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BOREL-SERRE SPACES AND SPACES OF SL(2)-ORBITS

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Contents

Introduction

Let $G_{\mathbf{R}}$ be the $\mathbf{R}$-valued points of a semi-simple $\mathbf{Q}$-group. Let $\mathcal{X}$ be the Hermitian symmetric space of maximal compact subgroups of $G_{\mathbf{R}}$, and let Γ be an arithmetic subgroup of $G_{\mathbf{R}}$. Borel and Serre [BS] constructed a compactification $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ ($\Gamma\backslash\overline{\mathcal{X}}$ in their notation) of $\Gamma\backslash\mathcal{X}$. Let D be a classifying space of polarized Hodge structures (cf. §1). Replacing $\mathcal{X}$ by D, we generalize in this paper their construction of $\mathcal{X}_{\mathrm{BS}}$ in two directions. The one is a Borel-Serre space D_{BS} (§2), and the other is a space of SL(2)-orbits $D_{\mathrm{SL}(2)}$ (§3).

Now let $G_{\mathbf{R}}$, $G_{\mathbf{Z}}$ be the groups in Notation below. The construction of the space D_{BS} is similar to that of $\mathcal{X}_{\mathrm{BS}}$ and is based on the theory of Iwasawa decomposition of the group $G_{\mathbf{R}}$, and the quotient $\Gamma\backslash D_{\mathrm{BS}}$, by a subgroup Γ of the $G_{\mathbf{Z}}$ of finite index, is compact Hausdorff space (§5). On the other hand, the construction of the space $D_{\mathrm{SL}(2)}$ is based on the theory of SL(2)-orbits in [CKS] and the theory of Cartan decomposition of the Lie algebra $\mathfrak{g}_{\mathbf{R}} = \mathrm{Lie}\, G_{\mathbf{R}}$. The space $D_{\mathrm{SL}(2)}$ is Hausdorff (§3) but not always locally compact (6.9). The quotient $\Gamma\backslash D_{\mathrm{SL}(2)}$ is Hausdorff (§5), and has a nice property for period maps (3.16) which is an advantage of $\Gamma\backslash D_{\mathrm{SL}(2)}$ while $\Gamma\backslash D_{\mathrm{BS}}$ does not have.

Two spaces D_{BS} and $D_{\mathrm{SL}(2)}$ are not related directly, because the subspaces in the family of weight filtrations associated to a point of $D_{\mathrm{SL}(2)}$ are generally not linearly ordered (§6). To rescue this situation, we introduce the projective limit $D_{\mathrm{BS,val}}$ (resp. $D_{\mathrm{SL}(2),\mathrm{val}}$)

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of the blowing-ups of $D_{\rm BS}$ (resp. $D_{{\rm SL}(2)}$) (2.7, 3.7). Then we have the following diagram of topological spaces (3.1 (1)).

$$\begin{array}{ccccc} D_{{\rm SL}(2),{\rm val}} & \hookrightarrow & D_{{\rm BS},{\rm val}} & & \\ \downarrow & & \downarrow & & \\ D_{{\rm SL}(2)} & & D_{\rm BS} & \to & \mathcal{X}_{\rm BS}. \end{array}$$

In the classical situation, that is, in the situation where D is a Hermitian symmetric space and its horizontal tangent bundle coincides with its tangent bundle (see, 6.4), we have $D_{{\rm SL}(2)} = D_{\rm BS} = \mathcal{X}_{\rm BS}$ and $D_{{\rm SL}(2),{\rm val}} = D_{{\rm BS},{\rm val}}$ (6.5). As a corollary, we have the canonical surjection from the Borel-Serre compactification $\Gamma\backslash D_{\rm BS}$ to the Satake compactification $\Gamma\backslash D_S$ (6.6), which was defined by Zucker [Z] in another method.

In the next paper [KU2], we will generalize the theory of toroidal compactifications of $\Gamma\backslash\mathcal{X}$ by Mumford et al. replacing $\mathcal{X}$ by general D (the summary is in [KU1]). The results of the present paper will be also used there.

Notation

Throughout this paper, we use the following notation.

Let H be a $\mathbf{Z}$-module. For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, we denote $H_A := A \otimes_{\mathbf{Z}} H$.

We fix a 4-tuple

$$\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$$

where w is an integer, $(h^{p,q})_{p,q\in\mathbf{Z}}$ is a set of non-negative integers satisfying

$$\begin{cases} h^{p,q} = 0 \text{ for almost all } p,\ q, \\ h^{p,q} = 0 \text{ if } p+q \neq w, \\ h^{p,q} = h^{q,p} \text{ for any } p,\ q, \end{cases}$$

H_0 is a free $\mathbf{Z}$-module of rank $\sum_{p,q} h^{p,q}$, and $\langle\ ,\ \rangle_0$ is a $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form $H_{0,\mathbf{C}}$ which is symmetric if w is even and anti-symmetric if w is odd. In the case w is even, say $w = 2t$, we assume that the signature (a, b) of $\langle\ ,\ \rangle_0$ satisfies

$$a \text{ (resp. } b) = \textstyle\sum_j h^{t+j,t-j}, \text{ where } j \text{ ranges over all even (resp. odd) integers.}$$

Let

$$G_{\mathbf{Z}} := {\rm Aut}(H_0, \langle\ ,\ \rangle_0),$$

and for $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let

$$G_A := {\rm Aut}(H_{0,A}, \langle\ ,\ \rangle_0),$$
$$\mathfrak{g}_A := {\rm Lie}\, G_A = \{N \in {\rm End}_A(H_{0,A}) \mid \langle Nx, y\rangle_0 + \langle x, Ny\rangle_0 = 0\ (\forall x, \forall y \in H_{0,A})\}.$$

§1. Classifying spaces of polarized Hodge structures

In this section, we recall polarized Hodge structures, the classifying space D of polarized Hodge structures, horizontal tangent bundles, polarized variations of Hodge structure and the associated period maps (cf. [G]). Let $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ be as in Notation.

1.1. A *Hodge structure* of weight w and of Hodge type $(h^{p,q}) = (h^{p,q})_{p,q\in\mathbf{Z}}$ is a pair $(H_{\mathbf{Z}}, F)$ consisting of a free $\mathbf{Z}$-module $H_{\mathbf{Z}}$ of rank $\sum_{p,q} h^{p,q}$ and of a decreasing filtration F of $H_{\mathbf{C}}$, which satisfy the following two conditions.

(i) $\dim_{\mathbf{C}} F^p = \sum_{r\ge p} h^{r,w-r}$ for all p.

(ii) $H_{\mathbf{C}} = \bigoplus_p H^{p,w-p}$, where $H^{p,w-p} := F^p \cap \overline{F}^{w-p}$.

Note that $\dim H^{p,w-p} = h^{p,w-p}$ for all p.

1.2. A *polarized Hodge structure* of weight w and of Hodge type $(h^{p,q})$ is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a Hodge structure $(H_{\mathbf{Z}}, F)$ and a $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form $\langle\ ,\ \rangle$ on $H_{\mathbf{C}}$, symmmetric for even w and anti-symmetric for odd w, which satisfy the following two conditions.

(i) $\langle F^p, F^q\rangle = 0$ for $p+q > w$.

(ii) The Hermitian form $H_{0,\mathbf{C}} \times H_{0,\mathbf{C}} \to \mathbf{C}$, $(x,y) \mapsto \langle C_F(x), \overline{y}\rangle$, is positive definite.

Here $\overline{y}$ is the complex conjugation of y with respect to $H_{0,\mathbf{R}}$, and C_F is the Weil operator which is defined by $C_F(x) := i^{2p-w}x$ for $x \in H^{p,w-p}$. The condition (i) (resp. (ii)) is called the Riemann-Hodge first (resp. second) bilinear relation.

1.3. Let X be a complex manifold. A *polarized variation of Hodge structure* on X of weight w and of Hodge type $(h^{p,q})$ is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a local system $H_{\mathbf{Z}}$ on X, of a locally constant $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form $\langle\ ,\ \rangle$ on $\mathbf{C} \otimes H_{\mathbf{Z}}$ and of a decreasing filtration F of $\mathcal{O}_X \otimes H_{\mathbf{Z}}$ by subbundles, which satisfy the following conditons (i), (ii).

(i) $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x, F(x))$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})$ $(\forall x \in X)$.

(ii) Griffiths transversality $\nabla F^p \subset \Omega^1_X \otimes F^{p-1}$ holds $(\forall p)$. (∇ is the Gauss-Manin connection of $\mathcal{O}_X \otimes H_{\mathbf{Z}}$.)

Definitions 1.4. *The* classifying space D of polarized Hodge structures of type Φ_0 *is the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})$.*

Note that, by the condition on the signatuture of $\langle\ ,\ \rangle_0$ in Notation, D is non-empty.

Definitions 1.5. *The* compact dual $\check{D}$ *of D is the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that the pair (H_0, F) is a Hodge structure of weight w and of Hodge type $(h^{p,q})$ and that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ satisfies the condition* 1.2 (i).

Note that D (resp. $\check{D}$) is homogeneous under $G_{\mathbf{R}}$ (resp. $G_{\mathbf{C}}$) and that D is an open subset of $\check{D}$.

Definition 1.6. *Let $F \in D$ and let $T_D(F)$ be the tangent space of D at F. The* horizontal tangent space $T^h_D(F)$ *of D at F is defined as follows:*

$$T^h_D(F) = F^{-1}(\mathfrak{g}_{\mathbf{C}})/F^0(\mathfrak{g}_{\mathbf{C}}) \subset T_D(F) = \mathfrak{g}_{\mathbf{C}}/F^0(\mathfrak{g}_{\mathbf{C}}),$$

where $F^r(\mathfrak{g}_{\mathbf{C}}) := \{N \in \mathfrak{g}_{\mathbf{C}} \mid N(F^p) \subset F^{p+r}\ (\forall p \in \mathbf{Z})\}$.

1.7. Let $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ be as in Notation. Let X be a connected complex manifold and let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X of weight w and of Hodge type $(h^{p,q})$ with $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x) \simeq (H_0, \langle\ ,\ \rangle_0)$ for some (hence any) point $x \in X$. Fix such a point x and identify $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x) = (H_0, \langle\ ,\ \rangle_0)$. Put $\Gamma := \mathrm{Im}(\pi_1(X) \to G_{\mathbf{Z}})$. Then we have the associated period map

$$\varphi : X \to \Gamma\backslash D. \tag{1}$$

Recall that Griffiths made a fundamental observation that the differential of the period map at $x \in X$, associated to a polarized variation of Hodge structure arising from geometry, factors through $T^h_D(\tilde\varphi(x))$ (Griffiths transversality, cf. [G]). Here $\tilde\varphi : U \to D$ is a local lifting of φ on a neighborhood U of x.

§2. Borel-Serre spaces

2.1. *Summary.* Let $\mathcal{X}$ be the set of all maximal compact subgroups of $G_{\mathbf{R}}$. Then $G_{\mathbf{R}}$ acts on $\mathcal{X}$ by inner automorphisms transitively. Since the normalizer of $G_{\mathbf{R}}$ at each $K \in \mathcal{X}$ is K itself, we have a $G_{\mathbf{R}}$-equivariant isomorphism

$$G_{\mathbf{R}}/K \overset{\sim}{\to} \mathcal{X}, \quad g \mapsto gKg^{-1},$$

for each fixed $K \in \mathcal{X}$. By using this isomorphism, we introduce a topology on $\mathcal{X}$. This topology does not depend on the choice of K. Borel and Serre constructed in [BS] a space $\mathcal{X}_{\mathrm{BS}}$ ($\overline{\mathcal{X}}$ in their notation) which contains $\mathcal{X}$ as an open dense subset. The action of $G_{\mathbf{Z}}$ on $\mathcal{X}$ extends to the action on $\mathcal{X}_{\mathrm{BS}}$. $\mathcal{X}_{\mathrm{BS}}$ has the following remarkable properties:

(i) If Γ is a subgroup of $G_{\mathbf{Z}}$ of finite index, then the quotient space $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is compact.

(ii) If Γ is a neat subgroup of $G_{\mathbf{Z}}$, then the projection $\mathcal{X}_{\mathrm{BS}} \to \Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is a local homeomorphism.

In this section, we enlarge D to get a topological space D_{BS}, which contains D as a dense open subspace, in the same way to enlarge $\mathcal{X}$ to $\mathcal{X}_{\mathrm{BS}}$. We also constructed a topological space $D_{\mathrm{BS,val}}$, as the projective limit of the blowing-ups of D_{BS}, which also contains D as a dense open subspace. These spaces are related by continuous proper surjective maps in the following way:

$$D_{\mathrm{BS,val}} \to D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}.$$

2.2. *Borel-Serre action.* Let P be a parabolic subgroup of $G_{\mathbf{R}}$ and P_u be its unipotent radical. Let S_P be the maximal $\mathbf{R}$-split torus of the center of P/P_u. Let A_P be the connected component of S_P containing the unity.

For $K \in \mathcal{X}$, let $\theta_K : G_{\mathbf{R}} \to G_{\mathbf{R}}$ be the Cartan involution associated to the maximal compact subgroup K, i.e., the unique automorphism of $G_{\mathbf{R}}$ characterized by $\theta_K^2 = \mathrm{id}$ and $K = \{g \in G_{\mathbf{R}} \mid \theta_K(g) = g\}$. By [BS], for each $K \in \mathcal{X}$ and $a \in S_P$, there exists a unique element $a_K \in P$ satisfying the following (i) and (ii).

(i) $(a_K \mod P_u) = a$.

(ii) $\theta_K(a_K) = a_K^{-1}$.

Then the map $S_P \to P$, $a \mapsto a_K$, is a homomorphism of algebraic groups over $\mathbf{R}$. We call a_K $(a \in S)$ the *Borel-Serre lifting of a at K.*

For $F \in D$, we use the following notation:

$$(1) \qquad \begin{aligned} K_F &:= \{g \in G_{\mathbf{R}} \mid g \text{ preserves the Hermitian inner product } \langle C_F(\ ), \bar{\ }\rangle_0\}, \\ K'_F &:= \{g \in G_{\mathbf{R}} \mid gF = F\} \subset K_F. \end{aligned}$$

Note that K_F is a maximal compact subgroup of $G_{\mathbf{R}}$ and the Cartan involution θ_{K_F} is given by

$$(2) \qquad \theta_{K_F} = \mathrm{Int}(C_F), \ \text{ where } C_F \text{ is the Weil operator in 1.2 (cf. [Sc, §8]).}$$

Note also that there is the canonical $G_{\mathbf{R}}$-equivariant continuous map

$$(3) \qquad D \to \mathcal{X}, \ \ F \mapsto K_F.$$

For $a \in A_P$ and $K \in \mathcal{X}$ (resp. $F \in D$), we define an action $\circ$ by

$$(4) \qquad a \circ K := \mathrm{Int}(a_K)K \quad (\text{resp. } a \circ F := a_{K_F}F).$$

We call this as the *Borel-Serre action.*

Lemma 2.3. *For $a \in A_P$ and $p \in P$, we have $a_{\mathrm{Int}(p)K} = \mathrm{Int}(p)a_K$.*

Proof. This follows from the fact

$$(1) \qquad \theta_{\mathrm{Int}(p)K} = \mathrm{Int}(p)\theta_K \,\mathrm{Int}(p)^{-1}$$

and the definition of the Borel-Serre liftings in 2.2. □

Lemma 2.4. *For $a \in A_P$, $p \in P$ and $F \in D$, we have $a \circ pF = p(a \circ F)$.*

Proof. By 2.3, we have

$$a \circ pF = a_{K_{pF}}pF = a_{\mathrm{Int}(p)K_F}pF = (\mathrm{Int}(p)a_{K_F})pF = pa_{K_F}F = p(a \circ F). \quad \square$$

By 2.4, we see that the Borel-Serre action is indeed an action of A_P on D. In fact, for $a, b \in A_P$ and $F \in D$, we have

$$a \circ (b \circ F) = a \circ (b_{K_F}F) = b_{K_F}(a \circ F) = b_{K_F}a_{K_F}F = (ba)_{K_F}F = (ab) \circ F.$$

It can be verified in a similar way that A_P acts on $\mathcal{X}$ via the Borel-Serre action.

Definition 2.5. *The* generalized Borel-Serre space D_{BS} (*resp.* Borel-Serre space $\mathcal{X}_{\mathrm{BS}}$) *is defined by*

$$D_{\mathrm{BS}}\ (\textit{resp. } \mathcal{X}_{\mathrm{BS}}) := \left\{ (P, Z) \,\middle|\, \begin{array}{l} P \textit{ is a } \mathbf{Q}\textit{-parabolic subgroup of } G_{\mathbf{R}}, \\ Z \textit{ is an } (A_P\circ)\textit{-orbit in } D \ (\textit{resp. } \mathcal{X}) \end{array} \right\}.$$

2.6. *Remark.* In the definition of $\mathcal{X}_{\mathrm{BS}}$ in [BS], the maximal $\mathbf{Q}$-split torus of the center of P/P_u was used. In our case, it coincides with the maximal $\mathbf{R}$-split torus S_P of the center of P/P_u.

Definition 2.7. *We define the space* $D_{\mathrm{BS,val}}$ *by*

$$D_{\mathrm{BS,val}} := \left\{ (T, Z, V) \,\middle|\, \begin{array}{l} T \textit{ is an } \mathbf{R}\textit{-split torus of } G_{\mathbf{R}}, \\ Z \textit{ is a } (T_{>0})\textit{-orbit in } D, \\ V \subset \textit{character group } X(T), \\ \textit{which satisfy the following } \text{(i)–(iii)} \end{array} \right\}.$$

(i) *If* $\chi, \chi' \in V$ *then* $\chi\chi' \in V$. $V \cup V^{-1} = X(T)$. $V \cap V^{-1} = \{1\}$.

(ii) *The parabolic subgroup* $P := P_{T,V}$ *of* $G_{\mathbf{R}}$ *associated to* (T, V) *is* $\mathbf{Q}$*-rational.*

(iii) *The image of* $T \to P/P_u$ *is contained in* S_P *and, at any* $F \in Z$, $\theta_{K_F}(t) = t^{-1}$ $(\forall t \in T)$.

Here, $T_{>0}$ *is the connected component of* T *containing the unity, and the definition of the parabolic subgroup* $P_{T,V}$ *of* $G_{\mathbf{R}}$ *is as follows. Let* $H_{0,\mathbf{R}} = \bigoplus_{\chi \in X(T)} H(\chi)$ *be the decomposition into eigen spaces* $H(\chi) := \{v \in H_{0,\mathbf{R}} \,|\, tv = \chi(t)v \ (\forall t \in T)\}$. *Define an increasing filtration* $M_{T,V} = (M_\chi)_{\chi \in X(T)}$ *of* $H_{0,\mathbf{R}}$ *and the associated* $P_{T,V}$ *by*

$$M_\chi := \bigoplus\nolimits_{\chi' \in \chi V^{-1}} H(\chi'), \quad P_{T,V} := \{g \in G_{\mathbf{R}} \,|\, g \textit{ preseves } M_{T,V}\}.$$

Note that $\mathrm{gr}^M_\chi \xleftarrow{\sim} H(\chi)$ for all $\chi \in X(T)$.

2.8. We have maps

$$\begin{array}{l} D_{\mathrm{BS,val}} \xrightarrow{\alpha} D_{\mathrm{BS}} \xrightarrow{\beta} \mathcal{X}_{\mathrm{BS}}, \quad \text{where} \\ \alpha : (T, Z, V) \mapsto (P_{T,V}, A_{P_{T,V}} \circ Z), \\ \beta : \text{the map induced by 2.2 (3).} \end{array}$$

2.9. For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$, we define

$$\begin{array}{l} D_{\mathrm{BS}}(P) := \{(Q, Z) \in D_{\mathrm{BS}} \,|\, Q \supset P\}, \\ \mathcal{X}_{\mathrm{BS}}(P) := \{(Q, Z) \in \mathcal{X}_{\mathrm{BS}} \,|\, Q \supset P\}, \\ D_{\mathrm{BS,val}}(P) := \{(T, Z, V) \in D_{\mathrm{BS,val}} \,|\, P_{T,V} \supset P\}. \end{array}$$

2.10. *Description of parabolic subgroups of $G_{\mathbf{R}}$.* There is a bijection:

$$\text{(parabolic subgroups of } G_{\mathbf{R}})$$
$$\updownarrow$$
$$\left\{ M = (M_j)_{0\le j\le m} \;\middle|\; \begin{array}{l} \text{an increasing filtration of } H_{0,\mathbf{R}},\ M_j \neq M_k\ (j\neq k), \\ M_0 = 0,\ M_m = H_{0,\mathbf{R}},\ M_{m-j} = M_j^{\perp}\ (\forall j) \end{array} \right\}$$

Here m varies, and $(\ )^{\perp}$ respects $\langle\ ,\ \rangle_0$. The correspondences are given by

$$\begin{aligned} &P := \{g \in G_{\mathbf{R}} \mid gM_j = M_j\ (\forall j)\}, \\ &M := \{M_j \mid \text{a subspace of } H_{0,\mathbf{R}} \text{ satisfying } gM_j = M_j\ (\forall g \in P)\}. \end{aligned}$$

Note that P is $\mathbf{Q}$-palabolic if and only if M_j $(0 \le j \le m)$ are $\mathbf{Q}$-rational.

2.11. *Description of S_P.* Let P be a parabolic subgroup of $G_{\mathbf{R}}$, and let $M = (M_j)_{0\le j\le m}$ be the associated filtration of $H_{0,\mathbf{R}}$ in 2.10. The unipotent radical P_u of P is described by

$$P_u = \{g \in P \mid g \text{ induces the identity on } \textstyle\bigoplus_j M_j/M_{j-1}\},$$

and the quotient P/P_u is given by

$$P/P_u \xrightarrow{\sim} \{(g_j)_{1\le j\le m} \mid g_j \in \mathrm{GL}(M_j/M_{j-1}),\ {}^t g_j = g_{m-j+1}^{-1}\ (\forall j)\},$$

where the transposed respects the pairing $(M_j/M_{j-1}) \times (M_{m-j+1}/M_{m-j}) \to \mathbf{R}$ induced by $\langle\ ,\ \rangle_0$. S_P in 2.2 is described by

$$S_P = \left\{ g \in P/P_u \;\middle|\; \begin{array}{l} g \text{ induces a scalar multiplication on } M_j/M_{j-1}\ (\forall j), \text{ and} \\ g \text{ is the identity on } M_{(m+1)/2}/M_{(m-1)/2} \text{ if } m \text{ is odd} \end{array} \right\}.$$

Put

$$r := \begin{cases} m/2 & \text{if } m \text{ is even,} \\ (m-1)/2 & \text{if } m \text{ is odd.} \end{cases} \tag{1}$$

Let a_j be the scalar induced by $a \in S_P$ on M_j/M_{j-1} $(1 \le j \le r)$. We fix from now on the following identification:

$$S_P \xrightarrow{\sim} \mathbf{G}_m(\mathbf{R})^r,\ \ a \mapsto (t_j)_{1\le j\le r},\ \ t_j := a_{j+1}/a_j\ (1 \le j \le r) \text{ where } a_{r+1} := 1. \tag{2}$$

Under this identification, we have $A_P = \mathbf{R}^r_{>0}$.

2.12. *Identification $D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \mathbf{R}^r_{\ge 0}$.* Let P, $M = (M_j)_{0\le j\le m}$ and r be as in 2.11. Assume P is $\mathbf{Q}$-rational. Then we have the following bijection which reverses the orders by inclusion:

$$\begin{aligned} &\text{(subsets of } \{1, \dots, \mathrm{r}\}) \longleftrightarrow (\mathbf{Q}\text{-parabolic subgroups of } G_{\mathbf{R}} \text{ containing } P), \\ &\qquad\qquad J \longleftrightarrow Q \\ &\text{where}\quad Q := \{g \in G_{\mathbf{R}} \mid gM_j = M_j\ (\forall j \in J)\}. \end{aligned} \tag{1}$$

Hence the set of all $\mathbf{Q}$-rational stable subspaces of $H_{0,\mathbf{R}}$ under Q is

$$M_j \ \ (j \in J), \ \ M_{m-j} \ \ (j \in J), \ \ M_0 = 0, \ \ M_m = H_{0,\mathbf{R}}.$$

In particular, we have

$$\begin{aligned} \text{the empty subset } \emptyset \text{ of } \{1, \dots, r\} &\longleftrightarrow G_{\mathbf{R}}, \\ \text{the subset } \{1, \dots, r\} \text{ of } \{1, \dots, r\} &\longleftrightarrow P. \end{aligned}$$

When $J \leftrightarrow Q$ under (1), we define a point

$$(2) \qquad e_Q = (b_j)_{1 \le j \le r} \in \mathbf{R}^r_{\ge 0} \ \text{ by } \ b_j := \begin{cases} 0 & (j \in J), \\ 1 & (j \notin J). \end{cases}$$

For example, we have $e_P = (0, \dots, 0)$, $e_{G_{\mathbf{R}}} = (1, \dots, 1)$.

We define and fix from now on the following identification:

$$(3) \qquad D_{\mathrm{BS}}(P) \overset{\sim}{\to} D \times^{A_P} \mathbf{R}^r_{\ge 0}, \ (Q, Z) \mapsto (F, e_Q), \text{ for any } F \in Z \text{ and } e_Q \text{ in (2)}.$$

Here $D \times^{A_P} \mathbf{R}^r_{\ge 0} := (D \times \mathbf{R}^r_{\ge 0})/A_P$ under the action $a(F, b) := (a \circ F, a^{-1}b)$ ($a \in A_P$, $(F, b) \in D \times \mathbf{R}^r_{\ge 0}$). The inverse of the map (3) is given by

$$(4) \qquad \begin{aligned} &D \times^{A_P} \mathbf{R}^r_{\ge 0} \overset{\sim}{\to} D_{\mathrm{BS}}(P), \ (F, (b_j)_{1 \le j \le r}) \mapsto (Q, Z), \quad \text{where} \\ &Q \text{ is the } \mathbf{Q}\text{-parabolic subgroup of } G_{\mathbf{R}} \text{ corresponding to the subset} \\ &\quad \{j \,|\, 1 \le j \le r, b_j = 0\} \text{ of } \{1, \dots, r\} \text{ under (1), and} \\ &Z := \{t \circ F \,|\, t = (t_j)_{1 \le j \le r} \in \mathbf{R}^r_{>0} = A_P, t_j = b_j \text{ if } b_j \ne 0\}. \end{aligned}$$

2.13. *A topological space* $(\mathbf{R}^r_{\ge 0})_{\mathrm{val}}$. We define a topological space $(\mathbf{R}^r_{\ge 0})_{\mathrm{val}}$ ($r \ge 0$) as follows. Let $r \ge 0$ be an integer, and define a topological space B by

$$(1) \qquad B := \varprojlim_{I \in \Psi} \mathrm{Bl}_I(\mathbf{G}_a(\mathbf{R})^r),$$

where Ψ denotes the set of all non-zero ideals $I = (f_1, \dots, f_n)$ of $\mathbf{R}[t_1, \dots, t_r] = \mathcal{O}(\mathbf{G}_a(\mathbf{R})^r)$ generated by some monomials $f_1, \dots, f_n$, and $\mathrm{Bl}_I(\mathbf{G}_a(\mathbf{R})^r)$ means the blowing-up $\mathrm{Proj}(\bigoplus_{s \ge 0} I^s)$ of $\mathbf{G}_a(\mathbf{R})^r$ along I. Define an order $I \le J$ in Ψ by $II' = J$ for some $I' \in \Psi$. For $I \le II' = J$ with $I = (f_1, \dots, f_n)$, $I' = (g_1, \dots, g_m)$, define a morphism

$$\begin{aligned} &\mathrm{Bl}_I(\mathbf{G}_a(\mathbf{R})^r) = \textstyle\bigcup_{1 \le k \le n} \mathrm{Spec}\left(\mathbf{R}\left[t_1, \dots, t_r, \frac{f_1}{f_k}, \dots, \frac{f_n}{f_k}\right]\right) \\ &\qquad \uparrow \\ &\mathrm{Bl}_J(\mathbf{G}_a(\mathbf{R})^r) = \textstyle\bigcup_{1 \le k \le n, 1 \le l \le m} \mathrm{Spec}\left(\mathbf{R}\left[t_1, \dots, t_r, \frac{f_1 g_1}{f_k g_l}, \dots, \frac{f_n g_m}{f_k g_l}\right]\right) \end{aligned}$$

by the inclusions of affine rings

$$\mathbf{R}\left[t_1, \dots, t_r, \tfrac{f_1}{f_k}, \dots, \tfrac{f_n}{f_k}\right] \hookrightarrow \mathbf{R}\left[t_1, \dots, t_r, \tfrac{f_1 g_1}{f_k g_l}, \dots, \tfrac{f_n g_m}{f_k g_l}\right] \quad (1 \le k \le n, 1 \le l \le m).$$

The projective limit in (1) is taken by this ordering.

Since the centers of the blowing-ups are outside $\mathbf{G}_m(\mathbf{R})^r \subset \mathbf{G}_a(\mathbf{R})^r$, we have an open immersion $\mathbf{G}_m(\mathbf{R})^r \hookrightarrow B$. Furthermore, there is a unique action of $\mathbf{G}_m(\mathbf{R})^r$ on B which is compatible with the standard action of $\mathbf{G}_m(\mathbf{R})^r$ on itself. Let $(\mathbf{R}^r_{\geq 0})_{\text{val}}$ be the closure of $\mathbf{R}^r_{>0}$ in B under the composite of open immersions $\mathbf{R}^r_{>0} \subset \mathbf{G}_m(\mathbf{R})^r \hookrightarrow B$. Then the canonical map $(\mathbf{R}^r_{\geq 0})_{\text{val}} \to \mathbf{R}^r_{\geq 0}$ is proper and surjective because so is $B \to \mathbf{G}_a(\mathbf{R})^r$. Furthermore the group $\mathbf{R}^r_{>0}$ acts on $(\mathbf{R}^r_{\geq 0})_{\text{val}}$.

Let $\mathbf{N} := \mathbf{Z}_{\geq 0}$. There exists a canonical bijection between $(\mathbf{R}^r_{\geq 0})_{\text{val}}$ and the set of all pairs (V, h) where V is a submonoid of $\mathbf{Z}^r$ containing $\mathbf{N}^r$ such that $V \cup (-V) = \mathbf{Z}^r$ and $h : V^\times \to \mathbf{R}_{>0}$ is a homomorphism of groups. Here $V^\times := V \cap (-V)$. In fact, for a point x of $(\mathbf{R}^r_{\geq 0})_{\text{val}} \subset B$, the corresponding pair (V, h) is defined by

$$(2) \qquad \begin{cases} V := \{m \in \mathbf{Z}^r \mid \prod_j t_j^{m(j)} \text{ is regular at } x\}, \\ h(m) := \big(\prod_j t_j^{m(j)}\big)(x). \end{cases}$$

The action of $a = (a_j)_{1 \leq j \leq r} \in \mathbf{R}^r_{>0}$ sends (V, h) to (V, ah), where ah is defined by

$$(ah)(m) := \big(\prod_j a_j^{m(j)}\big) h(m) \quad (m \in V^\times).$$

The inverse map $(V, h) \mapsto x = (x_I)_{I \in \Psi}$, $x_I \in \mathrm{Bl}_I(\mathbf{G}_a(\mathbf{R})^r)$, is given as follows. Let $I = (f_1, \ldots, f_n)$, take $1 \leq k \leq n$ such that the powers of $\frac{f_l}{f_k}$ $(1 \leq l \leq n)$ belong to V, and define a point x_I by an $\mathbf{R}$-algebra homomorphism

$$x_I : \mathbf{R}\big[t_1, \ldots, t_r, \tfrac{f_1}{f_k}, \ldots, \tfrac{f_n}{f_k}\big] \to \mathbf{R}, \quad x_I\big(\prod_j t_j^{m(j)}\big) := \begin{cases} h(m) & (m \in V^\times), \\ 1 & (m \in V,\ m \notin V^\times). \end{cases}$$

The generality of $(\ \ )_{\text{val}}$ is discussed in [K].

2.14. *Identification* $D_{\text{BS,val}}(P) \simeq D \times^{A_P} (\mathbf{R}^r_{\geq 0})_{\text{val}}$. We identify $A_P = \mathbf{R}^r_{>0}$ as before by 2.11 (2). The bijection

$$(1) \qquad D_{\text{BS,val}}(P) \to D \times^{A_P} (\mathbf{R}^r_{\geq 0})_{\text{val}}, \quad (T, Z, V') \mapsto (F, (V, h)),$$

is given by

$$(2) \qquad \begin{cases} F \in Z \ \text{ (any element)}, \\ V := (\text{the inverse image of } V' \text{ under } \mathbf{Z}^r = X(S_P) \to X(T)), \\ h : V^\times \to \mathbf{R}_{>0}, \text{ the trivial homomorphism } h(m) := 1 \ (\forall m \in V^\times). \end{cases}$$

Here the identification $\mathbf{Z}^r = X(S_P)$ is given by the isomorphism 2.11 (2) and $X(S_P) \to X(T)$ is induced from the composite of the embeddings $T \hookrightarrow S_{P_{T,V}} \hookrightarrow S_P$.

The inverse map $D \times^{A_P} (\mathbf{R}^r_{\geq 0})_{\text{val}} \to D_{\text{BS,val}}(P)$, $(F, (V, h)) \mapsto (T, Z, V')$, is given by

$$\begin{cases} T := \begin{pmatrix} \text{the image of the annihilator of } V^\times \text{ in } S_P \\ \text{under the Borel-Serre lifting } S_P \hookrightarrow G_{\mathbf{R}} \text{ at } K_F \end{pmatrix}, \\ Z := \{t \circ F \mid t = (t_j) \in \mathbf{R}^r_{>0} = A_P,\ \prod_j t_j^{m(j)} = h(m)\ (\forall m \in V^\times)\}, \\ V' := (\text{the image of } V \text{ under } X(S_P) \to X(T)) \ \text{ (so that } V' \simeq V/V^\times). \end{cases}$$

2.15. *Topologies of* D_{BS}, $\mathcal{X}_{\mathrm{BS}}$ *and* $D_{\mathrm{BS,val}}$. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. We fix from now on the following identifications:

$$\begin{aligned}
&D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \mathbf{R}^r_{\geq 0} \ \text{ (see 2.12)},\\
&\mathcal{X}_{\mathrm{BS}}(P) \simeq \mathcal{X} \times^{A_P} \mathbf{R}^r_{\geq 0} \ \text{ (analoguously as 2.12)},\\
&D_{\mathrm{BS,val}}(P) \simeq D \times^{A_P} (\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \ \text{ (see 2.14)}.
\end{aligned}$$

By using these identifications, we introduce a topology on $D_{\mathrm{BS}}(P)$ (resp. $\mathcal{X}_{\mathrm{BS}}(P)$, $D_{\mathrm{BS,val}}(P)$). We introduce the strongest topology on D_{BS} (resp. $\mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}$) for which the map $D_{\mathrm{BS}}(P) \hookrightarrow D_{\mathrm{BS}}$ (resp. $\mathcal{X}_{\mathrm{BS}}(P) \hookrightarrow \mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}(P) \hookrightarrow D_{\mathrm{BS,val}}$) is continuous for every $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$. Then, it can be shown as in [BS] that all these maps $D_{\mathrm{BS}}(P) \hookrightarrow D_{\mathrm{BS}}$, $\mathcal{X}_{\mathrm{BS}}(P) \hookrightarrow \mathcal{X}_{\mathrm{BS}}$, and $D_{\mathrm{BS,val}}(P) \hookrightarrow D_{\mathrm{BS,val}}$ are open embeddings.

2.16. *Relation with Iwasawa decomposition.* The local structures of the spaces $\mathcal{X}_{\mathrm{BS}}$, D_{BS}, $D_{\mathrm{BS,val}}$ are also described by the theory of Iwasawa decomposition.

In fact, let P be a minimal parabolic subgroup of $G_{\mathbf{R}}$ and K be a maximal compact subgroup of $G_{\mathbf{R}}$ so that we have the Iwasawa decomposition associated to (P, K)

$$(1) \qquad G_{\mathbf{R}} \simeq P_u \times \mathbf{R}^r_{>0} \times K \quad \text{(homeomorphism)},$$

where r is the dimension of a maximal $\mathbf{R}$-split torus of $G_{\mathbf{R}}$. This homeomorphism is defined in the following way. Regard S_P as a maxiamal $\mathbf{R}$-split torus of $G_{\mathbf{R}}$ via the Borel-Serre lifting at K, and identify $A_P \subset S_P$ with $\mathbf{R}^r_{>0}$ under 2.11 (2). Then the homeomorphism (1) is defined by $ptk \mapsto (p, t, k)$.

Now let Q be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and let $F \in D$. Take a a minimal parabolic subgroup P of $G_{\mathbf{R}}$ contained in Q and let $(M_j)_{0 \leq j \leq m}$ be the filtration of $H_{0,\mathbf{R}}$ corresponding to P in 2.10. Then r is given by 2.11 (1). Let

$$(2) \qquad \begin{aligned}
&\mathbf{R}^r_{\geq 0}(Q) := \{(b_j) \in \mathbf{R}^r_{\geq 0} \mid b_j \neq 0 \text{ if } M_j \text{ is not stable under } Q\},\\
&(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}(Q) : \text{the inverse image of } \mathbf{R}^r_{\geq 0}(Q) \text{ in } (\mathbf{R}^r_{\geq 0})_{\mathrm{val}}.
\end{aligned}$$

Then the Iwasawa decomposition (1) associated to (P, K_F) induces

$$(3) \qquad \begin{aligned}
&\mathcal{X} \simeq P_u \times \mathbf{R}^r_{>0}, \ \ \mathrm{Ad}(pt)K_F \leftrightarrow (p, t),\\
&D \simeq P_u \times \mathbf{R}^r_{>0} \times K_F/K'_F, \ \ ptkF \leftrightarrow (p, t, k).
\end{aligned}$$

Since $a \circ \mathrm{Ad}(pt)K_F = \mathrm{Ad}(pa_{K_F}t)K_F$ and $a \circ ptkF = pa_{K_F}tkF$ for $a \in A_Q (\subset A_P)$, (3) induces

$$(4) \qquad \begin{aligned}
&\mathcal{X}_{\mathrm{BS}}(Q) \simeq P_u \times \mathbf{R}^r_{\geq 0}(Q),\\
&D_{\mathrm{BS}}(Q) \simeq P_u \times \mathbf{R}^r_{\geq 0}(Q) \times K_F/K'_F,\\
&D_{\mathrm{BS,val}}(Q) \simeq P_u \times (\mathbf{R}^r_{\geq 0})_{\mathrm{val}}(Q) \times K_F/K'_F.
\end{aligned}$$

Theorem 2.17. (i) *The spaces* $\mathcal{X}_{\rm BS}$*,* $D_{\rm BS}$*,* $D_{\rm BS,val}$ *are Hausdorff and locally compact.*
(ii) *The maps* $\alpha : D_{\rm BS,val} \to D_{\rm BS}$*,* $\beta : D_{\rm BS} \to \mathcal{X}_{\rm BS}$ *in* 2.8 *are proper and surjective.*

Proof. (i) for $\mathcal{X}_{\rm BS}$ is proved in [BS]. (i) for $D_{\rm BS}$, $D_{\rm BS,val}$ follows from this together with the two facts that $\mathcal{X}_{\rm BS}(Q)$ (Q: $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$) form an open covering of $\mathcal{X}_{\rm BS}$ and that $D_{\rm BS}(Q) = \beta^{-1}(\mathcal{X}_{\rm BS}(Q))$, $D_{\rm BS,val}(Q) = (\beta\alpha)^{-1}(\mathcal{X}_{\rm BS}(Q))$ are Hausdorff and locally compact by the descriptions in 2.16 (4).

(ii) follows from the descriptions of $\mathcal{X}_{\rm BS}(Q)$, $D_{\rm BS}(Q)$, $D_{\rm BS,val}(Q)$ in 2.16 (4). □

§3. Spaces of SL(2)-orbits

3.1. *Summary.* Let n be a non-negative integer. We fix an embedding

$$\mathbf{G}_m^n \hookrightarrow \mathrm{SL}(2)^n, \quad (t_1, \dots, t_n) \mapsto \left(\begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix}, \dots, \begin{pmatrix} t_n^{-1} & 0 \\ 0 & t_n \end{pmatrix} \right),$$

throughout this paper. Let $\rho : \mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}$ be an injective homomorphism defined over $\mathbf{R}$. We denote by Y_j, $N_j = N_j^-$, N_j^+ the image under the Lie algebra homomorphism ρ_* of

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the j-th factor of $\mathfrak{sl}(2, \mathbf{C})^{\oplus n}$, respectively.

Put

$$\mathbf{i} := (i, \dots, i) \in \mathfrak{h}^n \subset \mathbf{P}^1(\mathbf{C})^n.$$

Here $\mathfrak{h}$ denotes the Siegel upper-half plane. Let (ρ, φ) be a pair of an injective homomorphism $\rho : \mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}$ defined over $\mathbf{R}$ and a map $\varphi : \mathbf{P}^1(\mathbf{C})^n \to \check{D}$. Such a pair (ρ, φ) is called an $\mathrm{SL}(2)^n$*-orbit* if it satisfies the following two conditions:

(i) $\varphi(gz) = \rho(g)\varphi(z)$ for all $g \in \mathrm{SL}(2, \mathbf{C})^n$ and all $z \in \mathbf{P}^1(\mathbf{C})^n$.

(ii) $\varphi(\mathbf{i}) \in D$, and the associated Lie algebra homomorphism $\rho_* : \mathfrak{sl}(2, \mathbf{C})^{\oplus n} \to \mathfrak{g}_{\mathbf{C}}$ is a homomorphism of type $(0, 0)$ with respect to the Hodge structures induced by the points $\mathbf{i} \in \mathfrak{h}^n$ and $\varphi(\mathbf{i}) \in D$, respectively.

In this section, we introduce spaces of SL(2)-orbits $D_{\mathrm{SL}(2)}$ and the projective limit $D_{\mathrm{SL}(2),\mathrm{val}}$ of the blowing-ups of $D_{\mathrm{SL}(2)}$. These spaces, together with the spaces in the previous section, will form the following diagram:

$$\begin{array}{ccccc} D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\rm BS,val} & & \\ \downarrow & & \downarrow & & \\ D_{\mathrm{SL}(2)} & & D_{\rm BS} & \to & \mathcal{X}_{\rm BS}. \end{array} \tag{1}$$

In general, there is no direct relation between $D_{\mathrm{SL}(2)}$ and $D_{\rm BS}$ (see, §6), and we need to introduce $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\rm BS,val}$.

3.2. *Weight filtrations.* For a nilpotent element $N \in \mathfrak{g}_{\mathbf{R}}$, the *weight filtration associated to* N (= N*-filtration*) is the increasing filtration $W = W(N)$ of $H_{0,\mathbf{R}}$ charcterized by the following conditions (i), (ii) ([D]).

(i) $NW_k \subset W_{k-2}$ for all $k \in \mathbf{Z}$;

(ii) $N^k : \mathrm{gr}_k^W \xrightarrow{\sim} \mathrm{gr}_{-k}^W$ for all $k \in \mathbf{Z}_{\geq 0}$.

3.3. *Cones.* We fix terminology concerning cones. Let V be an $\mathbf{R}$-vector space. A *cone* in V is a subset σ of V which is closed under addition and under multiplication by elements of $\mathbf{R}_{\geq 0}$ and satisfies $\sigma \cap (-\sigma) = 0$. A subset σ in $\mathfrak{g}_{\mathbf{R}}$ is a *nilpotent cone in* $\mathfrak{g}_{\mathbf{R}}$ if it is a finitely generated cone in $\mathfrak{g}_{\mathbf{R}}$ consisiting of mutually commutative nilpotent elements. Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$. For $A = \mathbf{R}, \mathbf{C}$, we denote by σ_A the A-linear span of σ in $\mathfrak{g}_A$.

Definition 3.4. *Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$. A subset Z of $\check{D}$ is a* σ-nilpotent orbit (*resp.* σ-nilpotent i-orbit) *if it satisfies the following conditions* (i)–(iii) *for some $F \in Z$.*

(i) $Z = \exp(\sigma_{\mathbf{C}})F$ (*resp.* $Z = \exp(i\sigma_{\mathbf{R}})F$).

(ii) $NF^p \subset F^{p-1}$ $(\forall p, \forall N \in \sigma)$.

(iii) $\exp(\sum_{1\leq j\leq r} iy_jN_j)F \in D$ $(\forall y_j \gg 0)$. *Here* $\sigma = \sum_{1\leq j\leq r} \mathbf{R}_{\geq 0}N_j$.

Note that, in 3.4, if the conditions (i)–(iii) are satisfied by one $F \in Z$ then they are satisfied by any $F \in Z$. The condition (ii) is called the *Griffiths transversality* and the condition (iii) is called the *positivity.*

3.5. *Weight filtrations associated to a nilotent orbit.* We recall here a result of Cattani and Kaplan.

Theorem-Definition ([CK2])**.** *Let (σ, Z) be a nilpotent i-orbit. Then, for any elements N, N' of the interior of σ, the filtrations $W(N)$ and $W(N')$ of $H_{0,\mathbf{R}}$ coincide. This common filtration is denoted by $W(\sigma)$.*

Note that, as in [CKS, §4], an $\mathrm{SL}(2)^n$-orbit (ρ, φ) defines an ordered family of nilpotent i-orbits $(\sigma_j, Z_j)_{1\leq j\leq n}$ by

$$\sigma_j := \mathbf{R}_{\geq 0}N_1 + \cdots + \mathbf{R}_{\geq 0}N_j, \quad Z_j := \exp(i\sigma_{j,\mathbf{R}})\varphi(\underbrace{0, \ldots, 0}_{j}, i, \ldots, i). \tag{1}$$

We have $W(N_1 + \cdots + N_j) = W(\sigma_j)$ for each $1 \leq j \leq n$.

Definition 3.6. *We define $D_{\mathrm{SL}(2),0} := D$ and, for a positive integer n, we define*

$$D_{\mathrm{SL}(2),n} := \left\{ (\rho, \varphi) \;\middle|\; \begin{matrix} (\rho, \varphi) \text{ is an } \mathrm{SL}(2)^n\text{-orbit,} \\ W(\sigma_j) \text{ is } \mathbf{Q}\text{-rational } (1 \leq j \leq n) \end{matrix} \right\} \Big/ \sim,$$

where $(\rho, \varphi) \sim (\rho', \varphi')$ if and only if there exists $t \in \mathbf{R}^n_{>0}$ such that $\rho' = \mathrm{Int}(\rho(t)) \circ \rho$ (where $\circ$ is the composite of maps) and $\varphi' = \rho(t) \cdot \varphi$. We define

$$D_{\mathrm{SL}(2)} := \bigcup_{n\geq 0} D_{\mathrm{SL}(2),n}, \quad D_{\mathrm{SL}(2),\leq r} := \bigcup_{0\leq n\leq r} D_{\mathrm{SL}(2),n}.$$

We denote by $[\rho, \varphi]$ the point of $D_{\mathrm{SL}(2)}$ represented by (ρ, φ).

Definition 3.7. *For a non-negative integer n, we define*

$$D_{\mathrm{SL}(2),\mathrm{val},n} := \left\{ ([\rho, \varphi], Z, V) \;\middle|\; \begin{matrix} [\rho, \varphi] \in D_{\mathrm{SL}(2),n},\ Z \subset \rho(\mathbf{R}^n_{>0})\varphi(\mathbf{i}), \\ V \subset \text{character group } X(\mathbf{G}^n_m), \\ \text{which satisfy the following (i)–(iii)} \end{matrix} \right\}.$$

(i) *If* $\chi, \chi' \in V$ *then* $\chi\chi' \in V$. $V \cup V^{-1} = X(\mathbf{G}_m^n)$.

(ii) *Let* $X(\mathbf{G}_m^n)_+$ *be the submonoid of* $X(\mathbf{G}_m^n)$ *generated by*

$$(t_1, \dots, t_n) \mapsto \tfrac{t_j}{t_{j+1}} \ \ (1 \le j \le n), \ \ \textit{where}\ t_{n+1} := 1,$$

and let $V^\times$ *be the group consisting of all invertible elements of* V. *Then,* $X(\mathbf{G}_m^n)_+ \subset V$ *and* $X(\mathbf{G}_m^n)_+ \cap V^\times = \{1\}$.

(iii) *Let*

$$T := \{t \in \mathbf{G}_m(\mathbf{R})^n \mid \chi(t) = 1 \ (\forall \chi \in V^\times)\}.$$

Then Z *is a* $\rho(T_{>0})$*-orbit in* D. *Here we denote by* $T_{>0}$ *the connected component of* T *containing the unity.*

We define

$$D_{\mathrm{SL}(2),\mathrm{val}} := \bigcup_{n \ge 0} D_{\mathrm{SL}(2),\mathrm{val},n}.$$

We have the canonical surjection $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$.

After preliminaries in Lemmas 3.8–3.11, we will relate $D_{\mathrm{SL}(2),\mathrm{val}}$ with $D_{\mathrm{BS},\mathrm{val}}$ in 3.12 below.

For $([\rho, \varphi], Z, V) \in D_{\mathrm{SL}(2),\mathrm{val},n}$, let $W = (W^{(j)})_{1 \le j \le n}$ be the family of weight filtrations $W^{(j)} := W(\sigma_j)$ of $H_{0,\mathbf{R}}$ associated to $[\rho, \varphi]$ (cf. 3.5), and let $G_{W,\mathbf{R}}$ be the subgroup of $G_{\mathbf{R}}$ consisting of all elements which preserve all the filtrations $W^{(1)}, \dots, W^{(n)}$. Let $V' := \rho_*(V/V^\times)$, which is regarded as a subset of the character group of $\rho(T)$. Let $P_{\rho(T),V'}$ be the parabolic subgroup in 2.7.

Lemma 3.8. *In the above notation,* $G_{W,\mathbf{R}} \subset P_{\rho(T),V'}$, *and* $P_{\rho(T),V'}$ *is* $\mathbf{Q}$*-rational.*

Proof. Put $W_\chi := \bigoplus_{\chi' \in \chi \cdot (X(\mathbf{G}_m^n)_+)^{-1}} H(\chi')$ and $M_\chi := \bigoplus_{\chi' \in \chi V^{-1}} H(\chi')$ (2.7). Then, by the condition $X(\mathbf{G}_m^n)_+ \subset V$ in 3.7 (ii), we have $M_\chi = \sum_{\chi' \in \chi V'^{-1}} W_{\chi'}$. Let l_j $(1 \le j \le n)$ be integers defined by $\chi(t_1, \dots, t_n) = \prod_{1 \le j \le n} \big(\frac{t_j}{t_{j+1}}\big)^{l_j}$ (t_{n+1} means 1). We have $W_\chi = \bigcap_{1 \le j \le n} W^{(j)}_{l_j}$. This implies that the W_χ are $\mathbf{Q}$-rational and preserved by $G_{W,\mathbf{R}}$, and hence that so are the M_χ. This proves the lemma. □

Lemma 3.9. *Let* (ρ, φ) *be an* $\mathrm{SL}(2)^n$*-orbit and put* $r = \varphi(\mathbf{i})$. *Then*

$$\theta_{K_r}(\rho(t)) = \rho(t)^{-1} \quad (\forall t \in \mathbf{G}_m(\mathbf{R})^n).$$

Proof. Let Y_j $(1 \le j \le n)$ be as in 3.1. It is enough to show $\theta_{K_r}(Y_j) = -Y_j$ for all j. We prove this. Here θ_{K_r} is regarded as the involution of $\mathfrak{g}_{\mathbf{C}}$ induced by the Cartan involution θ_{K_r} of $G_{\mathbf{R}}$ at K_r by abuse of the notation. Let

$$\mathfrak{g}_{\mathbf{C}} = \bigoplus_s \mathfrak{g}_r^{s,-s}, \quad \mathfrak{g}_r^{s,-s} := \{X \in \mathfrak{g}_{\mathbf{C}} \mid X H_r^{p,w-p} \subset H_r^{p+s,w-p-s} \ (\forall p)\}$$

be the Hodge structure on $\mathfrak{g}_{\mathbf{C}}$ induced by r. Then, by 2.2 (2) and the definition of the Weil operator C_r in 1.2, θ_{K_r} is given by

(1) $$\theta_{K_r}(X) = \textstyle\sum_s (-1)^s X^{s,-s} \quad \text{for} \quad X = \sum_s X^{s,-s} \in \mathfrak{g}_{\mathbf{C}} = \bigoplus_s \mathfrak{g}_r^{s,-s}.$$

On the other hand, the Hodge decomposition of $H_{\mathfrak{h},\mathbf{C}} = \mathbf{C}^2 = \mathbf{C}e_1 + \mathbf{C}e_2$ corresponding to $i \in \mathfrak{h}$ is

$$H_{\mathfrak{h},\mathbf{C}} = H_{\mathfrak{h}}^{1,0} \oplus H_{\mathfrak{h}}^{0,1} = \mathbf{C}(ie_1 + e_2) \oplus \mathbf{C}(-ie_1 + e_2) \ \text{(cf. 6.2 below)},$$

and this induces the Hodge decomposition

$$\mathfrak{sl}(2,\mathbf{C}) = \mathfrak{sl}(2)^{1,-1} \oplus \mathfrak{sl}(2)^{0,0} \oplus \mathfrak{sl}(2)^{-1,1} = \mathbf{C}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \oplus \mathbf{C}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \oplus \mathbf{C}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$$

of $\mathfrak{sl}(2,\mathbf{C})$. Since

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{i}{2}\left(\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} - \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right),$$

$Y_j \in \mathfrak{g}_r^{1,-1} \oplus \mathfrak{g}_r^{-1,1}$ by 3.1 (ii) for all j. Hence, by (1), $\theta_{K_r}(Y_j) = -Y_j$ for all j. □

Lemma 3.10. *Let* (ρ, φ) *and* r *be as in* 3.9. *Define*

$$(1) \quad \begin{aligned} &\tilde{\rho} := \rho \circ \mu : \mathbf{G}_m(\mathbf{R})^n \to G_{\mathbf{R}} \quad \text{the composite map of } \rho \text{ with} \\ &\mu : \mathbf{G}_m(\mathbf{R})^n \to \mathbf{G}_m(\mathbf{R})^n, \ \ (t_1, \dots, t_n) \mapsto \big(t_1 \cdots t_n, t_2 \cdots t_n, \dots, t_{n-1}t_n, t_n\big). \end{aligned}$$

Then we have the following (i) *and* (ii).

(i) *For* $\chi \in X(\mathbf{G}_m^n)$, $\chi \in X(\mathbf{G}_m^n)_+$ *if and only if* $\chi \circ \mu \in \mathbf{N}^n \subset \mathbf{Z}^n = X(\mathbf{G}_m^n)$.

(ii) *For* $1 \le j \le n$, *let* $W^{(j)} = W(\sigma_j)$ *be as just before* 3.8 *and let* P_j *be the* $\mathbf{Q}$-*parabolic subgroup of* $G_{\mathbf{R}}$ *preserving* $W^{(j)}$. *Then the* j-*th factor of* $\tilde{\rho}$,

$$t_j \mapsto \tilde{\rho}(1, \dots, 1, t_j, 1, \dots, 1),$$

coincides with the Borel-Serre lifting at K_r *of the* j-*th weight map*

$$\mathbf{G}_m(\mathbf{R}) \to P_j/P_{j,u}, \ \ t_j \mapsto (t_j^k \ \text{on} \ \mathrm{gr}_k^{W^{(j)}})_k.$$

Proof. (i) follows from the observation that the inverse of μ in (1) is given by

$$(t_1, \dots, t_n) \mapsto \big(\tfrac{t_1}{t_2}, \tfrac{t_2}{t_3}, \dots, \tfrac{t_{n-1}}{t_n}, t_n\big).$$

(ii) follows from 3.9 and the observation

$$\tilde{\rho}(1, \dots, 1, t_j, 1, \dots, 1) = \rho(\underbrace{t_j, \dots, t_j}_{j}, 1, \dots, 1) = \exp(\log(t_j)(Y_1 + \cdots + Y_j)). \quad \square$$

Lemma 3.11. *Let (ρ, φ), $r = \varphi(\mathbf{i})$ and W be as in* 3.9 *and just before* 3.8*. Then, an* $\mathrm{SL}(2)^n$*-orbit (ρ, φ) is determined by (W, r).*

Proof. By 3.10 (ii), $\tilde{\rho}$ is determined by (W, r), and Y_j $(1 \le j \le n)$ are determined by $\tilde{\rho}$. Let $T_D(r)$ (resp. $T_{\mathfrak{h}}(i)$) be the tangent space of D at r (resp. $\mathfrak{h}$ at i). Then we have the commutative diagram

(1)
$$\begin{array}{ccc} \mathfrak{g}_{\mathbf{C}} & \xrightarrow{\alpha_r} & T_D(r) \\ {\scriptstyle \rho_*}\uparrow & & {\scriptstyle d\varphi}\uparrow \\ \mathfrak{sl}(2,\mathbf{C}) & \xrightarrow{\alpha_i} & T_{\mathfrak{h}}(i), \end{array}$$

where α_r (resp. α_i) is the differential of $G_{\mathbf{C}} \to \check{D}, g \mapsto gr$, (resp. $\mathrm{SL}(2, \mathbf{C}) \to \mathbf{P}^1(\mathbf{C}), g \mapsto gi$,) at 1. Since $-2i\alpha_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \alpha_i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $-2i\alpha_r(N_j) = \alpha_r(Y_j)$. Since the restriction of α_r on $\mathrm{Lie}(P_{j,u})$ is injective, N_j $(1 \le j \le n)$ are determined. The N_j^+ are determined by the Y_j and the N_j. □

Theorem 3.12. *There is an injective map*

(1) $$D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}, \quad ([\rho, \varphi], Z, V) \mapsto (\rho(T), Z, V').$$

Here T is the subtorus of $\mathbf{G}_m(\mathbf{R})^n$ in 3.7 (iii) *and $V' := \rho_*(V/V^\times)$, which is regarded as a subset of the character group of $\rho(T)$.*

Proof. Let $([\rho, \varphi], Z, V) \in D_{\mathrm{SL}(2),\mathrm{val}}$ and let (T', Z, V') be its image under (1).

We check the conditions (i)–(iii) in 2.7 for (T', Z, V'). 2.7 (i) is evident, 2.7 (ii) is verified by 3.8, and 2.7 (iii) follows from the note in 2.7, 3.9 and 2.3 (1). Hence (1) is well-defined.

We prove that (1) is injective. It is sufficient to show that $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$ is determined by (T', Z, V'). Let $r \in Z$ and take a representative (ρ, φ) with $\varphi(\mathbf{i}) = r$. To prove that (ρ, φ) is determined by (T', Z, V') and r, it is sufficient to show, by 3.11, that the family of weight filtrations $(W(N_1 + \cdots + N_j))_{1 \le j \le n}$ associated to (ρ, φ) is determined by (T', Z, V'). Hence it is sufficient to prove that the family $(N_j)_{1 \le j \le n}$ is a unique family of elements of $\mathrm{Lie}(P_{T',V',u})$ having the following properties (i)–(iv). Here $P_{T',V',u}$ is the unipotent radical of $P_{T',V'}$.

(i) For $1 \le j \le n$, N_j is a non-zero eigen vector for the adjoint action of T'.

(ii) For $1 \le j \le n$, let $\chi_j : T' \to \mathbf{G}_m(\mathbf{R})$ be a character defined by $\mathrm{Ad}(t)N_j = \chi_j(t)N_j$ for $t \in T'$. Then the χ_j are non-trivial and different from each other.

(iii) In the notation (ii) above, $\chi_j\chi_{j+1}^{-1} \in V'$ $(1 \le j \le n-1)$.

(iv) Let $\alpha_r : \mathfrak{g}_{\mathbf{C}} \to T_D(r)$ be the canonical $\mathbf{C}$-linear map 3.11 (1). For $1 \le j \le n$, let $\mathrm{Lie}(\chi_j) : \mathrm{Lie}(T') \to \mathbf{R}$ be the map induced by χ_j. Then

$$-i\alpha_r(A) = \textstyle\sum_{1 \le j \le n} \mathrm{Lie}(\chi_j)(A)\alpha_r(N_j) \quad \text{for any } A \in \mathrm{Lie}(T').$$

We first show that the family $(N_j)_{1\le j\le n}$ associated to (ρ,φ) satisfies these (i)–(iv). For $t=(t_j)_{1\le j\le n}\in T'\subset \mathbf{G}_m(\mathbf{R})^n$, we have

$$\mathrm{Ad}(t)N_j = t_j^{-2}N_j \tag{2}$$

and hence (i) and (iii) are satisfied. Denote by χ_{n+1} the trivial character of T'. Suppose $\chi_j=\chi_k$ for some j, k with $1\le j\le k\le n+1$. Then the character $t\mapsto t_j/t_k$ is trivial on T' and hence belongs to $V^\times$. Since it also belongs to $X(\mathbf{G}_m)_+$ and since $X(\mathbf{G}_m)_+\cap V^\times=\{1\}$, we have $j=k$. Thus we have proved (ii). Note that

$$\alpha_r(Y_j) = -2i\alpha_r(N_j) \quad (1\le j\le n). \tag{3}$$

By (2) and (3), we have, for an element $A=\sum_{1\le j\le n} b_jY_j$ of $\mathrm{Lie}(T')$,

$$-i\alpha_r(A) = -2\textstyle\sum_{1\le j\le n} b_j\alpha_r(N_j) = \sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)\alpha_r(N_j).$$

Next we prove that a family $(N_j)_{1\le j\le n}$ of elements of $\mathrm{Lie}(P_{T',V',u})$ satisfying (i)–(iv) is unique. Since the restriction of α_r on $\mathrm{Lie}(P_{T',V',u})$ is injective, we have:

(4) $\sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)N_j$ for $A\in\mathrm{Lie}(T')$ is the unique element of $\mathrm{Lie}(P_{T',V',u})$ whose image under α_r coincides with $-i\alpha_r(A)$.

As sets, we have

$$\{\chi_j\}_{1\le j\le n} = \left\{\chi\in X(T')\ \middle|\ \begin{array}{l}\text{under the action of } \mathrm{Ad}(T') \text{ on } \mathfrak{g}_{\mathbf{C}}, \text{ the } \chi\text{-component}\\ \text{of } \sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)N_j\ (A\in\mathrm{Lie}(T')) \text{ is not zero}\end{array}\right\} \tag{5}$$

Since $(V')^\times=1$, (iii) determines the order and hence the family $(\chi_j)_{1\le j\le n}$. For each $1\le j\le n$, $\mathrm{Lie}(\chi_j)(A)N_j$ $(A\in\mathrm{Lie}(T'))$ is determined as the χ_j-component of $\sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)N_j$ $(A\in\mathrm{Lie}(T'))$ under the action of $\mathrm{Ad}(T')$. Since $\mathrm{Lie}(\chi_j)\neq 0$, N_j is determined. $\square$

3.13. *Topologies on* $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$. A family $(W^{(j)})_{1\le j\le n}$ of increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$ is called a *compatible family* if there exists a direct sum decomposition $H_{0,\mathbf{R}}=\bigoplus_{m\in\mathbf{Z}^n}H(m)$ such that $W^{(j)}_k=\bigoplus_{m\in\mathbf{Z}^n,\, m_j\le k}H(m)$ for any j and k. Note that, for $[\rho,\varphi]\in D_{SL(2),n}$, the family of weight filtrations $(W(\sigma_j))_{1\le j\le n}$ associated to $[\rho,\varphi]$ in 3.5 is a compatible family.

Let $W=(W^{(j)})_{1\le j\le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. We define the subset $D_{\mathrm{SL}(2)}(W)$ of $D_{\mathrm{SL}(2)}$ by

$$D_{\mathrm{SL}(2)}(W) := \bigcup_{0\le m\le n}\left\{x\in D_{\mathrm{SL}(2),m}\ \middle|\ \begin{array}{l}\exists s_j\in\mathbf{Z}\ (1\le j\le m) \text{ such that}\\ 1\le s_1<\cdots<s_m\le n \text{ and}\\ W(\sigma_j)=W^{(s_j)}\ (\forall j)\end{array}\right\}.$$

Here $(W(\sigma_j))_{1\le j\le m}$ is a family of weight filtrations associated to $x\in D_{\mathrm{SL}(2),m}$.

We define the subset $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ of $D_{\mathrm{SL}(2),\mathrm{val}}$ by the pull-back of $D_{\mathrm{SL}(2)}(W)$.

Definition 3.14. *We define the* topology on $D_{\mathrm{SL}(2),\mathrm{val}}$ *as the weakest one in which the following two families of subsets are open:*

(i) *The pull-backs on* $D_{\mathrm{SL}(2),\mathrm{val}}$ *of open subsets of* $D_{\mathrm{BS},\mathrm{val}}$*.*

(ii) *The subset* $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ *for any* n *and any compatible family of* $\mathbf{Q}$*-rational increasing filtrations* $W = (W^{(j)})_{1\le j\le n}$*.*

We induce the quotient topology on $D_{\mathrm{SL}(2)}$ *of the above one under the projection* $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$*.*

This topology of $D_{\mathrm{SL}(2)}$ has the following property (see 4.18 below). For an $\mathrm{SL}(2)^n$-orbit (ρ,φ), $[\rho,\varphi] \in D_{\mathrm{SL}(2)}$ is the limit of

$$\varphi(iy_1,\dots,iy_n) \in D, \ \text{ as } y_j > 0 \text{ and } \frac{y_j}{y_{j+1}} \to \infty \text{ for } 1 \le \forall j \le n \ (y_{n+1} \text{ denotes } 1).$$

Note that the space $D_{\mathrm{SL}(2),\mathrm{val}}$ is Hausdorff by 3.12 and 3.14.

Theorem 3.15. (i) *The canonical map* $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ *is proper and surjective.*
(ii) *The space* $D_{\mathrm{SL}(2)}$ *is Hausdorff.*

The proof of this theorem will be given in §4.

3.16 *Remark* (*Relation with period maps*). Let $\overline{X}$ be a connected complex manifold, let Y be a reduced divisor with simple normal crossings on $\overline{X}$, and let $X := \overline{X} - Y$. Let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X of weight w and of Hodge type $(h^{p,q})$. Then, as in 1.7, we have the associated period map

$$\varphi : X \to \Gamma\backslash D. \tag{1}$$

We will show, in the forthcoming paper [KU2], that φ extends continuously to $\varphi_{\mathrm{SL}(2)}$ and also to $\varphi^{\flat}_{\mathrm{SL}(2)}$ in the following diagram.

$$\begin{array}{ccc} \big(\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})\big)^{\log} & \xrightarrow{\varphi_{\mathrm{SL}(2)}} & \Gamma\backslash D_{\mathrm{SL}(2)} \\ \downarrow & & \downarrow \\ \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X}) & \xrightarrow{\varphi^{\flat}_{\mathrm{SL}(2)}} & \Gamma\backslash D^{\flat}_{\mathrm{SL}(2)}. \end{array} \tag{2}$$

Here, analogously as in 2.13, Ψ denotes the set of all non-zero $\mathcal{O}_X$-ideals which are locally of forms $I = (f_1,\dots,f_n)$ generated by some monomials $f_1,\dots,f_n$ in local equations of the irreducible components of Y, $\mathrm{Bl}_I(X)$ means the blowing-up of X along I, and the projective limit is taken with respect to the ordering of the set Ψ for which $J \ge I$ means $J = II'$ for some $I' \in \Psi$. $\big(\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})\big)^{\log}$ is a topological space defined by the method of [KN]. $D^{\flat}_{\mathrm{SL}(2)}$ is a space of Satake(-Baily-Borel)-Cattani-Kaplan type ([Sa], [BB], [CK]) which is defined as a quotient space of $D_{\mathrm{SL}(2)}$ under the following equivalence relation $\sim$. For $x \in D_{\mathrm{SL}(2),m}$, $y \in D_{\mathrm{SL}(2),n}$,

$$x \sim y \Leftrightarrow \begin{cases} m = n, \text{ and the associated families of weight} \\ \text{filtrations coincide, say } W, \text{ and } y \in G_{W,\mathbf{R},u}x. \end{cases}$$

Here $G_{W,\mathbf{R}}$ is the subgroup of $G_{\mathbf{R}}$ preserving W (see before 3.8) and $G_{W,\mathbf{R},u}$ is its unipotent radical. Since the centers of the blowing-ups are contained in Y, we have open immersions $X \hookrightarrow \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})$, $X \hookrightarrow \big(\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})\big)^{\log}$. Note that, when $\overline{X}$ is a unit disc and Y is the origin, we have $\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X}) = \overline{X}$.

§4. Proof of Theorem 3.15

4.1. *Summary.* This section is devoted to prove Theorem 3.15. To do so, we will introduce a new topology $\mathcal{T}$ on the set $D_{\mathrm{SL}(2)}(W)$ in 3.13 in terms of filters on D associated to points of $D_{\mathrm{SL}(2)}(W)$ by using Cartan decompositons (4.6, 4.8 below). Denote this new topological space by $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$, i.e., the underlying set coincides with the one of $D_{\mathrm{SL}(2)}(W)$ but whose topology is $\mathcal{T}$. We will show that the topological space $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ in 3.14 is homeomorphic to $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$ (4.14 below). From this, we have the homeomorphism $D_{\mathrm{SL}(2)}(W) \overset{\sim}{\to} D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ (4.15 below) and the proof of Theorem 3.15 (4.17 below).

4.2. First, we prove two Lemmas 4.3, 4.4 below. Let X_j $(1 \leq j \leq d)$ be indeterminates. A *convergent Lie power series in the X_j* is a power series with respect to the bracket product $[\ ,\ ]$ of the X_j with coefficients in $\mathbf{C}$ which converges if the X_j are elements of a finite-dimensional Lie aligebra over $\mathbf{C}$ and sufficiently near 0. The *order* of a convergent Lie power series $f(X_1, \dots, X_d)$ is the minimum of the degrees of the monomials in $f(X_1, \dots, X_d)$ whose coefficients are not zero.

Lemma 4.3. *Let X, Y be two indeterminates. Then there exist convergent Lie power serieses $f_-(X,Y)$ and $f_+(X,Y)$ which satisfy the following two conditions.*

(i) $\exp(X+Y) = \exp(f_-(X,Y))\exp(f_+(X,Y))$.

(ii) *Each monomial in $f_-(X,Y)$ (resp. $f_+(X,Y)$) is of odd (resp. even) degree in X.*

Proof. Let A be the convergent Lie power series of order ≥ 2 defined by

$$\exp(X+Y) = \exp(X+A)\exp(Y).$$

Devide $A = A_- + A_+$ so that each monomial in A_- (resp. A_+) is of odd (resp. even) degree in X. Let A' be the convergent Lie power series of order ≥ 3 defined by

$$\exp(X+Y) = \exp(X + A_- + A')\exp(A_+)\exp(Y).$$

Then $\exp(A_+)\exp(Y) = \exp(B_+)$ for some convergent Lie power series B_+ which is the sum of monomials of even degree in X. Devide $A' = A'_- + A'_+$ so that each monomial in A'_- (resp. A'_+) is of odd (resp. even) degree in X. Continuing this process, we obtain Lie power series $f_-(X,Y)$, $f_+(X,Y)$ which can be checked to be convergent Lie power series. □

Lemma 4.4. *Let $y \in D_{\mathrm{SL}(2),\mathrm{val},n}$ and let $[\rho,\varphi] \in D_{\mathrm{SL}(2),n}$ be the image of y. Let $P = P_{\rho(T),V'}$ be the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ associated to y in 3.7 and 2.7. Let $K = K_r$ be the maximal compact subgroup of $G_{\mathbf{R}}$ associated to the point $r := \varphi(\mathbf{i})$, and* *let $\theta = \theta_K$ be the associated Cartan involution. Let $\mathfrak{g}_{\mathbf{R}} = \bigoplus_l \mathfrak{g}(l)$ be the decomposition into the eigen spaces under the action of $\mathbf{G}_m(\mathbf{R})^n$ through $\mathrm{Ad}(\tilde\rho(\ ))$ (for $\tilde\rho$, see 3.10(1)). Let X_l, Y_l be indeterminates, where the index l runs over all $l \in \mathbf{Z}^n$ with $\mathfrak{g}(l) \neq 0$.*

Then, there exist convergent Lie power series f_P and f_K in the X_l and the Y_l which satisfy the following conditions. Let U be a sufficiently small neighborhood of 0 in $\mathfrak{g}_{\mathbf{R}}$. For any $x \in U$, writing $x = \sum_l x_l$, $x_l \in \mathfrak{g}(l)$, we have

(i) $f_P\big((x_l)_l, (\theta(x_l))_l\big) \in \mathrm{Lie}\, P$, $f_K\big((x_l)_l, (\theta(x_l))_l\big) \in \mathrm{Lie}\, K$, *and*

(ii) $\exp(x) = \exp\big(f_P\big((x_l)_l, (\theta(x_l))_l\big)\big) \exp\big(f_K\big((x_l)_l, (\theta(x_l))_l\big)\big)$.

Proof. Choose a submonoid V of the character group $\mathbf{Z}^n$ of $\mathbf{G}_m^n$ which has the following property.

$$V \supset \mathbf{Z}^n_{\geq 0},\ \ V \cup (-V) = \mathbf{Z}^n,\ \ V \cap (-V) = 0,\ \ \mathrm{Lie}(P) = \textstyle\bigoplus_{-l \in V} \mathfrak{g}(l).$$

We reform

$$\begin{aligned}\exp(x) &= \exp\big(\textstyle\sum_l x_l\big) = \exp\big(\sum_{-l\in V} x_l + \sum_{-l\notin V} x_l\big)\\ &= \exp\big(\textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V} \theta(x_l) + \sum_{-l\notin V}(x_l + \theta(x_l))\big)\\ &= \exp\big(\textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V} \theta(x_l) + A\big) \exp\big(\sum_{-l\notin V}(x_l + \theta(x_l))\big),\end{aligned}$$

where A is the convergent Lie power series of order ≥ 2 determined by the above. Then

$$\textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V} \theta(x_l) \in \mathrm{Lie}(P), \quad \sum_{-l\notin V}(x_l + \theta(x_l)) \in \mathrm{Lie}(K).$$

Reform this A in the same way as above, and repeat the same procedure. We finally obtain the desired convergent Lie power series f_P and f_K. □

4.5. *The filter on D associated to a point of $D_{\mathrm{SL}(2)}$.* We introduce here the filter $\mathcal{F}_x$ on D associated to a point $x \in D_{\mathrm{SL}(2)}$. We shall see later that this $\mathcal{F}_x$ coincides with the filter on D whose basis is given by $\{U \cap D \mid U$ is a neighborhood of x in $D_{\mathrm{SL}(2)}\}$ (4.15 below).

Definition 4.6. *Let $x \in D_{\mathrm{SL}(2),n}$ and (ρ, φ) be a representative of x. Put $r := \varphi(\mathbf{i})$. For*

(1)
$$\begin{cases} U \ \ : \text{a neighborhood of } r \text{ in } K_r r,\\ U' \ : \text{a neighborhood of } 0 \text{ in } \mathbf{R}^n_{\geq 0},\\ U'' : \text{a neighborhood of } 1 \text{ in } G_{\mathbf{R}},\end{cases}$$

we denote

(2) $A(U, U', U'') := \{g\tilde\rho(t)r' \mid r' \in U,\ t \in \mathbf{R}^n_{>0} \cap U',\ g \in U'',\ \theta_{\tilde\rho(t)r'}(g) = g^{-1}\}$,

where $\tilde\rho$ is as in 3.10 (1) and $\theta_{\tilde\rho(t)r'}$ is the Cartan involution of $G_{\mathbf{R}}$ associated to the maximal compact subgroup $K_{\tilde\rho(t)r'}$. We define $\mathcal{F}_x$ associated to x as the filter on D whose basis is given by the $A(U, U', U'')$ where U, U' and U'' run over all such neighborhoods as in (1).

Note that, since $\theta_{\tilde\rho(t)r'} = \theta_{\tilde\rho(t)r} = \mathrm{Int}(\tilde\rho(t))\theta_r \,\mathrm{Int}(\tilde\rho(t))^{-1}$, we have

(3) $A(U, U', U'') = \{\tilde\rho(t)gr' \mid r' \in U,\ t \in \mathbf{R}^n_{>0} \cap U',\ g \in \mathrm{Int}(\tilde\rho(t))^{-1}(U''),\ \theta_r(g) = g^{-1}\}$.

Lemma 4.7. *A basis of the filter $\mathcal{F}_x$ is also given by the following family of sets:*

$$B(U,U',U'') := \{\tilde\rho(t)gr' \mid r' \in U,\ t \in \mathbf{R}^n_{>0} \cap U',\ g \in U'',\ \operatorname{Int}(\tilde\rho(t))^{\pm 1}(g) \in U''\}, \tag{1}$$

where U, U' and U'' run over all such neighborhoods as 4.6 (1).

Proof. We prove that, for given U, U' and U'', there exist sufficiently small V, V' and V'', such as in 4.6 (1), which have the following two properties:

$$A(V,V',V'') \subset B(U,U',U''). \tag{2}$$

$$B(V,V',V'') \subset A(U,U',U''). \tag{3}$$

We prove (2). By the remark just after Definition 4.6, any element of $A(V,V',V'')$ can be written as

$$\tilde\rho(t)gr' \quad \text{such that} \quad r' \in V,\ t \in \mathbf{R}^n_{>0} \cap V',\ \operatorname{Int}(\tilde\rho(t))(g) \in V'',\ \theta_r(g) = g^{-1}.$$

Since V'' is sufficiently small, there exists $a \in \mathfrak{g}_{\mathbf{R}}$ near 0 such that

$$\operatorname{Int}(\tilde\rho(t))(g) = \exp(a),\ \ \big(\operatorname{Ad}(\tilde\rho(t))\theta_r \operatorname{Ad}(\tilde\rho(t))^{-1}\big)(a) = -a.$$

Decompose

$$a = \textstyle\sum_{l \in \mathbf{Z}^n} a_l \in \mathfrak{g}_{\mathbf{R}} = \bigoplus_{l \in \mathbf{Z}^n} \mathfrak{g}(l) \tag{4}$$

under the action of $\operatorname{Ad}(\tilde\rho(t))$ ($t \in \mathbf{G}_m(\mathbf{R})^n$). Then we have

$$\operatorname{Int}(\tilde\rho(t))(g) = \exp(a), \tag{5}$$

$$g = \operatorname{Int}(\tilde\rho(t))^{-1}(\exp(a)) = \exp\big(\textstyle\sum_l t_1^{-l_1} \dots t_n^{-l_n} a_l\big), \tag{6}$$

$$\operatorname{Int}(\tilde\rho(t))^{-1}(g) = \exp\big(\textstyle\sum_l t_1^{-2l_1} \dots t_n^{-2l_n} a_l\big). \tag{7}$$

We want to see g, $\operatorname{Int}(\tilde\rho(t))^{\pm 1}(g) \in U''$. $\operatorname{Int}(\tilde\rho(t))(g) \in U''$ is obvious by definition. In order to see $\operatorname{Int}(\tilde\rho(t))^{-1}(g) \in U''$, we compute as follows.

$$\begin{aligned} a &= -\big(\operatorname{Ad}(\tilde\rho(t))\theta_r \operatorname{Ad}(\tilde\rho(t))^{-1}\big)(a) = -\textstyle\sum_l \big(\operatorname{Ad}(\tilde\rho(t))\theta_r\big)(t_1^{-l_1} \dots t_n^{-l_n} a_l) \\ &= -\textstyle\sum_l \big(\theta_r \operatorname{Ad}(\tilde\rho(t))^{-1}\big)(t_1^{-l_1} \dots t_n^{-l_n} a_l) = -\sum_l t_1^{-2l_1} \dots t_n^{-2l_n} \theta_r(a_l). \end{aligned}$$

Since θ_r transforms $\mathfrak{g}(l)$ to $\mathfrak{g}(-l)$, we have $a_{-l} = -t_1^{-2l_1} \dots t_n^{-2l_n} \theta_r(a_l)$, i.e.,

$$t_1^{-2l_1} \dots t_n^{-2l_n} a_l = -\theta_r(a_{-l}) \quad (\forall l \in \mathbf{Z}^n). \tag{8}$$

Since a is sufficiently near 0, so is each component a_{-l} and hence so is each $-\theta(a_{-l})$. Therefore, by (8) and (7), we have $\operatorname{Int}(\tilde\rho(t))^{-1}(g) \in U''$. Now $g \in U''$ follows from $\operatorname{Int}(\tilde\rho(t))^{\pm 1}(g) \in U''$. In fact, take a basis $\{e_{l,j}\}$ of $\mathfrak{g}_{\mathbf{R}}$ subordinate to the decomposition (4) and write $a_l = \sum_j a_{l,j} e_{l,j}$. Since

$$t_1^{-l_1} \dots t_n^{-l_n} a_{l,j} = \sqrt{a_{l,j}(t_1^{-2l_1} \dots t_n^{-2l_n} a_{l,j})} \tag{9}$$

and since $a_{l,j}$ and $t_1^{-2l_1}\dots t_n^{-2l_n}a_{l,j}$ are sufficiently near 0, so is the left-hand-side of (9). Thus we have $g \in U''$ by (6).

We prove (3). By definition, any element of $B(V, V', V'')$ can be written as

$$\tilde{\rho}(t)gr' \quad \text{such that} \quad r' \in V,\ t \in \mathbf{R}^n_{>0} \cap V',\ g \in V'',\ \mathrm{Int}(\tilde{\rho}(t))^{\pm 1}(g) \in V''.$$

Since V'' is sufficiently small, there exists $b \in \mathfrak{g}_{\mathbf{R}}$ with

$$(10) \qquad g = \exp(b) \quad \text{such that } b \text{ and } \mathrm{Ad}(\tilde{\rho}(t))^{\pm 1}(b) \text{ are sufficiently near } 0.$$

Let

$$b = b^- + b^+ \in \mathfrak{g}_{\mathbf{R}} = \mathfrak{g}_{\mathbf{R}}^- \oplus \mathfrak{g}_{\mathbf{R}}^+, \quad \mathfrak{g}_{\mathbf{R}}^{\pm} := \{x \in \mathfrak{g}_{\mathbf{R}} \mid \theta_r(x) = \pm x\},$$

be the Cartan decomposition. Then, by 4.3, we have

$$g = \exp(b) = \exp(f_-(b^-, b^+)) \exp(f_+(b^-, b^+)), \quad f_{\pm}(b^-, b^+) \in \mathfrak{g}_{\mathbf{R}}^{\pm}.$$

Since

$$\tilde{\rho}(t)gr' = \tilde{\rho}(t) \exp(f_-(b^-, b^+))(\exp(f_+(b^-, b^+))r'),$$

it is enough to show $\exp(f_+(b^-, b^+))r' \in U$ and $\mathrm{Int}(\tilde{\rho}(t))(\exp(f_-(b^-, b^+))) \in U''$. Note that $\exp(f_+(b^-, b^+)) \in K_r$. Since b is sufficiently near 0, $\exp(f_+(b^-, b^+))$ is sufficiently near 1. Hence

$$\exp(f_+(b^-, b^+))r' \in U.$$

Since $b^{\pm} = (b \pm \theta_r(b))/2$, we have

$$\mathrm{Ad}(\tilde{\rho}(t))(b^{\pm}) = \frac{\mathrm{Ad}(\tilde{\rho}(t))(b) \pm \theta_r(\mathrm{Ad}(\tilde{\rho}(t))^{-1}(b))}{2}.$$

These are sufficiently near 0 by (10), and hence so is $f_-\big(\mathrm{Ad}(\tilde{\rho}(t))(b^-), \mathrm{Ad}(\tilde{\rho}(t))(b^+)\big)$. Thus

$$\mathrm{Int}(\tilde{\rho}(t))\big(\exp(f_-(b^-, b^+))\big) \in U''. \quad \square$$

4.8. *Topology $\mathcal{T}$ on $D_{\mathrm{SL}(2)}(W)$.* As in 3.13, let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. For $x \in D_{\mathrm{SL}(2)}(W)$, let $\mathcal{F}_x$ be the filter on D associated to x in 4.6. For an open set U of D, denote

$$\tilde{U} := \{x \in D_{\mathrm{SL}(2)}(W) \mid U \in \mathcal{F}_x\}.$$

We define the *topology $\mathcal{T}$ on* $D_{\mathrm{SL}(2)}(W)$ so that its basis of open sets is given by

$$\{\tilde{U} \mid U \text{ is an open set on } D\}.$$

We denote by $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ the topological space whose underlying set coincides with the one of $D_{\mathrm{SL}(2)}(W)$ but whose topology is $\mathcal{T}$. Hence, for $x \in D_{\mathrm{SL}(2)}(W)$, the filter $\mathcal{F}_x$ on D associated to x coincides with the filter whose basis is given by

$$\{V \cap D \mid V \text{ is a neighborhood of } x \text{ in } D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}\}.$$

Propositon 4.9. *The identity map $D_{\mathrm{SL}(2)}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ is continuous, i.e., the topology on $D_{\mathrm{SL}(2)}(W)$ in* 3.14 *is stronger than or equal to $\mathcal{T}$.*

Proof. Let $y \in D_{\mathrm{SL}(2),\mathrm{val},n}(W)$, $\overline{y}$ its image in $D_{\mathrm{SL}(2)}(W)$. Without loss of generality, we may assume $W^{(j)} = W(\sigma_j)$ $(1 \le j \le n)$, the family of weight filtrations of $H_{0,\mathbf{R}}$ associated to $\overline{y}$. Let U be an open subset of D satisfying $U \in \mathcal{F}_{\overline{y}}$. It is enough to prove that, for $y' \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$ sufficiently near y, its image $\overline{y}'$ in $D_{\mathrm{SL}(2)}(W)$ satisfies $U \in \mathcal{F}_{\overline{y}'}$.

Choose a subset $M = \{m_1, \dots, m_l\}$ $(m_1 < \cdots < m_l)$ of $\{1, \dots, n\}$, and let $W' = (W^{(m_j)})_{1 \le j \le l}$. Since there are only finitely many choices of M, we may (and do) fix M and assume that the family of the weight filtrations associated to $\overline{y}'$ coincides with W'.

Let $y = (\overline{y}, Z, V)$, and let P_y be the parabolic subgroup of $G_{\mathbf{R}}$ associated to y. Fix a representative (ρ, φ) of $\overline{y}$ such that $\varphi(\mathbf{i}) \in Z$. Put $r = \varphi(\mathbf{i})$. Let $\tilde{\rho}_M : \mathbf{G}_m(\mathbf{R})^l \to G_{\mathbf{R}}$ be the composite homomorphism $\mathbf{G}_m(\mathbf{R})^l \to \mathbf{G}_m(\mathbf{R})^n \to G_{\mathbf{R}}$ of the injection to the M-th components and $\tilde{\rho}$ in 3.10 (1). There is an minimal parabolic subgroup P of $G_{\mathbf{R}}$ such that

$$\rho(\mathbf{G}_m(\mathbf{R})^n) \subset P \subset P_y.$$

For such P, $\rho(\mathbf{G}_m(\mathbf{R})^n) \bmod P_u$ is contained in $S_P \subset P/P_u$ and, by 3.9, $\rho(t) \in P$ is the Borel-Serre lifting of $\rho(t) \bmod P_u$ at K_r for any $t \in \mathbf{G}_m(\mathbf{R})^n$. We fix P.

Assume $y' \in D_{\mathrm{BS},\mathrm{val}}(P_y)$, let (ρ', φ') be a representative of $\overline{y}'$, and let $r' = \varphi'(\mathbf{i})$. Write $r' = pkr$ $(p \in P,\ k \in K_r)$. We show

(1) $$\tilde{\rho}' = \mathrm{Int}(p)\tilde{\rho}_M,$$

(2) $$p \in G_{W',\mathbf{R}}.$$

In fact, since $\tilde{\rho}'$ and $\tilde{\rho}_M$ are two splittings of W', there exists $q \in G_{W',\mathbf{R},u}$ such that $\tilde{\rho}' = \mathrm{Int}(q)\tilde{\rho}_M$. Since $y' \in D_{\mathrm{BS},\mathrm{val}}(P_y)$, we have $G_{W',\mathbf{R},u} \subset P_{y,u}$. Since $P \subset P_y$, we have $P_{y,u} \subset P_u$. Hence $q \in P_u$. This shows $\tilde{\rho}'(\mathbf{G}_m(\mathbf{R})^l) \subset P$ and $\tilde{\rho}' \equiv \tilde{\rho}_M \bmod P_u$. By 3.9, $\tilde{\rho}'$ is the Borel-Serre lifting of $\tilde{\rho}' \bmod P_u$ at $K_{r'}$. Since $\tilde{\rho}_M$ is the Borel-Serre lifting of $\tilde{\rho}_M \bmod P_u = \tilde{\rho}' \bmod P_u$ at K_r, and since $p \in P$ and $K_{r'} = \mathrm{Int}(p)K_r$, we have $\tilde{\rho}' = \mathrm{Int}(p)\tilde{\rho}_M$. Hence $\mathrm{Int}(qp^{-1})\tilde{\rho}' = \tilde{\rho}'$ and hence $qp^{-1} \in G_{W',\mathbf{R}}$. This shows $p \in G_{W',\mathbf{R}}$.

Now we fix an open neighborhood U of r in $K_r r$, an open neighborhood U' of 0 in $\mathbf{R}^n_{\ge 0}$, and an open neighborhood U'' of 1 in $G_{\mathbf{R}}$. We prove that if y' is sufficiently near y then the set $B(U, U', U'')$ in 4.7 belongs to $\mathcal{F}_{\overline{y}'}$. Take a neighborhood U''_1 of $G_{\mathbf{R}}$ such that $ab \in U''$ for any $a, b \in U''_1$. Take a neighborhood U''_2 of $G_{\mathbf{R}}$ such that if $a \in U''_2 \cap G_{W',\mathbf{R}}$ then $\mathrm{Int}(\tilde{\rho}'(t))^j(a) \in U''_1$ for $j = 0, -1, -2$ and for any $t \in \mathbf{G}_m(\mathbf{R})^l$ in some neighborhood of 0 in $\mathbf{R}^l$. (Such U''_2 exists since $G_{W',\mathbf{R}}$ is generated by $G_{W',\mathbf{R},u}$ and elements which commute with all $\tilde{\rho}'(t)$ $(t \in \mathbf{G}_m(\mathbf{R})^l)$.) If y' is sufficiently near y then $y' \in D_{\mathrm{BS},\mathrm{val}}(P_y)$ and there are a representative (ρ', φ') of $\overline{y}'$, $p_0 \in P$, $t \in \mathbf{G}_m(\mathbf{R})^n$, and $k \in K_r$ satisfying the following conditions (3)–(6).

(3) Let $r' = \varphi'(\mathbf{i})$. Then $r' = p_0\tilde{\rho}(t)kr$.

(4) $kr \in U$.

(5) $tt' \in U'$ for any $t' \in \mathbf{G}_m(\mathbf{R})^l$ in some neighborhood of 0 in $\mathbf{R}^l$.
(Hence $\mathbf{G}_m(\mathbf{R})^l$ is embedded in the M-component of $\mathbf{G}_m(\mathbf{R})^n$.)

(6) $\mathrm{Int}(\tilde{\rho}(t))^j p_0 \in U_2''$ for $j = 0, -1, -2$.

It is sufficient to prove the following (7).

(7) Let $k' \in K_{r'}$, $t' \in \mathbf{G}_m(\mathbf{R})^l$, $g' \in G_{\mathbf{R}}$ and assume that k' is sufficiently near 1, t' is sufficiently near 0, and $\mathrm{Int}(\tilde{\rho}'(t'))^j g'$ for $j = 0, -1, -2$ are sufficiently near 1. Then $\tilde{\rho}'(t')g'k'r' \in B(U, U', U'')$.

Write $p = p_0\tilde{\rho}(t) \in P$. By (1), we have $\tilde{\rho}' = \mathrm{Int}(p)\tilde{\rho}_M$. Hence

$$\tilde{\rho}'(t')g'k'r' = \tilde{\rho}(tt')(\mathrm{Int}(\tilde{\rho}(tt'))^{-1}p_0)(p^{-1}g'p)(p^{-1}k'p)kr.$$

We have $p^{-1}k'p \in K_r$. Since $kr \in U$ and U is open, if k' is suficiently near 1 then $(p^{-1}k'p)kr \in U$. Next, for $j = 0, 1, -1$,

$$\mathrm{Int}(\tilde{\rho}(tt'))^j(p^{-1}g'p) = \mathrm{Int}(\tilde{\rho}(t))^j \,\mathrm{Int}(p)^{-1}\,\mathrm{Int}(\tilde{\rho}'(t'))^j(g')$$

and this belongs to U_1'' if $\mathrm{Int}(\tilde{\rho}'(t'))^j(g')$ is sufficiently near 1. Finally, for $j = 0, 1, -1$,

$$\mathrm{Int}(\tilde{\rho}(tt'))^j\,\mathrm{Int}(\tilde{\rho}(tt'))^{-1}p_0 = \mathrm{Int}(\tilde{\rho}(t'))^{j-1}\,\mathrm{Int}(\tilde{\rho}(t))^{j-1}p_0$$

and this belongs to U_1'' if t' is sufficiently near 0 because $\mathrm{Int}(\tilde{\rho}(t))^{j-1}p_0 \in U_2'' \cap G_{W',\mathbf{R}}$ by (2) and (6). □

We recall here the definition of 'regular spaces' and a property of a map into a regular space, which will be used in the proofs of 4.12, 4.14 below.

Definition 4.10 ([B, Ch. 1, §8, no. 4, Definition 2]). *A topological space is called* regular *if it is Hausdorff and satisfies the following axiom: Given any closed subset F of X and any point $x \notin F$, there is a neighborhood of x and a neighborhood of F which are disjoint.*

Lemma 4.11 ([B, Ch. 1, §8, no. 5, Theorem 1]). *Let X be a topological space, A a dense subset of X, $f : A \to Y$ a map of A into a regular space Y. A necessary and sufficient condition for f to extend to a continuous map $\overline{f} : X \to Y$ is that, for each $x \in X$, $f(y)$ tends to a limit in Y when y tends to x while remaining in A. The continuous extension $\overline{f}$ of f to X is then unique.*

Proposition 4.12. *Let $W = (W^{(j)})_{1\le j\le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. Fix a homomorphism $\nu : \mathbf{G}_m(\mathbf{R})^n \to G_{\mathbf{R}}$ which splits W, and fix a continuous map $\beta : D \to \mathbf{R}^n_{>0}$ such that*

$$\beta(\nu(t)x) = t\beta(x) \quad \textit{for all } x \in D \textit{ and all } t \in \mathbf{R}^n_{>0}.$$

Then the map β extends uniquely to a continuous map $\overline{\beta} : D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \to \mathbf{R}^n_{\ge 0}$.

Proof. Let $x \in D_{\mathrm{SL}(2)}(W)$, let (ρ, φ) be a representative of x, and let $r := \varphi(\mathbf{i})$. Let $W' = (W^{(m_1)}, \dots, W^{(m_l)})$ $(1 \le m_1 < \cdots < m_l \le n)$ be the family of weight filtrations associated to x, let $G_{W',\mathbf{R}}$ be the subgroup of $G_{\mathbf{R}}$ preserving $W^{(m_1)}, \dots, W^{(m_l)}$, and let $G_{W',\mathbf{R},u}$ be its unipotent radical. Put $M := \{m_1, \dots, m_l\}$, and denote by $\nu_M : \mathbf{G}_m(\mathbf{R})^l \to G_{\mathbf{R}}$ the composite homomorphism of the injection $\iota_M : \mathbf{G}_m(\mathbf{R})^l \to \mathbf{G}_m(\mathbf{R})^n$ to the M-th components and ν. Since both ρ and ν_M split W', there exists a unique

(1) $$u \in G_{W',\mathbf{R},u} \quad \text{such that} \quad \tilde{\rho} = \mathrm{Int}(u)\nu_M.$$

Since the target $\mathbf{R}^n_{\geq 0}$ is a regular space, it is sufficient to prove, by 4.11, that, for x, (ρ, φ), r as above, and for directed families $(t_\lambda)_\lambda$, $(g_\lambda)_\lambda$, $(r_\lambda)_\lambda$ such that $t_\lambda \in \mathbf{R}^l_{>0}$, $g_\lambda \in G_{\mathbf{R}}$, $r_\lambda \in K_r r$, $\lim_\lambda t_\lambda = 0$, $\lim_\lambda g_\lambda = 1$, $\lim_\lambda r_\lambda = r$, there exists a limit

$$(2) \qquad \lim_\lambda \beta(\tilde{\rho}(t_\lambda) g_\lambda r_\lambda) \in \mathbf{R}^n_{\geq 0}$$

Let W' and $u \in G_{W',\mathbf{R}}$ be as above, and let $u_\lambda := \nu_M(t_\lambda)^{-1} u \nu_M(t_\lambda)$. Then

$$\beta(\tilde{\rho}(t_\lambda) g_\lambda r_\lambda) = \beta(u \nu_M(t_\lambda) u^{-1} g_\lambda r_\lambda) = \beta(\nu_M(t_\lambda) u_\lambda u^{-1} g_\lambda r_\lambda) = \iota_M(t_\lambda) \beta(u_\lambda u^{-1} g_\lambda r_\lambda).$$

Since ν_M splits W' and $u \in G_{W',\mathbf{R},u}$, we have $\lim_\lambda u_\lambda = 1$. Hence $\lim_\lambda \beta(u_\lambda u^{-1} g_\lambda r_\lambda) = \beta(u^{-1} r)$. This proves the existence of the limit (2). □

The following property of proper maps will be used in the proof of 4.14 below.

Lemma 4.13 ([B, Ch. 1, §10, no. 1, Proposition 5 d)]). *Let $f : X \to X'$, $g : X' \to X''$ be two continuous maps. If $g \circ f$ is proper and X' is Hausdorff, then f is proper.*

Proposition 4.14. *Let $W = (W^{(j)})_{1 \leq j \leq n}$ be a compatible family of increasing $\mathbf{Q}$-rational filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. Fix $\nu : \mathbf{G}_m(\mathbf{R})^n \to G_{\mathbf{R}}$ and $\beta : D \to \mathbf{R}^n_{>0}$ as in 4.12, and let $\overline{\beta} : D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \to \mathbf{R}^n_{\geq 0}$ be the continuous extension of β in 4.12. Then there exists a unique homeomorphism*

$$D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}} \xrightarrow{\sim} D_{\mathrm{SL}(2),\mathrm{val}}(W)$$

which extends the identity map of D.

Proof. A map

$$(1) \qquad D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}} \to D_{\mathrm{SL}(2),\mathrm{val}}(W), \quad (x, (V, h)) \mapsto (x, Z, V'),$$

is defined as follows. First, let $V' \subset X(\mathbf{G}^n_m) = \mathbf{Z}^n$ be the pull-back of V under the isomorphism

$$\mathbf{G}^n_m \xrightarrow{\sim} \mathbf{G}^n_m, \quad (t_j)_j \mapsto t_{j+1}/t_j \quad (t_{n+1} \text{ means } 1).$$

Let (ρ, φ) be a representative of x, let W' be the family of weight filtrations associated to x, and let u be the unique element of $G_{W',\mathbf{R},u}$ such that $\tilde{\rho} = \mathrm{Int}(u)(\nu)$. Put $r := \varphi(\mathbf{i})$. Then, as is easily seen, the element $r_\beta(x) := \tilde{\rho}(\beta(u^{-1}r))^{-1} r \in D$ is independent of the choice of the representative (ρ, φ). The set Z is defined by

$$Z := \{\tilde{\rho}(t) r_\beta(x) \mid \textstyle\prod_j t_j^{m(j)} = h(m) \text{ for all } m \in V^\times\}.$$

The inverse map

$$(2) \qquad D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}, \quad (x, Z, V') \mapsto (x, (V, h)),$$

is defined as follows. The set V is produced from V' by inverting the process to get V' from V. Take a point of Z and express it in the form $\tilde{\rho}(t) r_\beta(x)$ for some $t \in \mathbf{R}^n_{>0}$. The homomorphism $h : V^\times \to \mathbf{R}_{>0}$ is defined by $h(m) := \prod_j t_j^{m(j)}$ for $m \in V^\times$. It is easy to see, by definition, that (1) and (2) are inverse each other.

To show that the map (1) is homeomorphic, it is enough to prove that (1) is continuous. In fact, suppose that (1) is continuous, then (1) is proper by 4.13, because $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ is continuous (3.14, 4.9), the left-hand-side of (1) is proper over $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ (2.13), and $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ is Housdorff (2.17, 3.14).

To prove that the map (1) is continuous, since the target $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ is a regular space, it is enough to show, by 4.11, the following: For $y \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$, we denote by x its image in $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$, and by z its image in $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$. If a directed family $(y_\lambda)_\lambda$ of points in D converges to a point $x \in D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ and if $(\beta(y_\lambda))_\lambda$ converges to z in $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$, then $(y_\lambda)_\lambda$ converges to y in $D_{\mathrm{SL}(2),\mathrm{val}}(W)$. We prove this.

We may assume without loss of generality that $W = (W^{(j)})_{1\leq j\leq n}$ is the family of weight filtrations of $H_{0,\mathbf{R}}$ associated to x. (Hence, the image of $z \in (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$ in $\mathbf{R}^n_{\geq 0}$ is 0.) Let (ρ, φ), r be as above. By the definition of the topology $\mathcal{T}$, y_λ is written as

$$y_\lambda = \tilde{\rho}(t_\lambda) g_\lambda r_\lambda, \quad \text{where } r_\lambda \in K_r r, \ g_\lambda \in G_{\mathbf{R}}, \ t_\lambda \in \mathbf{R}^n_{>0},$$

so that

$$\lim_\lambda r_\lambda = r, \ \lim_\lambda g_\lambda = \lim_\lambda \mathrm{Int}(\tilde{\rho}(t_\lambda))^{\pm 1}(g_\lambda) = 1, \ \lim_\lambda t_\lambda^{-1}\beta(y_\lambda) = \lim_\lambda \beta(g_\lambda r_\lambda) = \beta(r).$$

It follows from this, together with the assumption $\lim_\lambda \beta(y_\lambda) = z$, that

$$\lim_\lambda t_\lambda = \beta(r)^{-1} z.$$

Let P_y be the parabolic subgroup of $G_{\mathbf{R}}$ associated to y, and let P be a minimal parabolic subgroup of $G_{\mathbf{R}}$ such that $\rho(\mathbf{G}_m(\mathbf{R})^n) \subset P \subset P_y$. Then, by 4.4, we have

$$g_\lambda = h_\lambda k_\lambda, \text{ where } h_\lambda \in P, \ k_\lambda \in K_r, \text{ so that } \lim_\lambda h_\lambda = \lim_\lambda \mathrm{Int}(\tilde{\rho}(t_\lambda))(h_\lambda) = \lim_\lambda k_\lambda = 1.$$

Hence

$$y_\lambda = \tilde{\rho}(t_\lambda) h_\lambda k_\lambda r_\lambda = \tilde{\rho}(t_\lambda) h_\lambda \tilde{\rho}(t_\lambda)^{-1} \tilde{\rho}(t_\lambda) k_\lambda r_\lambda = \tilde{\rho}(t_\lambda) \circ_P (\tilde{\rho}(t_\lambda) h_\lambda \tilde{\rho}(t_\lambda)^{-1}) k_\lambda r_\lambda,$$

and this converges to y in $D_{\mathrm{SL}(2),\mathrm{val}}(W)$, as desired (cf. 3.14, 2.13). □

Corollary 4.15. *The topology $\mathcal{T}$ on $D_{\mathrm{SL}(2)}(W)$ coincides with the one as a subspace of the topological space $D_{\mathrm{SL}(2)}$ defined in* 3.14.

Proof. By 4.14, $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ is proper surjective, because so is $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}} \to \mathbf{R}^n_{\geq 0}$. Hence $\mathcal{T}$ coincides with the quotient topology of $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ which is the topology of $D_{\mathrm{SL}(2)}(W)$ by 3.14. □

The following property of proper maps will be used in the proof of 3.15 (4.17 below).

Lemma 4.16 ([B, Ch. 1, §10, no. 1, Corollary 2]). *Let $f : X \to Y$ be a proper map, where X is Hausdorff. Then the subspace $f(X)$ of Y is Hausdorff.*

4.17. *Proof of Theorem* 3.15. The assertion 3.15 (i) follows directly from 4.14 and 4.15. Applying 4.16 to $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$, the assertion 3.15 (ii) follows from 3.15 (i) and Hausdorffness of $D_{\mathrm{SL}(2),\mathrm{val}}$. □

4.18 *Remark.* We prove that, for an $\mathrm{SL}(2)^n$-orbit (ρ, φ), $[\rho, \varphi] = \lim \varphi(iy_1, \dots, iy_n)$ as $\frac{y_j}{y_{j+1}} \to \infty$ ($\forall j$, $y_{n+1} = 1$) in $D_{\mathrm{SL}(2)}$, which is stated after 3.14. Since

$$\varphi(iy_1, \dots, iy_n) = \tilde{\rho}\Big(\sqrt{\tfrac{y_2}{y_1}}, \dots, \sqrt{\tfrac{y_{n+1}}{y_n}}\Big)\varphi(\mathbf{i})$$

and $\sqrt{\frac{y_{j+1}}{y_j}} \to 0$, the right-hand-side converges to $[\rho, \varphi]$ in the $\mathcal{T}$-topology and hence in the topology of $D_{\mathrm{SL}(2)}$ (4.15).

§5. Actions of $G_{\mathbf{Z}}$

5.1. *Summary.* In this section, we will transport the good properties (i), (ii) in 2.1 of the quotient space $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ to other spaces along the diagram 3.1 (1). To do so, we will use the notion of 'proper action' of a group in [B] (see 5.3 below). The main result of this section is the following Theorem 5.2.

Theorem 5.2. (i) *For any subgroup Γ of $G_{\mathbf{Z}}$, all the quotient spaces $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS},\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{SL}(2)}$ are Hausdorff.*

(ii) *If Γ is a subgroup of $G_{\mathbf{Z}}$ of finite index, then $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS},\mathrm{val}}$ are compact.*

(iii) *If Γ is a neat subgroup of $G_{\mathbf{Z}}$, then all the projections $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$, $D_{\mathrm{BS},\mathrm{val}} \to \Gamma\backslash D_{\mathrm{BS},\mathrm{val}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ are local homeomorphisms.*

Before proving this theorem, we recall the notion of 'proper action' and some related results in [B] which are needed for our present purpose.

Definition 5.3 ([B, Ch. 3, §4, no. 1, Definition 1]). *Let G be a topological group acting continuously on a topological space X. G is said to act* properly *on X if the map*

$$G \times X \to X \times X, \quad (g, x) \mapsto (x, gx),$$

is proper.

Lemma 5.4 (cf. [B, Ch. 3, §4, no. 2, Proposition 3]). *If a topological group G acts properly on a topological space X, then the quotient space $G\backslash X$ is Hausdorff.*

Lemma 5.5 (cf. [B, Ch. 3, §4, no. 4, Corollary]). *If a discrete group G acts properly and freely on a Hausdorff space X, then the projection $X \to G\backslash X$ is a local homeomorphism.*

Lemma 5.6 (cf. [B, Ch. 3, §2, no. 2, Proposition 5]). *Let G be a topological group acting continuously on a topological spaces X and X'. Let $\psi : X \to X'$ be an equivariant continuous map.*

(i) *If ψ is surjective and proper, and if G acts properly on X, then G acts properly on X'.*

(ii) *If G acts properly on X' and if X is Hausdorff, then G acts properly on X.*

Now we come back to our situation.

Lemma 5.7. *If* Γ *is a neat subgroup of* $G_{\mathbf{Z}}$*, then* Γ *acts on* $D_{\mathrm{SL}(2)}$ *freely.*

Proof. Let $x \in D_{\mathrm{SL}(2),n}$ and let (ρ, φ) be a representative of x. Let $\gamma \in \Gamma$. Assume $\gamma(x) = x$, that is, $\gamma(\rho, \varphi) \sim (\rho, \varphi)$. Then $\mathrm{Ad}(\gamma)Y_j = Y_j$ $(1 \le j \le n)$. Here the Y_j are the semi-simple elements of $\mathfrak{g}_{\mathbf{R}}$ associated to ρ in 3.1. Put $Y := \sum_{1\le j\le n} Y_j$. Then γ preserves the l-eigen subspace $H(l) \subset H_{0,\mathbf{R}}$ of Y for all l. Put $\mathrm{gr}_k := \mathrm{gr}_k^{W^{(n)}}(H_{0,\mathbf{C}})$ and $\mathrm{gr} := \bigoplus_k \mathrm{gr}_k$. Let $F := \varphi(\mathbf{i}) \in D$ and $F(\mathrm{gr})$ be the filtration of gr induced by F. Then, by the assumption, the automorphism $\mathrm{gr}(\gamma)$ of gr induced by γ satisfies $\mathrm{gr}(\gamma)F(\mathrm{gr}) = F(\mathrm{gr})$. Thus we have the following (i)–(iv).

(i) $(W^{(n)}, F)$ is an $(N_1 + \cdots + N_n)$-polarized mixed Hodge structure ([Sc]).

(ii) $\gamma W^{(n)} = W^{(n)}$.

(iii) $\mathrm{gr}(\gamma)F(\mathrm{gr}) = F(\mathrm{gr})$.

(iv) If a is an eigen value of $\mathrm{gr}(\gamma)$ and if a is a root of 1, then $a = 1$.

We prove $\mathrm{gr}(\gamma) = 1$. Since $F(\mathrm{gr})$ is polarized, the isotropy group of $F(\mathrm{gr})$ is compact, and so $\mathrm{gr}(\gamma)$ is contained in the intersection of a discrete subgroup and a compact subgroup and hence is of finite order. Therefore $\mathrm{gr}(\gamma) = 1$ by (iv).

Now $\gamma = 1$ follows from $\mathrm{gr}(\gamma) = 1$ and the commutativity of γ and Y. □

5.8. *Proof of Theorem* 5.2. We prove (i). $G_{\mathbf{Z}}$ acts on $\mathcal{X}_{\mathrm{BS}}$ properly by [BS]. Since D_{BS}, $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$ are Hausdorff by 2.17 (i) and the note after Definition 3.14, it follows that $G_{\mathbf{Z}}$ acts on these spaces properly by 5.6 (ii). Since $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ is proper and surjective by 3.15 (i), it follows that $G_{\mathbf{Z}}$ acts on $D_{\mathrm{SL}(2)}$ properly by 5.6 (i). Hence, for any subgroup Γ of $G_{\mathbf{Z}}$, all the quotient spaces $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS,val}}$, $\Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{SL}(2)}$ are Hausdorff by 5.4.

We prove (ii). Let Γ be a subgroup of $G_{\mathbf{Z}}$ of finite index. Then $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is compact by [BS]. Since $D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}$ and $D_{\mathrm{BS,val}} \to D_{\mathrm{BS}}$ are proper by 2.17 (ii), $\Gamma\backslash D_{\mathrm{BS}} \to \Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ and $\Gamma\backslash D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS}}$ are proper. Hence $\Gamma\backslash D_{\mathrm{BS}}$ and $\Gamma\backslash D_{\mathrm{BS,val}}$ are compact.

We prove (iii). Let Γ be a neat subgroup of $G_{\mathbf{Z}}$. Since Γ acts on $\mathcal{X}_{\mathrm{BS}}$ freely by [BS], so does Γ on D_{BS}, on $D_{\mathrm{BS,val}}$, and on $D_{\mathrm{SL}(2),\mathrm{val}}$ by 3.12. Γ acts on $D_{\mathrm{SL}(2)}$ freely by 5.7. Moreover, by the above results in the proof of (i) (applied to $\Gamma = \{1\}$), all the spaces D_{BS}, $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)}$ are Hausdorff and acted by Γ properly. Hence all the projections $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$, $D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ are local homeomorphisms by 5.5. □

§6. Examples and comments

6.1. *Summary.* In this section, we will first give a criterion in Proposition 6.3 for the existence of the canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ by using the family of weight filtrations associated to a point of $D_{\mathrm{SL}(2)}$. This criterion explains the reason why we need to introduce the projective limits of blowing-ups $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$ of D_{BS}, $D_{\mathrm{SL}(2)}$, respectively, to relate D_{BS} and $D_{\mathrm{SL}(2)}$. Then we will give the list of 'classical situation' in 6.4, and in this situation we will show that $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}} = \mathcal{X}_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS,val}}$ in Theorem 6.5. As a corollary, we have in 6.6 the canonical surjection from the Borel-Serre space D_{BS} to the Satake space D_S in the 'classical situation'. This map was defined by Zucker [Z] by another method. Proposition 6.7 gives examples which do not have the canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$. We will give an example in 6.8 which shows

$D_{\mathrm{SL}(2)} \subsetneq D_{\mathrm{BS}}$ because the horizontal tangent bundle T_D^h is trivial. Proposition 6.9 gives examples which have the canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ but this is not isomorphic since $D_{\mathrm{SL}(2)}$ has a 'slit' because the isotropy subgroup K'_r is not maximal compact.

6.2. *The case of the upper-half plane* $\mathfrak{h}$. Let $H_{\mathfrak{h}} := \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, let $\langle\ ,\ \rangle_{\mathfrak{h}}$ be the anti-symmetric bilinear form on $H_{\mathfrak{h},\mathbf{C}} \times H_{\mathfrak{h},\mathbf{C}}$ characterized by $\langle e_2, e_1\rangle_{\mathfrak{h}} = 1$, and take $(H_{\mathfrak{h}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ as $(H_0, \langle\ ,\ \rangle_0)$. Then

$$\begin{aligned}&\check{D} = \mathbf{P}^1(\mathbf{C});\ F_z \leftrightarrow z = (z_1 : z_2),\\ &\text{where }\ F_z^0 = H_{0,\mathbf{C}},\ F_z^1 = \mathbf{C}(z_1e_1 + z_2e_2),\ F_z^2 = 0.\end{aligned}$$

Identify $z \in \mathbf{C}$ with $(z : 1) \in \mathbf{P}^1(\mathbf{C})$. Then $D \subset \check{D}$ is identified with the upper-half plane $\mathfrak{h} \subset \mathbf{P}^1(\mathbf{C})$. We have

$$G_{\mathbf{R}} = \mathrm{SL}(2, \mathbf{R}) \supset \mathrm{SO}(2, \mathbf{R}) = K_i = K'_i.$$

The map

$$\begin{aligned}&\mathbf{P}^1(\mathbf{Q}) \to \{P \,|\, \text{a } \mathbf{Q}\text{-parabolic subgroup of } G_{\mathbf{R}} \text{ with } P \neq G_{\mathbf{R}}\},\\ &z \mapsto P_z := \{g \in G_{\mathbf{R}} \,|\, gz = z\},\end{aligned}$$

is bijective. The Iwasawa decomposition of $G_{\mathbf{R}}$ associated to (P_∞, K_i) is given by

$$G_{\mathbf{R}} = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \middle|\, x \in \mathbf{R}\right\} \times \left\{\begin{pmatrix}a^{-1} & 0\\ 0 & a\end{pmatrix} \middle|\, a \in \mathbf{R}_{>0}\right\} \times \mathrm{SO}(2, \mathbf{R})$$

and this gives the identification

$$D_{\mathrm{BS}}(P_\infty) = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \middle|\, x \in \mathbf{R}\right\} \times \mathbf{R}_{\geq 0}.$$

Since the point $x + iy \in \mathfrak{h} = D$ $(x \in \mathbf{R}, y \in \mathbf{R}_{>0})$ corresponds to $\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, y^{-1/2}\right)$ in this identification, the point of $D_{\mathrm{BS}}(P_\infty)$ corresponding to $\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, 0\right)$ should be written as $x + i\infty$, and in this way we have the identification

$$D_{\mathrm{BS}}(P_\infty) = \{x + iy \,|\, x \in \mathbf{R}, 0 < y \leq \infty\}.$$

We have

$$D_{\mathrm{BS}} \xleftarrow{\sim} D_{\mathrm{BS,val}} \xleftarrow{\sim} D_{\mathrm{SL}(2),\mathrm{val}} \xrightarrow{\sim} D_{\mathrm{SL}(2)},$$

and $i\infty \in D_{\mathrm{BS}}(P_\infty)$ is identified with the class of the SL(2)-orbit $(\rho_{\mathfrak{h}}, \varphi_{\mathfrak{h}})$, which we call the *standard* SL(2)*-orbit*, defined by

$$\begin{aligned}&\rho_{\mathfrak{h}} = \mathrm{id} : \mathrm{SL}(2, \mathbf{C}) \to G_{\mathbf{C}},\\ &\varphi_{\mathfrak{h}}(z) = F_z \quad (z \in \mathbf{P}^1(\mathbf{C})).\end{aligned}$$

Proposition 6.3. *We use the notation in* 2.7, 3.7.

(i) *Let* $x \in D_{\mathrm{SL}(2),n}$ *and let* $W = (W^{(j)})_{1\le j\le n}$ *be the associated family of weight filtrations, where* $W^{(j)} = W(\sigma_j)$*. Then the following conditions* (a), ... , (e) *are equivalent.*

(a) *For any liftings* $y, y' \in D_{\mathrm{SL}(2),\mathrm{val}}$ *of* x*, their images in* D_{BS}*, via* $D_{\mathrm{BS},\mathrm{val}}$*, coincide.*

(b) *For any liftings* $y, y' \in D_{\mathrm{SL}(2),\mathrm{val}}$ *of* x*, the parabolic subgroups* $P_{T,V}$*,* $P_{T',V'}$*, associated to their images in* $D_{\mathrm{BS},\mathrm{val}}$*, coincide.*

(c) *The group* $G_{W,\mathbf{R}}$ *preserving* W *is parabolic.*

(d) *The subspaces* $W_i^{(j)}$ $(1 \le j \le n,\ i \in \mathbf{Z})$ *are linearly ordered by inclusion.*

(e) *For* $\chi, \chi' \in X(\mathbf{G}_m^n)$ *with* $H(\chi) \neq 0$*,* $H(\chi') \neq 0$*, either* $\chi{\chi'}^{-1}$ *or* $\chi'\chi^{-1}$ *is contained in* $X(\mathbf{G}_m^n)_+$*.*

(ii) *The equivalent conditions* (a), ... , (e) *in* (i) *hold at any point* $x \in D_{\mathrm{SL}(2)}$ *if and only if there exists a continuous map* $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ *which extends the identity map of* D*. In this case, this extended map is uniquely given by* $[\rho, \varphi] \mapsto (G_{W,\mathbf{R}}, A_{G_{W,\mathbf{R}}} \circ \varphi(\mathbf{i}))$*.*

The proof is straight forward by 2.7, 3.7, and we omit it.

6.4. *Classical situation.* Let $F \in D$, and let $T_D(F)$ and $T_D^h(F)$ be the tangent space and horizontal tangent space of D at F, respectively (1.6).

It can be proved that the following (i) and (ii) are equivalent.

(i) For any $F \in D$, $T_D^h(F) = T_D(F)$ and $K'_F = K_F$ (see 2.2 (1)).

(ii) One of the following (a), (b) is satisfied:

(a) There is $t \in \mathbf{Z}$ such that $w = 2t+1$ and $h^{p,w-p} = 0$ if $p \neq t, t+1$.

(b) There is $t \in \mathbf{Z}$ such that $w = 2t$, $h^{t+1,t-1} = h^{t-1,t+1} = 1$, $h^{t,t} \ge 1$, and $h^{p,q} = 0$ otherwise.

Note that the condition (i) is independent of the choice of $F \in D$ and that $K'_F = K_F$ in (i) implies $D = \mathcal{X}$. The equivalence of (i) and (ii) follows by computing dimensions of the subspaces $F^r(\mathfrak{g}_{\mathbf{C}})$ in1.6 and of the Lie algebras of the following (1).

$$
(1)\qquad
\begin{aligned}
G_{\mathbf{R}} &\simeq \begin{cases} \mathrm{Sp}(2g, \mathbf{R}) & \text{if } w = 2t+1, \\ O(a, b; \mathbf{R}) & \text{if } w = 2t, \end{cases} \\
K_r &\simeq \begin{cases} U(g) & \text{if } w = 2t+1, \\ O(a, \mathbf{R}) \times O(b, \mathbf{R}) & \text{if } w = 2t, \end{cases} \\
K'_r &\simeq \begin{cases} U(h^{w,0}) \times \cdots \times U(h^{t+1,t}) & \text{if } w = 2t+1, \\ U(h^{w,0}) \times \cdots \times U(h^{t+1,t-1}) \times O(h^{t,t}, \mathbf{R}) & \text{if } w = 2t, \end{cases}
\end{aligned}
$$

where $g := \operatorname{rank} H_0/2$ if $w = 2t+1$, and a (resp. b) $:= \sum_j h^{t+j,t-j}$ where j ranges over all even (resp. odd) integers as in Notation (cf. [U2]).

We say that we are in the *classical situation* if these equivalent conditions (i), (ii) are satisfied. The polarized Hodge structures in (ii) (a) are Tate twists of the first cohomology of polarized abelian varieties, and the primitive part of the second cohomology of a polarized K3 surface belongs to (ii) (b).

Theorem 6.5. *In the classical situation, there exists a homeomorphism* $D_{\mathrm{SL}(2)} \overset{\sim}{\to} D_{\mathrm{BS}}$ *which extends the identity map of* D*, and* $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ *is a homeomorphism.*

Proof. We have the following (1).

(1) Let $[\rho, \varphi] \in D_{\mathrm{SL}(2),n}$, and let $W = (W^{(j)})_{1\le j\le n}$ be the associated family of weight filtrations. Let $1 \le j \le n$. In the case 6.4 (ii) (a), we have

$$W^{(j)}_{-2} = 0,\ W^{(j)}_{1} = H_{0,\mathbf{R}}.$$

In the case 6.4 (ii) (b), we have one of the following (b1), (b2).

$$\text{(b1)}\qquad \mathrm{gr}^{W^{(j)}}_k = 0 \ \text{ unless } k = -1, 0, 1, \text{ and } \dim \mathrm{gr}^{W^{(j)}}_k = 2 \text{ for } k = \pm 1.$$

$$\text{(b2)}\qquad \mathrm{gr}^{W^{(j)}}_k = 0 \ \text{ unless } k = -2, 0, 2, \text{ and } \dim \mathrm{gr}^{W^{(j)}}_k = 1 \text{ for } k = \pm 2.$$

This follows from the facts that the filtration $\varphi(\mathbf{i})(\mathrm{gr}^{W^{(j)}}_k)$ induced on $\mathrm{gr}^{W^{(j)}}_k$ by $\varphi(\mathbf{i})$ is a Hodge structure of weight $w + k$ for each $k \in \mathbf{Z}$, and that if we denote the Hodge type of this Hodge structure by $(h^{p,q}_k)_{p,q\in\mathbf{Z}}$ then $h^{p,w-p} = \sum_k h^{p,w+k-p}_k$.

We next prove

(2) Let the notation be as in (1). In the case 6.4 (ii) (a), we have

$$0 \subsetneq W^{(1)}_{-1} \subsetneq W^{(2)}_{-1} \subsetneq \cdots \subsetneq W^{(n)}_{-1} = W^{(n)}_{0} \subsetneq \cdots \subsetneq W^{(2)}_{0} \subsetneq W^{(1)}_{0} \subsetneq H_{0,\mathbf{R}}.$$

In the case (ii) 6.4 (b) with $n \ge 2$, we have $n = 2$, $W^{(1)}$ is of type (b1), $W^{(2)}$ is of type (b2), and

$$0 \subsetneq W^{(2)}_{-2} = W^{(2)}_{-1} \subsetneq W^{(1)}_{-1} \subsetneq W^{(1)}_{0} \subsetneq W^{(2)}_{0} = W^{(2)}_{1} \subsetneq H_{0,\mathbf{R}},$$
$$\dim W^{(2)}_{-2} = \dim W^{(1)}_{-1}/W^{(2)}_{-1} = \dim W^{(2)}_{0}/W^{(1)}_{0} = \dim H_{0,\mathbf{R}}/W^{(2)}_{1} = 1.$$

In fact, in the case 6.4 (ii) (a), since $\mathrm{Ker}(a_1N_1 + \cdots + a_jN_j) = W^{(j)}_0$ for any $a_1, \ldots, a_j > 0$ ([CK2]), we have $W^{(j')}_0 \supset W^{(j)}_0$ for $1 \le j' \le j \le n$, and hence, by taking $(\ )^{\perp}$, we obtain $W^{(j')}_{-1} \subset W^{(j)}_{-1}$. Since $W^{(j')} \ne W^{(j)}$ for $j' \ne j$, this proves (2) in the case 6.4 (ii) (a).

We consider the case 6.4 (ii) (b). Assume $n \ge 2$. If $1 \le j \le n$ and $W^{(j)}$ is of type (b1), $(a_1N_1+\cdots+a_jN_j)^2 = 0$ for any $a_1, \ldots, a_j > 0$ and hence $(a_1N_1+\cdots+a_{j'}N_{j'})^2 = 0$ for any $j' \le j$ and any $a_1, \ldots, a_{j'} > 0$. Hence $W^{(j')}$ for $j' \le j$ is also of type (b1), and we have $W^{(j')}_0 \supsetneq W^{(j)}_0$ for $j' < j$ just as in the case 6.4 (ii) (a). This contradicts the statement about dimensions in (b1) if $j \ge 2$. Hence any $W^{(j)}$ with $j \ge 2$ is of type (b2). If $W^{(j)}$ is of type (b2), $\mathrm{Ker}((a_1N_1+\cdots+a_jN_j)^2) = W^{(j)}_1$ for any $a_1, \ldots, a_j > 0$. Hence, if $j' < j$ and $W^{(j')}$ and $W^{(j)}$ are of type (b2), we have $W^{(j')}_1 \supsetneq W^{(j)}_1$ which contradicts the statement about dimensions in (b2). Hence we have $n = 2$, $W^{(1)}$ is of type (b1), and $W^{(2)}$ is of type (b2). It follows that the codimensions of $\mathrm{Ker}\, N_1$ and of $\mathrm{Ker}(N_1 + N_2)$ in $H_{0,\mathbf{R}}$ coincide which are 2. On the other hand, since $\mathrm{Ker}(a_1N_1 + a_2N_2) = \mathrm{Ker}(N_1 + N_2)$ for any $a_1, a_2 > 0$ ([CK2]), $\mathrm{Ker}\, N_1 \supset \mathrm{Ker}(N_1 + N_2)$ and hence they coincide. Thus we have $W_0^{(1)} = \operatorname{Ker} N_1 = \operatorname{Ker}(N_1+N_2) \subset \operatorname{Ker}((N_1+N_2)^2) = W_1^{(2)}$, and $W_{-1}^{(1)} \supset W_{-2}^{(2)}$ by taking $(\ )^\perp$. This proves (2) in the case 6.4 (ii) (b).

By (1) and (2), we have

(3) In the classical situation, $G_{W,\mathbf{R}}$ is a parabolic subgroup of $G_\mathbf{R}$.

Hence, by 6.3, we have a continuous map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ which extends the identity map of D. We prove

(4) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is injective.

By 3.11, an $\mathrm{SL}(2)^n$-orbit (ρ, φ) is characterized by the associated (W, r). Assume that the points of $D_{\mathrm{SL}(2)}$ determined by (W, r), (W', r') are sent to the same point in D_{BS}. Then we have

$$G_{W',\mathbf{R}} = G_{W,\mathbf{R}}, \quad r' = a \circ r \ \ (\exists a \in A_{G_{W,\mathbf{R}}}).$$

It follows $W' = W$ from $G_{W',\mathbf{R}} = G_{W,\mathbf{R}}$ and (2). We see also, by (2), that the torus $S_{G_{W,\mathbf{R}}}$ in 2.2 coincides with the torus of the $\mathrm{SL}(2)^n$-orbit determined by (W, r). Hence $r' = a \circ r$ lies on the torus orbit containing r of this $\mathrm{SL}(2)^n$-orbit. Thus, the points in $D_{\mathrm{SL}(2)}$ determined by (W, r), (W', r') coincide, as desired. We prove

(5) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is surjective.

For this it is sufficient to prove

(6) For any $\mathbf{Q}$-parabolic subgroup P of $G_\mathbf{R}$, there exists a point of $D_{\mathrm{SL}(2)}$ whose family W of weight filtrations satisfies $G_{W,\mathbf{R}} = P$.

We deduce (5) from (6). Let $(P, Z) \in D_{\mathrm{BS}}$, take $x = [\rho, \varphi] \in D_{\mathrm{SL}(2)}$ whose family W of weight filtrations satisfies $G_{W,\mathbf{R}} = P$, and put $r = \varphi(\mathbf{i})$. Since $K_r = K'_r$, we have $D = Pr$ and hence there is $p \in P$ such that $pr \in Z$. The group $P = G_{W,\mathbf{R}}$ acts on $D_{\mathrm{SL}(2)}(W)$, and the image of $pr \in D_{\mathrm{SL}(2)}(W)$ in D_{BS} is (P, Z).

We prove (6). Let P be $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$ and let $M = (M_j)_{0 \le j \le m}$ be the corresponding $\mathbf{Q}$-rational increasing filtration of $H_{0,\mathbf{R}}$ (2.10). Let $n = m/2$ if m is even, and $n = (m-1)/2$ if m is odd.

We prove first the case 6.4 (ii) (a). Let $e(j) := \dim M_j/M_{j-1}$ for $1 \le j \le n$, and let $e := \sum_{1 \le j \le n} e(j)$. Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight 1 whose Hodge type $(h_1^{p,q})_{p,q \in \mathbf{Z}}$ is given by

$$h_1^{p,q} = g - e \ \text{ if } (p,q) = (1,0) \text{ or } (0,1), \quad h_1^{p,q} = 0 \ \text{ otherwise}.$$

Fix an isomorphism

$$\big((H_{\mathfrak{h},\mathbf{Q}})^{\oplus e} \oplus H_{1,\mathbf{Q}}, (\langle\ ,\ \rangle_\mathfrak{h})^{\oplus e} \oplus \langle\ ,\ \rangle_1\big) \simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0),$$

where $(H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_\mathfrak{h})$ is as in 6.2, and take this isomorphism as an identification. Let (ρ, φ) be the $\mathrm{SL}(2)^n$-orbit defined by

$$\rho(g_1, \dots, g_n) := \big(\textstyle\bigoplus_{1 \le j \le n} \rho_\mathfrak{h}(g_j)^{\oplus e(j)}\big) \oplus \mathrm{id},$$
$$\varphi(z_1, \dots, z_n) := \big(\textstyle\bigoplus_{1 \le j \le n} \varphi_\mathfrak{h}(z_j)^{\oplus e(j)}\big)(-t) \oplus F_1(-t),$$

where $(\rho_\mathfrak{h}, \varphi_\mathfrak{h})$ is the standard $\mathrm{SL}(2)$-orbit in 6.2, and $(-t)$ means the Tate twist. Then the family W of weight filtrations of (ρ, φ) satisfies $G_{W,\mathbf{R}} = P$.

Next we prove (6) in the case 6.4 (ii) (b). Since $G_{\mathbf{R}} \simeq O(h^{t,t}, 2; \mathbf{R})$, if $P \neq G_{\mathbf{R}}$ then we have one of the following (c), (d), (e).

(c) $n = 1$, $\dim M_1 = 2$.

(d) $n = 1$, $\dim M_1 = 1$.

(e) $n = 2$, $\dim M_1 = \dim M_2/M_1 = 1$.

In the case (d), since $G_{\mathbf{R}} \simeq O(h^{t,t}, 2; \mathbf{R})$ and $\dim M_1 < 2$, there is an element $l \in M_{m-1} \cap H_{0,\mathbf{Q}}$ such that $\langle l, l\rangle_0 < 0$. Fix such l. Take a $\mathbf{Q}$-subspace L of $H_{0,\mathbf{Q}}$ such that

$$\begin{aligned} &M_{m-1} = M_1 \oplus L_{\mathbf{R}} && \text{in the case (c)},\\ &M_{m-1} \cap (l^{\perp}) = M_1 \oplus L_{\mathbf{R}} && \text{in the case (d)},\\ &M_{m-2} = M_2 \oplus L_{\mathbf{R}} && \text{in the case (e)}. \end{aligned}$$

Then, in any case, the restriction of $\langle\ ,\ \rangle_0$ to $L_{\mathbf{C}}$ is non-degenerate. Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w satisfying the following conditions.

$$\begin{aligned} &\operatorname{rank} H_1 = \operatorname{rank} H_0 - \begin{cases} 4 & \text{in the cases (c), (e)},\\ 3 & \text{in the case (d)}, \end{cases}\\ &h_1^{p,q} = 0 \ \text{ for } (p,q) \neq (t,t),\\ &(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L, \text{the restriction of } \langle\ ,\ \rangle_0). \end{aligned}$$

In the case (c), fix also a polarized Hodge structure $(H_2, \langle\ ,\ \rangle_2, F_2)$ of weight 2 whose Hodge type $(h_2^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_2^{p,q} = 1 \ \text{ if } (p,q) = (1,0) \text{ or } (0,1), \quad h_2^{p,q} = 0 \ \text{ otherwise.}$$

Then $(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0)$ is isomorphic to

$$\begin{aligned} &\big(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus H_{1,\mathbf{Q}},\ \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus \langle\ ,\ \rangle_1\big) && \text{in the case (c)},\\ &\big(\operatorname{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}},\ -\langle l, l\rangle_0 \operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}}) \oplus \langle\ ,\ \rangle_1\big) && \text{in the case (d)},\\ &\big(H_{\mathfrak{h},\mathbf{Q}}^{\otimes 2} \oplus H_{1,\mathbf{Q}},\ \langle\ ,\ \rangle_{\mathfrak{h}}^{\otimes 2} \oplus \langle\ ,\ \rangle_1\big) && \text{in the case (e)}. \end{aligned}$$

Here $\operatorname{Sym}^k(\langle\ ,\ \rangle_{\mathfrak{h}})$ is defined by

$$\textstyle(\prod_{1\le j\le k} x_j, \prod_{1\le j\le k} y_j) \mapsto \sum_{\sigma\in\mathfrak{S}_k} \prod_{1\le j\le k} \langle x_j, \sigma y_j\rangle_{\mathfrak{h}},$$

and $-\langle l, l\rangle_0 \operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})$ means $-\langle l, l\rangle_0$ times $\operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})$. We fix this isomorphism and take it as an identification. Let (ρ, φ) be the SL(2)-orbit in the cases (c), (d), and the $\mathrm{SL}(2)^2$-orbit in the case (d), defined respectively by

$$\begin{aligned} &\rho(g) := \rho_{\mathfrak{h}}(g) \otimes \mathrm{id} \oplus \mathrm{id},\quad \varphi(z) := (\varphi_{\mathfrak{h}}(z) \otimes F_2)(1-t) \oplus F_1 && \text{in the case (c)},\\ &\rho(g) := \operatorname{Sym}^2(\rho_{\mathfrak{h}}(g)) \oplus \mathrm{id},\quad \varphi(z) := (\operatorname{Sym}^2(\varphi_{\mathfrak{h}}(z)))(1-t) \oplus F_1 && \text{in the case (d)},\\ &\begin{cases} \rho(g_1, g_2) := \rho_{\mathfrak{h}}(g_1) \otimes \rho_{\mathfrak{h}}(g_2) \oplus \mathrm{id},\\ \varphi(z_1, z_2) := (\varphi_{\mathfrak{h}}(z_1) \otimes \varphi_{\mathfrak{h}}(z_1))(1-t) \oplus F_1, \end{cases} && \text{in the case (e)}. \end{aligned}$$

Then the family W of weight filtrations of (ρ, φ) satisfies $G_{W,\mathbf{R}} = P$ as desired.

Finally we prove

(7) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ are homeomorphisms.

From the coincidence of the tori in the proof of (4), we see that, for $x \in D_{\mathrm{BS}}$, the map from the inverse image of x in $D_{\mathrm{SL}(2),\mathrm{val}}$ to the the inverse image of x in $D_{\mathrm{BS},\mathrm{val}}$ is bijective. Hence $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is bijective. By (3), this map is a homeomorphism. This shows that the bijection $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is also a homeomorphism. □

6.6. *Relation with Satake compactifications.* In the classical situation, we have a compactification $\Gamma\backslash D_S$ of $\Gamma\backslash D$ defined by Satake for a subgroup Γ of $G_{\mathbf{Z}}$ of finite index ([Sa]). The space D_S is the set of all pairs (W, F), where W is a $\mathbf{Q}$-rational increasing filtration of $H_{0,\mathbf{R}}$ and $F = (F_{(j)})_{j\in\mathbf{Z}}$ is a family of decreasing filtrations $F_{(j)}$ of the $\mathbf{C}$-vector spaces $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}^W_j$ $(j \in \mathbf{Z})$, satisfying the following condition (i).

(i) There exist an integer $n \geq 0$ and an element $[\rho, \varphi]$ of $D_{\mathrm{SL}(2),n}$ such that the n-th weight filtration $W(N_1 + \cdots + N_n)$ of $[\rho, \varphi]$ coincides with W, and such that, for some $\tilde{F} \in \varphi(\mathbf{C}^n) \subset \check{D}$, the filtration of $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}^W_j$ induced by $\tilde{F}$ (which is independent of the choice of $\tilde{F}$) coincides with $F_{(j)}$ for any $j \in \mathbf{Z}$.

By composing the evident surjection $D_{\mathrm{SL}(2)} \to D_S$ with the isomorphism $D_{\mathrm{SL}(2)} \simeq D_{\mathrm{BS}}$ in 6.5, we obtain a canonical surjection $D_{\mathrm{BS}} \to D_S$, which was defined by Zucker [Z] by another method.

Propositon 6.7. *Assume one of the following* (i), (ii) *is satisfied for some* $t \in \mathbf{Z}$*.*

(i) $w = 2t + 1$, $h^{t+1,t} \geq 2$, $h^{t+2,t-1} \neq 0$.

(ii) $w = 2t$, $h^{t,t} \geq 3$, $h^{t+1,t-1} \geq 2$, *and there is a* $\mathbf{Q}$*-vector subspace of* $H_{0,\mathbf{Q}}$ *of dimension* 3 *on which the restriction of* $\langle\ ,\ \rangle_0$ *is zero.*

Then there is no continuous map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ *which extends the identity map of* D*.*

Proof. First we consider the case (i). Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w whose Hodge type $(h_1^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t+1,t) \text{ or } (t,t+1), \\ 1 & \text{if } (p,q) = (t+2,t-1) \text{ or } (t-1,t+2), \\ 0 & \text{otherwise.} \end{cases}$$

Fix an isomorphism

$$\big(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} \mathrm{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \mathrm{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}}) \oplus \langle\ ,\ \rangle_1\big) \simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0),$$

where $(H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ is as in 6.2, and take this as an identification. Let (ρ, φ) be the $\mathrm{SL}(2)^2$-orbit defined by

$$\rho(g_1, g_2) := \rho_{\mathfrak{h}}(g_1) \otimes \mathrm{Sym}^2(\rho_{\mathfrak{h}}(g_2)) \oplus \mathrm{id},$$
$$\varphi(z_1, z_2) := \varphi_{\mathfrak{h}}(z_1) \otimes \mathrm{Sym}^2(\varphi_{\mathfrak{h}}(z_2)) \oplus F_1.$$

Then this $\mathrm{SL}(2)^2$-orbit does not satisfy the equivalent conditions in 6.3 (i). In fact, $W_0^{(1)}$ and $W_0^{(2)}$ of (ρ, φ) have no inclusion between them: $(e_1 \otimes e_2^2, 0)$ belongs to $W_0^{(1)}$

but does not belong to $W_0^{(2)}$, and $(e_2 \otimes e_1^2, 0)$ belongs to $W_0^{(2)}$ but does not belong to $W_0^{(1)}$.

Next we consider the case (ii). Let L be a $\mathbf{Q}$-vector subspace of $H_{0,\mathbf{Q}}$ of dimension 3 on which the restriction of $\langle\ ,\ \rangle_0$ is zero. Since $h^{t+1,t-1}+h^{t-1,t+1} > 3$, there is an element $l \in L^\perp \subset H_{0,\mathbf{Q}}$ such that $\langle l, l\rangle_0 < 0$. Let L' be a $\mathbf{Q}$-subspace of $(L+\mathbf{Q}l)^\perp \subset H_{0,\mathbf{Q}}$ such that $L \oplus L' = (L+\mathbf{Q}l)^\perp$. Then we have $\dim_{\mathbf{Q}} L' = \dim_{\mathbf{Q}} H_{0,\mathbf{Q}} - 7$, and the restriction of $\langle\ ,\ \rangle_0$ to $L'_{\mathbf{C}}$ is non-degenerate. Fix polarized Hodge structures $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w and $(H_2, \langle\ ,\ \rangle_2, F_2)$ of weight 1 having the following properties: The Hodge types $(h_j^{p,q})_{p,q\in\mathbf{Z}}$ of $(H_j, \langle\ ,\ \rangle_j, F_j)$ for $j = 1, 2$ are given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t,t), \\ 1 & \text{if } (p,q) = (t+1, t-1) \text{ or } (t-1, t+1), \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2^{p,q} = \begin{cases} 1 & \text{if } (p,q) = (1,0) \text{ or } (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L', \text{the restriction of } \langle\ ,\ \rangle_0).$$

Then there is an isomorphism

$$\begin{aligned}&\big(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus \mathrm{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}},\\ &\qquad \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus (-\langle l, l\rangle_0 \,\mathrm{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})) \oplus \langle\ ,\ \rangle_1\big)\\ &\simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0).\end{aligned}$$

We take this as an identification. Let (ρ, φ) be the $\mathrm{SL}(2)^2$-orbit defined by

$$\begin{aligned}\rho(g_1, g_2) &:= \rho_{\mathfrak{h}}(g_1) \otimes \mathrm{id} \oplus \mathrm{Sym}^2(\rho_{\mathfrak{h}}(g_2)) \oplus \mathrm{id},\\ \varphi(z_1, z_2) &:= (\varphi_{\mathfrak{h}}(z_1) \otimes F_2)(1-t) \oplus \mathrm{Sym}^2(\varphi_{\mathfrak{h}}(z_2))(1-t) \oplus F_1.\end{aligned}$$

Then this $\mathrm{SL}(2)^2$-orbit does not satisfy the equivalent conditions in 6.3 (i). In fact, $W_0^{(1)}$ and $W_0^{(2)}$ of (ρ, φ) have no inclusion between them: For any non-zero element $x \in H_2$, $(e_2 \otimes x, 0, 0)$ belongs to $W_0^{(1)}$ but does not belong to $W_0^{(2)}$, and $(0, e_2^2, 0)$ belongs to $W_0^{(2)}$ but does not belong to $W_0^{(1)}$. $\square$

The following 6.8 and 6.9 show that, for a subgroup Γ of $G_{\mathbf{Z}}$ of finite index, $\Gamma\backslash D_{\mathrm{SL}(2)}$ is not necessarily compact in general, and furthermore not necessarily locally compact in general.

6.8. *Example.* Consider the case $h^{5,0} = h^{0,5} = 1$ and $h^{p,q} = 0$ otherwise. This is satisfied by the polarized Hodge sructure associated to a modular form of weight 6. In this case, D is identified with the upper half plane $\mathfrak{h}$ which is the Griffiths domain of the case $h^{1,0} = h^{0,1} = 1$ and $h^{p,q} = 0$ otherwise. We have $D_{\mathrm{BS}} = \mathfrak{h}_{\mathrm{BS}}$, but $D_{\mathrm{SL}(2)} = \mathfrak{h}$, because of the fact $T_D^h = 0$ and the condition 3.1 (ii).

Propositon 6.9. *Assume one of the following* (i), (ii) *is satisfied for some* $t \in \mathbf{Z}$*.*

(i) $w = 2t+1$, $h^{t+1,t} \neq 0$, *and* $h^{s,w-s} \neq 0$ *for some* $s > t+1$.

(ii) $w = 2t$, $h^{t,t} \geq 2$, $h^{t+1,t-1} \geq 1$, *and there is a* $\mathbf{Q}$*-vector subspace of* $H_{0,\mathbf{Q}}$ *of dimension* 2 *on which the restriction of* $\langle\ ,\ \rangle_0$ *is zero.*

Then $D_{\mathrm{SL}(2)}$ *is not locally compact. More precisely, there are an open set* U *of* D_{BS} *and an ope set* V *of* $D_{\mathrm{SL}(2)}$ *such that the inverse image* U' *of* U *in* $D_{\mathrm{BS,val}}$ *and the inverse image* V' *of* V *in* $D_{\mathrm{SL}(2),\mathrm{val}}$ *satisfy*

$$U' \overset{\sim}{\to} U, \quad V' \overset{\sim}{\to} V, \quad V' = U' \cap D_{\mathrm{SL}(2),\mathrm{val}},$$

and such that there are integers $m > l \geq 0$ *and a commutative diagram*

$$\begin{array}{ccc} U' & \simeq & \mathbf{R}_{\geq 0} \times \mathbf{R}^m \\ \cup & & \cup \\ V' & \simeq & (\mathbf{R}_{>0} \times \mathbf{R}^m) \cup (0 \times \mathbf{R}^l) \\ \cup & & \cup \\ U' \cap D & \simeq & \mathbf{R}_{>0} \times \mathbf{R}^m. \end{array}$$

Note that the subspace $(\mathbf{R}_{>0} \times \mathbf{R}^m) \cup (0 \times \mathbf{R}^l)$ *of* $\mathbf{R}_{\geq 0} \times \mathbf{R}^m$ *is not locally compact.*

Proof. Fix a $\mathbf{Q}$-rational $\mathbf{R}$-subspace L of $H_{0,\mathbf{R}}$ satisfying the following condition. In the case (i), $\dim L = 1$. In the case (ii), $\dim L = 2$ and $\langle\ ,\ \rangle_0$ is zero on L. Let P be the $\mathbf{Q}$-parabolic subgroup $\{g \in G_{\mathbf{R}} \mid gL = L\}$ of $G_{\mathbf{R}}$, and let W be the $\mathbf{Q}$-rational filtration of $H_{0,\mathbf{R}}$ defined by

$$W_{-2} := 0 \subset W_{-1} := L \subset W_0 := L^{\perp} \subset W_1 := H_{0,\mathbf{R}}.$$

Then we have

$$G_{W,\mathbf{R}} = P, \quad D_{\mathrm{BS,val}}(P) \cap D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{SL}(2),\mathrm{val}}(W),$$
$$D_{\mathrm{BS,val}}(P) \overset{\sim}{\to} D_{\mathrm{BS}}(P), \quad D_{\mathrm{SL}(2),\mathrm{val}}(W) \overset{\sim}{\to} D_{\mathrm{SL}(2)}(W).$$

On the other hand, in the case (i), fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w whose Hodge type $(h_1^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 1 & \text{if } (p,q) = (t+1,t) \text{ or } (t,t+1), \\ 0 & \text{otherwise.} \end{cases}$$

In the case (ii), fix polarized Hodge structures $(H_j, \langle\ ,\ \rangle_j, F_j)$ $(j = 1,2)$ of weight w for $j = 1$ and of weight 1 for $j = 2$, respectively, having the following properties: The Hodge types $(h_j^{p,q})_{p,q\in\mathbf{Z}}$ of $(H_j, \langle\ ,\ \rangle_j, F_j)$ for $j = 1,2$ are given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t,t), \\ 1 & \text{if } (p,q) = (t+1,t-1) \text{ or } (t-1,t+1), \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2^{p,q} = \begin{cases} 1 & \text{if } (p,q) = (1,0) \text{ or } (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L', \text{the restriction of } \langle\ ,\ \rangle_0)$$

for some $\mathbf{Q}$-subspace L' of $H_{0,\mathbf{Q}}$ such that $L \oplus L'_{\mathbf{R}} = L^{\perp}$. Then we have

$$(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0) \simeq \begin{cases} (H_{\mathfrak{h},\mathbf{Q}} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \oplus \langle\ ,\ \rangle_1) & \text{in the case (i),} \\ (H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus \langle\ ,\ \rangle_1) & \text{in the case (ii),} \end{cases}$$

where $(H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ is as in 6.2. Fix this isomorphism and take it as an identification. Let (ρ, φ) be the SL(2)-orbit defined by

$$\begin{aligned} &\rho(g) := \rho_{\mathfrak{h}}(g) \oplus \mathrm{id}, \ \ \varphi(z) := \varphi_{\mathfrak{h}}(z) \oplus F_1, && \text{in the case (i),} \\ &\rho(g) := \rho_{\mathfrak{h}}(g) \otimes \mathrm{id} \oplus \mathrm{id}, \ \ \varphi(z) := (\varphi_{\mathfrak{h}}(z) \otimes F_2)(1-t) \oplus F_1, && \text{in the case (ii).} \end{aligned}$$

We claim

$$D_{\mathrm{SL}(2),\mathrm{val}}(W) = D \cup P[\rho, \varphi] \quad \text{in } D_{\mathrm{BS},\mathrm{val}}(P). \tag{1}$$

In fact, let (ρ', φ') be an SL(2)-orbit whose weight filtration is W. Since $\rho|_{\mathbf{G}_m}$ and $\rho'|_{\mathbf{G}_m}$ split W, there is an element $p \in P_u$ such that $\rho'|_{\mathbf{G}_m} = \mathrm{Int}(p)\rho|_{\mathbf{G}_m}$. The Hodge types $\varphi(i)(\mathrm{gr}^W_j)$ and of $\varphi'(i)(\mathrm{gr}^W_j)$ coincide for each j. (In the case (i) (resp. (ii)), it is $(t+1, t+1)$ (resp. $(t+1, t) + (t, t+1)$) for $j = 1$, (t, t) (resp. $(t, t-1) + (t-1, t)$) for $j = -1$, and $(h_1^{p,q})$ for $j = 0$.) Hence by [U1,Prop. 3.16 (iii)], there is an element $q \in G_{\mathbf{R}}$ which commutes with $\rho'|_{\mathbf{G}_m} = \mathrm{Int}(p)\rho|_{\mathbf{G}_m}$ and satisfies $\varphi'(i) = qp\varphi(i)$. By 3.11, we have $\rho' = \mathrm{Int}(qp)\rho$, $\varphi' = qp\varphi$. Since $hp \in P$, this proves (1).

Now take a minimal $\mathbf{R}$-parabolic subgroup P' of $G_{\mathbf{R}}$ contained in P, and take the Iwasawa decomposition of $G_{\mathbf{R}}$ associated to (P', K_r) where $r := \varphi(i)$. Then there is an identification

$$D_{\mathrm{BS},\mathrm{val}}(P) = P'_u \times \mathbf{R}_{\ge 0} \times \mathbf{R}^{s-1}_{>0} \times K_r/K'_r,$$

where $s := \dim S_{P'}$. Put $G_{1,\mathbf{R}} := \mathrm{Aut}(H_{1,\mathbf{R}}, \langle\ ,\ \rangle_1)$. Let K_{F_1} be the maximal compact subgroup of $G_{1,\mathbf{R}}$ associated to F_1, and let $K'_{F_1} := \{g \in G_{1,\mathbf{R}} \,|\, gF_1 = F_1\}$. Regard K_{F_1} as a subgroup of K_r acting trivially on $H_{\mathfrak{h},\mathbf{R}}$ (resp. on $H_{\mathfrak{h},\mathbf{R}} \otimes_{\mathbf{R}} H_{3,\mathbf{R}}$) in the case (i) (resp. (ii)). Then $K'_{F_1} = K' \cap K_{F_1}$. From (1), we obtain

$$D_{\mathrm{SL}(2),\mathrm{val}}(P) = D \cup (P'_u \times \mathbf{R}_{\ge 0} \times \mathbf{R}^{s-1}_{>0} \times K_{F_1}/K'_{F_1}) \quad \text{in } D_{\mathrm{BS},\mathrm{val}}(P).$$

(Note that $D = P'_u \times \mathbf{R}^s_{>0} \times K_r/K'_r$.) Hence, for the proof of 6.9, it is sufficient to prove

$$\dim K_{F_1}/K'_{F_1} < \dim K_r/K'_r. \tag{2}$$

In the case (i), we have, by 6.4 (1),

$$\begin{aligned} \dim K_r - \dim K_{F_1} &= \dim U(g) - \dim U(g-1) = 2g - 1, \\ \dim K'_r - \dim K'_{F_1} &= \dim U(h^{t+1,t}) - \dim U(h^{t+1,t} - 1) = 2h^{t+1,t} - 1. \end{aligned}$$

Since $g > h^{t+1,t}$, we obtain (2). In the case (ii), we have, by 6.4 (1),

$$\begin{aligned}
&\dim K_r - \dim K_{F_1} \\
&\qquad = \dim O(a,\mathbf{R}) + \dim O(b,\mathbf{R}) - \dim O(a-2,\mathbf{R}) - \dim O(b-2,\mathbf{R}) \\
&\qquad = 2(a+b) - 6, \\
&\dim K'_r - \dim K'_{F_1} \\
&\qquad = \dim U(h^{t+1,t-1}) + \dim O(h^{t,t}) - \dim U(h^{t+1,t-1}-1) - \dim O(h^{t,t}-2) \\
&\qquad = 2(h^{t+1,t-1} + h^{t,t}) - 4.
\end{aligned}$$

Since $a + b > h^{t+1,t-1} + h^{t,t} + 1$, we obtain (2). □

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