Title	Supplement to "Variation of mixed Hodge structure arising from family of logarithmic deformationsII: Classifying space"
Author(s)	Saito, Masahiko; Shimizu, Yuji; Usui, Sampei
Citation	Duke Mathematical Journal. 1985, 52(2), p. 529-534
Version Type	AM
URL	https://hdl.handle.net/11094/73419
rights	
Note	

SUPPLEMENT TO VARIATION OF MIXED HODGE STRUCTURE ARI~~Z~~SING FROM FAMILY OF LOGARITHMIC DEFORMATIONS II: CLASSIFYING SPACE

by Masahiko SAITO, Yuji SHIMIZU, Sampei USUI

In this note, we will define a graded polarization (abbreviated as GP) of the mixed Hodge structure (abbreviated as MHS) on $H^n(X-Y,Q)$, where X is a smooth projective variety over C and Y a ~~smooth~~ simple normal crossing divisor (abbreviated as SNCD) on X, and give some supplements to [U.2]. This note is based on the small meeting of the three authors at RIMS 1-6 X '84.

1. Graded polarization on $H^n(X-Y,Q)$: Let X and Y be as above and set $r := \dim X$. Choose a polarization $\omega \in H^{1,1}(X,Z)$ (i.e. the cohomology class of a very ample invertible sheaf) on X. Let

$$Y = \bigcup_{i\in I} Y_i$$

be the decomposition into irreducible components. We fix once for all an order of these components. We use the following notations:

$Y_J := \bigcap_{j\in J} Y_j$ for a subset $J \subset I$.

$\tilde{Y}^s := \coprod_J Y_J$ where the J run the subsets of I with $\#J = s$ and $\tilde{Y}^0 = X$.

$\omega_J \in H^{1,1}(Y_J,Z)$: the induced polarization on Y_J from ω.

$\omega_s := \bigoplus_{\#J=s} \omega_J \in H^{1,1}(\tilde{Y}^s,Z)$

$\nu_J : H^{2(r-s)}(Y_J,Z) \xrightarrow{\sim} Z$ with $\nu_J(\omega_J^{r-s}) = 1$, where $\#J = s$.

$\nu_s := \sum_{\#J=s} \nu_J : H^{2(r-s)}(\tilde{Y}^s,Z) \longrightarrow Z.$

L : the multiplication operator on the cohomology groups of $\tilde{Y}^s$ induced by the polarization ω_s.

$P^m(\tilde{Y}^s,Q) := \mathrm{Ker}(L^{(r-s)-m+1} : H^m(\tilde{Y}^s,Q) \longrightarrow H^{2(r-s)-m+2}(\tilde{Y}^s,Q))$ the primitive cohomology $(m \leqslant r - s)$.

Then, by the Lefschetz decomposition

$$(1.1)\qquad H^m(\tilde{Y}^s,Q) = \bigoplus_{a\geqslant 0} L^a P^{m-2a}(\tilde{Y}^s,Q),$$

we can define a polarization Q'_s of HS on $H^m(\tilde{Y}^s,Q)$ by

$$(1.2)\qquad Q'_s(u,v) := \sum_{\#J=s} \sum_a (-1)^{(m-2a)(m-2a+1)/2} \nu_s(u_{m-2a} \smile v_{m-2a} \smile \omega_s^{(r-s)-(m-2a)})$$

for $u = \Sigma L^a u_{m-2a}$, $v = \Sigma L^a v_{m-2a} \in H^m(\tilde{Y}^s,Q)$. Note that

$$\frac{(m - 2a)(m - 2a + 1)}{2} = \frac{m(m + 1)}{2} + a \qquad \mathrm{mod}(2).$$

Recall that the weight spectral sequence

$$(1.3)\qquad {}_W E_1^{-s,n+s} = H^{n-s}(\tilde{Y}^s) \Longrightarrow {}_W E^n = H^n(X-Y)$$

is nothing but the Leray spectral sequence for $X-Y \hookrightarrow X$ under the change of indices ${}_W E_k^{-s,n+s} = E_{k+1}^{n-s,s}$ of Leray ((3.2.4) in [D]), degenerates in ${}_W E_2 = {}_W E_\infty$ ((3.2.10) in [D]), and the d_1 are alternating sums of Gysin maps hence morphisms of HS of type (1,1) ((3.2.8) in [D], (5.21) in [G.S]). Set

$$B := \mathrm{Im}(H^{n-s-2}(\tilde{Y}^{s+1}) \xrightarrow{d_1} H^{n-s}(\tilde{Y}^s)) \quad \text{and}$$

$$Z := \mathrm{Ker}(H^{n-s}(\tilde{Y}^s) \xrightarrow{d_1} H^{n-s+2}(\tilde{Y}^{s-1})),$$

and define

$$(1.4)\qquad C := \{u \in Z \mid Q'_s(u,v) = 0 \quad (\forall v \in B)\}.$$

Then it is easy to verify:

(1.5) $C \xrightarrow{\sim} {}_W E_2^{-s,n+s} = {}_W E_\infty^{-s,n+s} = Gr_{n+s}^{W[n]} H^n(X-Y)$ as HS over Q.

The polarization Q'_s in (1.2) induces one on C over Q.

Thus, shifting the indices $Q_k := Q'_{k-n}$, we get:

Proposition(1.6). Let X be a smooth projective variety and Y a SNCD on X. Then a polarization ω on X induces a grade~~s~~d polarization $Q = \{Q_k\}$ of the MHS on $H^n(X-Y,Q)$.

2. Variation of GPMHS arizing from family of logarithmic deformations: The construction in §1 can be easily generalized to the relative case (cf. (3.4), (3.5) in [U.1]. There is a misprint in the latter, i.e. the constant $1/2\pi\sqrt{-1}$ should be omitted), and, instead of Theorem (1.7) in [U.2], we have:

Theorem (2.1). Let $f : \mathfrak{X} \longrightarrow S$ be a smooth, projective morphism of complex manifolds and $\mathfrak{Y}$ be a SNCD on $\mathfrak{X}$, flat over S. Then we have a VGPMHS $(S,\ R^n_Z(\mathring{f}),\ W[n],\ F,\ Q)$ in the sense of Definition (1.1) in [U.2], where $\mathring{f} := f|(\mathfrak{X}-\mathfrak{Y})$ and $R^n_Z(\mathring{f}) := R^n\mathring{f}_*Z$ modulo torsion.

3. Classifying space and period map of GPMHS: Let $(H_Z,\ W,\ F(0),\ Q)$ be a reference GPMHS. Recall the following notations in [U.2]:

$$f^p := \dim F(0)^p H_C.$$

$$f^p_k := \dim F(0)^p Gr^W_k H_C.$$

$$\check{\mathfrak{F}} := \{F = (F^\cdot) \in Flag(H_C;\ldots,f^p,\ldots) \mid \dim F^p Gr^W_k H_C = f^p_k \quad (\forall p, \forall k)\}.$$

$$GL_W(H_C) := \{g \in GL(H_C) \mid gW_k = W_k \quad (\forall k)\}.$$

HIC

$$\check{\pi}_k : \check{\mathfrak{F}} \longrightarrow \check{\mathfrak{F}}_k := Flag(Gr^W_k H_C;\ldots,f^p_k,\ldots).$$

$$\check{D}_k := \{F \in \check{\mathfrak{F}}_k \mid Q_k(F^p, F^{k-p+1}) = 0 \quad (\forall p)\}.$$

$$D_k := \{F \in \check{D}_k \mid i^{2p-k} Q_k(u,\bar{u}) > 0 \quad (0 \neq u \in F^p \cap \bar{F}^{k-p},\ \forall p)\}.$$

$$\check{D} := \bigcap_k \pi_k^{-1}(\check{D}_k) \subset \check{\mathfrak{F}}.$$

$$D := \bigcap_k \pi_k^{-1}(D_k) \subset \check{D}.$$

$$\check{\pi} : \check{D} \longrightarrow \prod_k \check{D}_k \quad \text{the projection.}$$

$\pi : D \longrightarrow \prod_k D_k$ the projection.

$G_{k,C} := \{g \in GL(Gr_k^W H_C) \mid Q_k(gu,gv) = Q_k(u,v) \quad (\forall u, \forall v \in Gr_k^W H_C)\}.$

$G_{k,R} := \{g \in G_{k,C} \mid gGr_k^W H_R = Gr_k^W H_R\}.$

$G_{k,Z} := \{g \in G_{k,R} \mid gGr_k^W H_Z = Gr_k^W H_Z\}.$

$G_C := \{g \in GL_W(H_C) \mid Gr_k^W(g) \in G_{k,C} \quad (\forall k)\}.$

$G_R := \{g \in G_C \mid gH_R = H_R\}.$

$G_Z := \{g \in G_R \mid gH_Z = H_Z\}.$

In case length $\geqslant 2$, G_R acts on D nontransitively (see (3.3) in [U.2]). Let $G_C = G_C'.G_C''$ be a Levi decomposition with G_C' = the unipotent radical of G_C and G_C'' = a semi-simple part of G_C. Instead of our previous G_R, Carlson in [C] takes

$G := G_C'.(G_R \cap G_C'').$

Then, it is obvious to see that G acts transitively on D. The isotropy subgroup of G at $0 \in D$ is not compact. Nevertheless he proved in [C] that G_Z acts on D properly discontituously.

Combining (2.10) in [U.1], (2.11), (2.16), (3.5), (3.6) and (4.2) in [U.2], and the result of §4 in [C], we get:

Theorem (3.1). (3.1.1) $\check{\pi} : \check{D} \longrightarrow \prod \check{D}_k$ is an algebraic homogeneous vector bundle with respect to the group G_C.

(3.1.2) G acts transitively on D.

(3.1.3) G_Z acts properly discontinuously on D.

(3.1.4) There is an extended horizontal subbundle $T^{eh}_{\check{D}}$ on $\check{D}$ which is compatible with the horizontal subbundle $\oplus T^h_{\check{D}_k}$ on $\prod \check{D}_k$ via $\check{\pi}$.

(3.1.5) The period map associated to the VGPMHS $(S, R^n_Z(\mathring{f}), W[n], F, Q)$ arising from geometry in (2.1) above has extended horizontal local liftings with respect to $T^{eh}_{\check{D}}$ and compatible with the period maps of Griffiths associated to the VPMHS HL $(S, Gr_k^{W[n]}R_Z^n(\mathring{f}), F$ $Q_k) = (S, R_Z^{2n-k}(\tilde{g}^{k-n}), F, Q_k)$, where $\tilde{g}^s : \tilde{\mathcal{Y}}^s \longrightarrow S$ is induced from f and $\tilde{\mathcal{Y}}^s$ is a relative version of $\tilde{Y}^s$ in §1.

4. Degeneration of VGPMHS associated to semi-stable degeneration of family of logarithmic deformations: We want to interpret the results in II.II of [E] and in §5 of [S.Z] into our language for our future use.

Consider a situation:

$\mathcal{X}$: a complex manifold.

Δ : the unit open disc in C.

$f : \mathcal{X} \longrightarrow \Delta$ a projective morphism, smooth over the punctured disc Δ^*.

(4.1) $\mathcal{Y} = \cup \mathcal{Y}_i$: a divisor on $\mathcal{X}$, the $\mathcal{Y}_i$ are irreducible components.

$X_0 := f^{-1}(0)$ is reduced.

$\mathcal{Y} \cup X_0$ is NCD on $\mathcal{X}$.

$\mathcal{Y}_{i_1} \cap \cdots \cap \mathcal{Y}_{i_p}$ $(p \geqslant 1)$ are flat over Δ and smooth over Δ^*.

In this situation, Elzein and Steenbrink-Zucker deal a trifiltered complex $(sA^{\cdot\cdot}, W, M, F)$ constructed as follows (II.II in [E], (5.5) in [S.Z]):

$$W(\mathcal{Y})_k \Omega^p_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) := \Omega^k_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) \wedge \Omega^{p-k}_{\mathcal{X}}(\log X_0).$$

$$W(X_0)_\ell \Omega^p_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) := \Omega^\ell_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) \wedge \Omega^{p-\ell}_{\mathcal{X}}(\log \mathcal{Y}).$$

$$W(\mathcal{Y}+X_0)_m \Omega^p_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) := \Omega^m_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) \wedge \Omega^{p-m}_{\mathcal{X}}.$$

$$A^{p,q} := \Omega^{p+q+1}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))/W(X_0)_q \quad (p,\ q \geqslant 0).$$

$W_k A^{p,q}$:= the image of $W(\mathcal{Y})_k \Omega^{p+q+1}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))$ in $A^{p,q}$.

$M_k A^{p,q}$:= the image of $W(\mathcal{Y}+X_0)_{2q+k+1}\Omega^{p+q+1}(\log(\mathcal{Y}+X_0))$ in $A^{p,q}$.

$$F^p A^{\cdot\cdot} := \bigoplus_{p' \geqslant p} A^{p',\cdot}.$$

$sA^{\cdot\cdot}$: the associated simple complex of $A^{\cdot\cdot}$.

$(sA^{\cdot\cdot}, W, M, F)$ [illegible] FCMH [illegible]

Note that there is a bifiltered quasi-isomorphism (cf. (4.16) in [St]):

$$\theta : (\Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))\otimes O_{X_0},\ W(\mathcal{Y}),\ F) \longrightarrow (sA^{\cdot\cdot},\ W,\ F).$$

Let

$$\Phi : \Delta^* \longrightarrow \langle T\rangle\backslash D$$

be the period map associated to the VGPMHS $(\Delta^*,\ R^n_Z \mathring{f}'),\ W(\mathcal{Y}-X_0)[n],\ F)$, where $\mathring{f}' := \mathrm{res}(f) : \mathcal{X} - \mathcal{Y} - X_0 \longrightarrow \Delta^*$ and T is the local monodromy. Since X_0 is reduced, T is unipotent. Set

$$(4.2)\qquad \begin{aligned} N &:= \log T. \\ \Psi(t) &:= \exp(-\log t/2\pi i.N)\Phi(t) \qquad (t \in \Delta^*). \end{aligned}$$

Then (4.1) in [E] and (3.13) in [S.Z] can be interpreted as:

Theorem (4.3) (Elzein and Steenbrink-Zucker). Assume the situation (4.1).

(4.3.1) The map Ψ in (4.2) extends to a holomorphic map

$$\tilde{\Psi} : \Delta \longrightarrow \check{D}$$

compatible with the extensions of Ψ_k associated to the period maps of Griffiths via $\pi : \check{D} \longrightarrow \prod \check{D}_k$.

(4.3.2) For each k, $(W[n]_k,\ M,\ \tilde{\Psi}(0))$ is a MHS and N gives a morphism of type $(-1,-1)$ with respect to this MHS.

(4.3.3) For each k, M induces the usual monodromy weight filtration on $Gr_k^{W[n]}$, and the GP Q_k in (2.1) induces the monodromy polarization on $(Gr_k^{W[n]},\ M,\ \tilde{\Psi}(0))$ (cf. (6.16) in [Sc]).

REFERENCES

[C] Carlson, J.A., The obstruction of splitting a mixed Hodge structure over the integers I. preprint received 23 V 1984.

[D] Deligne, P., Théorie de Hodge II, Publ. I.H.E.S. 40 (1971) 5-58.

[E] Elzein, F., Dégénérescence diagonale I; II, C.R. Acad. Sci. Paris 296 (1983) 51-54; 199-202.

[G.S] Griffiths, P. & W. Schmid, Recent developments in Hodge theory: A discussion of techniques and results, Proc. Internat. Coll., Bombay, Oxford Press (1971) 31-127.

[Sc] Schmid, W., Variation of Hodge structure: The singularities of period mapping, Inv. Math. 22 (1973) 211-319.

[St] Steenbrink, J., Limits of Hodge structures, Inv. Math. 31 (1976) 229-257.

[S.Z] Steenbrink, J. & S. Zucker, Variations of mixed Hodge structure I, preprint received 24 III 1984.

[U.1] Usui, S., Variation of mixed Hodge structure arising from family of logarithmic deformations, Ann. sci. Éc. Norm. Sup. 4-16 (1983) 91-107.

[U.2] Usui, S., Variation of mixed Hodge structure arising from family of logarithmic deformations II: Classifying space, to appear in Duke Math. J.

Yuji SHIMIZU
Department of Mathmatics
Faculty of Science
University of Tokyo
Tokyo, 113 Japan

Masahiko SAITO
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto, 606 Japan

Sampei USUI
Departmaent of Mathematics
Faculty of Sciences
Kochi University
Kochi, 780 Japan