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DEFORMATIONS AND LOCAL TORELLI THEOREM FOR CERTAIN SURFACES OF GENERAL TYPE

by Sampei USUI

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Introduction.

After the systematic investigation by Griffiths on the period maps [3], several researches have been made on the problem of their injectivity (Torelli-type problem) and of their surjectivity (for K3 surfaces, cf. [17], [10]; for Enriques surfaces, cf. [11]; for surfaces of general type, cf. II in [3], [15], [16], [20], [21], [12] etc.).

The purpose of this paper is to show the following: Let X' be a smooth projective surface of general type obtained by the normalization of a hypersurface X in a projective 3-space P only with ordinary singularities. Let D be the singular locus of X with reduced structure and let n be the degree of X in P. Then the period map is unramified at the origin of the parameter space of the Kuranishi family of the deformations of X' in one of the following cases:

(1) D is a complete intersection in P. (In this case we have few exceptions. For detail, see the theorem (3.5).)

(2) n is sufficiently large enough comparing to D. (See the theorem (5.8).)

The result in case (1) contains some examples of minimal surfaces of general type with non-ample canonical divisor, for which the local Torelli theorem holds. The result in case (2) gives some evidence that if there would be sufficiently many 2-forms on a given surface, their periods of integrals should determine the surface itself (cf. the remark (5.9)).

We recall here the definition of a surface X in a projective 3-space P only with ordinary singularities: Taking a suitable local coordinate system (x, y, z) at each point in P, the local equation of X in P is one of the following forms:

(i) 1,

(ii) z,

(iii) yz,

(iv) xyz,

(v) $xy^2 - z^2$.

These surfaces are attractive because every smooth projective surface can be obtained as the normalization of such a surface X. More precisely, in characteristic 0, via generic projection, every smooth projective surface can be projected onto such a surface X. These surfaces, especially their deformations, are intensively studed by Kodaira in [13] and, when their singular loci are smooth curves of complete intersections in the ambiant space, by Horikawa in [9] and by Tsuboi in [19].

We also recall that, given a smooth projective surface Y, the morphism

$$H^1(T_Y) \longrightarrow H^1(\Omega_Y^1) \otimes H^0(\Omega_Y^2)^\vee$$

induced from the contraction $T_Y \otimes \Omega_Y^2 \xrightarrow{\sim} \Omega_Y^1$ is called the infinitesimal period map at Y in the second cohomology (for the background, cf. [3], [20]).

This work was started on the joint research with Professor S. Tsuboi at the Research Institute for Mathematical Sciences in Kyoto previous year. The author expresses his hearty thanks to Professor S. Tsuboi and Professor K. Miyajima and the other professors at Kagoshima University who received him warmly in the previous summer.

March 10, 1978.

Notatios and conventions.

The category, which we treat, is schemes over the field $\mathbb{C}$ of complex numbers.

$h^i(F)=\dim_{\mathbb{C}} H^i(F)$ for a coherent $\underline{O}_X$-module F.

$\check{F}=\underline{\mathrm{Hom}}_{\underline{O}_X}(F, \underline{O}_X)$ for a coherent $\underline{O}_X$-module F.

$\check{V}=\mathrm{Hom}_{\mathbb{C}}(V, C)$ for a $\mathbb{C}$-vector space V.

$|L|$ denotes the complete linear system associated to an invertible $\underline{O}_X$-module L.

$\mathrm{Bs}|L|$ denotes the set of the base points of $|L|$.

S_a and M_a denote the set of homogeneous elements of degree a of a graded ring S and that of a graded module M respectively. $(a\in\mathbb{Z})$.

Ω_f denotes the sheaf of relative Kähler differentials for a morphism f of schemes.

$\Omega_X=\Omega_f$, where $f : X \longrightarrow \mathrm{Spec}\,\mathbb{C}$ is the structure morphism.

ω_X denotes the dualizing $\underline{O}_X$-module of a scheme X.

$$C(a)=\begin{cases} \dfrac{(a+3)(a+2)(a+1)}{6} & \text{if } a \text{ is a non-negative integer,} \\ 0 & \text{if } a \text{ is a negative integer.} \end{cases}$$

1. preliminaries.

In this section we summarize the preparatory results for the later use.

(1.1) Let P denote the projective 3-space. Let X be a hypersurface in P only with ordinary singularities, D be its double curve, that is, $D = \mathrm{Sing}(X)$ and T be the triple points of X, namely, $T = \mathrm{Sing}(D)$. Let $f : X' \longrightarrow X$ be the normalization. We set $D' = f^{-1}(D)$, $g = \mathrm{res}(f) : D' \longrightarrow D$, $T' = f^{-1}(T)$ and n, d and t being the degrees of X, of D and of T respectively.

(1.2) We also use the following notations.

$\omega_X, \omega_D, \omega_{X'}$: the dualizing sheaves of X, of D and of X' respectively. Note that $\omega_X = \underline{\mathrm{Ext}}^1_{\underline{O}_P}(\underline{O}_X, \omega_P) \simeq \underline{O}_X(n-4)$.

$\underline{O}_X(a-bD) = \mathrm{Im}\{\underline{O}_X(a) \otimes \mathcal{J}^b_D \longrightarrow \underline{O}_X(a)\}$ (a, $b \in \mathbf{Z}$ and $b > 0$), where $\mathcal{J}_D$ denotes the $\underline{O}_P$-ideal of D.

$\underline{O}_{X'}(a) = f^*\underline{O}_X(a)$ $(a \in \mathbf{Z})$.

$\underline{O}_{D'}(a) = g^*\underline{O}_D(a)$ $(a \in \mathbf{Z})$.

$\underline{O}_{X'}(a-bD') = \underline{O}_{X'}(a) \otimes \underline{O}_{X'}(-D)^{\otimes b}$ (a, $b \in \mathbf{Z}$).

The following lemma can be found in [18].

<u>Lemma</u> (1.3) (Roberts).

(1.3.1) D is locally Cohen-Macauley and of pure codimension 1 in X.

(1.3.2) $0 \longrightarrow \underline{O}_X \longrightarrow f_*\underline{O}_{X'} \longrightarrow \omega_D \otimes \check{\omega}_X \longrightarrow 0$ exact.

<u>Lemma</u> (1.4) (Kodaira).

(1.4.1) $\omega_{X'} \simeq \underline{O}_{X'}((n-4)-D')$.

(1.4.2) $f_*\underline{O}_{X'}(a-bD') \simeq \underline{O}_X(a-bD)$ (a, $b \in \mathbf{Z}$ and $b > 0$).

<u>Proof</u>. (1.4.1) is just the adjunction formula. (1.4.2) is obtained by a direct computation by using the local coordinates mentioned in the introduction. More precisely see Kodaira [13]. QED.

(1.5) Let $\mathcal{N}_{X/P}$ be the coherent $\underline{O}_{X'}$-module introduced by Horikawa (in [9], his notation is $\mathcal{T}_{X/P}$), which is defined for making the following sequence exact:

(1.5.1) $0 \longrightarrow T_{X'} \longrightarrow f^*(T_P \otimes \underline{O}_X) \longrightarrow \mathcal{N}_{X/P} \longrightarrow 0$.

Let $\mathcal{J}_X$ be the $\underline{O}_P$-ideal of X. Since $\mathcal{J}_X/\mathcal{J}_X^2 \simeq \underline{O}_X(-n)$, we have the exact sequence

(1.5.2) $$0 \longrightarrow \underline{O}_X(-n) \longrightarrow \Omega_P \otimes \underline{O}_X \longrightarrow \Omega_X \longrightarrow 0.$$

Dualizing (1.5.2), we get the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_P \otimes \underline{O}_X \longrightarrow \underline{O}_X(n) \xrightarrow{\alpha} \underline{\mathrm{Ext}}^1_{\underline{O}_X}(\Omega_X, \underline{O}_X) \longrightarrow 0,$$

where we denote $\underline{\mathrm{Hom}}_{\underline{O}_X}(\Omega_X, \underline{O}_X)$ by T_X. We define $\mathcal{N}_{X/P} = \mathrm{Im}\{T_P \otimes \underline{O}_X \xrightarrow{\alpha} \underline{O}_X(n)\}$, which is nothing but the coherent $\underline{O}_X$-module introduced by Kodaira (in [13], he uses the notation Ψ). Similarly we denote $\underline{\mathrm{Hom}}_{\underline{O}_D}(\Omega_D, \underline{O}_D)$ by T_D and $\mathrm{Coker}\{T_D \longrightarrow T_P \otimes \underline{O}_D\}$ by $\mathcal{N}_{D/P}$. $\mathcal{N}_{D/P}$ is just the sheaf N in Kodaira [13]. In case that D is smooth, we also use the notation $N_{D/P}$ for $\mathcal{N}_{D/P}$.

The following results can be found in [9] and in [13].

<u>Lemma</u> (1.6).

(1.6.1) $f_*\mathcal{N}_{X/P} \xleftarrow{\sim} \mathcal{N}_{X/P}$ (Horikawa).

(1.6.2) $0 \longrightarrow \underline{O}_X(n-2D) \longrightarrow \mathcal{N}_{X/P} \longrightarrow \mathcal{N}_{D/P} \longrightarrow 0$ exact (Kodaira).

The following formulae are calculated in [22] (cf. also [23]):

<u>Lemma</u> (1.7). X' has the following numerical characters:

(1.7.1) $c_1^2 = n(n-4)^2 - (5n-24)d - 4\chi(\underline{O}_D) + t.$

(1.7.2) $c_2 = n(n^2-4n-6) - (7n-24)d - 8\chi(\underline{O}_D) - t.$

(1.8) We summarize here the results concerning the spectral sequence of the Koszul complex introduced by Lieberman-Wilsker-Peters in [14].

Let Y be a complete smooth scheme. Let M be an invertible $\underline{O}_X$-module, V be a subspace of $H^0(M)$ and E be a locally free $\underline{O}_Y$-module. Choose a basis $f_1, \ldots, f_m$ of V and let $e_1, \ldots, e_m$ be the dual basis. For the triple (M, V, E) we denote by $K^{\cdot}(M, V, E)$ the Koszul complex consisting of the $\underline{O}_Y$-modules

$$K^p(M, V, E) = (E \underset{\underline{O}_Y}{\otimes} M^{\otimes p}) \underset{\mathbb{C}}{\otimes} \overset{p}{\wedge} V$$

together with the coboundary maps defined by

$$d(x) = \sum_{i_1 < \ldots < i_{p+1}} (\sum_j (-1)^j f_{i_j} x_{i_1 \ldots \hat{i}_j \ldots i_{p+1}}) e_{i_1} \wedge \cdots \wedge e_{i_{p+1}}$$

for $x = \sum_{i_1 < \ldots < i_p} x_{i_1 \ldots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p} \in K^p(M, V, E).$

The E_2 terms of the spectral sequence of the hypercohomology of this complex

$K^{\bullet}(M, V, E)$ with respect to the first filtlation are given by

$$'E_2^{p,q}(M, V, E) = \frac{\operatorname{Ker}\{H^q(E \otimes M^{\otimes p}) \otimes \overset{p}{\wedge} \check{V} \longrightarrow H^q(E \otimes M^{\otimes(p-1)}) \otimes \overset{p+1}{\wedge} \check{V}\}}{\operatorname{Im}\{H^q(E \otimes M^{\otimes(p-1)}) \otimes \overset{p-1}{\wedge} \check{V} \longrightarrow H^q(E \otimes M^{\otimes p}) \otimes \overset{p}{\wedge} \check{V}\}}.$$

Since $'E_2^{p,q}(M, V, E) = 0$ $(p < 0$ or $q < 0)$ by definition, we have the following well-known exact sequence:

$$(1.8.1) \qquad 0 \longrightarrow E_2^{1,0} \longrightarrow \mathbb{H}^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2.0} \longrightarrow \mathbb{H}^2.$$

Lemma (1.9) (Lieberman-Wilsker-Peters). If $\operatorname{codim}_Y \operatorname{Bs}|M| \geqslant 2$, then we have $\mathbb{H}^1(M, V, E) = 0$.

Proof. This lemma is proved easily by observing the E_2 terms of the spectral sequence with respect to the second filtration (for detail, cf. [14]). QED.

We conclude this section by adding one more lemma which is easy to prove and is rather useful.

Lemma (1.10). Let p be an integer with $p \neq m$. If $h^0(E \otimes M^{\otimes p}) \leqslant 1$, then we Have $'E_2^{p,0}(M, V, E) = 0$.

Proof. In case $h^0(E \otimes M^{\otimes p}) = 0$, it is trivial. We assume $h^0(E \otimes M^{\otimes p}) = 1$. Let t be a basis of $H^0(E \otimes M^{\otimes p})$ and let ρ be the map making the following diagram commutative:

$$\begin{tikzcd}
H^0(E \otimes M^{\otimes p}) \otimes \overset{p}{\wedge} \check{V} \arrow[r, "d"] & H^0(E \otimes M^{\otimes(p+1)}) \otimes \overset{p+1}{\wedge} \check{V}; \\
\overset{p}{\wedge} \check{V} \arrow[u, "t\otimes", "\wr"'] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

Explicitly ρ is given by

$$\rho(y) = \sum_{i_1 < \ldots < i_{p+1}} (\sum_j (-1)^j y_{i_1 \ldots \hat{i}_j \ldots i_{p+1}} t \cdot f_{i_j}) e_{i_1} \wedge \cdots \wedge e_{i_{p+1}} \qquad (y_{i_1 \ldots i_p} \in \mathbb{C}).$$

Since $M \xrightarrow{t\otimes} E \otimes M^{\otimes(p+1)}$ is injective, $H^0(M) \xrightarrow{t\cdot} H^0(E \otimes M^{\otimes(p+1)})$ is injective, and hence $t \cdot f_i$ $(1 \leqslant i \leqslant m)$ are linearly independent in $H^0(E \otimes M^{\otimes(p+1)})$. Therefore $\rho(y) = 0$ implies $y = 0$. This completes the proof. QED.

2. deformations.

In this section we study the small deformations of the surface X' in (1.1) when D is a smooth curve of complete intersection in P.1)

(2.1) If D is a curve of complete intersection in P, then the homogeneous equation of the hypersurface X in P is the following form:

$$AF^2 + 2BFG + CG^2,$$

where F, G, A, B and C are also homogeneous polynomials in $\mathbf{C}[X_0, \dots, X_3]$. In this case D is given by $F = G = 0$.

Let I be the homogeneous $\mathbf{C}[X_0, \dots, X_3]$- ideal generated by F and G. We set $n_1 = \deg F$ and $n_2 = \deg G$. We may assume $n_1 \geqslant n_2$ because of symmetry.

Note that $N_{D/P} \simeq \underline{O}_D(n_1) \oplus \underline{O}_D(n_2)$ and that $\omega_D \simeq \underline{O}_D(n_1+n_2-4)$.

From now on, except explicitly indicating the contrary, we assume that the surface X' is the normalization of a surface X just mentioned above.

(2.2) Let $P' \longrightarrow P$ be the blowing-up of P along D and let E' be the exceptional divisor. Then we have the following diagram:

$$\begin{tikzcd}
P' \arrow[d] & \supset & X' \arrow[d, "f"'] & \supset & D' \arrow[d, "g"'] & \subset & E' \arrow[dll, "\tilde{g}"] \\
P & \supset & X & \supset & D.
\end{tikzcd}$$

Since D is smooth, $\tilde{g} : E' = \operatorname{Proj}(\check{N}_{D/P}) \longrightarrow D$ is a $\mathbf{P}^1$-bundle. We denote by $L = \underline{O}_{P'}(-E') \otimes \underline{O}_{E'}$ the tautological invertible sheaf of the $\mathbf{P}^1$-bundle $\tilde{g}$. Note that $\tilde{g}_* L^{\otimes a} = S^a(\check{N}_{D/P})$ $(a \in \mathbf{Z})$, where $S^a(\ )$ denotes the a-th symmetric tensor product.

<u>Lemma</u> (2.3). The surface X' has the following numerical characters.

$$q = 0.$$

$$p_g = C(n-n_1-4) + C(n-n_2-4) - C(n-n_1-n_2-4).$$

$$c_1^2 = n(n-4)^2 + n_1 n_2\{2(n_1+n_2) + 16 - 5n\}.$$

We obtain the following table:

n	3	4	4	5	5	5	6	6	7	otherwise
n_1	1	1	2	1	2	2	2	3	3	
n_2	1	1	1	1	1	2	2	2	3	
p_g	0			2	1	0	2	1	2	$4 \leqslant$
c_1^2	+	0	+	0	-1	+	0			+
X'	not general type									general type

.

<u>Proof</u>. From (1.3.2), we have an exact sequence

$$H^1(\underline{O}_X) \longrightarrow H^1(f_*\underline{O}_{X'}) \longrightarrow H^1(\omega_D \otimes \check{\omega}_X) \xrightarrow{\beta} H^2(\underline{O}_X).$$

Since $H^1(\underline{O}_X)=0$ and the dual map of β is surjective, we see that $H^1(f_*\underline{O}_{X'})=0$, that is, $q=0$. By (1.4.1) and (1.4.2), we obtain that

$$H^0(\omega_{X'}) \overset{\sim}{\leftarrow} H^0(\underline{O}_X((n-4)-D) \simeq I_{n-4}.$$

From this, we get the formula for p_g. The formula for c_1^2 is the direct consequence of (1.7.1). As for the last line of the table we will check them case by case. In case $n=2n_1$, it is easy to see that $\omega_{X'} \simeq \underline{O}_{X'}(n-4-n_2)$ and hence X' is of general type if and only if $n-4-n_2>0$. In case $(n, n_1, n_2) = (3, 1, 1)$, $(4, 1, 1)$ or $(5, 2, 2)$, the direct calculation shows that the 2-ple genus of X' is zero and hence X' is not of general type by the criterion of Kodaira. In case $(n, n_1, n_2)=(5, 1, 1)$, $(6, 2, 2)$ or $(7, 3, 3)$, we have, by (1.4.1), $\omega_{X'} \simeq \underline{O}_{X'}(n_1-D')$ and hence $D_1'=f^{-1}(F=0) - D'$ and $D_2'=f^{-1}(G=0) - D'$ form a basis of the complete linear system $|\omega_{X'}|$. If there would exist an exceptional curve of the first kind on X', say C', $(C'\cdot D_1')=(C'\cdot D_2')=-1$ and hence $C' \subset D_1' \cap D_2'$. This is a contradiction, since it is easy to see that $D_1' \cap D_2' \neq \phi$ imposes the existence of a point x in P satisfying $A(x)=B(x)=C(x)=F(x) = G(x)=0$ and hence the existence of a point on X with its multiplicity $\geqslant 3$. Thus, in the considering case, X' is relatively minimal and its $c_1^2=0$ whence X' is not of general type by the criterion of Kodaira. In case $(n, n_1, n_2) = (5, 2, 1)$, $|\omega_{X'}|$ has only one element, namely, $D_2'=f^{-1}(G=0) - D'$, which is isomorphic to the line $A=G=0$ in P via res(f) and hence D_2' is the exceptional curve of the first kind. Contracting D_2', we get the relatively minimal model whose canonical invertible sheaf is trivial. The last assertion in the table, corresponding to "otherwise", can be easily verified. QED.

<u>Lemma</u> (2.4).

(2.4.1) $H^1(\underline{O}_{X'}(a-bD'))=0$ $(a \in \mathbb{Z},\ b=1$ or $2)$.

(2.4.2) $H^2(\underline{O}_{X'}(a-2D'))=0$ if and only if $a>\max\{n-4,\ 2n_1+n_2-4\}$.

<u>Proof</u>. We use the following exact sequences:

$$0 \longrightarrow \underline{O}_X(a-2D) \longrightarrow \underline{O}_X(a-D) \longrightarrow \check{N}_{D/P} \otimes \underline{O}_D(a) \longrightarrow 0.$$

$$0 \longrightarrow \underline{O}_X(a-D) \longrightarrow \underline{O}_X(a) \longrightarrow \underline{O}_D(a) \longrightarrow 0.$$

Since $H^0(\underline{O}_X(a-D)) \longrightarrow H^0(\check{N}_{D/P} \otimes \underline{O}_D(a))$ and $H^0(\underline{O}_X(a)) \longrightarrow H^0(\underline{O}_D(a))$ are surjective and since $H^1(\underline{O}_X(a)) = 0$, we obtain (2.4.1). By (2.4.1), we get that

$$\begin{aligned} h^2(\underline{O}_X(a-2D)) &= h^2(\underline{O}_X(a)) + h^1(\underline{O}_D(a)) + h^1(\check{N}_{D/P} \otimes \underline{O}_D(a)) \\ &= h^0(\underline{O}_X(n-4-a)) + h^0(\underline{O}_D(n_1+n_2-4-a)) + h^0(\underline{O}_D(2n_1+n_2-4-a)) \\ &\quad + h^0(\underline{O}_X(n_1+2n_2-4-a)). \end{aligned}$$

This proves (2.4.2). QED.

Lemma (2.5). Let δ be the connecting homomorphism

$$\delta : H^0(\mathcal{N}_{X'/P}) \longrightarrow H^1(T_{X'})$$

obtained from the exact sequence (1.5.1). We have the following table:

(2.5.1)

n	5	6	7	5	6	6	7	7	8	8	8	otherwise
n_1	1	2	3	2	3	2	3	3	4	4	4	
n_2	1	2	3	1	2	1	1	2	1	2	3	
$h^1(f^*(T_P \otimes \underline{O}_X))$	2			1								0
$h^1(\mathcal{N}_{X'/P})$	0								1			
δ	not surjective								surjective			

Proof. By the duality theorem and by (1.4), we get that

$$h^1(f^*(T_P \otimes \underline{O}_X)) = h^1(f^*(\Omega_P \otimes \underline{O}_X) \otimes \omega_{X'}) = h^1(\Omega_P \otimes \underline{O}_X((n-4)-D)).$$

Now we use the following exact sequence:

$$0 \longrightarrow \Omega_P \otimes \underline{O}_X((n-4)-D) \longrightarrow \underline{O}_X((n-5)-D)^{\oplus 4} \longrightarrow \underline{O}_X((n-4)-D) \longrightarrow 0.$$

By using (2.4.1), we get that

$$H^1(\Omega_P \otimes \underline{O}_X((n-4)-D)) \xleftarrow{\sim} \operatorname{Coker}\{I_{n-5}^{\oplus 4} \longrightarrow I_{n-4}\}.$$

From this, we can fill up the table concerning $h^1(f^*(T_P \otimes \underline{O}_X))$, by an elementary calculation. By (1.6) and by (2.4), we have, in the cases in the table (2.5.1) that

$$H^1(\mathcal{N}_{X'/P}) \xleftarrow{\sim} H^1(\mathcal{N}_{X/P}) \xrightarrow{\sim} H^1(N_{D/P}).$$

From this we can complete the table concerning $h^1(\mathcal{N}_{X'/P})$. As for the surjectivity of δ, Horikawa proved it in case $n = 2n_1$ and $n - n_2 - 4 > 0$ in [9]. QED.

(2.6) Let R denote the localization of $\mathbf{C}[T_1, \ldots, T_m]$ by the maximal ideal $\mathfrak{m}=(T_1, \dots , T_m)$, where

$$m = C(n_1)+C(n_2)+C(n-2n_1)+C(n-n_1-n_2)+C(n-2n_2).$$

Put $S = \operatorname{Spec} R$. Let $\tilde{F}$, $\tilde{G}$, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be the first order perturbations of F, of G, of A, of B and of C respectively, namely, $\tilde{F}=F+F_1$, where $F_1 = \sum_{1\leq i\leq C(n_1)} M_iT_i$ ($M_1, \dots , M_{C(n_1)}$ are the monomials of degree n_1 in $\mathbf{C}[X_0, \dots , X_3]$) etc. Let $\mathfrak{X}$ and $\mathfrak{D}$ be the subschemes in $P\times S$ defined by $\tilde{A}\tilde{F}^2+2\tilde{B}\tilde{F}\tilde{G}+\tilde{C}\tilde{G}^2=0$ and by $\tilde{F}=\tilde{G}=0$ respectively and let $\mathfrak{X}'$ be the blowing-up of $\mathfrak{X}$ along $\mathfrak{D}$ (or equivalently the normalization of $\mathfrak{X}$). Then we have the natural morphism:

(2.6.1) $\qquad \mathfrak{X}' \longrightarrow S.$

Theorem (2.7). Let X' be a surface of general type which is the normalizatio of such a hypersurface X as in (2.1). Except the cases

(2.7.1)

n	6	7	7
n_1	2	3	3
n_2	1	1	2

,

(2.6.1) gives a complete family of the deformations of X'. In particular, except the cases in the table (2.7.1), the parameter space of the Kuranishi family of the deformations of X' is smooth at the origin.[2]

Proof. The map $\tau : T_S\otimes k(\mathfrak{m}) \longrightarrow H^0(\mathcal{N}_{X/P})$ is given by

$$\tilde{A}\tilde{F}^2+2\tilde{B}\tilde{F}\tilde{G}+\tilde{C}\tilde{G}^2 \quad \mod \mathfrak{m}^2,$$

that is, for $s\in T_S\otimes k(\mathfrak{m})\simeq \mathbf{A}^m$,

$\tau(s) = (A_1F^2+2B_1FG+C_1G^2+2AFF_1+2BF_1G+2BFG_1+2CGG_1)\otimes k(\mathfrak{m}) \mod (AF^2+2BFG+CG^2)$,

and it is easily verified that τ is surjective. Tensoring non-zero $\omega\in H^0(\omega_{X'})$ gives an injective morphism $T_{X'} \xrightarrow{\otimes\,\omega} T_{X'}\otimes\omega_{X'}\simeq\Omega_{X'}$, and hence an injection $H^0(T_{X'}) \longrightarrow H^0(\Omega_{X'})$. Since $q=0$ by (2.3), we see $H^0(T_{X'})=H^0(\Omega_{X'})=0$. The other assertions in the theorem are the consequences of (2.5) and of $H^0(T_{X'})=0$ by the general theory of deformations. QED.

Remark (2.8). The argument in this section also holds in the case that D is globally Cohen-Macauley, that is,

$$H^0(\underline{O}_D(a)) \simeq (\mathbb{C}[X_0, \dots, X_3]/I)_a \qquad (a \in \mathbb{Z}),$$

where $I = \bigoplus_{a\in\mathbb{Z}} H^0(\mathcal{J}_D(a))$. Hence, in this case, the last statement of (2.7) is valid (several cases occur according to the degrees of the generators of I).

3. *local Torelli theorem*.

In this section we assume that X' is a surface of general type which is the normalization of such a surface X as in (2.1). We give a proof of the local Torelli theorem for such a surface X'.

Lemma (3.1).

(3.1.1) $|\underline{O}_{X'}(1)|$ is fixed points free.

(3.1.2) $|\underline{O}_{X'}(n_1-D')|$ is fixed components free. 3)

Proof. (3.1.1) is obvious. By (1.4.2), we have

$$H^0(\underline{O}_{X'}(n_1-D')) \cong H^0(\underline{O}_X(n_1-D)) \simeq I_{n_1}.$$

Put $D_1' = f^*(F=0) - D'$ and $D_2' = f^*(G=0) - D'$. Then

$$|\underline{O}_{X'}(n_1-D')| = \{C'+D_2' \mid C' = f^*(C),\ \ C \in |\underline{O}_X(n_1-n_2)|\} + \{D_1'\}.$$

Since $\bigcap_{C'} (C'+D_2') = D_2'$ and since $f^*(F=0) \cap f^*(G=0) = D'$, we get the assertion (3.1.2). QED.

Remark (3.2). In case $n = 2n_1$, it is easy to see that

$$\omega_{X'} \simeq \underline{O}_{X'}(n-n_2-4).$$

On the other hand, in case $n \neq 2n_1$, $\omega_{X'}$ is not a non-trivial power of an invertible sheaf.

The next lemma is the essential part in the proof of the local Torelli theorem. The proof of the lemma will be found in the next section.

Lemma (3.3).

(3.3.1) $$h^0(\Omega_{X'} \otimes \underline{O}_{X'}(1)) = \begin{cases} \leq 1 & \text{if } n = n_1+n_2+1, \\ 0 & \text{otherwise.} \end{cases}$$

(3.3.2) In case $n \neq 2n_1$, except the case $n = 2n_1+1$ and $n_2 = 1$, we have

$$H^0(T_{X'} \otimes \underline{O}_{X'}(2n_1-2D')) = 0.$$

Lemma (3.4). In case $n_1 > n_2+1$, the map

$$\mathrm{Im}(\delta) \longrightarrow H^1(T_{X'} \otimes \underline{O}_{X'}(n_2-D'))$$

is injective, where δ is the map in (2.5).

Proof. By (1.6.1), it is easy to see that

(3.4.1) $f_*(\mathcal{N}_{X/P}\otimes \underline{O}_{X'}(n_2-D')) \overset{\sim}{\leftarrow} \mathcal{N}_{X/P}\cdot\underline{O}_X(n_2-D)$,

where $\mathcal{N}_{X/P}\cdot\underline{O}_X(n_2-D) = \mathrm{Im}\{\mathcal{N}_{X/P}\otimes \underline{O}_X(n_2-D) \longrightarrow \mathcal{N}_{X/P}\otimes \underline{O}_X(n_2)\}$. By (1.5.1), (1.6.1) and (3.4.1), we have a commutative diagram:

$$\begin{tikzcd}
H^1(T_{X'}) \arrow[r] & H^1(T_{X'}\otimes \underline{O}_{X'}(n_2-D')) \\
\dfrac{H^0(\mathcal{N}_{X/P})}{\mathrm{Im}\,\{H^0(T_P\otimes \underline{O}_X) \longrightarrow H^0(\mathcal{N}_{X/P})\}} \arrow[u, "\begin{array}{l}\text{induced}\\ \text{from } \delta\end{array}"] \arrow[r, "\psi"] & \dfrac{H^0(\mathcal{N}_{X/P}\cdot\underline{O}_X(n_2-D))}{\mathrm{Im}\{H^0(T_P\otimes \underline{O}_X(n_2-D)) \longrightarrow H^0(\mathcal{N}_{X/P}\cdot\underline{O}_X(n_2-D))\}}. \arrow[u, "\gamma"']
\end{tikzcd}$$

Since γ is injective, it is enough to show that ψ is injective. The injectivity of ψ follows from the following assertion:

(3.4.2) Put $\Phi = AF^2 + 2BFG + CG^2$. If we assume that, for a given Ψ in $\mathbb{C}[X_0, \dots, X_3]_n$, there exist P_i in I_{n_2+1} $(0 \leq i \leq 3)$ satisfying $G\Psi \equiv \sum_{0\leq i\leq 3} P_i \frac{\partial\Phi}{\partial X_i} \bmod \Phi$, then there exist Q_i in $\mathbb{C}[X_0, \dots, X_3]_1$ $(0 \leq i \leq 3)$ such that $\Psi \equiv \sum_{0\leq i\leq 3} Q_i \frac{\partial\Phi}{\partial X_i} \bmod \Phi$.

The assertion (3.4.2) is trivially valid, because $I_{n_2+1} = G\mathbb{C}[X_0, \dots, X_3]_1$ by the assumption $n_1 > n_2 + 1$ and because G is a regular element modulo Φ. QED.

Theorem (3.5). If X' is a surface of general type which is the normalization of a hypersurface X defined in (2.1), then the local Torelli theorem holds for X' except the case $(n, n_1, n_2) = (7, 3, 1)$. As for this exception, still the map

$$\mathrm{Im}(\delta) \longrightarrow H^1(\Omega_{X'}) \otimes H^0(\omega_{X'})^{\vee}$$

is injective, where δ is the map in (2.5).

Proof. We use the notations in (1.8). By (3.3.1) and (1.10), we get

$$'E_2^{2,0}(\underline{O}_{X'}(1),\ H^0(\underline{O}_{X'}(1)),\ T_{X'}\otimes \underline{O}_{X'}(i-D')) = 0 \qquad (i < n-4),$$

and by (3.1.1) and (1.9), we have

$$\mathbb{H}^1(\underline{O}_{X'}(1),\ H^0(\underline{O}_{X'}(1)),\ T_{X'}\otimes \underline{O}_{X'}(i-D')) = 0 \qquad (\forall i),$$

and hence by (1.8.1) we see that

$$'E_1^{1,0}(\underline{O}_{X'}(1),\ H^0(\underline{O}_{X'}(1)),\ T_{X'}\otimes \underline{O}_{X'}(i-D')) = 0 \qquad (i < n-4).$$

Namely

$$(3.5.1) \qquad H^1(T_X,\otimes \underline{O}_X,(i-D')) \longrightarrow H^1(T_X,\otimes \underline{O}_X,(i+1-D'))\otimes H^0(\underline{O}_X,(1))^{\vee}$$

is injective $(i < n-4)$. In case $n \neq 2n_1$, except $n = 2n_1+1$ and $n_2 = 1$, a similar argument applied for $(M, V, E) = (\underline{O}_X,(n_1-D'), H^0(\underline{O}_X,(n_1-D')), T_X,)$ deduces, by using (3.3.2), (1.10), (3.1.2) and (1.9), that

$$(3.5.2) \qquad H^1(T_X,) \longrightarrow H^1(T_X,\otimes \underline{O}_X,(n_1-D'))\otimes H^0(\underline{O}_X,(n_1-D'))^{\vee}$$

is injective. In case $n = n_2+1$, the map

$$(3.5.3) \qquad \mathrm{Im}(\delta) \longrightarrow H^1(T_X,\otimes \underline{O}_X,(n_2-D'))\otimes H^0(\underline{O}_X,(n_2-D'))^{\vee}$$

is injective, which is an immediate consequence of (3.4), and, by (2.5), we see that $\mathrm{Im}(\delta) = H^1(T_X,)$ except the case $(n, n_1, n_2) = (7, 3, 1)$.

Combining the above results, we conclude the proof of the theorem as follows. In case $n = 2n_1$, by a successive use of (3.5.1) and by the remark (3.2), we have the following commutative diagram:

$$\begin{tikzcd}
H^1(T_X,) \arrow[r, hook] \arrow[dr, "\varphi"'] & H^1(T_X,\otimes \underline{O}_X,(n-n_2-4)) \otimes\{H^0(\underline{O}_X,(1))^{\vee}\}^{\otimes(n-n_2-4)} \\
& H^1(T_X,\otimes \underline{O}_X,(n-n_2-4))\otimes H^0(\underline{O}_X,(n-n_2-4))^{\vee}, \arrow[u]
\end{tikzcd}$$

where φ is the infinitesimal period map. In case $n \neq 2n_1$, except $n = 2n_1+1$ and $n_2 = 1$, by a successive use of (3.5.1) and by (3.5.2) we get the following diagram:

$$\begin{tikzcd}
H^1(T_X,) \arrow[r, hook] \arrow[dr, "\varphi"'] & H^1(T_X,\otimes \underline{O}_X,(n-4-D'))\otimes H^0(\underline{O}_X,(n_1-D'))^{\vee}\otimes \{H^0(\underline{O}_X,(1))^{\vee}\}^{\otimes(n-n_1-4)} \\
& H^1(T_X,\otimes \underline{O}_X,(n-4-D'))\otimes H^0(\underline{O}_X,(n-4-D'))^{\vee}. \arrow[u]
\end{tikzcd}$$

In case $n_1 = n_2+1$, by (3.5.3) and (3.5.1), we obtain

$$\begin{tikzcd}
\mathrm{Im}(\delta) \arrow[r, hook] \arrow[dr, "\varphi"'] & H^1(T_X,\otimes \underline{O}_X,(n-4-D'))\otimes H^0(\underline{O}_X,(n_2-D'))^{\vee}\otimes \{H^0(\underline{O}_X,(1))^{\vee}\}^{\otimes(n-n_2-4)} \\
& H^1(T_X,\otimes \underline{O}_X,(n-4-D'))\otimes H^0(\underline{O}_X,(n-4-D'))^{\vee}. \arrow[u]
\end{tikzcd}$$

Hence we get the injectivity of the infinitesimal period map φ in every case.

QED.

4. proof of the lemma (3.3).

We use the following well-known facts:

(4.1) $0 \longrightarrow \check{N}_{X/P'} \longrightarrow \Omega_{P'} \otimes \underline{O}_{X'} \longrightarrow \Omega_{X'} \longrightarrow 0$ exact.

(4.2) $0 \longrightarrow f^*(\Omega_P \otimes \underline{O}_X) \longrightarrow \Omega_{P'} \otimes \underline{O}_{X'} \longrightarrow \Omega_{\tilde{g}} \otimes \underline{O}_{D'} \longrightarrow 0$ exact.

(4.3) $\Omega_{\tilde{g}} \simeq L^{\otimes(-2)} \otimes \tilde{g}^*(\det \check{N}_{D/P})$.

By (4.1) and (4.2), we have the following diagram:

$$\begin{tikzcd}
H^0(\Omega_{\tilde{g}} \otimes \underline{O}_{D'}(1)) & & \\
H^0(\Omega_{P'} \otimes \underline{O}_{X'}(1)) \arrow[u] \arrow[r] & H^0(\Omega_{X'} \otimes \underline{O}_{X'}(1)) \arrow[r] & H^1(\check{N}_{X/P'} \otimes \underline{O}_{X'}(1)). \\
H^0(f^*(\Omega_P \otimes \underline{O}_X) \otimes \underline{O}_{X'}(1)) \arrow[u] & &
\end{tikzcd}$$

In order to prove (3.3.1), it is enough to show the following:

(4.4) $H^0(f^*(\Omega_P \otimes \underline{O}_X) \otimes \underline{O}_{X'}(1)) = 0.$

(4.5) $H^0(\Omega_{\tilde{g}} \otimes \underline{O}_{D'}(1)) = \begin{cases} \text{1-dimensional} & \text{if } n = 2n_1 \text{ and } n_1 = n_2 + 1, \\ 0 & \text{otherwise.} \end{cases}$

(4.6) $H^1(\check{N}_{X/P'} \otimes \underline{O}_{X'}(1)) = \begin{cases} \text{1-dimensional} & \text{if } n = 2n_1 + 1 \text{ and } n_1 = n_2, \\ 0 & \text{otherwise.} \end{cases}$

By (1.3.2), we have an exact sequence

$$0 \longrightarrow \Omega_P \otimes \underline{O}_X(1) \longrightarrow \Omega_P \otimes \underline{O}_X(1) \otimes f_*\underline{O}_{X'} \longrightarrow \Omega_P \otimes \underline{O}_D(1-n+n_1+n_2) \longrightarrow 0.$$

From this, (4.4) can be verified by an easy calculation.

Tensoring $\Omega_{\tilde{g}} \otimes \tilde{g}^*\underline{O}_D(1)$ to the exact sequence

(4.7) $0 \longrightarrow \underline{O}_{E'}(-D') \longrightarrow \underline{O}_{E'} \longrightarrow \underline{O}_{D'} \longrightarrow 0$

and taking the direct image, we get, by using the relative duality theorem, the exact sequence

$$0 \longrightarrow \tilde{g}_*(\Omega_{\tilde{g}} \otimes \underline{O}_{D'}(1)) \longrightarrow S^2(N_{D/P}) \otimes \underline{O}_D(1-n) \xrightarrow{\varepsilon} \underline{O}_D(1) \longrightarrow 0,$$

and hence we obtain

$$H^0(\Omega_{\tilde{g}} \otimes \underline{O}_{D'}(1)) \xrightarrow{\sim} \operatorname{Ker}\{H^0(\underline{O}_D(2n_1+1-n) \oplus \underline{O}_D(n_1+n_2+1-n) \oplus \underline{O}_D(2n_2+1-n) \xrightarrow{\varepsilon} H^0(\underline{O}_D(1))\}.$$

Note that the map ε, considered as a homomorphism of graded modules, is as follows:

$$\varepsilon : (Q_1, Q_2, Q_3) \longrightarrow AQ_1 + 2BQ_2 + CQ_3,$$

where A, B and C are the polynomials in (2.1). In case $n > 2n_1+1$, obviously $H^0(\Omega_{\tilde{g}} \otimes \underline{O}_{D'}(1)) = 0$. In case $n = 2n_1+1$, the map ε is injective and hence $H^0(\Omega_{\tilde{g}} \otimes \underline{O}_{D'}(1)) = 0$. In case $n = 2n_1$ and $n_1 > n_2+1$, the same assertion holds. In

(4.9) $H^0(\Omega_{\tilde{g}} \otimes \underline{O}_{X'}(2n_1+4-n-D')) = 0.$

(4.10) $H^1(\check{N}_{X/P'} \otimes \underline{O}_{X'}(2n_1+4-n-D')) = 0.$

From the exact sequence

$$0 \longrightarrow \Omega_P \otimes \underline{O}_X(2n_1+4-n-D) \longrightarrow \underline{O}_X(2n_1+3-n-D)^{\oplus 4} \longrightarrow \underline{O}_X(2n_1+4-n-D) \longrightarrow 0,$$

we have

$$0 \longrightarrow H^0(\Omega_P \otimes \underline{O}_X(2n_1+4-n-D)) \longrightarrow I^{\oplus 4}_{2n_1+3-n} \xrightarrow{\sigma} I_{2n_1+4-n}.$$

Since we assume that X' is of general type and $n \neq 2n_1$, the case $2n_1+3-n \geqslant n_1$ is decluded. Hence we have

$$I_{2n_1+3-n} = \mathbb{C}[X_0, \dots, X_3]_{2n_1+3-n-n_2}.$$

On the other hand, from the inequality $2n_1+3-n-n_2 \geqslant 0$, one of the following three cases occurs:

(4.11) $n = 2n_1+1$ and $n_2 = 1.$

(4.12) $n = 2n_1+1$ and $n_2 = 2.$

(4.13) $n = 2n_1+2$ and $n_2 = 1.$

In cases (4.12) and (4.13), the map σ is injective. In case (4.11), the map σ has 6-dimensional kernel consisting of the Koszul relations. Hence (4.8) is verified.

Tensoring $\Omega_{\tilde{g}} \otimes g^*\underline{O}_D(2n_1+4-n) \otimes L$ to the exact sequence (4.7) and taking the direct image, we obtain

$$g_*(\Omega_{\tilde{g}} \otimes \underline{O}_{X'}(2n_1+4-n-D')) \xrightarrow{\sim} R^1 g_*(\Omega_{\tilde{g}} \otimes L^{-1} \otimes g^*\underline{O}_D(2n_1+4-2n)) \simeq N_{D/P} \otimes \underline{O}_D(2n_1+4-2n).$$

Since X' is of general type with $n \neq 2n_1$, we can declude the case $3n_1+4-2n \geqslant 0$ and hence $H^0(N_{D/P} \otimes \underline{O}_D(2n_1+4-2n)) = 0$. This proves the assertion (4.9).

(4.10) follows from (2.4.1) by taking its dual. This completes the proof of (3.3.2).

5. appendix.

In this appendix, we prove that, in the situation in (1.1), the period map is unramified at the origin of the parameter space of the Kuranishi family of the deformations of X' provided that n is sufficiently large enough comparing to D. We use the notations at the end of the introduction and in the section 1.

case $n=2n_1$ and $n_1=n_2+1$, the map ε has the 1-dimensional kernel. Hence the assertion (4.5) is verified.

In order to prove (4.6), we may consider $H^1(\underline{O}_{X'},(2n-5-3D'))$ by the duality theorem. By the exact sequence

$$0 \longrightarrow \underline{O}_{X'}(2n-5-3D') \longrightarrow \underline{O}_{X'}(2n-5-2D') \longrightarrow \underline{O}_{X'}(2n-5-2D')\otimes \underline{O}_{D'} \longrightarrow 0,$$

and by (2.4.1), we get

$$H^1(\underline{O}_{X'}(2n-5-3D')) \xleftarrow{\sim} \mathrm{Coker}\{H^0(\underline{O}_{X'}(2n-5-2D')) \longrightarrow H^0(\underline{O}_{X'}(2n-5-2D')\otimes \underline{O}_{D'})\}.$$

Note that $\underline{O}_{X'}(2n-5-2D')\otimes \underline{O}_{D'} \simeq L^{\otimes 2}\otimes \underline{O}_{D'}(2n-5)$. Tensoring $L^{\otimes 2}\otimes \underline{O}_{D'}(2n-5)$ to the exact sequence (4.7), and taking the direct image, we get the exact sequence

$$0 \longrightarrow \underline{O}_D(n-5) \longrightarrow S^2(\check{N}_{D/P})\otimes \underline{O}_D(2n-5) \longrightarrow g_*(L^{\otimes 2}\otimes \underline{O}_{D'}(2n-5)) \longrightarrow 0.$$

Since $H^0(\underline{O}_X(2n-5-2D)) \longrightarrow H^0(S^2(\check{N}_{D/P})\otimes \underline{O}_D(2n-5))$ is surjective, we have

$$\begin{aligned} H^1(\underline{O}_{X'}(2n-5-3D)) &\xleftarrow{\sim} \mathrm{Coker}\{H^0(S^2(\check{N}_{D/P})\otimes \underline{O}_D(2n-5)) \longrightarrow H^0(g_*(L^{\otimes 2}\otimes \underline{O}_{D'}(2n-5)))\} \\ &\xrightarrow{\sim} \mathrm{Ker}\{H^1(\underline{O}_D(n-5)) \longrightarrow H^1(S^2(\check{N}_{D/P})\otimes \underline{O}_D(2n-5))\}, \end{aligned}$$

and hence, by the duality theorem, $H^1(\underline{O}_{X'}(2n-5-3D'))$ is dual to

$$\mathrm{Coker}\{H^0(\underline{O}_D(3n_1+n_2+1-2n)\oplus \underline{O}_D(2n_1+2n_2+1-2n)\oplus \underline{O}_D(n_1+3n_2+1-2n)) \xrightarrow{\varepsilon} H^0(\underline{O}_D(n_1+n_2+1-n))\}.$$

In case $n > n_1+n_2+1$, it is obvious that $H^1(\underline{O}_{X'}(2n-5-3D'))=0$. In case $n=2n_1$ and $n_1=n_2+1$, ε is surjective. In case $n=2n_1+1$ and $n_1=n_2$, ε has the 1-dimensional cokernel. This proves the assertion (4.6), and hence completes the proof of (3.3.1).

The proof of (3.3.2) is similar to that of (3.3.1). First note that $T_{X'}\otimes \underline{O}_{X'}(2n_1-2D') \simeq \Omega_{X'}\otimes \underline{O}_{X'}(2n_1+4-n-D')$. We use the following exact diagram obtained from (4.1) and (4.2):

$$\begin{tikzcd}
H^0(\Omega_{\tilde{g}}\otimes \underline{O}_{X'}(2n_1+4-n-D')) & & \\
H^0(\Omega_{P'}\otimes \underline{O}_{X'}(2n_1+4-n-D')) \arrow[u] \arrow[r] & H^0(\Omega_{X'}\otimes \underline{O}_{X'}(2n_1+4-n-D')) \arrow[r] & H^1(\check{N}_{X'/P'}\otimes \underline{O}_{X'}(2n_1+4-n-D')). \\
H^0(f^*(\Omega_P\otimes \underline{O}_X)\otimes \underline{O}_{X'}(2n_1+4-n-D')) \arrow[u] & &
\end{tikzcd}$$

We will show the following (assuming X' of general type with $n \neq 2n_1$):

$$(4.8) \qquad H^0(f^*(\Omega_P\otimes \underline{O}_X)\otimes \underline{O}_{X'}(2n_1+4-n-D')) = \begin{cases} \text{6-dimensional} & \text{if } n=2n_1+1 \ \& \ n_2=1, \\ 0 & \text{otherwise.} \end{cases}$$

(5.1) Let $q_1 : P_1 \longrightarrow P$ be the blowing-up of P with the center T, let E_1 be the exceptional divisor and set $\tilde{h}_1 : E_1 \longrightarrow T$. X_1 and D_1 denote the proper transforms of X and of D respectively and set $f_1 = \mathrm{res}(q_1) : X_1 \longrightarrow X$ and $g_1 = \mathrm{res}(q_1) : D_1 \longrightarrow D$. Set $T_1 = f^{-1}(T)$ and $h_1 = \mathrm{res}(f_1) : T_1 \longrightarrow T$.

Let $q_2 : P_2 \longrightarrow P_1$ be the blowing-up of P_1 along D_1, let E_2 be the exceptional divisor of q_2 and set $\tilde{g}_2 : E_2 \longrightarrow D_1$. X_2, T_2 and E_1' denote the proper transformations of X_1, of T_1 and of E_1 respectively and set $f_2 = \mathrm{res}(q_2) : X_2 \longrightarrow X_1$, $h_2 = \mathrm{res}(q_2) : T_2 \longrightarrow T_1$ and $p_2 = \mathrm{res}(q_2) : E_1' \longrightarrow E_1$. Set $D_2 = f_2^{-1}(D_1)$ and $g_2 = \mathrm{res}(f_2) : D_2 \longrightarrow D_1$.

It is easy to see that D_1, X_2 and D_2 are smooth and that T_2 consists of the exceptional curves of the first kind on X_2. The surface contracted T_2 on X_2 coincides with the normalization X' of X by virtue of the Zariski's Main Theorem. Set $f_3 : X_2 \longrightarrow X'$, $g_3 : D_2 \longrightarrow D'$.

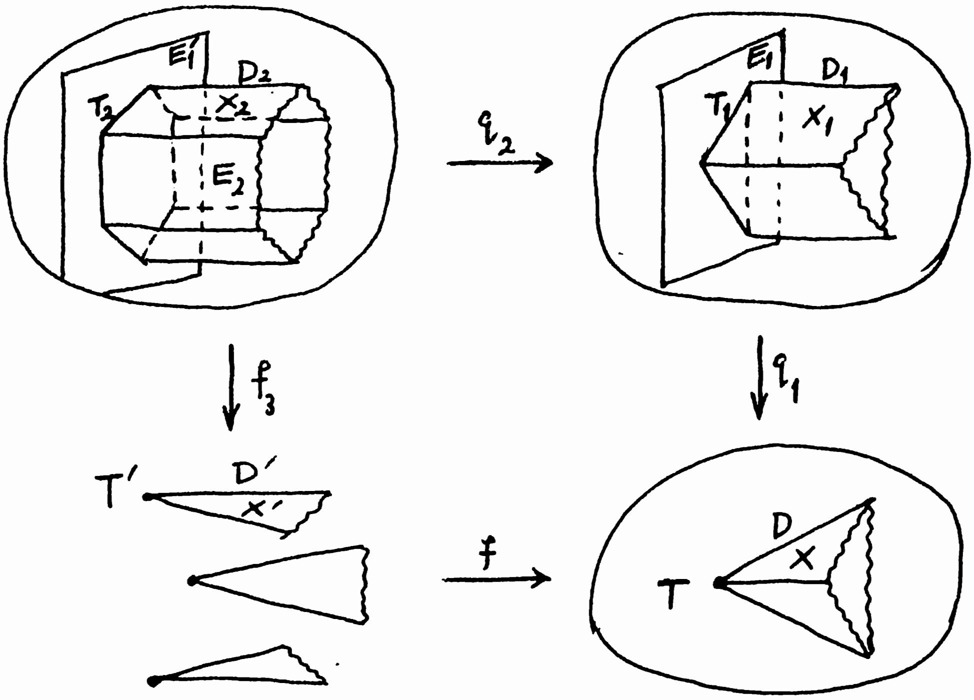
The above things form the following commutative diagram:

(5.1.1)

$$\begin{tikzcd}
E_1' \arrow[rrrrrrrrr, "p_2"] & & & & & & & & & E_1 \arrow[dddddddddl, "\tilde{h}_1"] \\
& T_2 \arrow[ul, hook] \arrow[rrrrrrr, "h_2"] \arrow[dddddddd] & & & & & & & T_2 \arrow[ur, hook] \arrow[dddddddd, "h_1"] & \\
& & E_2 \arrow[drrrrr, "\tilde{g}_2"] & & & & & & & \\
& & D_2 \arrow[u, hook] \arrow[rrrrr, "g_2"] \arrow[dr, hook] \arrow[ddddd, "g_3"] & & & & & D_1 \arrow[dl, hook] \arrow[ddddd, "g_1"] & & \\
& & & X_2 \arrow[rrr, "f_2"] \arrow[dr, hook] \arrow[ddd, "f_3"] & & & X_1 \arrow[dl, hook] \arrow[ddd, "f_1"] & & & \\
& & & & P_2 \arrow[r, "q_2"] & P_1 \arrow[d, "q_1"] & & & & \\
& & & & & P & & & & \\
& & & X' \arrow[rrr, "f"] & & & X \arrow[ul, hook] & & & \\
& & D' \arrow[ur, hook] \arrow[rrrrr, "g"] & & & & & D \arrow[ul, hook] & & \\
& T' \arrow[ur, hook] \arrow[rrrrrrr] & & & & & & & T \arrow[ul, hook] &
\end{tikzcd}$$

Note that $g_1 : D_1 \longrightarrow D$ is the normalization, $\tilde{h}_1 : E_1 \longrightarrow T$ is a $\mathbf{P}^2$-bundle, namely, a disjoint union of projective 2-spaces, $f_2 : X_2 \longrightarrow X_1$ is the normalization, $g_2 : D_2 \longrightarrow D_1$ is a ramified double covering, $\tilde{g}_2 : E_2 \longrightarrow D_1$ is a $\mathbf{P}^1$-bundle, $p_2 : E_1' \longrightarrow E_1$ is the blowing-up of the three points on $\mathbf{P}^2$ the component of E_1, $f_3 : X_2 \longrightarrow X'$ is the blowing-up of X' with the center T' and T_2 is its exceptional divisor, $g_3 : D_2 \longrightarrow D'$ is the normalization and that T' are the nodes on D' and by g each three nodes go down to a triple point of X.

In the neighborhood of a triple point of X, the figure of the above construction is as follows:



<u>Remark</u> (5.2) Let D be a curve in $P = \mathbf{P}^3$ only with singularities like coordinate axes in 3-space, that is, $D : yz = zx = xy = 0$ for a suitable local coordinates (x, y, z) in P. Given such a curve D, there exists an integer n_0 so that, for each integer $n \geq n_0$, there exist hypersurfaces X in P of degree n only with ordinary singularities and with $\mathrm{Sing}(X) = D$.

Actually, we can construct the following diagram as before in (5.1):

$$\begin{tikzcd}
E_1' \arrow[r, "p_2"] & E_1 \arrow[r, "\tilde{h}_1"] & T \arrow[d, phantom, "\cap"] \\
E_2 \arrow[r, "\tilde{g}_2"] \arrow[d, phantom, "\cap"] & D_1 \arrow[r, "g_1"] \arrow[d, phantom, "\cap"] & D \arrow[d, phantom, "\cap"] \\
P_2 \arrow[r, "q_2"] & P_1 \arrow[r, "q_1"] & P
\end{tikzcd}$$

The composite morphism $q_1 \circ q_2$ can be also obtained by once blowing-up (cf. (2.3.7) in [7]) and its exceptional divisor can be easily calculated as $E_2 + 2E_1'$. Thus we see that $\underline{O}_{P_2}(-E_2) \otimes q_2^* \underline{O}_{P_1}(-2E_1)$ is $(q_1 \circ q_2)$-very ample and hence there exists an integer n_0 such that, for $n \geqslant n_0$, $\underline{O}_{P_2}(-2E_2) \otimes q_2^* \underline{O}_{P_1}(-4E_1) \otimes (q_1 \circ q_2)^* \underline{O}_P(n)$ is very ample. Set

$$M_n = \underline{O}_{P_2}(-2E_2) \otimes q_2^* \underline{O}_{P_1}(-4E_1) \otimes (q_1 \circ q_2)^* \underline{O}_P(n) \quad \text{and}$$

$$N_n = \underline{O}_{P_2}(-2E_2) \otimes q_2^* \underline{O}_{P_1}(-3E_1) \otimes (q_1 \circ q_2)^* \underline{O}_P(n).$$

We will compute $\mathrm{Bs}|N_n|$. Tensoring N_n to the exact sequence

$$0 \longrightarrow \underline{O}_{P_2}(-E_1') \longrightarrow \underline{O}_{P_2} \longrightarrow \underline{O}_{E_1'} \longrightarrow 0$$

and using the Kodaira's vanishing theorem, we get the exact sequence

$$0 \longrightarrow H^0(M_n) \longrightarrow H^0(N_n) \longrightarrow H^0(N_n \otimes \underline{O}_{E_1'}) \longrightarrow 0.$$

Since M_n is very ample, the maximal fixed component of $|\mathrm{Im}(M_n \longrightarrow N_n)|$ is E_1'. Recall that E_1 is the disjoint union of $\mathbb{P}^2$ and set $E_1 = \coprod_{1 \leqslant i \leqslant t} E_1^{(i)}$ $(E_1^{(i)} = \mathbb{P}^2)$, $E_1' = \coprod_{1 \leqslant i \leqslant t} E_1'^{(i)}$ and $p_2^{(i)} = \mathrm{res}(p_2) : E_1'^{(i)} \longrightarrow E_1^{(i)}$. Note that

$$p_{2*}^{(i)}(N_n \otimes \underline{O}_{E_1'^{(i)}}) \simeq p_{2*}^{(i)}(\underline{O}_{E_1'^{(i)}}(-2E_2 \cdot E_1'^{(i)}) \otimes p_2^{(i)*} \underline{O}_{E_1^{(i)}}(3)) \simeq \mathcal{J}^2_{D \cdot E_1^{(i)}} \otimes \underline{O}_{E_1^{(i)}}(3),$$

where $\underline{O}_{E_1^{(i)}}(3) = \underline{O}_{\mathbb{P}^2}(3)$. $|\mathcal{J}^2_{D_1 \cdot E_1^{(i)}} \otimes \underline{O}_{E_1^{(i)}}(3)|$ contains only one member, say $T_1^{(i)}$, which is the uniquely determined three lines on $E_1^{(i)}$, and hence $|N_n \otimes \underline{O}_{E_1'^{(i)}}|$ consists of only one member, say $T_2^{(i)}$, which is the proper transform of $T_1^{(i)}$. Setting $T_2 = \coprod_{1 \leqslant i \leqslant t} T_2^{(i)}$, the above reasoning shows that $\mathrm{Bs}|N_n| = T_2$. Hence $|N_n|$ defines a birational morphism $P_2 - T_2 \longrightarrow \mathbb{P}^r$. Let X_2 be the closure in P_2 of the pull-back of a generic hyperplane by this morphism, then, since T_2 is of codimension 2 in P_2, we see that $X_2 \in |N_n|$ and that X_2 is smooth outside E_1'. It is easily seen that the image X of X_2 in P is just what we want.

(5.3) From now on we assume the following conditions:

(5.3.1) $H^1(\mathcal{J}_D \otimes \underline{O}_P(n-4)) = 0.$

(5.3.2) $H^1(\Omega_P \otimes \mathcal{J}_D \otimes \underline{O}_P(n-4)) = 0.$

(5.3.3) $H^0(\Omega_P \otimes \omega_D \otimes \underline{O}_D(5-n)) = 0.$

(5.3.4) $H^0(S^2(N_{D_1/P_1}) \otimes \underline{O}_{D_1}(4E_1 \cdot D_1) \otimes g_1^* \underline{O}_D(1-n)) = 0.$

(5.3.5) $H^0(\Omega_{P_1} \otimes \omega_{D_1} \otimes \underline{O}_{D_1}(2E_1 \cdot D_1) \otimes g_1^* \underline{O}_D(5-n)) = 0.$

(5.3.6) $\underline{O}_{P_2}(-2E_2) \otimes q_2^* \underline{O}_{P_1}(-4E_1) \otimes (q_1 \; q_2)^* \underline{O}_P(n-1)$ is ample.

(5.3.7) There exists an integer m satisfying the following conditions:

(5.3.7.1) $m \leqslant n-4.$

(5.3.7.2) $|\underline{O}_{X'}(m-D')|$ is fixed components free.

(5.3.7.3) $H^0(\underline{O}_{X'}(n-2m-3+D')) = 0.$

Note that these conditions are fulfilled by the Serre's theorem provided that the degree n of X is sufficiently large enough comparing to D.

Proposition (5.4). Under the conditions in (5.3), the parameter space of the Kuranishi family of the deformations of X' is smooth at the origin.

Proof. The conditions (5.3.1) and (5.3.2) imply $H^1(\mathcal{J}_D \otimes \underline{O}_P(n-5)) = 0.$ From this and from (5.3.2), it follows $H^1(f^*(T_P \otimes \underline{O}_X)) = 0$ by the same argument in the proof of (2.7) ((5.3.7.1) and (5.3.7.2) assure the existence of non-zero element in $H^0(\omega_{X'})$). The rest is the consequence of the general theorey of deformations. QED.

The following lemma can be proved in the same way as (1.3.2):

Lemma (5.5). $0 \longrightarrow \underline{O}_{X_1} \longrightarrow f_{2*}\underline{O}_{X_2} \longrightarrow \omega_{D_1} \otimes \check{\omega}_{X_1} \longrightarrow 0$ exact.

Since $f_3 : X_2 \longrightarrow X'$ is the blowing-up of the smooth scheme X' with the smooth center T', and T_2 is its exceptional divisor, we have the following lemma:

Lemma (5.6).

(5.6.1) $f_{3*}\underline{O}_{X_2}(aT_2) \simeq \underline{O}_{X'}, \qquad (a \geqslant 0).$

(5.6.2) $R^1 f_{3*}\underline{O}_{X_2}(aT_2) \simeq 0 \qquad (a \leqslant 1).$

Lemma (5.7). Under the conditions in (5.3), we have

(5.7.1) $H^0(\Omega_{X'} \otimes \underline{O}_{X'}(1)) = 0$ and

(5.7.2) $H^0(T_{X'} \otimes \underline{O}_{X'}(2m-2D')) = 0.$

Proof. Taking the direct image of the exact sequence

$$0 \longrightarrow f_3^*(\Omega_{X'} \otimes \underline{O}_{X'}(1)) \otimes \underline{O}_{X_2}(T_2) \longrightarrow \Omega_{X_2} \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \otimes \underline{O}_{X_2}(T_2)$$

$$\longrightarrow \Omega_{f_3} \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \otimes \underline{O}_{X_2}(T_2) \longrightarrow 0,$$

we get that, by (5.6.1),

$$\Omega_{X'} \otimes \underline{O}_{X'}(1) \simeq f_{3*}(\Omega_{X_2} \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \otimes \underline{O}_{X_2}(T_2)).$$

Hence to prove (5.7.1) is equivalent to prove

(5.7.3) $\quad H^0(\Omega_{X_2} \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \otimes \underline{O}_{X_2}(T_2)) = 0.$

By the exact sequence

$$0 \longrightarrow \check{N}_{X_2/P_2} \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \otimes \underline{O}_{X_2}(T_2) \longrightarrow \Omega_{P_2} \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \otimes \underline{O}_{X_2}(T_2)$$
$$\longrightarrow \Omega_{X_2} \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \otimes \underline{O}_{X_2}(T_2) \longrightarrow 0,$$

to prove (5.7.3) it is enough to show the following:

(5.7.4) $\quad H^0(\Omega_{P_2} \otimes \underline{O}_{X_2}(T_2) \otimes (f_1 \circ f_2)^* \underline{O}_X(1)) = 0.$

(5.7.5) $\quad H^1(\check{N}_{X_2/P_2} \otimes \underline{O}_{X_2}(T_2) \otimes (f_1 \circ f_2)^* \underline{O}_X(1)) = 0.$

Since $\check{N}_{X_2/P_2} \otimes \underline{O}_{X_2}(T_2) \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \simeq \underline{O}_{X_2}(D_2 + 2T_2)^{\otimes 2} \otimes (f_1 \circ f_2)^* \underline{O}_X(1-n)$, (5.7.5) follows the condition (5.3.6) by virtue of the Kodaira vanishing theorem.

Next we will prove (5.7.4). By the exact sequence

$$0 \longrightarrow f_2^*(\Omega_{P_1} \otimes f_1^* \underline{O}_X(1)) \otimes \underline{O}_{X_2}(T_2) \longrightarrow \Omega_{P_2} \otimes \underline{O}_{X_2}(T_2) \otimes (f_1 \circ f_2)^* \underline{O}_X(1)$$
$$\longrightarrow \Omega_{q_2} \otimes \underline{O}_{X_2}(T_2) \otimes (f_1 \circ f_2)^* \underline{O}_X(1) \longrightarrow 0,$$

it is enough to show the following:

(5.7.6) $\quad H^0(f_2^*(\Omega_{P_1} \otimes f_1^* \underline{O}_X(1)) \otimes \underline{O}_{X_2}(T_2)) = 0.$

(5.7.7) $\quad H^0(\Omega_{q_2} \otimes \underline{O}_{X_2}(T_2) \otimes (f_1 \circ f_2)^* \underline{O}_X(1)) = 0.$

We first prove (5.7.7). Tensoring $\Omega_{q_2} \otimes \underline{O}_{X_2}(T_2) \otimes (f_1 \circ f_2)^* \underline{O}_X(1)$ $\simeq \Omega_{\tilde{g}_2} \otimes \tilde{g}_2^* \underline{O}_{D_1}(E_1 \cdot D_1) \otimes (g_1 \circ \tilde{g}_2)^* \underline{O}_D(1)$ to the exact sequence

$$0 \longrightarrow \underline{O}_{E_2}(-D_2) \longrightarrow \underline{O}_{E_2} \longrightarrow \underline{O}_{D_2} \longrightarrow 0$$

and taking the direct image, we have the exact sequence

$$0 \longrightarrow \tilde{g}_{2*}\Omega_{\tilde{g}_2} \otimes \underline{O}_{D_1}(E_1 \cdot D_1) \otimes g_1^* \underline{O}_D(1) \longrightarrow S^2(N_{D_1/P_1}) \otimes \underline{O}_{D_1}(4E_1 \cdot D_1) \otimes g_1^* \underline{O}_D(1-n)$$
$$\longrightarrow \underline{O}_{D_1}(E_1 \cdot D_1) \otimes g_1^* \underline{O}_D(1) \longrightarrow 0$$

by the same argument as in the proof of (4.5). Hence (5.7.7) follows the condition (5.3.4), since $g_2 : D_2 \longrightarrow D_1$ is a finite morphism.

To prove (5.7.6), by using the exact sequence

$$0 \longrightarrow \Omega_{P_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1) \longrightarrow f_{2*}\underline{O}_{X_2} \otimes \Omega_{P_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1)$$
$$\longrightarrow \omega_{D_1} \otimes \check{\omega}_{X_1} \otimes \Omega_{P_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1) \longrightarrow 0$$

obtained from (5.5), it suffices to show the following:

$$(5.7.8) \qquad H^0(\Omega_{P_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1)) = 0.$$

$$(5.7.9) \qquad H^0(\omega_{D_1} \otimes \check{\omega}_{X_1} \otimes \Omega_{P_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1)) = 0.$$

To prove (5.7.8), we use the exact sequence

$$0 \longrightarrow f_1^*(\Omega_P \otimes \underline{O}_X(1)) \otimes \underline{O}_{X_1}(T_1) \longrightarrow \Omega_{P_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1)$$
$$\longrightarrow \Omega_{q_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1) \longrightarrow 0.$$

Since $f_{3*}f_2^*(f_1^*(\Omega_P \otimes \underline{O}_X(1)) \otimes \underline{O}_{X_1}(T_1)) \simeq f_{3*}(f_3^* f^*(\Omega_P \otimes \underline{O}_X(1)) \otimes \underline{O}_{X_2}(T_2)) \simeq f^*(\Omega_P \otimes \underline{O}_X(1))$ by (5.6.1), $H^0(f_1^*(\Omega_P \otimes \underline{O}_X(1)) \otimes \underline{O}_{X_1}(T_1)) = 0$ follows $H^0(f^*(\Omega_P \otimes \underline{O}_X(1))) = 0$, and the latter follows the exact sequence

$$0 \longrightarrow \Omega_P \otimes \underline{O}_X(1) \longrightarrow f_*\underline{O}_{X'} \otimes \Omega_P \otimes \underline{O}_X(1) \longrightarrow \omega_D \otimes \check{\omega}_X \otimes \Omega_P \otimes \underline{O}_X(1) \longrightarrow 0$$

obtained from (1.3.2) and the condition (5.3.3). On the other hand, E_1 is a disjoint union of $\mathbb{P}^2$ and since T_1 appears as three lines on each $\mathbb{P}^2$, setting $T_1 = \coprod_{1 \leqslant i \leqslant t} T_1^{(i)}$, we see that $\Omega_{q_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1) \simeq \Omega_{\tilde{h}_1} \otimes \underline{O}_{P_1}(E_1) \otimes \underline{O}_{E_1} \otimes \underline{O}_{T_1}$ is the disjoint union of $\Omega_{\mathbb{P}^2} \otimes \underline{O}_{\mathbb{P}^2}(-1) \otimes \underline{O}_{T_1^{(i)}}$ $(1 \leqslant i \leqslant t)$ and hence we can get $H^0(\Omega_{q_1} \otimes \underline{O}_{X_1}(T_1) \otimes f_1^* \underline{O}_X(1)) = 0$. Thus we have proven (5.7.8).

(5.7.9) follows the condition (5.3.5), since an easy computation shows that $\omega_{X_1} \simeq f_1^* \underline{O}_X(n-4) \otimes \underline{O}_{X_1}(-T_1)$. This completes the proof of (5.7.1).

Tensoring a non-zero element in $H^0(\underline{O}_{X'}(n-2m-3+D'))$ (such an element exists by the condition (5.3.7.3)) gives an injection

$$T_{X'} \otimes \underline{O}_{X'}(2m-2D') \longrightarrow T_{X'} \otimes \underline{O}_{X'}(n-3-D') \simeq \Omega_{X'} \otimes \underline{O}_{X'}(1)$$

and hence (5.7.2) follows (5.7.1). QED.

<u>Theorem</u> (5.8). In the case that the degree n of X in P is sufficiently large enough comparing to the singular locus D of X in the sense that the conditions in (5.3) are fulfilled, the local Torelli theorem holds for the

normalization X' of X.

<u>Proof</u>. We can derive this theorem from (5.3.7.1), (5.3.7.2) and (5.7) just in the same way as in proving (3.5). QED.

<u>Remark</u> (5.9). The moduli space of Gieseker ([6]) is divided by the Hilbert polynomial of $\omega_{X'}$, that is,

$$(\omega_{X'}^{\otimes s}) = \frac{1}{2} c_1^2 s - \frac{1}{2} c_1^2 s + \frac{1}{12}(c_1^2 + c_2).$$

(1.7) says that, fixing D and increasing n, c_1^2 is increasing and (1.4.1) says that $\omega_{X'}$ is getting "ampler and ampler". Hence (5.8) gives some evidevce to the naive feeling that if X' would have sufficiently many 2-forms, their periods of integrals should determine X' itself. (Note that the Ky̌nef's example [12] has $p_g = c_1^2 = 1$.)

Notes

1) If X has only ordinary singularities and its singular locus D is a complete intersection in P, D becomes automatically smooth. Actually, blowing-up P along D, the fact that D is a complete intersection imposes that the exceptional divisor becomes a $\mathbf{P}^1$-bundle. On the other hand, if X would have triple points, the fibres over such points are 2-dimensional.

2) By using the result (3.3.2) below, in cases $(n, n_1, n_2) = (6, 2, 1)$, $(7, 3, 2)$ we see that $H^2(T_{X'}) = 0$ by duality and also $H^0(T_{X'}) = 0$ and hence the parameter space of the Kuranishi family of deformations of X' is smooth at the origin. On the contrary, in case $(n, n_1, n_2) = (7, 3, 1)$ we see that $\dim H^2(T_{X'}) = 6$ by (3.3.2) and so the smoothness of the parameter space is still unknown.

3) Actually $|\underline{O}_{X'}(n_1 - D')|$ is fixed points free, but (3.1.2) is enough for our later use. Note also that, by (3.1.2), $|\omega_{X'}|$ is fixed components free and hence X' is minimal.

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