Soga, H.
Osaka J. Math.
17 (1980), 199-232

MIXED PROBLEMS FOR THE WAVE EQUATION WITH A SINGULAR OBLIQUE DERIVATIVE

HIDEO SOGA

(Received September 26, 1978)

Introduction. Let Ω be a domain in $\boldsymbol{R}^2$ with a compact C^∞ boundary Γ, and consider the mixed problem

$$(0.1)\qquad \begin{cases} \Box u \equiv \dfrac{\partial^2 u}{\partial t^2} - \Delta_x u = f(x, t) & \text{in } \Omega \times (0, t_0)\,, \\ \dfrac{\partial u}{\partial \nu}\Big|_\Gamma = g(x', t) & \text{on } \Gamma \times (0, t_0)\,, \\ u|_{t=0} = u_0(x) & \text{on } \Omega\,, \\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = u_1(x) & \text{on } \Omega\,, \end{cases}$$

where $\nu=\nu(x)$ is a non-vanishing real C^∞ vector field defined in a neighborhood of Γ. We say that (0.1) is C^∞ well-posed when there exists a unique solution $u(x, t)$ in $C^\infty(\bar{\Omega}\times[0, t_0])$ for any $(f, g, u_0, u_1)\in C^\infty(\bar{\Omega}\times[0, t_0])\times C^\infty(\Gamma\times[0, t_0])\times C^\infty(\bar{\Omega})\times C^\infty(\bar{\Omega})$ satisfying the compatibility condition of infinite order.

In the case where ν is not tangent to Γ anywhere, various results have been obtained. It has been well known for a long time that the problem (0.1) is C^∞ well-posed if ν is parallel anywhere to the normal vector n of Γ (the Neumann boundary condition). Ikawa [3] showed that (0.1) is C^∞ well-posed also if ν is oblique (i.e. not parallel to n) anywhere on Γ (the oblique boundary condition). When these two types are mixed, the shape of Ω has to be taken into consideration. Ikawa [4, 5, 6] examined it in detail.

In the present paper we shall study (0.1) in the case where ν is not necessarily non-tangent to Γ. We assume that ν is tangent to Γ at finite number of points (of Γ). And we call them singular points. At each singular point the Lopatinski condition is not satisfied; therefore, the mixed problem frozen there is not C^∞ well-posed (cf. Sakamoto [13]). We can classify the behavior of ν near each singular point into the following three types: As x' ($\in\Gamma$) passes the singular point in the direction of the tangential component of $\nu(x')$ to Γ,

(I) $\langle\nu(x'), n(x')\rangle$ changes sign from positive to negative;

(II) $\langle\nu(x'), n(x')\rangle$ changes sign from negative to positive;

(III) $\langle\nu(x'), n(x')\rangle$ does not change sign,
where $n(x')$ is the unit inner normal vector to Γ. Assuming that $\Omega=\boldsymbol{R}_+^2$, the author [15] has examined the problem (0.1) in the case (I) and (III). We want here to investigate (0.1) in a more general domain in each case.

One of our main results is as follows:

Theorem 1. *If the function* $\langle\nu(x'), n(x')\rangle$ $(\in C^\infty(\Gamma))$ *changes sign on* Γ (*the case* (I) *or* (II)), *then the mixed problem* (0.1) *is not* C^∞ *well-posed.*

As is seen from the proof of Theorem 1 (see §4), we may say that in the case (I) the uniqness does not hold and that in the case (II) the solvability is violated.

Another main result is the following

Theorem 2. *Assume the conditions* (a) *and* (b):

(a) $\langle\nu(x'), n(x')\rangle$ *does not change sign on* Γ (*the case* (III)) *and* $|\langle\nu(x'), n(x')\rangle|^{1/2}$ *is* C^∞ *smooth on* Γ;

(b) ν *is oblique anywhere.*

Then, the mixed problem (0.1) *is* C^∞ *well-posed, and domains of dependence are bounded, but it has not a finite propagation speed.*

Egorov-Kondrat'ev [1] considered an elliptic oblique derivative problem similar to the above problem (0.1):

$$(0.2)\qquad \begin{cases} A(x, D_x)u = f(x) & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu}\Big|_\Gamma = g(x') & \text{on } \Gamma\,, \end{cases}$$

where $A(x,D_x)$ is an elliptic differential operator of second order on $\bar{\Omega}$ and ν is a non-vanishing real vecter field tangent to Γ on its submanifold Γ_0. They assumed that $\dim\Gamma_0=\dim\Gamma-1$ $(\geqq 1)$ and that ν is transversal to Γ_0. Then the behavior of ν near Γ_0 can be classified into the three types (I)$\sim$(III) in the same way. On account of Egorov-Kondrat'ev [1], Maz'ja [11], the author [14], etc., in short, in the case (I) the kernel of (0.2) is infinite-dimensional, in the case (II) the cokernel of (0.2) is infinite-dimensional and in the case (III) the same results as in the coercive case are obtained.

As can be readily seen, our results (i.e. Theorem 1 and 2) are analogous to those of the above problem (0.2). Our methods, however, are little similar to those in the elliptic case.

Let us mention the procedure of the proofs of Theorem 1 and 2. Let $\mathscr{P}$ be the Poisson operator of the following Dirichlet problem considered in appropriate functional spaces:

$$\begin{cases} \Box u(x,t) = 0 & \text{in } \Omega\times(-\infty,\infty), \\ u|_\Gamma = h(x',t) & \text{on } \Gamma\times(-\infty,\infty). \end{cases}$$

Set $T h = \dfrac{\partial}{\partial \nu}\mathcal{P}h|_\Gamma$. Then the well-posedness of (0.1) can be reduced to that of the equation $T h=g$ considered on $\Gamma\times(-\infty,\infty)$. Although T is hard to deal with, T approximates to a pseudo-differential operator $\tilde{T}$ if the wave front of the h (or g) is near where the Lopatinskian vanishes. Analysing the (asymptotic) null solution of $\tilde{T}h=0$, we prove Theorem 1 in §4. In §5, deriving an estimate for $\tilde{T}$ in the same way as in the author [15], we verify Theorem 2 by the procedure similar to that of Ikawa [3].

1. Notations and properties of pseudo-differential operators

We denote by S^m ($m\in\boldsymbol{R}$) the set of functions $p(z,\omega)\in C^\infty(\boldsymbol{R}^2\times\boldsymbol{R}^2)$ satisfying for all multi-indices α,β

$$|\partial_z^\beta\partial_\omega^\alpha p(z,\omega)| \leqq C_{\alpha\beta}(1+|\omega|)^{m-|\alpha|},\ (z,\omega)\in\boldsymbol{R}^2\times\boldsymbol{R}^2,$$

where $\partial_z^\beta=\left(\dfrac{\partial}{\partial z}\right)^\beta$ and $\partial_\omega^\alpha=\left(\dfrac{\partial}{\partial \omega}\right)^\alpha$. For $p(z,\omega)\in S^m$ we define a pseudo-differential operator $p(z,D_z)$ by

$$pu = p(z,D_z)u(z) = \int e^{iz\omega}p(z,\omega)\hat{u}(\omega)đ\omega,\quad u(z)\in\mathcal{S},$$

where $đ\omega=(2\pi)^{-2}d\omega$, $\mathcal{S}$ is the space of rapidly decreasing functions and $\hat{u}(\omega)$ is the Fourier transform $\int e^{-iz\omega}u(z)dz$. We denote by $\mathcal{S}^m$ the set of these operators $p(z,D_z)$, and call $p(z,\omega)$ the symbol of $p(z,D_z)$. It is well known that the estimate

$$\|p(z,D_z)u\|_s \leqq C\|u\|_{s+m},\quad u\in\mathcal{S}\quad (s\in\boldsymbol{R})$$

holds for $p(z,\omega)\in S^m$, where the norm $\|\cdot\|_s$ is defined by

$$\|u\|_s^2 = \int(1+|\omega|^2)^s|\hat{u}(\omega)|^2đ\omega.$$

For $p(z,\omega)\in S^m$ and $q(z,\omega)\in S^{m'}$ we set

$$\sigma(p\circ q)(z,\omega) = \lim_{\varepsilon\to 0}\iint e^{-i\tilde{z}\tilde{\omega}}\chi(\varepsilon\tilde{\omega},\varepsilon\tilde{z})p(z,\omega+\tilde{\omega})q(z+\tilde{z},\omega)d\tilde{z}đ\tilde{\omega},$$

where $\chi(z,\omega)\in\mathcal{S}$ and $\chi(0,0)=1$. Then we have $\sigma(p\circ q)(z,\omega)\in S^{m+m'}$ and

$$\sigma(p\circ q)(z,D_z)u = p(z,D_z)(qu),\quad u\in\mathcal{S}.$$

Furthermore the asymptotic expansion formula

$$(1.1)\quad \sigma(p\circ q)(z,\omega)-\sum_{|\alpha|<N}\frac{1}{\alpha!}\left(\frac{\partial}{\partial\omega}\right)^{\alpha}p(z,\omega)\cdot D_z^{\alpha}q(z,\omega)\in S^{m+m'-N}\quad \left(D_z=-i\frac{\partial}{\partial z}\right)$$

is obtained for any integer N (>0). For $p(z,\omega)\in S^m$ there exists a symbol $p^*(z,\omega)\in S^m$ such that

$$(p(z,D_z)u,v)=(u,p^*(z,D_z)v),\quad u,v\in\mathcal{S}\,,$$

and the following asymptotic expansion formula holds for any N (>0):

$$(1.2)\qquad p^*(z,\omega)-\sum_{|\alpha|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\overline{\partial_\omega^\alpha D_z^\alpha p(z,\omega)}\in S^{m-N}\,.$$

These properties are described in Hörmander [2] or Kumano-go [7].

We introduce another class of pseudo-differential operators, whose symbols have a parameter $\tau=\sigma-i\gamma$ ($\sigma\in\boldsymbol{R}^1$, $\gamma\geqq 0$). Namely, the symbol $p(y,\eta,\tau)$ is a C^∞ function in $\boldsymbol{R}_y^1\times\boldsymbol{R}_\eta^1$ with the parameter τ and satisfies the following inequality for all non-negative integers α,β:

$$|\partial_y^\beta\partial_\eta^\alpha p(y,\eta,\tau)|\leqq C_{\alpha\beta}(|\eta|+|\tau|)^{m-\alpha},\ (y,\eta)\in\boldsymbol{R}^2,\ |\tau|\geqq 1\,,$$

where $m\in\boldsymbol{R}$ and $C_{\alpha\beta}$ is a constant independent of τ. We denote by $S_{(\tau)}^m$ the set of these symbols, and for $p(y,\eta,\tau)\in S_{(\tau)}^m$ define

$$pu=p(y,D_y,\tau)u=\int e^{iy\eta}p(y,\eta,\tau)\hat{u}(\eta)d\eta,\ u(y)\in\mathcal{S}\,.$$

Let us define a norm $|||\cdot|||_s$ ($s\in\boldsymbol{R}$) with the parameter τ by

$$|||u|||_s^2=\int(\eta^2+|\tau|^2)^s|\hat{u}(\eta)|^2d\eta\,.$$

Then, for $p(y,\eta,\tau)\in S_{(\tau)}^m$ the following estimate holds:

$$|||p(y,D_y,\tau)u|||_s\leqq C|||u|||_{s+m},\ u\in\mathcal{S},\ |\tau|\geqq 1\,.$$

The above constant C is uniform in τ. Hereafter, all the constants in estimates stated with the norm $|||\cdot|||_s$ are independent of τ. Obviously we obtain the same properties as for the class S^m. Let us note that if $p(y,\eta,\sigma)\in S^m$ ($z=(y,t)$, $\omega=(\eta,\sigma)$) then $p(y,\eta,\sigma)$ can be regarded as a symbol in $S_{(\sigma)}^m$ ($\tau=\sigma\geqq 1$). We say that a symbol $p(y,\eta,\tau)\in S_{(\tau)}^m$ has a homogeneous asymptotic expansion $\sum_{j=0}^{\infty}p_{m-j}(y,\eta,\tau)$ when $p_{m-j}(y,\lambda\eta,\lambda\tau)=\lambda^{m-j}\cdot p_{m-j}(y,\eta,\tau)$ for $\lambda\geqq 1$ ($\eta^2+|\tau|^2\geqq 1$, $j=0,1,\cdots$) and $p(y,\eta,\tau)-\sum_{j=0}^{N-1}p_{m-j}(y,\eta,\tau)\in S_{(\tau)}^{m-N}$ ($N=1,2,\cdots$). We call $p_m(y,\eta,\tau)$ the principal symbol of p and denote it by $\sigma_0(p)(y,\eta,\tau)$.

Proposition 1.1. *Let* $\chi(y,\eta,\tau)\in S_{(\tau)}^0$ *and* $p(y,\eta,\tau)\in S_{(\tau)}^m$. *Suppose that* $\operatorname{supp}_{\eta,\tau}\chi$ *is in an open conic set* Δ *and that the principal part* $p_m(y,\eta,\tau)$ (*i.e.* $p_m\in$

$S^m_{(\tau)}$ & $p-p_m \in S^{m-1}_{(\tau)}$) *satisfies*

$$|p_m(y,\eta,\tau)| \geqq \delta(|\eta|+|\tau|)^m \quad (\delta>0)$$

when $(\eta,\tau)\in\Delta$ *and* $|\eta|+|\tau|\geqq L$ (L *is a large constant*). *Then the following estimate is obtained for any constant* $N>0$:

$$|||\chi u|||_{m+s} \leqq C(|||pu|||_s + |||u|||_{s-N}),\ u\in\mathcal{S} \quad (s\in\boldsymbol{R}).$$

We can prove this proposition by constructing a parametrix for $p(y,D_y,\tau)$ available on Δ (cf. Hörmander [2]).

Proposition 1.2. *Let* $p(y,\eta,\tau)\in S^1_{(\tau)}$ *and its principal part* $p_1(y,\eta,\tau)$ *fulfil*

$$\operatorname{Im} p_1(y,\eta,\tau) \geqq \delta(\tau), \quad (\eta,\tau)\in\Delta\cap\{|\tau|\geqq 1\},$$

where $\delta(\tau)$ *is a positive function* ($\geqq 1$) *and* Δ *is an open conic set. Then, for any* $\chi(\eta,\tau)\in S^0_{(\tau)}$ *satisfying* supp $\chi\subset\Delta$ *there is a constant* C *such that*

$$\operatorname{Im}(p(y,D_y,\tau)\chi v,\chi v) \geqq \delta(\tau)|||\chi v|||_0^2 - C|||\chi v|||_0^2, \quad v(y)\in\mathcal{S} \quad (|\tau|\geqq 1).$$

Corollary. *In the above proposition, if* $\tilde{\chi}(y,\eta,\tau)$ ($\in S^0_{(\tau)}$) *depends on* y *and satisfies* $\operatorname{supp}_{\eta,\tau}\tilde{\chi}\subset\Delta$, *then we have for any* $N>0$

$$\operatorname{Im}(p(y,D_y,\tau)\tilde{\chi}v,\tilde{\chi}v) \geqq \frac{1}{2}\delta(\tau)|||\tilde{\chi}v|||_0^2 - C_1|||\tilde{\chi}v|||_0^2 - C_2|||v|||_{-N}^2, \quad v\in\mathcal{S}.$$

Proof. We set

$$q(y,\eta,\tau) = (\operatorname{Im} p_1(y,\eta,\tau) - \delta(\tau))\chi'(\eta,\tau),$$

where $\chi'(\eta,\tau)\in S^0_{(\tau)}$, $\chi'(\eta,\tau)=1$ on supp χ and supp $\chi'\subset\Delta$. Then it follows that

$$q(y,\eta,\tau)\geqq 0 \ \text{ for } (y,\eta)\in\boldsymbol{R}^2,\ |\tau|\geqq 1,$$
$$\operatorname{Im}((p-i\delta(\tau))\chi v,\ \chi v) \geqq \operatorname{Re}(q\chi v,\ \chi v) - C_1|||\chi v|||_0^2.$$

Let q_F denote the Friedrichs approximation of q (cf. Theorem 5.1 of Kumano-go [7]). Then, we have $q-q_F\in S^0_{(\tau)}$ and $(q_F v, v)\geqq 0$. Therefore we obtain

$$\operatorname{Im}(p\chi v,\chi v) - \delta(\tau)|||\chi v|||_0^2 = \operatorname{Im}((p-i\delta(\tau)\chi v,\chi v) \geqq -C_2|||\chi v|||_0^2.$$

Next, let us check the corollary. Let $\chi''(\eta,\tau)\in S^0_{(\tau)}, \chi''(\eta,\tau)=1$ on $\operatorname{supp}_{\eta,\tau}\tilde{\chi}$ and supp $\chi''\subset\Delta$. Then, from the above proposition it follows that

$$\begin{aligned}\operatorname{Im}(p\chi''\tilde{\chi}v,\ \chi''\tilde{\chi}v) &\geqq \delta(\tau)|||\chi''\tilde{\chi}v|||_0^2 - C_1|||\chi''\tilde{\chi}v|||_0^2 \\ &\geqq \frac{\delta(\tau)}{2}|||\tilde{\chi}v|||_0^2 - C_2|||\tilde{\chi}v|||_0^2 - C_3|||v|||_{-N}^2.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\operatorname{Im}(p\chi''\tilde{\chi}v, \chi''\tilde{\chi}v) &= \operatorname{Im}(p\tilde{\chi}v, \tilde{\chi}v)+\operatorname{Im}(p\chi''\tilde{\chi}v, (\chi''-1)\tilde{\chi}v)\\ &\quad +\operatorname{Im}(p(\chi''-1)\tilde{\chi}v, \tilde{\chi}v)\\ &\leqq \operatorname{Im}(p\tilde{\chi}v, \tilde{\chi}v)+C_4|||v|||^2_{-N}\,.\end{aligned}$$

Therefore the corollary is obtained. The proof is complete.

Now, let L be a differential operator of the from

$$L(y, D_x, D_y, \tau) = D_x^2 + \sum_{\substack{j+k+l\leqq 2\\ j=0,1}} a_{jkl}(y)\tau^l D_y^k D_x^j\,,$$

where $\tau=\sigma-i\gamma$ ($\sigma\in\boldsymbol{R}^1$, $\gamma\geqq 0$) and $a_{jkl}(y)\in\mathcal{B}^\infty(\boldsymbol{R}^1)=\{f\in C^\infty;\ \sup|\partial_y^\alpha f(y)|<+\infty$ for $\alpha=0,1,\cdots\}$. We denote by $\xi_0^\pm(y,\eta,\tau)$ the roots of the equation (in ξ)

$$L_0(y,\xi,\eta,\tau)\equiv\xi^2+\sum_{\substack{j+k+l=2\\ j=0,1}} a_{jkl}(y)\tau^l\eta^k\xi^j = 0\,.$$

Obviously, $\xi_0^\pm(y,\eta,\tau)$ are homogeneous of order one in (η,τ) and are smooth where $\xi_0^+(y,\eta,\tau)$ and $\xi_0^-(y,\eta,\tau)$ are distinct each other. We obtain the following factrization formula, which is proved in Kumano-go [9] (see Theorem 0 of [9]).

Proposition 1.3. *Let $\xi_0^+(y,\eta,\tau)$ and $\xi_0^-(y,\eta,\tau)$ be distinct on $\boldsymbol{R}_y^1\times\bar{\Delta}_{(\eta,\tau)}$ (Δ is an open conic set). Then there are symbols $\xi^\pm(y,\eta,\tau)\in S^1_{(\tau)}$ such that*

i) *$\xi^\pm(y,\eta,\tau)$ have homogeneous asymptotic expansions whose principal symbols $\sigma_0(\xi^\pm)$ satisfy*

$$\sigma_0(\xi^\pm)(y,\eta,\tau) = \xi_0^\pm(y,\eta,\tau) \quad \text{for } y\in\boldsymbol{R}^1,\ (\eta,\tau)\in\Delta;$$

ii) *Set $L^\pm=D_x-\xi^\pm(y,D_y,\tau)$. Then for any $\chi(y,\eta,\tau)\in S^m_{(\tau)}$ satisfying $\operatorname{supp}_{\eta,\tau}\chi\subset\Delta$, we have*

$$\begin{aligned}L\chi &= L^-L^+\chi+r_1D_x+r_2\,,\\ \chi L &= \chi L^-L^+ + r_3D_x+r_4\,,\end{aligned}$$

where $r_j=r_j(y,D_y,\tau)\in S^{-\infty}_{(\tau)}=\bigcap_{n\in\boldsymbol{R}} S^n_{(\tau)}$ ($j=1,\cdots,4$).

Let $\theta(y)$ be a real-valued C^∞ function on $\boldsymbol{R}^1$ satisfying $|\theta(y)|\leqq 1$ for $y\in\boldsymbol{R}^1$, $\theta(y)=y$ for $|y|\leqq\frac{1}{2}$ and $\theta(y)=1$ for $|y|\geqq 1$. For $p(y,\eta,\tau)\in S^m_{(\tau)}$ we set

$$p^{(\rho)}(y,\eta,\tau) = p\left(\rho\theta\left(\frac{y}{\rho}\right),\eta,\tau\right) \quad (\rho>0)\,. \tag{1.3}$$

Then $p^{(\rho)}(y,\eta,\tau)$ belongs to $S^m_{(\tau)}$. Moreover, $p^{(\rho)}(y,\eta,\tau)$ is equal to $p(y,\eta,\tau)$ if $|y|\leqq\frac{\rho}{2}$, and independent of y if $|y|\geqq\rho$.

Lemma 1.1. *Let Δ', Λ' be open sets of $S_+=\{(\eta,\tau): \eta^2+|\tau|^2=1, \gamma=-\mathrm{Im}\,\tau\geqq 0\}$ and $\bar{\Delta}'\subset\Lambda'$. Assume that $q(y,\eta,\tau)\in S^1_{(\tau)}$ has a homogeneous asymptotic expansion $\sum_{j=0}^{\infty} q_{1-j}(y,\eta,\tau)$ such that $q_1(y,\eta,\tau)$ is real-valued and satisfies*

$$|\partial_\eta q_1(y,\eta,\tau)|\geqq\delta\ (>0),\quad y\in \boldsymbol{R}^1,\ (\eta,\tau)\in S_+ . \tag{1.4}$$

Then, if $\rho>0$ is small enough for an integer $N>0$, there exists a symbol $\zeta(y,\eta,\tau)\in S^0_{(\tau)}$ such that

(i) $[q^{(\rho)}(y,D_y,\tau),\ \zeta(y,D_y,\tau)]\ (=q^{(\rho)}\zeta-\zeta q^{(\rho)})\in S^{-N}_{(\tau)}$,

(ii) $\operatorname*{supp}_{\eta,\tau}\zeta(y,\eta,\tau)\subset\Lambda,\quad 0\leqq\sigma_0(\zeta)\leqq 1$,

$\zeta(y,\eta,\tau)=1\ \text{ for } y\in\boldsymbol{R}^1, (\eta,\tau)\in\Delta\quad (\eta^2+|\tau|^2\geqq 1)$,

where Δ (resp. Λ)$=\{(\eta,\tau)=(\lambda\eta',\lambda\tau'): (\eta',\tau')\in\Delta'$ (resp. Λ'), $\lambda>0\}$.

This lemma in the case $N=1$ is due to Ikawa [3].

Proof. We take open sets $\Delta_1', \Delta_2', \cdots, \Delta_N'$ and $\Lambda_1', \Lambda_2', \cdots, \Lambda_N'$ in S_+ such that

$$\Delta'\subset\subset\Delta_N'\subset\subset\Delta_{N-1}'\subset\subset\cdots\subset\subset\Delta_1'\subset\subset\Lambda_1'\subset\subset\cdots\subset\subset\Lambda_N'\subset\subset\Lambda' ,$$

where $A\subset\subset B$ denotes $\bar{A}\subset B$. For $\Delta_1', \Delta_2', \cdots(\subset S_+)$ we set

$$\Delta_1=\{(\eta,\tau)=(\lambda\eta',\lambda\tau'): (\eta',\tau')\in\Delta_1', \lambda>0\}, \cdots$$

Assume that $\zeta(y,\eta,\tau)$ is of the form

$$\zeta(y,\eta,\tau)=\sum_{j=0}^{N-1}\zeta_{-j}(y,\eta,\tau)$$

where $\zeta_{-j}(y,\eta,\tau)\ (\in S^{-j}_{(\tau)})$ is homogeneous of order $-j$ in (η,τ) $(\eta^2+|\tau|^2\geqq 1)$. Then it follows from the formula (1.1) that the symbol of $[q^{(\rho)},\zeta]$ has the asymptotic expansion

$$\begin{aligned}\sum_{j=0}^{N-1}\{&\partial_\eta q_1^{(\rho)}(y,\eta,\tau)D_y\zeta_{-j}(y,\eta,\tau)-D_y q_1^{(\rho)}(y,\eta,\tau)\partial_\eta\zeta_{-j}(y,\eta,\tau)\\ &+\Phi_{-j}(y,\eta,\tau)\}+r_{-N}(y,\eta,\tau) .\end{aligned} \tag{1.5}$$

Here $r_{-N}(y,\eta,\tau)$ is a symbol belonging to $S^{-N}_{(\tau)}$ and $\Phi_{-j}(y,\eta,\tau)$ is defined by

$$\Phi_0=0 ,$$

$$\begin{aligned}\Phi_{-j}(y,\eta,\tau)=\sum_{\substack{l+k+i=j+1\\ 1\leqq k\\ 0\leqq i,l\\ 2\leqq k+l}}\frac{1}{k!}\{&\partial_\eta^k q_{1-l}^{(\rho)}(y,\eta,\tau)D_y^k\zeta_{-i}(y,\eta,\tau)\\ &-D_y^k q_{1-l}^{(\rho)}(y,\eta,\tau)\partial_\eta^k\zeta_{-i}(y,\eta,\tau)\}\ (j\geqq 1) .\end{aligned}$$

We shall choose $\zeta_0, \cdots, \zeta_{-N+1}$ so that each term in the summation (1.5) vanishes. Note that $\Phi_{-j}(y,\eta,\tau)$ is determined by only $\zeta_0, \zeta_{-1}, \cdots, \zeta_{-j+1}$ and homogeneous of order $-j$ in (η,τ) $(\eta^2+|\tau|^2\geqq 1)$. Let $\chi(\eta,\tau)$ be homogeneous of order 0 in (η,τ) $(\eta^2+|\tau|^2\geqq 1)$ and satisfy $0\leqq\chi\leqq 1$, supp $\chi\subset\Lambda_1$ and $\chi(\eta,\tau)=1$ on $\Delta_1\cap\{\eta^2+|\tau|^2\geqq 1\}$. Let us consider the following equation with the parameter τ:

$$(1.6)\qquad \begin{cases}\partial_\eta q_1^{(\rho)}(y,\eta,\tau)\partial_y\zeta_{-j}-\partial_y q_1^{(\rho)}(y,\eta,\tau)\partial_\eta\zeta_{-j}+i\Phi_{-j}(y,\eta,\tau)=0\,,\\ \zeta_0|_{y=0}=\chi(\eta,\tau),\quad \zeta_{-j}|_{y=0}=0\ (j\geqq 1)\,.\end{cases}$$

The characteristic curves of this equation are given by

$$(1.7)\qquad \begin{cases}\dfrac{d\tilde{y}}{ds}=\partial_\eta q_1^{(\rho)}(\tilde{y},\tilde{\eta},\tau)\,,\\ \dfrac{d\tilde{\eta}}{ds}=-\partial_y q_1^{(\rho)}(\tilde{y},\tilde{\eta},\tau)\,,\\ \tilde{y}|_{s=0}=0,\quad \tilde{\eta}|_{s=0}=\eta\quad (\eta^2+|\tau|^2\geqq 1)\,.\end{cases}$$

Since $\partial_\eta q_1^{(\rho)}(y,\eta,\tau)$ and $\partial_y q_1^{(\rho)}(y,\eta,\tau)$ are C^∞ real-valued functions on $\boldsymbol{R}^2_{(y,\eta)}$, we have a unique solution $(\tilde{y}(s;\eta,\tau),\tilde{\eta}(s;\eta,\tau))$ of (1.7) defined on $-\infty<s<\infty$. It follows from the definition (1.3) that

$$(1.8)\qquad \begin{cases}|\partial_\eta^\alpha q_1^{(\rho)}(y,\eta,\tau)|\leqq C_\alpha(|\eta|+|\tau|+1)^{1-\alpha},\quad (\alpha=1,2,\cdots)\,,\\ |\partial_y^\beta\partial_\eta^\gamma q_1^{(\rho)}(y,\eta,\tau)|\begin{cases}\leqq C_{\beta\gamma}\rho^{-\beta+1}(|\eta|+|\tau|+1)^{1-\gamma}\text{ if }|y|<\rho\,,\\ =0\text{ if }|y|\geqq\rho,\quad(\beta=1,2,\cdots;\ \gamma=0,\cdots)\,,\end{cases}\end{cases}$$

where C_α and $C_{\beta\gamma}$ are constants independent of y, η, τ and ρ. From these inequalities and the assumption (1.4) we obtain

$$(1.9)\qquad \delta|s|\leqq|\tilde{y}(s;\eta,\tau)|\leqq C_1|s|\,,$$

$$(1.10)\qquad |\tilde{\eta}(s;\eta,\tau)-\eta|\leqq(e^{C_2|s|}-1)(|\eta|+|\tau|+1)\,.$$

for constants C_1 and C_2 independent of s, η, τ and ρ. Combining (1.8), (1.9) and (1.10), we see that if ρ is small enough the following statements i)~iii) are valid:

i) $C_3^{-1}(|\eta|+|\tau|)\leqq|\tilde{\eta}(s;\eta,\tau)|+|\tau|\leqq C_3(|\eta|+|\tau|)$, $s\in\boldsymbol{R}$, $\eta^2+|\tau|^2\geqq 1$;

ii) If $(\eta,\tau)\in\Lambda_j-\bar{\Delta}_j$ $(\eta^2+|\tau|^2\geqq 1)$, then $(\tilde{\eta}(s;\eta,\tau),\tau)\in\Lambda_{j+1}-\bar{\Delta}_{j+1}$ for $s\in\boldsymbol{R}$ $(j=1,\cdots,N,\ \Lambda_{N+1}=\Lambda,\ \Lambda_{N+1}=\Delta)$;

iii) $$\left|\det\begin{pmatrix}\dfrac{\partial\tilde{y}(s;\eta,\tau)}{\partial s} & \dfrac{\partial\tilde{y}(s;\eta,\tau)}{\partial\eta}\\ \dfrac{\partial\tilde{\eta}(s;\eta,\tau)}{\partial s} & \dfrac{\partial\tilde{\eta}(s;\eta,\tau)}{\partial\eta}\end{pmatrix}\right|\geqq\frac{\delta}{2},\quad s\in\boldsymbol{R},\ \eta^2+|\tau|^2\geqq 1\,.$$

Therefore, we obtain the required solution $\zeta_{-j}(y,\eta,\tau)$ of (1.6). Noting that $\tilde{y}(s;\eta,\tau)$ and $\tilde{\eta}(s;\eta,\tau)$ are homogeneous of order 0 and 1 in (η,τ) respectively, we see that $\zeta_{-j}(y,\eta,\tau)$ is homogeneous of order $-j$ in (η,τ). Furthermore, from the above statement ii) it follows that

$$\begin{aligned}&\zeta_0(y,\eta,\tau)=1 \text{ if } (\eta,\tau)\in\Delta_2, \quad \operatorname*{supp}_{\eta,\tau}\zeta_0(y,\eta,\tau)\subset\Lambda_2\,,\\ &\operatorname*{supp}_{\eta,\tau}\zeta_{-j}(y,\eta,\tau)\subset\Lambda_{j+2}-\bar{\Delta}_{j+2} \quad (1\leqq j\leqq N-1)\,,\\ &\operatorname*{supp}_{\eta,\tau}\Phi_{-j-1}(y,\eta,\tau)\subset\Lambda_{j+2}-\bar{\Delta}_{j+2} \quad (1\leqq j\leqq N-1)\,.\end{aligned}$$

Hence the lemma is proved.

REMARK 1.1. We can make the assumption (1.4) in Lemma 1.1 weaker as follows:

$$(1.4)' \qquad |\partial_\eta q_1(y,\eta,\tau)|\geqq\delta(>0), \quad y\in\boldsymbol{R}^1,\ (\eta,\tau)\in\bar{\Lambda}'-\Delta'\,.$$

In fact: There exist symbols $q_\pm(y,\eta,\tau)\in S^1_{(\tau)}$ satisfying all the assumptions in Lemma 1.1 and equal to $q(y,\eta,\tau)$ on $\sum_\pm=\{(\eta,\tau)\in\bar{\Lambda}-\Delta;\ \pm\partial_\eta q_1(y,\eta,\tau)\geqq\delta\}$. Applying Lemma 1.1 to $q_\pm$, we have $\zeta_\pm(y,\eta,\tau)\in S^0_{(\tau)}$ such that

(i) $[q_\pm^{(p)},\zeta_\pm]\in S^{-N}_{(\tau)}$;

(ii) $\operatorname*{supp}_{\eta,\tau}\zeta_\pm(y,\eta,\tau)\subset\Lambda_\mp\cup\Lambda \quad (\bar{\Lambda}_+\cap\bar{\Lambda}_-=\phi,\ \sum_\pm\subset\subset\Lambda_\pm)$,

$0\leqq\sigma_0(\zeta_\pm)\leqq 1,\ \ \zeta_\pm(y,\eta,\tau)=1 \text{ if } (\eta,\tau)\in\sum_\mp\cup\Delta$.

$\zeta(y,\eta,\tau)=\zeta_+(y,\eta,\tau)\zeta_-(y,\eta,\tau)$ fulfills all the requirements.

2. Reduction to the problem in a half-space

Let $x=(x_1, x_2)$ be an orthogonal local coordinate system defined near a singular point $x_0'\in\Gamma$ such that $x_1=x_2=0$ denotes x_0' and the x_2-axis is tangent to Γ at x_0'. Let the curve Γ (near x_0') be expressed by the equation $x_1=\mu(x_2)$ and Ω (near x_0') by $x_1>\mu(x_2)$. We take another local coordinate system: $\tilde{x}=x_1-\mu(x_2)$, $\tilde{y}=x_2$. Then we have

i) Ω is mapped near x_0' to (a neighborhood of) a half space $\{(\tilde{x},\tilde{y}): \tilde{x}>0\}$, and Γ to $\{(\tilde{x},\tilde{y}): \tilde{x}=0\}$;

ii) $\Delta_x=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}$ is transformed near x_0' to

$$\tilde{\Delta}=(1+\mu'(\tilde{y})^2)\frac{\partial^2}{\partial\tilde{x}^2}-2\mu'(\tilde{y})\frac{\partial^2}{\partial\tilde{x}\partial\tilde{y}}+\frac{\partial^2}{\partial\tilde{y}^2}-\mu''(\tilde{y})\frac{\partial}{\partial\tilde{x}},$$

where $\mu'=\dfrac{d\mu}{d\tilde{y}}$ and $\mu''=\dfrac{d^2\mu}{d\tilde{y}^2}$ (note that $\mu'(0)=0$);

iii) $\dfrac{\partial}{\partial\nu}$ is transformed near x_0' to

$$\alpha(\tilde{y})\frac{\partial}{\partial\tilde{y}}+\beta(\tilde{y})\frac{\partial}{\partial\tilde{x}},$$

where $\alpha(\tilde{y})$ and $\beta(\tilde{y})$ are C^∞ functions defined near $\tilde{y}=0$ and satisfy $\alpha(0)\neq 0$ and $\beta(0)=0$.

Rewriting $\tilde{x}$, $\tilde{y}$ with x, y, we set

$$\begin{aligned} L(y,D_x,D_y,D_t) &= -(1+\mu'(y)^2)^{-1}(\Delta-\partial_t^2) \\ &\qquad (\equiv D_x^2+2a(y)D_xD_y+b(y)D_y^2+c(y)D_x-b(y)D_t^2)\,, \\ \psi(y) &= \alpha(y)^{-1}\beta(y)\,. \end{aligned}$$

For a C^∞ function $\varphi(y)$ defined near $y=0$ we define $\varphi^{(\rho)}(y)$ $(\rho>0)$ in the same way as (1.3), and write for $A=\sum_\gamma a_\gamma(y)D^\gamma_{(x,y,t)}$

$$A^{(\rho)} = \sum a_\gamma^{(\rho)}(y)D^\gamma_{(x,y,t)}\,.$$

From the statements i)~iii) stated earlier, it follows that (0.1) is equivalent to the following mixed problem if u has support in $\frac{\rho}{2}$-neighborhood of the singular point:

$$\text{(2.1)}\qquad \begin{cases} L^{(\rho)}(y,D_x,D_y,D_t)u = f(x,y,t) & \text{in } \boldsymbol{R}^2_+\times(0,t_0)\,, \\ (D_yu+\psi^{(\rho)}(y)D_xu)\,|_{x=0} = g(y,t) & \text{on } \boldsymbol{R}^1\times(0,t_0)\,, \\ u\,|_{t=0} = u_0(x,y) & \text{on } \boldsymbol{R}^2_+\,, \\ D_tu\,|_{t=0} = u_1(x,y) & \text{on } \boldsymbol{R}^2_+\,, \end{cases}$$

which we call the mixed problem localized at the singular point. The classification (I)~(III) stated in Introduction is rewrited respectively by the term $\psi^{(\rho)}(y)$ in (2.1) in the following way (let $\rho>0$ be small enough):

$$\text{(2.2)}\qquad \begin{cases} \text{(I)}\quad \psi^{(\rho)}(y)>0 \text{ for } y<0 \text{ and } \psi^{(\rho)}(y)<0 \text{ for } y>0; \\ \text{(II)}\quad \psi^{(\rho)}(y)<0 \text{ for } y<0 \text{ and } \psi^{(\rho)}(y)>0 \text{ for } y>0; \\ \text{(III)}\quad \psi^{(\rho)}(y)>0 \text{ (or } <0) \text{ for every } y\neq 0\,. \end{cases}$$

Hereafter we often abbreviate $L^{(\rho)}$, $\psi^{(\rho)}$, $\cdots$ to L, ψ, $\cdots$.

Proposition 2.1. i) *If the problem* (2.1) *localized at any singular point is C^∞ well-posed for a $\rho>0$, then* (0.1) *is C^∞ well-posed.*

ii) *If* (0.1) *is C^∞ well-posed, then the problem* (2.1) *localized at any singular point is C^∞ well-posed for any small $\rho>0$.*

We note that if (0.1) (or (2.1)) is C^∞ well-posed then so is also the mixed problem considered on $t_1\leqq t\leqq t_2$ (for any $t_1<t_2$) with the initial condition on $t=t_1$.

Proof of Proposition 2.1. Let us prove only i). ii) can also be verified in the same way.

Let $\{x'_j\}_{j=1,\cdots,N}$ be the singular points, and set for $\varepsilon>0$

$$U_j^{(\varepsilon)} = \{x \in \bar{\Omega};\ |x - x_j'| < \varepsilon\} . \tag{2.3}$$

We make ε so small that $U_i^{(\varepsilon)} \cap U_j^{(\varepsilon)} = \phi\ (i \neq j)$ and that in each $U_j^{(\varepsilon)}$ (0.1) is equivalent to the localized problem (2.1). From the results in the case where there is no singular point (cf. Ikawa [3]), we see that if the data in (0.1) vanish on $(\bigcup_{j=1}^{N} U_j^{(\varepsilon)}) \times [0, t_0]$ (t_0 is small enough for ε) there is a solution $u(x,t)$ with support in $(\bar{\Omega} - \bigcup_{j=1}^{N} U_j^{(\varepsilon/2)}) \times [0, t_0]$. Furthermore, we see that if $(x,t) \in (\bar{\Omega} - \bigcup_{j=1}^{N} U_j^{(\varepsilon)}) \times (0, t_0]$ there exists the bounded domain of dependence of the point (x,t), which is disjointed with $\bigcup_{j=1}^{N} U_j^{(\varepsilon/2)} \times [0, t_0]$.

Let $u(x,t)$ be a solution of (0.1) with null data (i.e. $f=0$, $g=0$, $u_0=u_1=0$). Then, from the above statement concerning the domain of dependence it follows that $\operatorname{supp} u \subset \bigcup_{j=1}^{N} U_j^{(\varepsilon)} \times [0, t_0]$. Since the uniqueness for each localized problem (2.1) is assumed, we have $u=0$. Therefore the solution of (0,1) is unique in $C^\infty(\bar{\Omega} \times [0, t_0])$.

Let us show existence of the solution of (0.1). Solving the Cauchy problem ignoring the boundary condition of (0.1), we may assume that $f=0$ and $u_0=u_1=0$. Then the compatibility condition implies that $D_t^k g|_{t=+0}=0$ for $k=0,1,\cdots$. Take a partition of unity $\{\phi_j(x)\}_{j=0,\cdots,N}$ on $\bar{\Omega}$ such that $\operatorname{supp} \phi_0 \subset \bar{\Omega} - \bigcup_{j=1}^{N} U_j^{(\varepsilon/2)}$ and $\operatorname{supp} \phi_j \subset U_j^{(\varepsilon)}$ $(j=1,\cdots,N)$. Obviously if $(f,g,u_0,u_1)=(0,g,0,0)$ is compatible, so is $(0,\phi_j g,0,0)$ $(j=0,\cdots,N)$. From the results in the non-singular case, we find a solution $u^{(0)}(x,t)$ satisfying

$$\begin{cases} \Box u^{(0)} = 0 & \text{in } \Omega \times (0, t_0)\,, \\ \dfrac{\partial u^{(0)}}{\partial \nu}\Big|_\Gamma = \phi_0 g & \text{on } \Gamma \times (0, t_0)\,, \\ u^{(0)}|_{t=0} = \partial_t u^{(0)}|_{t=0} = 0 & \text{on } \Omega\,. \end{cases}$$

Since each localized problem (2.1) is supposed C^∞ well-posed, for the data with support near the origin there is a unique solution of (2.1) with support near the origin (apply Theorem 3.1 in §3). Therefore, for $j=1,2,\cdots,N$ we have a solution $u^{(j)}$ satisfying

$$\begin{cases} \Box u^{(j)} = 0 & \text{in } \Omega \times (0, t_0)\,, \\ \dfrac{\partial u^{(j)}}{\partial \nu}\Big|_\Gamma = \phi_j g & \text{on } \Gamma \times (0, t_0)\,, \\ u^{(j)}|_{t=0} = \partial_t u^{(j)}|_{t=0} = 0 & \text{on } \Omega\,. \end{cases}$$

$u(x,t) = \sum_{j=0}^{N} u^{(j)}(x,t)$ $(\in C^\infty(\bar{\Omega} \times [0, t_0]))$ is the required solution. The proof is complete.

3. Domains of dependence

In this section, assuming that the solution of (0.1) is unique, we shall study the domain of dependence. We note that the solution is unique on $t_1 \leqq t \leqq t_2$ for any $t_1 < t_2$ if the uniqueness is guaranteed on $0 \leqq t \leqq t_0$ for some $t_0 > 0$ (because $\Box$, $\frac{\partial}{\partial \nu}$ are independent of t). From Theorem 3.1 and 3.2 stated later, it follows that the domain of dependence is bounded at any point although (0.1) has not a finite propagation speed. The results in this section are all valid also for the problem (2.1).

For a set S of $\boldsymbol{R}_x^2 \times [0, \infty)$ we set

$$\mathring{\sum}(S) = \bigcup_{X \in S} (\mathring{\sum} + X),$$

where $\mathring{\sum} = \{X = (x, t) \colon t \geqq |x|\}$. Then, as is well known, the solution of the Cauchy problem

$$\begin{cases} \Box u = f(x,t) & \text{in } \boldsymbol{R}^2 \times [0, \infty)\,, \\ u|_{t=0} = u_0(x) & \text{on } \boldsymbol{R}^2\,, \\ \partial_t u|_{t=0} = u_1(x) & \text{on } \boldsymbol{R}^2 \end{cases}$$

has support in $\mathring{\sum}(S)$ ($S = (\operatorname{supp} f) \cup (\operatorname{supp} u_0 \times \{t=0\}) \cup (\operatorname{supp} u_1 \times \{t=0\})$). Let Γ be given by

$$x = x'(s), \quad \left|\frac{dx'}{ds}(s)\right| = 1$$

($x'(s)$ is a periodic C^∞ function on $\boldsymbol{R}^1$), and for $x_0' \in \Gamma$ set

$$\kappa(s) = \kappa(s; x_0') = \int_{s_0}^{s} \frac{|\langle \nu(x'(\lambda)), n(x'(\lambda))\rangle|}{|\nu(x'(\lambda))|} d\lambda \quad (x_0' = x'(s_0))\,.$$

$\left(\frac{d\kappa}{ds}(s)\right)^{-1}$ is equal to the propagation speed of the mixed problem frozen at $x = x'(s)$ (let $x'(s)$ be a non-singular point) (cf. Appendix of Ikawa [3]). We set

$$\tilde{\sum}(x_0', t_0) = \{(x', t) \in \Gamma \times [0, \infty);\ x' = x'(s),\ t - t_0 \geqq |\kappa(s; x_0')|,\ s \in \boldsymbol{R}^1\}\,,$$

$$\sum{}'(S') = \bigcup_{X' \in S'} \mathring{\sum}(\tilde{\sum}(X')) \quad (S' \subset \Gamma \times [0, \infty))\,.$$

Theorem 3.1. *Assume that the solution of* (0.1) *is unique in* $C^\infty(\bar{\Omega} \times [0, t_0])$. *Let* S *be* $(\operatorname{supp} f) \cup (\operatorname{supp} u_0 \times \{t=0\}) \cup (\operatorname{supp} u_1 \times \{t=0\})$ *and* S' *be* $(\mathring{\sum}(S) \cap (\Gamma \times [0, t_0])) \cup \operatorname{supp} g$. *Then the solution* $u(x,t)$ *of* (0, 1) *has support in*

$$\sum(S) \equiv \mathring{\sum}(S) \cup \sum{}'(S')\,.$$

From this theorem it is seen that for any $\varepsilon > 0$ there is a constant $\tilde{t}(\varepsilon) > 0$

such that $\bigcup_{0\leq t\leq \tilde{t}(\varepsilon)} \operatorname{supp}_x [u(x,t)]$ is contained in ε-neighborhood of $\bigcup_{0\leq t\leq \tilde{t}(\varepsilon)} \operatorname{supp}_x$ [the data].

In the case where (0.1) has no singular point the above theorem has been obtained (cf. Ikawa [3]).

REMARK 3.1. If the uniqueness in the Sobolev space holds, the above theorem is valid for the solutions and data in that space.

Proof of Theorem 3.1. Because $\square$ and $\frac{\partial}{\partial\nu}$ (in (0.1)) do not depend on t, it suffices to show that supp $u\cap\{0\leqq t\leqq t_0\}\subset\sum(S)$ for a sufficiently small $t_0>0$. For each singular point x'_j $(j=1,\cdots N)$ we define $U_j^{(\varepsilon)}$ $(\varepsilon>0)$ by (2.3). Let ε be so small that $U_j^{(2\varepsilon)}\cap U_i^{(2\varepsilon)}=\phi$ if $i\neq j$ and take a small t_0 such that every $\sum(U_j^{(2\varepsilon)}-U_j^{(3\varepsilon/2)})\cap\{0\leqq t\leqq t_0\}$ $(j=1,\cdots,N)$ is disjointed with $\bigcup_{j=1}^{N} U_k^{(\varepsilon)}$.

Let $\phi_j(x)=1$ on $U_j^{(3\varepsilon/2)}$ and supp $\phi_j\subset U_j^{(2\varepsilon)}$, and set $u^{(j)}=\phi_j(x)u(x,t)$. Then $u^{(j)}$ satisfies

$$
(3.1)\qquad
\begin{cases}
\square u^{(j)}=[\square,\phi_j]u+\phi_j f\ (\equiv f^{(j)}) & \text{in } \bar{\Omega}\times(0,t_0)\,,\\
\dfrac{\partial u^{(j)}}{\partial\nu}\Big|_\Gamma=\left[\dfrac{\partial}{\partial\nu},\phi_j\right]u\big|_\Gamma+\phi_j g\ (\equiv g^{(j)}) & \text{on } \Gamma\times(0,t_0)\,,\\
u^{(j)}\big|_{t=0}=\phi_j u_0\ (\equiv u_0^{(j)}) & \text{on } \Omega\,,\\
\dfrac{\partial u^{(j)}}{\partial t}\Big|_{t=0}=\phi_j u_1\ (\equiv u_1^{(j)}) & \text{on } \Omega\,.
\end{cases}
$$

Obviously it follows that

$$
\Big(\bigcup_{k=1}^{N} U_k^{(\varepsilon)}\times[0,t_0]\Big)\cap\sum(S_j)\subset\sum(S)\,,
$$

where $S_j=\operatorname{supp}(f^{(j)}, g^{(j)}, u_0^{(j)}, u_1^{(j)})$. Set

$$
\tilde{t}_j=\inf\{t\colon (x'_j,t)\in\sum(S_j)\}\,.
$$

Then, for any $\tilde{t}$ $(0<\tilde{t}<\tilde{t}_j)$ we can solve (3.1) on $0\leqq t\leqq\tilde{t}$ by the methods in the non singular case (cf. Ikawa [3]), which implies that supp $u^{(j)}\cap\{0\leqq t\leqq\tilde{t}\}\subset\sum(S_j)$ (because the solution is unique). Next, consider the problem (3.1) on $\tilde{t}\leqq t\leqq t_0$ with the initial data $(u^{(j)}|_{t=\tilde{t}},\ \partial_t u^{(j)}|_{t=\tilde{t}})$ on $t=\tilde{t}$. Then, by the result concerning the domain of dependence in the non singular case, we see that supp $u^{(j)}\cap\{\tilde{t}_j\leqq t\leqq t_0\}\subset\sum(S_j)$. Therefore it is concluded that

$$
\operatorname{supp} u^{(j)}\cap\{0\leqq t\leqq t_0\}\subset\sum(S_j)\quad (j=1,\cdots,N)\,.
$$

This yields

$$
(\operatorname{supp} u)\cap\Big(\bigcup_{i=1}^{N} U_j^{(\varepsilon)}\times[0,t_0]\Big)\subset\bigcup_{j=1}^{N}\sum(S_j)\cap\Big(\bigcup_{j=1}^{N} U_k^{(\varepsilon)}\times[0,t_0]\Big)\subset\sum(S)\,.
$$

Take a C^∞ function $\varphi(x)$ such that $\varphi(x)=1$ on $\bar{\Omega}-\bigcup_{j=1}^{N} U_j^{(2\varepsilon/3)}$ and $\varphi(x)=0$ on $\bigcup_{j=1}^{N} U_j^{(\varepsilon/3)}$, and consider the following equation for a sufficiently small constant t_1 $(0<t_1\leqq t_0)$:

$$\begin{cases} \square(\varphi u) = [\square, \varphi]u+\varphi f & \text{in } \Omega\times(0,t_1)\,, \\ \dfrac{\partial(\varphi u)}{\partial\nu}\Big|_\Gamma = \left[\dfrac{\partial}{\partial\nu}, \varphi\right]u\,|_\Gamma+\varphi g & \text{on } \Gamma\times(0,t_1)\,, \\ (\varphi u)\,|_{t=0} = \varphi u_0 & \text{on } \Omega\,, \\ \partial_t(\varphi u)\,|_{t=0} = \varphi u_1 & \text{on } \Omega\,. \end{cases}$$

Then, by the result in the non singular case, we see that

$$\operatorname{supp}\,[\varphi u]\cap(\bar{\Omega}-\bigcup_{j=1}^{N} U_j^{(\varepsilon)})\times[0,t_1]\subset\textstyle\sum(S)\,.$$

Therefore we obtain the theorem.

The following theorem is another main result in this section:

Theorem 3.2. *Let the mixed problem* (0.1) *be* C^∞ *well-posed. Then* (0.1) *has not a finite propagation speed.*

Proof. We can prove this theorem by the same procedure as in the author [15] (see Theorem 4.1 of [15]). Let us mention an outline of the proof.

Obviously we have only to study near each singular point x_0'. For $v>0$ and $t_1>0$ set

$$D(x_0',t_1;v) = D = \{(x,t)\in\bar{\Omega}\times[0,t_1];\,|x-x_0'|\leqq(t_1-t)v\}\,.$$

Assume that (0.1) has a finite propagation speed not more than $v>0$. Then, for any $t_1\,(0<t_1\leqq t_0)$ it follows that if the equalities

$$(3.2)\qquad \begin{cases} \square u = 0 & \text{on } D(x_0',t_1;v)\,, \\ \dfrac{\partial u}{\partial\nu}\Big|_\Gamma = 0 & \text{on } D\cap(\Gamma\times[0,t_1])\,, \\ u\,|_{t=0} = \partial_t u\,|_{t=0} = 0 & \text{on } D\cap\{t=0\} \end{cases}$$

hold the solution $u(x,t)$ equals 0 on D. In the same way as in the proof of Theorem 4.1 of [15], we can construct an asymptotic solution

$$u_N(x,t;k) = \sum_{j=1}^{N} e^{ik\Phi(x,t)}v_j(x,t)\,(ik)^{-j}$$

such that $v_0(x_0',t_1)\neq0$ and

$$\begin{cases} \square u_N = e^{ik\Phi}\square v_N(ik)^{-N} & \text{in } \Omega\times(0,t_0)\,, \\ \dfrac{\partial u_N}{\partial\nu}\Big|_\Gamma = 0 & \text{on } D\cap(\Gamma\times(0,t_0))\,, \\ u_N\,|_{t=0} = \partial_t u_N\,|_{t=0} = 0 & \text{on } D\cap\{t=0\}\,. \end{cases}$$

Since (0.1) is supposed C^∞ well-posed, there exists a solution $w_N(x,t;k)$ satisfying

$$\begin{cases} \Box w_N = e^{ik\Phi}\Box v_N & \text{in } \Omega\times(0,t_0)\,, \\ \dfrac{\partial w_N}{\partial \nu}|_\Gamma = 0 & \text{on } \Gamma\times(0,t_0)\,, \\ w_N|_{t=0} = \partial_t w_N|_{t=0} = 0 & \text{on } \Omega\,, \end{cases}$$

and the estimate

$$|w_N|_{0,D} \leqq C_1 |e^{ik\Phi}\Box v_N|_{l,D'} \leqq C_2 k^l\,.$$

holds for constants C_1, C_2, l and a domain D' ($\supset D$) independent of k. Take N so that $l<N$, and set

$$u(x,t;k) = u_N(x,t;k) - (ik)^{-N} w_N(x,t;k)\,.$$

Then $u(x,t;k)$ satisfies (3.2), but $u(x_0',t_1;k) \neq 0$ for large k, which proves Theorem 3.2.

4. Proof of Theorem 1

If the assumption of Theorem 1 is fulfilled, the $\psi^{(p)}(y)$ in the problem (2.1) localized at a certain singular point satisfies the condition (I) or (II) of (2.2). To prove Theorem 1, it suffices from ii) of Proposition 2.1 to verify

Theorem 4.1. *Suppose that* $\psi^{(p)}(y)$ *in* (2.1) *satisfies the condition* (I) *or* (II) of (2.2). *Then the mixed problem* (2.1) *is not* C^∞ *well-posed.*

In the case (I) we can prove the theorem in the same way as in the author [15], namely, by constructing an appropriate asymptotic solution of (2.1) violating an ineuqality to be satisfied if the problem is C^∞ well-posed (see § 5 of [15]). But this method cannot be applied in the case (II). In this paper we employ a method applicable to both cases.

At first we shall construct an (approximate) Poisson operator of (2.1) by the methods of the Fourier integral operator. Consider the equation (in ξ)

$$L_0(y,\xi,\eta,\sigma) \equiv \xi^2 + 2a(y)\eta\xi + b(y)\eta^2 - b(y)\sigma^2 = 0$$

for $y\in \boldsymbol{R}^1$, $(\eta,\sigma)\in \boldsymbol{R}^2$. When $(\eta,\sigma)\in\Delta = \{(\eta,\sigma)\colon \sigma^2-\eta^2 > \delta(\sigma^2+\eta^2)\}$ (δ is a small positive constant), this equation has the distinct real roots

$$\xi_0^\pm(y,\eta,\sigma) = -a(y)\eta \mp \sqrt{b(y)(\sigma^2-\eta^2)+a^2\eta^2}\,.$$

Applying Proposition 1.3, we have symbols $\xi^\pm(y,\eta,\sigma)\in S^1$ ($\in S^1_{(\sigma)}$, $|\sigma|\geqq 1$) with the properties stated in i) and ii) of Proposition 1.3. Hereafter we denote by $\xi_0^\pm(y,\eta,\sigma)$ the principal symbols of $\xi^\pm(y,\eta,\sigma)$, and assume that $\xi_0^\pm(y,\eta,\sigma)$ are real-valued on whole $\boldsymbol{R}_y^1 \times \boldsymbol{R}_{(\eta,\sigma)}^2$. We set

$$\Delta_+ = \Delta \cap \{(\eta, \sigma): \sigma > 0\} .$$

Lemma 4.1. *Let $\tilde{\Delta}_+$ be a conic open set such that $\bar{\tilde{\Delta}}_+ \subset \Delta_+$, and let ρ in (2.1) be small enough. Then, for any $\chi^+(y,t,\eta,\sigma) \in S^0$ satisfying $\operatorname{supp}_{\eta,\sigma} \chi^+ \subset \tilde{\Delta}_+$ and $\operatorname{supp}_t \chi^+ \subset [\tilde{t}_0, \infty)$, there exists a bounded operator $\mathcal{P}^+(x)$ on $H_m(\boldsymbol{R}^2_{(y,t)})$ (the mapping: $x \to \mathcal{P}^+(x)$ $(0 \leqq x \leqq x_0)$ is C^∞ smooth in the operator-norm) such that*

i) $L^+\mathcal{P}^+(x) \in C_x^\infty(S^{-\infty})$[1] $(0 \leqq x \leqq x_0)$,
ii) $\mathcal{P}^+(0) = \chi^+(y,t,D_y,D_t)$,
iii) $L\mathcal{P}^+(x) \in C_x^\infty(S^{-\infty})$ $(0 \leqq x \leqq x_0)$,
iv) $\operatorname{supp}[\mathcal{P}^+(x)h] \subset \{(x,y,t): \tilde{t}_0 + \tilde{\delta}x \leqq t\}$ *for some constant* $\tilde{\delta} > 0$ $(h(y,t) \in \mathcal{S})$,
v) *defining* $\tilde{T}$ *by*

$$\tilde{T}h = B\mathcal{P}^+ h|_{x=0} \quad (B = D_y + \psi D_x) ,$$

we have $\tilde{T} \in S^1$ *and*

$$\sigma_0(\tilde{T}) = (\eta + \psi(y)\xi_0^+(y,\eta,\sigma))\chi^+(y,t,\eta,\sigma) .$$

Proof. We make the above operator $\mathcal{P}^+(x)$ in the same way as Kumano-go [8] constructed fundamental solutions for operators of the type $L_+ = D_x - \xi^+$ (see §3 of [8]).

As is described in Theorem 3.1 of [8], the eiconal equation

$$\begin{cases} \partial_x\phi - \xi_0^+(y, \nabla_{(y,t)}\phi) = 0, \ 0 \leqq x \leqq x_0 \quad (\nabla_{(y,t)}\phi = (\partial_y\phi, \partial_t\phi)) , \\ \phi|_{x=0} = y\eta + t\sigma \quad ((\eta,\sigma) \in \boldsymbol{R}^2) \end{cases}$$

has a unique solution $\phi(x,y,t,\eta,\sigma)$ satisfying $\phi - y\eta - t\sigma \in C_x^\infty(S^1)$. We assume that $\mathcal{P}^+(x)$ has the form

$$(\mathcal{P}^+h)(x,y,t) = \iint e^{i\phi(x,y,t,\eta,\sigma)} \sum_{j=0}^{-\infty} e_j(x,y,t,\eta,\sigma)\hat{h}(\eta,\sigma)\, d\eta d\sigma ,$$

$$e_j(x,y,t,\eta,\sigma) \in C_x^\infty(S^j) ,$$

and define $\{e_j\}$ inductively so that the requirements i) and ii) are satisfied. Then we obtain the transport equation of the form

$$(4.1) \qquad \begin{cases} D_xe_j - \partial_\eta\xi_0^+(y, \nabla_{(y,t)}\phi)D_ye_j - \partial_\sigma\xi_0^+(y, \nabla_{(y,t)}\phi)D_te_j \\ \qquad\qquad -ge_j - r_j = 0, \ 0 \leqq x \leqq x_0 , \\ e_0|_{x=0} = \chi^+(y,t,\eta,\sigma), \quad e_j|_{x=0} = 0 \quad (j \leqq -1) , \end{cases}$$

---

[1] $C_x^\infty(S^m)$ denotes the set of S^m-valued C^∞ functions.

where g is a function independent of $\{e_j\}$ and r_j is determined with only $e_0, e_{-1}, \cdots, e_{j+1}$. (4.1) has the solution $e_j(x,y,t,\eta,\sigma)\in C_x^\infty(S^j)$ $(j=0, -1, \cdots)$ (cf. proof of Theorem 3.2° of Kumano-go [8]). There exists a symbol $e(x,y,t,\eta,\sigma)$ $\in C_x^\infty(S^0)$ such that $e|_{x=0}=\chi^+$, supp $e\subset \bigcup_{j=0}^{-\infty}$supp e_j and $e(x,y,t,\eta,\sigma)-\sum_{j=0}^{-N+1} e_j(x,y,t,\eta,\sigma)\in C_x^\infty(S^{-N})$ $(N=1,2,\cdots)$ (cf. Theorem 2.7 of Hörmander [2]).

Now we set here

$$(4.2)\qquad (\mathscr{P}^+h)(x,y,t)=\iint \exp\{i\phi(x,y,t,\eta,\sigma)\}e(x,y,t,\eta,\sigma)\hat{h}(\eta,\sigma)d\eta d\sigma\,.$$

Then, obviously $\mathscr{P}^+$ satisfies i) and ii). Since $\operatorname{supp}_{\eta,\sigma} e\subset\tilde{\Delta}$, we obtain iii) by Proposition 1.3 (Proposition 1.3 is valid also when χ in ii) is a Fourier integral operator). From the definition it follows that

$$\begin{aligned} D_x\mathscr{P}^+h|_{x=0} &= \left(\iint e^{i\phi}(\partial_x\phi+D_xe)\hat{h}d\eta d\sigma\right)\Big|_{x=0} \\ &= \xi_0^+(y,D_y,D_t)\chi^+(y,t,D_y,D_t)h+r(y,t,D_y,D_t)h\,, \end{aligned}$$

where $r(y,t,\eta,\sigma)\in S^0$, which yields v). The bicharacteristic curve $\{q(x),p(x)\}_{x\geqq 0}$ $=\{(q_1,q_2),(p_1,p_2)\}$ through (y,t,η,σ) $(\eta^2+\sigma^2\geqq 1)$ is defined by

$$\begin{cases} \dfrac{dq_1}{dx}=-\partial_\eta\xi_0^+(q_1,p), & \dfrac{dp_1}{dx}=\partial_y\xi_0^+(q_1,p)\,, \\ \dfrac{dq_2}{dx}=-\partial_\sigma\xi_0^+(q_1,p), & \dfrac{dp_1}{dx}=\partial_t\xi_0^+(q_1,p)\ (=0)\,, \\ q|_{x=0}=(y,t)\,, & p|_{x=0}=(\eta,\sigma)\,. \end{cases}$$

It is seen that if $\rho>0$ is small enough $\{p(x)\}_{x\geqq 0}\subset\Delta_+$ follows from $p(0)=(\eta,\sigma)\in\tilde{\Delta}_+$ and that $-\partial_{\tilde{\sigma}}\xi_0^+(\tilde{y},\tilde{\eta},\tilde{\sigma})=-\dfrac{b(\tilde{y})\tilde{\sigma}}{\xi_0^+(\tilde{y},\tilde{\eta},\tilde{\sigma})+a(\tilde{y})\tilde{\sigma}}\geqq\tilde{\delta}\ (>0)$ for $\tilde{y}\in\boldsymbol{R}^1$, $(\tilde{\eta},\tilde{\sigma})\in\Delta_+$ $(\tilde{\eta}^2+\tilde{\sigma}^2\geqq 1)$. From these facts we have $\dfrac{dq_2}{dx}(x)\geqq\tilde{\delta}$ for $x\geqq 0$, which implies $t+\tilde{\delta}x\leqq q_2(x)$ for $x\geqq 0$. Therefore, noting that $\{q(x)\}_{x\geqq 0}$ is a characteristic curve of (4.1), we see that $\operatorname{supp}_{x,y,t} e_j\subset\{(x,y,t)\colon \tilde{t}_0+\tilde{\delta}x\leqq t\}$ $(j=0,-1,\cdots)$. This yields iv). The proof is complete.

Now, let us consider the Dirichlet problem

$$\begin{cases} Lw=0 & \text{in } \boldsymbol{R}_+^2\times(-\infty,t_0)\,, \\ w|_{x=0}=h & \text{on } \boldsymbol{R}^1\times(-\infty,t_0)\,. \end{cases}$$

This satisfies the uniform Lopatinski condition (cf. Sakamoto [12]). We set

$$C_+^\infty(M\times(-\infty,t_0]) = \{u\in C^\infty(M\times(-\infty,t_0]);\ \operatorname{supp}_t u\subset[t_1,t_0] \text{ for some } t_1\ (<t_0)\}\quad (M=\bar{\boldsymbol{R}}_+^2 \text{ or } \boldsymbol{R}^1)\,.$$

Then, for any $h(y,t)\in C_+^\infty(\boldsymbol{R}^1\times(-\infty,t_0])$ there exists a unique solution $w(x,y,t)$ in $C_+^\infty(\bar{\boldsymbol{R}}_+^2\times(-\infty,t_0])$, and $\operatorname{supp}_t w\subset[t_1,t_0]$ follows from $\operatorname{supp}_t h\subset[t_1,t_0]$. We define an operator T on $C_+^\infty(\boldsymbol{R}^1\times(-\infty,t_0])$ by

$$Th = Bw|_{x=0}\quad (=(D_y+\psi(y)D_x)w|_{x=0})\,.$$

As is easily seen, this operator $T=T_{t_0}$ does not depend on t_0, that is, for arbitrary t_0,t_0' $(t_0<t_0')$ $T_{t_0}h=T_{t_0'}h$ on $-\infty<t\leqq t_0$. It follows that the mixed problem (2.1) is C^∞ well-posed if and only if for any

(4.3) $g(y,t)\in C_+^\infty$ satisfying $\operatorname{supp}_t g\subset[0,t_0]$ there exists a unique solution $h(y,t)$ of $Th=g$ in $C_+^\infty(\boldsymbol{R}^1\times(-\infty,t_0])$ whose support is in $\boldsymbol{R}^1\times[0,t_0]$.

In fact: Ignoring the boundary condition of (2.1) and solving the Cauchy problem, we may assume that the data (f,g,u_0,u_1) in (2.1) satisfy $f=0$, $u_0=u_1=0$ and $\partial_t^j g|_{t=0}=0$ $(j=0,1,\cdots)$. If for any g $(\in C_+^\infty)$ with $\operatorname{supp}_t g\subset[0,t_0]$ there exists a solution $h(y,t)$ stated in (4.3), we have a function $w(x,y,t)$ $(\in C_+^\infty)$ such that $\operatorname{supp}_t w\subset[0,t_0]$, $w|_{x=0}=h$ and $Lw=0$ on $\boldsymbol{R}_+^2\times(-\infty,t_0]$. This w is a solution of (2.1) for the data $(0,g,0,0)$. Conversely, if $w(x,y,t)$ $(\in C_+^\infty)$ with $\operatorname{supp}_t w\subset[0,t_0]$ is a solution of (2.1) for the data $(0,g,0,0)$, $h(y,t)=w|_{x=0}$ satisfies $Th=g$.

The operator $\tilde{T}$ stated in Lemma 4.1 approximates to T in the following sense:

Lemma 4.2. *Let $\varphi(t)$ $(\in C^\infty)=1$ on $[2\tilde{t}_0,\infty)$ and $\operatorname{supp}\varphi\subset(\tilde{t}_0,\infty)$, and let $\tilde{\varphi}(t)$ $(\in C^\infty)$ satisfy $\operatorname{supp}\tilde{\varphi}\subset(-\infty,\tilde{t})$ $(0<2\tilde{t}_0<\tilde{t})$. Furthermore, let $\chi(\eta,\sigma)$ $(\in S^0)$ be homogeneous of order 0 $(\eta^2+\sigma^2\geqq1)$ and $\operatorname{supp}\chi\subset\tilde{\Delta}_+(\subset\subset\Delta_+)$, and assume that $\chi^+(y,t,\eta,\sigma)$ in Lemma 4.1 is equal to 1 on a neighborhood of $\operatorname{supp}[\varphi(t)\chi(\eta,\sigma)]$. Then, for any positive integer N we have*

i) $\|\tilde{\varphi}(T-\tilde{T})\varphi\chi h\|_N'\leqq C\|h\|_{-1}'$, $h(y,t)\in\mathcal{S}$;

ii) *(if ρ in (2.1) is small enough for N)*
$\|\chi\tilde{\varphi}(T-\tilde{T})\varphi h\|_N'\leqq C\|h\|_1'$, $h(y,t)\in\mathcal{S}$,

where $\|\cdot\|_N'$ is the norm of the Sobolev space $H_N(\boldsymbol{R}^2_{(y,t)})$.

Proof. By means of Corollary of Theorem 2 in Sakamoto [12] II, for $m=0,1,\cdots$ we have the estimate

$$\sum_{|\alpha|\leq m}\|D^\alpha_{(y,t)}u\|_{1,0<t<t_1}+\|D_xu\|'_{m,0<t<t_1} \leqq C_1\Big(\sum_{|\alpha|\leq m}\|D^\alpha_{(y,t)}Lu\|_{0,0<t<t_1}+\|u\|'_{m+1,0<t<t_1}\Big)\,, \tag{4.4}$$

where $D_t^j u|_{t=0}=0$ for $j=0,1,\cdots,m+1$, $||u||^2_{m,0<t<t_1}=\sum_{|\alpha|\leq m}\iiint_{x>0,0<t<t_1}|D^\alpha_{(x,y,t)}u|^2dxdydt$ and $||u||'^2_{m,0<t<t_1}=\sum_{|\alpha|\leq m}\iint_{0<t<t_1}|D^\alpha_{(y,t)}u|_{x=0}|^2dydt$. Let $w(x,y,t)$ $(\in C^\infty_+)$ be the solution of

$$\begin{cases}Lw=0 & \text{in } \boldsymbol{R}^2_+\times(-\infty,\tilde{t}),\\ w|_{x=0}=\varphi\chi h & \text{on } \boldsymbol{R}^1\times(-\infty,\tilde{t}),\end{cases}$$

and set

$$\tilde{w}(x,y,t)=\mathcal{P}^+(\varphi\chi h).$$

Then it follows that

$$\begin{aligned}||\tilde{\varphi}(T-\tilde{T})\varphi\chi h||'_N&\leqq||B(w-\tilde{w})||'_{N,0<t<\tilde{t}}\\&\leqq C_2(||w-\tilde{w}||'_{N+1,0<t<\tilde{t}}+||D_x(w-\tilde{w})||'_{N,0<t<\tilde{t}}).\end{aligned}$$

It is obvious from ii) of Lemma 4.1 that

$$||w-\tilde{w}||'_{N+1,0<t<\tilde{t}}\leqq||(1-\chi^+)\varphi\chi h||'_{N+1}\leqq C_3||h||'_{-1}.$$

Using (4.4) and iii) of Lemma 4.1, we obtain

$$\begin{aligned}||D_x(w-\tilde{w})||'_{N,0<t<\tilde{t}}&\leqq C_4\Big(\sum_{|\alpha|\leq N}||D^\alpha_{(y,t)}L\tilde{w}||_{0,0<x<\tilde{t}\delta^{-1}}+||w-\tilde{w}||'_{N+1,0<t<\tilde{t}}\Big)\\&\leqq C_5||h||'_{-1}.\end{aligned}$$

Therefore i) of Lemma 4.2 is derived.

Let us show ii) of Lemma 4.2. Let ρ in (2.1) be so small that by Lemma 1.1 we have a symbol $\zeta(y,\eta,\sigma)\in S^0(\in S^0_{(\sigma)})$ satisfying $[\zeta,\xi^-]\in S^{-N-1}$, $\zeta(y,\eta,\sigma)=1$ for $(\eta,\sigma)\in\tilde{\Delta}_+$ and $\operatorname{supp}_{\eta,\sigma}\zeta\subset\Delta_+$. Denote by $w(x,y,t)$ $(\in C^\infty_+)$ the solution of

$$\begin{cases}Lw=0 & \text{in } \boldsymbol{R}^2_+\times(-\infty,\tilde{t}+1),\\ w|_{x=0}=\varphi(t)h(y,t) & \text{on } \boldsymbol{R}^1\times(-\infty,\tilde{t}+1),\end{cases}$$

and take C^∞ functions $\varphi_1(t)$, $\tilde{\varphi}_1(t)$ such that $\operatorname{supp}\varphi_1\subset(\tilde{t}_0,\infty)$, $\operatorname{supp}\tilde{\varphi}_1\subset(-\infty,\tilde{t}+1)$ and $\varphi_1(t)=1$ on $\operatorname{supp}\varphi$, $\tilde{\varphi}_1(t)=1$ on $(-\infty,\tilde{t}]$. Then, using (4.4), we have

$$\begin{aligned}||\chi\tilde{\varphi}(T-\tilde{T})\varphi h||'_N&=||\chi\tilde{\varphi}B\{w-\mathcal{P}^+(\varphi h)\}||'_N\\&\leqq||\chi\tilde{\varphi}B\{\varphi_1\zeta\tilde{\varphi}_1w-\mathcal{P}^+(\varphi h)\}||'_N+C_1||\varphi h||'_1.\end{aligned}$$

Let us express $\varphi_1\zeta\tilde{\varphi}_1w$ by the Fourier integral operator. We write

$$\begin{aligned}L(\varphi_1\zeta\tilde{\varphi}_1w)&=\varphi_1\zeta L\tilde{\varphi}_1w+[L,\varphi_1]\zeta\tilde{\varphi}_1w+\varphi_1([L,\zeta]-[L^-L^+,\zeta])\tilde{\varphi}_1w\\&\quad+\varphi_1[L^-L^+,\zeta]\tilde{\varphi}_1w\equiv J_1+J_2+J_3+J_4.\end{aligned}$$

It is easily seen that

$$||D^{\alpha}_{(y,t)}J_i||_{0,0<t<\tilde{t}} \leqq C_2||w||_{1,0<t<\tilde{t}+1} \leqq C_3||\varphi h||'_1$$

for any α and $i=1,2$. In view of Proposition 1.3 we have for any α

$$||D^{\alpha}_{(y,t)}J_3||_{0,0<t<\tilde{t}} \leqq C_4||w||_{1,,0<t<\tilde{t}+1} \leqq C_5||\varphi h||'_1 .$$

From finiteness of propagation speed, there is a constant x_0 such that $\operatorname{supp}_x[\tilde{\varphi}_1 w] \subset [0,x_0)$. Let $\theta(x)$ $(\in C^\infty)=1$ on $(-\infty,\tilde{x}]$ and 0 on $[\tilde{x}+1,\infty)$, where $\tilde{x}$ is a constant larger than $x_0+(\tilde{t}+1)\tilde{\delta}^{-1}$ ($\tilde{\delta}$ is the constant in iv) of Lemma 4.1). We set

$$\tilde{v}(x,y,t) = \theta(x)\int_0^x \mathcal{P}^+(x-s)\ \{[\xi^+,\zeta]\tilde{\varphi}_1 w\}(s)ds .$$

Then, from Lemma 4.1 it follows that

$$\tilde{v}\,|_{x=0} = 0, \quad \operatorname{supp}_t \tilde{v} \subset [\tilde{t}_0,\infty) ,$$
$$||\chi\tilde{\varphi}B\tilde{v}||'_N \leqq C_6||\tilde{\varphi}_1 w||_1 \leqq C_6||\varphi h||'_0 ,$$
$$\begin{aligned} ||D^{\alpha}_{(y,t)}\{L^+\tilde{v}-[\xi^+,\zeta]\tilde{\varphi}_1 w\}||_0 &\leqq ||D^{\alpha}_{(y,t)}\{L^+\tilde{v}-[\xi^+,\zeta]\tilde{\varphi}_1 w\}||_{0,0<x<\tilde{x}} + ||D^{\alpha}_{(y,t)}L^+\tilde{v}||_{0,\tilde{x}<x<\tilde{x}+1} \\ &\leqq C_7||\tilde{\varphi}_1 w||_1 \leqq C_8||\varphi h||'_1 . \end{aligned}$$

Here, the inequality $||D^{\alpha}_{(y,t)}L^+\tilde{v}||_{0,\tilde{x}<x<\tilde{x}+1} \leqq C_9||\tilde{\varphi}_1 w||_1$ is derived from the fact that $\operatorname{supp}_{y,t} \tilde{\varphi}_1 w(s) \cap \operatorname{supp}_{y,t}\ e(x-s)=\phi$ if $s \leqq \tilde{x} \leqq x$ ($e(x)$ is the symbol in (4.2)). Noting $J_4=\varphi_1 L^-[\xi^+,\zeta]\tilde{\varphi}_1 w+\varphi_1[\xi^-,\zeta]L^+\tilde{\varphi}_1 w$ and $[\xi^-,\zeta]\in S^{-N-1}$, by Proposition 1.3 and the last of the above estimates we have for any α

$$\begin{aligned} ||D^{\alpha}_{(y,t)}(J_4-L\tilde{v})||_0 &\leqq ||D^{\alpha}(L^-L^+-L)\tilde{v}||_0 + ||D^{\alpha}\varphi_1 L^-(L^+\tilde{v}-[\xi^+,\zeta]\tilde{\varphi}_1 w)||_0 \\ &\quad + ||D^{\alpha}\varphi_1[\xi^-,\zeta]L^+\tilde{\varphi}_1 w||_0 \leqq C_{10}||\varphi h||'_1 . \end{aligned}$$

Thus we see that $\tilde{w}(x,y,t)=\tilde{v}+\mathcal{P}^+(\varphi_1\zeta\tilde{\varphi}_1\varphi h)$ is the required expression of $\varphi_1\zeta\tilde{\varphi}_1 w$: $\tilde{w}$ satisfies

$$\operatorname{supp}_t \tilde{w} \subset [\tilde{t}_0,\infty) ,$$
$$||D^{\alpha}_{(y,t)}L(\varphi_1\zeta\tilde{\varphi}_1 w-\tilde{w})||_{0,0<t<\tilde{t}} \leqq C_{11}||\varphi h||'_1 \ (0 \leqq |\alpha| \leqq N) ,$$
$$||\varphi_1\zeta\tilde{\varphi}_1 w-\tilde{w}||'_{N,0<t<\tilde{t}} \leqq C_{12}||\varphi h||'_1 .$$

From these and (4.4), it follows that

$$||B(\varphi_1\zeta\tilde{\varphi}_1 w-\tilde{w})||'_{N,0<t<\tilde{t}} \leqq C_{13}||\varphi h||'_1 .$$

On the other hand we have

$$||\chi\tilde{\varphi}(T-\tilde{T})\varphi h||'_N \leqq ||B(\varphi_1\zeta\tilde{\varphi}_1 w-\tilde{w})||'_{N,0<t<\tilde{\tau}}+||\chi\tilde{\varphi}B\{\tilde{w}-\mathcal{P}^+(\varphi h)\}||'_N + C_{14}||\varphi h||'_1 ,$$

and by Lemma 4.1

$$||\chi\tilde{\varphi}B\{\tilde{w}-\mathcal{P}^+(\varphi h)\}||'_N \leqq ||\chi\tilde{\varphi}B\mathcal{P}^+(\varphi_1\zeta\tilde{\varphi}_1-1)\varphi h||'_N+||\chi\tilde{\varphi}B\tilde{v}||'_N \leqq C_{15}||\varphi h||'_1 .$$

Therefore we obtain the estimate ii) of the lemma. The proof is complete.

Next, lec us construct an aymptotic null solution of $\tilde{T}h=0$ which is of the form

$$h_N(y,t;k)=\sum_{j=0}^{N} e^{ik\Phi(y,t)}v_{-j}(y,t)k^{-j} \quad (k>0),$$

where $\Phi(y,t)$ is a real-valued C^∞ function. As is stated in Lemma 4.1, the symbol of $\tilde{T}$ has a homogeneous asymptotic expansion $\sum_{j=0}^{\infty} q_{1-j}(y,t,\eta,\sigma)$ and its principal symbol q_1 is of the form stated in v) of Lemma 4.1. The following proposition plays a basic role on construction of the required solution.

Proposition 4.1. *Let $p(z,\omega)\in S^m$ and $h(z)\in C_0^\infty(\boldsymbol{R}^n)$. Assume that $l(z)$ is a real-valued C^∞ function and satisfies*

$$\inf_{z\in \operatorname{supp} h} |\nabla l(z)|>0 .$$

Then we have

i) $\sup_{z\in \boldsymbol{R}^n} |D_z^\alpha p(z,D_z)(e^{ikl}h)(z)| \leqq C_\alpha k^{m+|\alpha|}$;

ii) *if $p(z,\omega)$ is homogeneous of order m in $\omega(|\omega|\geqq 1)$, the following asymptotic expansion is obtained for any integer $N>0$:*

$$\begin{aligned} e^{-ikl}p(z,D_z)(e^{ikl}h)(z) &= \sum_{j=0}^{N-1} a_j(z)k^{m-j}+r_N(z;k)k^{m-N} \\ &= p(z,\nabla l(z))h(z)k^m \\ &\quad +\Big(\sum_{j=1}^{n}(\partial_{\omega_j}p)(z,\nabla l(z))D_{z_j}h(z) \\ &\quad -\frac{i}{2}\{\sum_{j,s=1}^{n}\partial_{\omega_j}\partial_{\omega_s}p(z,\nabla l(z))\partial_{z_j}\partial_{z_s}l(z)\}h(z)\Big)k^{m-1} \\ &\quad +\cdots , \end{aligned}$$

where $a_0(z),\cdots,a_{N-1}(z)$ and $r_N(z;k)$ $(\in C^\infty(\boldsymbol{R}_z^n))$ satisfy

$$\operatorname{supp} a_j \subset \operatorname{supp}\,[p(z,\nabla l(z))h(z)] ,$$

$$\sup_{\substack{z\in \boldsymbol{R}^n\\ k\geqq 1}} |D_z^\alpha r_N(z;k)| \leqq C_\alpha k^{|\alpha|} .$$

We can prove this proposition by the method of stationary phase (e.g., cf. §4 of Matsumura [10]).

Remark 4.1. In the above statement i), $p(e^{ikl}h)$ is computed also in the following way:

$$||p(z,D_z)(e^{ikl}h)(z)||_N \leqq C_N k^{m+N} \quad (N=0,1,\cdots).$$

By this proposition we can write

$$\begin{aligned} e^{-ik\Phi(z)}\tilde{T}h_N(z) &= k\{q_1(z,\nabla\Phi)v_0\}+\cdots \\ &\quad +k^{-l}\{q_1(z,\nabla\Phi)v_{-l-1}+\sum_{j=1}^{2}\partial_{z_j}q_1(z,\nabla\Phi)D_{z_j}v_{-l} \\ &\qquad\qquad +\gamma(z)v_{-l}-\psi_{-l}(z)\} \\ &\quad +\cdots \qquad (z=(y,t)), \end{aligned}$$

where $\gamma(z)=q_0(z,\nabla\Phi)-\frac{i}{2}\{\sum_{j,l=1}^{2}\partial_{\omega_j}\partial_{\omega_l}q_1(z,\nabla\Phi)\partial_{z_j}\partial_{z_l}\Phi(z)\}$ and $\psi_{-l}(z)$ is a function determined with only $v_0,\cdots,v_{-l+1}$. Let us solve the following two equation (corresponding to the eiconal and transport equations):

$$q_1(y,t,\nabla\Phi)=0, \tag{4.5}$$

$$\partial_\eta q_1(y,t,\nabla\Phi)D_y v_{-l}+\partial_\sigma q_1(y,t,\nabla\Phi)D_t v_{-l}+\gamma(z)v_{-l}=\psi_{-l}(y,t). \tag{4.6}$$

(4.5) is of the form

$$(\partial_y\Phi+\psi(y)\xi_0^+(y,\nabla\Phi))\chi^+=0.$$

It is easily seen that the function

$$\Phi(y,t)=\int_0^y\frac{\psi(s)b(s)^{1/2}}{(1-2a(s)\psi(s)+b(s)\psi(s)^2)^{1/2}}ds+t$$

is a solution of the equation

$$\partial_y\Phi+\psi(y)\xi_0^+(y,\nabla\Phi)=0,$$

and satisfies

$$\nabla\Phi(y,t)\in\tilde{\Delta}_+ \text{ and } |\nabla\Phi(y,t)|\geqq\frac{1}{2}, \quad (y,t)\in \boldsymbol{R}^2 \tag{4.7}$$

for a conic neighborhood $\tilde{\Delta}_+$ $(\subset\subset\Delta_+)$ of σ-axis $(\sigma>0)$ (if ρ in (2.1) is small enough). Put this $\Phi(y,t)$ into (4.6). Then, noting that (if ρ in (2.1) is small enough)

$$\partial_\eta q_1(y,t,\eta,\sigma) = 1+\psi(y)\partial_\eta \xi_0^+(y,\eta,\sigma) \geqq \delta\ (>0), \quad (\eta,\sigma)\in\Delta_+,\ t\geqq 2\tilde{t}_0,$$

$$\partial_\sigma q_1(y,t,\eta,\sigma) = \psi(y)\frac{b(y)\sigma}{\xi_0(y,\eta,\sigma)+a(y)\eta}, \quad (\eta,\sigma)\in\Delta_+,\ t\geqq 2\tilde{t}_0,$$

$$\frac{b(y)\sigma}{\xi_0(y,\eta,\sigma)+a(y)\eta} \leqq -\delta\ (<0), \quad (\eta,\sigma)\in\Delta_+,$$

we see that the characteristic curve $t=\tilde{t}(y)$ of (4.6) is of the following form:

i) if the condition (I) of (2.2) is satisfied, the curve is convex (i.e. $\frac{d\tilde{t}}{dy}(y)<0$ for $y<0$ and $\frac{d\tilde{t}}{dy}(y)>0$ for $y>0$);

ii) if the condition (II) of (2.2) is satisfied, the curve is concave (i.e. $\frac{d\tilde{t}}{dy}(y)>0$ for $y<0$ and $\frac{d\tilde{t}}{dy}(y)<0$ for $y>0$).

Since $\sigma_0(\tilde{T}^*)$ is of the same form (cf. (1.2)), the above statements are valid also for $\tilde{T}^*$.

Therefore, by choosing the solutions v_0, v_{-1}, $\cdots$ of (4.6) appropriately, we have

Lemma 4.3. i) *Let* ρ *in* (2.1) *be small enough to have* i) *of Lemma* 4.2. *Then, if the condition* (I) *of* (2.2) *holds, there is an asymptotic solution* $h_N(y,t;k)$ *for any integer* $N>0$ *such that*

$$\operatorname*{supp}_t h_N \subset [2\tilde{t}_0,\ 4\tilde{t}_0]^{1)},$$

$$\sup_{0\leqq t\leqq 3\tilde{t}_0} |h_N(0,t;k)| \geqq 1 \qquad \text{for large } k,$$

$$|Th_N|_{m,0\leqq t\leqq 3\tilde{t}_0} \leqq C_1 k^{m-N},$$

where the norm $|h|_{m,0\leqq t\leqq \tilde{t}}$ denotes $\sum_{|\alpha|\leqq m} \sup_{\substack{0\leqq t\leqq \tilde{t}\\ y\in R^1}} |D^\alpha h(y,t)|$.

ii) *For any integer* $N>0$ *let* ρ *in* (2.1) *be small enough to have* ii) *of Lemma* 4.2. *Then, if the condition* (II) *of* (2.2) *is satisfied, we have an asymptotic solution* $g_N(y,t;k)$ *such that*

$$\operatorname*{supp}_t g_N \subset [\tilde{t}_0,\ 3\tilde{t}_0],$$

$$\|g_N\|'_{0,5\tilde{t}_0/2<t<3\tilde{t}_0} \geqq 1 \qquad \text{for large } k,$$

$$\|\tilde{T}^* g_N\|'_{m,2\tilde{t}_0<t<4\tilde{t}_0} \leqq C_2 k^{m-N}.$$

Proof of Theorem 4.1. At first let us prove the theorem in the case (I). Assume that (2.1) is C^∞ well-posed. Then, for any compact set $D\subset \boldsymbol{R}_y^1$ there are

---

1) Assume that χ^+ in Lemma 4.1 satisfies $\operatorname*{supp}_t \chi^+ \subset [\tilde{t}_0, \infty)$ and $\chi^+(y,t,\eta,\sigma)=1$ for $(\eta,\sigma)\in\Delta_+$, $t\geqq 2\tilde{t}_0$.

an integer l and a compact set D' ($\supset D$) such that

$$|h|_{0,D\times[0,3\tilde{t}_0]}\leqq C|Th|_{l,D'\times[0,3\tilde{t}_0]}$$

where $D_t^j h|_{t=0}=0$ for $j=0,1,\cdots$ (cf. (4.3)).Putting $h_N(y,t;k)$ stated in i) of Lemma 4.3 into the above estimate, we have (by i) of Lemma 4.2 and 4.3)

$$1\leqq|h_N|_{0,D\times[0,3\tilde{t}_0]}\leqq C_1(|(T-\tilde{T})h_N|_{l,D'\times[0,3\tilde{t}_0]}+|\tilde{T}h_N|_{l,D'\times[0,3\tilde{t}_0]})$$
$$\leqq C_2(k^{l-N}+k^{-1}).$$

Let $N>l$. Then the above inequality does not hold when $k\to+\infty$.

Next, let us examine the case (II). Let (2.1) be C^∞ well-posed for a $\rho(>0)$. Then, it is so for any small ρ (>0). Furthermore, there are a constant $\tilde{t}_\rho$ (>0) for any small ρ (>0) and an integer l independent of ρ such that the estimate

$$\|h\|'_{1,0<t<4\tilde{t}_\rho}\leqq C\|T^{(\rho)}h\|'_{l,0<t<4\tilde{t}_\rho} \tag{4.8}$$

holds for $h(y,t)\in C_0^\infty(\boldsymbol{R}^1\times[0,4\tilde{t}_\rho])$ with $D_t^j h|_{t=0}=0$ ($j=0,1,\cdots$). In fact, fix $\rho=\rho_0$. Then, for any $\tilde{t}>0$ we have

$$|h|_{1,D\times[0,\tilde{t}]}\leqq C_1|T^{(\rho_0)}h|_{l_0,D'\times[0,\tilde{t}]} \tag{4.9}$$

for $h\in C_0^\infty(\boldsymbol{R}^1\times[0,\tilde{t}])$ with $D_t^j h|_{t=0}=0$ ($j=0,1,\cdots$), where l_0 is an integer independent of $\tilde{t}$, $D=[-1,1]$ and D' is a compact set containing D. Let $\alpha_0(y)$ and $\alpha_1(y)$ be C^∞ functions such that $\alpha_0(y)+\alpha_1(y)=1$, supp $\alpha_0\subset\left[-\frac{\rho}{3},\frac{\rho}{3}\right]$ and supp $\alpha_1\subset\left(-\infty,-\frac{\rho}{6}\right]\cup\left[\frac{\rho}{6},\infty\right)$, and let h_0 and h_1 be the solutions of $T^{(\rho)}h_0=\alpha_0(T^{(\rho)}h)$ and $T^{(\rho)}h_1=\alpha_1(T^{(\rho)}h)$ respectively. Then, $h=h_0+h_1$, and it follows from the result in §3 concerning domains of dependence that $\operatorname{supp}_y h_0\supset\left[-\frac{\rho}{2},\frac{\rho}{2}\right]$ and $\operatorname{supp}_y \eta_1\subset\left(-\infty,-\frac{\rho}{12}\right]\cup\left[\frac{\rho}{12},\infty\right)$ if $0\leqq t\leqq 4\tilde{t}_\rho$ ($\tilde{t}_\rho(>0)$ is a small constant depending on ρ). By the resuls in the non singular case (cf. Ikawa [3]), we have

$$\|h_1\|'_{1,0<t<4\tilde{t}_\rho}\leqq C_2\|T^{(\rho)}h_1\|'_{1,0<t<4\tilde{t}_\rho}.$$

Since $T^{(\rho)}h_0=T^{(\rho_0)}h_0$ if $0\leqq t\leqq 4\tilde{t}_\rho$, (4.9) yields

$$\|h_0\|'_{1,0<t<4\tilde{t}_\rho}\leqq C_3\|T^{(\rho)}h_0\|'_{l_0+1,0<t<4\tilde{t}_\rho}.$$

Therefore (4.8) is obtained. Let $\varphi(t)\in C^\infty$, supp $\varphi\subset(2\tilde{t}_\rho,\infty)$ and $\varphi(t)=1$ on $\left[\frac{5}{2}\tilde{t}_\rho,\infty\right)$, and let h be a solution of $T^{(\rho)}h=\varphi^2 g_N$, where g_N is the function stated in ii) of Lemma 4.3 (set $\tilde{t}_0=\tilde{t}_\rho$). Then, from ii) of Lemma 4.3 it follows that

$$1\leqq\|\varphi g_N\|_2'^2=(Th,g_N)'=(\varphi T\varphi h,g_N)',$$

where $\tilde{\phi}(t)$ $(\in C^\infty)=1$ for $t\leqq 3\tilde{t}_\rho$ and $\tilde{\phi}(t)=0$ for $t\geqq 4\tilde{t}_\rho$. We take a symbol $\chi(\eta,\sigma)$ $(\in S^0)$ such that $\chi(\eta,\sigma)=1$ on a conic neighborhood of σ-axis $(\sigma\geqq 1)$ and $\text{supp}\,\chi\subset\tilde{\Delta}_+(\tilde{\Delta}_+$ is the set in (4.7)), and write

$$\begin{aligned}(\tilde{\phi}T\tilde{\phi}h, g_N)' &= (\tilde{\phi}\tilde{T}\tilde{\phi}h, g_N)'+(\tilde{\phi}\tilde{T}\tilde{\phi}h, (\chi-1)g_N)' \\ &\quad +(\tilde{\phi}(T-\tilde{T})\tilde{\phi}h, \chi g_N)'+(\tilde{\phi}T\tilde{\phi}h, (1-\chi)g_N)' \\ &\equiv I_1+I_2+I_3+I_4\,.\end{aligned}$$

ii) of Proposi.ion 4.1 yields that for any $m>0$

$$||(1-\chi)g_N||_{L^2(D)}\leqq C_4 k^{-m}\,,$$

where D is a compact set in $\boldsymbol{R}^2$. Therefore, using (4.8), we have

$$\begin{aligned}|I_4| &\leqq C_5||h||'_{1,0<t<4\tilde{t}_\rho}||(1-\chi)g_N||'_{L^2(D)}\quad (D=\text{supp}\,\tilde{\phi}T\tilde{\phi}h) \\ &\leqq C_6 k^{-1}\end{aligned}$$

Similarly, it follows that

$$|I_2|\leqq C_7 k^{-1}\,.$$

(4.8) and ii) of Lemma 4.3 yield

$$\begin{aligned}|I_1| = |(\tilde{\phi}h, \tilde{T}^*g_N)'| &\leqq C_8||h||'_{0,2\tilde{t}_\rho<t<4\tilde{t}_\rho}||\tilde{T}^*g_N||'_{0,2\tilde{t}_\rho<t<4\tilde{t}_\rho} \\ &\leqq C_9 k^{l-N}\,.\end{aligned}$$

By means of ii) of Lemma 4.2 and Proposition 4.1 (Remark 4.1), we have

$$\begin{aligned}|I_3| &\leqq ||\chi\tilde{\phi}(T-\tilde{T})\tilde{\phi}h||'_N||g_N||'_{-N} \\ &\leqq C_{10}||\tilde{\phi}h||'_1\cdot C_{11}k^{-N} \\ &\leqq C_{12}k^{l-N}\,.\end{aligned}$$

We choose N beforehand so that $l<N$. Then it follows that

$$1\leqq \sum_{i=1}^{4}|I_i|\leqq C_{13}k^{-1}\,,$$

which is a contradiction when $k\to\infty$. The proof is complete.

5. Proof of Theorem 2

If the assumption (a) of Theorem 2 is satisfied, the $\psi(y)$ in the problem (2.1) is written by the form

$$\psi(y)=\varphi(y)^2\quad(\text{or}\quad -\varphi(y)^2)\,,$$

where $\varphi(y)$ is a real-valued C^∞ function defined near $y=0$ and satisfies $\varphi(0)=0$ and $\varphi(y)\neq 0$ for $y\neq 0$. Let us consider the problem

$$(5.1)\qquad \begin{cases} L^{(p)}(\tau)u \equiv L^{(p)}(y, D_x, D_y, \tau)u = f(x,y) & \text{in } \boldsymbol{R}^2_+ , \\ B^{(p)}_\varepsilon(y, D_x, D_y)u \equiv \{D_y u + (\varphi^{(p)}(y)^2 + \varepsilon)D_x u\}\,|_{x=0} = g(y) & \text{on } \boldsymbol{R}^1 . \end{cases}$$

Here $\tau=\sigma-i\gamma$ ($\sigma\in\boldsymbol{R}^1$, $\gamma\geqq 0$) and $0\leqq\varepsilon<\varepsilon_0$ (ε_0 is a samll constant). We define a norm $|||\cdot|||_m$ ($m=0,1,\cdots$) with the parameter τ by

$$|||u(x,y)|||^2_m = \sum_{\alpha+\beta\leqq m} |\tau|^{2(m-\alpha-\beta)} ||D^\alpha_x D^\beta_y u||^2_{L^2(\boldsymbol{R}^2_+)} .$$

Similarly, $|||\cdot|||'_s$ ($s\in\boldsymbol{R}$) is defined by

$$|||v(y)|||'^2_s = \int (\eta^2+|\tau|^2)^s |\hat{v}(\eta)|^2 d\eta .$$

We shall derive estimates with the norms $|||\cdot|||_m$, $|||\cdot|||'_s$ uniform in τ. A main task in this section is to prove

Theorem 5.1. *For any integer m ($\geqq 0$) there exist constants γ_0 and C independent of τ and ε such that if $\gamma=-\mathrm{Im}\,\tau\geq\gamma_0$*

$$\gamma|||u|||^2_{m+1} + \sum_{j=0}^{m+1} |||D^j_x u|||'^2_{m-j+1} \leqq C\gamma^{-1}(|||\Lambda^3 L(\tau)u|||^2_m + |||B_\varepsilon u|||'^2_{m+3}) ,$$
$$u(x,y)\in C^\infty_0(\bar{\boldsymbol{R}}^2_+)\quad (0\leqq\varepsilon<\varepsilon_0) ,$$

where $\Lambda=(D^2_y+|\tau|^2)^{1/2}$.

We note that the statements in this section are all valid also in the case where the boundary operator in (5.1) is of the form $D_y-(\varphi^{(p)}(y)^2+\varepsilon)D_x$.

Now, we consider the equation (in ξ)

$$(5.2)\qquad L_0(y,\xi,\eta,\tau) \equiv \xi^2 + 2a(y)\eta\xi + b(y)\eta^2 - b(y)\tau^2 = 0 ,$$
$$(y,\eta)\in\boldsymbol{R}^1\times\boldsymbol{R}^1,\ \gamma = -\mathrm{Im}\,\tau>0 .$$

This has two roots $\xi^\pm_0(y,\eta,\tau)$ of the form

$$(5.3)\qquad \xi^\pm_0(y,\eta,\tau) = -a(y)\eta \pm \sqrt[+]{b(y)\,(\tau^2-\eta^2)+a(y)^2\eta^2} ,$$

where $\sqrt[+]{\cdot}$ means the square root with positive imaginary part. From the hyperbolicity of L_0 the following estimate holds:

$$(5.4)\qquad \pm\mathrm{Im}\,\xi^\pm_0(y,\eta,\tau)\geqq\delta\gamma\quad (\delta>0) .$$

For $\sigma\in\boldsymbol{R}^1$ we define $\xi^\pm_0(y,\eta,\sigma)=\lim_{\gamma\to+0}\xi^\pm_0(y,\eta,\sigma-i\gamma)$, which coincide with $\xi^\pm_0(y,\eta,\sigma)$ defined in §4. Obviously $\xi^\pm_0(y,\eta,\tau)$ are homogeneous of order one in (η,τ). We set

$$(5.5)\qquad \begin{aligned} S_+ &= \{(\eta,\tau): \eta^2+|\tau|^2=1,\ \eta\in\boldsymbol{R},\ \gamma=-\mathrm{Im}\ \tau\geqq 0\}\ ,\\ \Delta'_d &= \{(\eta',\tau')\in S_+;\ |\eta'|<d\}\ \ (d>0)\ ,\\ \Delta_d &= \{(\eta,\tau)=(\lambda\eta',\lambda\tau'): (\eta',\tau')\in\Delta'_d,\ \lambda>0\}\ . \end{aligned}$$

Let d, d_1, d_2 be small positive constants ($d_2<d_1$). Then, if ρ in (5.1) is small enough, from the form (5.3) we have

$$(5.6)\qquad \xi_0^+(y,\eta',\tau')\neq\xi_0^-(y,\eta',\tau'),\quad y\in\boldsymbol{R}^1,\ (\eta',\tau')\in\bar\Delta'_{d_1}\ ,$$

$$(5.7)\qquad |\mathrm{Re}\ \partial_\eta\xi_0^-(y,\eta',\tau')|\geqq\delta'\ (>0),$$
$$y\in\boldsymbol{R}^1,\quad (\eta',\tau')\in(\bar\Delta'_{d_1}-\Delta'_{d_2})\cap\{0\leqq-\mathrm{Im}\ \tau'\leqq d\}\ .$$

Since $\xi_0^+(y,\eta,\tau)$ and $\xi_0^-(y,\eta,\tau)$ are distinct on $\bar\Delta'_{d_1}$, we can apply Proposition 1.3 to the operator $L(\tau)$ ($=L^{(\rho)}(\tau)$), and we have symbols $\xi^\pm(y,\eta,\tau)\in S^1_{(\tau)}$ such that $\sigma_0(\xi^\pm)(y,\eta,\tau)=\xi_0^\pm(y,\eta,\tau)$ on $\bar\Delta_{d_1}\cap\{\eta^2+|\tau|^2\geqq 1\}$ and $L^\pm=D_x-\xi^\pm(y,D_y,\tau)$ has the property ii) of Proposition 1.3. We set

$$P_\varepsilon = D_y+(\varphi(y)^2+\varepsilon)\xi^+(y,D_y,\tau)\ \ (0\leqq\varepsilon<\varepsilon_0)\,.$$

The following lemma plays an essential role on proof of Theorem 5.1.

Lemma 5.1. *Let $\chi(\eta,\tau)$ ($\in S^0_{(\tau)}$) be homogeneous of order 0 ($\eta^2+|\tau|^2\geqq 1$) and satisfy $\chi(\eta,\tau)=1$ on $\Delta_{d'}\cap\{\eta^2+|\tau|^2\geqq 1\}$ ($d'>0$) and $\mathrm{supp}\ \chi\subset\Delta_{d_1}$ (d_1 is the constant in (5.6)), and let $\zeta(y,\eta,\tau)$ ($\in S^0_{(\tau)}$) be equal to 1 on a neighborhood of $\boldsymbol{R}^1_y\times(\mathrm{supp}\ \chi)$. Then, for $s\in\boldsymbol{R}$ there are constants γ_0 and C independent of ε and τ sucn that if $\gamma=-\mathrm{Im}\ \tau\geqq\gamma_0$*

$$|||\chi v|||_s'^2\leqq C(\gamma^{-1}|||\zeta P_\varepsilon v|||'^2_{s+\frac12}+|||v|||'^2_{s-\frac12}),\quad v(y)\in\mathcal{S}\quad (0\leqq\varepsilon<\varepsilon_0)\,.$$

We shall prove this lemma later. By Sakamoto [12] I we have

Proposition 5.1. *For $m=0,1,\cdots$ there are constants C and γ_0 independent of τ such that if $\gamma=-\mathrm{Im}\ \tau\geqq\gamma_0$*

$$\gamma|||u|||^2_{m+1}+\sum_{j=0}^{m+1}|||D_x^ju|||'^2_{m-j+1}\leqq C(\gamma^{-1}|||L(\tau)u|||^2_m+|||u|||'^2_{m+1})\ ,$$
$$u(x,y)\in C_0^\infty(\bar{\boldsymbol{R}}^2_+)\,.$$

Combining this proposition with Lemma 5.1, we obtain

Lemma 5.2. *Let $\chi(\eta,\tau)$ ($\in S^0_{(\tau)}$) be the symbol stated in Lemma 5.1. Then, for $m=0,1,\cdots$ there are constants γ_0 and C independent of ε and τ such that if $\gamma=-\mathrm{Im}\ \tau\geqq\gamma_0$*

$$\gamma|||\chi(D_y,\tau)u|||_{m+1}^2+\sum_{j=0}^{m+1}|||D_x^j\chi(D_y,\tau)u|||'^2_{m-j+1}$$
$$\leqq C(\gamma^{-1}|||\Lambda^3L(\tau)u|||^2_m+\gamma^{-1}|||B_\varepsilon u|||'^2_{m+3}+|||u|||^2_{m+1})\ ,$$
$$u(x,y)\in C_0^\infty(\bar{\boldsymbol{R}}^2_+)\,.$$

Proof. Let $\chi'(\eta,\tau)$ $(\in S^0_{(\tau)})$ be homogeneous of order 0 $(\eta^2+|\tau|^2\geqq 1)$, supp $\chi'\subset\Delta_{d_1}$ and $\chi'(\eta,\tau)=1$ on a neighborhood of supp χ. At first, we show that for $s\geqq 0$ there is a constant γ_1 such that if $\gamma\geqq\gamma_1$

$$(5.8)\qquad |||\chi'v|||'_s\leqq C_1(|||\Lambda^s\chi_0L^-v|||_0+|||\Lambda^{s-4}v|||_1),\quad v(x,y)\in C_0^\infty(\bar{\boldsymbol{R}}^2_+),$$

where $\chi_0(\eta,\tau)$ $(\in S^0_{(\tau)})$ is homogeneous of order 0 $(\eta^2+|\tau|^2\geqq 1)$, $\chi_0(\eta,\tau)=1$ on $\Delta_{d_1}\cap\{\eta^2+|\tau|^2\geqq 1\}$ and supp $\chi_0\subset\Delta$ (Δ is the set in Proposition 1.3). We may assume that the principal symbols of $\xi^\pm$ satisfy the inequalities (5.4) for every y,η,τ:

$$(5.4)'\qquad \pm\mathrm{Im}\,\sigma_0(\xi^\pm)(y,\eta,\tau)\geqq\delta\gamma,\ (y,\eta)\in\boldsymbol{R}^1\times\boldsymbol{R}^1,\ \gamma>0\,.$$

Combining (5.4)′ and Proposition 1.2, we have

$$(5.9)\qquad \mathrm{Im}\,(\Lambda^s\chi'L^-v,\,\Lambda^s\chi'v)=\frac{1}{2}|||\chi'v|||'^2_s-\mathrm{Im}\,(\Lambda^s\chi'\xi^-v,\,\Lambda^s\chi'v)$$
$$\geqq\frac{1}{2}|||\chi'v|||'^2_s+\delta(\gamma-\gamma_2)|||\Lambda^s\chi'v|||_0^2-|([\Lambda^s\chi',\,\xi^-]v,\,\Lambda^s\chi'v)|\,.$$

Take symbols $\chi_1(\eta,\tau)$, $\chi_2(\eta,\tau)$ $(\in S^0_{(\tau)})$ homogeneous of order 0 $(\eta^2+|\tau|^2\geqq 1)$ such that $\chi_1(\eta,\tau)+\chi_2(\eta,\tau)=1$ on supp $\chi'\cap$ supp $(1-\chi')$, $S_+\cap$ supp $\chi_1\subset\Xi'\equiv(\Delta'_{d_1}-\bar{\Delta}'_{d'})\cap\{\gamma=-\mathrm{Im}\,\tau<d\}$ (d is the constant in (5.7)) and $S_+\cap$ supp $\chi_2\subset(\Delta'_{d_1}-\bar{\Delta}'_{d'})\cap\left\{\gamma>\dfrac{d}{2}\right\}$. Then it follows that

$$|([\Lambda^s\chi',\,\xi^-]v,\,\Lambda^s\chi'v)|\leqq C_2(|||\Lambda^s\chi'v|||_0^2+|||\Lambda^s\chi_1v|||_0^2$$
$$+|||\Lambda^s\chi_2v|||_0^2+|||\Lambda^{s-3}v|||_0^2)\,.$$

Therefore, we obtain (5.8) if the following estimates (5.10) and (5.11) hold when $\gamma=-\mathrm{Im}\,\tau$ is large enough:

$$(5.10)\qquad |||\Lambda^s\chi_1v|||_0^2\leqq C_3(|||\Lambda^s\chi_0L^-v|||_0^2+|||\Lambda^{s-4}v|||_1)\,,$$

$$(5.11)\qquad |||\Lambda^s\chi_2v|||_0^2\leqq C_4(|||\Lambda^{s-1}\chi_0L^-v|||_0^2+|||\Lambda^{s-3}v|||_0)\,.$$

Noting that $L^-=D_x-\xi^-$ is elliptic if (η,τ) is near supp χ_2 and that Im $\sigma_0(\xi^-)(y,\eta,\tau)$ is negative there (cf. (5.4)′), we see easily that the estimate (5.11) holds.

Let us derive (5.10). By the Taylor expansion we write

$$\sigma_0(\xi^-)(y,\eta,\sigma-i\gamma)=\sigma_0(\xi^-)(y,\eta,\sigma)+\kappa_0(y,\eta,\sigma-i\gamma)\gamma\,.$$

Then, if $(\eta,\tau)\in\Xi=\{(\eta,\tau)=(\mu\eta',\mu\tau'):\ \mu>0,\ (\eta',\tau')\in\Xi'\}$, $\sigma_0(\xi^-)(y,\eta,\sigma)$ and $\kappa_0(y,\eta,\sigma-i\gamma)$ belong to $S^1_{(\tau)}$ and $S^0_{(\tau)}$ respectively. Take a symbol $\tilde{\chi}_1(\eta,\tau)$ $(\in S^0_{(\tau)})$ homogeneous of order 0 and satisfying supp $\tilde{\chi}_1\subset\Xi$ and $\tilde{\chi}_1(\eta,\tau)=1$ on a conic neighborhood $\tilde{\Xi}$ of supp χ_1, and set

$$\lambda(y,\eta,\tau)=\{\sigma_0(\xi^-)(y,\eta,\sigma)+(\xi^-(y,\eta,\tau)-\sigma_0(\xi)(y,\eta,\tau))\}\tilde{\chi}_1(\eta,\tau)\,,$$

$$\kappa(y,\eta,\tau)=\kappa_0(y,\eta,\tau)\tilde{\chi}_1(\eta,\tau)\,,$$
$$\tilde{\xi}^-(y,\eta,\tau)=\lambda(y,\eta,\tau)+\kappa(y,\eta,\tau)\gamma\,.$$

Then we have $\lambda(y,\eta,\tau)\in S^1_{(\tau)}$, $\kappa(y,\eta,\tau)\in S^0_{(\tau)}$, and for any $p(y,\eta,\tau)\in S^m_{(\tau)}$ satisfying $\operatorname{supp}_{\eta,\tau} p\subset\tilde{\Xi}$

$$[\chi_1,\xi^-]p\equiv[\chi_1,\tilde{\xi}^-]p,\ [p,\xi^-]\equiv[p,\tilde{\xi}^-]\quad \text{mod } S^{-\infty}_{(\tau)}.$$

Applying Lemma 1.1 ($N=1$) to $\lambda(y,\eta,\tau)$ (cf. Remark 1.1 and (5.7)), we obtain a symbol $\zeta(y,\eta,\tau)\in S^0_{(\tau)}$ such that $[\lambda,\zeta]\in S^{-1}_{(\tau)}$, $\operatorname{supp}_{\eta,\tau}\zeta\subset\tilde{\Xi}$ and $\zeta(y,\eta,\tau)=1$ if $(\eta,\tau)\in\operatorname{supp}\chi_1$ ($\eta^2+|\tau|^2\geqq 1$) (let ρ in (5.1) be small enough). It is easy to see that for large $\mu>0$

$$\mu|||\Lambda^{-1}v|||_0\leqq 2|||(\zeta+i\mu\Lambda^{-1})v|||_0,\quad v(x,y)\in C_0^\infty(\bar{\boldsymbol{R}}^2_+)\,.$$

Noting that (for large μ)

$$\begin{aligned}|||[\zeta+i\mu\Lambda^{-1},\xi^-]v|||_0&\leqq|||[\zeta,\lambda]v|||_0+\gamma|||[\zeta,\kappa]v|||_0\\&\quad+\mu|||[\Lambda^{-1},\xi^-]v|||_0+C_5|||\Lambda^{-1}v|||_0\\&\leqq C_6(1+\gamma\mu^{-1})|||(\zeta+i\mu\Lambda^{-1})v|||_0\,,\end{aligned}$$

in the same way as in (5.9) we have

$$\begin{aligned}&\operatorname{Im}\,((\zeta+i\mu\Lambda^{-1})L^-v,\ (\zeta+i\mu\Lambda^{-1})v)\\&\quad\geqq\frac{1}{2}|||(\zeta+i\mu\Lambda^{-1})v|||_0'^2+(\delta\gamma-C_7)|||(\zeta+i\mu\Lambda^{-1})v|||_0^2\\&\qquad\qquad-C_6(1+\gamma\mu^{-1})|||(\zeta+i\mu\Lambda^{-1})v|||_0^2\\&\quad\geqq\Big(\frac{\delta}{2}\gamma-C_8\Big)|||(\zeta+i\mu\Lambda^{-1})v|||_0^2\quad(2C_6\delta^{-1}\leqq\mu)\,.\end{aligned}$$

Inductively, we obtain

$$\begin{aligned}&\operatorname{Im}\,((\zeta+i\mu\Lambda^{-1})^4\Lambda^sL^-v,\ (\zeta+i\mu\Lambda^{-1})^4\Lambda^sv)\\&\quad\geqq\Big(\frac{\delta}{4}\gamma-C_9\Big)|||(\zeta+i\mu\Lambda^{-1})^4\Lambda^sv|||_0^2\,.\end{aligned}$$

Therefore it follows that if γ is large enough

$$|||\Lambda^s\chi_1v|||_0^2\leqq C_{10}(|||(\zeta+i\mu\Lambda^{-1})^4\Lambda^sL^-v|||_0^2+|||\Lambda^{s-3}v|||_0^2)\,,$$

which proves (5.10).

From Lemma 5.1 and Proposition 5.1 it follows that

$$\begin{aligned}&\gamma|||\chi u|||_{m+1}^2+\sum_{j=0}^{m+1}|||D_x^j\chi u|||_{m-j+1}'^2\\&\quad\leqq C_{11}(\gamma^{-1}|||Lu|||_m^2+\gamma^{-1}|||\chi''P_\varepsilon u|||_{m+\frac{3}{2}}'^2+|||u|||_{m+1}^2)\,,\end{aligned}$$

where $\chi''(\eta,\tau)$ $(\in S^0_{(\tau)})=1$ on a neighborhood of supp χ and supp $\chi'' \subset\subset \{(\eta,\tau): \chi'(\eta,\tau)=1\}$. Noting that $P_\varepsilon=B_\varepsilon-(\varphi^2+\varepsilon)L^+$ and using (5.8) (set $v=L^+u$) and Proposition 1.3, we have

$$\begin{aligned}|||\chi''P_\varepsilon u|||'_{m+3} &\leqq C_{12}(|||\chi'L^+u|||'_{m+3}+|||B_\varepsilon u|||'_{m+3}+|||u|||_{m+1})\\ &\leqq C_{13}(|||\Lambda^{m+3}\chi_0L^-L^+u|||_0+|||B_\varepsilon u|||'_{m+3}+|||\Lambda^{m-1}L^+u|||_1+|||u|||_{m+1})\\ &\leqq C_{14}(|||\Lambda^3Lu|||_m+|||B_\varepsilon u|||'_{m+3}+|||u|||_{m+1})\,.\end{aligned}$$

Therefore Lemma 5.2 is obtained. The proof is complete.

Proof of Theorem 5.1. Let $\chi(\eta,\tau)$ be the symbol in Lemma 5.1. Then it follows that

$$\begin{aligned}|||(1-\chi)u|||'_{m+1} &\leqq C_1|||(1-\chi)D_yu|||'_m\\ &\leqq C_1\{|||B_\varepsilon^{(\rho')}u|||'_m+(|\varphi^{(\rho')^2}|_0+\varepsilon_0)|||D_xu|||'_m\}\\ &\qquad +C_2|||D_xu|||'_{m-1}\quad (0\leqq\varepsilon<\varepsilon_0)\,,\end{aligned}$$

where C_1 does not depend on ε or ρ'. Therefore, by Proposition 5.1 we have

$$\begin{aligned}\gamma|||(1-\chi)u|||^2_{m+1}+\sum_{j=0}^{m+1}|||D_x^j(1-\chi)u|||'^2_{m-j+1}\\ \leqq C_3(\gamma^{-1}|||L^{(\rho)}(\tau)u|||^2_m+|||B_\varepsilon^{(\rho')}u|||'^2_m+|||u|||^2_{m+1})\\ +C_4(|\varphi^{(\rho')^2}|_0+\varepsilon_0)^2|||D_xu|||'^2_m\,,\end{aligned}$$

where C_4 is independent of ε_0 and ρ'. Fix ρ in $L^{(\rho)}(\tau)$, and make only ρ' in $B_\varepsilon^{(\rho')}$ and ε_0 so small that $(|\varphi^{(\rho)^2}|_0+\varepsilon_0)^2\leqq\dfrac{1}{2C_4}$. Then, the following estimate holds:

$$\begin{aligned}\gamma|||(1-\chi)u|||^2_{m+1}+\sum_{j=0}^{m+1}|||D_x^j(1-\chi)u|||'^2_{m-j+1}\\ \leqq C_5(\gamma^{-1}|||L^{(\rho)}(\tau)u|||^2_m+|||B_\varepsilon^{(\rho')}u|||'^2_m+|||u|||^2_{m+1})+\frac{1}{2}|||D_xu|||'^2_m\,.\end{aligned}$$

Combining this inequality with Lemma 5.2, we obtain Theorem 5.1. The proof is complete.

Proof of Lemma 5.1. We shall prove this lemma by the same procedure as in the author [15] (cf. Lemma 3.2 of [15]). If ε and ρ (of $B_\varepsilon^{(\rho)}$) are small enough for $d'>0$, $P_\varepsilon=D_y+(\varphi^{(\rho')^2}+\varepsilon)\xi^-$ is elliptic on $(\Delta_{d'})^c$ ($\Delta_{d'}$ is defined in (5.5)). Therefore, in view of Proposition 1.1 we have only to derive the following estimate when γ is large enough:

$$|||\chi(D_y,\tau)v|||'^2_s\leqq C\gamma^{-1}|||P_\varepsilon(\chi v)|||'^2_{s+2},\quad v(y)\in\mathcal{S}\,. \tag{5.12}$$

The first step is to show that the estimate

(5.13) $$|||\varphi\chi v|||'^2_{s+1}+\varepsilon|||\chi v|||'^2_{s+1}\leqq C_1\gamma^{-1}(|||P_\varepsilon(\chi v)|||'^2_{s+2}+|||\chi v|||'^2_s),\quad v(y)\in\mathcal{S}$$

holds if γ is large enough. Let $\tilde{\chi}(\eta,\tau)$ $(\in S^0_{(\tau)})$ be homogeneous of order 0 $(\eta^2+|\tau|^2\geqq 1)$, $\tilde{\chi}(\eta,\tau)=1$ on a conic neighborhood Π of supp χ and supp $\tilde{\chi}\subset\Delta_{d_1}$, and set

$$\alpha(y,\eta,\tau)=\frac{a(y)^2-b(y)}{\overset{+}{\sqrt{}}b(y)\,(\tau^2-\eta^2)+a(y)^2\eta^2}\tilde{\chi}(\eta,\tau)\,,$$

$$\mathring{\xi}^+(y,\eta,\tau)=\int_0^\eta\alpha(y,\mu,\tau)\,\mu d\mu-\sqrt{b(y)}\,\tau\,.$$

Then we have

$$\alpha(y,\eta,\tau)\in S^{-1}_{(\tau)},\quad \mathring{\xi}(y,\eta,\tau)\in S^1_{(\tau)}\,,$$

(5.14) $$\partial_\eta\mathring{\xi}^+(y,\eta,\tau)=\alpha(y,\eta,\tau)\eta\,,$$

(5.15) $$\xi_0^+(y,\eta,\tau)=-a(y)\eta+\mathring{\xi}^+(y,\eta,\tau)\quad\text{if }(\eta,\tau)\in\Pi\,,$$

(5.16) $$\operatorname{Im}\mathring{\xi}^+(y,\eta,\tau)\geqq\delta\gamma\quad\text{if }(\eta,\tau)\in\Pi\,.$$

By (5.15) we may assume that $\sigma_0(\xi^+)\,(y,\eta,\tau)=-a(y)\eta+\mathring{\xi}(y,\eta,\tau)$ for every (y,η,τ). Set $\theta_\varepsilon(y)=\{1-(\varphi(y)^2+\varepsilon)a(y)\}^{-1}$. Then it follows that

$$\inf_{\substack{0\leqq\varepsilon<\varepsilon_0\\ y\in R^1}}\theta_\varepsilon(y)\geqq\delta_1\,(>0)\,,$$

$$\operatorname{Im}(\theta_\varepsilon P_\varepsilon v,v)'\geqq\operatorname{Im}((\varphi^2+\varepsilon)\theta_\varepsilon\mathring{\xi}^+v,v)'-C_2(|||v|||'^2_{-1}+|||\varphi v|||'^2_0+\varepsilon|||v|||'^2_0)\,.$$

Therefore, using Proposition 1.2 and its corollary (cf. (5.16)), we have

(5.17) $$\operatorname{Im}(\theta_\varepsilon P_\varepsilon(\Lambda^{s+1}\chi v),\Lambda^{s+1}\chi v)'\geqq(\delta_2\gamma-C_3)\,(|||\varphi\chi v|||'^2_{s+1}+\varepsilon|||\chi v|||'^2_{s+1})-|(\theta_\varepsilon[\varphi,\mathring{\xi}^+]\Lambda^{s+1}\chi v,\varphi\Lambda^{s+1}\chi v)'|-C_3|||\chi v|||'^2_s\,.$$

From (5.14) and $\partial_\eta\Lambda=\Lambda^{-1}D_y$, it is seen that $[\varphi,\mathring{\xi}^+]$ and $[\varphi,\Lambda^{s+1}]$ are of the form

$$[\varphi,\mathring{\xi}^+]=\mathring{\alpha}D_y+\mathring{\beta},\quad [\varphi,\Lambda^{s+1}]=\alpha_{s-1}D_y+\beta_{s-1}\,,$$

where $\mathring{\alpha}$, $\mathring{\beta}\in S^{-1}_{(\tau)}$ and α_{s-1}, $\beta_{s-1}\in S^{s-1}_{(\tau)}$. Therefore, noting that $D_y=P_\varepsilon-(\varphi^2+\varepsilon)\xi^+$, we obtain

$$\begin{aligned}
&|||\theta_\varepsilon[\varphi,\mathring{\xi}^+]\Lambda^{s+1}\chi v|||'_0\leqq|||\theta_\varepsilon\mathring{\alpha}\Lambda^{s+1}D_y\chi v|||'_0+|||\theta_\varepsilon\mathring{\beta}\Lambda^{s+1}\chi v|||'_0\\
&\quad\leqq C_4(|||D_y\chi v|||'_s+|||\chi v|||'_s)\\
&\quad\leqq C_5(|||P_\varepsilon(\chi v)|||'_s+|||\varphi(\chi v)|||'_{s+1}+\varepsilon|||\chi v|||'_{s+1}+|||\chi v|||'_s)\,,\\
&|(\theta_\varepsilon[P_\varepsilon,\Lambda^{s+1}]\chi v,\Lambda^{s+1}\chi v)'|\\
&\quad\leqq\varepsilon|(\theta_\varepsilon[\xi^+,\Lambda^{s+1}]\chi v,\Lambda^{s+1}\chi v)'|+|(\varphi^2[\xi^+,\Lambda^{s+1}]\chi v,\Lambda^{s+1}\chi v)'|
\end{aligned}$$

$$+|(\varphi[\varphi, \Lambda^{s+1}]\xi^+\chi v, \Lambda^{s+1}\chi v)'|+|([\varphi, \Lambda^{s+1}]\varphi\xi^+\chi v, \Lambda^{s+1}\chi v)'|$$
$$\leqq C_6(|||P_\varepsilon \chi v|||_s'^2+|||\varphi\chi v|||_{s+1}'^2+\varepsilon|||\chi v|||_{s+2}'^2+|||\chi v|||_s'^2)\,.$$

Combining these inequalities with (5.17), we have

$$|(\theta_\varepsilon\Lambda^{s+1}P_\varepsilon\chi v, \Lambda^{s+1}\chi v)'|\geqq(\delta_2\gamma-C_7)(|||\varphi\chi v|||_{s+1}'^2+\varepsilon|||\chi v|||_{s+1}'^2)$$
$$-C_8(|||\chi v|||_s'^2+|||P_\varepsilon\chi v|||_s'^2)\,,$$

which yields the estimate (5.13).

The second step is to derive

$$(5.18)\quad |||v|||_s'\leqq C(|||P_\varepsilon v|||_s'+|||\varphi v|||_{s+1}'+\varepsilon|||v|||_{s+1}'+|||v|||_{s-1}'),\quad v(y)\in\mathcal{S}\,.$$

Let $\psi(y)\in C_0^\infty(\boldsymbol{R}^1)$ and $\psi(y)=1$ near $y=0$. Then it follows that

$$|||v|||_0'\leqq C_1(|||\varphi v|||_0'+|||(1-\psi)v|||_0')$$
$$\leqq C_2(|||D_y v|||_0'+|||(D_y\psi)v|||_0'+|||(1-\psi)v|||_0')$$
$$\leqq C_3(|||P_\varepsilon v|||_0'+|||(\varphi^2+\varepsilon)\xi^+v|||_0'+|||\varphi v|||_0')\,.$$

From this inequality we have

$$|||v|||_s'\leqq C_4(|||P_\varepsilon v|||_s'+|||\varphi v|||_{s+1}'+\varepsilon|||v|||_{s+1}'+|||v|||_{s-1}'$$
$$+|||[P_\varepsilon, \Lambda^s]v|||_0'+|||[\varphi^2, \xi^+\Lambda^s]v|||_0')\,,$$

which yields (5.18).

It is easy to derive (5.12) from (5.13) and (5.18). The proof is complete.

Proof of Theorem 2. From i) of Proposition 2.1 it suffices to show that the mixed problem (2.1) with the boundary operator $D_y+\varphi^2D_x$ (or $D_y-\varphi^2D_x$) is C^∞ well-posed. Since the boundary condition of (5.1) is non degenerate if $\varepsilon>0$, by Ikawa [3] we have a solution u_ε of (5.1) in $H_{m+3}(\boldsymbol{R}_+^2)$ for any $(f,g)\in H_{m+3}(\boldsymbol{R}_+^2)\times H_{m+3}(\boldsymbol{R}^1)$ and $\varepsilon>0$ (if γ is large enough). Furthermore, by Theorem 5.1, this solution u_ε satisfies

$$\gamma|||u_\varepsilon|||_{m+1}^2\leqq C\gamma^{-1}(|||\Lambda^3 f|||_m^2+|||g|||_{m+3}'^2)\,,$$

which implies that $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is bounded in $H_{m+1}(\boldsymbol{R}_+^2)$ (for fixed (f,g)). Therefore, u_ε converges to some $u_0\in H_{m+1}(\boldsymbol{R}_+^2)$ weakly as $\varepsilon\to+0$. Then u_0 satisfies $L(\tau)u_0=f$ and $B_0u_0=g$. Hence, using the Laplace transformation in t, we see that (if γ is large enough) for any $(f(x,y,t),\ g(y,t))\in H_{m+3,\gamma}(\boldsymbol{R}_+^2\times\boldsymbol{R}^1)\times H_{m+3,\gamma}(\boldsymbol{R}^1\times\boldsymbol{R}^1)$ $(H_{m,\gamma}(M)=\{u\colon e^{-\gamma t}u\in H_m(M)\})$ there exists a unique solution $u(x,y,t)\in H_{m+1,\gamma}(\boldsymbol{R}_+^2\times\boldsymbol{R}^1)$ of the equation

$$\begin{cases}L(y,D_x,D_y,D_t)u=f(x,y,t) & \text{in }\ \boldsymbol{R}_+^2\times\boldsymbol{R}^1\,,\\ B_0(y,D_x,D_y)u=g(y,t) & \text{on }\ \boldsymbol{R}^1\times\boldsymbol{R}^1\,,\end{cases}$$

and that supp $u\subset\{t\geqq 0\}$ follows from supp $(f,g)\subset\{t\geqq 0\}$. Therefore we obtain the uniqueness and existence of the solution of (2.1) in the Sobolev space.

Combining this fact and the investigation in §3 concerning domains of dependence (cf. Remark 3.1), we see that the problem (2.1) is C^∞ well-posed. In fact: Let $\{\alpha_j(x,y)\}_{j=0,1,\cdots}$ be a partition of unity on $\bar{\boldsymbol{R}}^2_+$ such that $0\leqq\alpha_j\leqq 1$ and supp $\alpha_j\subset\{(x,y)\colon j-1\leqq|(x,y)|\leqq j+1\}$, and set $\beta_N(x,y)=\sum_{j=0}^{N}\alpha_j(x,y)$. Let u be a null solution of (2.1) (i.e. $f=0$, $g=0$, $u_0=u_1=0$). Then $\beta_N u$ satisfies

$$\begin{cases} L(\beta_N u)=[L,\beta_N]u & \text{in } \boldsymbol{R}^2_+\times(0,t_0)\,, \\ B_0(\beta_N u)=[B_0,\beta_N]u & \text{on } \boldsymbol{R}^1\times(0,t_0)\,, \\ \beta_N u|_{t=0}=D_t(\beta_N u)|_{t=0}=0 & \text{on } \boldsymbol{R}^2_+\,. \end{cases}$$

The data of this equation have support in $\{N-1\leqq(x^2+y^2)^{1/2}\leqq N+1\}$ and belong to the Sobolev space. From Theorem 3.1 (see Remark 3.1) it follows that $\beta_N u=0$ on $\{(x^2+y^2)^{1/2}\leqq C(N)\}$, where $C(N)\to\infty$ as $N\to\infty$. Hence the solution of (2.1) is unique in $C^\infty(\bar{\boldsymbol{R}}^2_+\times[0,t_0])$. Let us show the existence of the solution in $C^\infty(\bar{\boldsymbol{R}}^2_+\times[0,t_0])$. We may assume that $f=0$, $u_0=u_1=0$ and $D_t^j g|_{t=+0}=0$ $(j=0,1,\cdots)$. By the solvability in the Sobolev space we have a solution $u^{(j)}$ of (2.1) for the data $(0,\alpha_j g,0,0)$. From Theorem 3.1 (Remark 3.1), it is seen that $u=\sum_{j=0}^{\infty}u^{(j)}$ is the required solution. The proof is complete.

References

[1] Ju. V. Egorov and V.A. Kondrat'ev: *The oblique derivative problem,* Mat. Sb. **78** (120) (1969), 148–176=Math. USSR Sb. **7** (1969), 139–169.

[2] L. Hörmander: *Pseudo-differential operators and hypoelliptic equations,* Proc. Symposium on Singular Intergal, Amer. Math. Soc., **10** (1967), 138–183.

[3] M. Ikawa: *Mixed problem for the wave equation with an oblique derivative boundary condition,* Osaka J. Math. **7** (1970), 495–525.

[4] M. Ikawa: *Remarques sur les problèmes mixtes pour l'équation des ondes,* Colloque international du C.N.R.S., (1972), Astérisque 2 et 3, 217–221.

[5] M. Ikawa: *Problèmes mixtes pour l'équaution des ondes,* Publ. Research Inst. Math. Sci. Kyoto Univ. **12** (1976), 55–122.

[6] M. Ikawa: *On the mixed problems for the wave equation in an interior domain,* Comm Partial Differential Equations **3** (1978), 249–295.

[7] H. Kumano-go: *Algebras of pseudo-differential operators,* J. Fac. Sci. Univ. Tokyo **17** (1970), 31–50.

[8] H. Kumano-go: *A calculus of Fourier integral operators on $\boldsymbol{R}^n$ and the fundamental solutions for an opertor of hyperbolic type,* Comm. Partial Differential Equations **1** (1976), 1–44.

[9] H. Kumano-go: *Factorizations and fundamental solutions for differential operators of elliptic-hyperbolic type,* Proc. Japan Acad. **52** (1976), 480–483.

[10] M. Matsumura: *Asymptotic behavior at infinity for Green's functions of first order systems with characteristics of nonuniform multiplicity,* Publ. Rerearch Inst. Math. Sci. Kyoto Univ. **12** (1976), 317–377.

[11] V.G. Maz'ja: *On a degenerating problem with directional derivative,* Mat. Sb. **87** (129) (1972), 417–454=Math. USSR Sb. **16**. (1972), 429–469.

[12] R. Sakamoto: *Mixed problems for hyperbolic equations* I, II, J. Math. Kyoto Univ. **10** (1970), 349–373, 403–417.

[13] R. Sakamoto: *$\mathcal{E}$-well posedness for hyperbolic mixed problems with constant coefficients,* J. Math. Kyoto Univ. **14** (1974), 93–118.

[14] H. Soga: *Boundary value problems with oblique derivative*, Publ. Research Inst. Math. Sci. Kyoto Univ. **10** (1975), 619–778.

[15] H. Soga: *Mixed problems in a quarter space for the wave equation with a singular oblique derivative,* Publ. Research Inst. Math. Sci. Kyoto Univ., to appear.

Faculty of Education
Ibaraki University
Mito, Ibaraki 310, Japan