtained as follows. We identify σ with the cone $\mathbf{R}^2_{\ge 0}$ via $\mathbf{R}^2_{\ge 0} \simeq \sigma$, $(a_1, a_2) \mapsto a_1 N_1 + a_2 N_2$. For a rational subdivision S of $\mathbf{R}^2_{\ge 0}$, we have the corresponding subdivision of σ. If $B(S)$ denotes the toric variety corresponding to S with a proper birational morphism $B(S) \to \mathbf{C}^2$, we have an isomorphism $U(S) \simeq \Gamma \backslash D_\sigma(S)$, where $U(S)$ is the inverse image of U in $B(S) \times \mathbf{C}$ and $\Gamma \backslash D_\sigma(S)$ is a blow-up of $\Gamma \backslash D_\sigma$ corresponding to this subdivision of σ. The upper row of the diagram (1) is obtained as the projective limit of $B(S) \times \mathbf{C} \supset U(S) \simeq \Gamma\backslash D_\sigma(S)$.

Roughly speaking, in the first diagram, we are dividing the direction of degeneration $\exp(z_1 N_1 + z_2 N_2)$ with $\text{Im}(z_1), \text{Im}(z_2) \to \infty$ into narrower and narrower directions. A narrow direction that appears here is the direction $\exp(z_1(N_1 + sN_2) + z_2(N_1 + s'N_2))$ with $\text{Im}(z_1), \text{Im}(z_2) \to \infty$ for some $s, s' \in \mathbf{Q}_{\ge 0}$ or the direction $\exp(z_1(N_1 + sN_2) + z_2 N_2)$ with $\text{Im}(z_1)$, $\text{Im}(z_2) \to \infty$ for some $s \in \mathbf{Q}_{>0}$. When the directions become infinitely narrow, we obtain points at infinity in $\Gamma\backslash D_{\sigma,\text{val}}$.

The second diagram is obtained in a similar manner. For the third diagram, see [KU2, 2.14]. Note that $D_{\text{SL}(2),\text{val}} = D_{\text{BS},\text{val}}$ in this case [KU2, 6.7].

0.5.23

Upper half space (continued). Let the notation be as in 0.5.22 and consider $\exp(iy_1N_1 + iy_2N_2)F(0) \in D$ for $y_1, y_2 \in \mathbf{R}_{>0}$. We observe how this point converges or diverges when y_1 and y_2 move in special ways. This point corresponds to $(e^{-2\pi y_1}, e^{-2\pi y_2}, 0, 0, 0) \in \tilde{U}^{\log}_{\mathrm{val}}$ in the diagram (2), and to $(1, (y_2/y_1)^{1/2}, (1/y_2)^{1/2}) \in P_u \times (\mathbf{R}_{>0})^2$ in the diagram (3). From this, we have

(i) When $t \to \infty$, $\exp(it(2+\sin(t))N_1 + itN_2)F(0)$ converges in $D^\sharp_\sigma$ to the image of $(0, 0, 0, 0, 0) \in \tilde{U}^{\log}$, but diverges in $D^\sharp_{\sigma,\mathrm{val}}$, D_{BS}, $D_{\mathrm{BS,val}}$.

We show here the divergence in D_{BS}. The corresponding point is $p(t) := (1, (2+\sin(t))^{-1/2}, t^{-1/2}) \in P_u \times \mathbf{R}^2_{\geq 0}$, and

$$p(t) = \begin{cases} (1, 2^{-1/2}, t^{-1/2}) \to (1, 2^{-1/2}, 0) \in P_u \times \mathbf{R}^2_{\geq 0} \\ \qquad \text{when } t = \pi n,\ n = 1, 2, 3, \ldots, \\ (1, 1, t^{-1/2}) \to (1, 1, 0) \in P_u \times \mathbf{R}^2_{\geq 0} \\ \qquad \text{when } t = 2\pi n - \pi/2,\ n = 1, 2, 3, \ldots. \end{cases}$$

Hence $p(t)$ diverges in D_{BS}.

This (i) shows that the image of $(0, 0, 0, 0, 0) \in \tilde{U}^{\log}$ in $D^\sharp_\sigma$ has no neighborhood V such that the inclusion map $V \cap D \to D$ extends to a continuous map $V \to D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$.

We also see at the end of 0.5.26, concerning a map in the converse direction, that the image of $(1, 0, 0) \in P_u \times \mathbf{R}^2_{\geq 0}$ in D_{BS} has no neighborhood V such that the inclusion map $V \cap D \to D$ extends to a continuous map $V \to D^\sharp_\sigma$, and even that it has no neighborhood V such that the canonical map $V \cap D \to \Gamma\backslash D$ extends to a continuous map $V \to \Gamma\backslash D_\sigma$.

Thus, to connect the world of D_σ and $D^\sharp_\sigma$ to the world of $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$, we have to climb to the valuative space $D^\sharp_{\sigma,\mathrm{val}}$.

Similarly, by using the topological natures of $|\Delta|^2_{\mathrm{val}}$ and $(\mathbf{R}^2_{\geq 0})_{\mathrm{val}}$ explained in 0.5.21, we can show

(ii) When $t \to \infty$, $\exp(it^{c(2+\sin(t))}N_1 + itN_2)F(0)$, for a fixed $c > 1$ converges in $D^\sharp_{\sigma,\mathrm{val}}$ to the image of $((0, 0)_0, 0, 0, 0) \in \tilde{U}^{\log}_{\mathrm{val}}$, but diverges in $D_{\mathrm{BS,val}}$. Hence the image of $((0, 0)_0, 0, 0, 0)$ in $D^\sharp_{\sigma,\mathrm{val}}$ has no neighborhood V such that the inclusion map $V \cap D \to D$ extends to a continuous map $V \to D_{\mathrm{BS,val}}$. When $t \to \infty$, $\exp(i(t + \sin(t))N_1 + itN_2)F(0)$ converges in $D_{\mathrm{BS,val}}(P)$ to the image of $(1, (0, 0)_{1,1}) \in P_u \times (\mathbf{R}^2_{\geq 0})_{\mathrm{val}}$, but diverges in $D^\sharp_{\sigma,\mathrm{val}}$. Hence the image of $(1, (0, 0)_{1,1})$ in $D_{\mathrm{BS,val}}(P)$ has no neighborhood V such that the inclusion map $V \cap D \to D$ extends to a continuous map $V \to D^\sharp_{\sigma,\mathrm{val}}$.

Thus, the views of the infinity of various enlargements of D in the fundamental diagram are rather different from each other.

(iii) When $y_2 \to \infty$ and $y_1/y_2 \to \infty$, $\exp(iy_1N_1 + iy_2N_2)F(0)$ converges in $D^\sharp_{\sigma,\mathrm{val}}$ to the image of $((0, 0)_0, 0, 0, 0) \in \tilde{U}^{\log}_{\mathrm{val}}$, and also converges in $D_{\mathrm{BS}}(P)$

to the image of $(1, 0, 0) \in P_n \times \mathbf{R}^2_{\geq 0}$. As is explained in 0.5.26 below, these two limit points at infinity are related by the following continuous map $\psi : D^{\sharp}_{\sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$.

0.5.24

SL(2)-*orbit Theorem and continuous map* $D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$. The SL(2)-orbit theorem in several variables in [CKS] is interpreted as the relation between $D^{\sharp}_{\Sigma,\mathrm{val}}$ and $D_{\mathrm{SL}(2)}$.

Let $N_1, \dots, N_n \in \mathfrak{g}_{\mathbf{R}}$ be mutually commutative nilpotent elements, and $F \in D$. Assume that $(N_1, \dots, N_n, F)$ generates a nilpotent orbit, i.e., for $\sigma = \sum_{1\leq j\leq n}(\mathbf{R}_{\geq 0})N_j$, $\exp(\sigma_{\mathbf{C}})F$ is a σ-nilpotent orbit (0.4.7, 1.3.7). Cattani, Kaplan, and Schmid [CKS] defined an SL(2)-orbit (ρ, φ) in n variables associated to the family $(N_1, \dots, N_n, F)$ (cf. Section 6.1). Here the order of $N_1, \dots, N_n$ is important. They showed that two maps $\exp(\sum_{j=1}^n iy_jN_j)F$ and $\varphi(iy_1, \dots, iy_n)$ into D behave asymptotically when $y_j/y_{j+1} \to \infty$ (y_{n+1} means 1). Our geometric interpretation of the SL(2)-orbit Theorem is as follows.

There is a unique continuous map $\psi : D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ *which extends the identity map of* D *(we will prove this in* Chapter 6*),* $\exp(\sum_{j=1}^n iy_jN_j)F$ *converges in* $D^{\sharp}_{\Sigma,\mathrm{val}}$ *when* $y_j/y_{j+1} \to \infty$ $(1 \leq j \leq n)$ *(we saw this in* 0.5.23 (iii) *in a special case), and* ψ *sends this limit point to* $[\rho, \varphi]$.

This continuous map $\psi : D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ is the most important bridge in the fundamental diagram (3) in Introduction, which joins the four spaces D_Σ, $D_{\Sigma,\mathrm{val}}$, $D^{\sharp}_\Sigma$, and $D^{\sharp}_{\Sigma,\mathrm{val}}$ of orbits under nilpotent groups in the left-hand side and the four spaces $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, D_{BS}, and $D_{\mathrm{BS},\mathrm{val}}$ of orbits under tori in the right-hand side.

0.5.25

Fundamental Diagram. We thus have the diagram that relates D_Σ and D_{BS} ((3) in Introduction; see also 5.0.1):

$$
\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
\Gamma\backslash D_{\Sigma,\mathrm{val}} & \leftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
\Gamma\backslash D_\Sigma & \leftarrow & D^{\sharp}_\Sigma & & & &
\end{array}
$$

where all maps are continuous, and all vertical maps are proper surjective.

The theorems on these spaces introduced in this Section 0.5 are proved by starting at D_{BS} and moving in this diagram from the right to the left. The spaces D_{BS}, $D_{\mathrm{BS},\mathrm{val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, and $D_{\mathrm{SL}(2)}$ were already studied in [KU2]. By using the results on these spaces (which are reviewed in Chapter 5), in this book, we prove the results on the other spaces.

We now give explicit descriptions of $\psi : D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ in some examples. In the case of Example (i) (the case $D = \mathfrak{h}$) in 0.3.2, for any fan Σ in $\mathfrak{g}_{\mathbf{Q}}$, ψ is just the canonical map $D^{\sharp}_{\Sigma,\mathrm{val}} = D^{\sharp}_{\Sigma} \subset D^{\sharp}_{\Xi} \xrightarrow{\sim} D_{\mathrm{SL}(2)}$ in 0.5.2. In the following 0.5.26 and 0.5.27, we consider the cases of Example (ii) with $g = 2$ and Example (iii) in 0.3.2, respectively.

0.5.26

Upper half space (continued). Let $g = 2$ and $D = \mathfrak{h}_2$. In this case, $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$. Let N_1, N_2, and σ be as before (0.4.13). The triple $(N_1, N_2, F(0))$ generates a σ-nilpotent orbit, and the associated SL(2)-orbit in two variables (ρ, φ) is given (see 6.1.4) by

$$\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & e & 0 & f \\ c & 0 & d & 0 \\ 0 & g & 0 & h \end{pmatrix}, \qquad \varphi(z, w) = F\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}.$$

That is, the continuous map $\psi : D^{\sharp}_{\sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ sends the limit of $\exp(iy_1N_1 + iy_2N_2)F(0)$ for y_1/y_2, $y_2 \to \infty$ in $D^{\sharp}_{\sigma,\mathrm{val}}$ (see 0.5.23) to $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$. Furthermore, (ρ, φ) is of rank 2, and "the N_1 and N_2 of ρ" defined in 0.5.12 coincide with N_1 and N_2 here, respectively. Let $W = (W^{(1)}, W^{(2)})$ be the family of weight filtrations associated with $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$, that is, $W^{(1)} = W(N_1)$ and $W^{(2)} = W(N_1 + N_2)$. Then

$$0 = W^{(1)}_{-2} \subset W^{(1)}_{-1} = \mathbf{R}e_1 \subset W^{(1)}_{0} = \mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_4 \subset W^{(1)}_{1} = H_{0,\mathbf{R}},$$
$$0 = W^{(2)}_{-2} \subset W^{(2)}_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 = W^{(2)}_{0} \subset W^{(2)}_{1} = H_{0,\mathbf{R}}.$$

We have $G_{W,\mathbf{R}} = P$, $D_{\mathrm{SL}(2)}(W) = D_{\mathrm{BS}}(P)$. The element $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$ is also written as $(W, Z) \in D_{\mathrm{SL}(2)}$ with $Z = F\begin{pmatrix} i\mathbf{R}_{>0} & 0 \\ 0 & i\mathbf{R}_{>0} \end{pmatrix}$, and is also written as $(P, Z) \in D_{\mathrm{BS}}$ with the same Z.

In the homeomorphism $\tilde{U}^{\log}_{\mathrm{val}} \simeq D^{\sharp}_{\sigma,\mathrm{val}}$ in 0.5.22, the above limit point of $D^{\sharp}_{\sigma,\mathrm{val}}$ is the image of $((0,0)_0, 0, 0, 0) \in \tilde{U}^{\log}_{\mathrm{val}} \subset (|\Delta|^2)_{\mathrm{val}} \times \mathbf{R}^2 \times \mathbf{C}$. In the homeomorphism $P_u \times \mathbf{R}^2_{\geq 0} \simeq D_{\mathrm{BS}}(P)$, the above $[\rho, \varphi] \in D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ is the image of $(1, 0, 0)$.

We give an explicit description of $\psi : D^{\sharp}_{\sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ for some open neighborhood of the above limit point of $D^{\sharp}_{\sigma,\mathrm{val}}$. Let $(\tilde{U}^{\log}_{\mathrm{val}})'$ be the open set of $\tilde{U}^{\log}_{\mathrm{val}}$ consisting of all points (p, x_1, x_2, a) ($p \in (|\Delta|^2)_{\mathrm{val}}$, $x_1, x_2 \in \mathbf{R}$, $a \in \mathbf{C}$) such that $p \neq (0,0)_\infty$ and such that $p \neq (r, 0)$ for any $r \in |\Delta| - \{0\}$. Then $(\tilde{U}^{\log}_{\mathrm{val}})'$ contains the point $((0,0)_0, 0, 0, 0)$ of $\tilde{U}^{\log}_{\mathrm{val}}$. Define $N_3, N_4 \in \mathrm{Lie}(P_u)$ by

$$N_3 := e_{23} + e_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_4 := e_{43} - e_{12} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have a homeomorphism

$$\mathbf{R}^4 \xrightarrow{\sim} P_u, \quad (x_j)_{1\le j\le 4} \mapsto \exp\left(\sum\nolimits_{j=1}^{3} x_j N_j\right)\exp(x_4N_4).$$

The restriction of $\psi : D^{\sharp}_{\sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}(W) = D_{\mathrm{BS}}(P)$ to $(\tilde{U}^{\log}_{\mathrm{val}})'$ is explicitly described by the commutative diagram

$$\begin{array}{ccc} (|\Delta|^2)_{\mathrm{val}} \times \mathbf{R}^2 \times \mathbf{C} \supset (\tilde{U}^{\log}_{\mathrm{val}})' & \to & D^{\sharp}_{\sigma,\mathrm{val}} \\ & & \downarrow \psi \\ \downarrow & & D_{\mathrm{SL}(2)}(W) \\ & & \| \\ P_u \times \mathbf{R}^2_{\ge 0} & \xrightarrow{\sim} & D_{\mathrm{BS}}(P). \end{array}$$

Here the left vertical arrow sends $p = (r, x_1, x_2, x_3 + iy_3) \in (\tilde{U}^{\log}_{\mathrm{val}})'$ with $r \in (|\Delta|^2)_{\mathrm{val}}$ and $x_1, x_2, x_3, y_3 \in \mathbf{R}$ to the following element of $P_u \times \mathbf{R}^2_{\ge 0}$.

(1) When the image of r in $|\Delta|^2$ is not $(0, 0)$, if we write $r = (e^{-2\pi y_1}, e^{-2\pi y_2})$ with $0 < y_1 \le \infty$ and $0 < y_2 < \infty$, p is sent to

$$\left(\exp\left(\sum\nolimits_{j=1}^{3} x_jN_j\right)\exp\left(-\frac{y_3}{y_2}N_4\right), \frac{y_2}{y_1}\left(1 - \frac{y_3^2}{y_1y_2}\right)^{-1}, \frac{1}{y_2}\right) \in P_u \times \mathbf{R}^2_{\ge 0}.$$

(2) When the image of r in $|\Delta|^2$ is $(0, 0)$ and r has the form $(0, 0)_s$ or $(0, 0)_{s,z}$, p is sent to

$$\left(\exp\left(\sum_{j=1}^{3} x_jN_j\right), s, 0\right) \in P_u \times \mathbf{R}^2_{\ge 0}.$$

Note that, when $p \in (\tilde{U}^{\log}_{\mathrm{val}})'$ as in (1) converges to a point of $(\tilde{U}^{\log}_{\mathrm{val}})'$ as in (2) whose r has the form $(0, 0)_s$ or $(0, 0)_{s,z}$, then x_1, x_2, x_3, and y_3 converge, $y_1, y_2 \to \infty$, and $y_2/y_1 \to s$, and hence the terms y_3/y_2 and $1/y_2$ in (1) converge to 0 and the term $\frac{y_2}{y_1}\left(1 - \frac{y_3^2}{y_1y_2}\right)^{-1}$ in (1) converges to s.

Let $\Gamma = \exp(\mathbf{Z}N_1 + \mathbf{Z}N_2) = 1 + \mathbf{Z}N_1 + \mathbf{Z}N_2$. We show that, for any neighborhood V of $[\rho, \varphi] \in D_{\mathrm{SL}(2)}(W) = D_{\mathrm{BS}}(P)$, there is no continuous map $V \to \Gamma\backslash D_\sigma$ that extends the projection $V \cap D \to \Gamma\backslash D$. In fact, for any $c \in \mathbf{R}$, the image $p_c \in D^{\sharp}_{\sigma,\mathrm{val}}$ of $((0, 0)_0, 0, 0, ic) \in (\tilde{U}^{\log}_{\mathrm{val}})'$ is sent by ψ to $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$ which is independent of c. On the other hand, the image p'_c of p_c in $\Gamma\backslash D_\sigma$ is the class of the nilpotent orbit

$$\left(\sigma, \exp(\sigma_{\mathbf{C}})F\begin{pmatrix} 0 & ic \\ ic & 0 \end{pmatrix}\right)$$

modulo Γ, and we have $p'_c \ne p'_d$ if $c, d \in \mathbf{R}$ and $c \ne d$. Hence there is no V with $V \to \Gamma\backslash D_\sigma$ as above.

0.5.27

Example with $h^{2,0} = h^{0,2} = 2, h^{1,1} = 1$ (continued). In this example, the fundamental diagram becomes (see Criteria in 0.5.17 and in 0.5.18, and Theorem 10.1.6).

$$\begin{array}{ccccccc} & & & & D_{\mathrm{SL}(2),\mathrm{val}} & = & D_{\mathrm{BS},\mathrm{val}} \\ & & & & \| & & \| \\ D_{\Xi,\mathrm{val}} & \leftarrow & D^{\sharp}_{\Xi,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & = & D_{\mathrm{BS}}. \\ \| & & \| & & & & \\ D_{\Xi} & \leftarrow & D^{\sharp}_{\Xi}. & & & & \end{array}$$

Let W be the $\mathbf{Q}$-rational increasing filtration of $H_{0,\mathbf{R}}$ defined by $W_k = 0$ $(k \le -1)$, $W_0 = W_1 = \mathbf{R}e_4$, $W_2 = W_3 = \sum_{j=1}^4 \mathbf{R}e_j$, $W_k = H_{0,\mathbf{R}}$ $(k \ge 4)$. On the other hand, let the $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$ be as in 0.5.10. Then, we have $D_{\mathrm{SL}(2)}(W) = D_{\mathrm{BS}}(P)$ (cf. 0.5.10), and the map $\psi : D^{\sharp}_{\Xi} = D^{\sharp}_{\Xi,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ is injective.

Let $v \in \mathbf{S}^2 \cap (\sum_{j=1}^3 \mathbf{Q}e_j)$ and let $\sigma = (\mathbf{R}_{\ge 0})N_v$. Then $\psi : D^{\sharp}_{\Xi} \to D_{\mathrm{SL}(2)}$ sends $D^{\sharp}_{\sigma}$ into $D_{\mathrm{SL}(2)}(W)$. We consider the map $\psi : D^{\sharp}_{\sigma} = D^{\sharp}_{\sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}(W) = D_{\mathrm{BS}}(P)$.

Consider the exterior product $\times$ in $\sum_{j=1}^3 \mathbf{R}e_j$, i.e., the bilinear map $(\sum_{j=1}^3 \mathbf{R}e_j) \times (\sum_{j=1}^3 \mathbf{R}e_j) \to \sum_{j=1}^3 \mathbf{R}e_j$ characterized by

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2, \quad e_j \times e_k = -e_k \times e_j.$$

For $z \in Q$ and $u \in \sum_{j=1}^3 \mathbf{C}e_j$ such that u is not contained in $\mathbf{C}z + \sum_{j=1}^3 \mathbf{R}e_j$, we have $\exp(N_u)F(z) = \exp(N_b)s(t) \cdot \mathbf{r}(\theta(z))$ with $b = \mathrm{Re}(u) - \mathrm{Im}(u) \times \theta(z)$, $t = (-\langle \mathrm{Im}(u), \theta(z)\rangle_0)^{-1/2}$ (cf. Section 12.2).

Fix $v' \in \mathbf{S}^2$ which is orthogonal to v. From 0.5.6, 0.5.10, and 0.5.18, we have a commutative diagram

$$\begin{array}{cccc} ((\mathbf{R}_{\ge 0}) \times \mathbf{R}) \times \mathbf{C} \times Q \supset & \tilde{U}^{\log} & \rightarrow & D^{\sharp}_{\sigma} \\ & & & \downarrow \psi \\ & \downarrow & & D_{\mathrm{SL}(2)}(W) \\ & & & \| \\ & (\sum_{j=1}^3 \mathbf{R}e_j) \times (\mathbf{R}_{\ge 0}) \times \{\pm 1\} \times \mathbf{S}^2 & \xrightarrow{\sim} & D_{\mathrm{BS}}(P). \end{array}$$

The left vertical arrow sends $(r, x, a, z) \in \tilde{U}^{\log}$ with $r \neq 0$ to $(b, t, 1, \theta(z))$ where $b = xv + \mathrm{Re}(a)v' - (yv + \mathrm{Im}(a)v') \times \theta(z), t = (-\langle yv + \mathrm{Im}(a)v', \theta(z)\rangle_0)^{-1/2}$ with $y \in \mathbf{R}$ defined by $r = e^{-2\pi y/\ell}$ (ℓ is as in 0.4.18), and sends $(0, x, a, \theta^{-1}(v)) \in \tilde{U}^{\log}$ to $(xv + \mathrm{Re}(a)v' + \mathrm{Im}(a)v \times v', 0, 1, v)$.

We show the following two results.

(i) If we embed $D^{\sharp}_{\Xi}$ in $D_{\mathrm{SL}(2)}$ by the injection ψ, the topology of $D^{\sharp}_{\Xi}$ does not coincide with the topology as a subspace of $D_{\mathrm{SL}(2)}$.

(ii) Let us call the topology of $\tilde{U}^{\log}$ as a subspace of $((\mathbf{R}_{\geq 0}) \times \mathbf{R}) \times \mathbf{C} \times Q$ the naive topology. Then the composition $\tilde{U}^{\log} \to D^{\sharp}_{\sigma} \xrightarrow{\psi} D_{\mathrm{SL}(2)}(W) = D_{\mathrm{BS}}(P)$ is not continuous for the naive topology.

Let $p \in D^{\sharp}_{\sigma}$ be the image of $(0, 0, 0, \theta^{-1}(v)) \in \tilde{U}^{\log}$. Take $c > 0$ and consider the elements $f(s) = (\exp(-2\pi/(\ell s^{c})), 0, 0, \theta^{-1}((1-s^2)^{1/2}v + sv')) \in \tilde{U}^{\log}$ $(s > 0, s \to 0)$. In $\tilde{U}^{\log}$, when $s \to 0$, $f(s)$ converges to $(0, 0, 0, \theta^{-1}(v))$ for the naive topology. However, in U with the strong topology, the image $(\exp(-2\pi/(\ell s^{c})), 0, \theta^{-1}((1-s^2)^{1/2}v + sv'))$ of $f(s)$ does not converge to the image $(0, 0, \theta^{-1}(v))$ of $(0, 0, 0, \theta^{-1}(v))$ (0.4.15 (2)). Hence the image of $f(s)$ in $D^{\sharp}_{\sigma}$ does not converge to p (0.5.6).

The image of $f(s)$ in $(\sum_{j=1}^{3} \mathbf{R}e_j) \times (\mathbf{R}_{\geq 0}) \times \{\pm 1\} \times \mathbf{S}^2$ is

$$(s^{1-c}v' \times v,\ s^{c/2}(1-s^2)^{-1/4},\ 1,\ (1-s^2)^{1/2}v + sv').$$

This converges to $(0, 0, 1, v)$ if $c < 1$, but does not converge if $c > 1$ (by the existence of the term s^{1-c}).

For $c < 1$, this proves (i) because the image of $f(s)$ in $D^{\sharp}_{\sigma}$ does not converge to p but the image of $f(s)$ in $D_{\mathrm{SL}(2)}$ converges to $\psi(p)$.

For $c > 1$, this proves (ii) because $f(s)$ converges to $(0, 0, 0, \theta^{-1}(v))$ for the naive topology but the image of $f(s)$ in $D_{\mathrm{SL}(2)}$ does not converge to the image $\psi(p)$ of $(0, 0, 0, \theta^{-1}(v))$.

0.5.28

For an object X of $\mathcal{B}(\log)$, we will define a logarithmic local ringed space X_{val} over X in 3.6.18 and 3.6.23, by using the projective limit of blow-ups along the logarithmic structure. Though X_{val} need not belong to $\mathcal{B}(\log)$, we can define the topological space $(X_{\mathrm{val}})^{\log}$ (we denote it as $X^{\log}_{\mathrm{val}}$) in the same way as before. If $X = \mathbf{C}^2$ with the logarithmic structure associated with the normal crossing divisor $\mathbf{C}^2 - (\mathbf{C}^{\times})^2$, then

$$X_{\mathrm{val}} = (\mathbf{C})^2_{\mathrm{val}}, \quad X^{\log}_{\mathrm{val}} = (\mathbf{R}^2_{\geq 0})_{\mathrm{val}} \times (\mathbf{S}^1)^2.$$

For a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ and for a neat subgroup Γ of $G_{\mathbf{Z}}$ which is strongly compatible with Σ, we have

$$(\Gamma\backslash D_{\Sigma})_{\mathrm{val}} = \Gamma\backslash D_{\Sigma,\mathrm{val}}, \quad (\Gamma\backslash D_{\Sigma})^{\log}_{\mathrm{val}} = \Gamma\backslash D^{\sharp}_{\Sigma,\mathrm{val}}.$$

(See 8.4.3.)

In the classical situation 0.4.14, except for the unique exceptional case 0.5.17, the fundamental diagram and the fact that $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ in this situation show that there is a unique continuous map

$$(\Gamma\backslash D_{\Sigma})^{\log}_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{BS}}$$

which extends the identity map of D. Chikara Nakayama (unpublished) proved that in the classical situation 0.4.14, if we take Σ and Γ such that $\Gamma\backslash D_{\Sigma}$ is a toroidal compactification of $\Gamma\backslash D$, then $\Gamma\backslash D_{\Sigma,\mathrm{val}}$ coincides with the projective limit of all toroidal compactifications of $\Gamma\backslash D$.

In the general situation, we have

THEOREM 0.5.29 *Let X be a connected, logarithmically smooth, fs logarithmic analytic space, and let $U = X_{\mathrm{triv}} = \{x \in X \mid M_{X,x} = \mathcal{O}^{\times}_{X,x}\}$ be the open subspace of X consisting of all points of X at which the logarithmic structure of X is trivial. Let H be a variation of polarized Hodge structure on U with unipotent local monodromy along $X - U$. Fix a base point $u \in U$ and let $(H_0, \langle\ ,\ \rangle_0) = (H_{\mathbf{Z},u}, \langle\ ,\ \rangle_u)$. Let Γ be a subgroup of $G_{\mathbf{Z}}$ which contains the global monodromy group* Image$(\pi_1(U,u) \to G_{\mathbf{Z}})$ *and assume Γ is neat. Then the associated period map $\varphi : U \to \Gamma\backslash D$ extends to a continuous map*

$$X^{\log}_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2)}.$$

Here X_{val} is the projective limit of certain blow-ups of X at the boundary (see Section 3.6).

As is explained in Section 8.4, this theorem 0.5.29 is obtained from the period maps in 0.4.30 (ii) and the map $\psi : D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$. (The period map in 0.4.30 (ii) is obtained only locally on X, but the composition globalizes.)

0.5.30

$\flat$-*spaces.* Cattani and Kaplan [CK1] generalized Satake-Baily-Borel compactifications of $\Gamma\backslash D$ for a symmetric Hermitian domain D, to the case where D is a Griffiths domain of weight 2 under certain assumptions, and showed that period maps from a punctured disc Δ^* extend over the unit disc Δ. This was the first successful attempt to enlarge D beyond the classical situation 0.4.14. In Chapter 9 and Section 10.4, we consider the relationship between our theory and their theory and discuss related subjects.

We define the quotient topological spaces $D^{\flat}_{\mathrm{BS}} := D_{\mathrm{BS}}/\sim$, $D^{\flat}_{\mathrm{BS,val}} := D_{\mathrm{BS,val}}/\sim$ divided by the action of the unipotent radical of the parabolic subgroup of $G_{\mathbf{R}}$ associated to each point (9.1.1).

In the case of D being symmetric Hermitian domain, $D^{\flat}_{\mathrm{BS}}$ was studied by Zucker in [Z1, Z4], and is called the "reductive Borel-Serre space" by him.

Similarly we have the quotient space $D^{\flat}_{\mathrm{SL}(2),\le 1}$ of the part $D_{\mathrm{SL}(2),\le 1}$ of $D_{\mathrm{SL}(2)}$ of points of rank ≤ 1 by the unipotent part $G_{W,\mathbf{R},u}$ of $G_{W,\mathbf{R}}$ associated to each point of $D_{\mathrm{SL}(2),\le 1}$ (cf. 5.2.6). The space D^* of Cattani and Kaplan in [CK1] (defined for special D) is essentially this $D^{\flat}_{\mathrm{SL}(2),\le 1}$ (9.1.5).

Let X be an analytic manifold and U be the complement of a smooth divisor on X. Assume that we are given a variation of polarized Hodge structure H on U with unipotent local monodromy which has a Γ-level structure for a neat subgroup Γ of $G_{\mathbf{Z}}$ of finite index, then the associated period map $U \to \Gamma\backslash D$ extends to a continuous map $X \to \Gamma\backslash D^{\flat}_{\mathrm{SL}(2),\le 1}$ (see Section 9.4).

This is obtained as the composition of $X \to \Gamma\backslash D_{\Xi}$ (0.4.30 (i)) and the continuous map $\Gamma\backslash D_{\Xi} \to \Gamma\backslash D^{\flat}_{\mathrm{SL}(2),\le 1}$ which is induced from $\psi : D^{\sharp}_{\Xi} = D^{\sharp}_{\Xi,\mathrm{val}} \to D_{\mathrm{SL}(2),\le 1} \subset D_{\mathrm{SL}(2)}$.

0.6 PLAN OF THIS BOOK

The plan of this book is as follows.

Chapters 1–3 are preliminaries to state the main results of the present book, Theorems A and B in Chapter 4. In Chapter 1, we define the sets D_Σ and $D^\sharp_\Sigma$. In Chapter 2, we describe the theory of polarized logarithmic Hodge structures. In Chapter 3, we discuss the strong topology, logarithmic manifolds, the spaces E_σ, $\tilde{E}_\sigma$, $\check{E}_\sigma$, the categories $\mathcal{B}$, $\mathcal{B}(\log)$, and other enlargements of the category of analytic spaces. In Chapter 4, we state Theorems A and B without proofs. Theorems 0.4.19 and 0.5.5 are contained in Theorem A, and Theorem 0.4.27 is contained in Theorem B. Theorem 0.5.8 is contained in 5.1.10 and Theorem 5.1.14, Theorem 0.5.16 is contained in Theorem 5.2.15 and Proposition 5.2.16, and Theorem 0.5.20 is contained in Theorems 7.3,2, 7.4.2, 5.1.14, and 5.2.15. We also discuss, in Chapter 4, extensions of period maps over boundaries, and infinitesimal properties of extended period maps.

In Chapters 5–8, we prove Theorems A and B by moving from the right to the left in fundamental diagram (3) in Introduction (also in 0.5.25). In Chapter 5, we review the spaces $D_{\mathrm{SL}(2)}$, D_{BS}, $D_{\mathrm{SL}(2),\mathrm{val}}$, and $D_{\mathrm{BS},\mathrm{val}}$ defined in [KU2], and then we define $D_{\Sigma,\mathrm{val}}$ and $D^\sharp_{\Sigma,\mathrm{val}}$. By using the work [CKS] of Cattani, Kaplan, and Schmid on SL(2)-orbits in several variables, in Chapters 5 and 6 we connect the spaces $D^\sharp_{\Sigma,\mathrm{val}}$ and $D_{\mathrm{SL}(2)}$ as in Fundamental diagram (3) in Introduction (also in 0.5.25). In Chapter 7, we prove Theorem A, and in Chapter 8, Theorem B.

In Chapters 9–12, we give complements, examples, generalizations, and open problems. In Chapter 9, we consider the relationship of the present work with the enlargements of D studied by Cattani and Kaplan [CK1]. In Chapter 10, we describe local structures of $D_{\mathrm{SL}(2)}$. In Chapter 11, we consider the moduli of PLHs with coefficients. Although the case with coefficients is more general than the case without coefficients, we have chosen not to show the coefficients everywhere in this book (which would make the notation too complicated), but to describe the theory without coefficients except in Chapter 11, where we show that the results with coefficients can be simply deduced from those without. In Chapter 12, we give examples and discuss open problems.

Corrections to Previous Work

We indicate three mistakes in our previous work [KU1, KU2].

(i) In [KU1, (5.2)], there is a mistake in the definition of the notion of polarized logarithmic Hodge structures of type Φ. This mistake and its correction are explained in 2.5.16.

(ii) In [KU2, Lemma 4.7], the definition of $B(U, U', U'')$ is written as $\{g\tilde{\rho}(t)k \cdot \mathbf{r} \mid \cdots\}$, which is wrong. The correct definition is $\{\tilde{\rho}(t)gk \cdot \mathbf{r} \mid \cdots\}$. This point will be explained in 5.2.17.

(iii) In [KU2, Remarks 3.15, and 3.16], we indicated that we would consider a space $D^\flat_{\mathrm{SL}(2)}$ in this book. However, we actually consider only a part $D^\flat_{\mathrm{SL}(2),\leq 1}$ of $D^\flat_{\mathrm{SL}(2)}$ (Chapter 9). We realized that $D^\flat_{\mathrm{SL}(2)}$ is not necessarily Hausdorff and seems not to be a good object to consider, but that the part $D^{\flat}_{\mathrm{SL}(2),\leq 1}$ is Haussdorff and is certainly a nice object.

The present work was announced in [KU1] under the title "Logarithmic Hodge Structures and Classifying Spaces" and in [KU2] under the title "Logarithmic Hodge Structures and Their Moduli," but we have changed the title.

0.7 NOTATION AND CONVENTION

Throughout this book, we use the following notation and terminology.

0.7.1

As usual, **N**, **Z**, **Q**, **R**, and **C** mean the set of natural numbers $\{0, 1, 2, 3, \dots\}$, the set of integers, the set of rational numbers, the set of real numbers, and the set of complex numbers, respectively.

0.7.2

In this book, a *ring* is assumed to have the unit element 1, a *subring* shares 1, and a ring homomorphism respects 1.

0.7.3

Let L be a $\mathbf{Z}$-module. For $R = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, we denote $L_R := R \otimes_{\mathbf{Z}} L$.

We fix a 4-tuple

$$\Phi_0 = (w, (h^{p,q})_{p,q \in \mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$$

where w is an integer, $(h^{p,q})_{p,q \in \mathbf{Z}}$ is a set of non-negative integers satisfying

$$\begin{cases} h^{p,q} = 0 \text{ for almost all } p, q, \\ h^{p,q} = 0 \text{ if } p+q \neq w, \\ h^{p,q} = h^{q,p} \text{ for any } p, q, \end{cases}$$

H_0 is a free $\mathbf{Z}$-module of rank $\sum_{p,q} h^{p,q}$, and $\langle\ ,\ \rangle_0$ is a $\mathbf{Q}$-rational nondegenerate $\mathbf{C}$-bilinear form on $H_{0,\mathbf{C}}$ which is symmetric if w is even and antisymmetric if w is odd.

It is easily checked that the associated classifying space D of Griffiths (1.2.1) is nonempty if and only if either w is odd or $w = 2t$ and the signature (a, b) of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ satisfies

$$a \text{ (resp. } b) = \sum_j h^{t+j,t-j},$$

where j ranges over all even (resp. odd) integers.

Let

$$G_{\mathbf{Z}} := \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0),$$

and for $R = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let

$$\begin{aligned} G_R &:= \mathrm{Aut}(H_{0,R}, \langle\ ,\ \rangle_0), \\ \mathfrak{g}_R &:= \mathrm{Lie}\, G_R \\ &= \{N \in \mathrm{End}_R(H_{0,R}) \mid \langle Nx, y\rangle_0 + \langle x, Ny\rangle_0 = 0\ (\forall x, \forall y \in H_{0,R})\}. \end{aligned}$$

Following [BS], a *parabolic subgroup* of $G_\mathbf{R}$ means a parabolic subgroup of $(G^\circ)_\mathbf{R}$, where G° denotes the connected component of G in the Zariski topology containing the unity. (Note that $G^\circ = G$ if w is odd, and $G^\circ = \{g \in G \mid \det(g) = 1\}$ if w is even.)

0.7.4

We refer to a complex *analytic space* as an analytic space for brevity. We use the definition of analytic space (due to Grothendieck) in which the structure sheaf $\mathcal{O}_X$ of an analytic space X can have nonzero nilpotent sections. Precisely speaking, in this definition, an analytic space means a local ringed space over $\mathbf{C}$ which is locally isomorphic to the ringed space $(V, \mathcal{O}_U/(f_1, \dots, f_m))$, where U is an open set of $\mathbf{C}^n$ for some $n \ge 0$, $f_1, \dots, f_m$ are elements of $\Gamma(U, \mathcal{O}_U)$ for some $m \ge 0$, and $V = \{p \in U \mid f_1(p) = \cdots = f_m(p) = 0\}$.

We denote by

$$\mathcal{A}, \quad \mathcal{A}(\log)$$

the category of analytic spaces and the category of fs logarithmic analytic spaces, i.e., analytic spaces endowed with an fs logarithmic structure, respectively.

0.7.5

Throughout this book, *compact* spaces and *locally compact* spaces are already Hausdorff as in Bourbaki [Bn].

Throughout this book, *proper* means "proper in the sense of Bourbaki [Bn] and separated." Here, for a continuous map $f : X \to Y$, f is proper in the sense of Bourbaki [Bn] if and only if for any topological space Z the map $X \times_Y Z \to Z$ induced by f is closed. (For example, if X and Y are locally compact, f is proper if and only if for any compact subset K of Y, the inverse image $f^{-1}(K)$ is compact.) On the other hand, f is *separated* if and only if the diagonal map $X \to X \times_Y X$ is closed. That is, f is separated if and only if, for any $a, b \in X$ such that $f(a) = f(b)$, there are open sets U and V of X such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$.

In particular, in this book, a topological space X is compact if and only if the map from X to the one point set is proper.

0.7.6

For a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on Y, we denote the inverse image of $\mathcal{F}$ on X by $f^{-1}(\mathcal{F})$, not by $f^*(\mathcal{F})$. This is to avoid confusion with the module-theoretic inverse image. For f a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$

and a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules on Y, we denote by $f^*(\mathcal{F})$ the module-theoretic inverse image $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{F})$ on X.

0.7.7

Concerning monoids and cones, we use the following concise terminology in this book, for simplicity.

We call a commutative monoid just a *monoid*. A monoid is assumed (as usual) to have the neutral element 1. A *submonoid* is assumed to share 1, and a homomorphism of monoids is assumed to respect 1.

Concerning cones, as explained in Section 1.3, a convex cone in the sense of [Od] is called just a *cone* in this book. A convex polyhedral cone in [Od] is called a *finitely generated cone*. A strongly convex cone in [Od] is called a *sharp cone*.

In [Od], for a finitely generated cone σ (in our sense), the topological interior of σ in the vector space $\sigma_{\mathbf{R}}$ is called the relative interior of σ. (Here $\sigma_{\mathbf{R}}$ denotes the $\mathbf{R}$-vector space generated by σ.) We call the relative interior of σ just the *interior* of σ. The interior of σ coincides with the complement of the union of all proper faces of σ. (Here a proper face of σ is a face of σ that is different from σ.)

For an fs monoid $\mathcal{S}$ (0.2.11 and 2.1.4), similarly, we call the complement of the union of all proper faces of $\mathcal{S}$ the *interior* of $\mathcal{S}$.