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SOME REMARKS ON THE EQUATION $y_{tt}-\sigma(y_x)y_{xx}-y_{xtx}=f$

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1. Introduction

In [4], Greenberg, MacCamy and Mizel considered the following initial-boundary value problem which we denote by (Pr.I):

$$y_{tt}-\sigma(y_x)y_{xx}-y_{xtx}=f, \qquad (x,t)\in(0,1)\times(0,\infty)\,, \tag{1.1}$$

$$y(0,t)=y(1,t)=0, \qquad t\in(0,\infty)\,, \tag{1.2}$$

$$y(x,0)=y_0(x),\ y_t(x,0)=y_1(x), \qquad x\in(0,1)\,, \tag{1.3}$$

where y is an unknown function and y_0, y_1 and f are given functions. (For the physical meaning of this problem, see [4].) They established the existence, uniqueness and stability of smooth solutions of (Pr.I) under the assumptions that σ is a positive $C^2(-\infty,\infty)$ function and that initial data y_0 and y_1 are, respectively, $C^4[0,1]$ and $C^2[0,1]$ functions vanishing together with their second derivatives at zero and one. The method of proof used in [4] are rather complicate and heavily depends upon some special properties of the Green function of the heat equation. (See also Davis [1], Ebihara [2] and Greenberg [3].)

The main purpose of the present paper is to weaken the assumptions in [4] and give a simplified proof of the existence, uniqueness and stability of smooth solutions of (Pr.I). We assume that σ is a non-negative $C^1(-\infty,\infty)$ function and that initial data y_0 and y_1 are, respectively, $C^2[0,1]$ and $C[0,1]$ functions such that $y_0(0)=y_0(1)=y_{0,xx}(0)=y_{0,xx}(1)=0$ and $y_1(0)=y_1(1)=0$. Under these assumptions, we choose a Banach space $X_0=\{y\in C[0,1];\ y(0)=y(1)=0\}$ and regard y as a map from $[0,\infty)$ to X_0. Let $A=\partial^2/\partial x^2$. We can formally rewrite (Pr.I) in an abstract form:

$$\begin{cases} y_{tt}-Ay_t-By=f, & t\in(0,\infty)\,, \\ y(0)=y_0,\ y_t(0)=y_1\,, \end{cases} \tag{1.4}$$

where B is a nonlinear operator defined by $By(x)=\sigma(y_x(x))y_{xx}(x)$. Set $u=y_t$ and $v=Ay$. Then (1.4) is equivalent to the following:

$$(1.5)\qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} A & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} B(A^{-1}v) \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \ t \in (0, \infty), \\ \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} y_1 \\ Ay_0 \end{pmatrix}. \end{cases}$$

We can regard (1.5) as the Cauchy problem for a single equation in the product space $X_0 \times X_0$; so that the original problem (Pr.I) is reduced to the Cauchy problem for an abstract evolution equation. The existence result of (Pr.I) will follow from that of (1.5). Moreover, we can show that, if σ is positive and f tends rapidly to zero as $t \to \infty$, any solution of (Pr.I) decays exponentially to zero as $t \to \infty$.

In §2, we state main results; Theorem 1 (uniqueness), Theorem 2 (existence), Theorem 3 (dependence on data) and Theorem 4 (asymptotic behavior as $t \to \infty$). In §3, we prepare some abstract formulation of (Pr.I), which will justify the ideas in the preceding paragraph. We give some a priori estimates of smooth solutions of (Pr.I) in §4. §§5–8 are devoted to the proofs of Theorems 1, 2, 3 and 4, respectively.

2. Assumptions and results

First we shall prepare some notation which will be used later. Throughout this paper functions are all real. Let u and v be continuous functions on [0,1]. We put

$$|u|_\infty = \max_{0 \le x \le 1} |u(x)|,$$

$$(u, v) = \int_0^1 u(x)v(x)dx,$$

and

$$\|u\| = (u, u)^{1/2}.$$

Let X be any real Banach space. For any interval I of real numbers we denote by $C(I; X)$ the space of all X-valued functions u on I such that u is strongly continuous on I. Furthermore, we denote by $C^i(I; X)$ the space of all $u \in C(I; X)$ such that u is i times strongly continuously differentiable on I.

Now we consider the initial-boundary value problem (Pr.I). For the functions σ, y_0, y_1 and f appearing in (1.1) and (1.3), we make the following assumptions.

(A.1) σ is a non-negative $C^1(-\infty, \infty)$ function.

(A.2) y_0 is C^2 on $[0,1]$ and satisfies

$$y_0(0) = y_0(1) = y_{0,xx}(0) = y_{0,xx}(1) = 0,$$

or,

(A.2)′ y_0 is C^2 on $[0,1]$ and satisfies

$$y_0(0)=y_0(1)=0\,.$$

(A.3) y_1 is continuous on $[0,1]$ and satisfies

$$y_1(0)=y_1(1)=0\,.$$

(A.4) f is a continuous function in $(x,t)\in[0,1]\times[0,\infty)$ such that

$$f(0,t)=f(1,t)=0, \qquad t\geqq 0\,.$$

Furthermore, f satisfies

$$|f(\cdot,t)-f(\cdot,s)|_\infty \leqq L|t-s|^\theta, \qquad t,s\in[0,\infty)\,,$$

with some constants $L>0$ and $0<\theta\leqq 1$.

Under these assumptions we seek a smooth solution of (Pr.I) in the following sense.

DEFINITION 2.1. Let y be a function on $[0,1]\times[0,\infty)$. Then y is called a *solution* of (Pr.I) if, for each $T>0$, y has the following properties:

(i) $y\in C^1([0,1]\times[0,T])$,

(ii) $y_{xx}\in C([0,1]\times[0,T])$, $y_{xt}=y_{tx}$ and $y_{tt}\in C([0,1]\times(0,T])$,

(iii) $y_{txx}=y_{xtx}=y_{xxt}\in C([0,1]\times(0,T])$, and

(iv) y satisfies (1.1) on $[0,1]\times(0,T]$ and conditions (1.2) and (1.3).

We now state our main results. We have the following uniqueness result for solutions of (Pr.I).

Theorem 1. *Under assumptions* (A.1), (A.2)′, (A.3) *and* (A.4) *there exists at most one solution of* (Pr.I).

As to the existence of solutions of (Pr.I), we have

Theorem 2. *Under assumptions* (A.1), (A.2), (A.3) *and* (A.4) *there exists a* (*unique*) *solution of* (Pr.I) *such that*

$$y_{xx}(0,t)=y_{xx}(1,t)=0, \qquad t\in[0,\infty)\,,$$

and

$$y_{xtx}(0,t)=y_{xtx}(1,t)=0, \qquad t\in(0,\infty)\,.$$

In addition, assume that y_1 also satisfies (A.2) *and that f_t is continuous on* $[0,1]\times[0,\infty)$. *Then*

$$y\in C^2([0,1]\times[0,\infty)),\ y_{txx}=y_{xtx}=y_{xxt}\in C([0,1]\times[0,\infty))\,,$$

and y satisfies (1.1) *on* $[0,1]\times[0,\infty)$.

REMARK 2.2. Since the compatibility conditions at zero and one do not necessarily imply $y_{xx}(0)=y_{xx}(1)=0$, it is natural to seek a solution of (Pr.I) by assuming (A.2)′ rather than (A.2). However, by the technical reason, we shall prove the existence of a solution of (Pr.I) under assumption (A.2) (see also §3).

REMARK 2.3. Greenberg, MacCamy and Mizel [4, Theorem 2] established the existence and uniqueness of solutions of (Pr.I) under the assumptions that σ is a positive $C^2(-\infty, \infty)$ function and that initial data y_0 and y_1 are, respectively, $C^4[0,1]$ and $C^2[0,1]$ functions which vanish together with their second derivatives at zero and one. Therefore, our existence and uniqueness results (Theorems 1 and 2) generalize their results (see also Davis [1], Ebihara [2] and Greenberg [3].)

Next we present below the result on the dependence of solutions of (Pr.I) upon y_0, y_1 and f.

Theorem 3. *Let σ satisfy* (A.1) *and y_0, $\hat{y}_0\in C^2[0,1]$, y_1, $\hat{y}_1\in C[0,1]$ and $f, \hat{f}\in C([0,1]\times[0,\infty))$ satisfy* (A.2), (A.3) *and* (A.4), *respectively. Then for each $T>0$, the corresponding solutions $y, \hat{y}$ of* (Pr.I) *satisfy*

$$\begin{aligned}&|y(t)-\hat{y}(t)|_\infty+|y_t(t)-\hat{y}_t(t)|_\infty+|y_x(t)-\hat{y}_x(t)|_\infty+|y_{xx}(t)-\hat{y}_{xx}(t)|_\infty\\ &\leqq N(|y_{0,xx}-\hat{y}_{0,xx}|_\infty+|y_1-\hat{y}_1|_\infty+\sup_{0\leqq s\leqq T}|f(s)-\hat{f}(s)|_\infty), \qquad t\in[0,T],\end{aligned}$$

where N is a positive number depending continuously on T, $|y_{0,xx}|_\infty$, $|\hat{y}_{0,xx}|_\infty$, $|y_1|_\infty$, $|\hat{y}_1|_\infty$, $\sup_{0\leqq s\leqq T}|f(s)|_\infty$ and $\sup_{0\leqq s\leqq T}|\hat{f}(s)|_\infty$.

By Theorems 1, 2 and 3, the initial-boundary value problem (Pr.I) is well posed in the sense that there exists a unique solution which is stable with respect to perturbations in the given data.

Finally we give the stability result of solutions of (Pr.I).

Theorem 4. *In addition to* (A.1), (A.2), (A.3) *and* (A.4), *assume that σ is positive on $(-\infty, \infty)$ and that $|f(t)|_\infty$, $|f_t(t)|_\infty=0(e^{-\gamma t})$ with $\gamma>0$ as $t\to\infty$. Then there exists a positive constant δ (which depends on $\sigma(0)$ and γ) such that*

$$\begin{aligned}&|y(t)|_\infty+|y_t(t)|_\infty+|y_x(t)|_\infty+|y_{xx}(t)|_\infty\\ &+|y_{xt}(t)|_\infty+|y_{tt}(t)|_\infty+|y_{xtx}(t)|_\infty\\ &=0(e^{-\delta t}), \qquad \text{as } t\to\infty .\end{aligned}$$

REMARK 2.4. Greenberg, MacCamy and Mizel [4, Theorem 1] proved that y together with its derivatives appearing in Theorem 4 tends to zero as $t\to\infty$ if $f\equiv 0$. Theorem 4 gives the decay estimates of solutions of (Pr.I).

3. Reduction to abstract forms

In this section, we shall rewrite the original problem (Pr.I) in abstract forms to seek a solution of (Pr.I).

We first introduce the following real Banach space of all real continuous functions on [0,1]:

$$X=C[0,1]$$

with norm $|\cdot|_\infty$. Set

$$X_0=\{u\in X;\ u(0)=u(1)=0\}\ .$$

Then X_0 is also a real Banach space with norm $|\cdot|_\infty$. Define a closed linear operator A: $X_0\to X_0$ with a domain $D(A)$ by

$$\begin{cases} D(A)=\{u\in X_0;\ u_{xx}\in X_0\} \\ (Au)(x)=u_{xx}(x) \qquad \text{for } u\in D(A)\,. \end{cases} \tag{3.1}$$

It is well known that A generates an analytic semigroup of bounded linear operators $T(t)$, $t\geqq 0$, on X_0;

$$(T(t)u)(x)=\int_0^1 E(t,x,\xi)u(\xi)d\xi \qquad \text{for } u\in X_0\,,$$

where

$$E(t,x,\xi)=\frac{1}{2\sqrt{\pi t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left(-\frac{(x-\xi+2n)^2}{4t}\right)-\exp\left(-\frac{(x+\xi+2n)^2}{4t}\right)\right\}.$$

It is easily verified that $T(t)$ satisfies

$$|T(t)u|_\infty\leqq|u|_\infty \qquad \text{for } u\in X_0\,. \tag{3.2}$$

Note that A has a bounded inverse operator A^{-1} given by

$$(A^{-1}u)(x)=\int_0^x(x-\xi)u(\xi)d\xi+x\int_0^1(\xi-1)u(\xi)d\xi \qquad \text{for } u\in X_0\,. \tag{3.3}$$

Now we regard the function y in (1.1) as a map from $[0,\infty)$ to X_0. By (3.1) we can formally rewrite (Pr.I) in the following abstract Cauchy problem to the second-order equation;

$$y_{tt}(t)-Ay_t(t)-By(t)=f(t), \qquad t\in(0,\infty)\,, \tag{3.4}$$

$$y(0)=y_0,\ \ y_t(0)=y_1\,, \tag{3.5}$$

where B is a nonlinear operator defined by

$$(By)(x)=\sigma(y_x(x))y_{xx}(x) \tag{3.6}$$

with a domain $D(B)=D(A)$. By (Pr.II) we mean this Cauchy problem (3.4) and (3.5). We define a solution of (Pr.II) as follows.

DEFINITION 3.1. Let y be an X_0-valued function on $[0, \infty)$. Then y is called a *strong solution* of (Pr.II) if, for each $T>0$, it has the following properties:

(i) $y\in C^1([0,T]; X_0)\cap C^2((0,T]; X_0)$,

(ii) Ay and $By\in C([0,T]; X_0)$, $Ay_t\in C((0,T]; X_0)$, and

(iii) y satisfies (3.4) on $(0,T]$ and initial conditions (3.5).

If y is a strong solution of (Pr.II), then y is actually a solution of (Pr.I). To see this fact, we have only to note that by (3.3)

$$y(x,t) = \int_0^x (x-\xi)(Ay(t))(\xi)d\xi + x\int_0^1 (\xi-1)(Ay(t))(\xi)d\xi$$

holds for $0\leqq x\leqq 1$ and $t\geqq 0$. However, the converse is not necessarily true, for y_{xx} is in $C([0, \infty); X)$ (not in $C([0, \infty); X_0)$) when y is a solution of (Pr.I) in the sense of Definition 2.1.

In order to solve (Pr.II), we shall reduce the second-order equation to a system of the first-order equations (cf. Krein [5, chap. 3]). Let y be a strong solution of (Pr.II). Since A is closed,

$$\frac{d}{dt}Ay(t) = Ay_t(t)\,. \tag{3.7}$$

We introduce new functions $u(t)=y_t(t)$ and $v(t)=Ay(t)$. Since $u(t)$ and $v(t)$ are strongly continuously differentiable in t, we find in view of (3.4) and (3.7) that they satisfy

$$\begin{cases} u_t(t) = Au(t)+B(A^{-1}v(t))+f(t), & t\in(0,\infty)\,, \\ v_t(t) = Au(t)\,, & t\in(0,\infty)\,. \end{cases} \tag{3.8}$$

Set $U(t)={}^t(u(t),v(t))$. The system (3.8) may be considered as one equation in the product space $X_0\times X_0$; so that (Pr.II) is reduced to the following Cauchy problem which we denote by (Pr.III);

$$U_t(t) = \boldsymbol{A}U(t)+C(U(t))+F(t), \qquad t\in(0,\infty)\,, \tag{3.9}$$

$$U(0) = {}^t(y_1, Ay_0)\,, \tag{3.10}$$

where

$$\boldsymbol{A} = \begin{pmatrix} A & 0 \\ A & 0 \end{pmatrix},\ C(U) = \begin{pmatrix} B(A^{-1}v) \\ 0 \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}. \tag{3.11}$$

(For the properties of $C(U)$, see Lemma 6.1 (ii) and (iii).) It is easily seen that $\boldsymbol{A}$ is a closed linear operator in $X_0\times X_0$ with a dense domain $D(\boldsymbol{A})=D(A)\times X_0$ and generates an analytic semigroup of bounded linear operators $\boldsymbol{T}(t)$, $t\geqq 0$, on $X_0\times X_0$;

$$\boldsymbol{T}(t) = \begin{pmatrix} T(t) & 0 \\ T(t)-1 & 1 \end{pmatrix}, \tag{3.12}$$

where $T(t)$ is an analytic semigroup generated by A. Therefore, we can regard (Pr.III) as the Cauchy problem for an abstract semilinear evolution equation of parabolic type. We define a strong solution of (Pr.III) in the same way as Definition 3.1.

DEFINITION 3.2. Let $U={}^t(u,v)\colon [0,\infty)\to X_0\times X_0$. Then U is called a *strong solution* of (Pr.III) if, for each $T>0$, it has the following properties:

(i) $U\in C([0,T];\, X_0\times X_0)\cap C^1((0,T];\, X_0\times X_0)$,

(ii) $AU\in C((0,T];\, X_0\times X_0)$ and $C(U)\in C([0,T];\, X_0\times X_0)$, and

(iii) U satisfies (3.9) on $(0,T]$ and initial condition (3.10).

Then we have the following relations between strong solutions of (Pr.II) and (Pr.III).

Proposition 3.3. *Let* $y\colon [0,\infty)\to X_0$ *be a strong solution of* (Pr.II). *Define* $U={}^t(u,v)\colon [0,\infty)\to X_0\times X_0$ *by*

$$u(t)=y_t(t) \quad and \quad v(t)=Ay(t)\,. \tag{3.13}$$

Then U *is a strong solution of* (Pr.III).

Conversely, let $U={}^t(u,v)\colon [0,\infty)\to X_0\times X_0$ *be a strong solution of* (Pr.III). *Define* $y\colon [0,\infty)\to X_0$ *by*

$$y(t)=\int_0^t u(s)ds+y_0\,. \tag{3.14}$$

Then y *is a strong solution of* (Pr.II).

Proof. The first part of this proposition is evident from the above arguments.

We shall prove the latter half. Let U be a strong solution of (Pr.III) and define y by (3.14). It is clear from Definition 3.2 that y is in $C^1([0,\infty);\, X_0)$ and $C^2((0,\infty);\, X_0)$. Since $v_t(t)=Au(t)=Ay_t(t)\in C((0,\infty);\, X_0)$, we get for any $\varepsilon>0$

$$v(t)-v(\varepsilon)=\int_\varepsilon^t Ay_t(s)ds=A(y(t)-y(\varepsilon))\,,$$

where we have used the closedness of A. In view of $v\in C([0,\infty);\, X_0)$, the left-hand side tends to $v(t)-Ay_0$ as $\varepsilon\to 0$. Since $y(\varepsilon)\to y_0$ as $\varepsilon\to 0$, we see

$$v(t)=Ay(t) \qquad \text{on } [0,\infty)\,,$$

which implies $Ay\in C([0,\infty);\, X_0)$. Since $By=B(A^{-1}v)\in C([0,\infty);\, X_0)$, y clearly satisfies (3.9) on $(0,\infty)$. Thus we have shown that y satisfies all the properties in Definition 3.1. [q.e.d.]

By Proposition 3.3 we have established a one-to-one correspondence between strong solutions of (Pr.II) and (Pr.III): they are mutually combined by (3.13) and (3.14). In this sense, Cauchy problems (Pr.II) and (Pr.III) are equivalent. Since any strong solution of (Pr.II) is a solution of (Pr.I), we shall consider (Pr.II) or (Pr.III) to show the existence of a solution of (Pr.I).

REMARK 3.4. Greenberg, MacCamy and Mizel [4] considered (1.1) as two different inhomogeneous equations: one is the heat equation for y_t and the other is the ordinary differential equation for y_{xx}. They solved these equations separately to obtain the existence result of solutions of (Pr.I). Davis [1] and Ebihara [2] solved (Pr.I) by the Galerkin's method.

Our idea is different from theirs. By introducing two unknown functions u and v by (3.13), we regard (1.1) as a system of two differential equations (3.8). Hence, (3.8), or equivalently (3.9), can be treated as a single semilinear equation of evolution.

4. A priori estimates for solutions of (Pr.I)

In this section we assume that (A.1), (A.2)′, (A.3) and (A.4) always hold. We shall derive some a priori estimates for solutions of (Pr.I). These estimates will play an important role in the proofs of our theorems.

We first note the following result which will be of frequent use.

Lemma 4.1. *Let y be a $C^2[0,1]$ function which vanishes at zero and one. Then*

$$||y|| \leqq |y|_\infty \leqq ||y_x|| \leqq |y_x|_\infty \leqq ||y_{xx}|| \leqq |y_{xx}|_\infty .$$

Proof. It suffices to note the following equalities:

$$y(x) = \int_0^x y_x(\xi)d\xi$$

and

$$y_x(x) = \int_{x_0}^x y_{xx}(\xi)d\xi \qquad \text{for some } x_0 \in [0,1] .$$

[q.e.d.]

Lemma 4.2 (cf. [2, Lemma 4.1]). *Let y be a solution of* (Pr.I). *Then*

$$||y_t(t)||^2 + 2\int_0^1 \sum(y_x(x,t))dx + \int_0^t ||y_{tx}(s)||^2 ds$$
$$\leqq ||y_1||^2 + 2\int_0^1 \sum(y_{0,x}(x))dx + \int_0^t ||f(s)||^2 ds, \qquad t \geqq 0 ,$$

where $\sum$ is defined by

$$\sum(r)=\int_0^r\int_0^s\sigma(\tau)d\tau ds\geqq 0\,.$$

Proof. Since $y\in C^1([0,1]\times[0,\infty))$, we have

$$y(0,t)=y(1,t)=y_t(0,t)=y_t(1,t)=0,\qquad t\geqq 0\,. \tag{4.1}$$

Multiplying (1.1) by y_t and integrating over $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}||y_t(t)||^2+\frac{d}{dt}\int\sum(y_x(x,t))dx+||y_{tx}(t)||^2=(f(t),y_t(t)) \tag{4.2}$$

for $t>0$ (use (4.1)). By Lemma 4.1,

$$\begin{aligned}|(f(t),y_t(t))|&\leqq||f(t)||\cdot||y_t(t)||\leqq||f(t)||\cdot||y_{tx}(t)||\\ &\leqq\frac{1}{2}||f(t)||^2+\frac{1}{2}||y_{tx}(t)||^2\,.\end{aligned}$$

Hence, rearranging (4.2) and integrating the resulting expression over $(0,t)$, we obtain the conclusion. [q.e.d.]

Moreover, we have

Lemma 4.3 (cf. [2, Lemma 4.2]). *Let y be a solution of* (Pr.I). *Then*

$$\begin{aligned}&||y_{xx}(t)||^2+4\int_0^t\int_0^1\sigma(y_x(x,s))y_{xx}(x,s)^2dxds\\ &\leqq 4\Big[\Big\{||y_1||^2+||y_1||\cdot||y_{0,xx}||+\frac{1}{2}||y_{0,xx}||^2+2\int_0^1\sum(y_{0,x}(x))dx\\ &\quad+\int_0^t||f(s)||^2ds\Big\}^{1/2}+\int_0^t||f(s)||ds\Big]^2,\qquad t\geqq 0\,.\end{aligned}$$

Proof. Multiplying (1.1) by $-y_{xx}$ and integrating over (0,1), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||y_{xx}(t)||^2+\int_0^1\sigma(y_x(x,\cdot))y_{xx}(x,t)^2dx-\frac{d}{dt}(y_t(t),y_{xx}(t))-||y_{tx}(t)||^2\\ &=-(f(t),y_{xx}(t))\,.\end{aligned} \tag{4.3}$$

Integration of (4.3) over $(0,t)$ leads to the following:

$$\begin{aligned}&\frac{1}{2}||y_{xx}(t)||^2+\int_0^t\int_0^1\sigma(y_x(x,s))y_{xx}(x,s)^2dxds\\ &=\frac{1}{2}||y_{0,xx}||^2+(y_t(t),y_{xx}(t))-(y_1,y_{0,xx})+\int_0^t||y_{tx}(s)||^2ds-\int_0^t(f(s),y_{xx}(s))ds\\ &\leqq\frac{1}{4}||y_{xx}(t)||^2+\frac{1}{2}||y_{0,xx}||^2+||y_1||\cdot||y_{0,xx}||+||y_t(t)||^2\\ &\quad+\int_0^t||y_{tx}(s)||^2ds+\int_0^t||f(s)||\cdot||y_{xx}(s)||ds\,.\end{aligned}$$

Therefore, using Lemma 4.2 we get

$$\frac{1}{4}\|y_{xx}(t)\|^2+\int_0^t\int_0^1\sigma(y_x(x,s))y_{xx}(x,s))^2dxds$$
$$\leqq\|y_1\|^2+\|y_1\|\cdot\|y_{0,xx}\|+\frac{1}{2}\|y_{0,xx}\|^2+2\int_0^1\sum(y_{0,x}(x))dx$$
$$+\int_0^t\|f(s)\|^2ds+\int_0^t\|f(s)\|\cdot\|y_{xx}(s)\|ds\,.$$

In other words, we have

$$F(t)^2\leqq\int_0^t F(s)G(s)ds+H(t)\,, \tag{4.4}$$

where $F(t)=\frac{1}{2}\|y_{xx}(t)\|$, $G(t)=2\|f(t)\|$ and

$$H(t)=\|y_1\|^2+\|y_1\|\cdot\|y_{0,xx}\|+\frac{1}{2}\|y_{0,xx}\|^2+2\int_0^1\sum(y_{0,x}(x))dx+\int_0^t\|f(s)\|^2ds\,.$$

Since (4.4) implies

$$F(t)\leqq\frac{1}{2}\int_0^t G(s)ds+\sup_{0\leqq s\leqq t}H(s)^{1/2}\,,$$

we obtain the estimate of Lemma 4.3. [q.e.d.]

5. Proof of Theorem 1

In this section we shall prove Theorem 1. Let y and $\hat{y}$ be two solutions of (Pr.I). Let T be any fixed positive number. Then there exists a positive constant N such that

$$\|y_{xx}(t)\|\leqq N \quad\text{and}\quad \|\hat{y}_{xx}(t)\|\leqq N \qquad \text{for } t\in[0,T]\,, \tag{5.1}$$

(see also Lemma 4.3)). Set

$$K=\max\{\max_{|r|\leqq N}\sigma(r),\ \max_{|r|\leqq N}|\sigma'(r)|\}\,.$$

By Lemma 4.1 and (5.1), we have

$$\sigma(y_x(x,t))\leqq K,\qquad \sigma(\hat{y}_x(x,t))\leqq K,\qquad (x,t)\in[0,1]\times[0,T]\,, \tag{5.2}$$

and

$$|\sigma(y_x(x,t))-\sigma(\hat{y}_x(x,t))|\leqq K|y_x(x,t)-\hat{y}_x(x,t)|,\ (x,t)\in[0,1]\times[0,T]\,. \tag{5.3}$$

Now we put $z=y-\hat{y}$. Then z satisfies the following equation:

$$z_{tt}-\sigma(y_x)z_{xx}-z_{xtx}=(\sigma(y_x)-\sigma(\hat{y}_x))\hat{y}_{xx}\,. \tag{5.4}$$

Multiplying the both sides of (5.4) by z_t and integrating over (0,1), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||z_t(t)||^2+||z_{tx}(t)||^2\\
(5.5)\qquad &=(\sigma(y_x(t))z_{xx}(t)+\{\sigma(y_x(t))-\sigma(\hat{y}_x(t))\}\hat{y}_{xx}(t),\ z_t(t))\\
&\leqq K(||z_{xx}(t)||+|z_x(t)|_\infty\cdot||\hat{y}_{xx}(t)||)||z_t(t)||\,,
\end{aligned}
$$

where we have used (5.2) and (5.3). Next multiplying the both sides of (5.4) by $-\lambda z_{xx}$ with $0<\lambda<1$ and integrating over $(0,1)$, we have

$$
\begin{aligned}
&\lambda\left\{\frac{1}{2}\frac{d}{dt}||z_{xx}(t)||^2+(\sigma(y_x(t))z_{xx}(t),\ z_{xx}(t))-\frac{d}{dt}(z_t(t),\ z_{xx}(t))-||z_{tx}(t)||^2\right\}\\
(5.6)\qquad &=-\lambda(\{\sigma(y_x(t))-\sigma(\hat{y}_x(t))\}\hat{y}_{xx}(t),\ z_{xx}(t))\\
&\leqq K|z_x(t)|_\infty\cdot||\hat{y}_{xx}(t)||\cdot||z_{xx}(t)||\,.
\end{aligned}
$$

(In the last inequality of (5.6) we have used (5.3).) Hence, by virtue of (5.1), Lemma 4.1 and the non-negativity of σ, we see from (5.5) and (5.6) that

$$
\begin{aligned}
&\frac{1}{2}||z_t(t)||^2-\lambda(z_t(t),\ z_{xx}(t))+\frac{\lambda}{2}||z_{xx}(t)||^2+(1-\lambda)\int_0^t||z_{tx}(s)||^2ds\\
(5.7)\qquad &\leqq K\int_0^t\{(N+1)||z_t(s)||+N||z_{xx}(s)||\}||z_{xx}(s)||ds
\end{aligned}
$$

holds for every $0\leqq t\leqq T$ and $0<\lambda<1$. Note

$$
\begin{aligned}
&\frac{1}{2}||z_t(t)||^2-\lambda(z_t(t),\ z_{xx}(t))+\frac{\lambda}{2}||z_{xx}(t)||^2\\
&\geqq\frac{1-\sqrt{\lambda}}{2}||z_t(t)||^2+\frac{\lambda(1-\sqrt{\lambda})}{2}||z_{xx}(t)||^2\,.
\end{aligned}
$$

Hence (5.7) implies $||z_t(t)||^2+||z_{xx}(t)||^2\equiv0$ (i.e. $z\equiv0$) with the aid of Gronwall's inequality. Thus we have shown the uniqueness of solutions of (Pr.I). [q.e.d.]

6. Proof of Theorem 2

In order to show the existence of a solution of (Pr.I), we shall consider the Cauchy problem (Pr.III). Recall that (Pr.II) and (Pr.III) are equivalent in the sense of Proposition 3.3. Hence, if we can show the existence of a strong solution U of (Pr.III), the function y defined by (3.14) is a strong solution of (Pr.II) and, therefore, is a solution of (Pr.I). Thus a solution of (Pr.I) will be constructed.

We define the norm of the product space $X_0\times X_0$ as follows:

$$|U|_\infty=|u|_\infty+|v|_\infty\qquad\text{for }\ U={}^t(u,v)\in X_0\times X_0\,.$$

Then we have:

Lemma 6.1. *Let* $\boldsymbol{A}$, C *and* $\boldsymbol{T}(t)$ *be defined by* (3.11) *and* (3.12). *Then the following properties hold.*

(i) $|\boldsymbol{T}(t)U|_\infty \leqq 3|u|_\infty + |v|_\infty \leqq 3|U|_\infty$, $U={}^t(u,v)\in X_0\times X_0$, $t\geqq 0$.

(ii) $|C(U)|_\infty \leqq M(||v||)|v|_\infty \leqq M(|U|_\infty)|U|_\infty$, $U={}^t(u,v)\in X_0\times X_0$.

(iii) $|C(U)-C(\hat{U})|_\infty \leqq M(||v||)|v-\hat{v}|_\infty + |\hat{v}|_\infty M_1(||v||+||\hat{v}||)||v-\hat{v}||$
$\leqq \{M(|U|_\infty)+|\hat{U}|_\infty M_1(|U|_\infty+|\hat{U}|_\infty)\}|U-\hat{U}|_\infty$,
$U={}^t(u,v)$, $\hat{U}={}^t(\hat{u},\hat{v})\in X_0\times X_0$.

Here M *and* M_1 *are defined by*

$$M(r)=\max_{|s|\leqq r}\sigma(s) \qquad \text{and} \qquad M_1(r)=\max_{|s|\leqq r}|\sigma'(s)|\ .$$

Proof. Since $\boldsymbol{T}(t)$ is defined by (3.12), it is easy to show (i) by (3.2). Next we shall show (ii). From (3.3), (3.6) and (3.11) we have

$$C(U)(x) = {}^t(\sigma(w(x))v(x),\, 0),\ \ U={}^t(u,v)\in X_0\times X_0\,, \tag{6.1}$$

where

$$w(x)=\int_0^x v(\xi)d\xi+\int_0^1(\xi-1)v(\xi)d\xi\ . \tag{6.2}$$

Since there exists an $x_0\in[0,1]$ such that

$$w(x)=\int_{x_0}^x v(\xi)d\xi \qquad \text{for every}\ \ x\in[0,1]\,,$$

we see

$$|w|_\infty \leqq ||v|| \qquad \text{for}\ \ v\in X_0\,. \tag{6.3}$$

Hence it follows from (6.1) and (6.3) that property (ii) holds. Finally to prove (iii), we define $\hat{w}$ by (6.2) with $\hat{v}$ replacing v. Since

$$C(U)(x)-C(\hat{U})(x)={}^t(\sigma(w(x))v(x)-\sigma(\hat{w}(x))\hat{v}(x),\,0)\,,$$

we have only to estimate

$$\begin{aligned}&|\sigma(w(x))v(x)-\sigma(\hat{w}(x))\hat{v}(x)|\\ \leqq\ &|\sigma(w(x))(v(x)-\hat{v}(x))|+|(\sigma(w(x))-\sigma(\hat{w}(x)))\hat{v}(x)|\ .\end{aligned}$$

Using (6.2) and (6.3) we can derive (iii). [q.e.d.]

Now we are in a position to prove the existence of strong solutions of (Pr. III).

Proposition 6.2. *Under assumptions* (A.1), (A.2), (A.3) *and* (A.4) *there exists a unique strong solution* U *of* (Pr.III).

In addition, assume that y_1 *also satisfies* (A.2) *and that* f_t *is in* $C([0,\infty);X_0)$. *Then*

$$U\in C^1([0, \infty); X_0\times X_0),\quad AU\in C([0, \infty); X_0\times X_0),$$

and U satisfies (3.9) *on* $[0, \infty)$.

Proof. First suppose that (A.1), (A.2), (A.3) and (A.4) hold. Let U be a strong solution of (Pr.III). Then U satisfies the following integral equation

$$U(t)= \boldsymbol{T}(t)U_0+\int_0^t \boldsymbol{T}(t-s)(C(U(s))+F(s))ds, \tag{6.4}$$

where $U_0={}^t(y_1, Ay_0)$ (see e.g, Krein [5]). Conversely, if U is a strongly continuous function satisfying (6.4) (which we call a *mild solution* of (Pr.III)), then U is a strong solution of (Pr.III). In fact, to see this, we have only to use the result of Pazy [6, Theorem 5.2]. (Note that $\boldsymbol{T}(t)$ is an analytic semigroup.) Hence, in order to show the existence of a strong solution of (Pr.III), it suffices to prove the existence of a mild solution of (Pr.III).

Now we consider integral equation (6.4). Since $C(U)$ is locally Lipshitz continuous in U by Lemma 6.1, we can show, in a usual manner, by virtue of the fixed point theorem of a strictly contraction mapping that there exists locally (in time) a strongly continuous function U satisfying (6.4) (see e.g. Tanabe [7, chap. 6]). In order to extend this U to the interval $[0, \infty)$, we shall derive an a priori estimate of any mild solution U of (Pr.III).

Let T be an arbitrary fixed positive number. Let $U={}^t(u,v)$ be a mild solution (strong solution) of (Pr.III). By Proposition 3.3, the function y defined by (3.14) is a strong solution of (Pr.II) and, therefore, a solution of (Pr.I). Hence, by Lemma 4.3, there exists a positive constant N such that

$$||Ay(t)||\leqq N \qquad t\in[0, T],$$

which implies by (3.13)

$$||v(t)||\leqq N, \qquad t\in[0, T]. \tag{6.5}$$

Consequently, using Lemma 6.1 (ii) and (6.5), we get from (6.4)

$$\begin{aligned}|U(t)|_\infty &\leqq |\boldsymbol{T}(t)U_0|_\infty+\int_0^t |\boldsymbol{T}(t-s)|_\infty(|C(U(s))|_\infty+|F(s)|_\infty)ds\\ &\leqq 3\{|U_0|_\infty+\int_0^t(M(N)|U(s)|_\infty+|F(s)|_\infty)ds\},\ t\in[0, T],\end{aligned}$$

which, together with Gronwall's inequality, yields

$$|U(t)|_\infty\leqq 3(|U_0|_\infty+\int_0^T |f(s)|_\infty ds)\exp(3M(N)t), \qquad t\in[0, T]. \tag{6.6}$$

Since a priori estimate (6.6) is obtained, we can show the global existence and uniqueness of a mild solution of (Pr.III) in the standard way (see e.g. Pazy

[6, Theorem 3.1]).

Finally we shall prove the latter half of Proposition 6.2. In addition to (A.1), (A.2), (A.3) and (A.4), assume that $y_1 \in D(A)$ and $f_t \in C([0, \infty); X_0)$. Since $U_0 = {}^t(y_0, Ay_1) \in D(\boldsymbol{A}) = D(A) \times X_0$, $U_t(t)$ satisfies

$$\begin{aligned} U_t(t) = {} & \boldsymbol{T}(t)\boldsymbol{A}U_0 + \boldsymbol{T}(t)(C(U_0)+F(0)) \\ & + \int_0^t \boldsymbol{T}(t-s)\Big(\frac{d}{ds}C(U(s))+F_t(s)\Big)ds, \qquad t \geqq 0 . \end{aligned} \tag{6.7}$$

Here

$$\Big(\frac{d}{dt}C(U(t))\Big)(x) = {}^t(\sigma(w(x,t))v_t(x,t)+\sigma'(w(x,t))w_t(x,t)v(x,t), 0) , \tag{6.8}$$

where $w(x,t)$ is defined by (6.2) with $v(x)$ replaced by $v(x,t)$. Hence, it follows from (6.7) and (6.8) that U_t is strongly continuous on $[0, \infty)$. Thus, noting (3.9), we complete the proof. [q.e.d.]

Proof of Theorem 2. If U is a strong solution of (Pr.III), then y defined by (3.14) gives a solution of (Pr.I). Therefore, in view of Theorem 1 and Proposition 3.3, we obtain all the conclusions of Theorem 2 from Proposition 6.2. [q.e.d.]

7. Proof of Theorem 3

Suppose that $y_0, \hat{y}_0 \in C^2[0, 1]$, y_1, $\hat{y}_1 \in C[0, 1]$ and f, $\hat{f} \in C([0, 1] \times [0, \infty))$ satisfy (A.2), (A.3) and (A.4), respectively. Let y, $\hat{y}$ be the corresponding solutions of (Pr.I). (By Theorems 1 and 2, y and $\hat{y}$ satisfy (3.4).) Let T be any fixed positive number. Then, by Lemma 3.4,

$$||y_{xx}(t)|| \leqq N_1 \ \text{ and } \ ||\hat{y}_{xx}(t)|| \leqq N_1, \qquad t \in [0, T] , \tag{7.1}$$

where N_1 is a positive number depending continuously on T, $|y_{0,xx}|_\infty$, $|\hat{y}_{0,xx}|_\infty$, $|y_1|_\infty$, $|\hat{y}_1|_\infty$, $\sup_{0 \leqq t \leqq T} |f(t)|_\infty$ and $\sup_{0 \leqq t \leqq T} |\hat{f}(t)|_\infty$. In this section, we denote by N_i positive numbers depending continuously on the above quantities. Define U and $\hat{U}$ by (3.13), i.e.

$$U = {}^t(u,v) = {}^t(y_t, Ay) \ \text{ and } \ \hat{U} = {}^t(\hat{u},\hat{v}) = {}^t(\hat{y}_t, A\hat{y}) .$$

Then U and $\hat{U}$, respectively, satisfy the following integral equations:

$$U(t) = \boldsymbol{T}(t)U_0 + \int_0^t \boldsymbol{T}(t-s)(C(U(s))+F(s))ds$$

and

$$\hat{U}(t) = \boldsymbol{T}(t)\hat{U}_0 + \int_0^t \boldsymbol{T}(t-s)(C(\hat{U}(s))+\hat{F}(s))ds ,$$

where $F(t)={}^t(f(t), 0), \hat{F}(t)={}^t(\hat{f}(t),0)$, $U_0={}^t(y_1, Ay_0)$ and $\hat{U}_0=(\hat{y}_1, A\hat{y}_0)$ (see (6.4)). Consequently, we have

$$U(t)-\hat{U}(t) = \boldsymbol{T}(t)(U_0-\hat{U}_0)+\int_0^t \boldsymbol{T}(t-s)(C(U(s))-C(\hat{U}(s)))ds \tag{7.2}$$
$$+\int_0^t \boldsymbol{T}(t-s)(F(s)-\hat{F}(s))ds\,.$$

Note that, by (6.6),

$$|U(t)|_\infty \leqq N_2 \text{ and } |\hat{U}(t)|_\infty \leqq N_2, \qquad t\in[0,T]\,, \tag{7.3}$$

for some N_2. Hence, Lemma 6.1 (iii), together with (7.1) and (7.3), gives

$$\begin{aligned}|C(U(t))-C(\hat{U}(t))|_\infty &\leqq M(N_1)|v(t)-\hat{v}(t)|_\infty+N_2M_1(2N_1)||v(t)-\hat{v}(t)|| \\ &\leqq N_3|U(t)-\hat{U}(t)|_\infty, \qquad t\in[0,T]\,,\end{aligned} \tag{7.4}$$

for some N_3. Therefore, using Lemma 6,1 and (7.4) we get from (7.2)

$$|U(t)-\hat{U}(t)|_\infty \leqq 3|y_1-\hat{y}_1|_\infty+|Ay_0-A\hat{y}_0|_\infty+3N_3\int_0^t |U(s)-\hat{U}(s)|_\infty ds$$
$$+3\int_0^t |f(s)-\hat{f}(s)|_\infty ds\,,$$

which yields by Gronwall's inequality

$$|U(t)-\hat{U}(t)|_\infty \leqq (3|y_1-\hat{y}_1|_\infty+|Ay_0-A\hat{y}_0|_\infty+3\int_0^T |f(s)-\hat{f}(s)|_\infty ds)$$
$$\times \exp(3N_3t)\,,$$

for $0\leqq t\leqq T$. Thus using Lemma 4.1 we obtain the conclusion of Theorem 3. [q.e.d.]

8. Proof of Theorem 4

In this section, we assume, in addition to (A.1), (A.2), (A.3) and (A.4), that σ is positive on $(-\infty, \infty)$ and that both $|f(t)|_\infty$ and $|f_t(t)|_\infty$ decay like $e^{-\gamma t}$ with $\gamma>0$ as $t\to\infty$.

Let y be a solution of (Pr.I). Then Lemmas 4.2 and 4.3 imply that there exists a positive constant N such that

$$||y_t(t)||+||y_{xx}(t)||\leqq N, \qquad \text{for all } t\geqq 0\,. \tag{8.1}$$

First we shall prove the stronger result than (8.1):

$$||y_t(t)||+||y_{xx}(t)|| = 0(e^{-\beta t}) \text{ as } t\to\infty\,, \tag{8.2}$$

with some $\beta>0$. As in the proof of Lemma 4.2, multiplying (1.1) by $e^{\alpha t}y_t$

(where α is a positive number which will be specified later) and integrating over $[0,1]\times[0,t]$, we have

$$
\begin{aligned}
(8.3)\qquad & e^{\alpha t}\left\{\frac{1}{2}\|y_t(t)\|^2+\int_0^1\Sigma(y_x(x,t))dx\right\}+\int_0^t e^{\alpha s}\|y_{tx}(s)\|^2 ds \\
& -\alpha\int_0^t e^{\alpha s}\left\{\frac{1}{2}\|y_t(s)\|^2+\int_0^1\Sigma(y_x(x,s))dx\right\}ds \\
& =\frac{1}{2}\|y_1\|^2+\int_0^1\Sigma(y_{0,x}(x))dx+\int_0^t e^{\alpha s}(f(s),y_t(s))ds\,,
\end{aligned}
$$

where $\Sigma(r)=\int_0^r\int_0^s\sigma(\tau)d\tau ds$. If we put

$$m=\inf_{|s|\leq N}\sigma(s)>0 \quad\text{and}\quad M=M(N)\equiv\sup_{|s|\leq N}\sigma(s)\,,$$

then we get

$$(8.4)\qquad \frac{1}{2}my_x(x,t)^2\leqq\Sigma(y_x(x,t))\leqq\frac{1}{2}My_x(x,t)^2\,,$$

(note Lemma 4.1 and (8.1)). Making use of (8.4) we rearrange (8.3): then

$$
\begin{aligned}
(8.5)\qquad & \frac{1}{2}e^{\alpha t}\{\|y_t(t)\|^2+m\|y_x(t)\|^2\}+\int_0^t(1-\varepsilon)e^{\alpha s}\|y_{tx}(s)\|^2 ds \\
& -\frac{\alpha}{2}\int_0^t e^{\alpha s}\{\|y_t(s)\|^2+M\|y_x(s)\|^2\}ds \\
& \leqq\frac{1}{2}\left\{\|y_1\|^2+M\|y_{0,x}\|^2+\frac{1}{2\varepsilon}\int_0^t e^{\alpha s}\|f(s)\|^2 ds\right\}\,,
\end{aligned}
$$

for any $\varepsilon>0$ and $t\geqq 0$. Next, as in the proof of Lemma 4.3, multiplying (1.1) by $-\lambda e^{\alpha t}y_{xx}$ (where λ is another positive number which will be specified later) and integrating the resulting expression over $[0,1]\times[0,T]$, we have

$$
\begin{aligned}
& \frac{\lambda}{2}e^{\alpha t}\{\|y_{xx}(t)\|^2-2(y_t(t),y_{xx}(t))-\alpha\|y_x(t)\|^2\} \\
& +\lambda\int_0^t\int_0^1 e^{\alpha s}\sigma(y_x(x,s))y_{xx}(x,s)^2 dxds-\frac{\alpha\lambda}{2}\int_0^t e^{\alpha s}\|y_{xx}(s)\|^2 ds \\
(8.6)\qquad & +\frac{\alpha^2\lambda}{2}\int_0^t e^{\alpha s}\|y_x(s)\|^2 ds-\lambda\int_0^t e^{\alpha s}\|y_{tx}(s)\|^2 ds \\
& =\frac{\lambda}{2}\{\|y_{0,xx}\|^2-2(y_1,y_{0,xx})-\alpha\|y_{0,x}\|^2\}-\lambda\int_0^t e^{\alpha s}(f(s),y_{xx}(s))ds\,.
\end{aligned}
$$

Rearranging (8.6) we obtain

(3.4), so that we introduce

$$U(t)={}^t(u(t),\,v(t))={}^t(y_t(t),\,Ay(t))$$

and

$$U_0={}^t(y_1,\,Ay_0)\,.$$

Rewrite (3.9) in the following from:

$$U_t(t)=\boldsymbol{A}_1U(t)+C_1(U(t))+F(t)\,, \tag{8.11}$$

where

$$\boldsymbol{A}_1=\begin{pmatrix} A & \sigma(0)\\ A & 0\end{pmatrix} \text{ and } C_1(U)=\begin{pmatrix} B(A^{-1}v)-\sigma(0)v\\ 0\end{pmatrix}.$$

Since A is an infinitesimal generator of the analytic semigroup $T(t)$ on X_0, $\boldsymbol{A}_1$ also generates an analytic semigroup of bounded linear operators $\boldsymbol{T}_1(t)$, $t\geqq 0$, on $X_0\times X_0$ (see e.g. Krein [5]). We have the following lemma whose proof will be found at the end of this section.

Lemma 8.1. *Let $\boldsymbol{T}_1(t)$ be an analytic semigroup generated by $\boldsymbol{A}_1$. Then there exist some positive constants K and $\rho(<\sigma(0))$ such that*

$$|\boldsymbol{T}_1(t)U|_\infty\leqq Ke^{-\rho t}|U|_\infty$$

for $t\geqq 0$ and $U={}^t(u,v)\in X_0\times X_0$.

We shall continue the proof of Theorem 4. It follows from (8.11) that U satisfies

$$U(t)=\boldsymbol{T}_1(t)U_0+\int_0^t \boldsymbol{T}_1(t-s)\{C_1(U(s))+F(s)\}ds,\ t\geqq 0\,. \tag{8.12}$$

For the first conmponent of $C_1(U)$, we have:

$$\begin{aligned}|B(A^{-1}v)(x)-\sigma(0)v(x)| &= |\{\sigma(w(x))-\sigma(0)\}v(x)|\\ &\leqq M_1(|w|_\infty)|w(x)|\cdot|v(x)|\,,\end{aligned}$$

where w is defined by (6.2) and M_1 is defined as in Lemma 6.1. Hence, recalling $v(x,t)=y_{xx}(x,t)$, $w(x,t)=y_x(x,t)$ and (8.10), we get

$$|C_1(U(t))|_\infty\leqq M_1(N_3)N_3e^{-\beta t}|U(t)|_\infty\,. \tag{8.13}$$

Therefore, (8.12), combined with Lemma 8.1 and (8.13), gives

$$|U(t)|_\infty\leqq Ke^{-\rho t}|U_0|_\infty+K\int_0^t e^{-\rho(t-s)}(M_1(N_3)N_3e^{-\beta s}|U(s)|_\infty+N_4e^{-\gamma s})ds\,, \tag{8.14}$$

where we have used the assumption that $|F(t)|_\infty\leqq N_4e^{-\gamma t}$ with some $N_4>0$.

$$
(8.7)\qquad
\begin{aligned}
&\frac{\lambda}{2}e^{\alpha t}\{||y_{xx}(t)||^2-2||y_t(t)||\cdot||y_{xx}(t)||-\alpha||y_x(t)||^2\}\\
&+\lambda\int_0^t\Big(m-\frac{\alpha}{2}-\varepsilon\Big)e^{\alpha s}||y_{xx}(s)||^2ds\\
&+\frac{\alpha^2\lambda}{2}\int_0^t e^{\alpha s}||y_x(s)||^2ds-\lambda\int_0^t e^{\alpha s}||y_{tx}(s)||^2ds\\
&\leqq\frac{\lambda}{2}\Big\{||y_{0,xx}||^2+2||y_1||\cdot||y_{0,xx}||+\frac{1}{2\varepsilon}\int_0^t e^{\alpha s}||f(s)||^2ds\Big\},
\end{aligned}
$$

for any $\varepsilon>0$ and $t\geqq 0$. Addition of (8.5) and (8.7) leads to the following:

$$
(8.8)\qquad
\begin{aligned}
&\frac{1}{2}e^{\alpha t}\{||y_t(t)||^2-2\lambda||y_t(t)||\cdot||y_{xx}(t)||+\lambda||y_{xx}(t)||^2+(m-\alpha\lambda)||y_x(t)||^2\}\\
&+\lambda\int_0^t\Big(m-\frac{\alpha}{2}-\varepsilon\Big)e^{\alpha s}||y_{xx}(s)||^2ds+\int_0^t(1-\varepsilon-\lambda)e^{\alpha s}||y_{tx}(s)||^2ds\\
&+\frac{\alpha}{2}\int_0^t(\alpha\lambda-M)e^{\alpha s}||y_x(s)||^2ds-\frac{\alpha}{2}\int_0^t e^{\alpha s}||y_t(s)||^2ds\\
&\leqq\frac{1}{2}\{||y_1||^2+M||y_{0,x}||^2+\lambda(||y_{0,xx}||^2+2||y_1||\cdot||y_{0,xx}||)\}\\
&+\frac{1+\lambda}{4\varepsilon}\int_0^t e^{\alpha s}||f(s)||^2ds\\
&\leqq N_1\Big(1+\frac{1}{\varepsilon}\int_0^t e^{\alpha s}||f(s)||^2ds\Big),\quad t\geqq 0,
\end{aligned}
$$

where N_1 is a positive number independent of t. In what follows, we denote by N_i a positive number independent of t. In (8.8), put $\lambda=1/2$ and choose α such that

$$0<\alpha<\min\Big\{2\gamma,\ \frac{2m}{2M+1}\Big\}.$$

Then by taking a sufficiently small $\varepsilon>0$ we can show with the aid of Lemma 4.1

$$(8.9)\qquad e^{\alpha t}(||y_t(t)||^2+||y_{xx}(t)||^2)\leqq N_2,\quad t\geqq 0,$$

which implies (8.2) with $\beta=\alpha/2$. In particular, we have

$$(8.10)\qquad |y_x(t)|_\infty\leqq||y_{xx}(t)||\leqq N_3e^{-\beta t},\quad t\geqq 0,$$

for some N_3.

Next we shall prove that (8.2), or (8.9), holds with the L^2-norm $||\cdot||$ replaced by the maximum norm $|\cdot|_\infty$. As a map from $[0,\infty)$ to X_0, y satisfies

Now choose $\delta>0$ such that $\delta\leqq\rho$ and $\delta<\gamma$. Then it follows from (8.14) that

$$e^{\delta t}|U(t)|_\infty\leqq Ke^{(\delta-\rho)t}|U_0|_\infty+KN_4\int_0^t e^{(\delta-\rho)(t-s)}e^{(\delta-\gamma)s}ds$$
$$+KM_1(N_3)N_3\int_0^t e^{(\delta-\rho)(t-s)}e^{-\beta s}e^{\delta s}|U(s)|_\infty ds\,,$$

which yields with the aid of Gronwall's inequality

$$e^{\delta t}|U(t)|_\infty\leqq K\{|U_0|_\infty+N_4(\gamma-\delta)^{-1}\}\exp(KM_1(N_3)N_3\beta^{-1}) \tag{8.15}$$

for every $t\geqq 0$.

Next we shall estimate $U_t(t)$. Since $U(t)\in D(A_1)=D(A)\times X_0$ for $t>0$, we have

$$\begin{aligned}U_t(t)=&\ T_1(t-1)\{A_1U(1)+C_1(U(1))+F(1)\}\\ &+\int_1^t T_1(t-s)\left\{\frac{d}{ds}C_1(U(s))+F_t(s)\right\}ds,\quad t\geqq 1\,,\end{aligned} \tag{8.16}$$

(cf. (6.7)). The first component of $\dfrac{d}{dt}C_1(U(t))$ is estimated by

$$\begin{aligned}&\left|\frac{\partial}{\partial t}[\{\sigma(w(x,t))-\sigma(0)\}v(x,t)]\right|\\ &\leqq|\sigma'(w(x,t))w_t(x,t)v(x,t)|+|\{\sigma(w(x,t)-\sigma(0)\}v_t(x,t)|\\ &\leqq M_1(|w(t)|_\infty)\{|w_t(x,t)|\cdot|v(x,t)|+|w(x,t)|\cdot|v_t(x,t)|\}\,.\end{aligned}$$

By Lemma 4.1 and (6.3)

$$|w(t)|_\infty\leqq\|v(t)\|\leqq|v(t)|_\infty\quad\text{and}\quad|w_t(t)|_\infty\leqq|v_t(t)|_\infty\,,$$

so that we see, by making use of (8.10) and (8.15),

$$\left|\frac{d}{dt}C_1(U(t))\right|_\infty\leqq 2M_1(N_3)N_5e^{-\delta t}|U_t(t)|_\infty\,, \tag{8.17}$$

for some N_5. Therefore, it follows from (8.16), together with Lemma 8.1 and (8.17), that

$$\begin{aligned}|U_t(t)|_\infty\leqq&\ Ke^{-\rho(t-1)}(|A_1U(1)|_\infty+|C_1(U(1))|_\infty+|F(1)|_\infty)\\ &+K\int_1^t e^{-\rho(t-s)}(2M_1(N_3)N_5e^{-\delta s}|U_t(s)|_\infty+N_6e^{-\gamma s})ds,\quad t\geqq 1\,,\end{aligned} \tag{8.18}$$

for some $N_6>0$ (cf. (8.14)). From (8.18) we can show, in the same way as (8.15),

$$|U_t(t)|_\infty\leqq N_7e^{-\delta t}\qquad\text{for}\ \ t\geqq 1\,, \tag{8.19}$$

with some N_7. Thus, from (8.15) and (8.19) we get the assertion of Theorem 4 (use Lemma 4.1). [q.e.d.]

Finally we shall prove Lemma 8.1.

Proof of Lemma 8.1. For $U_0={}^t(u_0, v_0)\in X_0\times X_0$, put $U(t)\equiv{}^t(u(t), v(t)) = \boldsymbol{T}_1(t)U_0$. Define $y_0=A^{-1}v_0$, $y_1=u_0$ and

$$y(t) = y_0+\int_0^t u(s)ds\,.$$

Then we can show, as in §3, that y satisfies

$$\begin{cases} y_{tt}-Ay_t-\sigma(0)Ay = 0 \quad t>0\,, \\ y(0)=y_0,\ y_t(0)=y_1\,. \end{cases} \tag{8.20}$$

Therefore, applying the preceding arguments in this section to (8.20), we see that there exist some constants $K_1>0$ and $0<\rho<\sigma(0)$ (which are independent of U_0) such that

$$||y_t(t)||^2+||Ay(t)||^2\leqq K_1e^{-2\rho t}(||y_t(1)||^2+||Ay(1)||^2),\quad t\geqq 1\,, \tag{8.21}$$

(see (8.8) and (8.9)). Recall that $\boldsymbol{T}_1(t)$ is an analytic semigroup, so that

$$U_{tt}(t)-\boldsymbol{A}_1U_t(t)=0\,,\quad t>0\,,$$

and

$$U_{ttt}(t)-\boldsymbol{A}_1U_{tt}(t)=0,\quad t>0\,,$$

which imply

$$y_{ttt}(t)-Ay_{tt}(t)-\sigma(0)Ay_t(t)=0,\qquad t>0\,, \tag{8.22}$$

and

$$y_{tttt}(t)-Ay_{ttt}(t)-\sigma(0)Ay_{tt}(t)=0,\quad t>0\,, \tag{8.23}$$

respectively. Hence, using (8.21) we can show from (8.22) and (8.23)

$$||y_{tt}(t)||^2+||Ay_t(t)||^2\leqq K_1e^{-2\rho t}(||y_{tt}(1)||^2+||Ay_t(1)||^2),\quad t\geqq 1\,, \tag{8.24}$$

and

$$||y_{ttt}(t)||^2+||Ay_{tt}(t)||^2\leqq K_1e^{-2\rho t}(||y_{ttt}(1)||^2+||Ay_{tt}(1)||^2),\quad t\geqq 1\,. \tag{8.25}$$

Noting Lemma 4.1 we have from (8.24)

$$\begin{aligned} |u(t)|_\infty^2 = |y_t(t)|_\infty^2\leqq ||Ay_t(t)||^2 &\leqq K_1e^{-2\rho t}(||y_{tt}(1)||^2+||Ay_t(1)||^2) \\ &\leqq K_1e^{-2\rho t}|U_t(1)|_\infty^2,\quad t\geqq 1\,, \end{aligned} \tag{8.26}$$

and also from (8.25)

$$
\begin{aligned}
(8.27)\qquad |y_{tt}(t)|_\infty^2 &\leqq \|Ay_{tt}(t)\|^2 \leqq K_1 e^{-2\rho t}(\|y_{ttt}(1)\|^2+\|Ay_{tt}(1)\|^2)\\
&\leqq K_1 e^{-2\rho t}|U_{tt}(1)|_\infty^2, \quad t\geqq 1\,.
\end{aligned}
$$

On the other hand, since

$$\frac{d}{dt}\{e^{\sigma(0)t}Ay(t)\} = e^{\sigma(0)t}y_{tt}\,,$$

we get with the use of (8.27)

$$
\begin{aligned}
e^{\sigma(0)t}|Ay(t)|_\infty &\leqq e^{\sigma(0)}|Ay(1)|_\infty+\int_1^t e^{\sigma(0)s}|y_{tt}(s)|_\infty ds\\
&\leqq e^{\sigma(0)}|U(1)|_\infty + K_1^{1/2}|U_{tt}(1)|_\infty \int_1^t e^{(\sigma(0)-\rho)s}ds\,,
\end{aligned}
$$

which implies

$$(8.28)\qquad |v(t)|_\infty = |Ay(t)|_\infty \leqq K_2 e^{-\rho t}(|U(1)|_\infty+|U_{tt}(1)|_\infty), \quad t\geqq 1\,,$$

for some $K_2>0$ (note $\sigma(0)>\rho$). Since $\frac{d}{dt}\boldsymbol{T}_1(t)$ and $\frac{d^2}{dt^2}\boldsymbol{T}_1(t)$ are bounded operators for $t>0$ (see e.g. Krein [5, chap. 1, §3] or Tanabe [7, chap. 3, §3]), we get the conclusion from (8.26) and (8.28). [q.e.d.]

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