SUPPLEMENT TO VARIATION OF MIXED HODGE STRUCTURE ARISING FROM FAMILY OF LOGARITHMIC DEFORMATIONS II: CLASSIFYING SPACE

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In this note, we will define a graded polarization (abbreviated as GP) of the mixed Hodge structure (abbreviated as MHS) on $H^n(X-Y,Q)$, where X is a smooth projective variety over C and Y a simple ~~smooth~~ normal crossing divisor (abbreviated as SNCD) on X, and give some supplements to [U.2]. This note is based on the small meeting of the three authors at RIMS 1-6 X '84.

1. Graded polarization on $H^n(X-Y,Q)$: Let X and Y be as above and set $r := \dim X$. Choose a polarization $\omega \in H^{1,1}(X,Z)$ (i.e. the cohomology class of a very ample invertible sheaf) on X. Let

$$Y = \bigcup_{i \in I} Y_i$$

be the decomposition into irreducible components. We fix once for all an order of these components. We use the following notations:

$Y_J := \bigcap_{j \in J} Y_j$ for a subset $J \subset I$.

$\tilde{Y}^s := \coprod_J Y_J$ where the J run the subsets of I with $\#J = s$ and $\tilde{Y}^0 = X$.

$\omega_J \in H^{1,1}(Y_J,Z)$: the induced polarization on Y_J from ω.

$\omega_s := \bigoplus_{\#J=s} \omega_J \in H^{1,1}(\tilde{Y}^s,Z)$

$\nu_J : H^{2(r-s)}(Y_J,Z) \xrightarrow{\sim} Z$ with $\nu_J(\omega_J^{r-s}) = 1$, where $\#J = s$.

$\nu_s := \sum_{\#J=s} \nu_J : H^{2(r-s)}(\tilde{Y}^s,Z) \longrightarrow Z.$

L : the multiplication operator on the cohomology groups of $\tilde{Y}^s$ induced by the polarization ω_s.

$P^m(\tilde{Y}^s,Q) := \mathrm{Ker}(L^{(r-s)-m+1} : H^m(\tilde{Y}^s,Q) \longrightarrow H^{2(r-s)-m+2}(\tilde{Y}^s,Q))$ the primitive cohomology $(m \leqslant r - s)$.

Then, by the Lefschetz decomposition

$$(1.1) \qquad H^m(\tilde{Y}^s,Q) = \bigoplus_{a\geqslant 0} L^a P^{m-2a}(\tilde{Y}^s,Q),$$

we can define a polarization Q'_s of HS on $H^m(\tilde{Y}^s,Q)$ by

$$(1.2) \qquad Q'_s(u,v) := \sum_{\#J=s} \sum_a (-1)^{(m-2a)(m-2a+1)/2} \nu_s(u_{m-2a} \cup v_{m-2a} \cup \omega_s^{(r-s)-(m-2a)})$$

for $u = \Sigma\, L^a u_{m-2a}$, $v = \Sigma\, L^a v_{m-2a} \in H^m(\tilde{Y}^s,Q)$. Note that

$$\frac{(m - 2a)(m - 2a + 1)}{2} = \frac{m(m + 1)}{2} + a \qquad \mathrm{mod}(2).$$

Recall that the weight spectral sequence

$$(1.3) \qquad {}_W E_1^{-s,n+s} = H^{n-s}(\tilde{Y}^s) \Longrightarrow {}_W E^n = H^n(X-Y)$$

is nothing but the Leray spectral sequence for $X-Y \hookrightarrow X$ under the change of indices ${}_W E_k^{-s,n+s} = E_{k+1}^{n-s,s}$ of Leray ((3.2.4) in [D]), degenerates in ${}_W E_2 = {}_W E_\infty$ ((3.2.10) in [D]), and the d_1 are alternating sums of Gysin maps hence morphisms of HS of type (1,1) ((3.2.8) in [D], (5.21) in [G.S]). Set

$$B := \mathrm{Im}(H^{n-s-2}(\tilde{Y}^{s+1}) \xrightarrow{d_1} H^{n-s}(\tilde{Y}^s)) \quad \text{and}$$

$$Z := \mathrm{Ker}(H^{n-s}(\tilde{Y}^s) \xrightarrow{d_1} H^{n-s+2}(\tilde{Y}^{s-1})),$$

and define

$$(1.4) \qquad C := \{u \in Z \mid Q'_s(u,v) = 0 \quad (\forall v \in B)\}.$$

Then it is easy to verify:

(1.5) $C \xrightarrow{\sim} {}_W E_2^{-s,n+s} = {}_W E_\infty^{-s,n+s} = \mathrm{Gr}_{n+s}^{W[n]} H^n(X-Y)$ as HS over Q.

The polarization Q'_s in (1.2) induces one on C over Q.

Thus, shifting the indices $Q_k := Q'_{k-n}$, we get:

Proposition(1.6). Let X be a smooth projective variety and Y a SNCD on X. Then a polarization ω on X induces a grades polarization $Q = \{Q_k\}$ of the MHS on $H^n(X-Y,Q)$.

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2. Variation of GPMHS arizing from family of logarithmic deformations: The construction in §1 can be easily generalized to the relative case (cf. (3.4), (3.5) in [U.1]. There is a misprint in the latter, i.e. the constant $1/2\pi\sqrt{-1}$ should be omitted), and, instead of Theorem (1.7) in [U.2], we have:

Theorem (2.1). Let $f : \mathfrak{X} \longrightarrow S$ be a smooth, projective morphism of complex manifolds and $\mathfrak{Y}$ be a SNCD on $\mathfrak{X}$, flat over S. Then we have a VGPMHS $(S,\ R^n_Z(\mathring{f}),\ W[n],\ F,\ Q)$ in the sense of Definition (1.1) in [U.2], where $\mathring{f} := f|(\mathfrak{X}-\mathfrak{Y})$ and $R^n_Z(\mathring{f}) := R^n\mathring{f}_*Z$ modulo torsion.

3. Classifying space and period map of GPMHS: Let $(H_Z,\ W,\ F(0),\ Q)$ be a reference GPMHS. Recall the following notations in [U.2]:

$$f^p := \dim F(0)^p H_C.$$

$$f^p_k := \dim F(0)^p Gr^W_k H_C.$$

$$\check{\mathfrak{F}} := \{F = (F^\cdot) \in Flag(H_C;\dots,f^p,\dots) \mid \dim F^p Gr^W_k H_C = f^p_k \quad (\forall p, \forall k)\}.$$

$$GL_W(H_C) := \{g \in GL(H_C) \mid gW_k = W_k \quad (\forall k)\}.$$

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$$\check{\pi}_k : \check{\mathfrak{F}} \longrightarrow \check{\mathfrak{F}}_k := Flag(Gr^W_k H_C;\dots,f^p_k,\dots).$$

$$\check{D}_k := \{F \in \check{\mathfrak{F}}_k \mid Q_k(F^p, F^{k-p+1}) = 0 \quad (\forall p)\}.$$

$$D_k := \{F \in \check{D}_k \mid i^{2p-k} Q_k(u,\bar{u}) > 0 \quad (0 \neq u \in F^p \cap \bar{F}^{k-p},\ \forall p)\}.$$

$$\check{D} := \bigcap_k \pi_k^{-1}(\check{D}_k) \subset \check{\mathfrak{F}}.$$

$$D := \bigcap_k \pi_k^{-1}(D_k) \subset \check{D}.$$

$$\check{\pi} : \check{D} \longrightarrow \prod_k \check{D}_k \quad \text{the projection.}$$

$\pi : D \longrightarrow \prod_k D_k$ the projection.

$G_{k,C} := \{g \in GL(Gr_k^W H_C) \mid Q_k(gu,gv) = Q_k(u,v) \quad (\forall u, \forall v \in Gr_k^W H_C)\}.$

$G_{k,R} := \{g \in G_{k,C} \mid gGr_k^W H_R = Gr_k^W H_R\}.$

$G_{k,Z} := \{g \in G_{k,R} \mid gGr_k^W H_Z = Gr_k^W H_Z\}.$

$G_C := \{g \in GL_W(H_C) \mid Gr_k^W(g) \in G_{k,C} \quad (\forall k)\}.$

$G_R := \{g \in G_C \mid gH_R = H_R\}.$

$G_Z := \{g \in G_R \mid gH_Z = H_Z\}.$

In case length $\geqslant 2$, G_R acts on D nontransitively (see (3.3) in [U.2]). Let $G_C = G'_C . G''_C$ be a Levi decomposition with G'_C = the unipotent radical of G_C and G''_C = a semi-simple part of G_C. Instead of our previous G_R, Carlson in [C] takes

$$G := G'_C . (G_R \cap G''_C).$$

Then, it is obvious to see that G acts transitively on D. The isotropy subgroup of G at $0 \in D$ is not compact. Nevertheless he proved in [C] that G_Z acts on D properly discontituously.

Combining (2.10) in [U.1], (2.11), (2.16), (3.5), (3.6) and (4.2) in [U.2], and the result of §4 in [C], we get:

Theorem (3.1). (3.1.1) $\check{\pi} : \check{D} \longrightarrow \prod \check{D}_k$ is an algebraic homogeneous vector bundle with respect to the group G_C.

(3.1.2) G acts transitively on D.

(3.1.3) G_Z acts properly discontinuously on D.

(3.1.4) There is an extended horizontal subbundle $T^{eh}_{\check{D}}$ on $\check{D}$ which is compatible with the horizontal subbundle $\oplus T^h_{\check{D}_k}$ on $\prod \check{D}_k$ via $\check{\pi}$.

(3.1.5) The period map associated to the VGPMHS $(S, R^n_Z(\mathring{f}), W[n], F, Q)$ arising from geometry in (2.1) above has extended horizontal local liftings with respect to $T^{eh}_{\check{D}}$ and compatible with

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the period maps of Griffiths associated to the VPMHS $(S, Gr_k^{W[n]} R_Z^n(\mathring{f}), F$ $Q_k) = (S, R_Z^{2n-k}(\tilde{g}^{k-n}), F, Q_k)$, where $\tilde{g}^s : \tilde{\mathcal{Y}}^s \longrightarrow S$ is induced from f and $\tilde{\mathcal{Y}}^s$ is a relative version of $\tilde{Y}^s$ in §1.

4. Degeneration of VGPMHS associated to semi-stable degeneration of family of logarithmic deformations: We want to interpret the results in II.II of [E] and in §5 of [S.Z] into our language for our future use.

Consider a situation:

$\mathfrak{X}$: a complex manifold.

Δ : the unit open disc in C.

$f : \mathfrak{X} \longrightarrow \Delta$ a projective morphism, smooth over the punctured disc Δ^*.

(4.1) $\mathcal{Y} = \bigcup \mathcal{Y}_i$: a divisor on $\mathfrak{X}$, the $\mathcal{Y}_i$ are irreducible components.

$X_0 := f^{-1}(0)$ is reduced.

$\mathcal{Y} \cup X_0$ is NCD on $\mathfrak{X}$.

$\mathcal{Y}_{i_1} \cap \cdots \cap \mathcal{Y}_{i_p}$ $(p \geqslant 1)$ are flat over Δ and smooth over Δ^*.

In this situation, Elzein and Steenbrink-Zucker deal a trifiltered complex $(sA^{\cdot\cdot}, W, M, F)$ constructed as follows (II.II in [E], (5.5) in [S.Z]):

$W(\mathcal{Y})_k \Omega_{\mathfrak{X}}^p(\log(\mathcal{Y}+X_0)) := \Omega_{\mathfrak{X}}^k(\log(\mathcal{Y}+X_0)) \wedge \Omega_{\mathfrak{X}}^{p-k}(\log X_0)$.

$W(X_0)_\ell \Omega_{\mathfrak{X}}^p(\log(\mathcal{Y}+X_0)) := \Omega_{\mathfrak{X}}^\ell(\log(\mathcal{Y}+X_0)) \wedge \Omega_{\mathfrak{X}}^{p-\ell}(\log \mathcal{Y})$.

$W(\mathcal{Y}+X_0)_m \Omega_{\mathfrak{X}}^p(\log(\mathcal{Y}+X_0)) := \Omega_{\mathfrak{X}}^m(\log(\mathcal{Y}+X_0)) \wedge \Omega_{\mathfrak{X}}^{p-m}$.

$A^{p,q} := \Omega_{\mathfrak{X}}^{p+q+1}(\log(\mathcal{Y}+X_0))/W(X_0)_q$ $(p, q \geqslant 0)$.

$W_k A^{p,q}$:= the image of $W(\mathcal{Y})_k \Omega_{\mathfrak{X}}^{p+q+1}(\log(\mathcal{Y}+X_0))$ in $A^{p,q}$.

$M_k A^{p,q}$:= the image of $W(\mathcal{Y}+X_0)_{2q+k+1} \Omega^{p+q+1}(\log(\mathcal{Y}+X_0))$ in $A^{p,q}$.

$F^p A^{\cdot\cdot} := \bigoplus_{p' \geqslant p} A^{p',\cdot}$.

$sA^{\cdot\cdot}$: the associated simple complex of $A^{\cdot\cdot}$.

($sA^{\cdot\cdot}$, W, M, F) [illegible] FCMHC [illegible]

Note that there is a bifiltered quasi-isomorphism (cf. (4.16) in [St]):

$$\theta : (\Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))\otimes O_{X_0}, W(\mathcal{Y}), F) \longrightarrow (sA^{\cdot\cdot}, W, F).$$

Let

$$\Phi : \Delta^* \longrightarrow \langle T\rangle\backslash D$$

be the period map associated to the VGPMHS $(\Delta^*, R^n_Z(\mathring{f}'), W(\mathcal{Y}-X_0)[n], F)$, where $\mathring{f}' := \mathrm{res}(f) : \mathcal{X} - \mathcal{Y} - X_0 \longrightarrow \Delta^*$ and T is the local monodromy. Since X_0 is reduced, T is unipotent. Set

(4.2)
$$N := \log T.$$
$$\Psi(t) := \exp(-\log t/2\pi i.N)\Phi(t) \qquad (t \in \Delta^*).$$

Then (4.1) in [E] and (3.13) in [S.Z] can be interpreted as:

Theorem (4.3) (Elzein and Steenbrink-Zucker). Assume the situation (4.1).

(4.3.1) The map Ψ in (4.2) extends to a holomorphic map

$$\tilde{\Psi} : \Delta \longrightarrow \check{D}$$

compatible with the extensions of Ψ_k associated to the period maps of Griffiths via $\pi : \check{D} \longrightarrow \prod \check{D}_k$.

(4.3.2) For each k, $(W[n]_k, M, \tilde{\Psi}(0))$ is a MHS and N gives a morphism of type $(-1,-1)$ with respect to this MHS.

(4.3.3) For each k, M induces the usual monodromy weight filtration on $Gr^{W[n]}_k$, and the GP Q_k in (2.1) induces the monodromy polarization on $(Gr^{W[n]}_k, M, \tilde{\Psi}(0))$ (cf. (6.16) in [Sc]).

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