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SUPPLEMENT TO VARIATION OF MIXED HODGE STRUCTURE ARISING FROM
FAMILY OF LOGARITHMIC DEFORMATIONS II: CLASSIFYING SPACE

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In this note, we will define a graded polarization (abbreviated as GP) of the mixed Hodge structure (abbreviated as MHS) on $H^n(X-Y, \mathbb{Q})$, where X is a smooth projective variety over $\mathbb{C}$ and Y a ~~smooth~~ ^{simple} normal crossing divisor (abbreviated as SNCD) on X , and give some supplements to [U.2]. This note is based on the small meeting of the three authors at RIMS 1-6 X '84.

1. Graded polarization on $H^n(X-Y, \mathbb{Q})$: Let X and Y be as above and set $r := \dim X$. Choose a polarization $\omega \in H^{1,1}(X, \mathbb{Z})$ (i.e. the cohomology class of a very ample invertible sheaf) on X . Let

$$Y = \bigcup_{i \in I} Y_i$$

be the decomposition into irreducible components. We fix once for all an order of these components. We use the following notations:

$$Y_J := \bigcap_{j \in J} Y_j \quad \text{for a subset } J \subset I.$$

$$\tilde{Y}^s := \bigsqcup_J Y_J \quad \text{where the } J \text{ run the subsets of } I \text{ with } \#J = s \text{ and}$$

$$\tilde{Y}^0 = X.$$

$\omega_J \in H^{1,1}(Y_J, \mathbb{Z})$: the induced polarization on Y_J from ω .

$$\omega_s := \bigoplus_{\#J=s} \omega_J \in H^{1,1}(\tilde{Y}^s, \mathbb{Z})$$

$$\nu_J : H^{2(r-s)}(Y_J, \mathbb{Z}) \longrightarrow \mathbb{Z} \quad \text{with } \nu_J(\omega_J^{r-s}) = 1, \text{ where } \#J = s.$$

$$\nu_s := \sum_{\#J=s} \nu_J : H^{2(r-s)}(\tilde{Y}^s, \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

L : the multiplication operator on the cohomology groups of $\tilde{Y}^s$
induced by the polarization ω_s .
 $P^m(\tilde{Y}^s, Q) := \text{Ker}(L^{(r-s)-m+1} : H^m(\tilde{Y}^s, Q) \longrightarrow H^{2(r-s)-m+2}(\tilde{Y}^s, Q))$ the
primitive cohomology ($m \leq r - s$).

Then, by the Lefschetz decomposition

$$(1.1) \quad H^m(\tilde{Y}^s, Q) = \bigoplus_{a \geq 0} L^a P^{m-2a}(\tilde{Y}^s, Q),$$

we can define a polarization Q'_s of HS on $H^m(\tilde{Y}^s, Q)$ by

$$(1.2) \quad Q'_s(u, v) := \sum_{\#J=s} \sum_a (-1)^{(m-2a)(m-2a+1)/2} \cup_s (u_{m-2a} \cup v_{m-2a} \cup \omega_s^{(r-s)-(m-2a)})$$

for $u = \sum L^a u_{m-2a}$, $v = \sum L^a v_{m-2a} \in H^m(\tilde{Y}^s, Q)$. Note that

$$\frac{(m-2a)(m-2a+1)}{2} = \frac{m(m+1)}{2} + a \pmod{2}.$$

Recall that the weight spectral sequence

$$(1.3) \quad W_1^{E, -s, n+s} = H^{n-s}(\tilde{Y}^s) \iff W^E = H^n(X-Y)$$

is nothing but the Leray spectral sequence for $X-Y \hookrightarrow X$ under the change of

indices $W_k^{E, -s, n+s} = E_{k+1}^{n-s, s}$ of Leray ((3.2.4) in [D]), degenerates in $W_2^E = W_\infty^E$ ((3.2.10) in [D]), and the d_1 are alternating sums of Gysin maps hence morphisms of HS of type (1,1) ((3.2.8) in [D], (5.21) in [G.S]). Set

$$B := \text{Im}(H^{n-s-2}(\tilde{Y}^{s+1}) \xrightarrow{d_1} H^{n-s}(\tilde{Y}^s)) \quad \text{and} \\ Z := \text{Ker}(H^{n-s}(\tilde{Y}^s) \xrightarrow{d_1} H^{n-s+2}(\tilde{Y}^{s-1})),$$

and define

$$(1.4) \quad C := \{u \in Z \mid Q'_s(u, v) = 0 \quad (\forall v \in B)\}.$$

Then it is easy to verify:

$$(1.5) \quad C \cap W_2^{E, -s, n+s} = W_\infty^{E, -s, n+s} = \text{Gr}_{n+s}^{W[n]} H^n(X-Y) \quad \text{as HS over } Q.$$

The polarization Q'_s in (1.2) induces one on C over Q .

Thus, shifting the indices $Q_k := Q'_{k-n}$, we get:

Proposition(1.6). Let X be a smooth projective variety and Y a SNCD on X . Then a polarization ω on X induces a grades polarization $Q = \{Q_k\}$ of the MHS on $H^n(X-Y, \mathbb{Q})$.

2. Variation of GPMHS arising from family of logarithmic deformations: The construction in §1 can be easily generalized to the relative case (cf. (3.4), (3.5) in [U.1]. There is a misprint in the latter, i.e. the constant $1/2\pi\sqrt{-1}$ should be omitted), and, instead of Theorem (1.7) in [U.2], we have:

Theorem (2.1). Let $f : \mathcal{X} \longrightarrow S$ be a smooth, projective morphism of complex manifolds and $\mathcal{Y}$ be a SNCD on $\mathcal{X}$, flat over S . Then we have a VGPMHS $(S, R_Z^n(\overset{\circ}{f}), W[n], F, Q)$ in the sense of Definition (1.1) in [U.2], where $\overset{\circ}{f} := f|(\mathcal{X}-\mathcal{Y})$ and $R_Z^n(\overset{\circ}{f}) := R^n \overset{\circ}{f}_* \mathbb{Z}$ modulo torsion.

3. Classifying space and period map of GPMHS: Let $(H_Z, W, F(0), Q)$ be a reference GPMHS. Recall the following notations in [U.2]:

$$f^p := \dim F(0)^p H_C.$$

$$f_k^p := \dim F(0)^p \text{Gr}_k^W H_C.$$

$$\check{\mathcal{F}} := \{F = (F') \in \text{Flag}(H_C; \dots, f^p, \dots) \mid \dim F^p \text{Gr}_k^W H_C = f_k^p \quad (\forall p, \forall k)\}.$$

$$\text{GL}_W(H_C) := \{g \in \text{GL}(H_C) \mid gW_k = W_k \quad (\forall k)\}.$$

$$\pi_k : \check{\mathcal{F}} \longrightarrow \check{\mathcal{F}}_k := \text{Flag}(\text{Gr}_k^W H_C; \dots, f_k^p, \dots).$$

$$\check{D}_k := \{F \in \check{\mathcal{F}}_k \mid Q_k(F^p, F^{k-p+1}) = 0 \quad (\forall p)\}.$$

$$D_k := \{F \in \check{D}_k \mid i^{2p-k} Q_k(u, \bar{u}) > 0 \quad (0 \neq u \in F^p \cap \bar{F}^{k-p}, \forall p)\}.$$

$$\check{D} := \bigcap_k \pi_k^{-1}(\check{D}_k) \subset \check{\mathcal{F}}.$$

$$D := \bigcap_k \pi_k^{-1}(D_k) \subset \check{D}.$$

$$\tilde{\pi} : \check{D} \longrightarrow \prod_k \check{D}_k \quad \text{the projection.}$$

$\pi : D \longrightarrow \prod_k D_k$ the projection.

$$G_{k,C} := \{g \in GL(Gr_k^W H_C) \mid Q_k(gu, gv) = Q_k(u, v) \quad (\forall u, \forall v \in Gr_k^W H_C)\}.$$

$$G_{k,R} := \{g \in G_{k,C} \mid gGr_k^W H_R = Gr_k^W H_R\}.$$

$$G_{k,Z} := \{g \in G_{k,R} \mid gGr_k^W H_Z = Gr_k^W H_Z\}.$$

$$G_C := \{g \in GL_W(H_C) \mid Gr_k^W(g) \in G_{k,C} \quad (\forall k)\}.$$

$$G_R := \{g \in G_C \mid gH_R = H_R\}.$$

$$G_Z := \{g \in G_R \mid gH_Z = H_Z\}.$$

In case $\text{length} \geq 2$, G_R acts on D nontransitively (see (3.3) in [U.2]).

Let $G_C = G'_C \cdot G''_C$ be a Levi decomposition with G'_C = the unipotent radical of G_C and G''_C = a semi-simple part of G_C . Instead of our previous G_R , Carlson in [C] takes

$$G := G'_C \cdot (G_R \cap G''_C).$$

Then, it is obvious to see that G acts transitively on D . The isotropy subgroup of G at $0 \in D$ is not compact. Nevertheless he proved in [C] that G_Z acts on D properly discontinuously.

Combining (2.10) in [U.1], (2.11), (2.16), (3.5), (3.6) and (4.2) in [U.2], and the result of §4 in [C], we get:

Theorem (3.1). (3.1.1) $\tilde{\pi} : \tilde{D} \longrightarrow \prod \tilde{D}_k$ is an algebraic homogeneous vector bundle with respect to the group G_C .

(3.1.2) G acts transitively on D .

(3.1.3) G_Z acts properly discontinuously on D .

(3.1.4) There is an extended horizontal subbundle T_D^{eh} on $\tilde{D}$ which is compatible with the horizontal subbundle $\bigoplus T_{D_k}^h$ on $\prod \tilde{D}_k$ via $\tilde{\pi}$.

(3.1.5) The period map associated to the VGPMHS $(S, R_Z^n(\tilde{f}), W[n], F, Q)$ arising from geometry in (2.1) above has extended horizontal local liftings with

respect to T_D^{eh} and compatible with

the period maps of Griffiths associated to the VPMHS $(S, \text{Gr}_k^{W[n]} R_Z^n(\tilde{f}), F_{Q_k}) = (S, R_Z^{2n-k}(\tilde{g}^{k-n}), F, Q_k)$, where $\tilde{g}^S : \tilde{Y}^S \rightarrow S$ is induced from f and $\tilde{Y}^S$ is a relative version of $\tilde{Y}^S$ in §1.

4. Degeneration of VGPMHS associated to semi-stable degeneration of family of logarithmic deformations: We want to interpret the results in II.II of [E] and in §5 of [S.Z] into our language for our future use.

Consider a situation:

$\mathcal{X}$: a complex manifold.

Δ : the unit open disc in $\mathbb{C}$.

$f : \mathcal{X} \rightarrow \Delta$ a projective morphism, smooth over the punctured disc Δ^* .

(4.1) $\mathcal{Y} = \bigcup \mathcal{Y}_i$: a divisor on $\mathcal{X}$, the $\mathcal{Y}_i$ are irreducible components.

$X_0 := f^{-1}(0)$ is reduced.

$\mathcal{Y} \cup X_0$ is NCD on $\mathcal{X}$.

$\mathcal{Y}_{i_1} \cap \dots \cap \mathcal{Y}_{i_p}$ ($p \geq 1$) are flat over Δ and smooth over Δ^* .

In this situation, Elzein and Steenbrink-Zucker deal a trifiltered complex

$(sA^{\bullet\bullet}, W, M, F)$ constructed as follows (II.II in [E], (5.5) in [S.Z]):

$$W(\mathcal{Y})_k \Omega_{\mathcal{X}}^p(\log(\mathcal{Y}+X_0)) := \Omega_{\mathcal{X}}^k(\log(\mathcal{Y}+X_0)) \wedge \Omega_{\mathcal{X}}^{p-k}(\log X_0).$$

$$W(X_0)_\ell \Omega_{\mathcal{X}}^p(\log(\mathcal{Y}+X_0)) := \Omega_{\mathcal{X}}^\ell(\log(\mathcal{Y}+X_0)) \wedge \Omega_{\mathcal{X}}^{p-\ell}(\log \mathcal{Y}).$$

$$W(\mathcal{Y}+X_0)_m \Omega_{\mathcal{X}}^p(\log(\mathcal{Y}+X_0)) := \Omega_{\mathcal{X}}^m(\log(\mathcal{Y}+X_0)) \wedge \Omega_{\mathcal{X}}^{p-m}.$$

$$A^{p,q} := \Omega_{\mathcal{X}}^{p+q+1}(\log(\mathcal{Y}+X_0)) / W(X_0)_q \quad (p, q \geq 0).$$

$$W_k A^{p,q} := \text{the image of } W(\mathcal{Y})_k \Omega_{\mathcal{X}}^{p+q+1}(\log(\mathcal{Y}+X_0)) \text{ in } A^{p,q}.$$

$$M A^{p,q} := \text{the image of } W(\mathcal{Y}+X_0)_{2q+k+1} \Omega_{\mathcal{X}}^{p+q+1}(\log(\mathcal{Y}+X_0)) \text{ in } A^{p,q}.$$

$$F^p A^{\bullet\bullet} := \bigoplus_{p' \geq p} A^{p', \bullet\bullet}.$$

$sA^{\bullet\bullet}$: the associated simple complex of $A^{\bullet\bullet}$.

$$(sA^{\bullet\bullet}, W, M, F) \simeq FICMHS \text{ on } \mathcal{X} \text{ with } a_n \text{ and } \tau_n^* \delta_n.$$

Note that there is a bifiltered quasi-isomorphism (cf. (4.16) in [St]):

$$\theta : (\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log(\mathcal{Y}+X_0))) \otimes_{\mathcal{O}_{X_0}} W(\mathcal{Y}), F) \longrightarrow (SA^{\bullet}, W, F).$$

Let

$$\Phi : \Delta^* \longrightarrow \langle T \rangle \backslash D$$

be the period map associated to the VGPMHS $(\Delta^*, R_Z^n(\tilde{f}'), W(\mathcal{Y}-X_0)[n], F)$, where $\tilde{f}' := \text{res}(f) : \mathcal{X} - \mathcal{Y} - X_0 \longrightarrow \Delta^*$ and T is the local monodromy. Since X_0 is reduced, T is unipotent. Set

$$(4.2) \quad \begin{aligned} N &:= \log T. \\ \psi(t) &:= \exp(-\log t / 2\pi i \cdot N) \Phi(t) \quad (t \in \Delta^*). \end{aligned}$$

Then (4.1) in [E] and (3.13) in [S.Z] can be interpreted as:

Theorem (4.3) (Elzein and Steenbrink-Zucker). Assume the situation (4.1).

(4.3.1) The map ψ in (4.2) extends to a holomorphic map

$$\tilde{\psi} : \Delta \longrightarrow D$$

compatible with the extensions of ψ_k associated to the period maps of Griffiths via $\pi : D \longrightarrow \prod D_k$.

(4.3.2) For each k , $(W[n]_k, M, \tilde{\psi}(0))$ is a MHS, and N gives a morphism of type $(-1, -1)$ with respect to this MHS.

(4.3.3) For each k , M induces the usual monodromy weight filtration on $Gr_k^{W[n]}$, and the GP Q_k in (2.1) induces the monodromy polarization on $(Gr_k^{W[n]}, M, \tilde{\psi}(0))$ (cf. (6.16) in [Sc]).

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