TORELLI-TYPE PROBLEMS

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Abstract. Recently Log Geometry is used in Hodge Theory and there is a little progress in Torelli-type Problems by using Degenerations. This is an English translation of the survey appeared in Sugaku, Mathematical Society of Japan, **49-3** (1997) 235–252.

§1. Case of Elliptic Curves

As an introduction, let us consider the case of elliptic curves. This is the story in the 19-th century. Contrary to the history, we are starting with Weierstrass' $\mathfrak{p}$-functions. Let $L = \tau\mathbf{Z} + \mathbf{Z}$, $\operatorname{Im}\tau > 0$, be a lattice in the complex plane $\mathbf{C}$ and let z be a complex variable. Then the power series

$$\mathfrak{p}(z) := z^{-2} + \sum_{\omega \in L'} ((z-\omega)^{-2} - \omega^{-2}), \quad \text{ここで} \quad L' := L - \{0\}$$

converges absolutely, uniformly on any compact subset of $\mathbf{C}$ to a meromorphic function with double periods. Hence its differential $\mathfrak{p}'(z)$ can be obtained term-wisely. Comparing the terms of Laurent series with non-positive degrees, we get a relation:

$$\mathfrak{p}'(z)^2 = 4\mathfrak{p}(z)^3 - g_2\mathfrak{p}(z) - g_3, \quad g_2 := 60\sum_{\omega \in L'} \omega^{-4}, \quad g_3 := 140\sum_{\omega \in L'} \omega^{-6}.$$

Moreover, since $\mathfrak{p}(z)$ is a doubly periodic, even function, we see that the zeros of $\mathfrak{p}'(z)$ are $\{\tau/2, 1/2, (\tau+1)/2\}$ and the poles are $\{3\cdot 0\}$. Put $\infty = \mathfrak{p}(0)$, $e_1 := \mathfrak{p}(\tau/2)$, $e_2 := \mathfrak{p}(1/2)$, $e_3 := \mathfrak{p}((\tau+1)/2)$, and consider their cross ratio λ and its function J:

$$\lambda(\tau) := (\infty, e_1; e_2, e_3) = \frac{e_1 - e_3}{e_1 - e_2}, \quad J(\tau) := \frac{4}{27}\frac{(1-\lambda+\lambda^2)^3}{(\lambda(1-\lambda))^2}.$$

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Putting $x := \mathfrak{p}(z)$ and differentiating, we have $\psi := \pm(4x^3 - g_2x - g_3)^{-1/2}dx = dz$. From this together with the above results, we know that $\mathfrak{p} : \mathbf{C} \to \mathbf{P}^1$, $z \mapsto x$, is a branched covering which factorizing via the cubic curve $C := \{4x^3 - g_2x - g_3 - y^2 = 0\} \subset \mathbf{P}^2$, that $(\mathfrak{p}, \mathfrak{p}') : \mathbf{C} \to C$ is the universal covering and that $\mathrm{pr}_x : C \to \mathbf{P}^1$ is the double covering branched over the four points $\{e_1, e_2, e_3, \infty\}$.

The domain of definition of an integral $\int_\infty^x \psi$ should be C rather than $\mathbf{P}^1$ and its multi-valuedness occurs from the ambiguity of the choices of the paths, which are 1-cycles on C. Therefore, taking a symplectic basis α, β of $H_1(C, \mathbf{Z})$, taking the ratio $\tau := \int_\beta \psi / \int_\alpha \psi$ and dividing by $L := \tau\mathbf{Z} + \mathbf{Z}$, we obtain an isomorphism:

$$\int_\infty^{(\ )} \psi : C \tilde{\to} \mathbf{C}/L, \quad \text{its inverse is } (\mathfrak{p}, \mathfrak{p}').$$

Moreover, since

$$\begin{aligned} & C \mod PGL(2, \mathbf{C}) \quad \Leftrightarrow \quad \tau \mod SL(2, \mathbf{Z}) \\ \Leftrightarrow \ & \text{unordered set } \{\infty, e_1, e_2, e_3\} \mod PGL(1, \mathbf{C}) \\ \Leftrightarrow \ & \lambda(\tau) \mod (\text{permutation group of the ordered set } \{e_1, e_2, e_3\}) \\ \Leftrightarrow \ & J(\tau), \end{aligned}$$

the moduli of smooth cubic curves and its compactification are given by

$$\begin{array}{ccc} \mathfrak{h}/SL(2, \mathbf{Z}) & \overset{J}{\tilde{\to}} & \mathbf{C} \\ \cap & & \cap \\ \big(\mathfrak{h} \sqcup (\mathbf{Q} \sqcup \{\infty\})\big)/SL(2, \mathbf{Z}) & \overset{J}{\tilde{\to}} & \mathbf{C} \sqcup \{\infty\} = \mathbf{P}^1(\mathbf{C}). \end{array}$$

Here, $\mathbf{Q} \sqcup \{\infty\}$ is the easiest case of the rational boundary components of a Hermitian symmetric domain and the basis of the open neighborhoods of $0 \in \mathfrak{h} \sqcup \{\mathbf{Q} \sqcup \{\infty\}\}$ in the Satake topology are the family of $\{0\} \cup$ (open disc in $\mathfrak{h}$ tangent to the real line at 0). As $e_1 \to \infty$, we see $\lambda \to 1$ therefore $J \to \infty$. This means that a rational curve with one node sits over the point ∞ added in the compactification.
(For more details for this section, see, e.g., [HC].)

§2. Five approaches to Torelli-type problems

(2.1) Via theta divisors. The set $F := H^0(C, \Omega_C^1)$ of holomorphic 1-forms on a smooth algebraic curve C form a g-dimensional complex vector space. Here the number g is called the genus of C. Taking a base point $p_0 \in C$ and a basis $\omega_1, \dots, \omega_g \in F$, we consider an integral

$$\alpha(p) := \Big(\int_{p_0}^p \omega_1, \dots, \int_{p_0}^p \omega_g\Big), \quad p \in C.$$

This is multi-valued. Dividing the period of the integrals over the 1-cycles on C, we get a single-valued holomorphic map

$$\alpha : C \to J(C) := F^*/H_1(C, \mathbf{Z}).$$

$J(C)$ is a g-dimensional complex torus which is called the Jacobian variety associated to C, and α is called the Abel-Jacobi map. For a divisor $D = \sum_{1 \le i \le d} p_i \in S^d(C)$ of degree d, we extend α by $\alpha(D) := \sum_{1 \le i \le d} \alpha(p_i)$. Then

$$\Theta := \operatorname{Im}\big(\alpha : S^{g-1}(C) \to J(C)\big)$$

is the theta divisor of Riemann, which yields a principal polarization of $J(C)$. It is known that α is injective on the complement of $\alpha^{-1}(\operatorname{Sing}\Theta)$.

Torelli Theorem. *Let $(J(C), \Theta)$ and $(J(C'), \Theta')$ be the the principally polarized Jacobian varieties associated to smooth algebraic curves C and C', respectively. If $(J(C), \Theta)$ and $(J(C'), \Theta')$ are isomorphic, then C and C' are isomorphic.*

Several proofs are known for this theorem. Many of them use theta divisors. Let us see the idea of the proof given by Andreotti in the case of $g = 3$ because we can understand the geometric image in this case. Let $\mathbf{P}^{g-1}$ be the Proj of the symmetric algebra of F over $\mathbf{C}$. The canonical map of C is defined by

$$f : C \to \mathbf{P}^{g-1}, \quad f(p) := (\omega_1(p) : \cdots : \omega_g(p)).$$

The Gauss map of the theta divisor $\Theta \subset J(C)$ is defined by

$$\gamma : \Theta \to (\mathbf{P}^{g-1})^*, \quad \gamma := (\text{接空間 } T_\Theta(a) \subset T_{J(C)}(a) \simeq F^*).$$

Considering the geometric meaning of each definition, we obtain the following:

(2.1.1) The ratio of the entries of the differential of the Abel-Jacobi map α is the canonical map f.
(2.1.2) $\gamma \circ \alpha\big(\sum_{1 \le i \le g-1} p_i\big) = \big(\text{space spanned by } \{f(p_i) \,|\, 1 \le i \le g-1\}\big)$.

Case : C is a canonical curve. The canonical map f is an embedding and, in the case $g = 3$, we have $\deg \Omega_C^1 = 2 \cdot 3 - 2 = 4$. Hence C can be identified with the plane quartic curve $f(C)$. If a line L on $\mathbf{P}^2$ is not tangent to C then L intersects with C at different four points. If L is tangent to C then the intersection of L and C consists of three or less points. Considering L as a point of $(\mathbf{P}^2)^*$ and using (2.1.2), we can compute the number of points of the inverse by $\gamma \circ \alpha$ as follows:

$$\sharp(\gamma \circ \alpha)^{-1}(L) \begin{cases} = \binom{4}{2} = 6 & \text{if } L \text{ is not tangent to } C, \\ \le \binom{3}{2} = 3 & \text{if } L \text{ is tangent to } C. \end{cases}$$

From this we can recover the set of tangent lines of C as the branch locus of the Gauss map γ. This is nothing but the dual curve $C^* \subset (\mathbf{P}^2)^*$ of C and hence we can recover $C = C^{**}$.

Case : C is a hyperelliptic curve. In the case $g = 3$, via the canonical map f, C is a double covering over a plane conic C_0 branched at eight points p_i, $1 \leq i \leq 8$. In a similar argument as above, we can recover $C_0^* + p_1^* + \cdots p_8^*$ as the branch locus of the Gauss map γ. Hence we can recover $C_0 = C_0^{**}$ and the eight points $p_i = p_i^{**}$ on it and finally the double cover $f : C \to C_0$ branched at $\sum p_i$.
(For more details in this subsection, see [A] or [ACGH].)

(2.2) Via degenerations. The Jacobian variety $J(C)$ of a smooth algebraic curve C is an equivalent data of the Hodge structure $(H^1(C, \mathbf{Z}), H^1(C, \mathbf{C}) = H^{1,0}(C) \oplus H^{0,1}(C))$ of weight 1 over $\mathbf{Z}$ of C. Moreover, the theta divisor Θ up to translation is uniquely determined by the cohomology class $c_1[\Theta] \in H^2(J(C), \mathbf{Z}) \simeq \bigwedge^2 H^1(J(C), \mathbf{Z}) \simeq \bigwedge^2 H^1(C, \mathbf{Z})$, which corresponds to the cup product S on $H^1(C, \mathbf{Z})$ by the Poincaré duality. This yields a polarization of the Hodge structure, that is, the Riemann bilinear relations: For $\omega, \omega' \in H^{1,0}(C)$,

$$S(\omega, \omega') = \int_C \omega \wedge \omega' = 0, \quad \sqrt{-1} S(\omega, \bar{\omega}) = \sqrt{-1} \int_C \omega \wedge \bar{\omega} > 0 \quad (\omega \neq 0).$$

Taking a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \in H_1(C, \mathbf{Z})$ and a basis $\omega_1, \ldots, \omega_g$ of $F = H^0(C, \Omega_C^1) \simeq H^{1,0}(C)$ with $\int_{\alpha_i} \omega_j = \delta_{ij}$, the Riemann bilinear relations can be interpreted as the period matrix

$$\Omega := \left(\int_{\beta_i} \omega_j \right)_{1 \leq i, j \leq g}$$

is symmetric and its imaginary part is positive-definite, that is, a point of the Siegel upper-half space $\mathfrak{H}_g$.

Thus the correspondence $C \mapsto (J(C), \Theta)$ is regarded as the period mapping

$$\Phi : M_g \to \mathfrak{H}_g / \Gamma, \quad \Gamma := Sp(2g, \mathbf{Z}) \tag{2.2.1}$$

from the moduli space M_g of curves of genus g, and Torelli Theorem asserts that Φ is injective. Here dividing by Γ corresponds to the choice of symplectic bases. M_g has the compactification $\overline{M}_g$ of Deligne-Mumford ([DM]) added by the equivalence classes of stable curves. $\mathfrak{H}_g / \Gamma$ has the Satake compactification $(\mathfrak{H}_g / \Gamma)^S$ ([Sa1]), which carries an algebraic structure introduced by using automorphic forms by Baily-Borel ([BB]), and also has toroidal compactifications by Mumford et al. ([AMRT]). We denote here by $\overline{\mathfrak{H}_g / \Gamma}$ the Volonoi compactification by Yukihiko Namikawa which is one of toroidal compactifications. Then Φ can be extended as

$$\overline{\Phi} : \overline{M}_g \to \overline{\mathfrak{H}_g / \Gamma}. \tag{2.2.2}$$

Taking a general point on the boundary component of $\overline{M}_g$ corresponding to those curves of geometric genus $g - 1$ with one node and taking a neighborhood $B \simeq \Delta^{3g-3}$, $B^* := B \cap M_g = \Delta^* \times \Delta^{3g-4}$, of that point, and consider the family of curves $\{C_t\}_{t \in B}$

over it. Taking a locally constant symplectic frame $\alpha_1(t), \dots, \alpha_g(t), \beta_1(t), \dots, \beta_g(t) \in H_1(C_t, \mathbf{Z})$, $t \in B^*$, so that $\alpha_1(t)$ is a vanishing cycle and taking a holomorphic frame $\omega_1(t), \dots, \omega_g(t) \in F(C_t)$ with $\int_{\alpha_i(t)} \omega_j(t) = \delta_{ij}$, and consider a period mapping $\varphi(t) := (\varphi_{i,j}(t)) := \left(\int_{\beta_i(t)} \omega_j(t)\right)$. Then the Picard-Lefschetz formula for the local monodromy T becomes

$$T\beta_1(t) = \beta_1(t) + \alpha_1(t), \quad \text{other } \alpha_i(t),\ \beta_i(t) \text{ are invariant.}$$

$$T\varphi_{11}(t) = \varphi_{11}(t) + 1 \quad \text{hence } \varphi_{11}(t) = (2\pi\sqrt{-1})^{-1} \log t_1 + s(t),$$

$$\text{where } t = (t_1, t'), \text{ and } s(t) \text{ and the other } \varphi_{ij}(t) \text{ are invariant.}$$

By the hyperbolicity of $\mathfrak{H}_g$, $\varphi_{ij}(t)$, $(i,j) \neq (1,1)$, can be extended over B. (This has been generalized into the nilpotent orbit theorem by Schmid.)

Let $\tilde{C}_0$ be the normalization of C_0, let p, q and β_i be the pull-back to $\tilde{C}_0$ of the double point and the cycle $\beta_i(0)$, respectively, and let $\tilde{\omega}_1$, ω_j, $j \neq 1$, be the differential of the third kind and the differentials of the first kind on $\tilde{C}_0$ come from $\omega_j(0)$. Then the limit of the period matrix as $t' = 0$ and $t_1 \to 0$ becomes

(2.2.3)

$$\varphi(t) \ \text{mod T} = \begin{pmatrix} t_1 \exp 2\pi\sqrt{-1}s(t) & \dots \\ \vdots & \dots \end{pmatrix} \to \begin{pmatrix} 0 & \int_q^p \omega_2 & \dots & \int_q^p \omega_g \\ \int_{\beta_2} \tilde{\omega}_1 & & & \\ \vdots & & \varphi^1(0) & \\ \int_{\beta_g} \tilde{\omega}_1 & & & \end{pmatrix}$$

By using (2.2.3), we can prove Generic Torelli Theorem, i.e., the assertion of the mapping degree of Φ over its image being 1, by an induction on the genus g of the curve C. The outline of the proof is as follows. For the blocks of (2.2.3), we can observe (2.2.4)–(2.2.6) below.

(2.2.4)

$$\begin{aligned} &\text{modulus of } \tilde{C}_0 \ \Leftrightarrow\ \varphi^1(0) \\ \Leftrightarrow\ & gr^W \text{ of limit Hodge structure by monodromy weight filtration } W \\ \Leftrightarrow\ & (\text{point on } \mathfrak{H}_g/\Gamma)^S. \end{aligned}$$

Hence, by the induction assumption on g, $\tilde{C}_0$ can be recovered by $\varphi^1(0)$. Moreover, the dual of the differential of the (2, 2)-block of the extremely right matrix in (2.2.3) can be interpreted as the multiplication mapping $R_1 \times R_1 \to R_2$ of the canonical ring $R := \bigoplus H^0(\tilde{C}_0, (\Omega^1_C)^{\otimes n})$ and this mapping is injective provided that $\tilde{C}_0$ is non-hyperelliptic by the M. Neother theorem. This shows the injectivity of the differential of the period mapping in the directions of the moduli of $\tilde{C}_0$.

(2.2.5)

$$\begin{aligned} &\text{places of } p, q \in \tilde{C}_0 \\ \Leftrightarrow\ & \text{(1.2)-block of the extremely right matrix in (2.2.3)} \\ \Leftrightarrow\ & \text{(2.1)-block of the right matrix in (2.2.3) (by reciprocity)} \\ \Leftrightarrow\ & \text{extension data of the filtration } W \text{ of the limit mixed Hodge structure} \\ \Leftrightarrow\ & \text{point on the fiber of } \overline{\mathfrak{H}_g/\Gamma} \to (\mathfrak{H}_g/\Gamma)^S. \end{aligned}$$

The (1.2)-block of the extremely right matrix in (2.2.3) yields the Abel-Jacobi mapping and two points $p, q \in \tilde{C}_0$ can be recovered if $\tilde{C}_0$ is non-hyperelliptic. Moreover, the differential of this (1.2)-block by p or q yields the canonical mapping $f : \tilde{C}_0 \to \mathbf{P}^{g-2}$, which is injective if $\tilde{C}_0$ is a canonical curve. This shows the injectivity of the differential of the period mapping in the directions of the places of $p, q \in \tilde{C}_0$.

(2.2.6) Since $t_1 = 0$ is a local equation of the boundary component of $\overline{M}_g$ containing C_0, (1.1)-entry of the middle matrix in (2.2.3) shows the injectivity of the differential of the period mapping in the direction normal to this boundary component.

Thus we see that the period mapping Φ is proper, that $\Phi^{-1}\Phi(C_0)$ consists of one point and that the local mapping degree of Φ is 1. This proves Generic Torelli Theorem.

Problem. Taking the higher-order differentials into account, make the image $\operatorname{Im}\Phi$ of the period mapping to be normal and prove the injectivity of Φ by using the results in this subsection and Zariski Main Theorem. More generally, taking the higher-order differentials into account, make the image of the 'extended period mapping' to be normal.

(For more details in this subsection, see [N1], [CCK], the article of Friedman in [Get])

(2.3) Via special locus in moduli. We first formulate a period mapping and its differential after Griffiths ([G1], cf. also [Dl2]).

Let $H_{\mathbf{Z}}$ be a free $\mathbf{Z}$-module and set $H_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$. A direct sum decomposition $H_{\mathbf{C}} = \bigoplus_{p+q=w} H^{p,q}$ with $H^{q,p} = \overline{H^{p,q}}$ is called a Hodge decomposition or a Hodge structure of weight w. This is an equivalent data of a w-opposed decreasing filtration F, i.e., $\mathrm{gr}_F^p \mathrm{gr}_{\bar{F}}^q = 0$ if $p+q \neq w$. Their correspondence is as follows: $F^p = \sum_{p' \geq p} H^{p',q'}$, $H^{p,q} = F^p \cap \overline{F^q}$. Let S be a $(-1)^w$-symmetric bilinear form on $H_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$. S is called a polarization of the Hodge structure if

$$(2.3.1) \qquad S(\omega, \omega') = 0 \quad (\omega \in H^{p,q},\ \omega' \in H^{p',q'}) \quad \text{if } p + p' \neq w,$$

$$(2.3.2) \qquad (\sqrt{-1})^{p-q} S(\omega, \overline{\omega}) > 0 \quad (0 \neq \omega \in H^{p,q}).$$

In the geometric case, we can take as $H_{\mathbf{Z}}$ a summand in the Lefschetz decomposition of the cohomology group with coefficients in $\mathbf{Z}$ of a smooth projective variety. Then (2.3.1), (2.3.2) are fulfilled and called the Riemann-Hodge bilinear relations.

Put $f^p := \dim F^p$ and $f := (f^0, \ldots, f^w)$. The classifying space D and its 'compact dual' $\check{D}$ are defined by

$$\check{D} := \{F \in \mathrm{Flag}(H_{\mathbf{C}}, f) \,|\, \text{satisfies (2.3.1)}\}, \quad D := \{F \in \check{D} \,|\, \text{also satisfies (2.3.2)}\}.$$

Let Γ be a subgroup of $\mathrm{Aut}(H_{\mathbf{Z}}, S)$. Then it can be proved that Γ acts on D properly discontinuously. A holomorphic mapping

$$\Phi : M \to D/\Gamma$$

from an analytic manifold M is called a period mapping if at any point of M there exists a suitable neighborhood U and a local lifting $\tilde{\Phi} : U \to D$ which satisfies the Griffiths transversality $d\tilde{\Phi}T_U \subset \bigoplus_p \mathcal{H}om\,(\mathrm{gr}_F^p, \mathrm{gr}_F^{p-1})$. It is known that the differential of the period mapping associated to a smooth deformation $f : X \to M$ factorizes as $d\tilde{\Phi} = \kappa \circ \rho$. Here ρ is the Kodaira-Spenser mapping and κ is the mapping, up to non-zero constant, induced from the cup product $R^1 f_* T_X \otimes R^q f_* \Omega_X^p \to R^{q+1}\Omega_X^{p-1}$.

The following is Torelli Theorem for $K3$ surfaces proved by Piateckii-Shapiro & Shafarevich ([PS]) et al.:

(2.3.3) Theorem. *Let X, X' be Kähler $K3$ surfaces. Given an isomorphism $\psi : H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ preserving the cup products, the Hodge structures and the Kähler cones, there exists uniquely an isomorphism $f : X \tilde{\to} X'$ with $f^* = \psi$. Here the Kähler cone is the cone in $H^2(X, \mathbf{R})$ consisting of Kähler classes on X.*

In the proof of this theorem, the key role is played by special $K3$ surfaces called Kummer surfaces, which are obtained as the minimal resolution of singularities of the quotients of Abelian surfaces divided by the involution ± 1. We shall see here an outline of the proof. Five facts (2.3.4)–(2.3.8) below are proved first:

(2.3.4) Local Torelli Theorem for the Kuranishi family of $K3$ surfaces.

(2.3.5) 'Preserving the Kähler cones' is an open condition.

(2.3.6) Density in D of the image by the period mapping of the points corresponding to projective Kummer surfaces.

(2.3.7) Torelli Theorem of type (2.3.3) for a projective Kummer surface X and a $K3$ surface X'.

(2.3.8) Continuity of isomorphisms between Kähler $K3$ surfaces.

By using these facts the argument goes as follows. Let $p : Y \to S$ and $p' : Y' \to S'$ be the Kuranishi families of $X = Y_0$, $X' = Y'_0$ respectively. Extend the given ψ to the isomorphism of the constant sheaves $\Psi : R^2 p'_* \mathbf{Z}_{Y'} \to R^2 p_* \mathbf{Z}_Y$. Let L be the $K3$ lattice $H^2(X, \mathbf{Z})$ together with the cup product. Choose a marking $\alpha : R^2 p_* \mathbf{Z}_Y \tilde{\to} L \times S$ and put $\alpha' := \alpha \circ \Psi$. By these markings, we have period mappings $\Phi : S \to D$, $\Phi' : S' \to D$. By the construction and the condition in the theorem, we see $\Phi(0) = \Phi'(0)$. By Local Torelli Theorem (2.3.4), there exists an isomorphism $q : S \tilde{\to} S'$ such that $\Phi' \circ q = \Phi$. Replacing the Kuranishi family p' by $Y' \times_{S'} S \to S$, we may assume that $S' = S$, $q =$ (identity) and $\Phi' = \Phi$. By the construction, $\Psi(s) : H^2(Y'_s, \mathbf{Z}) \to H^2(Y_s, \mathbf{Z})$ preserves the Hodge structures at any point $s \in S$. Shrinking S if necessary, we may assume that $\Psi(s)$ preserves the Kähler cones. By Density (2.3.6), there exists a series of points $\{s_n\}$ in S converging 0 such that each $\Phi(s_n)$ is the period of a projective Kummer surface. By Torelli Theorem for projective Kummer surfaces (2.3.7), there exist isomorphisms $f_n : Y_{S_n} \to Y'_{S_n}$ with $f_n^* = \Psi(s_n)$. From this, by Continuity (2.3.8), we obtain an isomorphism $f : X \tilde{\to} X'$ with $f^* = \psi$ and the proof is finished.

Notice that Kummer surfaces are constructed from Abelian surfaces, which are characterized as $K3$ surfaces with mutually disjoint sixteen (-2)-curves among all $K3$ surfaces and that they have the density property (2.3.6) even if their moduli has only dimension 3 (the moduli of all $K3$ surfaces has dimension 20). The above proof is based on this lucky situation.
(For more details, see, e.g., [BPV])

(2.4) Via Jacobian rings. The following theorem was prepared by Griffiths and proved by Donagi.

(2.4.1) Theorem. *Generic Torelli Theorem* cf. (2.2) *holds for hypersurfaces of degree* d *in* $\mathbf{P}^{n+1}$ *except the following four cases:*
$d=3, n=2;\ d \mid n+2;\ d=4, n=4m\,(m \geq 1);\ d=6, n=6m+1\,(m \geq 1).$

Let $f \in S := \mathbf{C}[x_0, \ldots, x_{n+1}]$ be a homogeneous equation of a non-singular hypersurface of degree d and let $J(f) := (\partial f/\partial x_0, \ldots, \partial f/\partial x_{n+1})$. $R := S/J(f)$ is called the Jacobian ring of X which plays the key role in the proof of the theorem. Here is an outline of the proof.

(2.4.2) Put $t(p) := (n-p+1)d-(n+2)$. We have isomorphisms of $\mathbf{C}$-vector spaces

$$\lambda_p : R^{t(p)} \tilde{\to} H^{n-p}(X, \Omega_X^p), \quad \lambda_p(A) := \mathrm{Res}(A\Omega f^{-(n-p+1)}),$$
$$\text{where} \quad \Omega := \sum_{0 \leq i \leq n+1} (-1)^i x_i dx_0 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}.$$

(2.4.3) The differential of the period mapping Φ at the point corresponding to X can be interpreted as

$$\begin{array}{ccc} H^1(X, T_X) \times H^{n-p}(X, \Omega_X^p) & \longrightarrow & H^{n-p+1}(X, \Omega_X^{p-1}) \\ \lambda_d \times \lambda_{t(p)} \Big\uparrow \wr & & \lambda_{t(p)+d} \Big\uparrow \wr \\ R^d \times R^{t(p)} & \longrightarrow & R^{t(p)+d} \end{array}$$

Here the bottom arrow is the multiplication mapping of the ring R and the diagram is commutative up to non-zero constant.

(2.4.4) Put $\sigma := (d-2)(n+2)$. We see that the multiplication mapping $R^a \times R^{\sigma-a} \to R^\sigma \simeq \mathbf{C}$ is a perfect pairing.

(2.4.5) For a bilinear mapping of vector spaces $B : U \times V \to W$, the pair (T, B_-) defined below is called its symmetrizer:

$$T := \{P \in \mathrm{Hom}(U, V) \mid B(u, P(u')) = B(u', P(u)) \ \ (u, u' \in U)\}$$
$$B_- : T \times U \to V, \quad B_-(P, u) := P(u).$$

If $(d-2)(n-1) \geq 3$, $a \leq d-1$ and $b \leq d$, the symmetrizer of the multiplication mapping $B_{a,b} : R^a \times R^b \to R^{a+b}$ is the multiplication mapping $B_{b-a,a} : R^{b-a} \times R^a \to R^b$ for a general f (Symmetrizer Lemma).

(2.4.6) If $f, g \in S^d$ and f is general, then $J(f) = J(g)$ implies $f = cg$ $(\exists c \neq 0)$.

By using these facts, the theorem can be proved in a purely algebraic argument. (For more details, see [Dn] and also the article of Donagi in [Get].)

(2.5) Via mixed Hodge structures on open varieties. Surfaces of general type X with $p_g = (c_1)^2 = 1$ appeared as the first example for which the differential of the period mapping Φ is not injective, i.e., Infinitesimal Torelli Theorem does not hold

([Kn]). The fibers of the period mapping of such surfaces has dimension 0, 1, or 2 ([T], [U1]). This phenomenon can be explained by an effect of automorphisms of X ([U1]). In particular, we see the following:

$$\begin{aligned}&\text{Surface } X \text{ corresponding to a point on a 2-dimensional fiber of } \Phi\\ \Leftrightarrow\ & X \text{ has an automorphism } \sigma \text{ of order 2 such that } X/\langle\sigma\rangle \text{ is a } K3 \text{ surface.}\end{aligned}$$

Such a surface is called a Kunev surface. In order to rescue this situation, the author generalized the period mapping as the one assigning the mixed Hodge structure on the complement of the unique canonical divisor in X and proved the injectivity of the differential of this generalized period mapping, i.e., Infinitesimal Mixed Torelli Theorem ([U3]). Moreover, we have Generic Mixed Torelli Theorem as (2.5.1) below.

We state the theorem also for surfaces of general type X with $p_g = 1$, $(c_1)^2 = 2$, $\pi = \mathbf{Z}/(2)$, which have an automorphism σ of order 2 such that $X/\langle\sigma\rangle$ is a $K3$ surface, since we can deal with such surfaces in a similar way. The bi-canonical mapping f_{2K} of such surfaces X yields a branched double covering of $\mathbf{P}^2$ and a complete weighted projective space $\mathbf{Q}(2,1,1)$ respectively, and X can be recovered from the branch locus ([Ct], [CD]). Let $T_{(j)}$, $j = 1, 2$, be the parameter spaces of such branch loci respectively.

(2.5.1) Theorem ([SSU])**.** *Let $T := T_{(j)}$, $j = 1, 2$. Denote by X_i, C_i the surface of general type with $p_g = 1$ and its canonical curve corresponding to a point $t_i \in T$, $i = 1, 2$, respectively. Assume that the point t_1 is general and that a path γ from t_1 to t_2 in T induces an isomorphism γ^* of mixed Hodge structures on the σ-invariant parts $H^2(X_i - C_i, \mathbf{Z})^\sigma$. Then there exists an isomorphism $g : X_1 \tilde{\to} X_2$ such that $g^* = \gamma^*$. Such g is unique up to composition with an element of the group $\langle\sigma\rangle$.*

(For the proof of the theorem, see [SSU]. For this subsection, see also [Mr].)

We have surveyed the five approaches to Torelli-type Problems. There are some results in the approaches (2.3), (2.4) (cf. [Ref]). But, when we consider Torelli-type Problems for surfaces of general type for example, it seems possible to generalize the approach (2.2) and approach (2.5) which makes a good use, in some sense, of the idea to use theta divisor instead of merely cup product in the approach (2.1). In the remaining of this article, we shall survey those things related to the approach to Torelli-type Problem via mixed Hodge structures on the complements of divisors and via degenerations.

§3. Limits of Hodge structures

Here we survey the limits of Hodge structures after Cattani-Kaplan-Schmid ([CKS]), which generalized the results of Schmid ([Sm]).

We recall first the definitions of mixed Hodge structures etc. by Deligne et.al.

(3.1) Definition ([D2], [E])**.** *A* **Hodge Structure** *(HS for short) of weight w defined over $\mathbf{Q}$ is $(H_\mathbf{Q}, (H_\mathbf{C}, F))$ such that*

(0) *$H_\mathbf{Q}$ is a $\mathbf{Q}$-module of finite type and F is a decreasing filtration on $H_\mathbf{C} := \mathbf{C} \otimes H_\mathbf{Q}$.*

(i) *F and $\bar{F}$ are w-opposed, i.e., $\mathrm{gr}_F^p \mathrm{gr}_{\bar{F}}^q = 0$ unless $p + q = w$.*

A **Mixed HS** *(MHS for short) defined over* $\mathbf{Q}$ *is* $((H_{\mathbf{Q}}, W), (H_{\mathbf{C}}, F)) =: (H, W)$ *such that*

(0) *H is as above (HS.0) and W is an increasing filtration on $H_{\mathbf{Q}}$.*

(i) gr^W_k *is an HS of weight k.*

A **Filtered MHS** *(FMHS for short) defined over* $\mathbf{Q}$ *is* $((H_{\mathbf{Q}}, W, G), (H_{\mathbf{C}}, F)) =: (H, G)$ *such that*

(0) *H is as above (MHS.0) and G is an increasing filtration on $H_{\mathbf{Q}}$.*

(i) *$G_i H$ is an MHS for all i.*

Let D be the classifying space of Hodge structures on $H_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ of weight w, of Hodge numbers $\{h^{p,q}\}$ and polarized by S, and let $\check{D}$ be its 'compact dual'. Put $G_{\mathbf{C}} := \mathrm{Aut}(H_{\mathbf{C}}, S)$, and denote by $G_{\mathbf{R}}$ the subgroup preserving $H_{\mathbf{R}}$ and by $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{g}_{\mathbf{R}}$ their Lie algebras respectively. For mutually commutative nilpotent elements $N_1, \dots, N_m \in \mathfrak{g}_{\mathbf{R}}$, $C_{\underline{m}} := \sum_{i \le j \le m} \mathbf{R}_{>0} N_j$ is called a monodromy cone. For a nilpotent element $N \in \mathfrak{g}_{\mathbf{R}}$, there exists a unique increasing filtration $W = W(N)$ of $H_{\mathbf{R}}$ satisfying the following two conditions:

$$N W_k \subset W_{k-2}, \quad N^k : \mathrm{gr}^W_k \xrightarrow{\sim} \mathrm{gr}^W_{-k}.$$

BY [CK2], $W(N)$ is independent of the choice of $N \in C_{\underline{m}}$ and hence denoted by $W(C_{\underline{m}})$.

(3.2) Definition. . **(1)** *A pair $(C_{\underline{m}}, F)$, $F \in \check{D}$, is called a nilpotent orbit if it satisfies the following two conditions:*

(i) $NF^p \subset F^{p-1} \quad (\forall N \in C_{\underline{m}}, \forall p)$.

(ii) *There exists a real number α such that* $\exp(\sqrt{-1} y N) F \in D$ *if* $y > \alpha$.

(2) *A pair (ρ, F) of a homomorphism of groups $\rho : SL(2, \mathbf{R})^m \to G_{\mathbf{R}}$ and $F \in \check{D}$ is called an $SL(2)^m$-orbit if it satisfies the following condition:*

Putting $\tilde{N}_{\underline{m}} := \rho(n^-, \dots, n^-) \in \mathfrak{g}_{\mathbf{R}}$ *for* $n^- := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbf{R})$, *we have that* $\exp(\sqrt{-1} \tilde{N}_{\underline{m}}) F \in D$ *and that* $\rho_* : \mathfrak{sl}(2, \mathbf{C})^{\oplus m} \to \mathfrak{g}_{\mathbf{C}}$ *is a morphism of Hodge structures of type* $(0,0)$ *with respect to the Hodge structures induced by* $\sqrt{-1} \in \mathfrak{h}$ *and by* $\exp(\sqrt{-1} \tilde{N}_{\underline{m}}) F$ *respectively.*

(3) *A pair $(C_{\underline{m}}, F)$ in (1) is called a polarized mixed Hodge of weight w if it satisfies for each $N \in C_{\underline{m}}$ the following four conditions:*

(i) $N^{w+1} = 0$.

(ii) *(W, F) is a mixed Hodge structure. Here* $W := W(N)[-w]$.

(iii) $NF^p \subset F^{p-1}$.

(iv) *The Hodge structure on the primitive part* $P_{w+j} := \mathrm{Ker}(N^{j+1} : \mathrm{gr}^W_{w+j} \to \mathrm{gr}^W_{w-j-2})$ *is polarized by* $S_{(j)} := S(\,\cdot\,, N^j\ \cdot\,)$.

(4) *A mixed Hodge structure (W, F) is $\mathbf{R}$-split if* $H_{\mathbf{C}} = \oplus (F^p \cap \overline{F}^q \cap W_{p+q})$.

(3.3) Remark. An $SL(2)$-orbit induces a horizontal mapping $\tilde{\rho}$ which commutes with the group action:

$$\begin{array}{ccc} SL(2,\mathbf{C}) & \xrightarrow{\rho} & G_{\mathbf{C}} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\tilde{\rho}} & \check{D} \\ \sqrt{-1} & \longrightarrow & \exp(\sqrt{-1}N)F \end{array}$$

This is a generalization of '(H_1)-homomorphism' (see e.g. [Sa2]) in the present context.

For a mixed Hodge structure (W,F), define $L^{a,b}$ by

$$L^{a,b} := \{X \in \operatorname{End}(H_{\mathbf{C}}) \,|\, XW_k \subset W_{k+a+b}\ (\forall k), XF^p \subset F^{p+a}\ (\forall p), X\overline{F}^q \subset \overline{F}^{q+b}\ (\forall q)\}.$$

The following proposition on **R**-splitness plays an important role in the proofs of Theorems (3.5) below.

(3.4) Proposition. *Given a mixed Hodge structure (W,F) of weight w, we can find a unique $\delta \in L^{-1,-1} \cap \operatorname{End}(H_{\mathbf{R}})$ such that $(W,\tilde{F})$, $\tilde{F} := e^{-\sqrt{-1}\delta}F$, is an **R**-split mixed Hodge structure. We also have*

$$\operatorname{End}(W,F) = \{f \in \operatorname{End}(W,\tilde{F}) \,|\, [\delta, f] = 0\}$$

Moreover, if (N,F) is a polarized mixed Hodge structure, we see that $\delta \in \mathfrak{g}_{\mathbf{R}}$ and that $\tilde{F}$ is also polarized with respect to N.

Let (W,F), $(W,\tilde{F})$ be as in Proposition (3.4). We assume that $W = W(N)[-w]$ for a nilpotent element $N \in \mathfrak{g}_{\mathbf{R}}$. Define a semi-simple element $\tilde{Y} \in \mathfrak{g}_{\mathbf{R}}$ corresponding to the direct sum decomposition $H_{\mathbf{C}} = \oplus\big(\tilde{F}^p \cap \overline{\tilde{F}}^q \cap W_{p+q}\big)$ whose eigenvalue is $p+q-w$ on $\tilde{F}^p \cap \overline{\tilde{F}}^q \cap W_{p+q}$. Then $SL(2)$-orbit Theorem in one variable ([CKS, (3.25)]) asserts that there exists uniquely a real analytic, $G_{\mathbf{R}}$-valued function $\tilde{g}(y)$ having a convergent Taylor expansion in y^{-1} around $y = \infty$ with the following three properties:

$$\begin{aligned} &\exp(\sqrt{-1}yN)F = \tilde{g}(y)\exp(\sqrt{-1}yN)\tilde{F}, \\ &\tilde{h}(y)^{-1}\tilde{h}'(y) \perp \operatorname{Lie}\big(\text{Isotropy at } (e^{iN})\tilde{F}\big), \\ &\tilde{g}(\infty) \in \exp\big(L^{-1,-1} \cap \ker \operatorname{ad} N\big). \end{aligned}$$

Here $\tilde{h}(y) := \tilde{g}(y)\exp\big(-(1/2)\log y\tilde{Y}\big)$. Since $\tilde{g}(\infty)$ is a unipotent element, we can take $\zeta := \log\tilde{g}(\infty)$. Put $\tilde{F}_0 := g(\infty)\tilde{F} = e^{\zeta-\sqrt{-1}\delta}F$. Then it is known that $(W,\tilde{F}_0)$ is canonically determined by (W,F), i.e., independent of the choice of N ([CKS, (3.31)]). We call this the ***R**-split mixed Hodge structure associated to* (W,F).

Given a polarized mixed Hodge structure $(C_{\underline{m}},F)$ of weight w and an ordered set of generators $N_1,\dots,N_m$ of the cone $C_{\underline{m}}$. Put

$$C_{\underline{r}} := \sum_{1\le j\le r} \mathbf{R}_{>0}N_j, \qquad W^{\underline{r}} := W(C_{\underline{r}})[-w].$$

Define descending-inductively the associated Hodge filtrations $\tilde{F}_{\underline{r}}$, $m \geq r \geq 1$, by

$$(W^{\underline{r}}, \tilde{F}_{\underline{r}}) \text{ is the } \mathbf{R}\text{-split mixed Hodge structure associated to } \big(W^{\underline{r}}, e^{\sqrt{-1}N_{r+1}}\tilde{F}_{\underline{r+1}}\big).$$

Here we understand $e^{\sqrt{-1}N_{m+1}}\tilde{F}_{\underline{m+1}} := F$. Let $\tilde{Y}_{\underline{r}}$ be the semi-simple element in $\mathfrak{g}_{\mathbf{R}}$ defined by $(W^{\underline{r}}, \tilde{F}_{\underline{r}})$, and $\tilde{N}_r$ be the component in the subspace $\cap_{r>j\geq 1} \ker \operatorname{ad} \tilde{Y}_{\underline{j}}$ relative to the decomposition of $\mathfrak{g}_{\mathbf{R}}$ into the eigenspaces of the commuting set of semi-simple endmorphisms $\{\operatorname{ad} \tilde{Y}_{\underline{j}}\}_{r>j\geq 1}$. Denote

$$\tilde{Y}_r := \tilde{Y}_{\underline{r}} - \tilde{Y}_{\underline{r-1}}, \qquad \tilde{N}_{\underline{r}} := \sum_{r>j\geq 1} \tilde{N}_j.$$

A rough description of the main results in [Sm] and [CKS] is as follow.

(3.5) Theorem. *We use the above notation.*

(i) *For a pair $(C_{\underline{m}}, F)$ of a monodromy cone and a filtration $F \in \check{D}$, we have the equivalence:*

a nilpotent orbit $\Leftrightarrow$ a polarized mixed Hodge structure.

In the case of $m = 1$, we have moreover the equivalence:

an $SL(2)$-orbit $\Leftrightarrow$ a polarized $\mathbf{R}$-split mixed Hodge structure.

(ii) *A lifting of a period mapping $\tilde{\varphi} : \mathfrak{h}^m \to D$ is approximated by the nilpotent orbit*

$$\tilde{\varphi}(z) \sim \exp\Big(\sum_{1\leq j\leq m} z_j N_j\Big)F \quad \text{as } z \to \big(\sqrt{-1}\infty, \dots, \sqrt{-1}\infty\big).$$

Here $F := \psi(0)$, $\psi : \Delta^m \to \check{D}$ is a map induced by $\tilde{\psi} := \exp\big(-\sum_{1\leq j\leq m} \mu_j z_j N_j\big)\tilde{\varphi}(\mu z)$. The nilpotent imaginary orbit is approximated by the $SL(2)^m$ imaginary orbit

$$\exp\Big(\sqrt{-1}\sum_{1\leq j\leq m} y_j N_j\Big)F \sim \exp\Big(\sqrt{-1}\sum_{1\leq j\leq m} y_j \tilde{N}_j\Big)\tilde{F}_{\underline{m}}$$

as $y_j/y_{j+1} \to \infty \quad (m \geq j \geq 1,\ y_{m+1} := 1)$.

(For more details, see [Sm], [CKS].)

For a nilpotent element $N \in \mathfrak{g}_{\mathbf{R}}$ compatible with an increasing filtration W of $H_{\mathbf{R}}$, an increasing filtration $M := W(N, W)$ is uniquely determined, if exists, by the following two conditions:

$$NM_k \subset M_{k-2}, \quad N^k : \operatorname{gr}^M_{j+k} \operatorname{gr}^W_j \xrightarrow{\sim} \operatorname{gr}^M_{j-k} \operatorname{gr}^W_j .$$

$M := W(N, W)$ is called a *W-relative N-filtration.*

(3.6) Corollary. *For subsets $I, J \subset \{1, \cdots, m\}$, we have $W(C_{I\cup J}) = W(C_I, W(C_J))$.*

(3.7) Remark. As a corollary of Theorem (3.4), one can describe the asymptotic behavior of the Hodge metric $S(C_F\,\cdot\,, \bar{\cdot}\,)$ near the boundary with respect to the flat frame and also to the frame of the canonical extension, i.e., norm estimates ([CKS, §5]). Kashiwara also obtained the norm estimates in the method of distributive family of filtrations ([Ks]).

§4. Compactifications of D/Γ

We survey in this section some compactifications of the quotients D/Γ and extensions of period mappings.

(4.1) We first give an elementary remark. Let D be the classifying space of Hodge structures of weight w of Hodge type $\{h^{p,q}\}$ and polarized by S. D is then a homogeneous space under the automorphism group $G_{\mathbf{R}} = \mathrm{Aut}(H_{\mathbf{R}}, S)$, and $G_{\mathbf{R}}$ and its isotropy subgroup I at some point of D are described as follows:

$$G_{\mathbf{R}} \simeq \begin{cases} O(k, 2h), \\ Sp(2h, \mathbf{R}), \end{cases} \quad I \simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}) & \text{if } w = 2t, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w = 2t+1. \end{cases}$$

Here $k := \sum_j h^{t+2j,t-2j}$, $h := (\dim_{\mathbf{C}} H_{\mathbf{C}} - k)/2$ if $w = 2t$, and $h := \dim_{\mathbf{C}} H_{\mathbf{C}}/2$ if $w = 2t+1$. Hence I is a compact subgroup of $G_{\mathbf{R}}$ but not maximal compact in general. Notice that I is maximal compact if and only if $D \simeq G_{\mathbf{R}}/I$ is a Hermitian symmetric domain. In the case that the horizontal subbundle T^h_X is not 0, I is maximal compact only in the following three cases:

$w = 2t+1$, and $h^{p,q} = 0$ unless $p = t+1, t$;

$w = 2t$, $h^{p,q} = 1$ for $p = t \pm 1$, $h^{t,t}$ is arbitrary, and $h^{p,q} = 0$ otherwise;

$w = 2t$, $h^{p,q} = 1$ for $p = t \pm (a-1), t \pm a$ for some $a \geq 2$, and $h^{p,q} = 0$ otherwise.

(4.2) Case: D is a Hermitian symmetric domain. In this case, as we mentioned in §2, we have the Satake compactifications $(D/\Gamma)^S$ ([Sa1]), which have algebraic structures induced via automorphic forms by Baily-Borel ([BB]), toroidal compactifications $(D/\Gamma)^M$ by Mumford et.al ([AMRT]) ·

(4.2.1) Extensions to the Satake-Baily-Borel compactifications ([B])**.**
Any holomorphic mapping $\Phi : (\Delta^*)^k \times \Delta^{n-k} \to D/\Gamma$ can be extended over Δ^n to the Satake-Baily-Borel compactification $\Phi^S : \Delta^n \to (D/\Gamma)^S$.

Under the condition in the above theorem, let $\tilde{\Phi}^S : \mathfrak{h}^k \times \Delta^{(n-k)} \to D$ be a lifting of Φ^S and B be the boundary component containing $\tilde{\Phi}^S\big(\underline{\sqrt{-1}\infty, 0}\big)$, then $\mathrm{Im}\,\tilde{\Phi}^S \subset \bigcup_{\bar{B}' \supset B} B'$.

(4.2.2) Extensions to the toroidal compactifications ([AMRT])**.**
In the above notation, let $C(B)$, $U(B)$ be the cone and the unipotent subgroup corresponding to the boundary component B. Then every local monodromy T_i belongs to $\overline{C}(B) \cap U(B)_{\mathbf{Z}}$. Hence Φ can be lifted to $\Phi' : (\Delta^*)^k \times \Delta^{n-k} \to D/U(B)_{\mathbf{Z}}$ and it is proved that the following three conditions are equivalent:

(i) *There exists a cone $\sigma_\alpha \subset C(B)$ which contains all the T_i.*

(ii) *Φ' can be extended holomorphically to ${\Phi'}^M : \Delta^n \to (D/U(B)_{\mathbf{Z}})_{\{\sigma_\alpha\}}$.*

(iii) *Φ can be extended holomorphically to $\Phi^M : \Delta^n \to (D/\Gamma)^M$.*

(For more details, see [AMRT]).

(4.3) Case: D is general. There is a speculation of Griffiths in [G2]. We survey some experimental results obtained since then.

Cattani and Kaplan obtained the following theorem from the Schmid theory in one-variable case which we have seen in §3.

(4.3.1) Case of weight 2 [CK1]). *In the case of Hodge structures of weight* 2*, we can construct a partial compactification* $(D/\Gamma)^C$ *of Satake type, which is a topological space having the following properties:*

(i) *Any period mapping* $\Phi : \Delta^* \to D/\Gamma$ *from a punctured disc can be extended continuously to* $\Phi^C : \Delta \to (D/\Gamma)^C$.

(ii) *Conversely, for any point* $b \in (D/\Gamma)^C$*, there exists a period mapping* $\Phi : \Delta^* \to D/\Gamma$ *with* $\Phi^C(0) = b$.

The partial compactification $(D/\Gamma)^C$ is constructed in the following way:
For an $SL(2)$-orbit (ρ, F) in one-variable case, let $N := \rho(n^-)$, $W := W(N)[-2]$. The corresponding boundary component $B(\rho)$, the boundary bundle $\mathcal{B}(W)$, and their unions D^C, D^* are defined by

$$\begin{aligned}
&B(\rho) := \text{classifying space of} \\
&\qquad (S_{(0)}\text{-polarized HS on } P_0 \subset \mathrm{gr}_0^W) \times (S_{(-1)}\text{-polarized HS on } \mathrm{gr}_{-1}^W), \\
&\mathcal{B}(L) := \bigcup \{B(\rho) \mid \text{ preserving only } W\}, \\
&D^C := \bigcup_{\rho:\,\text{rational}} B(\rho) \underset{\text{dense}}{\subset} D^* := \bigcup_{W:\,\text{rational}} \mathcal{B}(L).
\end{aligned}$$

Then the normalizer $N(\mathcal{B}(L))$ is a parabolic subgroup of $G_{\mathbf{R}}$, and we can induce a Satake topology on D^* on which Γ acts properly discontinuously. Now we define $(D/\Gamma)^C := D^C/\Gamma$.

In the approach of Torelli-type Problems via degenerations, a desirable partial compactification of D/Γ should be one of type of a toroidal copmactificaiton with complex structure, whose boundary points contain extension data of the limit Hodge structures (cf. (2.2.5)). In this direction, the author obtained the following experimental result.

We restrict the monodromy weight filtrations of the following type:

$$(*) \qquad 0 = W_{w-2} \subset W_{w-1} \subset W_w \subset W_{w+1} := H_{\mathbf{Q}}, \quad \dim W_{w-1} = \begin{cases} 1 & (w \text{ is odd}), \\ 2 & (w \text{ is even}). \end{cases}$$

A nilpotent orbit (C, F), $C := \mathbf{R}_{>0}N$, is *rational* if the C-filtration $W(C)$ is defined over $\mathbf{Q}$.

(4.3.2) Theorem ([U7])**.** *There exists a partial compactification* $\overline{D/\Gamma}$ *with the following three properties:*

(i) *As point sets,* $|\overline{D/\Gamma}| = |D/\Gamma| \cup \{$*rational nilpotent orbits* (C, F) *satisfying* $(*)\}/\Gamma$.

(ii) $\overline{D/\Gamma}$ *is a Hausdorff space with at most finite quotient singularities which may not be locally compact but carries a 'complex structure'.*

(iii) *Any period mapping* $\Phi : \Delta^* \to D/\Gamma$ *from a punctured disc with the property* $(*)$ *can be extended to* $\overline{\Phi} : \Delta \to \overline{D/\Gamma}$. *Conversely, for any point* $\xi \in \overline{D/\Gamma}$*, there exists a period mapping* $\Phi : \Delta^* \to D/\Gamma$ *with the property* $(*)$ *such that* $\overline{\Phi}(0) = \xi$.

(4.3.3) Problem. $\overline{D/\Gamma}$ in Theorem (4.3.2) is a partial compactification added the boundary components with one-dimensional monodromy cone. Add boundary components with higher dimensional monodromy cone.

(4.4) Mild degenerations of surfaces of general type on Noether lines. Surfaces of general type on the Noether lines are divided into two types according to $(c_1)^2$ being even or odd: Type (I) $(c_1)^2 = 2p_g - 4$; Type (II) $(c_1)^2 = 2p_g - 3$. These two series of surfaces are deeply studied by Horikawa ([H]) and the author observed moreover that these surfaces are joined by smooth deformations and mild degenerations:

$$
\begin{array}{cccccccccccc}
\text{(II)}: & (4,5) & & (5,7) & & \cdots & & (p,2p-3) & & (p+1,2p-1) & & \cdots \\
 & \downarrow & \nwarrow & \downarrow & \nwarrow & \cdots & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \cdots \\
\text{(I)}: & (4,4) & & (5,6) & & \cdots & & (p,2p-4) & & (p+1,2p-2) & & \cdots
\end{array}
$$

Here the pairs of numbers indicate $(p_g, (c_1)^2)$, $\downarrow$ means a degeneration of a surface collapsing a rational curve with self-intersection -4, and $\nwarrow$ means a degeneration of a surface with one simple elliptic singular point of type $\tilde{E}_8$.

We can apply Theorem (4.3.2) to these degenerations by using the Gysin exact sequence and the Clemens-Schmid exact sequence (5.4) below for their semi-stable reductions. For more details, see [U7].

We remark here that Ashikaga and Konno have observed that the mild degenerations as above occur widely in the geography of surfaces of general type ([AK]).

(4.4.1) Problem. Classify the degenerations of surfaces of general type with finite local monodromy. (This problem is solved for surfaces with $p_g = (c_1)^2 = 1$ in [U4]. See also [Fr2].) Adding the points corresponding to these degenerations, we can make the period mapping proper (cf. Monodromy Criteria in (5.4.3) below). Classify also those mild degenerations which we have seen in this subsection. Or, more modestly, classify the degenerations of the varieties corresponding to the boundary points of $\overline{D/\Gamma}$ in Theorem (4.3.2).

§5. Log Geometry

We restrict ourselves mainly to the log structures associated to pairs of compact complex manifolds and their divisors with normal crossings.

(5.1) Log Riemann-Hilbert Correspondence. We survey here Log Riemann-Hilbert Correspondence by K. Kato and C. Nakayama ([KtNk]) only in our present case.

A commutative semi-group with unity is called a monoid. Let X be a d-dimensional complex manifold and D a divisor with normal crossings on it. Regarding $\mathcal{O}_X$ as a monoid by multiplication, we call a subsheaf of monoids

$$\mathcal{M}_X := \{f \in \mathcal{O}_X \mid f \text{ is invertible outside } D\} \overset{\alpha}{\hookrightarrow} \mathcal{O}_X$$

the *fs (= fine saturated) log structure associated to the pair* (X, D)

A morphism of log complex manifolds $(f, \varphi) : (Y, \mathcal{M}_Y) \to (Z, \mathcal{M}_Z)$ is a pair of a morphism of complex manifolds $f : Y \to Z$ and a morphism of monoids $\varphi : f^{-1}\mathcal{M}_Z \to \mathcal{M}_Y$ satisfying $\alpha_Y \circ \varphi = f^* \circ \alpha_Z$.

Let T be a log point defined by

$$T := (\operatorname{Spec}\mathbf{C}, \mathbf{R}_{\geq 0}\times\mathbf{C}_1), \quad \alpha : \mathbf{R}_{\geq 0}\times\mathbf{C}_1 \to \mathbf{C}, \quad \alpha(r,u) := ru.$$

Here $\mathbf{C}_1 := \{u \in \mathbf{C} \mid |u| = 1\}$. For a log complex manifold $(X, \mathcal{M}_X)$, we consider the set $X^{\log}$ of T-valued points in the category of log complex manifolds and the projection τ_X:

$$X^{\log} := \operatorname{Hom}(T, (X, \mathcal{M}_X)) \xrightarrow{\tau_X} X, \quad \text{forgetting the log structure.}$$

Let $\tilde{x} \in X^{\log}$, $x := \tau_X(\tilde{x})$, and let $\prod_{1\leq i\leq r(x)} z_i^{m(i)}$, $m(i) \geq 1$, be a local equation of D in $\mathcal{O}_{X,x}$. Then we have

$$\mathcal{M}_{X,x} = \bigcup \{\mathcal{O}_{X,x}^\times \prod_{1\leq i\leq r} z_i^{m(i)a(i)} \mid a \in \mathbf{N}^{r(x)}\} \simeq \mathcal{O}_{X,x}^\times \oplus \mathbf{N}^{r(x)}, \quad \mathbf{N} := \mathbf{Z}_{\geq 0},$$

$$\begin{aligned} X^{\log} &\overset{\text{locally}}{\simeq} \{(z_i, u_i)_{1\leq i\leq r(x)} \in \mathbf{C}^{r(x)} \times (\mathbf{C}_1)^{r(x)} \mid z_i = |z_i|\, u_i,\ (\forall i)\} \times \mathbf{C}^{d-r(x)} \\ &\tilde{\to} (\mathbf{R}_{\geq 0})^{r(x)} \times (\mathbf{C}_1)^{r(x)} \times \mathbf{C}^{d-r(x)}, \\ &\big((z_i,u_i)_{1\leq i\leq r(x)}, (z_j)_{r(x)+1\leq j\leq d}\big) \mapsto \big((|z_i|, u_i)_{1\leq i\leq r(x)}, (z_j)_{r(x)+1\leq j\leq d}\big). \end{aligned}$$

Hence $X^{\log}$ can be regarded as a real blowing-up along D (cf. [Mj]), and it can be also regarded as a product of a real analytic manifold with corners, real compact torus and a complex manifold (cf. [AMRT]). Define a sheaf of rings $\mathcal{O}_X^{\log}$ on the topological space $X^{\log}$ by

$$(\mathcal{O}_X^{\log})_{\tilde{x}} := \mathcal{O}_{X,x}[l_1, \dots, l_{r(x)}], \quad l_i := (2\pi\sqrt{-1})^{-1}\log z_i.$$

This is not a local ring.

(5.1.1) Theorem ([KtNk]). *The following two categories $L_{\text{unip}}(X^{\log})$ and $D_{\text{nilp}}(X)$ are equivalent.*

$L_{\text{unip}}(X^{\log})$: *The category of locally constant sheaves L on $X^{\log}$ of finite dimensional* $\mathbf{C}$*-vector spaces with the following property. There exists locally on $X^{\log}$ an increasing filtration $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ consisting of locally constant* $\mathbf{C}$*-subsheaves of L such that each L_i/L_{i-1} is a pull-back of a locally constant sheaf on X of* $\mathbf{C}$*-vector spaces.*

$D_{\text{nilp}}(X)$: *A category of locally free $\mathcal{O}_X$-modules $\mathcal{V}$ of finite rank on X endowed with an integrable connection $\nabla : \mathcal{V} \to \omega_X^1 \otimes \mathcal{V}$ with the following property. Locally on X, there exists a finite filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$ consisting of $\mathcal{O}_X$-submodules of $\mathcal{V}$ such that $\nabla\mathcal{V}_i \subset \omega_X^1 \otimes \mathcal{V}_i$, $\mathcal{V}_i/\mathcal{V}_{i-1}$ is locally free and ∇ has no poles for all i.*

The equivalence $L_{\text{unip}}(X^{\log}) \to D_{\text{nilp}}(X)$, $L \mapsto \mathcal{V}$, and its inverse $\mathcal{V} \mapsto L$ are defined as follows:

$$\begin{aligned} \mathcal{V} &:= \tau_*(\mathcal{O}_X^{\log} \otimes_{\mathbf{C}} L), \\ L &:= \operatorname{Ker}(\tau^*\mathcal{V} \xrightarrow{\nabla} \omega_X^{1,\log} \otimes_{\mathcal{O}_X^{\log}} \tau^*\mathcal{V}). \end{aligned}$$

Here $\tau^*(\ ) := \mathcal{O}_X^{\log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}(\ )$, $\omega_X^{1,\log} := \tau^*(\Omega_X^1(\log D))$

(5.2) Recovering of vanishing cycles. Consider now a relative case. Let $f : X \to \Delta$ be a proper analytic morphism from a complex manifold to an open disc of relative dimension d. Assume that f is smooth over the punctured disc $\Delta^* := \Delta - \{0\}$, that the central fiber $X_0 := f^{-1}(0)$ is a divisor with simple normal crossings, that Y is a divisor on X flat with respect to f and that $X_0 + Y$ is also a divisor with simple normal crossings. Then, as in (5.1), we can construct a continuous map $f^{\log} : X^{\log} \to \Delta^{\log}$ and a subspace $Y^{\log}$ of $X^{\log}$ over the given ones and we have the following commutative diagram:

(5.2.1)
$$\begin{array}{ccc} (X^{\log}, Y^{\log}) & \xrightarrow{\tau_X} & (X, Y) \\ {\scriptstyle f^{\log}}\downarrow & & {\scriptstyle f}\downarrow \\ \Delta^{\log} & \xrightarrow{\tau_\Delta} & \Delta. \end{array}$$

The following theorem is a generalization of the local differentiable triviality of a smooth deformation of compact complex manifolds.

(5.2.2) Theorem ([U8]). *In the above notation, the family of open spaces*

$$\mathring{f}^{\log} : (X^{\log} - Y^{\log}) \to \Delta^{\log}$$

is locally piecewise C^∞ trivial over its base. In particular, this is a family which recovers the vanishing cycles of the family $f : (X - Y) \to \Delta$.

In the classical case of proper smooth family, the local triviality is proved by using the existence and uniqueness of the solutions of linear differential equations. In the proof of Theorem (5.2.2), we use an argument which generalizes the one by Clemens ([Cl]). We construct a family of C^∞ global defining equations of $X_0 + Y$, with which we define an action of the monoid $S := [0, 1] \times \mathbf{C}_1$ on the diagram (5.2.1) so that $[0, 1]$ acts as shrinking and $\mathbf{C}_1$ acts rounding. (We have to rescue and generalize the argument in [Cl], because there is a gap there. For more details, see [U8].)

(5.3) Limits of Hodge structures in geometric origin. Given a diagram (5.2.1). Assume moreover that $X \subset \mathbf{P}^m \times \Delta$ and f is the second projection and that X_0 is reduced. By Theorem (5.2.2), $L_{\mathbf{C}} := R^q(\mathring{f}^{log})_*\mathbf{C}$ is a locally constant sheaf on $\Delta^{\log}$ of $\mathbf{C}$-vector spaces. On the other hand, by the result of Steenbrink and Zucker ([SZ]) $\mathcal{V} := R^q f_* \Omega^\bullet_{X/\Delta}(\log(X_0 + Y))$ is a locally free $\mathcal{O}_\Delta$-module with Gauss-Manin connection ∇ and with $W(Y)$-relative monodromy filtration M. Hence, by Theorem (5.1.1), it corresponds to a locally constant sheaf on $\Delta^{\log}$ of $\mathbf{C}$-vector spaces. By construction, this locally constant sheaf coincides with $L_{\mathbf{C}}$ on Δ^* and hence on the whole $\Delta^{\log}$. Thus we have

$$L_{\mathbf{C}} \simeq \operatorname{Ker}\left(\nabla : (\tau_\Delta)^*\mathcal{V} \to \omega_\Delta^{1,\log} \otimes_{\mathcal{O}_X^{\log}} (\tau_\Delta)^*\mathcal{V}\right) \quad \text{on } \Delta^{\log},$$

$$\mathcal{V} \simeq (\tau_\Delta)_*\left(\mathcal{O}_\Delta^{\log} \otimes_{\mathbf{C}} L_{\mathbf{C}}\right) \quad \text{on } \Delta.$$

From this, we have the $\mathbf{Z}$-structure of the degenerate variation of mixed Hodge structure in [SZ] as in the following theorem. Let F be the Hodge filtration of $\mathcal{V}$ coming from the stupid filtration of $\Omega^\bullet_{X/\Delta}(\log(X_0 + Y))$.

(5.3.1) Theorem. *$(\mathcal{V}, M, W(Y), F)$ is a degenerate variation of mixed Hodge structure on Δ and its $\mathbf{Z}$-structure is given in the following two ways:*

(i) $$(\tau_\Delta)^*\mathcal{V} \simeq \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} R^q(\mathring{f}^{\log})_*\mathbf{Z} \quad \text{on } \Delta^{\log}.$$

(ii) $$\mathcal{V} \simeq \mathcal{O}_\Delta \otimes_{\mathbf{Z}} (\tau_\Delta)_* R^q(\mathring{f}^{\log})_*(\mathring{f}^{\log})^{-1}\big(\mathbf{Z}[(2\pi\sqrt{-1})^{-1}\log t]\big) \quad \text{on } \Delta.$$

In (i)*, the local monodromy is induced by the action of $\mathbf{C}_1$ on $\Delta^{\log}$. In* (ii)*, the monodromy logarithm coincides with $-2\pi\sqrt{-1}\,\mathrm{Res}_0(\nabla)$.*

Take a point $\eta \in \Delta^{\log}$ lying over the origin $0 \in \Delta$. As the limit of variation of mixed Hodge structure, $H^q((X^{\log} - Y^{\log})_\eta, \mathbf{Z})$ carries a $W(Y)$-filtered mixed Hodge structure (M, F). Notice that there are examples of variations of mixed Hodge structure not arising from geometry, which do not have limits (cf. [SZ]).
(For the $\mathbf{Z}$-structures in Theorem (5.3.1), see [U8]. See also [St2], [Fj], [Mt]).

(5.3.2) Remark. The author was communicated by Morihiko Saito that there is a correction of the proof of [St1, (5.9)] in [SM1, 4.2].

(5.4) Clemens-Schmid sequences. We use the notation in (5.3).

(5.4.1) Theorem. ([Cl]) *Assume that $Y = \emptyset$. The following diagram of (co)homology groups with coefficients in $\mathbf{Q}$ is a commutative diagram of mixed Hodge structures with exact rows.*

$$\begin{array}{ccccccccc} H^q(X_0) & \to & H^q(X_\eta^{\log}) & \xrightarrow{N} & H^q(X_\eta^{\log}) & \to & H_{2d-q}(X_0) & \to & H^{q+2}(X_0) \\ & & & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ & & & & H^{q+1}(X^*) & \to & H^{q+2}(X, X^*) & \to & H^{q+2}(X) \end{array}$$

(5.4.2) Remark. In the above theorem, the assertions other than the exactness are valid even in the case of $Y \neq \emptyset$. Moreover, under a suitable condition, the exactness also holds for $q \leq 2$ ([U5]). The exactness is not known for general q.

(5.4.3) Corollary (Monodromy Criteria)**.** (i) *Let $X_0 = \bigcup X_i$ be the irreducible decomposition of the central fiber. Then $p_g(X_t) \geq \sum_i p_g(X_i)$, $t \neq 0$.*
(ii) *$N = 0$ on $H^d(X_\eta^{\log})$ $\Rightarrow$ the equality holds in* (i) *$\Rightarrow$ $N^{d-1} = 0$ on $H^d(X_\eta^{\log})$.*
(iii) *For $q = 1, 2$, $N^q = 0$ on $H^q(X_\eta^{\log})$ $\Leftrightarrow$ $H^q($dual graph of $X_0) = 0$.*

(For this corollary, see the article of Morrison in [Get].)

(5.5) Log smooth deformations. The results in this subsection were obtained by Kawamata and Yoshinori Namikawa ([KwNm]) and arranged by F. Kato ([KtF1], [KtF2]).

We survey first a Log Geometric characterization of the notion of Friedman's d-semi-stability. $(\mathrm{Spec}\,\mathbf{C}, \mathbf{C}^\times \oplus \mathbf{N})$, $\mathbf{C}^\times \oplus \mathbf{N} \ni (c, n) \mapsto c \cdot 0^n \in \mathbf{C}$, is called a canonical log point. Let X be a normal crossing variety. A log structure of *semi-stable type* on X is an fs log structure $\mathcal{M}_X^s$ on X which makes the following diagram commutative: .

(5.5.1) $$\begin{array}{ccc} \mathcal{M}_{X,x}^s = \mathcal{O}_{X,x}^\times \oplus \mathbf{N}^{r(x)} & \xrightarrow{\alpha_X^s} & \mathcal{O}_{X,x} = \mathbf{C}\{z_1, \ldots, z_d\}/(z_1 \cdots z_{r(x)}) \\ {\scriptstyle\varphi^s}\uparrow & & \uparrow \\ \mathbf{C}^\times \oplus \mathbf{N} & \longrightarrow & \mathbf{C} \end{array}$$

Here $\alpha^s_X(e_i) := z_i$, $1 \le i \le r(x)$, and $\mathbf{N} \to \mathbf{N}^{r(x)}$ in φ^s is the diagonal map. $\mathcal{T}^1_X := \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \simeq (\mathcal{I}_X/\mathcal{I}^2_X)^\vee \otimes \mathcal{O}_D$, $D := \mathrm{Sing}(X)$, is called an *infinitesimal normal bundle* of X, and X is called *d-semi-stable* if $\mathcal{T}^1_X \simeq \mathcal{O}_D$ ([Fr1]).

(5.5.2) Theorem ([KwNm]). *For a normal crossing variety X, we have the equivalence:*

X has a log structure of semi-stable type $\Leftrightarrow$ X is d-semi-stable.

In the remaining of this subsection, we survey the existence of versal deformations for log analytic spaces associated to d-semi-stable normal crossing varieties. It will be obtained in the manner of the Schlessinger theory ([Sl]) as follows.

Let X be a d-semi-stable compact normal crossing variety. Let $D = \bigsqcup_{1 \le i \le m} D_i$ be the decomposition of the singular locus of X into connected components. For each i, $X - \bigsqcup_{j \ne i} D_j$ is d-semi-stable with connected singular locus and hence, by (5.5.2), there exists a log structure of semi-stable type uniquely up to isomorphism. Let $\varphi_i : \mathbf{C}^\times \oplus \mathbf{N} \to \mathcal{M}_i$ be the morphism as in (5.5.1) of this log structure of semi-stable type extended trivially over $\bigsqcup_{j \ne i} D_i$. The log structure defined by the following diagram is called the *canonical log structure* of d-semi-stable compact normal crossing variety X:

(5.5.3)
$$\begin{array}{ccc} \mathcal{M}^c_X := \bigoplus^{1 \le i \le m}_{\mathcal{O}^\times_X} \mathcal{M}_i & \longrightarrow & \mathcal{O}_X \\ {\scriptstyle \varphi^c := (\varphi_1, \dots, \varphi_m)} \big\uparrow & & \big\uparrow \\ \mathbf{C}^\times \oplus \mathbf{N}^m \simeq \bigoplus^m_{\mathbf{C}^\times} (\mathbf{C}^\times \oplus \mathbf{N}) & \longrightarrow & \mathbf{C} \end{array}$$

Here $M \oplus_G N$ is the co-fiber product, i.e., push-out, of $M \overset{\mu}{\leftarrow} G \overset{\nu}{\to} N$. This means the quotient of $M \oplus N$ by the equivalence relation $\sim$: $(m, n) \sim (m', n') \Leftrightarrow m\mu(g) = m'\mu(g'), n\nu(g') = n'\nu(g)$ $(\exists g, g' \in G)$

(5.5.4) Definition ([KtK1]). *A morphism $f : X \to Y$ of fine log analytic spaces is called log smooth if it satisfies the following two conditions:*

(i) *As a morphism of analytic spaces, it is locally finitely generated.*

(ii) *Let $i : S_0 \to S$ be a thickening of order 1, i.e., as a morphism of analytic spaces, it is a closed embedding with $\mathcal{I}^2 = 0$ for the ideal $\mathcal{I}$ of S_0 in S and their log structures are related as $f^{-1}\mathcal{M}_S = \mathcal{M}_{S_0}$. For a given commutative diagram:*

$$\begin{array}{ccc} S_0 & \longrightarrow & X \\ {\scriptstyle i} \big\downarrow & & {\scriptstyle f} \big\downarrow \\ S & \longrightarrow & Y, \end{array}$$

There exists a morphism $g : S \to X$ of log analytic spaces which makes the resulting diagram commutative.

Let $\mathcal{A}_{\Lambda(m)}$ be the category of local Artinian algebras over the formal power series ring $\Lambda(m) := \mathbf{C}[[t_1, \dots, t_m]]$. Let $f : (X, \mathcal{M}^c_X) \to (\mathrm{Spec}\, \mathbf{C}, \mathbf{C}^\times \oplus \mathbf{N}^m)$ be the morphism of fs log analytic spaces corresponding to the canonical log structure (5.5.3), which is known to be log smooth. For this f, we consider a functor $LD_f : \mathcal{A}_{\Lambda(m)} \to$ (Sets) defined by

$$LD_f(A) := \{\text{isomorphism class of a lifting of } f \text{ to } (\operatorname{Spec} A, A^\times \oplus \mathbf{N}^m)\}$$

for $A \in \operatorname{Obj}(\mathcal{A}_{\Lambda(m)})$.

(5.5.5) Theorem ([KwNm])**.** *Let X be a semi-stable compact normal crossing variety. In the above notation, LD_f has a hull in the sense of* [Sl].

(For the proof of this theorem, see [KtF1], and also [KtF2].)

(5.5.6) Theorem ([KwNm])**.** *Under the same condition as in Theorem (5.5.5), if moreover $H^2(X, T_X(-\log X)) = 0$, then X is smoothable by flat deformation.*

(5.5.7) Problem. For a semi-stable degeneration $f : X \to \Delta$, formulate the differential of an extended period mapping. Does it become as follows?

$$H^1(X_0, T_{X_0}(-\log X_0)) \to \bigoplus_{p+q=d} \operatorname{Hom}(H^q(X_0, \Omega^p_{X_0}(\log X_0)), H^{q+1}(X_0, \Omega^{p-1}_{X_0}(\log X_0)))$$

(For more details in this subsection, see [KtK1], [KwNm], [KtF1], [KtF2] etc.)

(5.6) Remark. Kazuya Kato introduces a notion of 'log Hodge structure' in order to generalize Hodge structures so as to include their limits and is trying to construct partial compactifications of D/Γ ([KtK3]).

Supplement

Although the author knows well that the following topics are closely related to the present article, he decided to omit them because of space.

(1) (Mixed) Hodge modules ([SM1], [SM2]). The author expects that Dr. Morihiko Saito himself will write a survey in this journal.

(2) L^2 cohomology theory. There is a survey by Dr. Takeo Ohsawa [O] in this journal.

(3) Singularity theory.

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