# IMAGES OF EXTENDED PERIOD MAPS

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Abstract. As a geometric application of polarized log Hodge structures, we show the following. Let $M_H^{\mathrm{sm}}$ be a projective variety which is a compactification of the coarse moduli space of surfaces of general type constructed by Kawamata, Kollár, Shepherd-Barron, Alexeev, Mori, Karu, et al., and let $\Gamma\backslash D_\Sigma$ be a log manifold which is the fine moduli space of polarized log Hodge structures constructed by Kato and Usui. If we take a suitable finite cover $M' \to M_i$ of any irreducible component $M_i$ of $M_H^{\mathrm{sm}}$, and if we assume the existence of a suitable fan $\Sigma$, then there is an extended period map $\psi : M' \to \Gamma\backslash D_\Sigma$ and its image is the analytic subspace associated to a separated compact algebraic space. The point is that, although $\Gamma\backslash D_\Sigma$ is a "log manifold" with slits, the image $\psi(M')$ is not affected by these slits and is a classical familiar object: a separated compact algebraic space.

## Contents



## Introduction

The present paper is the first step of geometric applications of the theory of polarized log Hodge structures.

After fundamental works by Kawamata, Kollár, Shepherd-Barron ([Kaw88], [KSB88], [Ko1,90]), Alexeev constructed finally a compactification of the coarse moduli space of surfaces of general type over **C** with fixed numerical invariants as a projective variety in [Al94]. The construction is based on Mori theory. Then, as we will explain in §3,

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assuming MMP(n+1) below, Karu constructed in [Kar00] a projective variety $M_H^{\mathrm{sm}}$ which coarsely represents a functor $\mathcal{M}_H^{\mathrm{sm}}$ : (schemes/$\mathbf{C}$) $\to$ (sets) defined by

$$\mathcal{M}_H^{\mathrm{sm}}(S) := \begin{pmatrix}\text{isomorphism classes of stable smoothable} \\ n\text{-folds over } S \text{ with Hilbert polynomial } H\end{pmatrix},$$

*i.e.*, $M_H^{\mathrm{sm}}$ is a compactification of the coarse moduli space of $n$-folds of general type over $\mathbf{C}$ with Hilbert polynomial $H$.

*Assumption* MMP(n+1). Let $X$ be a normal irreducible $\mathbf{Q}$-Gorenstein $(n+1)$-fold with at most canonical singularities, and let $f : X \to Y$ be a morphism of one of the following:

(1) $f$ is birational and for some morphism $g : Y \to C$ to a nonsingular curve $C$, the composite $g \circ f : X \to C$ is semistable.

(2) $f$ is a flat projective morphism to a nonsingular curve $Y$ with fibers of general type.

Then the relative canonical ring

$$R_{X/Y} := \bigoplus_{l \geq 0} f_* \mathcal{O}_X(lmK_X) \quad (m : \text{index})$$

is a finitely generated $\mathcal{O}_Y$-algebra. $\square$

The outline of the construction of $M_H^{\mathrm{sm}}$ is as follows. By Matsusaka's Theorem [Mat86], there exists an integer $\nu_0 > 0$ such that all normal varieties with rational Gorenstein singularities, with ample canonical divisor, and with a given Hilbert polynomial, are $\nu_0$-canonically embedded in the same projective space. Let $B_0$ be the closure, with reduced structure, in the Hilbert scheme of the parameter space of these embedded varieties, let $f_0 : X_0 \to B_0$ be the universal family, and construct the following diagram:

$$\begin{array}{ccccccc}
X_0 & \longleftarrow & X & \xrightarrow{\text{rational map}} & \tilde{X} = \operatorname{Proj}_B R_{X/B} & & \\
{\scriptstyle f_0}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle \tilde{f}}\big\downarrow & & \\
B_0 & \longleftarrow & B & = & B & \xrightarrow{\pi} & M_H^{\mathrm{sm}},
\end{array}$$

where $f : X \to B$ is a weakly semistable reduction by Abramovich and Karu ([AK97e]) of $f_0 : X_0 \to B_0$, $R_{X/B}$ is the relative canonical ring which is proved, under Assumption MMP(n+1), to be a finitely generated $\mathcal{O}_B$-algebra, and $\pi : B \to M_H^{\mathrm{sm}}$ is the quotient by the equivalence of isomorphisms.

On the other hand, as we will explain in §2, for a given data $\Phi = \big(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0, \Gamma, \Sigma\big)$ consisting of an integer $w$ called weight, a collection of Hodge numbers $(h^{p,q})_{p+q=w}$, a free $\mathbf{Z}$-module $H_0$ of rank $\sum h^{p,q}$, a non-degenerate biliniear form $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$, a global monodromy $\Gamma \subset G_{\mathbf{Z}} = \operatorname{Aut}(H_0, \langle\ ,\ \rangle_0)$, and a fan $\Sigma$ in $\mathfrak{g}_{\mathbf{Q}} = \operatorname{Lie} G_{\mathbf{Q}}$, which satisfy certain conditions, Kato and the author constructed in [KU1,99], [KU3,03p] a Hausdorff log manifold (§1) $\Gamma \backslash D_\Sigma$ which represents a functor $\underline{\mathrm{PLH}}_\Phi : \mathcal{B}(\log) \to$ (sets) from the category $\mathcal{B}(\log)$ (§1) defined by

$$\underline{\mathrm{PLH}}_\Phi(S) := \begin{pmatrix}\text{isomorphism classes of polarized} \\ \text{Hodge structures on } S \text{ of type } \Phi\end{pmatrix}.$$

To relate $M_H^{\mathrm{sm}}$ and $\Gamma\backslash D_\Sigma$, we apply in §4 a theorem of polarized log Hodge structures for higher direct images by Kato, Matsubara and Nakayama ([KMN02]) to the analytic morphism $f^{\mathrm{an}} : X_i^{\mathrm{an}} \to B_i^{\mathrm{an}}$ induced from the above weakly semistable family $f : X \to B$ when we take any irreducible component $M_i$ of $M_H^{\mathrm{sm}}$, a connected component $B_i \subset B$ which dominates $M_i$, and put $X_i := f^{-1}(B_i) \subset X$. This is possible by Kato's characterization of log smooth morphism ([Kk1,89]), and we have the following log period map $\varphi$:

$$\begin{array}{ccc} B_i^{\mathrm{an}} & \xrightarrow{\varphi} & \Gamma\backslash D_\Sigma. \\ {\scriptstyle\pi}\big\downarrow & & \\ M_i^{\mathrm{an}} & & \end{array}$$

Forgetting log structures, we will investigate the image of $\varphi$ in §5. The following elementary lemma will play the key role:

**Lemma.** *Let $Z$ be an analytic space and $S \subset Z$ a subset endowed with the strong topology in $Z$ (§1). Let $C$ be a compact subset of $S$. Then the topologies on $S$ and on $Z$ induce the same one on $C$.* $\square$

By this lemma and a theorem of Grauert ([GR84]) for coherency, we will first prove that the image $\mathrm{Im}(\varphi) \subset \Gamma\backslash D_\Sigma$ is an analytic subspace, and then prove the main theorem in the present paper:

**Main Theorem.** *Assume* MMP(n+1) *and assume the existence of a fan $\Sigma$ in §4. Let $\pi : B_i^{\mathrm{an}} \xrightarrow{\pi'} M' \xrightarrow{\pi''} M_i^{\mathrm{an}}$ be the Stein factorization of $\pi$, where $\pi'$ is a projective morphism with connected fibers, i.e., $\pi'_*\mathcal{O}_{B_i}^{\mathrm{an}} = \mathcal{O}_{M'}$, and $\pi''$ is finite. Then there exists a unique morphism $\psi : M' \to \Gamma\backslash D_\Sigma$ of ringed spaces such that $\psi\circ\pi' = \varphi$ as morphisms of ringed spaces, and the image* $\mathrm{Im}(\psi) = \mathrm{Im}(\varphi) \subset \Gamma\backslash D_\Sigma$ *is the analytic subspace associated to a compact algebraic space, which is Hausdorff.* $\square$

The important observation is that, although $\Gamma\backslash D_\Sigma$ is a "log manifold" with slits, the image $\psi(M')$ is not affected by these slits and is a classical familiar object: a separated compact algebraic space.

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## §0. Example: degeneration of elliptic curves and the associated PLH

As an introduction, in this section we describe concretely the fundamental objects concerning polarized log Hodge structures, such as fs log structures $M_S$, ringed spaces $(S^{\log}, \mathcal{O}_S^{\log})$ of "real oriented blowing-up", polarized log Hodge structures $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$

of type $\Phi = \big(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0, \Gamma, \Sigma\big)$, log period maps etc., associated to degenerating elliptic curves. For precise definition of these, see [Kk1,89], [KkNc99], [KU1,99], [KU3,03p].

Let $\Delta = \{q \in \mathbf{C} \mid |q| < 1\}$ be a unit disc. Let $X = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1 z_2| < 1\}/\sim$, where $\sim$ is the following equivalence relation. Let $(z_1, z_2), (z_1', z_2') \in \mathbf{C}^2$ with $|z_1 z_2| < 1$, $|z_1' z_2'| < 1$. Put $q = z_1 z_2$. In the case $q \neq 0$, $(z_1, z_2) \sim (z_1', z_2')$ if and only if there exists $n \in \mathbf{Z}$ such that $z_1' = q^n z_1$ and $z_2' = q^{-n} z_2$. In the case $q = 0$, $(z_1, z_2) \sim (z_1', z_2')$ if and only if either $(z_1, z_2) = (z_1', z_2')$ or there exists $t \in \mathbf{C}^\times$ such that $\{(z_1, z_2), (z_1', z_2')\} = \{(t, 0), (0, t^{-1})\}$. Let $f : X \to \Delta$ be the morphism $(z_1, z_2) \mapsto q = z_1 z_2$. For $q \in \Delta - \{0\}$, the fiber $f^{-1}(q)$ of $f$ is identified with the elliptic curve $\mathbf{C}^\times/q^{\mathbf{Z}}$ via $\mathbf{C}^\times/q^{\mathbf{Z}} \ni (z \bmod q^{\mathbf{Z}}) \mapsto (z, qz^{-1}) \in f^{-1}(q)$, and this $f : X \to \Delta$ is a family of degenerating elliptic curves.

Endow $\Delta$ with the fs log structure associated to the divisor $\{0\}$, that is, $M_\Delta := (j_\Delta)_* \mathcal{O}_{\Delta^*}^\times \cap \mathcal{O}_\Delta = \mathcal{O}_\Delta^\times \cdot q^{\mathbf{N}} \hookrightarrow \mathcal{O}_\Delta$, where $\Delta^* := \Delta - \{0\}$ and $j_\Delta : \Delta^* \hookrightarrow \Delta$ is the inclusion. Similarly, endow $X$ with the fs log structure $M_X \hookrightarrow \mathcal{O}_X$ associated to the divisor $f^{-1}(0)$, i.e., $M_X := (j_X)_* \mathcal{O}_{X^*}^\times \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$, where $X^* := X - f^{-1}(0)$ and $j_X : X^* \hookrightarrow X$ is the inclusion. Then, $\mathcal{O}_\Delta \to f_* \mathcal{O}_X$ induces $M_\Delta \to f_* M_X$, and $f : X \to \Delta$ becomes a morphism of fs log analytic spaces over $\mathbf{C}$.

Let $f^{\log} : X^{\log} \to \Delta^{\log}$ be the continuous map induced from $f : X \to \Delta$ by real oriented blowing-up of $X$ (resp. $\Delta$) along the divisor $f^{-1}(0)$ (resp. $\{0\}$). We describe the ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ by using the uniformizing parameters $z_1$, $z_2$. Note that, by construction, $\{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1| \leq 1, |z_2| < 1\}$ is a fundamental domain of $X$, i.e., $X$ is obtained from $\{(z_1, z_2) \in \mathbf{C}^2 \mid |z_j| \leq 1\ (j = 1, 2), |z_1 z_2| \neq 1\}$ by identifying boundary points $(z_1, z_2)$ and $(z_1^2 z_2, z_1^{-1})$ for $|z_1| = 1$. Hence the topological space $X^{\log}$ is obtained from $\{(r_1, r_2, u_1, u_2) \mid 0 \leq r_j \leq 1, u_j \in \mathbf{S}^1\ (j = 1, 2), r_1 r_2 \neq 1\}$ by identifying boundary points $(1, r_2, u_1, u_2)$ and $(r_2, 1, u_1^2 u_2, u_1^{-1})$. The canonical map $\tau_X : X^{\log} \to X$ is given by $(r_1, r_2, u_1, u_2) \mapsto (z_1, z_2) = (r_1 u_1, r_2 u_2)$. The sheaf of rings $\mathcal{O}_X^{\log}$ on $X^{\log}$ is defined so that the stalk at $y \in X^{\log}$ is given by

$$\mathcal{O}_{X,y}^{\log} = \mathcal{O}_{X,x}[\log(z_1), \log(z_2)] \subset (j_X^{\log})_* (\mathcal{O}_{X^*})_y,$$

where $x := \tau_X(y)$, and $j_X^{\log} : X^* \hookrightarrow X^{\log}$ is the evident extension of the inclusion map $j_X : X^* \hookrightarrow X$. Here the $\log(z_j)$ are considered to be taken some branches, and $\mathcal{O}_{X,y}^{\log}$ is regarded as a polynomial ring over $\mathcal{O}_{X,x}$ of $\operatorname{rank}(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$-variables. Similarly, we have that $\Delta^{\log} = \{(s, v) \mid 0 \leq s < 1, v \in \mathbf{S}^1\}$, $\tau_\Delta : \Delta^{\log} \to \Delta$, $(s, v) \mapsto q = sv$, and $\mathcal{O}_{\Delta,p}^{\log} = \mathcal{O}_{\Delta,q}[\log(q)] \subset (j_\Delta^{\log})_* (\mathcal{O}_{\Delta^*})_p$ for $p \in \Delta^{\log}$ and $q = \tau_\Delta(p)$ where $j_\Delta^{\log} : \Delta^* \hookrightarrow \Delta^{\log}$. The continuous map $f^{\log} : X^{\log} \to \Delta^{\log}$ is given by $(r_1, r_2, u_1, u_2) \mapsto (s, v) = (r_1 r_2, u_1 u_2)$, and we have a commutative diagram of ringed spaces:

$$\begin{array}{ccc} X^{\log} & \xrightarrow{\tau_X} & X \\ {\scriptstyle f^{\log}}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \Delta^{\log} & \xrightarrow{\tau_\Delta} & \Delta. \end{array}$$

The PLH $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ on $\Delta$ of type $\Phi$ associated to $f : X \to \Delta$ is described explicitly in the following way.

The sheaf $H_{\mathbf{Z}} = R^1(f^{\log})_*\mathbf{Z}$ is locally isomorphic to $\mathbf{Z}^2$. For $q \in \Delta^*$, we have a canonical isomorphism

$$H_{\mathbf{Z},q} \simeq \mathrm{Hom}(H_1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z}), \mathbf{Z}).$$

Let $|\Delta| := \{s \in \mathbf{R} \mid 0 \le s < 1\}$. Then $\Delta^{\log} = |\Delta| \times \mathbf{S}^1$. We have a global section $e_1$ of $H_{\mathbf{Z}}$ and a basis $(e_1, e_2(a))$ of the restriction of $H_{\mathbf{Z}}$ to $(|\Delta|, \exp(2\pi i a)) \subset \Delta^{\log}$ characterized as follows. For $s \in |\Delta^*| := |\Delta| - \{0\}$ and $a \in \mathbf{R}$, via the isomorphisms

$$\exp : \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}) \xrightarrow{\sim} \mathbf{C}^\times/q^{\mathbf{Z}}, \quad H_1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z}) \simeq \mathbf{Z}\tau + \mathbf{Z},$$

where $q = s \cdot \exp(2\pi i a)$, $\tau = a + iy$ with $y = -\frac{\log(s)}{2\pi}$, let $(e_1^*(s,a), e_2^*(s,a))$ be the $\mathbf{Z}$-basis of $H_1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z})$ corresponding to the $\mathbf{Z}$-basis $(\tau, 1)$ of $\mathbf{Z}\tau + \mathbf{Z}$. For each $a \in \mathbf{R}$, the restriction of $H_{\mathbf{Z}}$ to $(|\Delta|, \exp(2\pi i a)) \subset \Delta^{\log}$ is a constant sheaf and $(e_1, e_2(a))$ is a basis of this constant sheaf characterized by the property that the germ of $(e_1, e_2(a))$ at $s \cdot \exp(2\pi i a)$ is the dual basis of $(e_1^*(s,a), e_2^*(s,a))$ for any $s \in |\Delta^*|$.

Let 0 be the origin of $\Delta$ endowed with the inverse image of the log structure $M_\Delta$. Then the inclusion map $0^{\log} \to \Delta^{\log}$ induces an isomorphism $\pi_1(0^{\log}) \xrightarrow{\sim} \pi_1(\Delta^{\log})$. Let $\gamma \in \pi_1(\Delta^{\log})$ be the image of the generator of $\pi_1(0^{\log})^+ \simeq \mathbf{N}$. Then

$$\gamma(e_1) = e_1, \quad \gamma(e_2(a)) = e_2(a+1) = e_2(a) - e_1. \tag{1}$$

This follows from the fact that $e_1$ is a global section of $H_{\mathbf{Z}}$, from

$$\gamma(e_1^*(s,a)) = e_1^*(s, a+1) = e_1^*(s,a) + e_2^*(s,a), \quad \gamma(e_2^*(s,a)) = e_2^*(s,a)),$$

and from the fact that $\gamma(e_2(a))$ sends $\gamma(e_1^*(s,a))$ to 0 and $\gamma(e_2^*(s,a))$ to 1.

The Hodge filtration $F$ of $\mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ is given as $F^0 = \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$, $F^2 = 0$ and $F^1$ described as follows: For $s \in |\Delta|$ and $a \in \mathbf{R}$, the stalk of $F^1$ at $p = (s, \exp(2\pi i a))$ is

$$F_p^1 = \mathcal{O}_{\Delta,p}^{\log}(\ell_{s,a} e_1 + 2\pi i e_2(a)), \tag{2}$$

where $\ell_{s,a}$ is the branch of $\log(q)$ whose value at $2\pi i\tau \in (\mathfrak{h} \cup (\mathbf{R} \times i\infty))$ is $2\pi i\tau$ where $\tau = a + \frac{\log(s)}{2\pi i}$. In fact, $\mathcal{M}^1 = f_*(\omega^1_{X/\Delta})$ is a free $\mathcal{O}_\Delta$-module of rank 1 generated by $d\log(z_1)$, and we have

$$F^1 = \mathcal{O}_\Delta^{\log} d\log(z_1). \tag{3}$$

In the case $s \neq 0$, the integration of $d\log(z_1)$ along $e_1^*(s,a)$ (resp. $e_2^*(s,a)$) coincides with $2\pi i\tau$ (resp. $2\pi i$). This shows that

$$d\log(z_1) = \ell_{s,a} e_1 + 2\pi i e_2(a) \quad \text{in } (\mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}})_p \tag{4}$$

if $s \neq 0$. Since $\mathcal{O}_\Delta^{\log} \to (j^{\log})_*\mathcal{O}_{\Delta^*}$ is injective, (4) holds also in the case $s = 0$.

$\langle\ ,\ \rangle$ is the anti-symmetric form $H_{\mathbf{Z}} \times H_{\mathbf{Z}} \to \mathbf{Z}$ characterized by $\langle e_2(a), e_1 \rangle = 1$.

$\Phi = \big(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0, \Gamma, \Sigma\big)$ is given as follows. Fix a point $q_0 \in \Delta^*$ and denote by $(e_1, e_2)$ the germ of $(e_1, e_2(a))$ at $q_0$.

$$\begin{aligned} &w = 1, \quad h^{1,0} = h^{0,1} = 1 \text{ and } h^{p,q} = 0 \text{ otherwise}, \\ &H_0 := H_{\mathbf{Z},q_0}, \quad \langle\ ,\ \rangle_0 := \langle\ ,\ \rangle_{q_0}, \\ &\Gamma = \gamma^{\mathbf{Z}} = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}, \quad \Sigma = \left\{ \{0\}, \begin{pmatrix} 0 & \mathbf{R}_{\ge 0} \\ 0 & 0 \end{pmatrix} \right\}. \end{aligned}$$

We define a global section of the sheaf $\Gamma\backslash\, \underline{\mathrm{Isom}}\,\big((H_{\mathbf{Z}}, \langle\ ,\ \rangle), (H_0, \langle\ ,\ \rangle_0)\big)$ on $\Delta^{\log}$ by sending $e_1$ to $e_1$ and $e_2(a)$ to $e_2$.

Then $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ is a PLH of type $\Phi$. The corresponding log period map $\varphi : \Delta \to \Gamma\backslash D_\Sigma$ and the map $\varphi^{\log} : \Delta^{\log} \to \Gamma\backslash D_\Sigma^\sharp$ are the isomorphisms which send $q \in \Delta^*$ to $(\frac{1}{2\pi i}\log(q) \mod \mathbf{Z}) \in \Gamma\backslash D = \mathbf{Z}\backslash\mathfrak{h}$.

## §1. Log manifolds and the associated ringed spaces

In this section, we explane rather unfamilar notion "log manifold" which is introduced in [KU3,03p] to describe the fine moduli spaces $\Gamma\backslash D_\Sigma$ of polarized log Hodge structures of type $\Phi$. The main feature of log maniflod is to allow slits and hence not locally compact in general. We also recall the definition of the associated ringed spaces $(S^{\log}, \mathcal{O}_S^{\log})$, which is introduced by Kato and Nakayama for fs log analytic spaces $S$ ([KkNc99]) and generalized for an object $S$ in $\mathcal{B}(\log)$ ([KU3,03p]).

The underlying local ringed space over $\mathbf{C}$ of $\Gamma\backslash D_\Sigma$ in §2 is not necessarily an analytic space in general. Sometimes, it can be something like

$$S := \{(x, y) \in \mathbf{C}^2 \mid x \neq 0\} \cup \{(0, 0)\}$$

endowed with a topology which is stronger than the topology as a subspace of $\mathbf{C}^2$, called the "strong topology". We first describe the idea of log manifold by using the above example $S \subset \mathbf{C}^2$. Let $Z = \mathbf{C}^2$ with coordinate functions $x$, $y$, and endow $Z$ with the log structure associated to $\mathbf{N} \to \mathcal{O}_Z$, $n \mapsto x^n$. Then, the sheaf $\omega_Z^1$ of log differential 1-forms on $Z$ (= the sheaf of differential 1-forms with log poles along $x = 0$) is a free $\mathcal{O}_Z$-module with basis $(d\log(x), dy)$. For each $z \in Z$, let $\omega_z^1$ be the module of log differential 1-forms on the point $z$ which is regarded as an fs log analytic space endowed with the ring $\mathbf{C}$ and with the inverse image of $M_Z$. Then, if $z$ does not belong to the part $x = 0$ of $Z$, $z$ is just a usual point $\mathrm{Spec}(\mathbf{C})$ with the trivial log structure, and $\omega_z^1 = 0$. If $z$ is in the part $x = 0$, $z$ is a point $\mathrm{Spec}(\mathbf{C})$ with the induced log structure $M_z = \bigsqcup_{n\geq 0} \mathbf{C}^\times x^n \simeq \mathbf{C}^\times \times \mathbf{N}$, and hence $\omega_z^1$ is a one-dimensional $\mathbf{C}$-vector space generated by $d\log(x)$. Thus $\omega_z^1$ is not equal to the fiber of $\omega_Z^1$ at $z$ which is a 2-dimensional $\mathbf{C}$-vector space with basis $(d\log(x), dy)$. Now the the above set $S$ has a presentation

$$S = \{z \in Z \mid \text{the image of } y\,d\log(x) \text{ in } \omega_z^1 \text{ is zero}\}.$$

Recall that the zeros of a holomorphic function on $Z$ form a closed analytic subset of $Z$. Here we discovered that $S$ is the set of "zeros" of the differential form $y\,d\log(x)$ on $Z$, but the meaning of "zero" is not that the image of $y\,d\log(x)$ in the fiber of $\omega_Z^1$ is zero (the latter "zeros" form the closed analytic subset $y = 0$ of $Z$). The "zeros in the new sense" of a differential 1-form with log poles is the idea of a "log manifold".

The precise definition is as follows. In [Kk1], Kato introduced a notion of "log smooth" fs log analytic spaces and characterized that. An fs log analytic space is *log smooth* if and only if it has an open covering whose each member is isomorphic to an open set $Z$ of a toric variety $\mathrm{Spec}(\mathbf{C}[\mathcal{S}])^{\mathrm{an}}$ ($\mathcal{S}$ an fs monoid) whose log structure is associated to the inclusion homomorphism $\mathcal{S} \hookrightarrow \mathbf{C}[\mathcal{S}]$. For example, $Z = \mathbf{C}^2$ with the above log structure is log smooth.

By a *log manifold* $S$, we mean a log local ringed space over $\mathbf{C}$ which has an open covering $(U_\lambda)_\lambda$ with the following property: For each $\lambda$, there exist a log smooth fs log analytic space $Z_\lambda$, a finite subset $I_\lambda$ of $\Gamma(Z_\lambda, \omega^1_{Z_\lambda})$, and an isomorphism of log local ringed spaces over $\mathbf{C}$ between $U_\lambda$ and an open subset of

$$S_\lambda = \{z \in Z_\lambda \mid \text{the image of } I_\lambda \text{ in } \omega^1_z \text{ is zero}\},$$

where $S_\lambda$ is endowed with the strong topology in $Z_\lambda$ and with the inverse images of $\mathcal{O}_{Z_\lambda}$ and $M_{Z_\lambda}$. Here, for a subset $E$ of an analytic space $X$, the *strong topology* on $E$ in $X$ is the strongest topology on $E$ among those in which, for any analytic morphism $f : Y \to X$ from any analytic space $Y$ with $f(Y) \subset E$, the map $f : Y \to E$ is continuous.

In the theoretical treatment of polarized log Hodge structures, the following category $\mathcal{B}(\log)$ is the most convenient one, which contains the category of fs log analytic spaces and the category of log manifolds (see [KU3,03p]).

Let $\mathcal{B}$ be the category of local ringed spaces $X$ over $\mathbf{C}$ which have an open covering $(U_\lambda)_\lambda$ satisfying the following condition: For each $\lambda$, there exist an analytic space $Z_\lambda$ and a subset $S_\lambda$ of $Z_\lambda$ such that, as local ringed spaces over $\mathbf{C}$, $U_\lambda$ is isomorphic to $S_\lambda$ which is endowed with the strong topology in $Z_\lambda$ and the inverse image of $\mathcal{O}_{Z_\lambda}$.

Let $\mathcal{B}(\log)$ be the category of objects of $\mathcal{B}$ endowed with an fs log structure.

As an example in §0, we can associate the ringed space $(S^{\log}, \mathcal{O}^{\log}_S)$ to an object $S$ in $\mathcal{B}(\log)$. We recall here the precise definition for the readers' convenience ([KkNc99], [KU3,03p]).

Let $S \in \mathcal{B}(\log)$. As a set, $S^{\log}$ is defined to be the set of all pairs $(s, h)$ consisting of a point $s \in S$ and an *argument function* $h$ which is a homomorphism $M_{S,s} \to \mathbf{S}^1$ whose restriction to $\mathcal{O}^\times_{S,s}$ is $u \mapsto u(s)/|u(s)|$. Here $\mathbf{S}^1 := \{z \in \mathbf{C} \mid |z| = 1\}$.

The topology of $S^{\log}$ is defined as follows. We work locally on $S$. Take a chart $\mathcal{S} \to M_S$, then we have an injective map

$$S^{\log} \hookrightarrow S \times \mathrm{Hom}(\mathcal{S}^{\mathrm{gp}}, \mathbf{S}^1), \ \ (s, h) \mapsto (s, h_{\mathcal{S}}),$$

where $h_{\mathcal{S}}$ denotes the composite map $\mathcal{S}^{\mathrm{gp}} \to M^{\mathrm{gp}}_{S,s} \to \mathbf{S}^1$. We endow $S^{\log}$ with the topology as a subset of $S \times \mathrm{Hom}(\mathcal{S}^{\mathrm{gp}}, \mathbf{S}^1)$. This topology is independent of the choice of chart, and hence is globally well-defined. The canonical map

$$\tau = \tau_S : S^{\log} \to S, \ \ (s, h) \mapsto s,$$

is surjective, continuous and proper. For $s \in S$, the inverse image $\tau^{-1}(s)$ is homeomorphic to $(\mathbf{S}^1)^r$ where $r := \mathrm{rank}_{\mathbf{Z}}(M^{\mathrm{gp}}_S/\mathcal{O}^\times_S)_s$.

We define the sheaf of rings $\mathcal{O}^{\log}_S$ on $S^{\log}$ as follows. We define first the *sheaf of logarithms* $\mathcal{L}$ of $M^{\mathrm{gp}}_S$ on $S^{\log}$ as the fiber product of

$$\begin{array}{ccc} & & \tau^{-1}(M^{\mathrm{gp}}_S) \\ & & \downarrow \\ \mathrm{Cont}(\ \ , i\mathbf{R}) & \xrightarrow{\exp} & \mathrm{Cont}(\ \ , \mathbf{S}^1), \end{array}$$

where Cont( $\ ,Y$), for a topological space $Y$, denotes the sheaf on $S^{\log}$ of continuous maps to $Y$, and $\tau^{-1}(M_S^{\mathrm{gp}}) \to \mathrm{Cont}(\ \ , \mathbf{S}^1)$ comes from the definition of $S^{\log}$. We define

$$\mathcal{O}_S^{\log} := (\tau^{-1}(\mathcal{O}_S) \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L}))/\mathfrak{a},$$

where $\mathrm{Sym}_{\mathbf{Z}}(\mathcal{L})$ denotes the symmetric algebra of $\mathcal{L}$ over $\mathbf{Z}$, and $\mathfrak{a}$ is the ideal of $\tau^{-1}(\mathcal{O}_S) \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L})$ generated by the image of

$$\tau^{-1}(\mathcal{O}_S) \to \tau^{-1}(\mathcal{O}_S) \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L}), \quad f \mapsto f \otimes 1 - 1 \otimes \iota(f).$$

Here the map $\iota : \tau^{-1}(\mathcal{O}_S) \to \mathcal{L}$ is the one induced by

$$\begin{aligned} &\tau^{-1}(\mathcal{O}_S) \to \mathrm{Cont}(\ \ , i\mathbf{R}), \quad f \mapsto \tfrac{1}{2}(f - \bar{f}), \quad \text{and} \\ &\tau^{-1}(\mathcal{O}_S) \xrightarrow{\exp} \tau^{-1}(\mathcal{O}_S^{\times}) \subset \tau^{-1}(M_S^{\mathrm{gp}}). \end{aligned}$$

In the above, $\bar{\ }$ means the complex conjugation.

We denote the projection $\mathcal{L} \to \tau^{-1}(M_S^{\mathrm{gp}})$ by exp, and the inverse $\tau^{-1}(M_S^{\mathrm{gp}}) \to \mathcal{L}/(2\pi i\mathbf{Z})$ by log. Then we have a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z} \arrow[r, "2\pi i"] \arrow[d, equal] & \tau^{-1}(\mathcal{O}_S) \arrow[r, "\exp"] \arrow[d, "\iota"', "\cap"] & \tau^{-1}(\mathcal{O}_S^{\times}) \arrow[r] \arrow[d, "\cap"] & 1 \\ 0 \arrow[r] & \mathbf{Z} \arrow[r, "2\pi i"] & \mathcal{L} \arrow[r, "\exp"] & \tau^{-1}(M_S^{\mathrm{gp}}) \arrow[r] & 1. \end{tikzcd}$$

A morphism $f : S \to T$ in $\mathcal{B}(\log)$ induces a morphism

$$f^{\log} : (S^{\log}, \mathcal{O}_S^{\log}) \to (T^{\log}, \mathcal{O}_T^{\log})$$

of ringed spaces over $\mathbf{C}$ in the evident way.

For $t \in S^{\log}$, the stalk $\mathcal{O}_{S,t}^{\log}$ is as follows. Let $s = \tau(t) \in S$ and $r = \mathrm{rank}_{\mathbf{Z}}(M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s$. Let $(\ell_j)_{1 \le j \le r}$ be a family of elements of $\mathcal{L}_t$ whose images in $(M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s$ form a system of free generators. Then we have an isomorphism of $\mathcal{O}_{S,s}$-algebras

$$\mathcal{O}_{S,s}[X_1, \dots, X_r] \xrightarrow{\sim} \mathcal{O}_{S,t}^{\log}, \quad X_j \mapsto \ell_j,$$

where the $X_j$ are $r$ indeterminates. Note that $\mathcal{O}_{S,t}^{\log}$ is *not* a local ring if $r \ge 1$.

Let $s \in S$ and let $M_s \to \mathcal{O}_s = \mathbf{C}$ be the fs log structure on $s$ induced from that on $S$. Let $(s^{\log}, \mathcal{O}_s^{\log})$ be the associated ringed space. For $t \in s^{\log} = \tau^{-1}(s)$, we define

$$\mathrm{sp}(t) := \mathrm{Hom}_{\mathbf{C}\text{-alg}}(\mathcal{O}_{s,t}^{\log}, \mathbf{C}).$$

## §2. Polarized log Hodge structures

In this section, we recall the definition of the functor $\underline{\mathrm{PLH}}_\Phi : \mathcal{B}(\log) \to (\text{sets})$ of polarized log Hodge structures of type $\Phi$, and state the main result in [KU3,03p], that is, the functor $\underline{\mathrm{PLH}}_\Phi$ is represented by a Hausdorff log manifold $\Gamma\backslash D_\Sigma$.

Let $\Phi = \big(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0, \Gamma, \Sigma\big)$ be a data consisting of an integer $w$ (called weight), a collection of non-negatgive integers $(h^{p,q})_{p+q=w}$ satisfying $h^{p,q} = h^{q,p}$ for any $p$, $q$, and $h^{p,q} = 0$ except finite of them (called Hodge numbers), $H_0$ is a free $\mathbf{Z}$-module of rank $\sum h^{p,q}$, a non-degenerate bilinear form $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$ which is symmetric for even $w$ and anti-symmetric for odd $w$ (called polarization), a neat subgroup $\Gamma$ of $G_\mathbf{Z} = \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0)$ (called global monodromy), and a fan $\Sigma$ in $\mathfrak{g}_\mathbf{Q} = \mathrm{Lie}\, G_\mathbf{Q}$, which are *strongly compatible*; i.e., if $\gamma \in \Gamma$ and $\sigma \in \Sigma$ then $\gamma\sigma\gamma^{-1} \in \Sigma$, and $\sigma$ is generated by $\log(\exp(\sigma) \cap \Gamma)$ as a cone over $\mathbf{R}_{\geq 0}$.

Define a functor $\underline{\mathrm{PLH}}_\Phi : \mathcal{B}(\log) \to (\text{sets})$ by

$$\underline{\mathrm{PLH}}_\Phi(S) := (\text{isomorphism classes of PLH on } S \text{ of type } \Phi).$$

Here PLH is the abbreviation of polarized log Hodge structure. The definition of PLH of type $\Phi$ is as follows.

**Definition** ([KU3,03p]). *Let $S$ be an object of $\mathcal{B}(\log)$. A* polarized log Hodge structure *on $S$ of type $\Phi$* (PLH on $S$ of type $\Phi$, *for short) is a 4-tuple $(H_\mathbf{Z}, \langle\ ,\ \rangle, F, \mu)$ on $S^{\log}$ consisting of a locally constant sheaf of free $\mathbf{Z}$-modules $H_\mathbf{Z}$ of rank $\sum_{p,q} h^{p,q}$ on $S^{\log}$, of a non-degenerate $\mathbf{Q}$-bilinear form $\langle\ ,\ \rangle$ on $H_\mathbf{Q} = \mathbf{Q} \otimes H_\mathbf{Z}$, symmetric for even $w$ and anti-symmetric for odd $w$, of a decreasing filtration $F$ of the $\mathcal{O}_S^{\log}$-module $\mathcal{O}_S^{\log} \otimes H_\mathbf{Z}$, and of a global section of the sheaf $\Gamma\backslash\, \underline{\mathrm{Isom}}((H_\mathbf{Z}, \langle\ ,\ \rangle), (H_0, \langle\ ,\ \rangle_0))$ on $S^{\log}$, which satisfy the following conditions.*

(1) *There exist an $\mathcal{O}_S$-module $\mathcal{M}$ on $S$ and a decreasing filtration $(\mathcal{M}^p)_{p\in\mathbf{Z}}$ of $\mathcal{M}$ by $\mathcal{O}_S$-submodules such that $\mathcal{M}$, $\mathcal{M}^p$ and $\mathcal{M}/\mathcal{M}^p$ are locally free of finite rank, $\mathcal{O}_S^{\log} \otimes_\mathbf{Z} H_\mathbf{Z} = \tau^*(\mathcal{M})$ and $F^p = \tau^*(\mathcal{M}^p)$, and $\mathrm{rank}_{\mathcal{O}_S}(\mathcal{M}^p) = \sum_{r\geq p} h^{r,w-r}$ for all $p$.*
*Here $\tau^*$ is the module theoretic inverse image $\mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\ )$.*
(2) *$\langle F^p, F^q\rangle = 0$ if $p + q > w$.*

(3) *For any $s \in S$ and any $t \in s^{\log} = \tau^{-1}(s)$, if $\tilde{\mu}_t : (H_{\mathbf{Z},t}, \langle\ ,\ \rangle_t) \xrightarrow{\sim} (H_0, \langle\ ,\ \rangle_0)$ denotes a representative of the stalk of $\mu$ at $t$, then there exists $\sigma \in \Sigma$ such that the image of the composite map*

$$\pi_1^-(s^{\log}) \hookrightarrow \pi_1(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},t}, \langle\ ,\ \rangle_t) \overset{\text{by } \tilde{\mu}_t}{\xrightarrow{\sim}} \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0)$$

*is contained in $\exp(\sigma)$.*
(4) *Let $s$, $t$, $\tilde{\mu}_t$ be as in (3). Take the smallest $\sigma \in \Sigma$ having the property (3). Then $\{\tilde{\mu}_t(F(c)) \mid c \in \mathrm{sp}(t)\}$ is a $\sigma$-nilpotent orbit; i.e., for any fixed $c_0 \in \mathrm{sp}(t)$, $\{\tilde{\mu}_t(F(c)) \mid c \in \mathrm{sp}(t)\} = \exp(\sigma_\mathbf{C})\tilde{\mu}_t(F(c_0))$, $N\tilde{\mu}_t(F(c_0)^p) \subset \tilde{\mu}_t(F(c_0)^{p-1})$ for all $p$ and $N \in \sigma$, and $\exp(\sum_{1\leq j\leq r} iy_jN_j)\tilde{\mu}_t(F(c_0)) \in D$ for $y_j \gg 0$. Here $\sigma = \sum_{1\leq j\leq r} \mathbf{R}_{\geq 0}N_j$.* □

$D$ in the above (4) is the classifying space of polarized Hodge structures associated to $\big(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0\big)$.

Then we have:

**Theorem** ([KU3,03p]). *The functor* $\underline{\mathrm{PLH}}_\Phi$ *is represented by a Hausdorff log manifold* $\Gamma\backslash D_\Sigma$. $\square$

## §3. Stable smoothable $n$-folds

In this section, we recall the definition of the functor $\mathcal{M}_H^{\mathrm{sm}}$ : (schemes/$\mathbf{C}$) $\to$ (sets) of stable smoothable $n$-folds with Hilbert polynomial $H$ introduced by Kollár ([Ko1,90]), and state the main result of Alexeev and Karu ([Al94], [Kar00]), that is, under Assumption MMP(n+1), the functor $\mathcal{M}_H^{\mathrm{sm}}$ is bounded and hence coarsely represented by a projective scheme $M_H^{\mathrm{sm}}$.

Define a functor $\mathcal{M}_H^{\mathrm{sm}}$ : (schemes/$\mathbf{C}$) $\to$ (sets) by

$$\mathcal{M}_H^{\mathrm{sm}}(S) := \begin{pmatrix} \text{isomorphism classes of stable smoothable} \\ n\text{-folds over } S \text{ with Hilbert polynomial } H \end{pmatrix}.$$

Here "stable smoothable $n$-fold over $S$ with Hilbert polynomial $H$" is defined as follows:

**Definition** ([KSB88], [Ko1,90], [Kar00]).

(1) *A* stable *$n$-fold $X$ with Hilbert polynomial $H$ is a connected projective variety (not necessarily irreducible) of dimension $n$ over* $\mathbf{C}$ *with "slc (= semi-log-canonical) singularities" and ample canonical divisor $K_X$ such that $h^0(X, lmK_X) = H(lm)$ for $\forall l \gg 0$, where $m$ is the index of $X$ (i.e., the least positive integer $m$ such that $mK_X$ is a Cartier divisor).*

*A* stable *$n$-fold $X$ over $S$ with Hilbert polynomial $H$ is a flat projective* $\mathbf{Q}$*-Gorenstein morphism $f : X \to S$ such that every geometric fiber $X_s$ is a stable $n$-fold with Hilbert polynomial $H$ and that $\mathcal{O}_{X_s}(lmK_{X_s}) \simeq \mathcal{O}_X(lmK_{X/S})|_{X_s}$ for $\forall l \gg 0$.*

(2) *A stable $n$-fold $X$ is* smoothable *if there exists a flat projective* $\mathbf{Q}$*-Gorenstein one-parameter family $\pi : Y \to C$ of stable $n$-folds such that the special fiber of $\pi$ is $Y_0 \simeq X$ and the general fiber $Y_t$ is a normal $n$-fold with at most rational Gorenstein singularities.*

*A stable $n$-fold $f : X \to S$ over $S$ is* smoothable *if the following condition is satisfied for all points $s \in S$ : Let $\pi : Y \to D$ be a versal deformation of the fiber $X_s \simeq \pi^{-1}(0)$; so we get a morphism $g : (s \in S) \to (0 \in D)$ of formal germs. Let $D^{\mathrm{st}}$ be a locally closed subscheme of $D$ over which $\pi$ is a stable family (hence the morphism $g$ factors through $D^{\mathrm{st}}$), and let $D^{\mathrm{sm}}$ be the open subscheme of $D^{\mathrm{st}}$ over which $\pi$ is a family of normal $n$-folds with at most rational Gorenstein singularities. Then the morphism $g$ has to factor through the closure $\overline{D^{\mathrm{sm}}} \subset D^{\mathrm{st}}$.* $\square$

Here "slc singularities" are defined as follows:

**Definition** ([Kaw88], [KSB88], [Kar00]). *A* $\mathbf{Q}$*-Gorenstein variety $X$ has* slc (= semi-log-canonical) *singularities if*

(1) *$X$ has Serre's property $S_2$;*

(2) *$X$ has normal crossing singularities in codimension 1; and*

(3) *for any birational morphism $f : Y \to X$ from a normal* $\mathbf{Q}$*-Gorenstein variety $Y$, we have*

$$K_Y \equiv f^*K_X + \textstyle\sum a_i E_i \quad \text{with } \forall a_i \geq -1.$$

((3) $\iff$ *the pair* $(X^\nu, \mathrm{cond}(\nu))$ *has log canonical singularities, where* $X^\nu \to X$ *is the normalization and* $\mathrm{cond}(\nu)$ *is the inverse image of the double divisor with reduced structure.*) $\square$

Slc surface singularities are completely classified in [Kaw88] and [KSB88], which are roughly as follows:
semismooth (i.e., smooth, normal crossing double, pinch) points,
DuVal singularities,
simple elliptic singularities,
cusps and degenerate cusps,
quotients of the above by certain cyclic group actions. $\square$

Then we have:

**Theorem** ([Al94], [Kar00]). *Assuming* MMP(n+1)*, there exists a family* $\tilde{X} \to B$ *over a projective scheme* $B$ *in* $\mathcal{M}_H^{\mathrm{sm}}(B)$ *whose geometric fibers include all stable smoothable n-folds with Hilbert polynomial* $H$. $\square$

**Corollary** ([Ko1,90], [Al94], [Kar00]). *Assuming* MMP(n+1)*, the functor* $\mathcal{M}_H^{\mathrm{sm}}$ *is coarsely represented by a separated projective scheme* $M_H^{\mathrm{sm}}$*, i.e., there exists a morphism of functors* $\theta : \mathcal{M}_H^{\mathrm{sm}} \to \mathrm{Mor}(\ , M_H^{\mathrm{sm}})$ *which satisfies the following two conditions:*
(1) $\theta(\mathrm{Spec}(\mathbf{C})) : \mathcal{M}_H^{\mathrm{sm}}(\mathrm{Spec}(\mathbf{C})) \xrightarrow{\sim} \mathrm{Mor}(\mathrm{Spec}(\mathbf{C}), M_H^{\mathrm{sm}})$ *is bijective.*
(2) *For any scheme* $B$ *and any morphism of functors* $\chi : \mathcal{M}_H^{\mathrm{sm}} \to \mathrm{Mor}(\ , B)$*, there exists a unique morphsim of functors* $\psi : \mathrm{Mor}(\ , M_H^{\mathrm{sm}}) \to \mathrm{Mor}(\ , B)$ *satisfying* $\chi = \psi \circ \theta$. $\square$

Here "Assumption MMP(n+1)" is as follows:

*Assumption* MMP(n+1). Let $X$ be a normal irreducible $\mathbf{Q}$-Gorenstein $(n+1)$-fold with at most canonical singularities, and let $f : X \to Y$ be a morphism of one of the following:

(1) $f$ is birational and for some morphism $g : Y \to C$ to a nonsingular curve $C$, the composite $g \circ f : X \to C$ is semistable.

(2) $f$ is a flat projective morphism to a nonsingular curve $Y$ with fibers of general type.

Then the relative canonical ring

$$R_{X/Y} := \bigoplus_{l \geq 0} f_* \mathcal{O}_X(lmK_X) \quad (m : \mathrm{index})$$

is a finitely generated $\mathcal{O}_Y$-algebra. $\square$

## §4. Weakly semistable reduction and the associated PLH

In this section, we recall the weakly semistable reduction theorem of Abramovich and Karu ([AK97e]), and apply a theorem for direct images of Kato, Matsubara and Nakayama ([KMN02]) to a weakly semistable family $f : X \to B$ which appears in the construction of the family $\tilde{f} : \tilde{X} \to B$ showing the boundedness of the functor $\mathcal{M}_H^{\mathrm{sm}}$ (Theorem in §3) in [Kar00]. We thus have an extended period map $\varphi : B_i^{\mathrm{an}} \to \Gamma \backslash D_\Sigma$.

**Definition** ([AK97e], [Kar00]). *A toroidal morphism $f : (U_X \subset X) \to (U_B \subset B)$ without horizontal divisors is* weakly semistable *if*

(1) *$f$ is equidimensional;*
(2) *$f$ has reduced fibers; and*
(3) *$B$ is nonsingular (hence $B - U_B$ is a reduced normal crossing divisor* (cf. [Kf1,96, 4.9])).

*The morphism $f$ is* semistable *if $X$ is also nonsingular.* $\square$

**Theorem** ([AK97e]). *Let $X \to B$ be a surjective morphism of projective varieties with geometrically integral generic fiber. Then, there exist a generically finite proper surjective morphism $B' \to B$ and a proper birational morphism $X' \to X \times_B B'$ such that the induced morphism $f' : X' \to B'$ is weakly semistable.* $\square$

*Idea of proof of boundedness* ([Kar00]). By Matsusaka's Theorem [Mat86], there exists an integer $\nu_0 > 0$ such that if $X$ is a normal variety with rational Gorenstein singularities, with ample canonical divisor $K_X$, and with a given Hilbert polynomial $H(l) = h^0(X, lK_X)$ for large $l$, then $\nu_0 K_X$ is very ample and has no higher cohomology. Let $B_0$ be the closure, with reduced structure, in $\mathrm{Hilb}_{H(\nu_0 t)}(\mathbf{P}^{H(\nu_0)-1})$ of the parameter space of $\nu_0$-canonical embeddings of the above varieties. Let $f_0 : X_0 \to B_0$ be the induced universal family and construct the following commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[d, "f_0"'] & X_1 = X_0 \times_{B_0} B \arrow[l] \arrow[d, "f_1"'] & X \arrow[l] \arrow[r, "\text{rational map}"] \arrow[d, "f"'] & \tilde{X} = \mathrm{Proj}_B R_{X/B} \arrow[d, "\tilde{f}"'] \\
B_0 & B \arrow[l] \arrow[r, equal] & B \arrow[r, equal] & B,
\end{tikzcd}$$

where $f : X \to B$ is a weakly semistable reduction of $f_0 : X_0 \to B_0$ and $R_{X/B}$ is the relative canonical ring. By using Assumption MMP(n+1), deformation invariance of plurigenera etc., [Kar00, 3.1] shows that $R_{X/B}$ is a finitely generated $\mathcal{O}_B$-algebra. Thus, we get the desired family $\tilde{f} : \tilde{X} \to B$ which shows the boundedness of the functor $\mathcal{M}_H^{sm}$. $\square$

Since, by a theorem of Kato ([Kk1,89]), a log smooth morphism between log smooth fs log analytic spaces is nothing but a toroidal morphism between toroidal analytic spaces, we can apply to certain subfamilies of the weakly semistatble family $f^{\mathrm{an}} : X^{\mathrm{an}} \to B^{\mathrm{an}}$ the following theorem:

**Theorem** ([KMN02]). *Let $f : X \to S$ be a projective, vertical, log smooth morphism between log smooth fs log analytic spaces such that* $\mathrm{Coker}((M_S/\mathcal{O}_S^\times)_{f(x)} \to (M_X/\mathcal{O}_X^\times)_x)$ *is torsion free. Take and fix a point $t_0 \in S^{\log}$ and put $(H_0, \langle\ ,\ \rangle_0) := (H_{\mathbf{Z},t_0}, \langle\ ,\ \rangle_{t_0})$. Then, for each $w \in \mathbf{Z}$,*

$$H_{\mathbf{Z}} := R^w f_*^{\log}\mathbf{Z}/\text{torsion}, \quad \mathcal{M} := R^w f_*(\omega^\bullet_{X/S}), \quad \mathcal{M}^p := R^w f_*(\omega^{\geq p}_{X/S}),$$
$$\Gamma := \mathrm{Im}\left(\pi_1(S^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},t_0}, \langle\ ,\ \rangle_{t_0}) = \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0)\right),$$

*and a fan $\Sigma$ in $\mathfrak{g}_{\mathbf{Q}}$ below form a PLH on $S$ of type $\Phi$, provided that $\Gamma$ is neat and such a fan $\Sigma$ exists.* $\square$

In general, the following problem is open:

*Problem of a fan* $\Sigma$. For $s \in S$, $t \in s^{\log}$ and $\tilde{\mu}_t : (H_{\mathbf{Z},t}, \langle\ ,\ \rangle_t) \xrightarrow{\sim} (H_0, \langle\ ,\ \rangle_0)$, set

$$\Gamma(s) := \mathrm{Im}\left(\pi_1^-(s^{\log}) \hookrightarrow \pi_1(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},t}, \langle\ ,\ \rangle_t) \overset{\text{by } \tilde{\mu}_t}{\xrightarrow{\sim}} \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0)\right),$$
$$\sigma_s : \text{the nilpotent cone in } \mathfrak{g}_{\mathbf{Q}} \text{ generated by } \log \Gamma(s).$$

Is there a fan $\Sigma$ in $\mathfrak{g}_{\mathbf{Q}}$ strongly compatible with $\Gamma$, which can *accept* a collection of cones $\{\sigma_s\}_{s \in S}$, i.e., for $\forall s \in S$, $\exists \sigma \in \Sigma$ such that $\sigma_s \subset \sigma$? □

We apply this theorem to the weakly semistable family $f : X \to B$ appeared in "Idea of proof of boundedness". Let $M_i$ be any irreducible component of $M_H^{\mathrm{sm}}$, let $B_i$ be a connected component of $B$ which dominates $M_i$, and put $X_i := f^{-1}(B_i) \subset X$. Since there is a neat subgroup of $\Gamma$ with finite index, after taking a finite covering of $B_i$ if necessary, we may assume $\Gamma$ for $S = B_i^{\mathrm{an}}$ is neat. Assume also the existence of an above fan $\Sigma$ for $S = B_i^{\mathrm{an}}$. Then we have a diagram:

$$\begin{array}{ccc} B_i^{\mathrm{an}} & \xrightarrow{\varphi} & \Gamma\backslash D_\Sigma, \\ {\scriptstyle \pi}\big\downarrow & & \\ M_i^{\mathrm{an}} & & \end{array}$$

where $\varphi$ is the log period map associated, by the Theorem in §2, to the PLH on $B_i^{\mathrm{an}}$ of type $\Phi$ arising from the weakly semistable family $f^{\mathrm{an}} : X_i^{\mathrm{an}} \to B_i^{\mathrm{an}}$, and $\pi$ is the analytic morphism associated to the surjective projective algebraic morphism $B_i \to M_i$.

*Problem.* • Prove the existence of the fan $\Sigma$.

• Formulate a local version of the log period map $\varphi$. □

## §5. Main Theorem

In this section, we forget log structures and investigate only structures of ringed spaces. We prove here the main theorem in the present paper, which asserts that the log period map $\varphi : B_i^{\mathrm{an}} \to \Gamma\backslash D_\Sigma$ constructed in §4 drops to a morphism $\psi : M' \to \Gamma\backslash D_\Sigma$ of ringed spaces, where $\pi : B_i^{\mathrm{an}} \to M' \to M_i^{\mathrm{an}}$ is the Stein factorization, and the image $\mathrm{Im}(\psi) = \mathrm{Im}(\varphi) \subset \Gamma\backslash D_\Sigma$ is the analytic subspace associated to a compact separated algebraic space.

We begin by an easy but fundamental lemma:

**Lemma.** *Let $Z$ be an analytic space and $S \subset Z$ a subset endowed with the strong topology in $Z$ (§1). Let $C$ be a compact subset of $S$. Then the topologies on $S$ and on $Z$ induce the same one on $C$.*

*Proof.* By the definition of strong topology, an open set in the latter topology is open in the former topology. To see the converse, let $W$ be an open set on $C$ in the former topology. Then $C - W$ is a compact subset of $S$ and also a compact subset of $Z$. In particular, it is closed in $Z$ and hence the complement $Z - (C - W)$ is open in $Z$. It follows $C \cap (Z - (C - W)) = W$ is open in the latter topology. □

**Theorem.** *Let $h : Y \to M$ be a morphism from analytic space $Y$ to the underlying ringed space of a log manifold $M$ (§1). If $Y$ is compact, then the image* $\mathrm{Im}(h) \subset M$ *is a compact analytic subspace.*

*If, moreover, $Y$ is a compact Moishezon space (equivalently, the analytic space associated to a compact algebraic space), then so is the image* $\mathrm{Im}(h)$.

*Proof.* Assume $Y$ is compact. Since $h$ is continuous, the image $C := \mathrm{Im}(h)$ is a compact subset of $M$. We use local ambient analytic spaces of the log manifold $M$. By definition of log manifold (§1), for any point $p \in \mathrm{Im}(h)$, there exists an open neighborhood $U$ in $M$ and a log smooth fs log analytic space $Z$ with a continuous injective map $j : U \hookrightarrow Z$. Since the topological space $M$ is normal, there exists a closed neighborhood $V$ of $p$ in $M$ contained in $U$. Then $C \cap V$ is a compact subset of $U$. By the lemma applied to $C \cap V \subset U \overset{j}{\hookrightarrow} Z$, the topologies on $U$ and on $Z$ induce the same one on $C \cap V$. Hence there exist an open neighborhood $U'$ of $p$ in $M$ contained in $V$ and an open subspace $Z'$ of $Z$ containing $j(U')$ such that $C \cap U' = C \cap Z'$ via $j : U' \hookrightarrow Z'$. Since $C \cap V$ is compact, its inverse image $h^{-1}(C \cap V)$ is compact and hence the composite continuous map $h^{-1}(C \cap V) \xrightarrow{h} C \cap V \subset U \overset{j}{\hookrightarrow} Z$ is proper. It follows that a restriction $g : h^{-1}(U') \xrightarrow{h} U' \overset{j}{\hookrightarrow} Z'$ is proper. By the definition of log manifold (§1), $g$ is also an analytic morphism of complex spaces. Applying a theorem of Grauert [GR84, 10.5.6], the direct image $g_*\mathcal{O}_{h^{-1}(U')}$ is a coherent sheaf and hence the ideal $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{Z'} \to g_*\mathcal{O}_{h^{-1}(U')})$ is also coherent. Thus we know the image $\mathrm{Im}(g) = C \cap U' = C \cap Z'$ (via $j : U' \hookrightarrow Z'$) is an analytic subspace of $Z'$ defined by the coherent ideal $\mathcal{I}$. $\mathrm{Im}(g) = C \cap U'$ can be also regarded as an analytic subspace of the underlying space of the log manifold $U'$ defined by the coherent ideal $j^*\mathcal{I}$ of $\mathcal{O}_{U'}$. The first case is proved.

The second case follows from the first by applying a theorem of Moishezon [Mo69] or a theorem of Artin [Ar70] to $h : Y \to C$. □

**Main Theorem.** *We use the notation in the previous sections. Assume* MMP(n+1) *in* §3 *and assume the existence of a fan $\Sigma$ in* §4. *Let $\pi : B_i^{\mathrm{an}} \xrightarrow{\pi'} M' \xrightarrow{\pi''} M_i^{\mathrm{an}}$ be the Stein factorization of $\pi$ in* §4, *where $\pi'$ is a projective morphism with connected fibers, i.e., $\pi'_*\mathcal{O}_{B_i}^{\mathrm{an}} = \mathcal{O}_{M'}$, and $\pi''$ is finite. Then there exists a unique morphism $\psi : M' \to \Gamma\backslash D_\Sigma$ of ringed spaces such that $\psi \circ \pi' = \varphi$ as morphisms of ringed spaces, and the image* $\mathrm{Im}(\psi) = \mathrm{Im}(\varphi) \subset \Gamma\backslash D_\Sigma$ *is the analytic subspace associated to a compact algebraic space, which is Hausdorff.*

*Proof.* Since $B_i$ is a projective variety (§4), the previous theorem, applied to $\varphi : B_i^{\mathrm{an}} \to \Gamma\backslash D_\Sigma$, shows that the image $C := \mathrm{Im}(\varphi) \subset \Gamma\backslash D_\Sigma$ is the analytic subspace associated to a compact algebraic space. By the definition of the morphisms of analytic spaces $\varphi : B_i^{\mathrm{an}} \to C$ and $\pi : B_i^{\mathrm{an}} \to M_i^{\mathrm{an}}$, $\varphi$ is constant on every connected fiber component of $\pi$. Hence there exists a unique morphism $\psi : M' \to C \subset \Gamma\backslash D_\Sigma$ of ringed spaces such that $\psi \circ \pi' = \varphi$ (cf. [GR84, 10.6.1]). Since $\Gamma\backslash D_\Sigma$ is Hausdorff, so is $\mathrm{Im}(\psi) = \mathrm{Im}(\varphi)$. □

*Remark.* At present, Assumption MMP(n+1) is known for $n \leq 2$.

*Problem.* • Is $\mathrm{Im}(\psi) = \mathrm{Im}(\varphi) \subset \Gamma\backslash D_\Sigma$ projective (cf. [Ko1,90])?

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