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LOGARITHMIC HODGE STRUCTURES AND CLASSIFYING SPACES

KAZUYA KATO, SAMPEI USUI

INTRODUCTION

This work is an attempt to add ‘points at infinity’ to the classifying space of polarized Hodge structures (PH, for short) of arbitrary weight. It is based on the following two ideas:

(0.1) We introduce the notion of *polarized logarithmic Hodge structures (PLH, for short)* (§5). Polarized logarithmic Hodge structures are defined by using the theory of logarithmic structures of Fontaine-Illusie (cf. [I], [K1]), and they work well in the analysis of degenerations of polarized Hodge structures. Our principle is that we enlarge the classifying space of polarized Hodge structures in the following way (for precise definitions, see Definitions in (2.1), (5.2)):

$$\begin{aligned} D &:= (\text{Classifying space of PH}) \subset \\ D_\Sigma &:= (\text{Space of nilpotent orbits in directions in } \Sigma), \\ \Gamma \backslash D_\Sigma &= (\text{Classifying space of PLH with local monodromies} \\ &\quad \text{in directions in } \Sigma \text{ and with global monodromy } \Gamma). \end{aligned}$$

Here Γ is a discrete subgroup of $\text{Aut}(D)$ satisfying certain conditions.

(0.2) We introduce the notions of *generalized analytic spaces* and *generalized fs logarithmic analytic spaces* (see (4.2)). Our extended space $\Gamma \backslash D_\Sigma \supset \Gamma \backslash D$ is not in general an analytic space. For instance, it can be something like

$$S := \{(x, y) \in \mathbf{C}^2; x \neq 0\} \cup \{(0, 0)\}.$$

The space $\Gamma \backslash D_\Sigma$ and the above space S admit the structure of generalized analytic space or even generalized fs logarithmic analytic space. In order to give S the structure of generalized analytic space, the essential point is to endow it with the so-called *strong topology* (see (4.1)), which is strictly stronger than the topology as a subspace of \mathbf{C}^2 .

Our main result says that the generalized fs logarithmic analytic space $\Gamma \backslash D_\Sigma$ is a fine moduli space of polarized logarithmic Hodge structures (§6). In a way, this realizes one of the dreams of Griffiths in [G].

We would like to add a remark. D is always a homogeneous complex manifold. Assume that the horizontal tangent bundle of D is not trivial. Then, it is a Hermitian symmetric domain if and only if the set of Hodge numbers is one of the following three cases:

- (i) $w = 2t + 1$, $h^{p,q} = 0$ unless $p = t + 1$, t .
- (ii) $w = 2t$, $h^{p,q} = 1$ for $p = t + 1$, $t - 1$, $h^{t,t}$ is arbitrary, $h^{p,q} = 0$ otherwise.
- (iii) $w = 2t$, $h^{p,q} = 1$ for $p = t + a$, $t + a - 1$, $t - a + 1$, $t - a$ for some $a \geq 2$, $h^{p,q} = 0$ otherwise.

Familiar (analytic) compactifications of $\Gamma \backslash D$ such as the Satake-Baily-Borel, the Borel-Serre, and toroidal are constructed only for those D which are Hermitian symmetric domains. The only partially successful attempts to go beyond the symmetric case are the work [CK] of Cattani-Kaplan and the work [U2]. By the construction of the present article, we can talk about, for example, the extended period mapping associated to a degeneration of surfaces of general type, or of Calabi-Yau manifolds. One of the remaining

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open problems is to find ‘big fans’ Σ so that the spaces $\Gamma \backslash D_\Sigma$ contain all possible boundary points. This work is a continuation of the study [U2], which treats the part of $\Gamma \backslash D_\Sigma$, and is also a continuation of the study [KkNc] on logarithmic complex geometry.

This article is an abbreviated account. A paper [KU] with complete proofs will appear elsewhere.

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§1. PRELIMINARIES

(1.1) *Classifying spaces of Hodge structures.*

Let $(h^{p,q})_{p+q=w}$, with $h^{p,q} = h^{q,p}$, be a set of Hodge numbers of weight w . Let H_0 be a free \mathbf{Z} -module of rank $\sum h^{p,q}$ and let $\langle \cdot, \cdot \rangle_0$ be a non-degenerate \mathbf{Q} -bilinear form on $H_{0,\mathbf{Q}} := \mathbf{Q} \otimes H_0$ which is symmetric if w is even and anti-symmetric if w is odd. We denote by D the set of all decreasing filtrations F on $H_{0,\mathbf{C}} := \mathbf{C} \otimes H_0$ such that $(H_0, \langle \cdot, \cdot \rangle_0, F)$ is a polarized Hodge structure of Hodge type $(h^{p,q})_{p+q=w}$. We denote by \check{D} the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that $\dim F^p = \sum_{r \geq p} h^{r, w-r}$ and $\langle F^p, F^q \rangle_0 = 0$ for $p+q > w$.

Let $G_{\mathbf{Z}} := \text{Aut}(H_0, \langle \cdot, \cdot \rangle_0)$. For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, we denote $G_A := \text{Aut}(H_{0,A}, \langle \cdot, \cdot \rangle_0)$ and $\mathfrak{g}_A := \text{Lie } G_A$. Then \check{D} is a compact homogeneous space under $G_{\mathbf{C}}$, D is an open subset of \check{D} that is homogeneous under $G_{\mathbf{R}}$. Let Γ be a subgroup of $G_{\mathbf{Z}}$. Then $\Gamma \backslash D$ is the classifying space of polarized Hodge structures on H_0 with the given Hodge numbers $(h^{p,q})_{p+q=w}$ and with the given global monodromy Γ .

(1.2) *Fans.*

Definitions. (1.2.1) *A subset $\sigma \subset \mathfrak{g}_{\mathbf{Q}}$ is a nilpotent cone if it satisfies the following two conditions:*

- (i) σ is a finitely generated cone in $\mathfrak{g}_{\mathbf{Q}}$ consisting of mutually commuting nilpotent elements.
- (ii) $\sigma \cap (-\sigma) = \{0\}$.

(1.2.2) *A fan in $\mathfrak{g}_{\mathbf{Q}}$ is a non-empty set Σ of nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$ satisfying the following two conditions:*

- (i) If $\sigma \in \Sigma$ then any face of σ belongs to Σ .
- (ii) If $\sigma, \sigma' \in \Sigma$ then $\sigma \cap \sigma'$ is a face of σ and of σ' .

(1.2.3) *A subgroup $\Gamma \subset G_{\mathbf{Z}}$ is strongly compatible with a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ if it satisfies the following two conditions:*

- (i) If $\gamma \in \Gamma$ and $\sigma \in \Sigma$ then $\gamma\sigma\gamma^{-1} \in \Sigma$.
- (ii) If $\sigma \in \Sigma$ and if we define

$$\Gamma(\sigma) := \{\gamma \in \Gamma; \gamma: \text{unipotent, } \log \gamma \in \sigma\},$$

then $\log \Gamma(\sigma)$ generates the cone σ .

Example. Let

$$\Xi := \{\sigma; \sigma \text{ is a nilpotent cone in } \mathfrak{g}_{\mathbf{Q}}, \text{rank}(\sigma) \leq 1\}.$$

Then Ξ is a fan. For any subgroup Γ of $G_{\mathbf{Z}}$ with finite index, Γ is automatically strongly compatible with Ξ . Whereas, $\Gamma = \{1\}$ is not strongly compatible with Ξ , i.e., the condition (1.2.3) (ii) is not fulfilled.

(1.3) *Nilpotent orbits and nilpotent i -orbits.*

Definition. Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{Q}}$. For $A = \mathbf{R}, \mathbf{C}$, we denote by σ_A the A -linear span of σ in \mathfrak{g}_A . A subset Z of \check{D} is a σ -nilpotent orbit (resp. σ -nilpotent i -orbit) if it satisfies the following three conditions for some $F \in Z$.

- (i) $Z = \exp(\sigma_{\mathbf{C}})F$ (resp. $Z = \exp(i\sigma_{\mathbf{R}})F$).
- (ii) $NF^p \subset F^{p-1}$ ($\forall p, \forall N \in \sigma$), (Griffiths transversality).
- (iii) $\exp(\sum_{1 \leq j \leq r} iy_j N_j)F \in D$ ($\forall y_j \gg 0$), (positivity). Here $(N_j)_{1 \leq j \leq r}$ is a system of generators of σ .

(1.4) *Logarithmic structures.*

Let X be a ringed space over \mathbf{C} with structure sheaf \mathcal{O}_X . We recall briefly the notion of logarithmic structures on X of Fontaine-Illusie (for details, cf. [I], [K1]).

A *pre-logarithmic structure* on X is a sheaf of commutative monoids M_X together with a homomorphism $\alpha : M_X \rightarrow \mathcal{O}_X$ of monoids, where \mathcal{O}_X is regarded as a sheaf of monoids by multiplication. A *logarithmic structure* on X is a pre-logarithmic structure (M_X, α) on X which satisfies $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$.

Let (M_X, α) be a pre-logarithmic structure on X . The *associated logarithmic structure* $(\tilde{M}_X, \tilde{\alpha})$ is defined as the push-forward \tilde{M}_X of $\mathcal{O}_X^\times \leftarrow \alpha^{-1}(\mathcal{O}_X^\times) \rightarrow M_X$, namely the diagram

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^\times) & \longrightarrow & M_X \\ \downarrow & & \downarrow \\ \mathcal{O}_X^\times & \longrightarrow & \tilde{M}_X. \end{array}$$

is co-cartesian, together with the homomorphism $\tilde{\alpha} : \tilde{M}_X \rightarrow \mathcal{O}_X$ induced by $\alpha : M_X \rightarrow \mathcal{O}_X$ and the inclusion $\mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$.

Let $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and (M, α) be a logarithmic structure on Y . Then the sheaf-theoretic inverse image $f^{-1}M$ together with the composite morphism $f^{-1}M \rightarrow f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ form a pre-logarithmic structure on X . The *inverse image* $f^*(M, \alpha)$ of (M, α) is defined as the logarithmic structure on X associated to the above pre-logarithmic structure.

A monoid \mathcal{S} is *fs* (= finitely-generated, integral and saturated) if it satisfies the following three conditions.

- (i) \mathcal{S} is finitely generated.
- (ii) If $a, b, c \in \mathcal{S}$ and $ab = ac$, then $b = c$.
- (iii) If $a \in \mathcal{S}^{\text{gp}}$ and $a^n \in \mathcal{S}$ for some positive integer n , then $a \in \mathcal{S}$.

Here \mathcal{S}^{gp} is the group generated by \mathcal{S} . A logarithmic structure (M_X, α) on X is *fs* if there exists an open covering $(U_\lambda)_\lambda$ of X such that each restriction $(M_X, \alpha)|_{U_\lambda}$ is the logarithmic structure associated to a pre-logarithmic structure of the form $(\mathcal{S}_{U_\lambda}, \alpha_\lambda)$ where \mathcal{S}_{U_λ} is a constant sheaf of monoids on U_λ induced by an fs monoid \mathcal{S} . In this case, $(\mathcal{S}_{U_\lambda}, \alpha_\lambda)$ is called a *chart* of (M_X, α) on U_λ .

(1.5) *The ringed space $(X^{\text{log}}, \mathcal{O}_X^{\text{log}})$.*

Let X be an fs logarithmic local ringed space (= a local ringed space endowed with an fs logarithmic structure) over \mathbf{C} which satisfies the following two conditions:

- (i) $\mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbf{C}$.
- (ii) For any open set $U \subset X$ and any $f \in \mathcal{O}_X(U)$, the evaluation map $U \rightarrow \mathbf{C}, x \mapsto f(x)$, is continuous.

We recall briefly the associated ringed space $(X^{\text{log}}, \mathcal{O}_X^{\text{log}})$ introduced in [KkNc] (cf. also [KyNy]).

As a set, X^{log} is the set of all pairs (x, h) , consisting of a point $x \in X$ and a homomorphism $h : M_{X,x} \rightarrow \mathbf{S}^1$ whose restriction to \mathcal{O}_X^\times is $u \mapsto u(x)/|u(x)|$. Here $\mathbf{S}^1 := \{t \in \mathbf{C}; |t| = 1\}$.

The topology of X^{log} is defined as follows. We work locally on X . Take a chart $\mathcal{S} \rightarrow M_X$, then we have an injective map

$$X^{\text{log}} \hookrightarrow X \times \text{Hom}(\mathcal{S}^{\text{gp}}, \mathbf{S}^1), (x, h) \mapsto (x, h_{\mathcal{S}}),$$

where $h_{\mathcal{S}}$ denotes the composite map $\mathcal{S}^{\text{gp}} \rightarrow M_{X,x}^{\text{gp}} \rightarrow \mathbf{S}^1$. We endow X^{log} with the topology as a subset of $X \times \text{Hom}(\mathcal{S}^{\text{gp}}, \mathbf{S}^1)$. It can be seen that this topology is independent of the choice of chart and hence is globally well-defined, and that the canonical map $\tau : X^{\text{log}} \rightarrow X, (x, h) \mapsto x$ is surjective, continuous and proper.

The sheaf of \mathbf{C} -algebras \mathcal{O}_X^{\log} is defined as follows. Let \mathcal{L} be the fiber product of

$$\begin{array}{ccc} & \tau^{-1}(M_X^{\text{gp}}) & \\ & \downarrow & \\ \text{Cont}(\cdot, i\mathbf{R}) & \xrightarrow{\text{exp}} & \text{Cont}(\cdot, \mathbf{S}^1), \end{array}$$

where $\text{Cont}(\cdot, T)$, for a topological space T , denotes the sheaf on X^{\log} of continuous maps to T , and $\tau^{-1}(M_X^{\text{gp}}) \rightarrow \text{Cont}(\cdot, \mathbf{S}^1)$ comes from the definition of X^{\log} . We define

$$\mathcal{O}_X^{\log} := (\tau^{-1}(\mathcal{O}_X) \otimes_{\mathbf{Z}} \text{Sym}_{\mathbf{Z}}(\mathcal{L})) / \mathfrak{a},$$

where $\text{Sym}_{\mathbf{Z}}(\mathcal{L})$ denotes the symmetric algebra of \mathcal{L} over \mathbf{Z} , and \mathfrak{a} is the ideal of $\tau^{-1}(\mathcal{O}_X) \otimes_{\mathbf{Z}} \text{Sym}_{\mathbf{Z}}(\mathcal{L})$ generated by the image of

$$\tau^{-1}(\mathcal{O}_X) \rightarrow \tau^{-1}(\mathcal{O}_X) \otimes_{\mathbf{Z}} \text{Sym}_{\mathbf{Z}}(\mathcal{L}), \quad f \mapsto f \otimes 1 - 1 \otimes \iota(f).$$

Here the map $\iota : \tau^{-1}(\mathcal{O}_X) \rightarrow \mathcal{L}$ is the one induced by

$$\begin{aligned} \tau^{-1}(\mathcal{O}_X) &\rightarrow \text{Cont}(\cdot, i\mathbf{R}), \quad f \mapsto \frac{1}{2}(f - \bar{f}), \quad \text{and} \\ \tau^{-1}(\mathcal{O}_X) &\xrightarrow{\text{exp}} \tau^{-1}(\mathcal{O}_X^{\times}) \subset \tau^{-1}(M_X^{\text{gp}}). \end{aligned}$$

Let $y \in X^{\log}$ and $x := \tau(y) \in X$. There exist a family $(\ell_j)_{1 \leq j \leq n}$ of elements of \mathcal{L}_y , whose images in $(M_X^{\text{gp}} / \mathcal{O}_X^{\times})_x$ form a system of free generators, and which are algebraically independent over $\mathcal{O}_{X,x}$ and generate the stalk $\mathcal{O}_{X,y}^{\log} = \tau^{-1}(\mathcal{O}_{X,x})[\ell_1, \dots, \ell_n]$. Note that this is *not* a local ring if $(M_X^{\text{gp}} / \mathcal{O}_X^{\times})_x \neq \{1\}$.

Example. Let $X := \mathbf{C}$ be the complex plane and let z be the coordinate of X . The fs logarithmic structure on X associated to the divisor $\{0\}$ is defined by

$$M_X := \{f \in \mathcal{O}_X; f \text{ is invertible outside } \{0\}\} \hookrightarrow \mathcal{O}_X.$$

We can take a chart $\mathbf{N} \rightarrow M_X = \bigsqcup_{n \geq 0} z^n \mathcal{O}_X^{\times}$, $n \mapsto z^n$. We have an isomorphism

$$X^{\log} \xrightarrow{\sim} \mathbf{R}_{\geq 0} \times \mathbf{S}^1, \quad (x, h) \mapsto (|z(x)|, h(z)).$$

Let $\ell := \log z$, then we have $\mathcal{O}_X^{\log} = \tau^{-1}(\mathcal{O}_X)[\ell]$.

§2. SPACES OF NILPOTENT ORBITS

Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ .

(2.1) *The set D_{Σ} .*

Definition. *As a set, we define the space D_{Σ} of nilpotent orbits in the directions in Σ by*

$$D_{\Sigma} := \{(\sigma, Z); \sigma \in \Sigma, Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit}\}.$$

Remark. Note that the set D is contained in D_{Σ} by $F \mapsto (\{0\}, \{F\})$.

Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{Q}}$. Then, $\{\text{faces of } \sigma\}$ is a fan, and we denote

$$D_{\sigma} := D_{\{\text{faces of } \sigma\}}.$$

(2.2) *The toric variety $\text{toric}(\Gamma(\sigma))$.*

In order to fix notations, we review here toric varieties (cf. [AMRT]). For $\sigma \in \Sigma$, we see that $\Gamma(\sigma)$ is an fs monoid with $\Gamma(\sigma)^\times = \{1\}$. The associated toric variety is defined by

$$\text{toric}(\Gamma(\sigma)) := \text{Spec}(\mathbf{C}[\Gamma(\sigma)^\vee])_{\text{an}}.$$

Here $\Gamma(\sigma)^\vee := \text{Hom}(\Gamma(\sigma), \mathbf{N})$. Note that the set of complex points of $\text{toric}(\Gamma(\sigma))$ can be identified with $\text{Hom}(\Gamma(\sigma)^\vee, \mathbf{C}^{\text{mult}})$. Here \mathbf{C}^{mult} means the set \mathbf{C} of complex numbers with multiplication.

We have bijections

$$(2.2.1) \quad \text{toric}(\Gamma(\sigma)) \xleftarrow{\alpha} \bigsqcup_{\tau \prec \sigma} (\Gamma(\sigma)^{\text{gp}}/\Gamma(\tau)^{\text{gp}}) \otimes \mathbf{C}^\times \xrightarrow{\beta} \bigsqcup_{\tau \prec \sigma} \exp(\sigma_{\mathbf{C}})/(\exp(\tau_{\mathbf{C}})\Gamma(\sigma)^{\text{gp}})$$

Here $\tau \prec \sigma$ means that τ is a face of σ , and α is defined as follows: For $a \in (\Gamma(\sigma)^{\text{gp}}/\Gamma(\tau)^{\text{gp}}) \otimes \mathbf{C}^\times$ and $h \in \Gamma(\sigma)^\vee$,

$$\alpha(\tau, a)(h) := \begin{cases} 0 & \text{if } h(\Gamma(\tau)) \neq 0, \\ h(a) & \text{if } h(\Gamma(\tau)) = 0. \end{cases}$$

In the second case in the above, we regard h as the induced homomorphism $h : (\Gamma(\sigma)^{\text{gp}}/\Gamma(\tau)^{\text{gp}}) \otimes \mathbf{C}^\times \rightarrow \mathbf{Z} \otimes \mathbf{C}^\times = \mathbf{C}^\times$.

And β is defined by

$$\beta(\gamma \otimes \exp(2\pi iz)) := \exp(z \log \gamma) \quad (\gamma \in \Gamma(\sigma)^{\text{gp}}, z \in \mathbf{C}).$$

(2.3) The subset $E_\sigma \subset \text{toric}(\Gamma(\sigma)) \times \check{D}$.

Definition. For $\sigma \in \Sigma$, we define the set E_σ by

$$E_\sigma := \{(\theta, F) \in \text{toric}(\Gamma(\sigma)) \times \check{D}; \exp(\tau_{\mathbf{C}})gF \text{ is a } \tau\text{-nilpotent orbit}\}$$

Here $(\tau, g \bmod \exp(\tau_{\mathbf{C}})\Gamma(\sigma)^{\text{gp}})$ denotes the pair corresponding to θ under the composite map (2.2.1).

Define the map $E_\sigma \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ by $(\theta, F) \mapsto (\tau, \exp(\tau_{\mathbf{C}})gF)$.

Define the action of $\sigma_{\mathbf{C}}$ on E_σ over $\Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ by

$$a \cdot (\theta, F) := (\exp(2\pi ia)\theta, \exp(-a)F) \quad (a \in \sigma_{\mathbf{C}}).$$

Here, for $b \in \sigma_{\mathbf{C}}$, $\exp(b)\theta$ is the element of $\text{Hom}_{\mathbf{C}\text{-alg}}(\mathbf{C}[\Gamma(\sigma)^\vee], \mathbf{C})$ defined by

$$(\exp(b)\theta)(h) := \exp(h(\exp(b)))\theta(h) \quad (h \in \Gamma(\sigma)^\vee).$$

Then, $E_\sigma \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ is surjective (more precisely, a $\sigma_{\mathbf{C}}$ -torsor, see (6.1.1) below).

Example. In the case $\text{rank}(\sigma) = 1$, we have $\text{toric}(\Gamma(\sigma)) \simeq \mathbf{C}$. Let γ be the generator of $\Gamma(\sigma)$ and let $N := \log(\gamma)$. Then we have

$$E_\sigma := \{(\theta, F) \in \mathbf{C} \times \check{D}; \text{If } \theta \neq 0, \exp((\log(\theta)/2\pi i)N)F \in D. \\ \text{If } \theta = 0, \exp(\sigma_{\mathbf{C}})F \text{ is a } \sigma\text{-nilpotent orbit.}\}$$

Here the cases $\theta \neq 0$, $\theta = 0$ correspond $\tau = 0$, $\tau = \sigma$, respectively. $\sigma_{\mathbf{C}} \simeq \mathbf{C} \ni a$ acts on E_σ over $\Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ by

$$a \cdot (\theta, F) = (\exp(2\pi ia)\theta, \exp(-aN)F).$$

§3. SPACES OF NILPOTENT i -ORBITS

Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ as before.

(3.1) *The set D_{Σ}^{\sharp} .*

Definition. *As a set, we define the space D_{Σ}^{\sharp} of nilpotent i -orbits in the directions in Σ by*

$$D_{\Sigma}^{\sharp} := \{(\sigma, Z); \sigma \in \Sigma, Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent } i\text{-orbit}\}.$$

Define the map

$$D_{\Sigma}^{\sharp} \rightarrow D_{\Sigma}, \quad (\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z).$$

Remark. Note that the set D is contained in D_{Σ}^{\sharp} by $F \mapsto (\{0\}, \{F\})$.

For a nilpotent cone σ in $\mathfrak{g}_{\mathbf{Q}}$, we denote

$$D_{\sigma}^{\sharp} := D_{\{\text{faces of } \sigma\}}^{\sharp}.$$

(3.2) *The real toric variety $|\text{toric}|(\sigma)$.*

In order to fix notations, we review here real toric varieties (for details, see [AMRT, I §1], [HZ, 2.1]). For $\sigma \in \Sigma$, we define

$$|\text{toric}|(\sigma) := \text{Hom}(\sigma^{\vee}, (\mathbf{R} \cup \{\infty\})^{\text{add}}) \supset \sigma_{\mathbf{R}} = \text{Hom}(\sigma^{\vee}, \mathbf{R}^{\text{add}}).$$

Here $\sigma^{\vee} := \text{Hom}(\sigma, \mathbf{Q}_{\geq 0}^{\text{add}})$. We have the bijection

$$(3.2.1) \quad |\text{toric}|(\sigma) \xrightarrow{\alpha} \bigsqcup_{\tau \prec \sigma} \sigma_{\mathbf{R}} / \tau_{\mathbf{R}},$$

defined by

$$\alpha(\tau, H \bmod \tau_{\mathbf{R}})(h) := \begin{cases} h(H) & \text{if } h(\tau) = 0, \\ \infty & \text{if } h(\tau) \neq 0, \end{cases} \quad (h \in \sigma^{\vee}).$$

By using the isomorphism

$$(\mathbf{R} \cup \{\infty\})^{\text{add}} \xrightarrow{\sim} \mathbf{R}_{\geq 0}^{\text{mult}}, \quad y \mapsto e^{-2\pi y},$$

we have

$$|\text{toric}|(\sigma) \simeq \text{Hom}(\sigma^{\vee}, \mathbf{R}_{\geq 0}^{\text{mult}}) \simeq \text{Hom}(\Gamma(\sigma)^{\vee}, \mathbf{R}_{\geq 0}^{\text{mult}}).$$

If we identify $\text{toric}(\Gamma(\sigma))$ with $\text{Hom}(\Gamma(\sigma)^{\vee}, \mathbf{C}^{\text{mult}})$, we see that the maps

$$\mathbf{C}^{\text{mult}} \rightarrow \mathbf{R}_{\geq 0}^{\text{mult}}, \quad q \mapsto |q|, \quad \text{and} \quad \mathbf{R}_{\geq 0}^{\text{mult}} \rightarrow \mathbf{C}^{\text{mult}}, \quad r \mapsto r,$$

induce the maps

$$(3.2.2) \quad \text{toric}(\Gamma(\sigma)) \rightleftarrows |\text{toric}|(\sigma).$$

(3.3) *The subset $E_{\Sigma}^{\sharp} \subset |\text{toric}|(\sigma) \times \check{D}$.*

Definition. *For $\sigma \in \Sigma$, we define the set E_{σ}^{\sharp} by*

$$E_{\sigma}^{\sharp} := \{(b, F) \in |\text{toric}|(\sigma) \times \check{D}; \exp(i\tau_{\mathbf{R}})\exp(iH)F \text{ is a } \tau\text{-nilpotent } i\text{-orbit}\}.$$

Here $(\tau, H \bmod \tau_{\mathbf{R}})$ is the point in $\bigsqcup_{\tau \prec \sigma} \sigma_{\mathbf{R}} / \tau_{\mathbf{R}}$ corresponding to b in $|\text{toric}|(\sigma)$ under the bijection (3.2.1).

Define the map $E_{\sigma}^{\sharp} \rightarrow D_{\sigma}^{\sharp}$ by $(b, F) \mapsto \exp(i\tau_{\mathbf{R}})\exp(iH)F$.

Define the action of $\sigma_{\mathbf{R}}$ on E_{σ}^{\sharp} over D_{σ}^{\sharp} by

$$a \cdot ((\tau, H \bmod \tau_{\mathbf{R}}), F) := ((\tau, (H + a) \bmod \tau_{\mathbf{R}}), \exp(-ia)F) \quad (a \in \sigma_{\mathbf{R}}).$$

Then, $E_{\sigma}^{\sharp} \rightarrow D_{\sigma}^{\sharp}$ is surjective (more precisely, a $\sigma_{\mathbf{R}}$ -torsor, see (6.1.1) below). The maps in (3.2.2) induce the maps

$$(3.3.1) \quad E_{\sigma} \rightleftarrows E_{\sigma}^{\sharp},$$

whose composite $E_{\sigma}^{\sharp} \rightarrow E_{\sigma} \rightarrow E_{\sigma}^{\sharp}$ is the identity.

§4. TOPOLOGY, GENERALIZED FS LOGARITHMIC ANALYTIC SPACES

(4.1) *The strong topology.*

Definition. Let X be an analytic space and S be a subset on X . The strong topology on S is the strongest topology on S for which the map $\lambda : Y \rightarrow S$ is continuous for any analytic space Y and for any analytic morphism $\lambda : Y \rightarrow X$ with $\lambda(Y) \subset S$.

Remark. In the notation of the above definition, if S is an analytic subspace of X then the strong topology on S coincides with the usual topology, i.e., the topology as a subspace of X .

Example. Let $X := \mathbf{C}^2$ be the complex two-dimensional plane and let $S := \mathbf{C}^2 - \{0\} \times \mathbf{C} + \{(0, 0)\} \subset X$. For the strong topology on S and the usual topology on S as a subspace of X , we can observe the following:

If $k \in \mathbf{Z}_{\geq 1}$ and $z \in \mathbf{C}$ converges to 0 then (z^k, z) converges to $(0, 0)$ in the strong topology on S .

If $a \in \mathbf{R}_{>1}$ and $s \in \mathbf{R}$, $s \rightarrow \infty$, then $(1/a^s, 1/s)$ does not converge to $(0, 0)$ in the strong topology on S , whereas $(1/a^s, 1/s)$ converges to $(0, 0)$ in the usual topology on S as a subspace of X .

(4.2) *Generalized (fs logarithmic) analytic spaces.*

Definitions. (4.2.1) Let \mathcal{C} be a category and let \mathcal{A} be a full subcategory of \mathcal{C} . For an object S of \mathcal{C} , let $h_{\mathcal{A}}^S$ be the contravariant functor

$$\mathcal{A} \rightarrow \text{Sets}, \quad Y \mapsto h_{\mathcal{A}}^S(Y) := \text{Mor}(Y, S).$$

Denote by $\bar{\mathcal{A}} \supset \mathcal{A}$ the full subcategory of \mathcal{C} consisting of all objects S satisfying

$$\text{Mor}(S, Z) \xrightarrow{\sim} \text{Mor}(h_{\mathcal{A}}^S, h_{\mathcal{A}}^Z) \quad \text{for any object } Z \text{ of } \mathcal{C}.$$

(4.2.2) For

$$\mathcal{C} := (\text{local ringed spaces over } \mathbf{C}) \supset \mathcal{A} := (\text{analytic spaces}),$$

we call an object of $\bar{\mathcal{A}}$ a generalized analytic space.

(4.2.3) For

$$\begin{aligned} \mathcal{C} &:= (\text{fs logarithmic local ringed spaces over } \mathbf{C}) \\ &\supset \mathcal{A} := (\text{fs logarithmic analytic spaces}), \end{aligned}$$

we call an object of $\bar{\mathcal{A}}$ a generalized fs logarithmic analytic space.

It can be proved that, if we forget logarithmic structures, generalized fs logarithmic analytic spaces become generalized analytic spaces. Generalized analytic spaces satisfy the conditions (1.5) (i), (ii).

Remark. In the example of (4.1), S , endowed with the strong topology and the pull-back sheaf of rings \mathcal{O} , is a generalized analytic space. S , endowed moreover with the pull-back fs logarithmic structure M from those on X , is a generalized fs logarithmic analytic space. Here the fs logarithmic structure on X is the one associated to the divisor $\{0\} \times \mathbf{C} \subset X$.

§5. POLARIZED LOGARITHMIC HODGE STRUCTURES

(5.1) *Polarized logarithmic Hodge structures.*

Definition. (5.1.1) Let X be a generalized fs logarithmic analytic space. A pre-polarized logarithmic Hodge structure (pre-PLH for short) on X , of weight w and of Hodge type $(h^{p,q})_{p+q=w}$ is a triple $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F)$ consisting of a locally constant sheaf of free \mathbf{Z} -modules $H_{\mathbf{Z}}$ on X^{\log} , of a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \rightarrow \mathbf{Q}$, symmetric for even w and anti-symmetric for odd w and of a decreasing filtration F of the \mathcal{O}_X^{\log} -module $\mathcal{O}_X^{\log} \otimes H_{\mathbf{Z}}$, which satisfy the following three conditions.

(i) Each F^p is the pull-back under $\tau : (X^{\log}, \mathcal{O}_X^{\log}) \rightarrow (X, \mathcal{O}_X)$ of a locally free \mathcal{O}_X -module of rank $\sum_{r \geq p} h^{r, w-r}$.

- (ii) Each F^p is, locally on X^{\log} , an \mathcal{O}_X^{\log} -direct summand of $\mathcal{O}_X^{\log} \otimes H_{\mathbf{Z}}$.
- (iii) $\langle F^p, F^q \rangle = 0$ if $p + q > w$.

Let X be an fs logarithmic analytic space. Let Ω_X^1 be the sheaf of Kähler differentials on X . The sheaf of logarithmic differential forms on X is defined by

$$\omega_X^1 := (\Omega_X^1 \oplus M_X^{\text{gp}})/N,$$

where N is the \mathcal{O}_X -submodule generated by $\{(-d\alpha(x), \alpha(x) \otimes x); x \in M_X\}$. The derivation $d : \mathcal{O}_X \rightarrow \Omega_X^1$ induces the one $d : \mathcal{O}_X \rightarrow \omega_X^1$ via the map $\Omega_X^1 \rightarrow \omega_X^1$. The sheaf $\omega_X^{1, \log}$ on X^{\log} is defined by

$$\omega_X^{1, \log} := \tau^*(\omega_X^1) = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_X^1).$$

The derivation

$$d : \mathcal{O}_X^{\log} \rightarrow \omega_X^{1, \log}$$

is induced by $d : \mathcal{O}_X \rightarrow \omega_X^1$ and by $\mathcal{L} \xrightarrow{\text{exp}} \tau^{-1}(M_X^{\text{gp}}) \xrightarrow{d\log} \tau^{-1}(\omega_X^1)$, where $d\log : x \mapsto (0, 1 \otimes x)$.

Let now x be an fs logarithmic analytic space whose underlying analytic space is $\text{Spec}(\mathbf{C})$ and let $r := \text{rank}_{\mathbf{Z}}(M_x^{\text{gp}}/\mathcal{O}_x^{\times})$. Then $x^{\log} = (\mathbf{S}^1)^r$. Let $y \in x^{\log}$. We define

$$\begin{aligned} \text{sp}(y) &:= \text{Hom}_{\mathbf{C}\text{-alg}}(\mathcal{O}_{x,y}^{\log}, \mathbf{C}), \\ \text{sp}(x) &:= \{t \in \text{Hom}(M_x, \mathbf{C}^{\text{mult}}); t|_{\mathbf{C}^{\times}} = id\}, \quad \text{and} \\ \text{sp}(y) &\rightarrow \text{sp}(x), \quad s \mapsto t := (M_x \ni a \mapsto \exp(s(\log(a)))) \in \mathbf{C}^{\times} \subset \mathbf{C}. \end{aligned}$$

Note that the structure homomorphism $\alpha : M_x \rightarrow \mathbf{C}$ of the logarithmic structure of x is an element of $\text{sp}(x)$, which does not belong to the image of the canonical map $\text{sp}(y) \rightarrow \text{sp}(x)$ if $r > 0$. Note also that $\text{sp}(x) \simeq \mathbf{C}^r$, whereas $\text{sp}(y)$ ($y \in x^{\log}$) form a $\mathbf{C} \otimes_{\mathbf{Z}} \pi_1(x^{\log})$ -torsor over x^{\log} (cf. (5.2) below).

For a pre-PLH $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F)$ on x , for $y \in x^{\log}$ and for $s \in \text{sp}(y)$, we have a decreasing filtration

$$F(s) := (\mathbf{C} \otimes_{\mathcal{O}_{x,y}^{\log}} F_y^p)_{p \in \mathbf{Z}}, \quad \text{where } s : \mathcal{O}_{x,y}^{\log} \rightarrow \mathbf{C},$$

on the \mathbf{C} -vector space $H_{\mathbf{C},y} := \mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z},y}$.

Definition. (5.1.2) Let x be an fs logarithmic analytic space whose underlying analytic space is $\text{Spec}(\mathbf{C})$. A pre-PLH $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F)$ on x is called a polarized logarithmic Hodge structure (PLH for short) on x if it satisfies the following two conditions.

- (i) $(d \otimes 1)(F^p) \subset \omega_x^{1, \log} \otimes_{\mathcal{O}_x^{\log}} F^{p-1}$ for all p .

Here $d \otimes 1$ means $d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_x^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \rightarrow \omega_x^{1, \log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$.

- (ii) For $y \in x^{\log}$ and for $s \in \text{sp}(y)$ whose image in $\text{sp}(x)$ is sufficiently near to the structure homomorphism $\alpha \in \text{sp}(x)$, $(H_{\mathbf{Z},y}, \langle \cdot, \cdot \rangle_y, F(s))$ is a polarized Hodge structure in the usual sense.

Note that the validity of the condition (5.1.2) (ii) is independent of the choice of y .

Definition. (5.1.3) Let X be a generalized fs logarithmic analytic space. A pre-PLH $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F)$ on X is called a polarized logarithmic Hodge structure (PLH for short) on X if, for any $x \in X$, the inverse image of $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F)$ on x is a PLH. Here we regard x as an fs logarithmic analytic space with ring \mathbf{C} and with the inverse image of the logarithmic structure of X .

(5.2) Polarized logarithmic Hodge structures of type Φ .

Let x be an fs logarithmic analytic space whose underlying analytic space is $\text{Spec}(\mathbf{C})$. Then $x^{\log} \simeq (\mathbf{S}^1)^r$, where $r = \text{rank}(M_x^{\text{gp}}/\mathcal{O}_x^{\times})$, as before. Let $y \in x^{\log}$. We define an action of $\mathbf{C} \otimes \pi_1(x^{\log})$ on $\text{sp}(y)$ as follows. Let \mathcal{L} be the sheaf constructed in (1.5). The exact sequence

$$0 \rightarrow \mathbf{Z} \xrightarrow{2\pi i} \mathcal{L} \xrightarrow{\text{exp}} \tau^{-1}(M_x^{\text{gp}}) \rightarrow 1$$

on x^{\log} induces an exact sequence of cohomology groups

$$\mathcal{O}_x \xrightarrow{\text{exp}} M_x^{\text{gp}} \rightarrow H^1(x^{\log}, \mathbf{Z}) \simeq \text{Hom}(\pi_1(x^{\log}), \mathbf{Z}).$$

Hence we have

$$\pi_1(x^{\log}) \simeq \text{Hom}(M_x^{\text{gp}}/\mathcal{O}_x^\times, \mathbf{Z}) \leftarrow \text{Hom}(\mathcal{L}_y/\mathcal{O}_x^{\text{add}}, \mathbf{Z}),$$

where the last isomorphism is induced by exp. On the other hand, the restriction of $s \in \text{sp}(y)$ to \mathcal{L}_y yields a bijection

$$\text{sp}(y) \xrightarrow{\sim} (\text{splittings of the group homomorphism } \iota : \mathbf{C}^{\text{add}} = \mathcal{O}_x^{\text{add}} \rightarrow \mathcal{L}_y).$$

Now we define the action of $\varphi \in \mathbf{C} \otimes_{\mathbf{Z}} \pi_1(x^{\log}) \xrightarrow{\nu} \text{Hom}(\mathcal{L}_y/\mathcal{O}_x^{\text{add}}, \mathbf{C}^{\text{add}})$ on $s \in \text{sp}(y)$ by

$$(\varphi s)|_{\mathcal{L}_y} := (\varphi - \text{id}) + s|_{\mathcal{L}_y}.$$

Note that $(\varphi - \text{id}) : \mathcal{L}_y \rightarrow \mathbf{C}^{\text{add}}$ is the lifting of $\nu(\varphi)$. Then, the $\text{sp}(y)$ ($y \in x^{\log}$) form a $\mathbf{C} \otimes_{\mathbf{Z}} \pi_1(x^{\log})$ -torsor over $x^{\log} \simeq (\mathbf{S}^1)^r$ under this action.

The following proposition gives a connection between a PLH and nilpotent orbits.

Proposition. *For $x \in X$, $y \in x^{\log}$, $s \in \text{sp}(y)$ and $\varphi = \sum_{1 \leq j \leq n} z_j \otimes \gamma_j \in \mathbf{C} \otimes \pi_1(x^{\log})$, we have*

$$F(\varphi s) = \exp\left(- \sum_{1 \leq j \leq n} z_j \log(\gamma_j)\right) F(s).$$

Let

$$(5.2.1) \quad \Phi := (w, (h^{p,q})_{p+q=w}, H_0, \langle \cdot, \cdot \rangle_0, \Gamma, \Sigma)$$

be a 6-tuple consisting of a weight, a set of Hodge numbers, a free \mathbf{Z} -module H_0 of rank $\sum h^{p,q}$, a \mathbf{Q} -bilinear form $\langle \cdot, \cdot \rangle_0$ on H_0, \mathbf{Q} as in (1.1), a subgroup of $G_{\mathbf{Z}}$ and a fan with which the subgroup Γ is strongly compatible.

Definition. *Let X be a generalized fs logarithmic analytic space. A PLH of type Φ on X is a 4-tuple $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F, \mu)$ consisting of a pre-PLH $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F)$ on X of weight w and of Hodge type $(h^{p,q})_{p,q}$, and of a global section μ of the sheaf*

$$\Gamma \backslash \underline{\text{Isom}}((H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle), (H_0, \langle \cdot, \cdot \rangle_0))$$

on X^{\log} which satisfy condition (i) below. Here H_0 with $\langle \cdot, \cdot \rangle_0$ is regarded as a constant sheaf on X^{\log} , $\underline{\text{Isom}}$ is the sheaf of isomorphisms, and $\gamma \in \Gamma$ acts on $\underline{\text{Isom}}(\cdot, \cdot)$ by $h \mapsto \gamma \circ h$.

(i) For any $x \in X$ and any $y \in x^{\log} = \tau^{-1}(x)$, if

$$\tilde{\mu}_y : (H_{\mathbf{Z},y}, \langle \cdot, \cdot \rangle_y) \xrightarrow{\sim} (H_0, \langle \cdot, \cdot \rangle_0)$$

denotes a representative of the stalk of μ at y , then there exists $\sigma \in \Sigma$ such that the image of the composite map

$$\text{Hom}(M_{X,x}/\mathcal{O}_{X,x}^\times, \mathbf{Z}_{\leq 0}) \hookrightarrow \pi_1(x^{\log}) \rightarrow \text{Aut}_{\mathbf{Z}}(H_{\mathbf{Z},y}, \langle \cdot, \cdot \rangle_y) \xrightarrow{\text{by } \tilde{\mu}_y} \text{Aut}_{\mathbf{Z}}(H_0, \langle \cdot, \cdot \rangle_0)$$

is contained in $\Gamma(\sigma)$ and that the filtrations $\tilde{\mu}_y(F|_{x^{\log}}(s))$ ($s \in \text{sp}(y)$) on H_0, \mathbf{C} form a σ -nilpotent orbit (cf. Definition in (1.3)).

Remark. Using the above Proposition, we can prove that the conditions (i), (ii) in Definition (5.1.2) follow from the conditions (ii) (Griffiths transversality), (iii) (positivity) in Definition (1.3), respectively. Hence if $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F, \mu)$ is a PLH on X of type Φ in the above definition, then $(H_{\mathbf{Z}}, \langle \cdot, \cdot \rangle, F)$ is actually a PLH on X in the sense of (5.1.2).

§6. MAIN RESULT

(6.1) Let $\Phi := (w, (h^{p,q})_{p+q=w}, H_0, \langle \cdot, \cdot \rangle_0, \Gamma, \Sigma)$ be as in (5.2.1) and let $\sigma \in \Sigma$.

We endow $E_\sigma \subset \check{E}_\sigma := \text{toric}(\Gamma(\sigma)) \times \check{D}$ with the strong topology, the pull-back sheaf of rings \mathcal{O} and the pull-back fs logarithmic structure M from those on \check{E}_σ . Here the fs logarithmic structure on \check{E}_σ is the pull-back of the one on $\text{toric}(\Gamma(\sigma))$ associated to the normal crossing divisor $\text{toric}(\Gamma(\sigma)) - \Gamma(\sigma)^{\text{gp}} \otimes \mathbf{C}^\times$.

Via $\pi_\sigma : E_\sigma \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma \rightarrow \Gamma \backslash D_\Sigma$, we endow $\Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ with the quotient topology, and $\Gamma \backslash D_\Sigma$ with the strongest topology for which π_σ is continuous for all $\sigma \in \Sigma$. For any open set U of $\Gamma \backslash D_\Sigma$, let $U_\sigma := \pi_\sigma^{-1}(U)$ and define

$$\mathcal{O}(U) \text{ (resp. } M(U)) := \{\text{map } f : U \rightarrow \mathbf{C}; f \circ \pi_\sigma \in \mathcal{O}(U_\sigma) \text{ (resp. } \in M(U_\sigma)) \text{ (}\forall \sigma \in \Sigma)\}.$$

Here we regard $M(U_\sigma)$ as a subset of $\mathcal{O}(U_\sigma)$ via $\alpha : M(U_\sigma) \rightarrow \mathcal{O}(U_\sigma)$ which is injective.

We introduce the quotient topology on E_σ^\sharp via the surjection $E_\sigma \rightarrow E_\sigma^\sharp$ in (3.3.1). Via $\pi_\sigma^\sharp : E_\sigma^\sharp \rightarrow D_\sigma^\sharp \rightarrow \Gamma \backslash D_\Sigma^\sharp$, we endow D_σ^\sharp with the quotient topology, and D_Σ^\sharp with the strongest topology for which π_σ^\sharp is continuous for all $\sigma \in \Sigma$.

A subgroup Γ of $G_{\mathbf{Z}}$ is called *neat* if the subgroup of \mathbf{C}^\times , generated by all the eigenvalues of all $\gamma \in \Gamma$, is torsion free. It is known that there exist neat subgroups of $G_{\mathbf{Z}}$ with finite index.

Proposition. *Assume Γ is neat. Then, in the above notation, we have*

(6.1.1) $E_\sigma \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ (resp. $E_\sigma^\sharp \rightarrow D_\sigma^\sharp$) is a $\sigma_{\mathbf{C}}$ -torsor (resp. $\sigma_{\mathbf{R}}$ -torsor) in the category of topological spaces.

(6.1.2) The maps $\Gamma(\sigma)^{\text{gp}} \backslash D_\sigma \rightarrow \Gamma \backslash D_\Sigma$ and $D_\sigma^\sharp \rightarrow \Gamma \backslash D_\Sigma^\sharp$ are open.

(6.2) Our main result is

Theorem. *Let $\Phi := (w, (h^{p,q})_{p+q=w}, H_0, \langle \cdot, \cdot \rangle_0, \Gamma, \Sigma)$ be as in (5.2.1). We assume Γ is neat. Then we have*

(6.2.1) $\Gamma \backslash D_\Sigma$ is a generalized fs logarithmic analytic space, which is Hausdorff, but not locally compact in general. There is a homeomorphism of the topological spaces $(\Gamma \backslash D_\Sigma)^{\text{log}}$ and $\Gamma \backslash D_\Sigma^\sharp$, which is compatible with the canonical map $\tau : (\Gamma \backslash D_\Sigma)^{\text{log}} \rightarrow \Gamma \backslash D_\Sigma$ and the map $\Gamma \backslash D_\Sigma^\sharp \rightarrow \Gamma \backslash D_\Sigma$ induced by $D_\Sigma^\sharp \rightarrow D_\Sigma$ in (3.1).

(6.2.2) Let $\bar{\mathcal{A}} :=$ (generalized fs logarithmic analytic spaces) and define the contravariant functor $\underline{\text{PLH}}_\Phi : \bar{\mathcal{A}} \rightarrow \text{Sets}$ by

$$\underline{\text{PLH}}_\Phi(X) := (\text{isomorphism classes of PLH on } X \text{ of type } \Phi).$$

Then this functor $\underline{\text{PLH}}_\Phi$ is represented by $\Gamma \backslash D_\Sigma$.

Remark. In the case where D is a Hermitian symmetric domain, the strong topology on $E_\sigma \subset \check{E}_\sigma$ coincides with the usual topology on E_σ as a subspace of \check{E}_σ , and hence they coincide on $\Gamma \backslash D_\Sigma$. Thus our space $\Gamma \backslash D_\Sigma$ becomes an fs logarithmic analytic space whose underlying analytic space is nothing but a toroidal compactification of Mumford et al. for a ‘big fan’ Σ . Even in this case, our construction has the advantage that it carries the universal family of PLH.

§7. EXAMPLES

(7.1) *Example of weight 3.*

In the case of Hodge structures of weight 2, Cattani and Kaplan constructed a partial compactification $\Gamma \backslash D_{CK}$ ($\Gamma \backslash \bar{D}$ in their notation in [CK]) as a topological space, which is a generalization of the Satake type. For the example of weight 3 below, their argument is applicable to obtain $\Gamma \backslash D_{CK}$. Consider the case of weight 2, or weight 3 of the type below. Let Ξ be the fan in Example in (1.2) and let Γ be a subgroup of $G_{\mathbf{Z}}$ with finite index. Then, there is the standard map $\Gamma \backslash D_\Xi \rightarrow \Gamma \backslash D_{CK}$ from our space to the Cattani-Kaplan, which is continuous.

In the following, we give an example which shows that the strong topology is needed. Let $w = 3$ and $h^{p,q} = 1$ ($p + q = 3$, $p, q \geq 0$), $h^{p,q} = 0$ otherwise. Let e_1, e_2, f_1, f_2 be a free \mathbf{Z} -basis of H_0 and let $\langle \cdot, \cdot \rangle_0$ be an anti-symmetric bilinear form on H_0 defined by

$$\langle e_1, e_2 \rangle_0 = \langle f_1, f_2 \rangle_0 = 1, \quad \langle e_j, e_k \rangle_0 = 0 \quad (\forall j, \forall k).$$

Let N be a nilpotent endomorphism of H_0 defined by

$$Nf_1 = f_2, \quad Nf_2 = Ne_j = 0 \quad (\forall j).$$

For $s \in \mathbf{C}$, we define a family of Hodge filtrations $F_s \in \check{D}$ by

$$\begin{aligned} F_s^3 &:= \{sf_1 + e_1 - ie_2\}_{\mathbf{C}} \subset F_s^2 := \{f_1, e_1 - ie_2\}_{\mathbf{C}} \\ &\subset F_s^1 := \{f_1, e_1 - ie_2, se_2 - f_2\}_{\mathbf{C}} \subset F_s^0 := H_{0, \mathbf{C}}, \end{aligned}$$

where $\{\dots\}_{\mathbf{C}}$ means the \mathbf{C} -linear span. Using this family, we can prove

Proposition. (7.1.1) *The following are equivalent:*

- (a) (N, F_s) generates a nilpotent orbit.
- (b) (N, F_s) satisfies Griffiths transversality.
- (c) $s = 0$.

(7.1.2) *We use the notation in (7.1.1). For $y \in \mathbf{R}$, we see $e^{iyN}F_s \in D$ if and only if $0 < y < |s|^{-2}$. (Here we understand $|s|^{-2} = \infty$ for $s = 0$).*

(7.1.3) *Let Ξ be the fan in Example in (1.2). If we endow E_σ ($\sigma \in \Xi$) with the usual topology as a subset of \check{E}_σ and endow $\Gamma \backslash D_\Xi$ with the quotient topology via $\bigsqcup_{\sigma \in \Xi} E_\sigma \rightarrow \Gamma \backslash D_\Xi$, then $\Gamma \backslash D_\Xi$ is not Hausdorff and the standard map $\Gamma \backslash D_\Xi \rightarrow \Gamma \backslash D_{\text{CK}}$ is not continuous.*

The example in this subsection is a degeneration of the Hodge structures of weight 3 of the mirrors of quintic hypersurfaces in \mathbf{P}^4 (cf., for example, [M]).

(7.2) *Example of weight 2.*

Let Ξ be the fan in Example in (1.2). In this subsection, we give an example of $\Gamma \backslash D_\Xi$ which is locally isomorphic to $\mathbf{C} \times (\mathbf{C}^2 - \{0\}) \times \mathbf{C} + \{(0, 0)\}$ (cf. Example in (4.1)).

Put

$$\begin{aligned} S &:= \mathbf{C} \times \mathbf{C} - \{0\} \times \mathbf{C} + \{(0, 0)\}, \\ \tilde{S} &:= \mathbf{R}_{\geq 0} \times \mathbf{C} - \{0\} \times \mathbf{C} + \{(0, 0)\}. \end{aligned}$$

We define a topology \mathcal{T} (resp. $\tilde{\mathcal{T}}$) on S (resp. on \tilde{S}) as follow. On $S - \{(0, 0)\}$ (resp. on $\tilde{S} - \{(0, 0)\}$), \mathcal{T} (resp. $\tilde{\mathcal{T}}$) coincides with the usual topology, and, at the point $(0, 0)$, a fundamental system of neighborhoods is given by

$$\begin{aligned} W(\varepsilon) &:= \bigcup_{n \geq 1} \{(v, z); |v| < \varepsilon_n, |z| < \varepsilon_n, |z|^n < |v|\}. \\ \left(\text{resp. } \tilde{W}(\varepsilon) &:= \bigcup_{n \geq 1} \{(r, z); r < \varepsilon_n, |z| < \varepsilon_n, |z|^n < r\}. \right) \end{aligned}$$

Here $\varepsilon = (\varepsilon_n)_{n \geq 1}$ runs over all families of positive real numbers. Then, \mathcal{T} coincides with the strong topology of $S \subset \mathbf{C}^2$ considered in Example in (4.1).

Let $w = 2$, $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ and $h^{p,q} = 0$ otherwise. Let e_j ($1 \leq j \leq 5$) be a free \mathbf{Z} -basis of H_0 and let $\langle \cdot, \cdot \rangle_0$ be a symmetric bilinear form on H_0 defined by

$$\langle e_j, e_j \rangle_0 = 1 \quad (1 \leq j \leq 3), \quad \langle e_4, e_5 \rangle_0 = \langle e_5, e_4 \rangle_0 = 1, \quad \langle e_j, e_k \rangle_0 = 0 \quad \text{otherwise.}$$

Then $\dim_{\mathbf{C}} D = 3$ and we can prove

Proposition. *In the above notation, we have the following.*

- (i) D_Ξ contains all nilpotent orbits.
- (ii) *Let $p \in \Gamma \backslash D_\Xi - \Gamma \backslash D$. Then, there exist an open neighborhood U of p in $\Gamma \backslash D_\Xi$ and an open set \tilde{U} in D_Ξ^\sharp , in the topologies introduced in (6.1), which fit in the following commutative diagram:*

$$\begin{array}{ccccc} \Gamma \backslash D_\Xi & \supset & U & \xrightarrow{\alpha} & \mathbf{C} \times S \\ \uparrow & & \uparrow & & \beta \uparrow \\ D_\Xi^\sharp & \supset & \tilde{U} & \hookrightarrow & \mathbf{C} \times \mathbf{R} \times \tilde{S} \end{array}$$

Here $\alpha : p \mapsto (0, 0, 0)$, $\beta : (w, a, r, z) \mapsto (w, re^{ai}, z)$. If we endow $\mathbf{C} \times S$ (resp. $\mathbf{C} \times \mathbf{R} \times \tilde{S}$) with the product topology of the usual topology of \mathbf{C} (resp. $\mathbf{C} \times \mathbf{R}$) and \mathcal{T} (resp. $\tilde{\mathcal{T}}$), then both \hookrightarrow are open immersions.

REFERENCES

- [AMRT] A. Ash, D. Mumford, M. Rapoport and Y. S. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, Brookline, 1975.
- [BB] L. Baily and A. Borel, *Compactifications of arithmetic quotients of bounded symmetric domains*, Ann. of Math. **84** (1966), 442–528.
- [BS] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491.
- [CCK] J. A. Carlson, E. Cattani and A. Kaplan, *Mixed Hodge structures and compactifications of Siegel’s space*, in: Journées de géométrie algébrique d’Angers 1979 (A. Beauville, ed.), Sijthoff & Noordhoff, 1980, pp. 77–105.
- [CK] E. Cattani and A. Kaplan, *Extension of period mappings for Hodge structures of weight 2*, Duke Math. J. **44** (1977), 1–43.
- [CKS] E. Cattani and A. Kaplan and W. Schmid, *Degenerations of Hodge structures*, Ann. of Math. **123** (1986), 457–535.
- [G] P. A. Griffiths, *Periods of integrals on algebraic manifolds: Summary of main results and discussions of open problems*, Bull. Amer. Math. Soc. **76** (1970), 228–296.
- [HZ] M. Harris and S. Zucker, *Boundary cohomology of Shimura varieties I. – Coherent cohomology on toroidal compactifications*, Ann. scient. Éc. Norm. Sup. **27** (1994), 249–344.
- [I] L. Illusie, *Logarithmic spaces (according to K. Kato)*, in: Barsotti Symposium in Algebraic Geometry (V. Cristante and W. Messing, eds.), Perspectives in Math. 15, Academic Press, 1994, pp. 183–203.
- [K1] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in: Algebraic analysis, geometry, and number theory (J.-I. Igusa, ed.), Perspectives in Math., Johns Hopkins University Press, Baltimore, 1989, pp. 191–224.
- [K2] ———, *Toric singularity*, Amer. J. Math. **116** (1994), 1073–1099.
- [KkNc] K. Kato and C. Nakayama, *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over \mathbf{C}* , Prepublications, Univ. Tokyo, UTMS 95-16 (1995).
- [KyNy] Y. Kawamata and Y. Namikawa, *Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties*, Invent. Math. **118** (1994), 395–409.
- [KU] K. Kato and S. Usui, *Logarithmic Hodge structures and classifying spaces*, in preparation.
- [M] D. R. Morrison, *Mirror symmetry and rational curves on quintic threefolds: A guide for mathematicians*, J. Amer. Math. Soc. **6-1** (1993), 223–247.
- [Sa] I. Satake, *On compactification of the quotient spaces for arithmetically defined discontinuous groups*, Ann. of Math. **72** (1960), 555–580.
- [Sc] W. Schmid, *Variation of Hodge structure; the singularities of the period mappings*, Invent. Math. **22** (1973), 211–319.
- [U1] S. Usui, *A numerical criterion for admissibility of semi-simple elements*, Tôhoku Math. J. **45** (1993), 471–484.
- [U2] ———, *Complex structures on partial compactifications of arithmetic quotients of classifying spaces of Hodge structures*, Tôhoku Math. J. **47** (1995), 405–429.

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