SL(2)-ORBIT THEOREM FOR DEGENERATION OF MIXED HODGE STRUCTURE

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Abstract. In [CKS], Cattani, Kaplan and Schmid established the SL(2)-orbit theorem in several variables for degeneration of polarized Hodge structure. The aim of the present paper is to generalize it for degeneration of mixed Hodge structure whose graded quotients by the weight filtration are polarized. As an application, we obtain a mixed Hodge version of the estimate of the Hodge metric for degeneration of polarized Hodge structure in [Sc], [CKS], [K1].

Contents

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§0. Introduction

Let $\Delta = \{q \in \mathbf{C} \mid |q| < 1\}$, and let $f : X \to \Delta$ be a projective morphism of complex analytic manifolds which is smooth over $\Delta^* = \Delta \setminus \{0\}$ and which is of semi-stable degeneration at 0. Let $q \in \Delta^*$ and $X_q = f^{-1}(q)$. Then for $m \geq 0$, $H^m(X_q, \mathbf{Z})$ carries a Hodge structure pure of weight m depending on q. Let $F(q)$ be its Hodge filtration. It is a basic problem in Hodge theory to describe the asymptotic behavior of such a variation of $F(q)$ when q tends to 0.

In general, there are two steps in analyzing this kind of asymptotic behaviors, which are summed up by the following picture:

degeneration of Hodge structure	$\Rightarrow$	a nilpotent orbit appears	$\Rightarrow$	an SL(2)-orbit appears

In this picture, the first $\Rightarrow$ shows that nilpotent orbits often appear when variations of polarized Hodge structure degenerate ([Sc], [St], [Fs], [KMN], etc.). This is illustrated in the above situation as follows. As is shown by Schmid [Sc] and Steenbrink [St], we have so called limit Hodge filtration $F := \lim_{q\to 0} \exp(-zN)F(q)$, where $q = \exp(2\pi iz)$ and N is the logarithm of the monodromy. The pair (N, F) generates a nilpotent orbit $z \mapsto \exp(zN)F$ ($z \in \mathbf{C}$, $\mathrm{Im}(z) \to \infty$), and $F(q)$ and $\exp(zN)F$ are near. (See 0.1 below for the precise definition of a nilpotent orbit.)

The second $\Rightarrow$ is called SL(2)-orbit theorem of Cattani, Kaplan and Schmid ([Sc], [CKS]). In 0.1 below, we explain this step in detail. In the above situation, it gives another Hodge filtration $\mathbf{r}$ and a real multiplicative operator $t(y)$ such that $\exp(iyN)F$ and $t(y)\mathbf{r}$ are near for $y > 0$, $y \to \infty$. Since $t(y)$ is real, it preserves real structures.

In conclusion, $F(q)$ is approximated by more understandable $\exp(xN)t(y)\mathbf{r}$ ($z = x + iy$, x, $y \in \mathbf{R}$), and this fact has many applications, for example, the estimates of the Hodge metric for degeneration of polarized Hodge structure in [Sc], [CKS], [K1].

Next we proceed to the mixed situation, that is, the situation where we allow a horizontal divisor $E \subset X$, and consider the variation of mixed Hodge structure on $H^m((X \setminus E)_q, \mathbf{Z})$. Then we can consider the mixed version of the above picture:

degeneration of mixed Hodge structure	$\Rightarrow$	a mixed nilpotent orbit appears	$\Rightarrow$	a mixed SL(2)-orbit appears

The first $\Rightarrow$ in this picture shows that mixed nilpotent orbits (or IMHM in the sense of Kashiwara's [K2]) often appear when variations of mixed Hodge structure with polarized graded quotients degenerate ([SZ], [K2], [Sa], [P2], etc.). See 12.10 for a review of this step. (See 0.2 below for the precise definition of a mixed nilpotent orbit.)

The present work fits into this picture as the second $\Rightarrow$, that is, our aim is to generalize the SL(2)-orbit theorem (in several variables) on degeneration of polarized Hodge structure to the mixed version for degeneration of mixed Hodge structure whose graded quotients for the weight filtration are polarized. In 0.2 below, we explain the details. As an application, we generalize the estimate of the Hodge metric for degeneration of polarized Hodge structure mentioned above, to a mixed Hodge version.

A work in this direction was also done by G. Pearlstein in [P3] by a different method.

0.1. We review the SL(2)-orbit theorem in [CKS] shortly.

Let V be a finite dimensional $\mathbf{R}$-vector space, let w be an integer, and let $\langle\ ,\ \rangle : V \times V \to \mathbf{R}$ be a non-degenerate $\mathbf{R}$-bilinear form which is symmetric if w is even and anti-symmetric if w is odd.

We denote by D the set of all decreasing filtrations $F = (F^p)_{p\in\mathbf{Z}}$ on $V_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} V$ such that $(F, \langle\ ,\ \rangle)$ is a polarized $\mathbf{R}$-Hodge structure of weight w. This D is a disjoint union of classifying spaces of polarized Hodge structures defined by Griffiths [G] where Hodge numbers $h^{p,w-p} = \dim_{\mathbf{C}}(F^p/F^{p+1})$ are fixed. Let $D^\vee$ be the set of all decreasing filtrations F on $V_{\mathbf{C}}$ such that the annihilator of F^p with respect to $\langle\ ,\ \rangle$ is F^{w+1-p} for any p. Then $D^\vee$ is a complex analytic manifold and D is an open set of $D^\vee$.

Let $G_{\mathbf{R}}$ be the algebraic group over $\mathbf{R}$ of all automorphisms of the $\mathbf{R}$-vector space V preserving $\langle\ ,\ \rangle$, and let $\mathfrak{g}_{\mathbf{R}} = \mathrm{Lie}(G_{\mathbf{R}})$ which we identify with the set of all $\mathbf{R}$-linear operators $N : V \to V$ satisfying $\langle N(x), y\rangle + \langle x, N(y)\rangle = 0$ for all $x, y \in V$.

Consider an $(n+1)$-ple $(N_1, \ldots, N_n, F)$, where $N_j \in \mathfrak{g}_{\mathbf{R}}$ $(1 \le j \le n)$ and $F \in D^\vee$, satisfying the following conditions (i)–(iii).

(i) The operators $N_j : V \to V$ are nilpotent for all j, and $N_j N_k = N_k N_j$ for all j, k.

(ii) If $y_j \gg 0$ $(1 \le j \le n)$, then $\exp(\sum_{j=1}^n iy_jN_j)F \in D$.

(iii) $N_jF^p \subset F^{p-1}$ for all j and p. (Griffiths transversality.)

Then the map

$$\mathbf{C}^n \to D^\vee; (z_1, \ldots, z_n) \mapsto \exp(\textstyle\sum_{j=1}^n z_jN_j)F$$

is usually called a nilpotent orbit. To avoid the confusions with the mixed case explained in 0.2 below, we call it in the present paper a *pure nilpotent orbit*, and we say that $(N_1, \ldots, N_n, F)$ (or $(V, w, \langle\ ,\ \rangle, N_1, \ldots, N_n, F)$) *generates a pure nilpotent orbit*.

Pure nilpotent orbits often appear when variations of polarized Hodge structure degenerate. In the situation at the beginning of this §0, $(V, w, \langle\ ,\ \rangle, N_1, \ldots, N_n, F) = (H^m(X_q, \mathbf{R}), m, \langle\ ,\ \rangle, N, F)$ (with $n = 1$ and with $\langle\ ,\ \rangle$ induced from a polarization of X) generates a pure nilpotent orbit. Many N_j appear when variations of polarized Hodge structure on Δ^n $(n \ge 2)$ degenerate, as the logarithms of monodromy operators.

Assume that $(N_1, \ldots, N_n, F)$ generates a pure nilpotent orbit. Then the theory of SL(2)-orbit in several variables in [CKS] shows that an SL(2)-orbit in n variables (ρ, φ)

$$\rho : \mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}, \quad \varphi : \mathbf{P}^1(\mathbf{C})^n \to D^\vee$$

(cf. 2.1) is associated to $(N_1, \ldots, N_n, F)$. Here ρ is a homomorphism of algebraic groups, which is defined over $\mathbf{R}$, and φ is a holomorphic map satisfying

$$\varphi(gz) = \rho(g)\varphi(z) \text{ for any } g \in \mathrm{SL}(2, \mathbf{C})^n \text{ and } z \in \mathbf{P}^1(\mathbf{C})^n,$$
$$\varphi(\mathfrak{h}^n) \subset D, \text{ where } \mathfrak{h} \text{ is the upper half plane } \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}.$$

Let $\mathbf{i} = (i, \ldots, i) \in \mathfrak{h}^n$ and $\mathbf{r} = \varphi(\mathbf{i}) \in D$, and for $y = (y_1, \ldots, y_n)$ $(y_j > 0)$, let

$$t(y) = \rho\left(\begin{pmatrix} \sqrt{y_1} & 0 \\ 0 & 1/\sqrt{y_1} \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{y_n} & 0 \\ 0 & 1/\sqrt{y_n} \end{pmatrix}\right) \in G_{\mathbf{R}}.$$

Then Theorem 4.20 of [CKS] (the SL(2)-orbit theorem in n variables) shows that there exist $c > 0$ and $a_m, b_m \in \mathfrak{g}_{\mathbf{R}}$ $(m \in \mathbf{N}^n)$ such that $a_0 = b_0 = 0$, that

$$\sum_{m\in\mathbf{N}^n} a_m \prod_{j=1}^n \lambda_j^{m(j)}, \quad \sum_{m\in\mathbf{N}^n} b_m \prod_{j=1}^n \lambda_j^{m(j)/2}$$

absolutely converge when $0 \le \lambda_j < c$ for $1 \le j \le n$, and such that whenever $y_{j+1}/y_j < c$ for $1 \le j \le n$ (y_{n+1} denotes 1), if we put

$$g(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} a_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)}),$$
$${}^e g(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} b_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)/2}),$$

then we have ${}^e g(y) = t(y)^{-1} g(y) t(y)$ and the following presentation of $\exp(\sum iy_j N_j)F$:

$$\exp(\textstyle\sum_{j=1}^n iy_j N_j)F = g(y)\varphi(iy_1, \dots, iy_n) = g(y)t(y)\mathbf{r} = t(y) \cdot {}^e g(y)\mathbf{r}.$$

0.2. Now let V be a finite dimensional $\mathbf{R}$-vector space endowed with an increasing filtration $W = (W_w)_{w\in\mathbf{Z}}$ such that $W_w = V$ for $w \gg 0$ and such that $W_w = 0$ for $w \ll 0$. Assume that for each $w \in \mathbf{Z}$, we are given a non-degenerate $\mathbf{R}$-bilinear form $\langle\ ,\ \rangle_w : \mathrm{gr}_w^W(V) \times \mathrm{gr}_w^W(V) \to \mathbf{R}$ which is symmetric if w is even and anti-symmetric if w is odd.

For each $w \in \mathbf{Z}$, let D_w, $\check{D}_w$, $G_{w,\mathbf{R}}$, $\mathfrak{g}_{w,\mathbf{R}}$ be the "D", "$D^\vee$", "$G_{\mathbf{R}}$", "$\mathfrak{g}_{\mathbf{R}}$" in 0.1 for $(\mathrm{gr}_w^W(V), w, \langle\ ,\ \rangle_w)$, respectively.

We denote by D (resp. $D^\vee$) the set of all decreasing filtrations $F = (F^p)_{p\in\mathbf{Z}}$ on $V_{\mathbf{C}}$ such that $F(\mathrm{gr}_w^W) \in D_w$ (resp. $\check{D}_w$) for all w. Here $F(\mathrm{gr}_w^W)$ denotes the decreasing filtration on $\mathrm{gr}_w^W(V_{\mathbf{C}})$ induced by F. Then $D^\vee$ is a complex analytic manifold and D is an open subset of $D^\vee$. This D is a disjoint union of classifying spaces of mixed Hodge structures with polarized graded quotients defined in [U] where the Hodge numbers $h^{p,w-p} = \dim_{\mathbf{C}}(F^p(\mathrm{gr}_w^W)/F^{p+1}(\mathrm{gr}_w^W))$ of the graded quotients $F(\mathrm{gr}_w^W)$ are fixed. (See also [SSU], [P1] for this classifying space.) Let $G_{\mathbf{R}}$ be the algebraic group over $\mathbf{R}$ of all automorphisms g of V preserving W such that $\mathrm{gr}_w^W(g) \in G_{w,\mathbf{R}}$ for all $w \in \mathbf{Z}$, and let $\mathfrak{g}_{\mathbf{R}} = \mathrm{Lie}(G_{\mathbf{R}})$ which we identify with the set of all $\mathbf{R}$-linear operators $N : V \to V$ such that $N(W_w) \subset W_w$ for all w and such that $\mathrm{gr}_w^W(N) \in \mathfrak{g}_{w,\mathbf{R}}$ for all w.

Consider an $(n+1)$-ple $(N_1, ..., N_n, F)$, where $N_j \in \mathfrak{g}_{\mathbf{R}}$ $(1 \le j \le n)$ and $F \in D^\vee$, satisfying the following conditions (i)–(iv).

(i) The operators $N_j : V \to V$ are nilpotent for all j, and $N_j N_k = N_k N_j$ for all j, k.

(ii) If $y_j \gg 0$ $(1 \le j \le n)$, then $\exp(\sum_{j=1}^n iy_j N_j)F \in D$.

(iii) $N_j F^p \subset F^{p-1}$ for all j and p. (Griffiths transversality.)

(iv) Let J be any subset of $\{1, \dots, n\}$. Then for $y_j \in \mathbf{R}_{>0}$ $(j \in J)$, the relative monodromy filtration $M(\sum_{j\in J} y_j N_j, W)$ exists. Furthermore, this filtration is independent of the choice of $y_j \in \mathbf{R}_{>0}$. (The definition of the relative monodromy filtration is reviewed in 5.1.)

Then we call the map $(z_1, \dots, z_n) \mapsto \exp(\sum_{j=1}^n z_j N_j)F$ a *mixed nilpotent orbit*, and we say that $(N_1 \dots, N_n, F)$ (or $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, F)$) *generates a mixed nilpotent orbit.*

In the terminology in Kashiwara [K2] 4.3, $(V; W_{\mathbf{C}}; F, \overline{F}; N_1, \dots, N_n)$, with $\overline{F}$ the complex conjugate of F, is an infinitesimal mixed Hodge module (IMHM).

Mixed nilpotent orbits (or IMHM) often appear when variations of mixed Hodge structure with polarized graded quotients degenerate. In this case, similarly as in the pure situation, $N_1, \dots, N_n$ are the logarithms of the monodormy operators, and F is the limit Hodge filtration. See 12.10 for a review of this appearance.

For such $(N_1, \dots, N_n, F)$, the mixed Hodge version of the associated SL(2)-orbit (ρ, φ) in 0.1 is, in the formulation of the present paper, the pair

$$(\text{the collection of SL(2)-orbits } (\rho_w, \varphi_w) \text{ for } w \in \mathbf{Z},\ \ s),$$

where (ρ_w, φ_w) is the SL(2)-orbit in n variables associated to $(\mathrm{gr}^W_w(N_1), \dots, \mathrm{gr}^W_w(N_n), F(\mathrm{gr}^W_w))$ for each $w \in \mathbf{Z}$, and s is a certain splitting of W, i.e., an isomorphism $s : \mathrm{gr}^W(V) = \bigoplus_{w \in \mathbf{Z}} \mathrm{gr}^W_w(V) \xrightarrow{\sim} V$ such that $(s(x) \bmod W_{w-1}) = x$ for any $w \in \mathbf{Z}$ and $x \in \mathrm{gr}^W_w(V)$, explained in Theorem 0.5 (1) below.

0.3. Before we state our Main Theorem 0.5 for the situation 0.2, we review shortly the canonical $\mathbf{R}$-splitting of the weight filtration associated to a mixed Hodge structure defined in [CKS] (see §1 of the present paper for details). In fact, there are a few ways of associating an $\mathbf{R}$-splitting to a given mixed Hodge structure. But, in the present paper, one of them is more important than the others, and we call it the canonical splitting.

Let V be a finite dimensional $\mathbf{R}$-vector space with an increasing filtration W such that $W_w = V$ for $w \gg 0$ and such that $W_w = 0$ for $w \ll 0$. Let F be a decreasing filtration on $V_{\mathbf{C}}$ such that (W, F) is a mixed $\mathbf{R}$-Hodge structure, that is, $F(\mathrm{gr}^W_w)$ is an $\mathbf{R}$-Hodge structure of weight w for all $w \in \mathbf{Z}$. Then an associated splitting $s : \mathrm{gr}^W(V) \xrightarrow{\sim} V$ of W, $\mathbf{R}$-linear maps $\delta(W, F)$, $\zeta(W, F) : V \to V$, and a $\mathbf{C}$-linear map $\varepsilon(W, F) : V_{\mathbf{C}} \to V_{\mathbf{C}}$ are defined so as to satisfy

$$\begin{aligned}
&\delta(W, F)(W_w) \subset W_{w-2}, \ \ \zeta(W, F)(W_w) \subset W_{w-2}, \ \ \varepsilon(W, F)(W_{w,\mathbf{C}}) \subset W_{w-2,\mathbf{C}} \ \ (\forall w),\\
&\exp(\varepsilon(W, F)) = \exp(i\delta(W, F)) \exp(-\zeta(W, F)),\\
&F = \exp(\varepsilon(W, F)) s(F(\mathrm{gr}^W)),
\end{aligned}$$

where $s(F(\mathrm{gr}^W))$ is the image under $s : \mathrm{gr}^W(V_{\mathbf{C}}) \xrightarrow{\sim} V_{\mathbf{C}}$ of the filtration $F(\mathrm{gr}^W)$ on $\mathrm{gr}^W(V_{\mathbf{C}})$ induced by F. This $s(F(\mathrm{gr}^W))$ is denoted by $\tilde{F}_0$ in [CKS] (3.31). We denote it by $\hat{F}$ in the present paper, and call it the canonical $\mathbf{R}$-split mixed Hodge structure associated to (W, F). We have

$$\varepsilon(gW, gF) = g\varepsilon(W, F)g^{-1} \quad \text{for} \quad g \in \mathrm{Aut}_{\mathbf{R}}(V).$$

See §1 for the precise definitions of $\delta(W, F)$, $\zeta(W, F)$, $\varepsilon(W, F)$ and s.

0.4. To state Theorem 0.5, we introduce some notation. Let $(N_1, \dots, N_n, F)$ generate a mixed nilpotent orbit as in 0.2.

For each $w \in \mathbf{Z}$, let (ρ_w, φ_w) be the SL(2)-orbit in n variables associated to $(\mathrm{gr}^W_w(N_1), \dots, \mathrm{gr}^W_w(N_n), F(\mathrm{gr}^W_w))$. Let

$$\exp(\textstyle\sum_{j=1}^n iy_j \,\mathrm{gr}^W_w(N_j))F(\mathrm{gr}^W_w) = g_w(y)\varphi_w(iy_1, \dots, iy_n) = g_w(y)t_w(y)\mathbf{r}_w$$

be a presentation of $\exp(\sum_{j=1}^n iy_j \operatorname{gr}_w^W(N_j))F(\operatorname{gr}_w^W)$ $(y_j/y_{j+1} \gg 0)$ introduced in 0.1. In fact, $g(y)$ in 0.1 is not unique, and so $g_w(y)$ for each w is not unique here. However Theorem 0.5 holds for any $g_w(y)$ which appears in the result on $(\operatorname{gr}_w^W(N_1), \dots, \operatorname{gr}_w^W(N_n), F(\operatorname{gr}_w^W))$ introduced in 0.1.

Our main result is the following theorem.

Theorem 0.5. *Assume that $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, F)$ generates a mixed nilpotent orbit as in 0.2. For $y = (y_j)_{1\le j\le n}$ with $y_j \gg 0$ $(1 \le j \le n)$, let $s(y) : \operatorname{gr}^W(V) \overset{\sim}{\to} V$ be the canonical splitting of W associated to the mixed $\mathbf{R}$-Hodge structure $(W, \exp(\sum_{j=1}^n iy_jN_j)F)$.*

(1) If $y_j > 0$ $(1 \le j \le n)$ and y_j/y_{j+1} tends to ∞ for $1 \le j \le n$ (y_{n+1} denotes 1), then $s(y)$ converges to a splitting s of W.

(2) More precisely, there exist $c > 0$ and $u_m \in \operatorname{Ker}(\mathfrak{g}_\mathbf{R} \to \prod_w \mathfrak{g}_{w,\mathbf{R}})$ $(m \in \mathbf{N}^n)$ satisfying the following conditions (i)–(iv).

(i) $u_0 = 0$.

(ii) $u_mM(N_1 + \cdots + N_j, W)_k \subset M(N_1 + \cdots + N_j, W)_{k+m(j)}$ for any $m \in \mathbf{N}^n$, $1 \le j \le n$ and $k \in \mathbf{Z}$.

(iii) $\sum_{m\in\mathbf{N}^n} u_m \prod_{j=1}^n \lambda_j^{m(j)}$ absolutely converges when $0 \le \lambda_j < c$ for $1 \le j \le n$.

(iv) Whenever $y_{j+1}/y_j < c$ for $1 \le j \le n$ (y_{n+1} means 1), we have

$$s(y) = u(y)s \quad \textit{with} \quad u(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} u_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)}).$$

(3) Define

$$g(y), {}^e g(y) \in G_\mathbf{R} \quad (y_j/y_{j+1} \gg 0), \quad t(y) \in \operatorname{Aut}(V, W) \quad (y_j > 0), \quad \mathbf{r} \in D$$

by

$$g(y) = u(y)s(\textstyle\bigoplus_w g_w(y))s^{-1}, \quad t(y) = s(\bigoplus_w y_1^{-w/2} t_w(y))s^{-1},$$
$${}^e g(y) = t(y)^{-1} g(y) t(y), \quad \mathbf{r} = s(\textstyle\bigoplus_{w\in\mathbf{Z}} \mathbf{r}_w).$$

Then the $\mathbf{R}$-split mixed Hodge structures associated to $(W, \exp(\sum_{j=1}^n iy_jN_j)F)$ and to $(W, t(y)^{-1}g(y)^{-1}\exp(\sum_{j=1}^n iy_jN_j)F)$ for $y_j/y_{j+1} \gg 0$ are

$$g(y)t(y)\mathbf{r} = t(y) \cdot {}^e g(y)\mathbf{r} \quad \textit{and} \quad \mathbf{r}, \quad \textit{respectively.}$$

Furthermore, there exist $c > 0$ and $b_m \in \mathfrak{g}_\mathbf{R}$ $(m \in \mathbf{N}^n)$ such that $b_0 = 0$, that $\sum_{m\in\mathbf{N}^n} b_m \prod_{j=1}^n \lambda_j^{m(j)/2}$ absolutely converges when $0 \le \lambda_j < c$ for $1 \le j \le n$, and that whenever $y_{j+1}/y_j < c$ for $1 \le j \le n$, we have

$${}^e g(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} b_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)/2}).$$

(4) For $y_j/y_{j+1} \gg 0$, let

$$\varepsilon(y) = \varepsilon(W, t(y)^{-1}g(y)^{-1}\exp(\textstyle\sum_{j=1}^n iy_jN_j)F),$$

so that

$$\exp(\textstyle\sum_{j=1}^n iy_jN_j)F = g(y)t(y)\exp(\varepsilon(y))\mathbf{r} = t(y)\cdot {}^e g(y)\exp(\varepsilon(y))\mathbf{r}.$$

Then there exist $c > 0$ and $\varepsilon_m \in \mathfrak{g}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$ ($m \in \mathbf{N}^n$) such that when $0 \le \lambda_j < c$ for $1 \le j \le n$, $\sum_{m\in\mathbf{N}^n} \varepsilon_m \prod_{j=1}^n \lambda_j^{m(j)/2}$ absolutely converges, and such that whenever $y_{j+1}/y_j < c$ for $1 \le j \le n$, we have

$$\varepsilon(y) = \textstyle\sum_{m\in\mathbf{N}^n} \varepsilon_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)/2}.$$

(Here ε_0 need not be 0.)

As an application of Theorem 0.5, we will generalize the norm estimates in [Sc], [CKS], [K1], which are the results on the asymptotic behavior of the Hodge metric in degeneration of polarized Hodge structure, to degeneration of mixed Hodge structure with polarized graded quotients (Theorem 12.4).

0.6. The key of our proof of Theorem 0.5 is the idea that any mixed nilpotent orbit in 0.2 can be regarded as a quotient of (and also a part of) a pure nilpotent orbit in 0.1 whose number of the operators N_j is increased by one. By this, we can reduce our theorem for the mixed nilpotent orbit to the theorem in [CKS] for the pure nilpotent orbit.

It may sound rather strange that a mixed object is a quotient or a part of a pure object. But, in algebraic geometry, a mixed object is often found to be embedded into a pure object, for example, in the following way. Let $f\colon X \to \Delta$, m, $F(q)$, and (N, F) be as at the beginning of this §0. Let $V = H^m(X_q, \mathbf{R})$ as in 0.1. As explained there, with the intersection form $\langle\ ,\ \rangle$ coming from a polarization of X, $(V, m, \langle\ ,\ \rangle, N, F)$ generates a *pure* nilpotent orbit of weight m. On the other hand, let $X_0 = f^{-1}(0)$. Then $V' = H^m(X_0, \mathbf{R})$ carries a mixed Hodge structure, and there is a homomorphism $V' \to V$.

Now the image of $V' \to V$ coincides with $\mathrm{Ker}(N : V \to V)$ by the local invariant cycle theorem. If the homomorphism $V' \to V$ is injective (this happens for example in the case $m = 1$), the Hodge filtration of V' coincides with the restriction of F on V, the weight filtration of V' coincides with the restriction of (the $-m$ shift of) the monodromy filtration on V defined by N, and so we have a situation that a mixed Hodge structure is a part of a pure nilpotent orbit as in 0.1 with one N.

In fact, we can show that a mixed Hodge structure is always a part of a pure nilpotent orbit as in 0.1 with one N. See 3.5.

(Pure objects live inside mixed objects as subquotients, but thus conversely a mixed object lives inside a pure object. The authors like to compare this fact with a phrase in a poem “Nostalgia” by a Japanese poet Tatsuji Miyoshi, which says that the Chinese character representing “sea” contains as its part the Chinese character representing “mother” and conversely the French word mère contains the French word mer.)

0.7. In [P3] Theorem 4.2, Pearlstein proved the SL(2)-orbit theorem for degeneration of mixed Hodge structure in the case $n = 1$ assuming either one of the following conditions (I), (II) is satisfied.

(I) There is $k \in \mathbf{Z}$ such that $\mathrm{gr}^W_w(V) = 0$ if $w \neq k, k-1$.

(II) There is $k \in \mathbf{Z}$ such that the Hodge numbers $h^{p,q}$ of F are zero unless $p + q = 2k - 1$, or $(p, q) = (k, k)$, $(k - 1, k - 1)$.

His SL(2)-orbit theorem has slightly different form from ours. Also our method differs from his. In §11, we reprove a part of his SL(2)-orbit theorem by using our Theorem 0.5. In [P3] Theorem 4.7, in these cases, he studied also the asymptotic behavior of the Hodge metric in the degeneration of mixed Hodge structure by a method different from ours.

0.8. In our forthcoming paper, we will use the results of the present paper for the construction of the moduli space of mixed log Hodge structures whose graded quotients for the weight filtration are polarized. There we add points at infinity to the space D in 0.2 corresponding to degenerations. The role played by our results there is the same as the role played by the results of [CKS] in the construction in [KU2] of the moduli of polarized pure log Hodge structures.

0.9. We give here one example to show how the convergence of the canonical splitting in Theorem 0.5 (1) is delicate and how the condition (iv) in 0.2 is important.

Let V be a 3 dimensional $\mathbf{R}$-vector space with basis (e_1, e_2, e_3), let W be the increasing filtration on V defined by $W_0 = V$, $W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2$ and $W_{-2} = 0$. Hence $\mathrm{gr}^W_w = 0$ unless $w = 0, -1$. Define $\langle\ ,\ \rangle_w$ by $\langle e_3, e_3\rangle_0 = 1$, $\langle e_2, e_1\rangle_{-1} = 1$. Take $a, b \in \mathbf{R}$, and let N be the element defined by $N(e_3) = ae_1 + be_2$, $N(e_2) = e_1$, $N(e_1) = 0$. Define $F^{-1} = V_{\mathbf{C}}$, $F^0 = \mathbf{C}e_2 + \mathbf{C}e_3$, $F^1 = 0$. Then (N, F) always satisfies the conditions (i)–(iii) in 0.2, but satisfies (iv) if and only if $b = 0$.

On the other hand, we have

$$\begin{aligned}\exp(iyN)F^0 &= \mathbf{C}(e_2 + iye_1) + \mathbf{C}(e_3 + iyae_1 + iybe_2 - \tfrac{y^2}{2}be_1)\\ &= \mathbf{C}(e_2 + iye_1) + \mathbf{C}(e_3 - ae_2 + \tfrac{by^2}{2}e_1).\end{aligned}$$

Note that $e_3 - ae_2 + \frac{by^2}{2}e_1$ is real. By this (see §1), the canonical splitting $s(y)$ of W associated to the mixed Hodge structure $(W, \exp(iyN)F)$ for $y \gg 0$ is given by

$$s(y)(e_3 \bmod W_{-1}) = e_3 - ae_2 + \tfrac{by^2}{2}e_1.$$

Therefore $s(y)$ converges if and only if $b = 0$.

See §13 for various examples of $s(y)$, $u(y)$, $\varepsilon(y)$ in Theorem 0.5.

0.10. The plan of the present paper is as follows.

After preliminary sections §1–§3, we prove Theorem 0.5 in §4 assuming two propositions 4.1 and 4.2. Proposition 4.1 shows that an object $(N_1, \ldots, N_n, F)$ in 0.2 which generates a mixed nilpotent orbit is regarded as a quotient of an object $(N'_0, \ldots, N'_n, F')$ in 0.1 which generates a pure nilpotent orbit. Proposition 4.2 is a result complementary to the SL(2)-orbit theorem of Cattani-Kaplan-Schmid. In §4, using Proposition 4.1, we deduce Theorem 0.5 for $(N_1, \dots, N_n, F)$ from the SL(2)-orbit theorem and Proposition 4.2 for $(N'_0, \dots, N'_n, F')$, by passing to the quotient. We prove Proposition 4.1 in §5–§7, and we prove Proposition 4.2 in §8 and §9. For the proof of Proposition 4.2, we use the relation (8.4) of the SL(2)-orbit theorem in [CKS] and the theory of Borel-Serre in [BS], which is obtained in [KU1]. In §10, we give complementary results to Theorem 0.5, such as more detail information about ε, δ, ζ (10.4, 10.6, 10.7), real analytic dependence of constructions in 0.5 on parameters (10.8), etc. In §11, we explain the relationship between the present work and the work [P3] of Pearlstein. In §12, we generalize norm estimates in [Sc], [CKS] and [K1] for degeneration of polarized Hodge structure to degeneration of mixed Hodge structure. In §13, we give examples.

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§1. Review of canonical splitting of weight filtration of mixed Hodge structure

In the present paper, by a mixed Hodge structure, we mean an **R**-mixed Hodge structure (we do not consider **Z**-structure nor **Q**-structure), except in the beginning of the introduction and in the last paragraph of 13.2.

In this section, we review the canonical splitting of the weight filtration of a mixed Hodge structure constructed in [CKS].

1.1. ***R**-split mixed Hodge structure.* Let V be a finite dimensional **R**-vector space. Let (W, F) be a mixed Hodge structure on V. A *splitting* of (W, F) is a bigrading $V_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{R}} V = \bigoplus J^{p,q}$ such that $W_{k,\mathbf{C}} = \bigoplus_{p+q \le k} J^{p,q}$ and $F^p = \bigoplus_{r \ge p, q \in \mathbf{Z}} J^{r,q}$.

We say (W, F) is **R**-*split* if it admits a splitting $(J^{p,q})$ satisfying $\overline{J^{p,q}} = J^{q,p}$ (called an **R**-*splitting*). Note that an **R**-splitting is unique if it exists and is given by $J^{p,q} = F^p \cap \overline{F}^q \cap W_{p+q,\mathbf{C}}$. If (W, F) is **R**-split with the **R**-splitting $(J^{p,q})$, each $J^{p,q}$ is called the (p, q)-*Hodge component* of (W, F).

For a fixed increasing filtration W of V, an **R**-split mixed Hodge structure (W, F) is equivalent to a pair $((F(\mathrm{gr}^W_w))_w, s)$ of a family of Hodge structures $F(\mathrm{gr}^W_w)$ on gr^W_w of weight w for all w and a splitting $s : \mathrm{gr}^W(V) \xrightarrow{\sim} V$ of W. In fact, given (W, F), we have the induced Hodge structures $F(\mathrm{gr}^W_w)$ on gr^W_w for all w, and **R**-linear isomorphisms $\mathrm{gr}^W_w \xrightarrow{\sim} \bigoplus_{p+q=w}(F^p \cap \overline{F}^q \cap W_{p+q})$ for all w which give a splitting s of W. Conversely, given $((F(\mathrm{gr}^W_w))_w, s)$, we have $F = s(\bigoplus_w F(\mathrm{gr}^W_w))$.

For any mixed Hodge structure (W, F), there is a unique splitting $(I^{p,q})$ satisfying $\overline{I^{p,q}} \equiv I^{q,p} \mod (\bigoplus_{r<p, s<q} I^{r,s})$ for any $p, q \in \mathbf{Z}$ ([CKS] (2.13)), which we call *Deligne's splitting.* This is defined explicitly by

$$I^{p,q} := (F^p \cap W_{p+q,\mathbf{C}}) \cap (\overline{F}^q \cap W_{p+q,\mathbf{C}} + \textstyle\sum_{j \ge 0} \overline{F}^{q-1-j} \cap W_{p+q-2-j,\mathbf{C}}).$$

Clearly, (W, F) is **R**-split if and only if its Deligne's splitting satisfies $\overline{I^{p,q}} = I^{q,p}$.

1.2. *Canonical splitting of W associated to F.* Let (W, F) be a mixed Hodge structure on V. Then an $\mathbf{R}$-split mixed Hodge structure $(W, \hat{F})$ on V is associated to (W, F). We call the splitting $s : \mathrm{gr}^W(V) \xrightarrow{\sim} V$ of W by $\hat{F}$ the *canonical splitting of W associated to F*. This $\hat{F}$ is $\tilde{F}_0$ in [CKS] (3.31). It is defined by $\hat{F} = \exp(-\varepsilon)F$, where $\varepsilon = \varepsilon(W, F)$ is the unique nilpotent linear map $V_{\mathbf{C}} \to V_{\mathbf{C}}$ such that $\exp(\varepsilon) = \exp(i\delta)\exp(-\zeta)$, and $\delta = \delta(W, F)$ and $\zeta = \zeta(W, F)$ are $\mathbf{R}$-linear maps $V \to V$ defined in 1.3 and 1.4 below, respectively. By the Campbell-Hausdorff formula $\exp(x)\exp(y) = \exp(H(x, y))$ with Hausdorff series $H(x, y) = x + y + \frac{1}{2}[x, y] + \cdots$, ε is written explicitly as $\varepsilon = H(i\delta, -\zeta)$. On the other hand, δ and ζ are recovered from ε as

$$\delta = (2i)^{-1}H(\varepsilon, -\bar{\varepsilon}), \quad \zeta = -H(-i\delta, \varepsilon), \tag{1}$$

where $\bar{\varepsilon}$ is the complex conjugate of ε.

1.3. δ. There is a unique $(\tilde{F}, \delta)$ with $\tilde{F}$ being a decreasing filtration on $V_{\mathbf{C}}$ and δ being an $\mathbf{R}$-linear map $V \to V$ such that $(W, \tilde{F})$ is an $\mathbf{R}$-split mixed Hodge structure, $F = \exp(i\delta)\tilde{F}$, and for each p, q, the homomorphism $\delta : V_{\mathbf{C}} \to V_{\mathbf{C}}$ sends the (p, q)-Hodge component $J^{p,q}$ of $\tilde{F}$ into the sum of (p', q')-Hodge components with $p' < p$, $q' < q$. See [CKS] (2.20).

In terms of Deligne's splitting $(I^{p,q})$, the $\mathbf{R}$-linear map δ is characterized by the following two properties.

(i) For any (p, q), δ sends $I^{p,q}$ into the sum of $I^{p',q'}$ with $p' < p$, $q' < q$.

(ii) For any (p, q), $\exp(-2i\delta)I^{p,q} = \overline{I^{q,p}}$.

1.4. ζ. ζ is determined by δ as follows. Let $(\tilde{I}^{p,q})$ be the $\mathbf{R}$-splitting of $\tilde{F}$, and let $\delta_{p,q}$ be the (p, q)-Hodge component of δ with respect to $(\tilde{I}^{p,q})$, that is, $\delta = \sum_{p,q \in \mathbf{Z}} \delta_{p,q}$, $\delta_{p,q}(\tilde{I}^{r,s}) \subset \tilde{I}^{r+p,s+q}$ for any r, s. Then, ζ is defined as the universal Lie polynomial in the $\delta_{p,q}$ with coefficients in $\mathbf{Q}(i)$ in [CKS] (6.60).

For example (see also Appendix), some Hodge components of ζ are

$$\zeta_{-1,-1} = 0, \quad \zeta_{-1,-2} = -\tfrac{i}{2}\delta_{-1,-2}, \quad \zeta_{-2,-3} = -\tfrac{3i}{8}\delta_{-2,-3} - \tfrac{1}{8}[\delta_{-1,-1}, \delta_{-1,-2}].$$

We review here the definition of this universal Lie polynomial.

Let $b^l_{p,q}$ ($p, q, l \in \mathbf{Z}$, $p, q, l \geq 0$) be the integers determined by $(1 - x)^p(1 + x)^q = \sum_l b^l_{p,q}x^l$, so that $b^l_{p,q} = 0$ unless $p + q \geq l$.

Define non-commutative polynomials $P_k = P_k(X_2, \ldots, X_{k+1})$ over $\mathbf{Q}$ by $P_0 = 1$, $P_k = -\frac{1}{k}\sum_{j=1}^{k} P_{k-j}X_{j+1}$ ($k \geq 1$). (So $P_1 = -X_2$, $P_2 = \frac{1}{2}X_2^2 - \frac{1}{2}X_3$, $P_3 = -\frac{1}{6}X_2^3 + \frac{1}{6}X_3X_2 + \frac{1}{3}X_2X_3 - \frac{1}{3}X_4$, etc.)

Let A be the ring of non-commutative polynomials in variables $\delta_{-p,-q}$ ($p \geq 1, q \geq 1$) over $\mathbf{Q}(i)$. For $p, q \geq 1$, let $S_{-p,-q}$ be the part of A consisting of linear combinations over $\mathbf{Q}(i)$ of products of the form $\delta_{-p_1,-q_1} \cdots \delta_{-p_k,-q_k}$ with $p = \sum_j p_j$, $q = \sum_j q_j$. Then A is the direct sum of the $S_{-p,-q}$ and $\mathbf{Q}(i)$ as a $\mathbf{Q}(i)$-module.

In [CKS] (6.60), it is proved that there exist a unique family of elements $\zeta_{-p,-q}$ and $\eta_{-p,-q}$ of $S_{-p,-q}$ ($p, q \geq 1$) satisfying the following conditions (i) and (ii).

(i) Let $\hat{A}$ be the formal completion $\varprojlim_k A/I^k$, where I^k denotes the sum of $S_{-p,-q}$ such that $p+q \geq k$. Let $\zeta = \sum_{p,q} \zeta_{-p,-q}$, $\eta = \sum_{p,q} \eta_{-p,-q} \in \hat{A}$. Then we have an identity in $\hat{A}$

$$\exp(-\zeta)\exp(i\delta) = \textstyle\sum_{k\geq 0} P_k(C_2, \dots, C_{k+1}),$$

where $C_{l+1} = i \sum\limits_{p,q\geq 1} b^{l-1}_{p-1,q-1}\eta_{-p,-q}$.

(ii) By the unique ring homomorphism $A \to A$ which sends i to $-i$ and $\delta_{-p,-q}$ to $\delta_{-q,-p}$, $\zeta_{-p,-q}$ is sent to $\zeta_{-q,-p}$, and $\eta_{-p,-q}$ is sent to $\eta_{-q,-p}$.

This is the definition of ζ.

For example, the identity in (i) shows $-\zeta_{-1,-1}+i\delta_{-1,-1} = -i\eta_{-1,-1}$. By this and by (ii), we have $-\zeta_{-1,-1}-i\delta_{-1,-1} = i\eta_{-1,-1}$. From these two equalities, we have $\zeta_{-1,-1} = 0$ and $\eta_{-1,-1} = -\delta_{-1,-1}$.

For the fact that the $\zeta_{-p,-q}$ are Lie polynomials and for some further computations of the $\zeta_{-p,-q}$, see Appendix.

Example 1.5. Let (W, F) be a mixed Hodge structure, and assume that there is $k \in \mathbf{Z}$ such that $W_k = V$ and $W_{k-2} = 0$. Then $\delta = \zeta = 0$ since $\delta_{-p,-q} = 0$ unless $p > 0, q > 0$. Hence $\hat{F} = F$ and F is $\mathbf{R}$-split.

Assume further that $k = 0$ and gr^W_0 is of one dimensional. We fix an isomorphism $\mathrm{gr}^W_0 \cong \mathbf{R}$. Then the associated $\mathbf{R}$-splitting is described as follows. There is a unique element $v \in F^0 \cap V$ which lifts $1 \in \mathbf{R} = \mathrm{gr}^W_0$, and the $\mathbf{R}$-splitting of W is characterized by the property that this v is pure of weight 0. The element v is obtained as follows. Take a lifting $e \in V$ of $1 \in \mathbf{R}$. Since $F(\mathrm{gr}^W_0)$ is of type $(0, 0)$, there is $a \in W_{-1,\mathbf{C}}$ such that F^0 is generated by $F^0 \cap W_{-1,\mathbf{C}}$ and $e + a$. Since the restriction of F to $W_{-1,\mathbf{C}}$ is pure of weight -1, we have $W_{-1,\mathbf{C}} = W_{-1} \oplus (F^0 \cap W_{-1,\mathbf{C}})$ as $\mathbf{R}$-vector spaces. Hence we can write $a = b + c$ with $b \in W_{-1}$ and $c \in F^0 \cap W_{-1,\mathbf{C}}$. Then $v = e + b$ is the desired element. The uniqueness reduces to that of b.

The following lemmas will be used later.

Lemma 1.6. *Let $f\colon (V, W, F) \to (V', W', F')$ be a homomorphism of mixed Hodge structures.*

(1) The canonical splittings $s\colon \mathrm{gr}^W \cong V$ and $s'\colon \mathrm{gr}^{W'} \cong V'$ commute with f, that is, $f \circ s = s' \circ \mathrm{gr}(f)$.

(2) $f \circ \delta_{p,q} = \delta'_{p,q} \circ f$, $f \circ \zeta_{p,q} = \zeta'_{p,q} \circ f$, $f \circ \varepsilon_{p,q} = \varepsilon'_{p,q} \circ f$ for any p, q, where $\delta_{p,q}$, $\zeta_{p,q}$, $\varepsilon_{p,q}$ are the (p, q)-parts of δ, ζ, ε for (W, F), respectively, and $\delta'_{p,q}$, $\zeta'_{p,q}$, $\varepsilon'_{p,q}$ are those for (W', F').

Proof. We may assume that f is surjective or injective. We prove the lemma assuming that f is surjective. The proof for an injective f is similar and omitted.

First, by the explicit construction of Deligne's splitting in 1.1, we have

(3) $f(I^{p,q}_{(W,F)}) = I^{p,q}_{(W',F')}$ for any p, q.

Next we show

(4) $f \circ \delta = \delta' \circ f$,

where $\delta' = \delta(W', F')$. For this, it is sufficient to prove $\exp(-2i\delta') \circ f = f \circ \exp(-2i\delta)$ because δ and δ' are nilpotent. Let $x \in I^{p,q}_{(W,F)}$ and let $y = \exp(-2i\delta)x$. Then the characterization of δ by (i) and (ii) in 1.3 shows that y is the unique element of $\overline{I^{q,p}_{(W,F)}}$ such that $x \equiv y \mod W_{p+q-1,\mathbf{C}}$. Since $f(x) \equiv f(y) \mod W'_{p+q-1,\mathbf{C}}$ and $f(x) \in I^{p,q}_{(W',F')}$, $f(y) \in \overline{I^{q,p}_{(W',F')}}$ by (3), the characterization of δ' by (i) and (ii) in 1.3 shows that $\exp(-2i\delta')f(x) = f(y)$. Hence $\exp(-2i\delta') \circ f = f \circ \exp(-2i\delta)$.

Now, (3) and (4) imply

(5) $f \circ \delta_{p,q} = \delta'_{p,q} \circ f$ for any p, q.

Since ζ is defined by the universal Lie polynomial of the $\delta_{p,q}$, (5) implies

(6) $f \circ \zeta_{p,q} = \zeta'_{p,q} \circ f$ and $f \circ \varepsilon_{p,q} = \varepsilon'_{p,q} \circ f$ for any p, q.

This proves (2).

Hence $f(\hat{F}) = \hat{F}'$, which proves (1). $\square$

The following 1.7 is proved for δ in [CKS] §2, and the results for ζ, ε follow from it.

Lemma 1.7. *Let (W, F) be a mixed Hodge structure on a finite dimensional $\mathbf{R}$-vector space V, let $r \in \mathbf{Z}$, and let $f : V \to V$ be a linear map satisfying $f(F^p) \subset F^{p+r}$ and $f(W_k) \subset W_{k+2r}$ for all $p, k \in \mathbf{Z}$. Then:*

$$f \circ \delta_{p,q} = \delta_{p,q} \circ f, \quad f \circ \zeta_{p,q} = \zeta_{p,q} \circ f, \quad f \circ \varepsilon_{p,q} = \varepsilon_{p,q} \circ f \quad \textit{for any } p, q.$$

§2. Review of SL(2)-orbit theorem in pure case

Fix $(V, w, \langle\ ,\ \rangle)$ as in 0.1. Let D, $D^\vee$, $G_{\mathbf{R}}$, $\mathfrak{g}_{\mathbf{R}}$ be as in 0.1.

In this section, we review the theory of SL(2)-orbits in [Sc] and [CKS].

2.1. SL(2)-*orbits.* An SL(2)-orbit in n variables is a pair (ρ, φ) of a homomorphism of algebraic groups $\mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}$, which is defined over $\mathbf{R}$, and a holomorphic map $\mathbf{P}^1(\mathbf{C})^n \to D^\vee$ satisfying the following conditions 2.1.1–2.1.3.

2.1.1. $\varphi(gz) = \rho(g)\varphi(z)$ for any $g \in \mathrm{SL}(2, \mathbf{C})^n, z \in \mathbf{P}^1(\mathbf{C})^n$.

2.1.2. $\varphi(\mathfrak{h}^n) \subset D$. Here $\mathfrak{h}$ is the upper half plane.

2.1.3. The homomorphism of Lie algebras $\mathfrak{sl}(2, \mathbf{C})^n \to \mathfrak{g}_{\mathbf{C}}$ induced by ρ sends $F^p_z(\mathfrak{sl}(2, \mathbf{C})^n)$ into $F^p_{\varphi(z)}\mathfrak{g}_{\mathbf{C}}$ for any $z \in \mathbf{P}^1(\mathbf{C})^n$ and $p \in \mathbf{Z}$, where $F_z(\mathfrak{sl}(2, \mathbf{C})^n)$ and $F_{\varphi(z)}\mathfrak{g}_{\mathbf{C}}$ are the decreasing filtrations defined as follows.

For $\alpha \in \mathbf{P}^1(\mathbf{C})$, let F_α be the decreasing filtration on $\mathbf{C}^2$ defined by : $F^{-1}_\alpha = \mathbf{C}^2$, $F^1_\alpha = 0$, and F^0_α is the one dimensional $\mathbf{C}$-subspace of $\mathbf{C}^2$ corresponding to α ($\alpha \in \mathbf{C} \subset \mathbf{P}^1(\mathbf{C})$ corresponds to $\mathbf{C}(\alpha e_1 + e_2)$ and $\infty \in \mathbf{P}^1(\mathbf{C})$ corresponds to $\mathbf{C}e_1$). Define

$$F^p_z(\mathfrak{sl}(2, \mathbf{C})^n) = \{X = (X_j)_j \in \mathfrak{sl}(2, \mathbf{C})^n \mid X_j F^q_{z_j} \subset F^{q+p}_{z_j} \ \ (q \in \mathbf{Z}, 1 \le j \le n)\},$$
$$F^p_{\varphi(z)}(\mathfrak{g}_{\mathbf{C}}) = \{X \in \mathfrak{g}_{\mathbf{C}} \mid X\varphi(z)^q \subset \varphi(z)^{q+p} \ \ (q \in \mathbf{Z})\}.$$

In the present paper, we use the condition 2.1.3 only implicitly, but, for example, it is used in the proof of a result in 8.4 which plays an important role in the present paper (the proof of 8.4 was given in [KU1]).

2.2. We introduce some notation concerning SL(2)-orbits. We first review some basic facts about representations of $\mathbf{G}_m$ and of SL(2). Let U be a finite dimensional vector space over a field K.

2.2.1. A homomorphism $h : \mathbf{G}_m \to \mathrm{Aut}(U)$ of algebraic groups over K corresponds in one to one manner to a direct sum decomposition $U = \bigoplus_{\mu\in\mathbf{Z}} U^{[\mu]}$, by the following rule: For $\mu \in \mathbf{Z}$, $U^{[\mu]}$ is the part of U on which $h(\lambda)$ for $\lambda \in \mathbf{G}_m$ acts as the multiplication by λ^μ.

2.2.2. Assume that K is of characteristic 0. Then a homomorphism $\rho : \mathrm{SL}(2) \to \mathrm{Aut}(U)$ of algebraic groups over K corresponds in one to one manner to a pair of a direct sum decomposition $U = \bigoplus_{\mu\in\mathbf{Z}} U^{[\mu]}$ and a linear map $N : U \to U$ such that $N(U^{[\mu]}) \subset U^{[\mu-2]}$ for any $\mu \in \mathbf{Z}$ and such that N^μ induces an isomorphism $N^\mu : U^{[\mu]} \xrightarrow{\sim} U^{[-\mu]}$ for any $\mu \geq 0$. In fact, the direct sum decomposition of U corresponding to ρ is given by $\mathbf{G}_m \to \mathrm{Aut}(U); \lambda \mapsto \rho\left(\begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$ as in 2.2.1, and the corresponding N is the image of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ under the homomorphism of Lie algebras $\mathfrak{sl}(2) \to \mathrm{End}(U)$ induced by ρ.

If U' is another finite dimensional vector space over K and $\rho' : \mathrm{SL}(2) \to \mathrm{Aut}(U')$ is a homomorphism of algebraic groups over K corresponding to $U' = \bigoplus_\mu U'^{[\mu]}$ and N', and if $f : U \to U'$ is a linear map, then ρ and ρ' are compatible via f if and only if $f(U^{[\mu]}) \subset U'^{[\mu]}$ for any $\mu \in \mathbf{Z}$ and $N'f = fN$.

2.2.3. For a nilpotent linear map $N : U \to U$, let $W(N)$ be the monodormy filtration on U associated to N, which is characterized by the two properties; $NW(N)_k \subset W(N)_{k-2}$ for all $k \in \mathbf{Z}$, and $N^k : \mathrm{gr}_k^{W(N)} \xrightarrow{\sim} \mathrm{gr}_{-k}^{W(N)}$ for all $k \geq 0$ (Deligne [D2] 1.6).

2.2.4. Assume that K is of characteristic 0, let $\rho : \mathrm{SL}(2) \to \mathrm{Aut}(U)$ be a homomorphism of algebraic groups over K, and consider the corresponding $U = \bigoplus_{\mu\in\mathbf{Z}} U^{[\mu]}$ and N. Then $W(N)_k = \bigoplus_{\mu\leq k} U^{[\mu]}$, and $(U^{[\mu]})_\mu$ gives a splitting of $W(N)$.

Let

$$\Delta : \mathbf{G}_m^n \to \mathrm{SL}(2)^n \ ; \ (\lambda_1, \ldots, \lambda_n) \mapsto \left(\begin{pmatrix} 1/\lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} 1/\lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix}\right),$$

$$\Delta^{(j)} : \mathbf{G}_m \to \mathrm{SL}(2)^n \ ; \ \Delta^{(j)}(\lambda) = \Delta(\{\lambda\}^j \times \{1\}^{n-j}) \quad \text{for } 1 \leq j \leq n.$$

2.2.5. Now let (ρ, φ) be an SL(2)-orbit in n variables.

For $1 \leq j \leq n$, let $\hat{N}_j \in \mathfrak{g}_{\mathbf{R}}$ be the image of the j-th $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $\mathfrak{sl}(2,\mathbf{R})^n$ under the homomorphism of Lie algebras $\mathfrak{sl}(2,\mathbf{R})^n \to \mathfrak{g}_{\mathbf{R}}$ induced by ρ. Then $\hat{N}_j$ are nilpotent and $\hat{N}_j\hat{N}_k = \hat{N}_k\hat{N}_j$ for any j, k. Let

$$W^{(j)} = W(\hat{N}_1 + \cdots + \hat{N}_j)[-w] \quad \text{for} \quad 1 \leq j \leq n.$$

(That is, $W_k^{(j)} = W(\hat{N}_1 + \cdots + \hat{N}_j)_{k-w}$ for $k \in \mathbf{Z}$). By 2.2.4 applied to $\mathrm{SL}(2) \to \mathrm{SL}(2)^n \xrightarrow{\rho} \mathrm{Aut}(V)$, where the first arrow is the diagonal embedding into the first j factors of $\mathrm{SL}(2)^n$, we have a splitting $s^{(j)}$ of $W^{(j)}$ characterized by the property that $\rho(\Delta^{(j)}(\lambda))$ acts on $s^{(j)}(\mathrm{gr}_k^{W^{(j)}})$ as the multiplication by λ^{k-w} for any $\lambda \in \mathbf{R}^\times$ and $k \in \mathbf{Z}$.

We call $s^{(j)}$ the splitting of $W^{(j)}$ associated to ρ.

By 2.2.2, the homomorphism $\rho : \mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}$ is determined by $\hat{N}_j$ and $s^{(j)}$ $(1 \le j \le n)$. Note that once the homomorphism ρ is determined, a holomorphic map φ satisfying 2.1.1 is determined by $\varphi(\mathbf{0}_n)$, where $\mathbf{0}_n = (0, \ldots, 0) \in \mathbf{P}^1(\mathbf{C})^n$.

We have a direct sum decomposition $\mathfrak{g}_{\mathbf{R}} = \bigoplus_{\mu \in \mathbf{Z}^n} \mathfrak{g}_{\mathbf{R}}^{[\mu]}$, where $\mathfrak{g}_{\mathbf{R}}^{[\mu]}$ is the part of $\mathfrak{g}_{\mathbf{R}}$ on which $\mathrm{Ad}(\rho(\Delta^{(j)}(\lambda)))$ $(\lambda \in \mathbf{R}^\times)$ acts as the multiplication by $\lambda^{\mu(j)}$ for any $1 \le j \le n$.

We will use the following notation. For $y = (y_1, \ldots, y_n)$ with $y_j > 0$, let

$$t(y) = \rho\left(\begin{pmatrix} \sqrt{y_1} & 0 \\ 0 & 1/\sqrt{y_1} \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{y_n} & 0 \\ 0 & 1/\sqrt{y_n} \end{pmatrix}\right) = \rho(\Delta(\sqrt{y_1}, \ldots, \sqrt{y_n}))^{-1},$$

For $1 \le j \le n$ and $\lambda > 0$, let

$$t^{(j)}(\lambda) = t(\{\lambda\}^j \times \{1\}^{n-j}) = \rho(\Delta^{(j)}(\sqrt{\lambda}))^{-1}.$$

We have

$$t(y) = \textstyle\prod_{j=1}^n t^{(j)}(y_j/y_{j+1}) \quad (y_{n+1} \text{ denotes } 1).$$

2.3. *The associated* SL(2)*-orbit in one variable.* Let $N \in \mathfrak{g}_{\mathbf{R}}$, $F \in D^\vee$, and assume that $(V, w, \langle\ ,\ \rangle, N, F)$ generates a pure nilpotent orbit. Let $W = W(N)[-w]$. Then (W, F) is a mixed Hodge structure by [Sc].

The SL(2)-orbit (ρ, φ) in one variable associated to (N, F) is the SL(2)-orbit characterized by the following properties 2.3.1–2.3.3. (The existence of the SL(2)-orbit satisfying 2.3.1–2.3.3 is shown in [Sc] §5 and [CKS] §3. The uniqueness is shown easily (cf. 2.2).)

2.3.1. The homomorphism of Lie algebras $\mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}_{\mathbf{R}}$ induced by ρ sends $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to N.

2.3.2. The splitting of W associated to ρ in 2.2 coincides with the canonical splitting of W associated to the mixed Hodge structure (W, F).

2.3.3. $\varphi(0) = \hat{F}$, where $(W, \hat{F})$ is the $\mathbf{R}$-split mixed Hodge structure associated to (W, F).

In this case, $\delta(W, F)$ and $\zeta(W, F)$ belong to the Lie algebra $\mathfrak{g}_{\mathbf{R}}$ (cf. the last paragraph of Section 2 in [CKS]), so that $\varepsilon(W, F)$ belongs to $\mathfrak{g}_{\mathbf{C}}$, and they commute with N (1.7).

2.4. The associated SL(2)-orbit in one variable in 2.3 is the SL(2)-orbit characterized also by the properties 2.3.1 and the following 2.4.1.

2.4.1. There exist $c > 0$ and $a_m \in \mathfrak{g}_{\mathbf{R}}$ $(m \ge 1)$ such that $\sum_{m \ge 1} a_m \lambda^m$ absolutely converges when $0 \le \lambda < c$, that

$$\exp(iyN)F = \exp\left(\textstyle\sum_{m \ge 1} a_m y^{-m}\right)\varphi(iy),$$

when $y^{-1} < c$, and that $a_m W(N)_k \subset W(N)_{k+m-1}$ for any m and k.

The fact that the associated SL(2)-orbit (ρ, φ) has the above property 2.4.1 is shown in [Sc] Theorem 5.13.

The uniqueness of the SL(2)-orbit (ρ, φ) satisfying 2.3.1 and 2.4.1 is not explicitly stated in [Sc] nor in [CKS]. In 9.10, we will give a proof of the uniqueness.

2.5. *The associated* SL(2)*-orbit in several variables.* (See [CKS] (4.20).)

Let $N_1, \dots, N_n \in \mathfrak{g}_{\mathbf{R}}$, $F \in D^{\vee}$, and assume that $(V, w, \langle\ ,\ \rangle, N_1, \dots, N_n, F)$ generates a pure nilpotent orbit.

Then by Cattani-Kaplan [CK], $W(\sum_{j=1}^{n} a_j N_j)$ for $a_j \geq 0$ depends only on the set $\{j \mid a_j > 0\}$. (This implies that $(N_1, \dots, N_n, F)$ generates a mixed nilpotent orbit for the trivial filtration W defined by $W_k = V$ for $k \geq w$ and $W_k = 0$ for $k < w$ and with respect to the intersection form $\langle\ ,\ \rangle$ on $\mathrm{gr}^W_w = V$.)

Let

$$W^{(j)} = W(N_1 + \cdots + N_j)[-w] \quad \text{for} \quad 1 \leq j \leq n.$$

The SL(2)-orbit in n variables associated to $(N_1, \dots, N_n, F)$ is as follows.

First $(W^{(n)}, F)$ is a mixed Hodge structure. Let $(W^{(n)}, \hat{F}_{(n)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to it. Then $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$ is a mixed Hodge structure. Let $(W^{(n-1)}, \hat{F}_{(n-1)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to it. Then $(W^{(n-2)}, \exp(iN_{n-1})\hat{F}_{(n-1)})$ is a mixed Hodge structure, and so on. This process continues until we obtain the $\mathbf{R}$-split mixed Hodge structure $(W^{(1)}, \hat{F}_{(1)})$.

The SL(2)-orbit (ρ, φ) associated to $(N_1, \dots, N_n, F)$ is the SL(2)-orbit characterized by the following properties 2.5.1–2.5.4. (The existence of the SL(2)-orbit satisfying 2.5.1–2.5.4 is shown in [CKS] §4. The uniqueness is shown easily (cf. 2.2).)

For $1 \leq j \leq n$, let $\hat{N}_j \in \mathfrak{g}_{\mathbf{R}}$ be the element associated to ρ as in 2.2.

2.5.1. For $1 \leq j \leq n$, $W^{(j)}$ of (ρ, φ) (2.2) coincides with the above $W^{(j)}$.

2.5.2. For $1 \leq j \leq n$, $s^{(j)}$ coincides with the splitting of $W^{(j)}$ by the $\mathbf{R}$-split mixed Hodge structure $(W^{(j)}, \hat{F}_{(j)})$.

2.5.3. Let $1 \leq j \leq n$ and write $N_j = \sum_{\mu \in \mathbf{Z}^n} N_j^{[\mu]}$ with $N_j^{[\mu]} \in \mathfrak{g}_{\mathbf{R}}^{[\mu]}$ (2.2). Then $\hat{N}_j$ is the sum of $N_j^{[\mu]}$ for all μ such that $\mu(k) = 0$ for any $1 \leq k < j$. In particular, $\hat{N}_1 = N_1$.

2.5.4. $\varphi(\mathbf{0}_n) = \hat{F}_{(n)}$.

Furthermore, the following 2.5.5 and 2.5.6 hold.

2.5.5. $\varphi(\mathbf{0}_j, \mathbf{i}_{n-j}) = \hat{F}_{(j)}$ for $1 \leq j \leq n$. Here $\mathbf{0}_j := (0, \dots, 0) \in \mathbf{C}^j$ and $\mathbf{i}_k := (i, \dots, i) \in \mathfrak{h}^k \subset \mathbf{C}^k$.

2.5.6. For $1 \leq j \leq n$, $(N_1, \dots, N_j, \hat{F}_{(j)})$ generates a pure nilpotent orbit, and $\exp(\sum_{k=1}^{j} iy_k N_k)\hat{F}_{(j)}$ belong to D for all $y_k > 0$ $(1 \leq k \leq j)$.

We add two lemmas to this review, which are used later.

Lemma 2.6. *Assume that* $(V, w, \langle\ ,\ \rangle, N_1, \dots, N_n, F)$ *and* $(V', w, \langle\ ,\ \rangle', N'_1, \dots, N'_n, F')$ *generate pure nilpotent orbits (they have the common* w *and* n*), and let* $f : V \to V'$ *be a linear map such that* $N'_j f = f N_j$ *for any* $1 \le j \le n$ *and such that* $f(F^p) \subset (F')^p$ *for any* $p \in \mathbf{Z}$*. (We do not put any assumption on the relation between* $\langle\ ,\ \rangle$ *and* $\langle\ ,\ \rangle'$*.) Let* (ρ, φ) *and* (ρ', φ') *be the associated* SL(2)*-orbits in* n *variables, respectively. Then*:

(1) $\rho'(g)f = f\rho(g)$ *for any* $g \in \mathrm{SL}(2, \mathbf{C})^n$.

(2) For any $z \in \mathbf{P}^1(\mathbf{C})^n$ *and* $p \in \mathbf{Z}$*, we have* $f(\varphi(z)^p) \subset \varphi'(z)^p$.

Proof. Since $N'_j f = f N_j$ for $1 \le j \le n$, we have $f(W^{(j)}_k) \subset {W'}^{(j)}_k$ for any j, k. We have a morphism of mixed Hodge structures $f : (W^{(n)}, F) \to ({W'}^{(n)}, F')$. By Lemma 1.6, we have ${s'}^{(n)} \operatorname{gr}(f) = f s^{(n)}$, and $f(\hat{F}^p_{(n)}) \subset \hat{F'}^p_{(n)}$ for any p with the notation in 2.5. We have a morphism of mixed Hodge structures $f : (W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)}) \to ({W'}^{(n-1)}, \exp(iN'_n)\hat{F'}_{(n)})$. Again by Lemma 1.6, we have ${s'}^{(n-1)} \operatorname{gr}(f) = f s^{(n-1)}$. In this way, we obtain inductively ${s'}^{(j)} \operatorname{gr}(f) = f s^{(j)}$ for $1 \le j \le n$. By this and by 2.5.3, we have $\hat{N}'_j f = f \hat{N}_j$ for $1 \le j \le n$. These prove $\rho'(g)f = f\rho(g)$ for $g \in \mathrm{SL}(2, \mathbf{C})^n$. This and $f(\varphi(\mathbf{0}_n)^p) \subset \varphi'(\mathbf{0}_n)^p$ $(p \in \mathbf{Z})$ show $f(\varphi(z)^p) \subset \varphi'(z)^p$ for any $z \in \mathbf{P}^1(\mathbf{C})^n$ by 2.1.1. $\square$

Lemma 2.7. *Assume that* $(V, w, \langle\ ,\ \rangle, N_1, \dots, N_n, F)$ *generates a pure nilpotent orbit, and let* (ρ, φ) *be the associated* SL(2)*-orbit in* n *variables. Then with the notation in 2.2, we have*

$$\operatorname{Ad}(t^{(j)}(\lambda))(N_k) = \lambda N_k \quad \textit{for } 1 \le k \le j \le n,\ \lambda > 0.$$

Proof. In the case $n = j = 1$, this follows from 2.3.1 and

$$\operatorname{Ad}\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We consider the general case. Let $W^{(j)}$ and $\hat{F}_{(j)}$ be as in 2.5. For $y = (y_1, \dots, y_j)$ $(y_k > 0)$, let $N_y = y_1 N_1 + \dots + y_j N_j$. Then $(N_y, \hat{F}_{(j)})$ generates a pure nilpotent orbit by 2.5.6. Let (ρ', φ') be the SL(2)-orbit in one variable associated to $(N_y, \hat{F}_{(j)})$, and let $t'(\lambda) = \rho'(\Delta(\sqrt{\lambda}))^{-1}$. Then $\operatorname{Ad}(t'(\lambda))(N_y) = \lambda N_y$ by the case $n = j = 1$. Since the mixed Hodge structure $(W^{(j)}, \hat{F}_{(j)})$ is $\mathbf{R}$-split and $W^{(j)} = W(N_y)[-w]$ ([CK]), we have $\varphi'(0) = \hat{F}_{(j)}$ and from this we have $t'(\lambda) = t^{(j)}(\lambda)$. Hence $\operatorname{Ad}(t^{(j)}(\lambda))(N_y) = \lambda N_y$. Since y is arbitrary, this proves $\operatorname{Ad}(t^{(j)}(\lambda))(N_k) = \lambda N_k$ for $k = 1, \dots, j$. $\square$

§3. Mixed object is quotient (and also part) of pure object: Idea

3.1. As in 2.3, if a mixed Hodge structure is of the form $(W(N)[-w], F)$ for some $(V, w, \langle\ ,\ \rangle, N, F)$ generating a pure nilpotent orbit, then the associated $\mathbf{R}$-split Hodge structure $\hat{F}$ is given as $\varphi(0)$ for the associated SL(2)-orbit (ρ, φ).

In this §3, we explain how the definition of the associated $\mathbf{R}$-split Hodge filtration $\hat{F}$ for an arbitrary mixed Hodge structure (W, F) (which need not come from an (N, F)

generating a pure nilpotent orbit) is still given in terms of SL(2)-orbit without using the universal Lie polynomial.

The point is that a mixed Hodge structure can be regarded as a quotient (and also as a part) of a pure object in 0.1 with $n = 1$ which generates a pure nilpotent orbit. See 0.6 for an explanation of this statement by a geometric example.

In fact, this idea "a mixed situation is a quotient of a pure situation" is the key idea of the present paper, and the constructions in the proof of 3.2 below will be repeatedly used in the rest of the present paper.

Lemma 3.2. *Let (W, F) be an $\mathbf{R}$-split mixed Hodge structure on a finite dimensional $\mathbf{R}$-vector space V whose graded quotient $F(\mathrm{gr}^W_w)$ for each $w \in \mathbf{Z}$ is polarized by a non-degenerate $(-1)^w$-symmetric $\mathbf{R}$-bilinear form $\langle\, ,\, \rangle_w : \mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$. Let k be an integer such that all the weights of (W, F) are not less than k.*

(1) Then there are a finite dimensional $\mathbf{R}$-vector space V', $(-1)^k$-symmetric non-degenerate $\mathbf{R}$-bilinear form $\langle\, ,\, \rangle$ on V', an element N of $\mathfrak{g}'_{\mathbf{R}} = \mathrm{Lie}(\mathrm{Aut}(V', \langle\ ,\ \rangle))$, a decreasing filtration F' on $V'_{\mathbf{C}}$, and an isomorphism $V \cong \mathrm{Coker}(N)$ satisfying the following conditions:

(i) $(V', k, \langle\ ,\ \rangle, N, F')$ generates a pure nilpotent orbit and the mixed Hodge structure $(W(N)[-k], F')$ is $\mathbf{R}$-split.

(ii) $W(N)[-k]$ induces W on V and F' induces F on $V_{\mathbf{C}}$.

(2) Furthermore, for any $\delta \in \mathfrak{g}_{\mathbf{R}} = \mathrm{Lie}(\mathrm{Aut}(V, W, (\langle\ ,\ \rangle_w)_w))$ which sends the (p, q)-Hodge component of (W, F) to the sum of (p', q')-Hodge components with $p' < p, q' < q$ for any (p, q), there exists $\delta' \in \mathfrak{g}'_{\mathbf{R}}$ such that

(i)′ δ' sends the (p, q)-Hodge component of $(W(N)[-k], F')$ to the sum of (p', q')-Hodge components with $p' < p, q' < q$ for any (p, q),

(ii)′ δ' commutes with N, and

(iii)′ δ' induces δ on V.

Proof. (1) For $w \in \mathbf{Z}$, let $S_w \subset V$ be the lifting of gr^W_w with respect to the $\mathbf{R}$-split mixed Hodge structure (W, F). We endow S_w with the Hodge structure of weight w induced by $F(\mathrm{gr}^W_w)$.

Let $V' := \bigoplus_{l \geq 0} \bigoplus_{0 \leq m \leq l} S_{k+l}(m)$, where (m) means the Tate twist. We define the Hodge filtration F' on $V'_{\mathbf{C}}$ as the evident direct sum using these twists.

We define N as follows. N sends $x \in S_{k+l}(m)$ to $x(1) \in S_{k+l}(m+1)$ if $0 \leq m < l$, and to 0 if $m = l$. We have an evident isomorphism $V = \bigoplus_{l \geq 0} S_{k+l} \simeq \mathrm{Coker}(N)$:

$$V' = \begin{pmatrix} & S_{k+2} & & \\ & {\scriptstyle N}\downarrow\simeq & S_{k+1} & \\ \cdots & S_{k+2}(1) & {\scriptstyle N}\downarrow\simeq & S_k \\ & {\scriptstyle N}\downarrow\simeq & S_{k+1}(1) & \\ & S_{k+2}(2) & & \end{pmatrix} \to V = \begin{pmatrix} \ddots & & & \\ & S_{k+2} & & \\ & & S_{k+1} & \\ & & & S_k \\ & & & \end{pmatrix}.$$

We define a $(-1)^k$-symmetric pairing $\langle\ ,\ \rangle : V'\times V' \to \mathbf{R}$ as follows. For $x \in S_{k+l}(m)$, $y \in S_{k+l'}(m')$, we define $\langle x, y\rangle = 0$ unless $l = l'$ and $m + m' = l$. For $x \in S_{k+l}(m)$, $y \in S_{k+l}(l-m)$, we define $\langle x, y\rangle = (-1)^m\langle x(-m), y(m-l)\rangle_{k+l}$. (Note that this pairing $\langle\ ,\ \rangle : V'\times V' \to \mathbf{R}$ is the direct sum of the pairings $S_k \times S_k \to \mathbf{R}$, $S_{k+1}\times S_{k+1}(1) \to \mathbf{R}$, $S_{k+2} \times S_{k+2}(2) \to \mathbf{R}$, $S_{k+2}(1) \times S_{k+2}(1) \to \mathbf{R}$, ... with suitable signs.)

Then we can easily check that N is in $\mathfrak{g}'_{\mathbf{R}}$, that $(W(N)[-k], F')$ is an $\mathbf{R}$-split mixed Hodge structure, and that the condition (ii) in (1) is satisfied. It remains to prove that (N, F') generates a pure nilpotent orbit. To see this, by [KK] proposition (1.2.2) or by [CKS] (4.66), it is enough to show that the following (a), (b), and (c) are satisfied.

(a) (N, F') satisfies the Griffiths transversality.

(b) For $l \geq 0$, the primitive part of $\mathrm{gr}_l^{W(N)}$ with the Hodge filtration induced by F' is polarized by $\langle \bullet, N^l\bullet\rangle$.

(c) For any $p \in \mathbf{Z}$, the annihilator of F'^p in $V'_{\mathbf{C}}$ with respect to $\langle\ ,\ \rangle$ coincides with F'^{k-p+1}.

(a) is clear, (c) is easily checked, and (b) is seen once we note that the primitive part coincides with S_{k+l}.

(2) Let δ'' be the endomorphism on V' whose $\mathrm{Hom}(S_{k+l}(m), S_{k+l-a}(m))$-component is $\delta(m)$ $(l \geq a > 0)$, and whose other components are zero. Then, it is clear that δ'' commutes with N and that δ'' induces δ on V. Next, let ${}^t\delta''$ be the transpose of δ'' with respect to $\langle\ ,\ \rangle$. Then, since ${}^t\delta''$ has only $\mathrm{Hom}(S_{k+l}(m), S_{k+l+a}(m + a))$-components for $a > 0$, it induces 0 on V. Further, it commutes with N because

$$ {}^t\delta'' N = -{}^t\delta''{}^tN = -{}^t(N\delta'') = -{}^t(\delta'' N) = -{}^tN{}^t\delta'' = N{}^t\delta'', $$

where we used the fact ${}^tN = -N$. Hence $\delta' := \delta'' - {}^t\delta''$ satisfies the desired properties. □

3.3. We explain the definition of $\hat{F}$ mentioned in 3.1.

Let (W, F) be any mixed Hodge structure.

As is well known, any $\mathbf{R}$-Hodge structure is polarizable. Take polarizations on all graded pieces of (W, F). We can apply 3.2 (1) to the $\mathbf{R}$-split mixed Hodge structure $(W, \tilde{F} := \exp(-i\delta)F)$ in 1.3, and we can apply 3.2 (2) to δ in 1.3.

Let $(N, \tilde{F}')$ and δ' be the lifts which 3.2 gives. Let $F' := \exp(i\delta')\tilde{F}'$. Then (N, F') generates a pure nilpotent orbit by [CKS] (4.66), which induces (W, F) on $\mathrm{Coker}(N) \cong V$. Now let (ρ, φ) be the SL(2)-orbit which is associated to (N, F').

Proposition 3.4. *$\hat{F}$ in 1.2 is the image on $V_{\mathbf{C}}$ of the filtration $\varphi(0)$ on $V'_{\mathbf{C}}$.*

This gives a characterization of $\hat{F}$ in 1.2 without using the universal Lie polynomials. (The associated SL(2)-orbit is characterized without using the universal Lie polynomials as in 2.4, and also as in 8.7.)

Proof of Proposition 3.4. This is deduced from Lemma 1.6 as follows. Since $\tilde{F}'$ is $\mathbf{R}$-split and the (p, q)-Hodge components of δ' is zero unless $p, q < 0$, the unique pair in 1.3 for $(W(N)[-k], F')$ is nothing but $(\tilde{F}', \delta')$. Further, since ζ for F and ζ for F' are defined by the same universal Lie polynomial of $(\delta_{p,q})$ and $(\delta'_{p,q})$ respectively, ζ for F' on V' (which we denote by ζ') induces ζ for F on V. Hence $\hat{F} := \exp(\zeta)\tilde{F}$ is the image of $\exp(\zeta')\tilde{F}'$, which is $\varphi(0)$. $\square$

3.5. In the above we showed that a mixed Hodge structure is a quotient of a pure nilpotent orbit as in 0.1 with one N. We can also show that a mixed Hodge structure is a part of a pure nilpotent orbit as in 0.1 with one N. In fact, we can prove the variant of Lemma 3.2 in which the part "not less than k" is replaced by "not bigger than k", and $V \cong \mathrm{Coker}(N)$ is replaced by $V \cong \mathrm{Ker}(N)$. The proof of this variant is the evident modification of the above proof of 3.2.

§4. Reduction of Main Theorem to two propositions

In this section, we prove our main result Theorem 0.5 assuming the following two propositions 4.1 and 4.2. These propositions will be proved in later sections.

In §3, we showed that any mixed Hodge structure is a quotient of (and also a part of) a pure nilpotent orbit in 0.1 with one N. Proposition 4.1 says that a mixed nilpotent orbit in 0.2 is a quotient of (and also a part of) a pure nilpotent orbit in 0.1 which has one more N.

Proposition 4.1. *Assume that* $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, F)$ *generates a mixed nilpotent orbit. Then:*

(1) There exist $(V', w, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ *generating a pure nilpotent orbit, a surjective* $\mathbf{R}$*-linear map* $p : V' \to V$*, and non-negative real numbers* a_{jk} $(1 \le k \le j \le n)$ *with* $a_{jj} = 1$ *such that* W *is the image of* $W(N'_0)[-w]$*,* F *is the image of* F'*,* $p \circ N'_0 = 0$*,* $p \circ N'_j = \sum_{k=1}^{j} a_{jk} N_k \circ p$ *for* $1 \le j \le n$*.*

(2) There exist $(V', w, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ *generating a pure nilpotent orbit, an injective* $\mathbf{R}$*-linear map* $q : V \to V'$*, and non-negative real numbers* a_{jk} $(1 \le k \le j \le n)$ *with* $a_{jj} = 1$ *such that* W *is the pull-back of* $W(N'_0)[-w]$*,* F *is the pull-back of* F'*,* $N'_0 \circ q = 0$*,* $N'_j \circ q = q \circ \sum_{k=1}^{j} a_{jk} N_k$ *for* $1 \le j \le n$*.*

Here we do not put any relation between $(\langle\ ,\ \rangle_w)_w$ and $\langle\ ,\ \rangle$.

The proof of Proposition 4.1 will be given in §6 and §7 below.

Remark. The authors do not know whether we can take all the a_{jk} in 4.1 to be 0 unless $j = k$, that is, whether we can put the stronger condition $p \circ N'_j = N_j \circ p$ (resp. $N'_j \circ q = q \circ N_j$) for $1 \le j \le n$ in (1) (resp. (2)). See 5.9 for a comment on this point.

The following proposition is a complementary result to the SL(2)-orbit theorem of Cattani-Kaplan-Schmid [CKS] for pure nilpotent orbits as in 0.1.

Proposition 4.2. *Assume that* $(V, w, \langle\ ,\ \rangle, N_1, \dots, N_n, F)$ *generates a pure nilpotent orbit, and let* (ρ, φ) *be the associated* SL(2)*-orbit in* n *variables. For* $1 \le j \le n$*, let* $W^{(j)} = W(N_1 + \cdots + N_j)[-w]$*, and let* $s^{(j)}$ *be the splitting of* $W^{(j)}$ *associated to* ρ *(2.2, 2.5.1). Fix* k *such that* $1 \le k \le n$*. For* $y = (y_{k+1}, \dots, y_n)$ *such that* $y_j \gg 0$ $(k < j \le n)$*, let* $s(y) : \mathrm{gr}^{W^{(k)}}(V) \xrightarrow{\sim} V$ *be the canonical splitting of* $W^{(k)}$ *associated to the mixed Hodge structure* $(W^{(k)}, \exp(\sum_{j=k+1}^{n} iy_j N_j)F)$*. Then* $s(y)$ *converges to* $s^{(k)}$

as y_j/y_{j+1} tends to ∞ for $k < j \le n$. Furthermore there are $c > 0$ and $u_m \in \mathfrak{g}_{\mathbf{R}}$ ($m \in \mathbf{N}^{n-k}$) such that $u_0 = 0$, $\sum_{m\in\mathbf{N}^{n-k}} u_m \prod_{j=1}^{n-k} \lambda_j^{m(j)}$ absolutely converges when $0 \le \lambda_j < c$, and such that the following (i)–(iii) are satisfied.

(i) Whenever $y_{j+1}/y_j < c$ ($k < j \le n$), we have

$$s(y) = u(y)s^{(k)} \quad \text{with} \quad u(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^{n-k}} u_m \prod_{j=k+1}^{n} (\frac{y_{j+1}}{y_j})^{m(j-k)}).$$

(ii) $u_m W_l^{(k)} \subset W_{l-1}^{(k)}$ for any $m \in \mathbf{N}^{n-k}$ and any $l \in \mathbf{Z}$.

(iii) $u_m W_l^{(j)} \subset W_{l+m(j-k)}^{(j)}$ (resp. $u_m W_l^{(j)} \subset W_l^{(j)}$) for any $m \in \mathbf{N}^{n-k}$ and any $l \in \mathbf{Z}$ if $k < j \le n$ (resp. $1 \le j \le k$).

The proof of Proposition 4.2 will be given in §8 and §9 below.

4.3. In the rest of this section, we assume 4.1 and 4.2.

Let $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, F)$ be as in the hypothesis of Theorem 0.5.

Let $(V', w, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ be as in 4.1.

For the proof of 0.5, we may replace N_j ($1 \le j \le n$) by $\sum_{k=1}^{j} a_{jk} N_k$, where the a_{jk} are as in 4.1. Hence we may assume $N_j \circ p = p \circ N'_j$ for $1 \le j \le n$. Note $p \circ N'_0 = 0$.

Let (ρ', φ') be the SL(2)-orbit in $n+1$ variables associated to $(N'_0, \dots, N'_n, F')$. For $0 \le j \le n$, let $W'^{(j)} = W(N'_0 + \cdots + N'_j)[-w]$, and let $s'^{(j)}$ be the splitting of $W'^{(j)}$ associated to ρ'. We denote $W'^{(0)} = W(N'_0)[-w]$ simply by W', and $s'^{(0)}$ simply by s'. Note that W is the image of W' on V.

By 4.2 applied to $(V', w, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ (we take $W' = W'^{(0)}$ as $W^{(k)}$ of 4.2), we have the following facts.

For $y = (y_1, \dots, y_n)$ with $y_j \gg 0$ ($1 \le j \le n$), let $s'(y) : \mathrm{gr}^{W'}(V') \xrightarrow{\sim} V'$ be the canonical splitting of W' associated to the mixed Hodge structure $(W', \exp(\sum_{j=1}^{n} iy_j N'_j)F')$. Then $s'(y)$ converges to s' as y_j/y_{j+1} tends to ∞ for $1 \le j \le n$. Furthermore there are $c > 0$ and $u'_m \in \mathfrak{g}'_{\mathbf{R}}$ ($m \in \mathbf{N}^n$; here $\mathfrak{g}'_{\mathbf{R}}$ is the $\mathfrak{g}_{\mathbf{R}}$ in 0.1 defined for $(V', w, \langle\ ,\ \rangle)$) such that $u'_0 = 0$, $\sum_{m\in\mathbf{N}^n} u'_m \prod_{j=1}^{n} \lambda_j^{m(j)}$ absolutely converges when $0 \le \lambda_j < c$, and such that the following (i)–(iii) are satisfied.

(i) Whenever $y_{j+1}/y_j < c$ ($1 \le j \le n$), we have

$$s'(y) = u'(y)s' \quad \text{with} \quad u'(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} u'_m \prod_{j=1}^{n} (\frac{y_{j+1}}{y_j})^{m(j)}).$$

(ii) For any $m \in \mathbf{N}^n$ and any $l \in \mathbf{Z}$, u'_m sends W'_l into W'_{l-1}.

(iii) Let $1 \le j \le n$. Then for any $m \in \mathbf{N}^n$ and $l \in \mathbf{Z}$, u'_m sends $W'^{(j)}_l$ into $W'^{(j)}_{l+m(j)}$.

Lemma 4.4. *For $g = (g_1, \dots, g_n) \in \mathrm{SL}(2, \mathbf{C})^n$, define $\rho''(g) = \rho'(1, g_1, \dots, g_n)$. Then $\rho''(g)$ preserves $W'_{\mathbf{C}}$. Let $\rho''_k(g)$ for $k \in \mathbf{Z}$ be the automorphism of $\mathrm{gr}^{W'}_{k,\mathbf{C}}$ induced by $\rho''(g)$. Then $\rho''_k(g)$ ($g \in \mathrm{SL}(2, \mathbf{C})^n$) commutes with $\rho_k(g)$ on $\mathrm{gr}^{W}_{k,\mathbf{C}}$ via $\mathrm{gr}_k(p) : \mathrm{gr}^{W'}_{k,\mathbf{C}} \to \mathrm{gr}^{W}_{k,\mathbf{C}}$.*

Proof. We apply 2.6 by taking $\Phi' = (\mathrm{gr}^{W'}_k(V'), \mathrm{gr}^{W'}_k(N'_1), \dots, \mathrm{gr}^{W'}_k(N'_n), F'(\mathrm{gr}^{W'}_k))$ as $(V, N_1, \dots, N_n, F)$ in 2.6, and taking $\Phi = (\mathrm{gr}^{W}_k(V), \mathrm{gr}^{W}_k(N_1), \dots, \mathrm{gr}^{W}_k(N_n), F(\mathrm{gr}^{W}_k))$ as

$(V', N_1', \ldots, N_n', F')$ in 2.6. For this, we need to define an intersection form $\langle\ ,\ \rangle_k'$ on $\mathrm{gr}_k^{W'}(V')$ for which Φ' generates a pure nilpotent orbit. First assume $k \geq w$. Then, we have $\mathrm{gr}_k^{W'}(V') = \bigoplus_{l\geq 0} A_l$, where A_l is the intersection of the image of $(N_0')^l$: $\mathrm{gr}_{k+2l}^{W'} \to \mathrm{gr}_k^{W'}$ and the kernel of $(N_0')^{l+(k-w)+1} : \mathrm{gr}_k^{W'} \to \mathrm{gr}_{2w-k-2l-2}^{W'}$ ([D2] 1.6). The intersection form $\langle\ ,\ \rangle_k'$ on $\mathrm{gr}_k^{W'}(V')$ is defined to be $\langle \bullet, N^l \bullet\rangle$ on A_l for even l, and to be $-\langle \bullet, N^l \bullet\rangle$ on A_l for odd l. The intersection form $\langle\ ,\ \rangle_k'$ in the case $k \leq w$ is induced from $\langle\ ,\ \rangle_{2w-k}'$ via $(N_0')^{w-k} : \mathrm{gr}_{2w-k}^{W'} \xrightarrow{\sim} \mathrm{gr}_k^{W'}$. By [KK] (1.2.2) or by [CKS] (4.66), with this intersection form, Φ' generates a pure nilpotent orbit.

Hence 4.4 is deduced from 2.6. □

4.5. We prove (1) of 0.5, i.e., that $s(y)$ converges to a splitting of W.

By Lemma 1.6 (1), we have $ps'(y) = s(y)\,\mathrm{gr}(p)$. By 4.3, $ps'(y)$ converges to ps'. Hence $s(y)\,\mathrm{gr}(p)$ converges to ps'. Let $q : \mathrm{gr}^W(V) \to \mathrm{gr}^{W'}(V')$ be an $\mathbf{R}$-linear map such that $\mathrm{gr}(p)q$ is the identity. Then $s(y) = s(y)\,\mathrm{gr}(p)q$ and hence $s(y)$ converges to $s := ps'q$, which is a splitting of W since it is a limit of splittings of W. □

Lemma 4.6. *Let s be as in 0.5 (1), and define $t(y) = s(\bigoplus_{k\in\mathbf{Z}} y_1^{-k/2} t_k(y))s^{-1}$ for $y = (y_1, \ldots, y_n)$, $y_j > 0$ $(1 \leq j \leq n)$. For $0 \leq j \leq n$ and $\lambda > 0$, let $(t')^{(j)}(\lambda) = \rho'(\Delta(a_0, \ldots, a_n))^{-1}$ with $a_l = \sqrt{\lambda}$ for $0 \leq l \leq j$ and $a_l = 1$ for $j < l \leq n$. Then for $y = (y_1, \ldots, y_n)$ $(y_j > 0)$, $t(y)$ on V commutes with $\prod_{j=1}^n (t')^{(j)}(y_j/y_{j+1})$ on V'.*

Proof. In fact, $\prod_{j=1}^n (t')^{(j)}(y_j/y_{j+1}) = \rho'(\Delta(a_0, \ldots, a_n))^{-1}$, where $a_0 = \sqrt{y_1}$, $a_j = \sqrt{y_j}$ for $1 \leq j \leq n$. Since $\rho'(\Delta(1, a_1, \ldots, a_n))$ is compatible with $\rho_k(\Delta(a_1, \ldots, a_n))$ by 4.4, we have 4.6. □

4.7. Define $u(y) = s(y)s^{-1} \colon V \to V$. We have $s\,\mathrm{gr}(p) = ps'$, $pu'(y) = u(y)p$.

4.8. We prove (2) of 0.5. Let $q : \mathrm{gr}^W(V) \to \mathrm{gr}^{W'}(V')$ be as in 4.5. Then $u(y) = ps'(y)qs^{-1} = pu'(y)s'qs^{-1}$. Hence by the presentation of $u'(y)$ in 4.3 (i), if we put $\lambda_j = y_{j+1}/y_j$ for $1 \leq j \leq n$, the map $\lambda = (\lambda_1, \ldots, \lambda_n) \mapsto u(y) \in G_{\mathbf{R}}$ defined when $0 < \lambda_j < c$ $(1 \leq j \leq n)$ is extended to a real analytic function in the λ_j $(1 \leq j \leq n)$ defined on the area $-c < \lambda_j < c$. Hence we have the presentation of $u(y)$ of the form $u(y) = \exp(\sum_{m\in\mathbf{N}^n} u_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)})$ for y_{j+1}/y_j sufficiently small. Furthermore $u_0 = 0$ because $u(y)$ converges to 1 as $y_j/y_{j+1} \to \infty$. By $u(y)p = pu'(y)$, we have $u_m p = pu_m'$ for any m. By 4.6, the property (ii) of u_m follows from 4.3 (iii). □

4.9. We prove (3) of 0.5. The statement for the $\mathbf{R}$-splitting is shown as follows. The $\mathbf{R}$-split mixed Hodge structure associated to $(W, \exp(\sum_{j=1}^n iy_jN_j)F)$ is

$$\begin{aligned} &s(y)(\exp(\textstyle\sum_{j=1}^n iy_j\,\mathrm{gr}^W(N_j))F(\mathrm{gr}^W)) \\ &= u(y)s(\textstyle\bigoplus_w g_w(y)t_w(y)\mathbf{r}_w) = g(y)s(\bigoplus_w t_w(y))s^{-1}\mathbf{r}. \end{aligned}$$

Let $c(y_1)$ be the $\mathbf{R}$-linear map $V \to V$ whose restriction to $s(\mathrm{gr}_w^W)$ is given by the multiplication by $y_1^{-w/2}$ for any $w \in \mathbf{Z}$. Then $c(y_1)\mathbf{r} = \mathbf{r}$. Since $t(y) = s(\bigoplus_w t_w(y))s^{-1}c(y_1)$, we have

$$g(y)s(\textstyle\bigoplus_w t_w(y))s^{-1}\mathbf{r} = g(y)s(\bigoplus_w t_w(y))s^{-1}c(y_1)\mathbf{r} = g(y)t(y)\mathbf{r}.$$

We prove the expression of ${}^e g(y)$ as the exponential of power series.

Since $g(y) = u(y)s(\bigoplus_w g_w(y))s^{-1}$ and $t(y) = s(\bigoplus_w y_1^{-w/2} t_w(y))s^{-1}$, we have

$$ {}^e g(y) = t(y)^{-1}g(y)t(y) = t(y)^{-1}u(y)t(y)s(\bigoplus_w {}^e g_w(y))s^{-1}. $$

Hence it is sufficient to prove that $t(y)^{-1}u(y)t(y)$ is the exponential of a convergent power series in y_{j+1}/y_j $(1 \le j \le n)$ without constant term.

By (iii) in 4.3, if we write u'_m as the sum of $u'^{[\mu]}_m$, where $\mathrm{Ad}(t'^{(j)}(\lambda))$ $(1 \le j \le n, \lambda > 0)$ acts on $u'^{[\mu]}_m$ as the multiplication by $\lambda^{-\mu(j)/2}$, we have $u'^{[\mu]}_m = 0$ unless $\mu \le m$. Hence by $u_m p = pu'_m$ and 4.6, we have:

Claim 1. Write u_m as the sum of $u^{[\mu]}_m$ for $\mu \in \mathbf{Z}^n$, where $\mathrm{Ad}(t(y))$ $(y = (y_1, \dots, y_n)$, $y_j > 0)$ acts on $u^{[\mu]}_m$ as the multiplication by $\prod_{j=1}^n (y_j/y_{j+1})^{\mu(j)/2}$. Then $u^{[\mu]}_m = 0$ unless $\mu \le m$.

We have

$$ t(y)^{-1}u(y)t(y) = \exp(\textstyle\sum_{m \in \mathbf{N}^n, \mu \in \mathbf{Z}^n} u^{[\mu]}_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)-\mu(j)/2}). $$

Since $m > \mu/2$ (that is, $m \ge \mu/2$ and $m \ne \mu/2$) for any $m \in \mathbf{N}^n \setminus \{0\}$ and $\mu \in \mathbf{Z}^n$ satisfying $m \ge \mu$, Claim 1 shows that the infinite sum inside exp is a convergent power series without the constant term. □

4.10. We prove (4) of 0.5. Let

$$ t'(y_0, \dots, y_n) = \rho'(\Delta(\sqrt{y_0}, \dots, \sqrt{y_n}))^{-1} = \textstyle\prod_{j=0}^n t'^{(j)}(\frac{y_j}{y_{j+1}}). $$

By applying the SL(2)-orbit theorem of Cattani-Kaplan-Schmid [CKS] (4.20) (see 0.1) to $(N'_0, \dots, N'_n, F')$, we see the following: As a function in the $\lambda_j := (y_{j+1}/y_j)^{1/2}$ $(0 \le j \le n)$ with values in $D' = ($"D" of $(V', w, \langle\ ,\ \rangle))$, $t'(y_0, \dots, y_n)^{-1} \exp(\sum_{j=0}^n iy_j N'_j)F'$ (defined when the $\lambda_j > 0$ $(0 \le j \le n)$ are small) is extended to a real analytic function in the λ_j defined on the area $-c < \lambda_j < c$ $(0 \le j \le n)$ for some $c > 0$. Fix $\lambda_0 > 0$ which is smaller than c, and consider the image under $p : V'_{\mathbf{C}} \to V_{\mathbf{C}}$. By 4.6, we see that as a function in the λ_j $(1 \le j \le n)$ with values in D, $t(y_1, \dots, y_n)^{-1} \exp(\sum_{j=1}^n iy_j N_j)F$ (defined when the $\lambda_j > 0$ $(1 \le j \le n)$ are small) is extended to a real analytic function with values in D defined on the area $-c < \lambda_j < c$ $(1 \le j \le n)$ for some $c > 0$. By real analyticity of $\varepsilon(W, ?)$ ([CKS] section 2), this shows that the function $\varepsilon(W, {}^e g(y)^{-1} t(y)^{-1} \exp(\sum_{j=1}^n iy_j N_j)F)$ in the λ_j $(1 \le j \le n)$ with values in $\mathfrak{g}_{\mathbf{C}}$, where $y = (y_1, \dots, y_n)$, is extended to a real analytic function defined on the area $-c < \lambda_j < c$ $(1 \le j \le n)$ for some $c > 0$. Hence, when the $\lambda_j > 0$ $(1 \le j \le n)$ are sufficiently small, $\varepsilon(W, t(y)^{-1} g(y)^{-1} \exp(\sum_{j=1}^n iy_j N_j)F)$ can be written as a convergent power series in $\lambda_1, \dots, \lambda_n$. □

§5. Review of mixed nilpotent orbit

In this section, we review some results by Kashiwara in [K1], [K2] on mixed nilpotent orbits and discuss some related results.

5.1. We first review the definition of relative monodromy filtration ([D2] 1.6).

Let V be a finite dimensional $\mathbf{R}$-vector space, W an increasing filtration of V, and $N : V \to V$ a nilpotent $\mathbf{R}$-linear map such that $NW_w \subset W_w$ for any $w \in \mathbf{Z}$. An increasing filtration M on V is called a relative monodromy filtration of N with respect to W if the following (i) and (ii) are satisfied.

(i) $NM_k \subset M_{k-2}$ for any $k \in \mathbf{Z}$.

(ii) $N^l : \mathrm{gr}^M_{k+l}\,\mathrm{gr}^W_k \xrightarrow{\sim} \mathrm{gr}^M_{k-l}\,\mathrm{gr}^W_k$ for all $k \in \mathbf{Z}$ and $l \geq 0$.

A relative monodromy filtration need not exist, but it is unique if it exists. If it exists, it is written as $M(N, W)$.

We recall that in the definition of mixed nilpotent orbit in 0.2, the condition (iv), that is the existence of the relative monodromy filtration, is essential and controls the convergence of the canonical splitting in our main theorem. See the example 0.9.

5.2. By a real infinitesimal mixed Hodge module (we abbreviate it as $\mathbf{R}$-IMHM), we mean $(V, W, N_1, \ldots, N_n, F)$ such that for some non-degenerate $(-1)^w$-symmetric bilinear forms $\langle\ ,\ \rangle_w : \mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$ given for all $w \in \mathbf{Z}$, $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \ldots, N_n, F)$ generates a mixed nilpotent orbit in the sense of 0.2. We do not review the definition of "infinitesimal mixed Hodge module (IMHM)" in [K2], but just tell that if $(V, W, N_1, \ldots, N_n, F)$ is an $\mathbf{R}$-IMHM, then $(V_{\mathbf{C}}; W_{\mathbf{C}}; F, \bar{F}; N_1, \ldots, N_n)$, where $\bar{F}$ is the complex conjugate of F, is an IMHM. (In the definition of IMHM, the real vector space V does not appear; the complex vector space $V_{\mathbf{C}}$ appears but the real structure V is not fixed, and $\bar{F}$ is taken independently of F (satisfying a certain condition). Roughly speaking, an $\mathbf{R}$-IMHM is an IMHM with an $\mathbf{R}$-structure.)

Following the terminology in [K2] but adding $\mathbf{R}$, by a pre-$\mathbf{R}$-IMHM, we mean $(V, W, N_1, \ldots, N_n, F)$ which satisfies all the conditions of $\mathbf{R}$-IMHM except the condition (iv) in 0.2.

The following results 5.3–5.6 on $\mathbf{R}$-IMHM are deduced from results on IMHM in [K1], [K2].

From now on we fix n in $(V, W, N_1, \ldots, N_n, F)$. The meaning of a homomorphism $(V, W, N_1, \ldots, N_n, F) \to (V', W', N'_1, \ldots, N'_n, F')$ of $\mathbf{R}$-IMHMs (resp. pre-$\mathbf{R}$-IMHMs) is clear.

5.3. The direct sums, the duals, and the tensor products are defined in the category of $\mathbf{R}$-IMHM and also in the category of pre-$\mathbf{R}$-IMHM, in the natural way. For example, the tensor product of $(V, W, N_1, \ldots, N_n, F)$ and $(V', W', N'_1, \ldots, N'_n, F')$ is $(V'', W'', N''_1, \ldots, N''_n, F'')$, where $V'' = V \otimes V'$, $W''_w = \sum_{k+l=w} W_k \otimes W'_l$, $N''_j = N_j \otimes 1 + 1 \otimes N'_j$, and $F''^p = \sum_{k+l=p} F^k \otimes F'^l$.

Furthermore ([K2] Proposition 5.2.6), the category of $\mathbf{R}$-IMHM and the category of pre-$\mathbf{R}$-IMHM are abelian. In both categories, the kernel of a homomorphism

$$(V, W, N_1, \ldots, N_n, F) \to (V', W', N'_1, \ldots, N'_n, F')$$

is $(V'', W'', N''_1, \ldots, N''_n, F'')$ with $V'' = \mathrm{Ker}(V \to V')$ and with the restrictions W'', N''_j, F'' of W, N_j, F to V'', respectively. The description of the cokernel is similar. Any homomorphism is strict for the filtrations. That is, if V'' denotes the image of $V \to V'$, the image of W_k (resp. F^p) in V'' coincides with the restriction of W'_k (resp. $(F')^p$) to V'' for any k (resp. p). Furthermore, a sequence is exact in this category if and only if the sequence of the underlying vector spaces is exact. If the sequence is exact, then the sequences of W_k of each object and the sequence of F^p of each object are exact for any k and p.

Theorem 5.4 **(**[K2] Theorem 4.4.1**).** *Let $H = (V, W, N_1, \dots, N_n, F)$ be a pre-$\mathbf{R}$-IMHM. Assume that there exists the relative monodromy weight filtration $M(N_j, W)$ for each $j = 1, \dots, n$. Then H is an $\mathbf{R}$-IMHM.*

Proposition 5.5 **(**[K2] Proposition 5.2.4**).** *Let $(V, W, N_1, \dots, N_n, F)$ be an $\mathbf{R}$-IMHM. For $1 \le j \le n$, let $W^{(j)} = M(N_1 + \cdots + N_j, W)$ ($W^{(0)}$ is defined to be W). Then the filtrations $W^{(0)}_{\mathbf{C}}, \dots, W^{(n)}_{\mathbf{C}}, F$ are distributive.*

The distributivity here means that if Φ denotes the smallest set of $\mathbf{C}$-subspaces of $V_{\mathbf{C}}$ containing $W^{(j)}_{\mathbf{C},k}$ $(0 \le j \le n, k \in \mathbf{Z})$ and F^p $(p \in \mathbf{Z})$ which is stable under the operations $(A, B) \mapsto A + B$ and $(A, B) \mapsto A \cap B$, then the distributive laws

$$(A \cap B) + C = (A + C) \cap (B + C), \quad (A + B) \cap C = (A \cap C) + (B \cap C) \quad \text{for all } A, B, C \in \Phi$$

hold. (See [K1].)

Proposition 5.6 **(**[K2] Proposition 5.2.5, Corollary 5.5.4**).** *Let $(V, W, N_1, \dots, N_n, F)$ be an $\mathbf{R}$-IMHM. Then for $0 \le j \le n$, $(V, W^{(j)}, N_{j+1}, \dots, N_n, F)$ is an $\mathbf{R}$-IMHM. In particular (for $j = n$), $(W^{(n)}, F)$ is a mixed Hodge structure.*

The fact that $(W^{(n)}, F)$ is a mixed Hodge structure in Proposition 5.6 was proved first by Deligne ([SZ], [K2] 5.2.1, 5.2.3).

Proposition 5.7. *Assume that $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, F)$ generates a mixed nilpotent orbit (0.2). Let $(W^{(n)}, \hat{F})$ be the $\mathbf{R}$-split mixed Hodge structure associated to $(W^{(n)}, F)$ (5.6). Then $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, \hat{F})$ also generates a mixed nilpotent orbit.*

Proof. By 1.7, N_j commutes with $\delta = \delta(W^{(n)}, F)$ and $\zeta = \zeta(W^{(n)}, F)$. Hence 0.2 (iii) is satisfied. Further, in a similar way as in the proof of Lemma 1.6, it is straightforward to show that Deligne's splitting is compatible with taking gr^W_j for each j. From this, again as in the proof of Lemma 1.6, we know that δ and the $\delta_{p,q}$ preserve W and are compatible with taking gr^W_j. Since ζ is a Lie polynomial in the $\delta_{p,q}$, it also preserves W and is compatible with taking gr^W_j. Thus we reduce 0.2 (ii) to the pure case, which is a consequence of [CKS] (4.66). □

Lemma 5.8. *Let $H = (V, W, N_1, \dots, N_n, F) \to H' = (V', W', N'_1, \dots, N'_n, F')$ be a homomorphism of pre-$\mathbf{R}$-IMHMs. Assume that $V \to V'$ is surjective. Assume that H is an $\mathbf{R}$-IMHM. Then H' is also an $\mathbf{R}$-IMHM.*

Proof. Recall that if $H \to H'$ is a surjective morphism of $\mathbf{R}$-IMHMs then $M(N'_j, W')$ is the image of $M(N_j, W)$ ([K2] 5.2.6 (ii)). Conversely, by Theorem 5.4, if $M(N'_j, W')$ exists then H' is an $\mathbf{R}$-IMHM. Hence, it is necessary and sufficient to prove that the image M' of $M(N_1, W)$ is the relative monodromy filtration of N'_1 with respect to W'.

Since $N'_1(M'_k) \subset M'_{k-2}$ for any k, it suffices to show that M' induces $W(\mathrm{gr}_k^{W'}(N'_1))[-k]$ on $\mathrm{gr}_k^{W'}$ for any k. Since a homomorphism of the category of pre-**R**-IMHMs is strict for the filtrations (5.3), W' is the image of W. Hence the filtration induced by M' on $\mathrm{gr}_k^{W'}$ coincides with the image of that induced by $M(N_1, W)$ on gr_k^W, that is, $W(\mathrm{gr}_k^W(N_1))[-k]$. Thus the desired statement reduces to the statement in a pure situation that $W(\mathrm{gr}_k^{W'}(N'_1))[-k]$ is the image of $W(\mathrm{gr}_k^W(N_1))[-k]$ by the surjection $\mathrm{gr}_k^W \to \mathrm{gr}_k^{W'}$ of pure nilpotent orbits, which is a special case of the fact recalled at the beginning of this proof. □

5.9. As remarked after 4.1, the authors do not know whether we can take all the a_{jk} in 4.1 to be 0 unless $j = k$. Assume that we can prove 4.1 (1) with this stronger condition. Then by 5.6 and 5.8, we would have the following characterization of **R**-IMHM without using relative monodromy filtrations:

A pre-**R**-IMHM $(V, W, N_1, \dots, N_n, F)$ is an **R**-IMHM if and only if there exist a pure nilpotent orbit $(V', w, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ and a surjective morphism

$$(V', W(N'_0)[-w], N'_1, \dots, N'_n, F') \to (V, W, N_1, \dots, N_n, F)$$

of pre-**R**-IMHMs.

§6. Mixed object is quotient (and also part) of pure object: Proof of Proposition 4.1, in special case

In this section, we prove a special case of Proposition 4.1 (1).

Assume that $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, F)$ generates a mixed nilpotent orbit. In this section, we assume $W_0 = V, W_{-2} = 0$, and $\dim_{\mathbf{R}}(\mathrm{gr}_0^W) = 1$. We prove 4.1 (1) under these assumptions.

Fix an isomorphism $\mathrm{gr}_0^W(V) \cong \mathbf{R}$.

Let $W^{(j)}$ be as in 5.5. Let $(W^{(n)}, \hat{F})$ be the **R**-split mixed Hodge structure associated to $(W^{(n)}, F)$.

Lemma 6.1. *There is an element e of V which lifts $1 \in \mathbf{R}$ such that $e \in W_0^{(j)}$ for any $1 \le j \le n$ and such that e belongs to the $(0,0)$-Hodge component of the **R**-split mixed Hodge structure $(W^{(n)}, \hat{F})$.*

Proof. First we prove

Claim 1. There exists an element a of $\bigcap_{j=1}^n W_{0,\mathbf{C}}^{(j)} \cap F^0$ whose image in $\mathrm{gr}_{0,\mathbf{C}}^W = \mathbf{C}$ coincides with 1.

Proof of Claim 1. Since $W_0^{(j)} + W_{-1} = V$ for any j and $F^0 + W_{-1,\mathbf{C}} = V_{\mathbf{C}}$, by the distributive property (5.5), we have $(\bigcap_{j=1}^n W_{0,\mathbf{C}}^{(j)} \cap F^0) + W_{-1,\mathbf{C}} = \bigcap_{j=1}^n (W_{0,\mathbf{C}}^{(j)} + W_{-1,\mathbf{C}}) \cap (F^0 + W_{-1,\mathbf{C}}) = V_{\mathbf{C}}$. Hence $\bigcap_{j=1}^n W_{0,\mathbf{C}}^{(j)} \cap F^0 \to \mathrm{gr}_{0,\mathbf{C}}^W$ is surjective, which shows the existence of a. Thus Claim 1 is proved.

Claim 2. There exists an element a of $\bigcap_{j=1}^{n} W^{(j)}_{0,\mathbf{C}} \cap \hat{F}^0$ whose image in $\mathrm{gr}^W_{0,\mathbf{C}} = \mathbf{C}$ coincides with 1.

Proof of Claim 2. By Proposition 5.7, we can replace F in Claim 1 by $\hat{F}$, which proves Claim 2.

Let a be as in Claim 2. Let b be the $(0,0)$-Hodge component of a with respect to $(W^{(n)}, \hat{F})$, and let e be the real part $(b+\bar{b})/2$ of b. We will prove that b and e have the same properties as a, that is, they belong to $\bigcap_{j=1}^{n} W^{(j)}_{0,\mathbf{C}} \cap \hat{F}^0$ and their images in $\mathrm{gr}^W_{0,\mathbf{C}} = \mathbf{C}$ coincide with 1. Note that b is furthermore of type $(0,0)$, and e is furthermore of type $(0,0)$ and real. This e is the element which we are looking for.

Claim 3. If $(p,q) \neq (0,0)$, the (p,q)-Hodge component of $(W^{(n)}, \hat{F})$ is contained in $W_{-1,\mathbf{C}}$.

Proof of Claim 3. We have the canonical homomorphism of mixed Hodge structures $(W^{(n)}, \hat{F}) \to (W^{(n)}(\mathrm{gr}^W_0), \hat{F}(\mathrm{gr}^W_0))$ in which the latter is pure of weight 0 and of Hodge type $(0,0)$. This proves Claim 3.

Claim 3 shows that the image of b in $\mathrm{gr}^W_{0,\mathbf{C}}$ coincides with that of a.

Next we have to prove that b belongs to $W^{(j)}_{0,\mathbf{C}}$ for all j.

Claim 4. If $a \in W^{(j)}_{k,\mathbf{C}}$, then the (p,q)-Hodge component of a with respect to $(W^{(n)}, \hat{F})$ also belongs to $W^{(j)}_{k,\mathbf{C}}$ for any p and q.

Proof of Claim 4. For $t = (t_1, t_2) \in (\mathbf{C}^\times)^2$, let $w(t)$ be the linear operator which acts on the (p,q)-part with respect to $(W^{(n)}, \hat{F})$ by the multiplication by $t_1^p t_2^q$. Then $\mathrm{Ad}(w(t))N_j = t_1^{-1}t_2^{-1}N_j$ for any j. Let $N = N_1 + \cdots + N_j$. Then $W^{(j)} = M(N, W)$. Since $w(t)W_{\mathbf{C}} = W_{\mathbf{C}}$ by Claim 3 and $\mathrm{Ad}(w(t))N = t_1^{-1}t_2^{-1}N$, we have $w(t)M(N,W)_{\mathbf{C}} = M(\mathrm{Ad}(w(t))N, w(t)W)_{\mathbf{C}} = M(t_1^{-1}t_2^{-1}N, W)_{\mathbf{C}} = M(N,W)_{\mathbf{C}}$. This formula $w(t)W^{(j)}_{\mathbf{C}} = W^{(j)}_{\mathbf{C}}$ proves Claim 4.

Finally the property of e stated in 6.1 is now easily seen. □

We fix e as in Lemma 6.1. The following construction is a variant of what appeared in §3.

6.2. Let V' be the direct sum of W_{-1} and a 2 dimensional $\mathbf{R}$-vector space with basis e_0, e_{-2}.

We consider the projection $V' \to V$ which is the identity on W_{-1}, which kills e_{-2}, and which sends e_0 to e.

We define an anti-symmetric $\mathbf{R}$-bilinear form $\langle\ ,\ \rangle : V' \times V' \to \mathbf{R}$ as follows. On W_{-1} it is $\langle\ ,\ \rangle_{-1}$; $\langle e_{2l}, W_{-1}\rangle = 0$ for $l = 0, -1$; $\langle e_0, e_{-2}\rangle = 1$.

6.3. We define $N'_j : V' \to V'$ $(0 \leq j \leq n)$ as follows.

N'_0 kills W_{-1}, sends e_0 to e_{-2}, and kills e_{-2}.

Assume $1 \leq j \leq n$. Then N'_j kills e_{-2}, $N'_j(e_0) = N_j(e) \in W_{-1}$, the $(W_{-1} \to W_{-1})$-component of N'_j is the restriction of N_j to W_{-1}, the $(W_{-1} \to \mathbf{R}e_0)$-component of N'_j

is zero, and the $(W_{-1} \to \mathbf{R}e_{-2})$-component of N'_j is -1 times the transpose of the $(\mathbf{R}e_0 \to W_{-1})$-component of N'_j with respect to $\langle\ ,\ \rangle$ in 6.2.

6.4. Define $\hat{F}'$ and F' as follows.

First, $\hat{F}'$ is the direct sum of the restriction of $\hat{F}$ to $W_{-1,\mathbf{C}}$ and the Hodge filtration of type (l,l) on $\mathbf{C}e_{2l}$ for $l = 0, -1$.

Let $\varepsilon = \varepsilon(W^{(n)}, F)$ (§1). We define a linear map $\varepsilon' : V'_{\mathbf{C}} \to V'_{\mathbf{C}}$ as follows. ε' kills e_{-2}, $\varepsilon'(e_0) = \varepsilon(e) \in W_{-1,\mathbf{C}}$, the $(W_{-1,\mathbf{C}} \to W_{-1,\mathbf{C}})$-component of ε' is the restriction of ε to $W_{-1,\mathbf{C}}$, the $(W_{-1,\mathbf{C}} \to \mathbf{C}e_0)$-component of ε' is zero, and the $(W_{-1,\mathbf{C}} \to \mathbf{C}e_{-2})$-component of ε' is -1 times the transpose of the $(\mathbf{C}e_0 \to W_{-1,\mathbf{C}})$-component of ε'.

Finally let $F' = \exp(\varepsilon')\hat{F}'$. Then, as is easily seen, F' belongs to $D^\vee$ of $(V', -1, \langle\ ,\ \rangle)$.

Lemma 6.5. *(1) $\langle N'_j(x), y\rangle + \langle x, N'_j(y)\rangle = 0$ for any $x, y \in V'$ and any j.*

(2) $N'_j N'_k = N'_k N'_j$ for any j, k.

(3) $N'_j \varepsilon' = \varepsilon' N'_j$ for any j.

Proof. (1) For $j = 0$, this is easy. For $j > 0$, the crucial is the case when $x \in W_{-1}$ and $y = e_0$. But, by the definition of N'_j with the transpose, we have $\langle N'_j(x), e_0\rangle = -\langle x, N'_j(e_0)\rangle$.

(2) Let $j > 0$. It is easy to see $N'_j N'_0 = 0 = N'_0 N'_j$. Let $j, k > 0$ and we will prove that N'_j and N'_k commute. By the commutativity of N_j and N_k, it is enough to show $\langle N_j(e), N_k(e)\rangle_{-1} = \langle N_k(e), N_j(e)\rangle_{-1}$. Since e belongs to the $(0,0)$-Hodge component of $(W^{(n)}, \hat{F})$, $N_j(e)$ and $N_k(e)$ belong to the $(-1,-1)$-Hodge component of $(W^{(n)}, \hat{F})$. Hence the both sides of the above equality in question are zero.

(3) Since the N_j and ε commute by [CKS, 3], as in (2), the desired commutativity reduces to $\langle N_j(e), \varepsilon(e)\rangle_{-1} = \langle \varepsilon(e), N_j(e)\rangle_{-1}$. We write N_j as N in the rest. Then $N(V) = N(M(N,W)_0 + W_{-1}) \subset M(N,W)_{-2} + N(W_{-1}) = N(W_{-1})$. Take $v \in W_{-1}$ such that $N(e) = N(v)$. Then we have $\langle N(e), \varepsilon(e)\rangle_{-1} = \langle N(v), \varepsilon(e)\rangle_{-1} = -\langle v, N\varepsilon(e)\rangle_{-1} = -\langle v, \varepsilon N(e)\rangle_{-1} = -\langle v, \varepsilon N(v)\rangle_{-1} = 0$. Here the last equality is by the fact that $\langle\ ,\ \rangle_{-1}$ is anti-symmetric, together with $\langle v, \varepsilon N(v)\rangle_{-1} = \langle \varepsilon N(v), v\rangle_{-1}$. We have also $\langle \varepsilon(e), N(e)\rangle_{-1} = -\langle N(e), \varepsilon(e)\rangle_{-1} = 0$. □

Thus, the special case of 4.1 (1) mentioned in the beginning of this section is verified if the following proposition is proved. (Recall that N'_0 induces 0 on V.)

Proposition 6.6. *There exist $a_{jk} \in \mathbf{R}$ $(0 \le k \le j \le n)$ such that $a_{jk} \ge 0$, $a_{jj} = 1$, and such that if we put $N''_j = \sum_{0\le k\le j} a_{jk} N'_k$, then $(V', -1, \langle\ ,\ \rangle, N''_0, \dots, N''_n, F')$ generates a pure nilpotent orbit.*

6.7. To prove this, we use the following lemma 6.8. For $y = (y_0, \dots, y_n)$ $(y_j > 0)$, let $t'(y)$ be the isomorphism $V' \xrightarrow{\sim} V'$ preserving the bilinear form $\langle\, ,\, \rangle$ and the grading $V' = W_{-1} \oplus \mathbf{R}e_{-2} \oplus \mathbf{R}e_0$ defined as follows. Let $(\rho_{-1}, \varphi_{-1})$ be the SL(2)-orbit in n variables associated to $(\mathrm{gr}^W_{-1}(N_1), \dots, \mathrm{gr}^W_{-1}(N_n), F(\mathrm{gr}^W_{-1}))$, where ρ_{-1} is a homomorphism of algebraic groups over $\mathbf{R}$ from $\mathrm{SL}(2)^n$ to the automorphism group G_{-1} of $(\mathrm{gr}^W_{-1}, \langle\ ,\ \rangle_{-1})$, and φ_{-1} is an holomorphic map $\mathbf{P}^1(\mathbf{C})^n \to$ ($D^\vee$ of gr^W_{-1}). On W_{-1}, the action of $t'(y)$ is $\rho_{-1}(\Delta(\sqrt{y_1}, \dots, \sqrt{y_n}))^{-1}$ (2.2). The actions of $t'(y)$ on e_0, e_{-2} are given by $t'(y)e_0 = \sqrt{y_0}^{\,-1} e_0$, $t'(y)e_{-2} = \sqrt{y_0}\, e_{-2}$.

Lemma 6.8. *If $y_j \in \mathbf{R}_{>0}$ and y_j/y_{j+1} tends to ∞ for $0 \le j \le n$ (y_{n+1} denotes 1), then*

$$(1) \qquad t'(y)^{-1}\exp(\textstyle\sum_{0\le j\le n} iy_jN'_j)F' \quad (\in D^\vee \text{ of } (V', -1, \langle\ ,\ \rangle))$$

converges to a point of D of $(V', -1, \langle\ ,\ \rangle)$.

Proof. For $1 \le j \le n$, let $\hat{N}_j$ be the linear map $V' \to V'$ which coincides on W_{-1} with the image of the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of the j-th factor of $\mathfrak{sl}(2,\mathbf{R})^n$ under the homomorphism $(\rho_{-1})_*: \mathfrak{sl}(2,\mathbf{R})^n \to \mathfrak{g}_{-1}$ associated to ρ_{-1}, and which kills e_0 and e_{-2}.

On the other hand, let $\hat{N}_0 = N'_0$. It is sufficient to prove the following Claim 1 and Claim 2.

Claim 1. (1) converges to $\exp(\sum_{0\le j\le n} i\hat{N}_j)\hat{F}'$.

Claim 2. $\exp(\sum_{0\le j\le n} i\hat{N}_j)\hat{F}'$ is a polarized Hodge structure of weight -1 (with respect to $\langle\ ,\ \rangle$).

We prove Claim 1. Since

$$\begin{aligned} t'(y)^{-1}\exp(\textstyle\sum_{j=0}^n iy_jN'_j)F' &= t'(y)^{-1}\exp(\textstyle\sum_{j=0}^n iy_jN'_j)\exp(\varepsilon')\hat{F}' \\ = \exp(\textstyle\sum_{j=0}^n i\,\mathrm{Ad}(t'(y))^{-1}&(y_jN'_j))\exp(\mathrm{Ad}(t'(y))^{-1}(\varepsilon'))t'(y)^{-1}\hat{F}', \end{aligned}$$

it is sufficient to prove the following Claims 1.a, 1.b and 1.c.

Claim 1.a. Let $0 \le j \le n$. Then $\mathrm{Ad}(t'(y))^{-1}(y_jN'_j)$ converges to $\hat{N}_j$.

Proof of Claim 1.a. For $j = 0$, this is clear with $\mathrm{Ad}(t'(y))^{-1}(y_0N'_0) = N'_0 = \hat{N}_0$. Assume $1 \le j \le n$, and write $N'_j = P_j + Q_j + R_j$, where P_j is the $(W_{-1} \to W_{-1})$-component of N'_j, Q_j is the $(\mathbf{R}e_0 \to W_{-1})$-component of N'_j, and R_j is the $(W_{-1} \to \mathbf{R}e_{-2})$-component of N'_j. So $P_j : W_{-1} \to W_{-1}$ coincides with $\mathrm{gr}^W_{-1}(N_j)$, and R_j is the transpose of $-Q_j$. It is sufficient to show that

(2) $\mathrm{Ad}(t'(y))^{-1}(y_jP_j)$ converges to $\hat{N}_j$, and

(3) $\mathrm{Ad}(t'(y))^{-1}(y_jQ_j)$ and $\mathrm{Ad}(t'(y))^{-1}(y_jR_j)$ converge to 0.

We prove (2). Consider the decomposition $\mathfrak{g}_{-1,\mathbf{R}} = \bigoplus_{\mu\in\mathbf{Z}^n} \mathfrak{g}^{[\mu]}_{-1,\mathbf{R}}$ (2.2), where $\mathrm{Ad}(\rho_{-1}(\Delta^{(j)}(\lambda)))$ ($1 \le j \le n, \lambda \in \mathbf{R}^\times$) (2.2) acts on $\mathfrak{g}^{[\mu]}_{-1,\mathbf{R}}$ as the multiplication by $\lambda^{\mu(j)}$. Write $P_j = \sum_{\mu\in\mathbf{Z}^n} P_j^{[\mu]}$ according to this decomposition. By Lemma 2.7 applied to $(\mathrm{gr}^W_{-1}(N_1), \dots, \mathrm{gr}^W_{-1}(N_n), F(\mathrm{gr}^W_{-1}))$, which generates a pure nilpotent orbit, $P_j^{[\mu]} = 0$ unless $\mu(l) \le 0$ for $1 \le l < j$ and $\mu(l) = -2$ for $j \le l \le n$. Hence we have

$$\mathrm{Ad}(t'(y))^{-1}(y_jP_j) = \textstyle\sum_{\mu\in\mathbf{Z}^n} \prod_{l=1}^{j-1} (y_l/y_{l+1})^{\mu(l)/2} P_j^{[\mu]}.$$

When $y_l/y_{l+1} \to \infty$ for $1 \le l < j$, the μ-component of this converges to 0 (resp. is constantly $P_j^{[\mu]}$) unless (resp. if) $\mu(l) = 0$ for $1 \le l < j$. Hence $\mathrm{Ad}(t'(y))^{-1}(y_jP_j)$ converges to $\hat{N}_j$ by 2.5.3.

We prove (3). The proof for R_j is similar to that for Q_j and so we give here only the proof for Q_j.

Consider the decomposition $W_{-1} = \bigoplus_{\mu \in \mathbf{Z}^n} W_{-1}^{[\mu]}$ (2.2), where $\rho_{-1}(\Delta^{(j)}(\lambda))$ $(1 \le j \le n, \lambda \in \mathbf{R}^\times)$ acts on $W_{-1}^{[\mu]}$ as the multiplication by $\lambda^{\mu(j)}$. $Q_j(e_0) = N_j(e)$ by definition. Write this as $N_j(e) = \sum_{\mu \in \mathbf{Z}^n} N_j(e)^{[\mu]}$ according to this decomposition. Since $e \in W_0^{(l)}$ for $1 \le l \le n$, $N_j(e)$ belongs to $W_0^{(l)}$ if $1 \le l < j$, and to $W_{-2}^{(l)}$ if $j \le l \le n$. Hence $N_j(e)^{[\mu]} = 0$ unless $\mu(l) \le 0$ for $1 \le l < j$ and $\mu(l) \le -2$ for $j \le l \le n$. Since $t'(y)e_0 = y_0^{-1/2} e_0$, $\mathrm{Ad}(t'(y))^{-1}(y_j Q_j)$ sends e_0 to

$$\textstyle\sum_{\mu \in \mathbf{Z}^n} y_0^{-1/2} \prod_{l=1}^{j-1} (\frac{y_l}{y_{l+1}})^{\mu(l)/2} \prod_{l=j}^{n} (\frac{y_l}{y_{l+1}})^{(2+\mu(l))/2} N_j(e)^{[\mu]}.$$

This converges to 0.

Thus Claim 1.a is proved.

Claim 1.b. $\mathrm{Ad}(t'(y))^{-1}(\varepsilon')$ converges to 0.

Proof of Claim 1.b. Write $\varepsilon' = P+Q+R$, where P is the $(W_{-1,\mathbf{C}} \to W_{-1,\mathbf{C}})$-component, Q is the $(\mathbf{C}e_0 \to W_{-1,\mathbf{C}})$-component, and R is the $(W_{-1,\mathbf{C}} \to \mathbf{C}e_{-2})$-component of ε'. Hence R is the transpose of $-Q$. It is sufficient to show that

(4) $\mathrm{Ad}(t'(y))^{-1}(P)$ converges to 0, and

(5) $\mathrm{Ad}(t'(y))^{-1}(Q)$ and $\mathrm{Ad}(t'(y))^{-1}(R)$ converge to 0.

We prove (4). We use the decomposition $\mathfrak{g}_{-1,\mathbf{C}} = \bigoplus_{\mu \in \mathbf{Z}^n} \mathfrak{g}_{-1,\mathbf{C}}^{[\mu]}$. Write $P = \sum_{\mu \in \mathbf{Z}^n} P^{[\mu]}$ according to this decomposition. Since ε preserves $W_{\mathbf{C}}^{(l)}$ for $1 \le l < n$ and $\varepsilon(W_{k,\mathbf{C}}^{(n)}) \subset W_{k-2,\mathbf{C}}^{(n)}$ for any k, $P^{[\mu]} = 0$ unless $\mu(l) \le 0$ for $1 \le l \le n-1$ and $\mu(n) \le -2$. Hence

$$\textstyle\mathrm{Ad}(t'(y))^{-1}(P) = \sum_{\mu \in \mathbf{Z}^n} \prod_{j=1}^{n} (\frac{y_j}{y_{j+1}})^{\mu(j)/2} P^{[\mu]}$$

converges to 0.

We prove (5). The proof for R is similar to that for Q and so we give here only the proof for Q. We use the decomposition $W_{-1,\mathbf{C}} = \bigoplus_{\mu \in \mathbf{Z}^n} W_{-1,\mathbf{C}}^{[\mu]}$. $Q(e_0) = \varepsilon(e)$ by definition. Write this as $\varepsilon(e) = \sum_{\mu \in \mathbf{Z}^n} \varepsilon(e)^{[\mu]}$ according to this decomposition. By the property of e, $\varepsilon(e)$ belongs to $W_{0,\mathbf{C}}^{(l)}$ for $1 \le l \le n$. Hence $\varepsilon(e)^{[\mu]} = 0$ unless $\mu(l) \le 0$ for $1 \le l \le n$. Since $t'(y)e_0 = y_0^{-1/2} e_0$, $\mathrm{Ad}(t'(y))^{-1}(Q)$ sends e_0 to

$$\textstyle\sum_{\mu \in \mathbf{Z}^n} y_0^{-1/2} \prod_{j=1}^{n} (\frac{y_j}{y_{j+1}})^{\mu(j)/2} \varepsilon(e)^{[\mu]}.$$

This converges to 0.

Thus Claim 1.b is proved.

Claim 1.c. $\hat{F}'$ does not move under the action of $t'(y)$.

Proof of Claim 1.c. Since the restriction of $\hat{F}$ to W_{-1} is $\varphi_{-1}(\mathbf{0}_n)$ (this follows from 2.5.4 and 1.6), the assertion is reduced to $\rho(\Delta((\mathbf{R}^\times)^n))\varphi_{-1}(\mathbf{0}_n) = \varphi_{-1}(\mathbf{0}_n)$. Claim 1.c is thus proved.

The proof of Claim 2 is also reduced to the fact in the pure situation that $\varphi_{-1}(\mathbf{i}_n)$ is a polarized Hodge structure.

This completes the proof of Lemma 6.8. □

6.9. *Proof of Proposition 6.6.* The Griffiths transversality (the condition 0.1 (iii)) is reduced by Lemma 6.5 (3) to that for $\hat{F}'$. We have to show that for each $0 \le j \le n$ and each $p \in \mathbf{Z}$, $N'_j(\hat{F}')^p \subset (\hat{F}')^{p-1}$. The case $j = 0$ is easy. Assume $j > 0$. It suffices to prove $N_j(e) \in \hat{F}^{-1}$ and $\langle \hat{F}^1 W_{-1}, N_j(e)\rangle_{-1} = 0$. The former follows from $e \in \hat{F}^0$, and the latter follows from the former.

We prove that the condition 0.1 (ii) is satisfied after replacing N'_j with N''_j as in the statement of 6.6. By Lemma 6.8 and by the fact that D of $(V', -1, \langle\ ,\rangle)$ is open in $D^\vee$, $t'(y)^{-1}\exp(\sum_{j=0}^n iy_jN'_j)F'$ belongs to D if $y_j/y_{j+1} \gg 0$ for $0 \le j \le n$. Since the operator $t'(y)$ is real, $\exp(\sum_{j=0}^n iy_jN'_j)F'$ belongs to D for such $y_0, \dots, y_n$. Take $a_j \ge 0$ $(j = 0, \dots, n)$ such that $\exp(\sum_{j=0}^n iy_jN'_j)F'$ belongs to D for any $y_0, \dots, y_n$ satisfying $y_j > a_jy_{j+1}$ $(j = 0, \dots, n)$. For $0 \le k < j$, let $a_{jk} = a_ka_{k+1}\cdots a_{j-1}$. Then 0.1 (ii) is satisfied with respect to N''_j.

This completes the proof of 6.6. □

§7. Mixed object is quotient (and also part) of pure object: Proof of Proposition 4.1, in general case

In this section, we complete the proof of 4.1. Since 4.1 (2) is proved in the same way as 4.1 (1), we give here only the proof of 4.1 (1). We will reduce it to its special case proved in the previous section.

7.1. First, we prove the case where there exists $m \in \mathbf{Z}$ such that $W_m = V, W_{m-2} = 0$. In this case, we show that we can find $(V', m-1, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$, a surjective linear map $V' \to V$, and real numbers a_{jk} $(1 \le k \le j \le n)$ satisfying the conditions in 4.1 (1) and the further conditions $W(N'_0)_1 = V', W(N'_0)_{-2} = 0$.

First, we construct an object $({}^{(1)}V, {}^{(1)}W, ({}^{(1)}\langle\ ,\ \rangle_w)_w, {}^{(1)}N_1, \dots, {}^{(1)}N_n, {}^{(1)}F)$ which generates a mixed nilpotent orbit and which is of the type considered in §6, as follows. Let $A = \mathrm{gr}^W_m(V)$, $B = \mathrm{gr}^W_{m-1}(V)$. We regard A and B as $\mathbf{R}$-IMHM of pure weights m and $m-1$, respectively, in the evident way. Define the $\mathbf{R}$-IMHM ${}^{(1)}V = ({}^{(1)}V, {}^{(1)}W, {}^{(1)}N_1, \dots, {}^{(1)}N_n, {}^{(1)}F)$ as the fiber product of

$$A^* \otimes V \to A^* \otimes A \leftarrow \mathbf{R}$$

in the category of $\mathbf{R}$-IMHM (5.3), where A^* is the dual of A, $\mathbf{R}$ is regarded as an $\mathbf{R}$-IMHM of weight zero in the trivial way, $\otimes$ are the tensor product in the category of $\mathbf{R}$-IMHM (5.3), and $\mathbf{R} \to A^* \otimes A$ is the evident map. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & A^*\otimes B & \to & {}^{(1)}V & \to & \mathbf{R} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A^*\otimes B & \to & A^*\otimes V & \to & A^*\otimes A & \to & 0,
\end{array}$$

and we have ${}^{(1)}W_0 = {}^{(1)}V, {}^{(1)}W_{-1} = A^* \otimes B, {}^{(1)}W_{-2} = 0$, and $\mathrm{gr}_0^{{}^{(1)}W} = \mathbf{R}$. We define the intersection forms ${}^{(1)}\langle\ ,\ \rangle_w$ on $\mathrm{gr}_w^{{}^{(1)}W}$ as follows. For $w = 0$, this is $\mathbf{R} \times \mathbf{R} \to \mathbf{R}; (x, y) \mapsto xy$. For $w = -1$, this is the tensor product $\langle\ ,\ \rangle_m^* \otimes \langle\ ,\ \rangle_{m-1}$ on $A^* \otimes B$. Then clearly $({}^{(1)}V, {}^{(1)}W, ({}^{(1)}\langle\ ,\ \rangle_w)_w, {}^{(1)}N_1, \dots, {}^{(1)}N_n, {}^{(1)}F)$ generates a mixed nilpotent orbit.

By §6, there exist $({}^{(2)}V, -1, {}^{(2)}\langle\ ,\ \rangle, {}^{(2)}N_0, \dots, {}^{(2)}N_n, {}^{(2)}F)$ which generates a pure nilpotent orbit, a surjective linear map $f : {}^{(2)}V \to {}^{(1)}V$, real numbers $a_{jk} \in \mathbf{R}$ $(1 \le k \le j \le n)$ such that ${}^{(1)}W$ is the image of $W({}^{(2)}N_0)[1]$, ${}^{(1)}F$ is the image of ${}^{(2)}F$, $f \circ {}^{(2)}N_0 = 0$, $f \circ {}^{(2)}N_j = \sum_{k=1}^{j} a_{jk}{}^{(1)}N_k \circ f$ for $1 \le j \le n$, $a_{jk} \ge 0$, $a_{jj} = 1$, $W({}^{(2)}N_0)_1 = {}^{(2)}V$, $W({}^{(2)}N_0)_{-2} = 0$.

Now the desired object $(V', m-1, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ is defined as follows; $V' = A \otimes {}^{(2)}V$, $\langle\ ,\ \rangle = \langle\ ,\ \rangle_m \otimes {}^{(2)}\langle\ ,\ \rangle$, $N'_j = 1 \otimes {}^{(2)}N_j + \sum_{k=1}^{j} a_{jk}\, \mathrm{gr}_m^W(N_k) \otimes 1$, and F is the tensor product of $F(\mathrm{gr}_m^W)$ and ${}^{(2)}F$. Let $p : V' \to V$ be the composition $A \otimes {}^{(2)}V \to A \otimes {}^{(1)}V \to A \otimes A^* \otimes V \to V$. Then p is surjective as is easily seen. This $(V', m-1, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ is the tensor product of two pure polarized $\mathbf{R}$-IMHM, and hence generates a pure nilpotent orbit.

7.2. We consider the general case. We may assume that there exists $m \ge 2$ such that $\mathrm{gr}_w^W(V)$ is zero if w does not belong to the closed interval $[0, m]$. We prove that we can find $(V', 0, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ of weight 0.

We prove by induction on m.

Let ${}^1V = V/W_{m-2}$, ${}^1W$ the filtration on ${}^1V$ induced by W, ${}^1F$ the filtration on ${}^1V_{\mathbf{C}}$ induced by F, and ${}^1N_j : {}^1V \to {}^1V$ $(1 \le j \le n)$ the homomorphism induced by N_j. Then by 7.1, there exist an object $({}^2V, m-1, {}^2\langle\ ,\ \rangle, {}^2N_0, \dots, {}^2N_n, {}^2F)$ generating a pure nilpotent orbit, a surjective linear map ${}^2V \to {}^1V$, and real numbers a_{jk} $(1 \le k \le j \le n)$ such that ${}^1W$ is the image of $W({}^2N_0)[1-m]$, ${}^1F$ is the image of ${}^2F$, ${}^2N_j$ on ${}^2V$ $(0 \le j \le n)$ commutes with 0 (resp. $\sum_{k=1}^{j} a_{jk}{}^1N_k$) on ${}^1V$ if $j = 0$ (resp. $1 \le j \le n$), $a_{jk} \ge 0$, $a_{jj} = 1$, and $W({}^2N_0)_1 = {}^2V$, $W({}^2N_0)_{-2} = 0$.

We may and do replace N_j by $\sum_{k=1}^{j} a_{jk}N_k$, and replace ${}^1N_j$ by $\sum_{k=1}^{j} a_{jk}\ {}^1N_k$. Hence ${}^2N_j$ now commutes with ${}^1N_j$ for $1 \le j \le n$.

Define $({}^3V, {}^3N_0, {}^3N_1, \dots, {}^3N_n, {}^3F)$ as the fiber product of

$$(V, 0, N_1, \dots, N_n, F) \to ({}^1V, 0, {}^1N_1, \dots, {}^1N_n, {}^1F) \leftarrow ({}^2V, {}^2N_0, {}^2N_1, \dots, {}^2N_n, {}^2F).$$

Define the weight filtration ${}^3W$ of ${}^3V$ as follows.

${}^3W_k = {}^3V$ for $k \ge m-1$. ${}^3W_k$ is $W_k \times \{0\}$ on the fiber product for $k \le m-2$.

Then $\mathrm{gr}_{m-1}^{{}^3W} = {}^2V$.

7.3. Let ${}^4N_j = {}^3N_j + {}^3N_0$. ${}^4V = {}^3V$, ${}^4W = {}^3W$, ${}^4F = {}^3F$. Define ${}^4\langle\ ,\ \rangle_w$ on $\mathrm{gr}_w^{{}^4W}$ for each $w \in \mathbf{Z}$ as follows. For $w \le m-2$, ${}^4\langle\ ,\ \rangle_w$ is $\langle\ ,\ \rangle_w$ of gr_w^W. For $w = m-1$, ${}^4\langle\ ,\ \rangle_{m-1}$ is ${}^2\langle\ ,\ \rangle$ of ${}^2V$.

Lemma 7.4. *$({}^4V, {}^4W, ({}^4\langle\ ,\ \rangle_w)_w, {}^4N_0, \dots, {}^4N_n, {}^4F)$ generates a mixed nilpotent orbit.*

Proof. By Theorem 5.4, it is enough to show that the relative monodromy filtration $M({}^4N_j, {}^4W)$ exists for $0 \le j \le n$.

Assume first $j = 0$. Then we have $M({}^4N_0, {}^4W)$ as the fiber product of

$$W \to {}^1W \leftarrow W({}^2N_0)[1-m].$$

In fact, let M be this fiber product. Then the condition (i) of the relative monodromy filtration in 5.1 is clearly satisfied. We check (ii). On $\mathrm{gr}^{{}^4W}_{m-1} = {}^2V$, M induces the filtration $W({}^2N_0)[1-m]$, and hence (ii) for the case $k = m-1$ is clear. For $k \geq m$, $\mathrm{gr}^{{}^4W}_k = 0$. For $k < m-1$, $\mathrm{gr}^{{}^4W}_k = \mathrm{gr}^W_k$ and M induces there the trivial filtration (M_k is the total space and M_{k-1} vanishes there) and (ii) is satisfied since ${}^4N_0$ induces the zero map there.

Next assume $1 \leq j \leq n$. We have $M({}^4N_j, {}^4W)$ as the fiber product of

$$M(N_j, W) \to M({}^1N_j, {}^1W) \leftarrow M({}^2N_j, W({}^2N_0)[1-m]).$$

In fact, let M be this fiber product. Then the condition (i) of the relative monodromy filtration in 5.1 is clearly satisfied. We check (ii). On $\mathrm{gr}^{{}^4W}_{m-1} = {}^2V$, M induces the filtration $M({}^2N_j, W({}^2N_0)[1-m])$ which coincides with $W({}^2N_0 + {}^2N_j)[1-m]$ by [CK]. Since ${}^4N_j$ induces ${}^2N_0 + {}^2N_j$ there, the condition (ii) for $k = m-1$ is satisfied. For $k \geq m$, $\mathrm{gr}^{{}^4W}_k = 0$. For $k < m-1$, the condition (ii) is satisfied because the restriction of M to ${}^4W_{m-2} = W_{m-2}$ coincides with the restriction of $M(N_j, W)$ to W_{m-2}. □

7.5. Since the weights of $({}^4V, {}^4W)$ belong to the interval $[0, m-1]$, by the hypothesis of the induction on m, there exist$({}^5V, 0, {}^5\langle\ ,\ \rangle, {}^5N_{-1}, {}^5N_0, \dots, {}^5N_n, {}^5F)$ generating a pure nilpotent orbit, a surjective linear map $f : {}^5V \to {}^4V$, real numbers b_{jk} $(0 \leq k \leq j \leq n)$ such that ${}^4W$ is the image of $W({}^5N_{-1})$, ${}^4F$ is the image of ${}^5F$, $f \circ {}^5N_{-1} = 0$, $f \circ {}^5N_j = \sum_{k=0}^{j} b_{jk}\ {}^4N_k \circ f$ for $0 \leq j \leq n$, $b_{jk} \geq 0$, $b_{jj} = 1$.

7.6. Finally, let $V' = {}^5V$, $\langle\ ,\ \rangle = {}^5\langle\ ,\ \rangle$, $N'_0 = {}^5N_{-1} + {}^5N_0$, and let $N'_j = {}^5N_j$ for $1 \leq j \leq n$, $F' = {}^5F$. Then $(V', 0, \langle\ ,\ \rangle, N'_0, \dots, N'_n, F')$ generates a pure nilpotent orbit.

Let $p : V' \to V$ be the composition $V' = {}^5V \to {}^4V = {}^3V \to V$.

It is evident that F is the image of F', and conditions on N'_j $(0 \leq j \leq n)$ stated in 4.1 (1) are satisfied. It remains to prove that W is the image of $W(N'_0)$.

For $1 \leq j \leq n$, replace N_j by $\sum_{k=1}^{j} b_{jk} N_k$, ${}^1N_j$ by $\sum_{k=1}^{j} b_{jk}\ {}^1N_k$, and ${}^4N_j$ by $\sum_{k=0}^{j} b_{jk}\ {}^4N_k$. Then for each $1 \leq j \leq n$, N'_j, ${}^4N_j$, and N_j commute via projections.

Note that $W(N'_0) = M({}^5N_0, W({}^5N_{-1}))$ [CK]. By Proposition 5.6, $({}^4V, M({}^4N_0, {}^4W), {}^4N_1, \dots, {}^4N_n, {}^4F)$ is an **R**-IMHM, and we have homomorphisms of **R**-IMHM

$$(V', W(N'_0), N'_1, \dots, N'_n, F') = ({}^5V, M({}^5N_0, W({}^5N_{-1})), {}^5N_1, \dots, {}^5N_n, {}^5F)$$
$$\to ({}^4V, M({}^4N_0, {}^4W), {}^4N_1, \dots, {}^4N_n, {}^4F) \to (V, W, N_1, \dots, N_n, F).$$

Since $V' \to V$ is surjective, W coincides with the image of $W(N'_0)$ by 5.3.

§8. Borel-Serre theory and SL(2)-orbits

The goal of §8 and §9 is to prove Proposition 4.2. In fact, a shorter proof of 4.2 can be given if we use [KU2] Proposition 6.2.2 (which is essentially the same as Proposition

9.2 below). Since [KU2] is not yet published, we give here a self-contained proof of 4.2 including arguments in [KU2] which were used for the proof of [KU2] 6.2.2.

In this section, we present the relationship between Borel-Serre theory in [BS] and the theory of SL(2)-orbit, studied in [KU1], [KU2].

8.1. *Borel-Serre theory.* In general, let G be a semi-simple algebraic group over $\mathbf{R}$, let P be a parabolic subgroup of G, and let K be a maximal compact subgroup of G. Let P_u be the unipotent radical of P, and let T be the maximal $\mathbf{R}$-split torus of the center of P/P_u. Then by Borel-Serre [BS], there is a unique homomorphism of algebraic groups over $\mathbf{R}$

$$\mathrm{BS}_{P,K} : T \to P$$

characterized by the following properties (i) and (ii).

(i) The composition $T \xrightarrow{\mathrm{BS}_{P,K}} P \to P/P_u$ coincides with the inclusion map $T \to P/P_u$.

(ii) Let $\theta_K : G \to G$ be the Cartan involution of G associated to K, that is, the unique automorphism θ_K of G such that θ_K^2 is the identity and $K = \{g \in G \mid \theta_K(g) = g\}$. Then $\theta_K(t) = t^{-1}$ for any $t \in \mathrm{BS}_{P,K}(T)$.

The following holds.

(iii) If $g \in P$, $\mathrm{BS}_{P,gKg^{-1}} = \mathrm{Int}(g)(\mathrm{BS}_{P,K})$. Here $\mathrm{Int}(g)$ means the inner automorphism by g.

8.2. In the rest of §8, fix $(V, w, \langle\ ,\ \rangle)$ as in 0.1. Let $D, D^\vee, G_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}$ be as in 0.1.

The algebraic group $G_{\mathbf{R}}$ over $\mathbf{R}$ is semi-simple except the case $\dim(V) = 2$ and w is even. In this exceptional case, there is no non-zero nilpotent operator in $\mathfrak{g}_{\mathbf{R}}$. Since we are going to consider pure nilpotent orbits in §8 and §9, we will assume in these sections that we are not in this exceptional case. We apply 8.1 to the semi-simple algebraic group $G_{\mathbf{R}}$.

Each $F \in D$ determines a maximal compact subgroup of $G_{\mathbf{R}}$: For $F \in D$, let K_F be the maximal compact subgroup of $G_{\mathbf{R}}$ consisting of all elements which preserve the Hodge metric $(\ ,\ )_F$ on $V_{\mathbf{C}}$ associated to F. Here $(\ ,\ )_F$ is the positive definite Hermitian form $V_{\mathbf{C}} \times V_{\mathbf{C}} \to \mathbf{C}$ defined by

$$(x, y)_F = i^{2p-w}\langle x, \bar{y}\rangle \quad \text{for } x \in F^p \cap \bar{F}^{w-p} \text{ and } y \in V_{\mathbf{C}}.$$

Here $y \mapsto \bar{y}$ denotes the complex conjugation.

8.3. Let $W = (W_k)_{k\in\mathbf{Z}}$ be an increasing filtration on V such that the annihilator of W_k in V with respect to $\langle\ ,\ \rangle$ coincides with W_{2w-1-k} for any $k \in \mathbf{Z}$.

For $F \in D$, we define a splitting $\mathrm{BS}(W, F)$ of W, which we call the Borel-Serre splitting of W associated to F, as follows.

Let P be the parabolic subgroup of $G_{\mathbf{R}}$ associated to W. That is, P is the connected component as an algebraic group of $G_{W,\mathbf{R}} = \{g \in G_{\mathbf{R}} \mid gW = W\}$ containing the unit element. Then P_u is identified with the kernel of the canonical homomorphism $P \to \mathrm{Aut}(\mathrm{gr}^W(V))$ and hence P/P_u acts on $\mathrm{gr}^W(V)$. Let T be the maximal $\mathbf{R}$-split torus of the center of P/P_u, and $a : \mathbf{G}_m \to T$ be the homomorphism of algebraic groups over $\mathbf{R}$ characterized by the property that for $y \in \mathbf{R}^\times$, $a(y)$ acts on gr_k^W as the multiplication by y^{k-w} for any $k \in \mathbf{Z}$. Let $K = K_F$. Define $\mathrm{BS}(W,F)$ to be the splitting of W associated to $\mathrm{BS}_{P,K} \circ a : \mathbf{G}_m \to P$. That is, $\mathrm{BS}(W,F)$ is the unique splitting s of W satisfying $\mathrm{BS}_{P,K}(a(\lambda)) = s \circ a(\lambda) \circ s^{-1}$ for any $\lambda \in \mathbf{R}^\times$.

By 8.1 (iii), the following holds.

(i) If $g \in P$, $\mathrm{BS}(W, gF) = g\,\mathrm{BS}(W,F)$.

8.4. The following relationship between SL(2)-orbits and Borel-Serre theory was proved in [KU1] 3.9:

Let (ρ,φ) be an SL(2)-orbit in n variables (for $(V, w, \langle\ ,\ \rangle)$), and let the filtration $W^{(j)}$ and the splitting $s^{(j)}$ of $W^{(j)}$ $(1 \le j \le n)$ for (ρ,φ) be as in 2.2. Then

$$\mathrm{BS}(W^{(j)}, \varphi(z)) = s^{(j)} \quad \text{for } 1 \le j \le n \text{ and for } z \in (i\mathbf{R}_{>0})^n.$$

Proposition 8.5. *Assume that $(V, w, \langle\ ,\ \rangle, N_1, \dots, N_n, F)$ generates a pure nilpotent orbit, and let (ρ,φ) be the associated* SL(2)*-orbit in n variables. Fix k such that $1 \le k \le n$, and let $s^{(k)}$ be the splitting of $W^{(k)}$ associated to ρ. Then there exist $c > 0$ and $v_m \in \mathfrak{g}_{\mathbf{R}}$ $(m \in \mathbf{N}^n)$ such that $v_0 = 0$, the conditions (1)–(3) below are satisfied, $\sum_{m\in\mathbf{N}^n} v_m \prod_{j=1}^n \lambda_j^{m(j)}$ absolutely converges when $0 \le \lambda_j < c$ for $1 \le j \le n$, and such that whenever $y_{j+1}/y_j < c$ for $1 \le j \le n$, we have*

$$\exp(\textstyle\sum_{j=1}^n iy_jN_j)F \in D \quad \textit{and} \quad \mathrm{BS}(W^{(k)}, \exp(\sum_{j=1}^n iy_jN_j)F) = v(y)s^{(k)},$$
$$\textit{with} \quad v(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} v_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)}).$$

(1) $v_m W_l^{(k)} \subset W_{l-1}^{(k)}$ for all m and l.

(2) $v_m W_l^{(j)} \subset W_{l+m(j)}^{(j)}$ for all m, j, l.

(3) $v_m = 0$ if $m(k) = 0$ and there is j such that $1 \le j < k$ and $m(j) \ne 0$.

8.6. Proposition 8.5 will be proved in §9 together with Proposition 4.2.

As we will see in §9, 8.5 and 4.2 are related as follows: the v_m $(m \in \mathbf{N}^n)$ in 8.5 and the u_m $(m \in \mathbf{N}^{n-k})$ in 4.2 are related by $u_m = v_{m'}$ $(m \in \mathbf{N}^{n-k}$, $m' \in \mathbf{N}^n)$, where $m'(j) = 0$ if $j \le k$, and $m'(j) = m(j-k)$ if $k < j \le n$.

8.7. *A characterization of the associated* SL(2)*-orbit by Borel-Serre theory.* Assume that $(N_1, \dots, N_n, F)$ generates a pure nilpotent orbit. Then 8.5 shows that the associated SL(2)-orbit (ρ,φ) in n variables is the SL(2)-orbit characterized by the properties 2.5.1, 2.5.3 and the following 8.7.1 and 8.7.2.

8.7.1. For $1 \le k \le n$, the splitting $s^{(k)}$ of $W^{(k)}$ associated to ρ is the limit of $\mathrm{BS}(W^{(k)}, \exp(\sum_{j=1}^n iy_jN_j)F)$ for $y_j/y_{j+1} \to \infty$.

8.7.2. $\varphi(\mathbf{0}_n) = s^{(n)}(F(\mathrm{gr}^{W^{(n)}}))$.

§9. Proof of Proposition 4.2

In this section, we prove Proposition 4.2. We prove it and the propositions 8.5, 9.2, and 9.3 together.

Let $(V, w, \langle\ ,\ \rangle)$, $G_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}$ be as in 0.1.

9.1. Recall 2.2 that for an SL(2)-orbit in n variables for $(V, w, \langle\ ,\ \rangle)$, we have a direct sum decomposition

$$\mathfrak{g}_{\mathbf{R}} = \bigoplus_{\mu \in \mathbf{Z}^n} \mathfrak{g}_{\mathbf{R}}^{[\mu]}.$$

The action of $\mathrm{Ad}(t(y))$ ($y = (y_1, \dots, y_n)$, $t(y) = \rho(\Delta(\sqrt{y_1}, \dots, \sqrt{y_n}))^{-1}$ as in 2.2) on $\mathfrak{g}_{\mathbf{R}}^{[\mu]}$ coincides with the multiplication by $\prod_{j=1}^{n} (y_{j+1}/y_j)^{\mu(j)/2}$.

For $1 \le j \le n$, we have

$$\mathrm{Lie}(G_{W^{(j)},\mathbf{R}}) = \bigoplus_{\mu \in \mathbf{Z}^n, \mu(j) \le 0} \mathfrak{g}_{\mathbf{R}}^{[\mu]}.$$

The following Propositions 9.2 and 9.3 are complementary results to the SL(2)-orbit Theorem of Cattani-Kaplan-Schmid in [CKS].

Proposition 9.2. *Assume that $(N_1, \dots, N_n, F)$ generates a pure nilpotent orbit. Consider the associated* SL(2)*-orbit (ρ, φ) in n variables.*

Let A be an $\mathbf{R}$-subspace of $\mathfrak{g}_{\mathbf{R}}$ such that $\mathfrak{g}_{\mathbf{R}} = A \oplus \mathrm{Lie}(K_{\mathbf{r}})$ and such that if $x \in A$, then $x^{[\mu]} + x^{[-\mu]} \in A$ for any $\mu \in \mathbf{Z}^n$.

(1) There exist $c > 0$ and $h_m \in A$, $k_m \in \mathrm{Lie}(K_{\mathbf{r}})$ ($m \in \mathbf{N}^n$) such that $h_0 = k_0 = 0$,

$$\sum_{m \in \mathbf{N}^n} h_m \prod_{j=1}^{n} \lambda_j^{m(j)/2} \quad \textit{and} \quad \sum_{m \in \mathbf{N}^n} k_m \prod_{j=1}^{n} \lambda_j^{m(j)/2}$$

absolutely converge when $0 \le \lambda_j < c$ for $1 \le j \le n$, $h_m^{[\mu]} = k_m^{[\mu]} = 0$ unless $\mu(j) \equiv m(j)$ mod 2 *for $1 \le j \le n$, and such that whenever $y_{j+1}/y_j < c$ for $1 \le j \le n$, we have*

$$\exp(\textstyle\sum_{1 \le j \le n} iy_j N_j) F = t(y) h(y) k(y) \mathbf{r}, \quad \textit{with}$$
$$h(y) = \exp(\textstyle\sum_{m \in \mathbf{N}^n} h_m \prod_{j=1}^{n} (\frac{y_{j+1}}{y_j})^{m(j)/2}),$$
$$k(y) = \exp(\textstyle\sum_{m \in \mathbf{N}^n} k_m \prod_{j=1}^{n} (\frac{y_{j+1}}{y_j})^{m(j)/2}).$$

Here $t(y)$ and $\mathbf{r}$ are for (ρ, φ).

(2) Let (h_m) be as in (1). Write $h_m = \sum_{\mu \in \mathbf{Z}^n} h_m^{[\mu]}$, where $h_m^{[\mu]} \in \mathfrak{g}_{\mathbf{R}}^{[\mu]}$.
Then we have:

(2.1) $h_m^{[\mu]} = 0$ unless $|\mu(j)| \le m(j)$ for all j.

(2.2) Let $m \in \mathbf{N}^n$, $m \ne 0$, and let k be the largest integer in $\{1, \dots, n\}$ such that $m(k) \ne 0$. Let $\mu \in \mathbf{Z}^n$ and assume $h_m^{[\mu]} \ne 0$. Then $|\mu(k)| < m(k)$.

An example of A as in 9.2 is given as follows. Note that the Cartan involution $\theta_{K_{\mathbf{r}}} : \mathfrak{g}_{\mathbf{R}} \to \mathfrak{g}_{\mathbf{R}}$ associated to $K_{\mathbf{r}}$ sends $\mathfrak{g}_{\mathbf{R}}^{[\mu]}$ to $\mathfrak{g}_{\mathbf{R}}^{[-\mu]}$ for any $\mu \in \mathbf{Z}^n$ and satisfies $\mathrm{Lie}(K_{\mathbf{r}}) = \{x \in \mathfrak{g}_{\mathbf{R}} \mid \theta_{K_{\mathbf{r}}}(x) = x\}$. For example, we can take $A = \{x \in \mathfrak{g}_{\mathbf{R}} \mid \theta_{K_{\mathbf{r}}}(x) = -x\}$. For this A, Proposition 9.2 is identical with [KU2] Proposition 6.2.2.

Proposition 9.3. *Assume that $(N_1, \dots, N_n, F)$ generates a pure nilpotent orbit. Consider the associated* $\mathrm{SL}(2)$*-orbit (ρ, φ) in n variables.*

Fix an integer k such that $1 \le k \le n$. Then there exist $c > 0$ and $p_m \in \mathrm{Lie}(G_{W^{(k)},\mathbf{R}})$, $k_m \in \mathrm{Lie}(K_{\mathbf{r}})$ ($m \in \mathbf{N}^n$) such that $p_0 = k_0 = 0$,

$$\textstyle\sum_{m\in\mathbf{N}^n} p_m \prod_{j=1}^n \lambda_j^{m(j)} \quad \textit{and} \quad \sum_{m\in\mathbf{N}^n} k_m \prod_{j=1}^n \lambda_j^{m(j)/2}$$

absolutely converge when $0 \le \lambda_j < c$ for $1 \le j \le n$, $k_m^{[\mu]} = 0$ unless $\mu(j) \equiv m(j) \mod 2$ for $1 \le j \le n$, the conditions (1) and (2) below are satisfied, and such that whenever $y_{j+1}/y_j < c$ for $1 \le j \le n$, we have

$$\begin{aligned} &\exp(\textstyle\sum_{1\le j\le n} iy_jN_j)F = p(y)t(y)k(y)\mathbf{r}, \quad \textit{with} \\ &p(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} p_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)}), \\ &k(y) = \exp(\textstyle\sum_{m\in\mathbf{N}^n} k_m \prod_{j=1}^n (\frac{y_{j+1}}{y_j})^{m(j)/2}). \end{aligned}$$

Here $t(y)$ and $\mathbf{r}$ are for (ρ, φ).

(1) $p_m^{[\mu]} = 0$ unless $\mu \le m$.

(2) $p_m^{[\mu]} = 0$ if $m(j) = 0$ for $k \le j \le n$ and $\mu(k) \ne 0$.

Lemma 9.4. *Let C be a finite dimensional Lie algebra over a field K of characteristic 0, and assume that we have a direct sum decomposition $C = A \oplus B$ of C as a vector space over K. Let $R = K[[T_1, \dots, T_n]]$ and let $R_+ = \mathrm{Ker}(R \to K; T_j \mapsto 0$ for all $j)$. Let $\psi = (\psi_1, \psi_2) : R_+ \otimes_K C \to (R_+ \otimes_K A) \times (R_+ \otimes_K B)$ be the bijection characterized by $c = H(\psi_1(c), \psi_2(c))$, $c \in R_+ \otimes_K C$, where H is the Hausdorff series (for this terminology, cf. 1.2). Assume that a direct sum decomposition $C = \bigoplus_{\mu\in\mathbf{Z}^n} C^{[\mu]}$ is given. Assume that $[C^{[\mu]}, C^{[\mu']}] \subset C^{[\mu+\mu']}$ for any $\mu, \mu' \in \mathbf{Z}^n$. Assume that if $a \in A$, $b \in B$, then $a^{[\mu]} + a^{[-\mu]} \in A$ and $b^{[\mu]} + b^{[-\mu]} \in B$ for any $\mu \in \mathbf{Z}^n$. Let $c = \sum_{m\in\mathbf{N}^n} c_mT^m \in R_+ \otimes_K C$ ($c_m \in C$, $c_0 = 0$, $T^m = \prod_{j=1}^n T_j^{m(j)}$) and let $a = \psi_1(c) = \sum_{m\in\mathbf{N}^n} a_mT^m \in R_+ \otimes_K A$, $b = \psi_2(c) = \sum_{m\in\mathbf{N}^n} b_mT^m \in R_+ \otimes_K B$ ($a_m \in A$, $b_m \in B$, $a_0 = b_0 = 0$).*

Then we have:

(1) If $c_m^{[\mu]} = 0$ for any (μ, m) which does not satisfy $\mu \equiv m \mod 2$, then $a_m^{[\mu]} = b_m^{[\mu]} = 0$ for any (μ, m) which does not satisfy $\mu \equiv m \mod 2$.

(2) Fix k such that $1 \le k \le n$. If $c_m^{[\mu]} = 0$ for any (μ, m) such that $|\mu(k)| > m(k)$, then $a_m^{[\mu]} = b_m^{[\mu]} = 0$ for any (μ, m) such that $|\mu(k)| > m(k)$.

(3) Fix k such that $1 \le k \le n$. If $c_m^{[\mu]} = 0$ for any (μ, m) such that $m(j) = 0$ for $k < j \le n$ and such that $|\mu(k)| \ge m(k)$, then $a_m^{[\mu]} = b_m^{[\mu]} = 0$ for any (μ, m) such that $m(j) = 0$ for $k < j \le n$ and such that $|\mu(k)| \ge m(k)$.

(4) If $K = \mathbf{R}$ or $\mathbf{C}$, ψ_1, ψ_2 send convergent series to convergent series.

Proof. (1)–(3) are proved by induction on $\sum_{j=1}^n m(j)$. (4) follows from the fact that the Hausdorff series induces an analytic isomorphism from an open neighborhood of the origin in $A \times B$ to an open neighborhood of the origin in C, for the differential of the Hausdorff series at the origin gives the isomorphism $A \times B \to C$; $(a, b) \mapsto a + b$. □

In the rest of §9, we assume that $(N_1, \dots, N_n, F)$ generates a pure nilpotent orbit. For simplicity, when $n = 1$, we write N in place of N_1. Let (ρ, φ) be the associated SL(2)-orbit in n variables, and let t, $t^{(j)}$ be as in 2.2.

9.5. Assume $n = 1$. Then we have the following expression of $\exp(iyN)F$:

There exist $c > 0$ and $b_m \in \mathfrak{g}_{\mathbf{R}}$ $(m \geq 1)$ such that $\sum_{m \geq 1} b_m \lambda^{m/2}$ absolutely converges when $0 \leq \lambda < c$, that

$$\exp(iyN)F = t(y) \exp\big(\textstyle\sum_{m \geq 1} b_m y^{-m/2}\big)\mathbf{r}$$

when $y^{-1} < c$, and such that the component $b_m^{[\mu]}$ of b_m in $\mathfrak{g}_{\mathbf{R}}^{[\mu]}$ $(\mu \in \mathbf{Z})$ is zero unless $|\mu| \leq m - 1$ and $\mu \equiv m \mod 2$.

Proof. Let $a_m \in \mathfrak{g}_{\mathbf{R}}$ $(m \geq 1)$ be as in 2.4.1. Then

$$\begin{aligned} \exp\big(\textstyle\sum_{m \geq 1} a_m y^{-m}\big)t(y) &= t(y) \exp\big(\textstyle\sum_{m \geq 1, \mu \in \mathbf{Z}} a_m^{[\mu]} y^{-(2m-\mu)/2}\big) \\ &= t(y) \exp\big(\textstyle\sum_{m \geq 1} b_m y^{-m/2}\big), \end{aligned}$$

where $b_m = \sum_k a_k^{[2k-m]}$. We have $a_k^{[2k-m]} = 0$ unless $2k - m < k$. Since $k > 0$, the condition $2k - m < k$ is equivalent to the condition $|2k - m| < m$. □

9.6. We deduce the case $n = 1$ of 9.2.

Let b_m $(m \geq 1)$ be as in 9.5. We apply the case $n = 1$ of 9.4 by taking $B = \mathrm{Lie}(K_{\mathbf{r}})$. Let $\sum_{m \geq 1} h_m T^m$ (resp. $\sum_{m \geq 1} k_m T^m$) be the image of $\sum_{m \geq 1} b_m T^m$ under the map ψ_1 (resp. ψ_2) of 9.4. Let $h(y) = \exp(\sum_{m \geq 1} h_m y^{-m/2})$ and $k(y) = \exp(\sum_{m \geq 1} k_m y^{-m/2})$. Then by the case $n = 1$ of 9.4, we obtain the case $n = 1$ of 9.2.

9.7. We prove the case $n = 1$ of 9.3. That is, there exist $c > 0$ and $p_m \in \mathrm{Lie}(G_{W,\mathbf{R}})$ with $W = W(N)[-w]$, $k_m \in \mathrm{Lie}(K_{\mathbf{r}})$ $(m \geq 1)$ such that $\sum_{m \geq 1} p_m \lambda^m$ and $\sum_{m \geq 1} k_m \lambda^{m/2}$ absolutely converge when $0 \leq \lambda < c$, that $k_m^{[\mu]} = 0$ $(m \in \mathbf{N}, \mu \in \mathbf{Z})$ unless $\mu \equiv m$ mod 2, that $p_m^{[\mu]} = 0$ $(m \in \mathbf{N}, \mu \in \mathbf{Z})$ unless $\mu \leq m$, and that

$$\begin{aligned} &\exp(iyN)F = p(y)t(y)k(y)\mathbf{r} \quad \text{with} \\ &p(y) = \exp(\textstyle\sum_{m \geq 1} p_m y^{-m}), \quad k(y) = \exp(\textstyle\sum_{m \geq 1} k_m y^{-m/2}) \end{aligned}$$

when $y^{-1} < c$. (This result is almost the same as [Sc] 5.25.)

Take A and $B = \mathrm{Lie}(K_{\mathbf{r}})$ in 9.4 such that $A \subset \mathrm{Lie}(G_{W,\mathbf{R}})$. Such A exists. Indeed, take an $\mathbf{R}$-subspace A_0 of $\mathfrak{g}_{\mathbf{R}}^{[0]}$ such that $\mathfrak{g}_{\mathbf{R}}^{[0]} = A_0 \oplus (\mathrm{Lie}(K_{\mathbf{r}}) \cap \mathfrak{g}_{\mathbf{R}}^{[0]})$. Let $A = (\bigoplus_{\mu < 0} \mathfrak{g}_{\mathbf{R}}^{[\mu]}) \oplus A_0$. Then $\mathfrak{g}_{\mathbf{R}} = A \oplus B$ and $A \subset \mathrm{Lie}(G_{W,\mathbf{R}})$.

Let $h(y)$ be as in 9.6 with respect to the above A and B, and let $p(y) = t(y)h(y)t(y)^{-1}$. Then $p(y) = \exp(\sum_{m \in \mathbf{Z}} p_m y^{-m})$, where $p_m = \sum_{k \geq 1} h_k^{[2m-k]}$. We prove $p_m = 0$ for

$m \le 0$. Assume $m \le 0$. Then since $|2m-k| \ge k$ for any k, we have $h_k^{[2m-k]} = 0$ for any k by (2) of the case $n = 1$ of 9.2, and this proves $p_m = 0$. The fact that $p_m^{[\mu]} = 0$ ($m \in \mathbf{N}, \mu \in \mathbf{Z}$) unless $\mu \le m$ is similarly proved by (2.1) of the case $n = 1$ of 9.2.

9.8. We prove the case $n = 1$ of 8.5.

Let $p(y)$ be as in 9.7. Write $p(y) = v(y)p'(y)$ with $v(y), p'(y) \in G_{W,\mathbf{R}}$ ($W = W(N)[-w]$), where $p'(y) = s\,\mathrm{gr}^W(p(y))s^{-1}$ with s the splitting of W associated to ρ, and $v(y)$ acts trivially on $\mathrm{gr}^W(V)$. Then

$$\begin{aligned}\mathrm{BS}(W, \exp(iyN)F) &= \mathrm{BS}(W, v(y)p'(y)t(y)k(y)\mathbf{r}) \\ &= v(y)\,\mathrm{BS}(W, p'(y)t(y)\mathbf{r}) = v(y)\,\mathrm{BS}(W, \mathbf{r})\end{aligned}$$

by 8.3 (i), 8.4, and $p'(y)t(y)s^{(1)} = s^{(1)}$.

9.9. From 9.8, we obtain the case $n = 1$ of 8.7. That is, in the case $n = 1$, the splitting of $W = W(N)[-w]$ associated to ρ is the limit of $\mathrm{BS}(W, \exp(iyN)F)$ for $y \to \infty$.

9.10. We prove the statement in 2.4 about the uniqueness of (ρ, φ) satisfying 2.3.1 and 2.4.1. In fact, the arguments in 9.5–9.9 show that an SL(2)-orbit (ρ, φ) in one variable satisfying 2.3.1 and 2.4.1 has the characterizing property in 9.9 described by Borel-Serre theory, and hence is unique.

9.11. We prove (1) of 9.2.

Let $b_m \in \mathfrak{g}_{\mathbf{R}}$ ($m \in \mathbf{N}^n$) be as in 0.1 (b_m appeared in the expression of ${}^e g(y)$).

By using 9.4 with $B = \mathrm{Lie}(K_{\mathbf{r}})$, let $\sum_{m \in \mathbf{N}^n} h_m T^m$ (resp. $\sum_{m \in \mathbf{N}^n} k_m T^m$) be the image of $\sum_{m \in \mathbf{N}^n} b_m T^m$ under ψ_1 (resp. ψ_2). Let

$$\begin{aligned} h(y) &= \exp(\textstyle\sum_{m \in \mathbf{N}^n} h_m \prod_{j=1}^n (y_{j+1}/y_j)^{m(j)/2}), \\ k(y) &= \exp(\textstyle\sum_{m \in \mathbf{N}^n} k_m \prod_{j=1}^n (y_{j+1}/y_j)^{m(j)/2}). \end{aligned}$$

Then we have

$$\exp(\textstyle\sum_{1 \le j \le n} iy_j N_j)F = t(y)\ {}^e g(y)\mathbf{r} = t(y)h(y)k(y)\mathbf{r}.$$

Since $b_m^{[\mu]} = 0$ unless $m \equiv \mu \mod 2$ as is seen by the method in 9.5, $h_m^{[\mu]} = 0$ and $k_m^{[\mu]} = 0$ unless $m \equiv \mu \mod 2$ by 9.4 (1).

9.12. We prove 9.2 (2.1).

We fix k such that $1 \le k \le n$. We prove that $h_m^{[\mu]} = 0$ unless $|\mu(k)| \le m(k)$.

First we show that the validity of 9.2 (2.1) is independent of the choice of A. Assume that A' satisfies the same condition as A, let $\psi' = (\psi'_1, \psi'_2)$ be the map $\psi = (\psi_1, \psi_2)$ in 9.4 defined for the pair $(A', \mathrm{Lie}(K_{\mathbf{r}}))$, and define $\sum_{m \in \mathbf{N}^n} h'_m T^m$ and $\sum_{m \in \mathbf{N}^n} k''_m T^m$ as the images of $\sum_{m \in \mathbf{N}^n} h_m T^m$ under the maps ψ'_1 and ψ'_2, respectively. Let $h'(y) = \exp(\sum_{m \in \mathbf{N}^n} h'_m \prod_{j=1}^n (y_{j+1}/y_j)^{m(j)/2})$, $k''(y) = \exp(\sum_{m \in \mathbf{N}^n} k''_m \prod_{j=1}^n (y_{j+1}/y_j)^{m(j)/2})$, $k'(y) = k''(y)k(y)$. Then we have $h(y)k(y) = h'(k)k'(y)$. Hence by 9.4 (2), if 9.2 (2.1) is true for A, then it is true for A'.

By this independence, we may and will assume $A \subset \mathrm{Lie}(G_{W^{(k)},\mathbf{R}})$. Let P be the connected component as an algebraic group of $G_{W^{(k)},\mathbf{R}}$ containing the unit element. Fix $y_j > 0$ for $1 \le j \le n$ such that $y_j/y_{j+1} \gg 0$ $(1 \le j \le n)$. Then the pair $\big(\sum_{1\le j\le k} y_jN_j, \exp(\sum_{k<j\le n} iy_jN_j)F\big)$ generates a pure nilpotent orbit. Let (ρ', φ') be the SL(2)-orbit in one variable associated to this pair.

By the case $n = 1$ of 9.3 proved in 9.7, we have $f_s \in \mathrm{Lie}(P)$ $(s \ge 1)$ such that $\sum_{s\ge1} f_s\lambda^s$ absolutely converges for $\lambda \ge 0$ sufficiently small, and such that

$$\exp\big(i\lambda\big(\textstyle\sum_{j=1}^k y_jN_j\big) + \sum_{j=k+1}^n iy_jN_j\big)F \in f(\lambda)t'(\lambda)K_{\mathbf{r}'}\cdot\mathbf{r}' \quad \text{with} \quad f(\lambda) = \exp\big(\textstyle\sum_{s\ge1} f_s\lambda^{-s}\big) \tag{1}$$

for $\lambda \gg 0$, where $\mathbf{r}' := \varphi'(i)$ and t' is the "t" for ρ'.

Since P is a parabolic subgroup of $G_{\mathbf{R}}$ and $K_{\mathbf{r}}$ is a maximal compact subgroup of $G_{\mathbf{R}}$, we have $G_{\mathbf{R}} = PK_{\mathbf{r}}$ ([B] §11). That is, there exists $g \in P$ such that $\mathbf{r}' \in gK_{\mathbf{r}}\cdot\mathbf{r}$. We have $K_{\mathbf{r}'} = \mathrm{Int}(g)(K_{\mathbf{r}})$. Hence by 8.4 and 8.1 (iii), we have for $\lambda > 0$

$$t'(\lambda) = \mathrm{BS}_{P,K_{\mathbf{r}'}}(a) = \mathrm{Int}(g)(\mathrm{BS}_{P,K_{\mathbf{r}}}(a)) = \mathrm{Int}(g)(t^{(k)}(\lambda)),$$
$$\text{where} \quad a : \textstyle\bigoplus_k \mathrm{gr}_k^{W^{(k)}}(V) \to \bigoplus_k \mathrm{gr}_k^{W^{(k)}}(V)\ ;\ (x_k)_k \mapsto (\lambda^{(w-k)/2}x_k)_k.$$

This shows
$$f(\lambda)t'(\lambda)K_{\mathbf{r}'}\cdot\mathbf{r}' = t^{(k)}(\lambda)\,\mathrm{Int}(t^{(k)}(\lambda))^{-1}(f(\lambda)g)K_{\mathbf{r}}\cdot\mathbf{r}.$$

On the other hand, the left hand side of (1) belongs to

$$t^{(k)}(\lambda)t(y)h(y,\lambda)K_{\mathbf{r}}\cdot\mathbf{r},$$
$$\text{where} \quad h(y,\lambda) = \exp\Big(\textstyle\sum_{m\in\mathbf{N}^n} h_m \prod_{1\le j\le n}(\frac{y_{j+1}}{y_j})^{m(j)/2}\lambda^{-m(k)/2}\Big)$$

$(h_m \in A,\ h_0 = 0)$. Hence

$$h(y,\lambda)(P\cap K_{\mathbf{r}}) = t(y)^{-1}\,\mathrm{Int}(t^{(k)}(\lambda))^{-1}(f(\lambda)g)(P\cap K_{\mathbf{r}}). \tag{2}$$

We can write $g \in P$ as $g = g'g_0$, $g' = \exp(\sum_{s\ge1} g_{-s})$, where $g_0 = s^{(k)}\,\mathrm{gr}^{W^{(k)}}(g)(s^{(k)})^{-1} \in P$ and g_{-s} is an element of $\mathrm{Lie}(P)$ such that $g_{-s}^{[\mu]}$ for $\mu \in \mathbf{Z}^n$ is 0 unless $\mu(k) = -s$. Then g_0 commutes with $t^{(k)}(\lambda)$. We have

$$\mathrm{Int}(t^{(k)}(\lambda))^{-1}(f(\lambda)g') = \exp\big(\textstyle\sum_{s\ge1,\mu\in\mathbf{Z}^n} f_s^{[\mu]}\lambda^{(\mu(k)/2)-s}\big)\exp\big(\sum_{s\ge1} g_{-s}\lambda^{-s/2}\big). \tag{3}$$

Since $f_s \in \mathrm{Lie}(P)$, $f_s^{[\mu]} = 0$ unless $\mu(k) \le 0$. Hence (3) shows that $\mathrm{Int}(t^{(k)}(\lambda))^{-1}(f(\lambda)g)$ converges to g_0 as $\lambda \to \infty$. On the other hand, the left hand side of (2) converges to

$$h(y,\infty)(P\cap K_{\mathbf{r}}) = \exp(\textstyle\sum_{m\in\mathbf{N}^n,m(k)=0} h_m \prod_{1\le j\le n}(\frac{y_{j+1}}{y_j})^{m(j)/2})(P\cap K_{\mathbf{r}})$$

in $P/(P\cap K_{\mathbf{r}})$. Hence the limit of (2) for $\lambda \to \infty$ shows $g_0 \in t(y)h(y,\infty)(P\cap K_{\mathbf{r}})$. Hence the right hand side of (2) is written as $t(y)^{-1}\,\mathrm{Int}(t^{(k)}(\lambda))^{-1}(f(\lambda)g')t(y)h(y,\infty)(P\cap K_{\mathbf{r}})$. Note that

(4) $\exp(a)K_{\mathbf{r}}\cdot\mathbf{r}=\exp(b)K_{\mathbf{r}}\cdot\mathbf{r}$ implies $a=b$ if a,b belong to a sufficiently small neighborhood of 0 in A.

Since $y_j/y_{j+1}\gg 0$ $(1\le j\le n)$, $t(y)^{-1}\,\mathrm{Int}(t^{(k)}(\lambda))^{-1}(f(\lambda)g')t(y)h(y,\infty)$ for $\lambda\gg 0$ is near to 1. By (3), this element has the form $\exp(\sum_{s\ge 1}d_s\lambda^{-s/2})$ with $d_s\in\mathrm{Lie}(P)$ such that $d_s^{[\mu]}=0$ unless $|\mu(k)|\le s$. Let $\sum_{s\ge1}d'_sT^s=\psi_1(\sum_{s\ge1}d_sT^s)$ in 9.4 with $B=\mathrm{Lie}(K_{\mathbf{r}})$ and $n=1$. Then by 9.4 (2), ${d'}_s^{[\mu]}=0$ unless $|\mu(k)|\le s$. By (2) and (4), we have

$$\textstyle\sum_{m\in\mathbf{N}^n}h_m\prod_{j=1}^n(\frac{y_{j+1}}{y_j})^{m(j)/2}\lambda^{-m(k)/2}=\sum_{s\ge0}d'_s\lambda^{-s/2},$$

and this shows that $h_m^{[\mu]}=0$ unless $|\mu(k)|\le m(k)$. $\square$

9.13. We prove 9.3 except the following parts: the part $p_0=0$ and 9.3 (2).

We take $A\subset\mathrm{Lie}(G_{W^{(k)},\mathbf{R}})$.

Let $p(y)=t(y)h(t)t(y)^{-1}$. Then

$$p(y)=\exp(\textstyle\sum_{m\in\mathbf{Z}^n}p_m\prod_{j=1}^n(y_{j+1}/y_j)^{m(j)}),\quad p_m^{[\mu]}=h_{2m-\mu}^{[\mu]},\quad h_m^{[\mu]}=p_{(m+\mu)/2}^{[\mu]}.$$

By 9.2 (2.1) which we have already proved, $h_{2m-\mu}^{[\mu]}=0$ unless $|\mu(j)|\le 2m(j)-\mu(j)$ for any j, that is, unless $m(j)\ge0$ and $m(j)\ge\mu(j)$ for all j. Hence we have $p_m=0$ unless $m\ge0$, and we have $p_m^{[\mu]}=0$ unless $\mu(j)\le m(j)$ for any j.

9.14. We prove 8.5 except the following parts: the part $v_0=0$ and 8.5 (3).

Let $p(y)$ be as in 9.13, and write $p(y)=v(y)p'(y)$ with $v(y),p'(y)\in G_{W^{(k)},\mathbf{R}}$, where $p'(y)=s^{(k)}\,\mathrm{gr}^{W^{(k)}}(p(y))(s^{(k)})^{-1}$ and $v(y)$ acts trivially on $\mathrm{gr}^{W^{(k)}}(V)$. Then by 8.1 (iii) and 8.4, we have

$$\begin{aligned}&\mathrm{BS}(W^{(k)},\exp(\textstyle\sum_{j=1}^n iy_jN_j)F)=\mathrm{BS}(W^{(k)},p(y)t(y)k(y)\mathbf{r})\\&=v(y)\,\mathrm{BS}(W^{(k)},\mathbf{r})=v(y)s^{(k)}.\end{aligned}$$

Write $v(y)=\exp(\sum_{m\in\mathbf{N}^n}v_m\prod_{j=1}^n(y_{j+1}/y_j)^{m(j)})$. Then the property 8.5 (1) of v_m is clear and the property 8.5 (2) follows from the property 9.3 (1) of p_m.

9.15. We prove 4.2 except the part $u_0=0$, and prove also 8.5 (3).

Let $v(y)$ and v_m be as in 9.14. Then for $\lambda>0$, we have

$$\mathrm{BS}(W^{(k)},\exp(\textstyle\sum_{j=1}^k i\lambda y_jN_j+\sum_{j=k+1}^n iy_jN_j)F)=v(y')s^{(k)}\tag{1}$$

with $y'=(\lambda y_1,\dots,\lambda y_k,y_{k+1},\dots,y_n)$. Note

$$v(y')=\exp(\textstyle\sum_{m\in\mathbf{N}^n}v_m\cdot(\frac{y_{k+1}}{\lambda y_k})^{m(k)}\cdot\prod_{1\le j\le n,j\ne k}(\frac{y_{j+1}}{y_j})^{m(j)}).$$

By the case $n=1$ of 8.5 proved in 9.8, when $\lambda\to\infty$, the left hand side of (1) converges to the canonical splitting of $W^{(k)}$ associated to the mixed Hodge structure

$(W^{(k)}, \exp(\sum_{j=k+1}^n iy_jN_j)F)$, which is independent of $y_1, \dots, y_k$. On the other hand, the right hand side of (1) converges to

$$\exp(\textstyle\sum_{m\in\mathbf{N}^m, m(k)=0} v_m \prod_{1\le j\le n, j\ne k}(\frac{y_{j+1}}{y_j})^{m(j)})s^{(k)}.$$

Hence this is independent of $y_1, \dots, y_k$, and this proves 8.5 (3).

Let $u_m = v_{m'}$, where $m' \in \mathbf{N}^n$ is defined by $m'(j) = 0$ for $j \le k$ and $m'(j) = m(j-k)$ for $j > k$. We have shown:

(2) The canonical splitting of $W^{(k)}$ associated to $\exp(\sum_{j=k+1}^n iy_jN_j)F$ is $u(y)s^{(k)}$ with $u(y) = \exp(\sum_{m\in\mathbf{N}^{n-k}} u_m \prod_{j=k+1}^n (\frac{y_{j+1}}{y_j})^{m(j-k)})$.

The property 4.2 (ii) of u_m $(m \in \mathbf{N}^{n-k})$ is clear and the property 4.2 (iii) follows from the property 8.5 (2) of v_m $(m \in \mathbf{N}^n)$.

Lemma 9.16. *Let $k \le l \le n$. Then the canonical splitting of $W^{(k)}$ associated to the mixed Hodge structure $(W^{(k)}, \exp(\sum_{k<j\le l} iy_jN_j)\exp(iN_{l+1})\hat{F}_{(l+1)})$ is*

$$\exp(\textstyle\sum'_{m,\mu} u_m^{[\mu]} \cdot (\prod_{k<j<l}(\frac{y_{j+1}}{y_j})^{m(j-k)}) \cdot (\frac{1}{y_l})^{m(l-k)}) \cdot s^{(k)},$$

where $\sum'_{m,\mu}$ is the sum for all $m \in \mathbf{N}^{n-k}$ and $\mu \in \mathbf{Z}^n$ such that $m(j) = 0$ for $l-k < j \le n-k$ and such that $\mu(j) = 0$ for $l < j \le n$. Here in the case $l = n$, N_{l+1} means 0 and $\hat{F}_{(l+1)}$ means F.

Proof. We prove this by downward induction. In the case $l = n$, this is 9.15 (2). Write the statement 9.16 for l as $9.16^{(l)}$. Assume $l < n$ and assume that $9.16^{(l+1)}$ is true. Consider the mixed Hodge structure

$$(1) \qquad (W^{(k)}, t^{(l+1)}(\lambda)^{-1}\exp(\textstyle\sum_{k<j\le l} i\lambda y_jN_j)\exp(i\lambda N_{l+1})\exp(iN_{l+2})\hat{F}_{(l+2)}).$$

We have

$$\begin{aligned} &t^{(l+1)}(\lambda)^{-1}\exp(\textstyle\sum_{k<j\le l} i\lambda y_jN_j)\exp(i\lambda N_{l+1})\exp(iN_{l+2})\hat{F}_{(l+2)} \\ &= \exp(\textstyle\sum_{k<j\le l} iy_jN_j)\exp(iN_{l+1})t^{(l+1)}(\lambda)^{-1}\exp(iN_{l+2})\hat{F}_{(l+2)} \end{aligned}$$

by 2.7, and

$$t^{(l+1)}(\lambda)^{-1}\exp(iN_{l+2})\hat{F}_{(l+2)} \to \hat{F}_{(l+1)} \quad \text{when } \lambda \to \infty.$$

Hence we have:

(2) When $\lambda \to \infty$, the canonical splitting of $W^{(k)}$ associated to (1) converges to the canonical splitting of $W^{(k)}$ associated to the mixed Hodge structure $(W^{(k)}, \exp(\sum_{k<j\le l} iy_jN_j)\exp(iN_{l+1})\hat{F}_{(l+1)})$.

On the other hand, by $9.16^{(l+1)}$, the canonical splitting of $W^{(k)}$ associated to (1) coincides with

$$\begin{aligned} &\mathrm{Int}(t^{(l+1)}(\lambda))^{-1}\big(\exp(\textstyle\sum''_{m,\mu} u_m^{[\mu]}(\prod_{k<j<l}(\frac{y_{j+1}}{y_j})^{m(j-k)})(\frac{1}{y_l})^{m(l-k)}(\frac{1}{\lambda})^{m(l+1-k)})\big)s^{(k)} \\ &= \exp\big(\textstyle\sum''_{m,\mu} u_m^{[\mu]}(\prod_{k<j<l}(\frac{y_{j+1}}{y_j})^{m(j-k)})(\frac{1}{y_l})^{m(l-k)}\lambda^{(\mu(l+1)/2)-m(l+1-k)}\big)s^{(k)}, \end{aligned}$$

where $\sum''_{m,\mu}$ is the sum for all $m \in \mathbf{N}^{n-k}$ and $\mu \in \mathbf{Z}^n$ such that $m(j) = 0$ for $l+1-k < j \le n-k$ and such that $\mu(j) = 0$ for $l+1 < j \le n$. By 4.2 (iii) which we have proved in 9.15, we have $u_m^{[\mu]} = 0$ unless $\mu(l+1) \le m(l+1-k)$. Hence if $(\mu(l+1)/2) - m(l+1-k) \ge 0$ and $u_m^{[\mu]} \ne 0$, then $m(l+1-k) = \mu(l+1) = 0$. From this, we have:

(3) When $\lambda \to \infty$, the canonical splitting of $W^{(k)}$ associated to (1) converges to $\exp(\sum'_{m,\mu} u_m^{[\mu]} \prod_{k<j<l} (\frac{y_{j+1}}{y_j})^{m(j-k)} \cdot (\frac{1}{y_l})^{m(l-k)}) \cdot s^{(k)}$.

By (2) and (3), we have $9.16^{(l)}$. □

9.17. Write the statement 9.2 (2.2) for a fixed k as 9.2 $(2.2)_k$. By 9.4 (3), the validity of 9.2 $(2.2)_k$ is independent of the choice of A as is seen by the argument at the beginning of 9.12.

9.18. Denote by $9.3'$ the part "9.3 without the statement $p_0 = 0$" of 9.3. We complete the proofs of 4.2, 8.5, 9.2, and $9.3'$.

We prove 4.2, 8.5, 9.2 $(2.2)_k$, and $9.3'$ together, by downward induction on k.

Take $A \subset \mathrm{Lie}(G_{W^{(k)},\mathbf{R}})$. Let h_m, p_m, v_m, u_m be as in 9.11–9.15. We first prove

Claim 1. $p_0^{[\mu]} = v_0^{[\mu]} = u_0^{[\mu]} = 0$ unless $\mu(j) = 0$ for any j such that $k < j \le n$.

Note $p_0 = \sum_{m \in \mathbf{N}^n} h_m^{[-m]}$. By the hypothesis of our downward induction, $h_m^{[-m]} = 0$ unless $m(j) = 0$ for any j such that $k < j \le n$. This proves the statement for p_0. The statement for v_0 follows from that for p_0, and the statement for u_0 follows from that for v_0.

Now by Claim 1, $9.16^{(k)}$ is read as

$$s^{(k)} = \exp(u_0) s^{(k)}.$$

This proves

$$u_0 = 0.$$

This proves 4.2.

We obtain $v_0 = 0$ from $u_0 = 0$. This proves 8.5.

By $v_0 = 0$ and by 8.5 (3), we have $v_m = 0$ if $m(j) = 0$ for any j such that $k \le j \le n$. This proves 9.3 (2), and proves $9.3'$.

We prove 9.2 $(2.2)_k$. Assume $h_m^{[\mu]} \ne 0$ and assume $m(j) = 0$ for $k < j \le n$, and $m(k) \ne 0$. We have $\mu(j) = 0$ for $k < j \le n$ because $|\mu(j)| \le m(j)$ by 9.3 (2.1). If $|\mu(k)| = m(k)$, then $\mu(k) = -m(k)$ since $\mu(k) \le 0$ by $h_m \in \mathrm{Lie}(G_{W^{(k)},\mathbf{R}})$. We have $h_m^{[\mu]} = p_{(m+\mu)/2}^{[\mu]}$ (9.13). Since $(m(j) + \mu(j))/2 = 0$ for $k \le j \le n$, we have $\mu(k) = 0$ by 9.3 (2). But this contradicts $\mu(k) = -m(k) \ne 0$.

9.19. Let p_m and h_m be as above. We complete the proof of 9.3 by showing $p_0 = 0$. We have $p_0 = \sum_{m \in \mathbf{N}^n} h_m^{[-m]}$. But $h_m^{[-m]} = 0$ by 9.2 (2.2).

§10. Complements to Main Theorem

We give complementary results to the main theorem 0.5.

10.1. Let the notation be as in Theorem 0.5.

The following facts 10.1.1–10.1.3 are shown by the reduction to the pure situation by 4.1, by using 2.5 and 1.6. In particular, we have a description of the limit splitting s in Theorem 0.5 (1) by finite algebraic steps, not as a limit.

10.1.1. Let $W^{(j)} = M(N_1 + \cdots + N_j, W)$ for $0 \le j \le n$ (in particular, $W^{(0)} = W$). Let $(W^{(n)}, \hat{F}_{(n)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(n)}, F)$. Then $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$ is a mixed Hodge structure. Let $(W^{(n-1)}, \hat{F}_{(n-1)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to it. Then $(W^{(n-2)}, \exp(iN_{n-1})\hat{F}_{(n-1)})$ is a mixed Hodge structure, and so on. This process continues until we obtain the $\mathbf{R}$-split mixed Hodge structure $(W^{(0)}, \hat{F}_{(0)})$.

10.1.2. $\hat{F}_{(0)} = \mathbf{r}$ $(= s(\bigoplus_w \mathbf{r}_w))$, and the splitting s of W in Theorem 0.5 (1) is the splitting of $W = W^{(0)}$ given by the $\mathbf{R}$-split mixed Hodge structure $(W^{(0)}, \hat{F}_{(0)})$.

10.1.3. For $1 \le j \le n$, $\hat{F}_{(j)} = s(\bigoplus_w \varphi_w(\mathbf{0}_j, \mathbf{i}_{n-j}))$.

Define a homomorphism ρ of algebraic groups, which is defined over $\mathbf{R}$, and a holomorphic map φ

$$\rho : \mathbf{C}^\times \times \mathrm{SL}(2, \mathbf{C})^n \to \mathrm{Aut}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}}), \quad \varphi : \mathbf{P}^1(\mathbf{C})^n \to D^\vee,$$

by

$$\rho(\lambda, g) = s(\textstyle\bigoplus_{w\in\mathbf{Z}} \lambda^w \rho_w(g))s^{-1}, \quad \varphi(z) = s(\textstyle\bigoplus_{w\in\mathbf{Z}} \varphi_w(z))$$

$(\lambda \in \mathbf{C}^\times$, $g \in \mathrm{SL}(2, \mathbf{C})^n)$. Then 10.1.3 and the first part of 10.1.2 are written as

10.1.4. For $0 \le j \le n$, $\hat{F}_{(j)} = \varphi(\mathbf{0}_j, \mathbf{i}_{n-j})$.

We have:

10.1.5. $\rho(\lambda, g)\varphi(z) = \varphi(gz)$ for $\lambda \in \mathbf{C}^\times$, $g \in \mathrm{SL}(2, \mathbf{C})^n$, $z \in \mathbf{P}^1(\mathbf{C})^n$.

10.1.6. $\varphi(\mathfrak{h}^n) \subset D$.

Thus this (ρ, φ) is similar to the (ρ, φ) in the pure case (0.1).

We have

$$t(y_1, \ldots, y_n) = \rho\left(1/\sqrt{y_1}, \begin{pmatrix} \sqrt{y_1} & 0 \\ 0 & 1/\sqrt{y_1} \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{y_n} & 0 \\ 0 & 1/\sqrt{y_n} \end{pmatrix}\right),$$
$$\varphi(iy_1, \ldots, iy_n) = t(y_1, \ldots, y_n)\mathbf{r},$$

where $t(y_1, \ldots, y_n)$ is as in Theorem 0.5.

A difference from the pure case is that in the mixed case, in general, it is not possible to have a result of the form as in 0.1

$$\exp(\textstyle\sum_{j=1}^n iy_jN_j)F = f(y)\varphi(iy_1, \ldots, iy_n)$$

($f(y) \in \mathrm{Aut}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}})$, $y_j/y_{j+1} \gg 0$, $y_{n+1} = 1$) such that $f(y) \to 1$ when $y_j/y_{j+1} \to \infty$ $(1 \le j \le n)$. See Example 13.2.

10.2. Let the notation be as in Theorem 0.5.

Let $s^{(j)}$ $(0 \le j \le n)$ be the splitting of $W^{(j)}$ given by the $\mathbf{R}$-split mixed Hodge structure $(W^{(j)}, \hat{F}_{(j)})$. In particular, $s^{(0)}$ is the splitting s of W in Theorem 0.5 (1).

Then by 10.1.3, for $0 \le j \le n$ and for $w \in \mathbf{Z}$, $s^{(j)}(\mathrm{gr}_w^{W^{(j)}})$ coincides with the part of V on which

$$\rho\left(\lambda, \left\{\begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\}^j \times \{1\}^{n-j}\right) \quad (\lambda \in \mathbf{R}^\times)$$

acts as the multiplication by λ^w. Here ρ is as in 10.1. From this, we have the following facts.

For $\theta \in \mathbf{Z}^{\{0,\dots,n\}}$, let

$$V^{[\theta]} = \textstyle\bigcap_{j=0}^n s^{(j)}(\mathrm{gr}_{\theta(j)}^{W^{(j)}}).$$

Then

$$V = \textstyle\bigoplus_{\theta \in \mathbf{Z}^{\{0,\dots,n\}}} V^{[\theta]}, \quad W_k^{(j)} = \textstyle\bigoplus_{\theta \in \mathbf{Z}^{\{0,\dots,n\}}, \theta(j) \le k} V^{[\theta]} \quad (0 \le j \le n,\ k \in \mathbf{Z}).$$

This explains the distributive property of $W^{(0)}, \dots, W^{(n)}$ (5.5).

For $\mu \in \mathbf{Z}^n$, let

$$V^{[\mu]} = \textstyle\bigcap_{j=1}^n s^{(j)}(\mathrm{gr}_{\mu(j)}^{W^{(j)}}).$$

That is, $V^{[\mu]}$ is the direct sum of $V^{[\theta]}$ for all $\theta \in \mathbf{Z}^{\{0,\dots,n\}}$ such that $\theta(j) = \mu(j)$ for $1 \le j \le n$. Then

$$V = \textstyle\bigoplus_{\mu \in \mathbf{Z}^n} V^{[\mu]}, \quad W_k^{(j)} = \textstyle\bigoplus_{\mu \in \mathbf{Z}^n, \mu(j) \le k} V^{[\mu]} \quad (1 \le j \le n,\ k \in \mathbf{Z}),$$

and $V^{[\mu]}$ coincides with the part of V on which $t(y)$ acts as the multiplication by $\prod_{j=1}^n (y_{j+1}/y_j)^{\mu(j)/2}$.

For $\theta \in \mathbf{Z}^{\{0,\dots,n\}}$ and for $\mu \in \mathbf{Z}^n$, let

$$\mathfrak{g}_{\mathbf{R}}^{[\theta]} = \{X \in \mathfrak{g}_{\mathbf{R}} \mid XV^{[\theta']} \subset V^{[\theta'+\theta]} \text{ for any } \theta' \in \mathbf{Z}^{\{0,\dots,n\}}\},$$
$$\mathfrak{g}_{\mathbf{R}}^{[\mu]} = \{X \in \mathfrak{g}_{\mathbf{R}} \mid XV^{[\mu']} \subset V^{[\mu'+\mu]} \text{ for any } \mu' \in \mathbf{Z}^n\}.$$

Then

$$\mathfrak{g}_{\mathbf{R}} = \textstyle\bigoplus_{\theta \in \mathbf{Z}^{\{0,\dots,n\}}} \mathfrak{g}_{\mathbf{R}}^{[\theta]} = \textstyle\bigoplus_{\mu \in \mathbf{Z}^n} \mathfrak{g}_{\mathbf{R}}^{[\mu]},$$
$$\mathrm{Lie}(G_{W^{(j)},\mathbf{R}}) = \textstyle\bigoplus_{\theta \in \mathbf{Z}^{\{0,\dots,n\}}, \theta(j) \le 0} \mathfrak{g}_{\mathbf{R}}^{[\theta]} \ \text{ for } \ 0 \le j \le n,$$
$$\mathrm{Lie}(G_{W^{(j)},\mathbf{R}}) = \textstyle\bigoplus_{\mu \in \mathbf{Z}^n, \mu(j) \le 0} \mathfrak{g}_{\mathbf{R}}^{[\mu]} \ \text{ for } \ 1 \le j \le n,$$

and $\mathfrak{g}_{\mathbf{R}}^{[\mu]}$ for $\mu \in \mathbf{Z}^n$ coincides with the part of $\mathfrak{g}_{\mathbf{R}}$ on which $\mathrm{Ad}(t(y))$ acts as the multiplication by $\prod_{j=1}^n (y_{j+1}/y_j)^{\mu(j)/2}$.

For $1 \le j \le n$, write $N_j = \sum_{\theta \in \mathbf{Z}^{\{0,\dots,n\}}} N_j^{[\theta]} = \sum_{\mu \in \mathbf{Z}^n} N_j^{[\mu]}$ with $N_j^{[\theta]} \in \mathfrak{g}_{\mathbf{R}}^{[\theta]}$ and $N_j^{[\mu]} \in \mathfrak{g}_{\mathbf{R}}^{[\mu]}$. Define

$$\hat{N}_j = \textstyle\sum'_{\theta \in \mathbf{Z}^{\{0,\dots,n\}}} N_j^{[\theta]}, \quad N_j^{\Delta} = \textstyle\sum'_{\mu \in \mathbf{Z}^n} N_j^{[\mu]},$$

where $\sum'_\theta$ (resp. $\sum'_\mu$) is the sum for all $\theta \in \mathbf{Z}^{\{0,\dots,n\}}$ (resp. $\mu \in \mathbf{Z}^n$) such that $\theta(k) = 0$ for $0 \le k < j$ (resp. $\mu(k) = 0$ for $1 \le k < j$). In particular, $N_1^{\Delta} = N_1$. We have $\hat{N}_j = s(\bigoplus_{w \in \mathbf{Z}} (\hat{N}_j \text{ of } \rho_w))s^{-1}$ ($s = s^{(0)}$ = the limit splitting of W in Theorem 0.5), and $\hat{N}_j$ coincides with the component of N_j^{Δ} purely of weight 0 with respect to the splitting s of W.

10.3. Let the notation be as in Theorem 0.5. For $1 \le j \le n$ and $\lambda > 0$, let

$$t^{(j)}(\lambda) = s(\textstyle\bigoplus_{w \in \mathbf{Z}} \lambda^{-w/2} t_w^{(j)}(\lambda))s^{-1}.$$

So, $t(y) = \prod_{j=1}^n t^{(j)}(y_j/y_{j+1})$. We have

$$\mathrm{Ad}(t^{(j)}(\lambda))(N_k) = \lambda N_k \quad \text{for} \quad 1 \le k \le j.$$

This is shown by the reduction to the pure case 2.7 by 4.1, by using 4.4.

Proposition 10.4. *Let the notation be as in Theorem 0.5.*

(1) For $0 \le j \le n$, *we have* $\hat{F}_{(j)} = \exp(\sum_{k=j+1}^n i\hat{N}_k)\hat{F}_{(n)}$.
In particular, $\exp(\sum_{j=1}^n i\hat{N}_j)\hat{F}_{(n)} = \mathbf{r}$.

(2) $\exp(\sum_{j=1}^n iN_j^{\Delta})\hat{F}_{(n)} = \exp(\varepsilon_0)\mathbf{r}$.
Here ε_0 *is a member of* $(\varepsilon_m)_m$ *in Theorem 0.5.*

Proof. (1) follows from 10.1.3.

We prove (2). Write $\varepsilon^{(n)} = \varepsilon(W^{(n)}, F)$ so that $F = \exp(\varepsilon^{(n)})\hat{F}_{(n)}$. Consider

$$\begin{aligned} &t(y)^{-1}\exp(\textstyle\sum_{j=1}^n iy_jN_j)F \\ &= \exp(\mathrm{Ad}(t(y))^{-1}\textstyle\sum_{j=1}^n iy_jN_j)\exp(\mathrm{Ad}(t(y))^{-1}\varepsilon^{(n)})t(y)^{-1}\hat{F}_{(n)}. \end{aligned}$$

Since $N_j^{[\mu]} = 0$ unless $\mu(k) = -2$ for $j \le k \le n$ by 10.3, we have

$$\mathrm{Ad}(t(y))^{-1}(y_jN_j) = \textstyle\sum_{\mu \in \mathbf{Z}^n} \prod_{k=1}^{j-1} (\frac{y_k}{y_{k+1}})^{\mu(k)/2} N_j^{[\mu]}.$$

This converges to N_j^{Δ}.

Next we have

$$\mathrm{Ad}(t(y))^{-1}\varepsilon^{(n)} = \textstyle\sum_{\mu \in \mathbf{Z}^n} \prod_{k=1}^{n} (\frac{y_k}{y_{k+1}})^{\mu(k)/2} (\varepsilon^{(n)})^{[\mu]}.$$

Since $(\varepsilon^{(n)})^{[\mu]} = 0$ unless $\mu(n) \le -2$ and $\mu(k) \le 0$ for any $1 \le k \le n$, this converges to 0.

We have $t(y)^{-1}\hat{F}_{(n)} = \hat{F}_{(n)}$ by 10.1.3.

Hence

$$t(y)^{-1}\exp(\textstyle\sum_{j=1}^n iy_jN_j)F \to \exp(\sum_{j=1}^n iN_j^{\Delta})\hat{F}_{(n)} \quad \text{as} \ \ y_j/y_{j+1} \to \infty \ \ (1 \le j \le n).$$

On the other hand, $t(y)^{-1}\exp(\sum_{j=1}^n iy_jN_j)F = {}^e g(y)\exp(\varepsilon(y))\mathbf{r}$ as in Theorem 0.5, and this converges to $\exp(\varepsilon_0)\mathbf{r}$. This proves (2). □

Lemma 10.5. *Let the notation be as in Theorem 0.5. We have*

$$\delta(W, \exp(\textstyle\sum_{j=1}^n iy_jN_j)F),\ \zeta(W, \exp(\textstyle\sum_{j=1}^n iy_jN_j)F) \in \mathbf{R}(y) \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}},$$
$$\varepsilon(W, \exp(\textstyle\sum_{j=1}^n iy_jN_j)F) \in \mathbf{C}(y) \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}.$$

Here $\mathbf{R}(y)$ *denotes the rational function field* $\mathbf{R}(y_1, \dots, y_n)$ *in* n *variables* $y_1, \dots, y_n$ *over* $\mathbf{R}$*, and* $\mathbf{C}(y) = \mathbf{C}(y_1, \dots, y_n)$*.*

Proof. Since N_j are nilpotent and commute with each other, $\exp(\sum_{j=1}^n iy_jN_j)$ belongs to $\mathbf{C}[y_1, \dots, y_n] \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}} \subset \mathbf{C}(y) \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$. We can regard Deligne's splitting $(I_y^{p,q})$ of $(W, \exp(\sum_{j=1}^n iy_jN_j)F)$ as a direct sum decomposition of the vector space $\mathbf{C}(y) \otimes_{\mathbf{R}} V$ over $\mathbf{C}(y)$. The proof in [CKS] of the unique existence of δ having the properties 1.3 (i) and (ii) shows that there is a unique $\delta_y \in \mathbf{R}(y) \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$ having the properties 1.3 (i) and (ii) for $(I_y^{p,q})$. We obtain elements $\delta_{y,p,q} \in \mathbf{R}(y) \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$ and they determine $\zeta_y \in \mathbf{R}(y) \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$. We have $\delta(W, \exp(\sum_{j=1}^n iy_jN_j)F) = \delta_y$, $\zeta(W, \exp(\sum_{j=1}^n iy_jN_j)F) = \zeta_y$. By the definition of ε (1.2), we have $\varepsilon(W, \exp(\sum_{j=1}^n iy_jN_j)F) \in \mathbf{C}(y) \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$. □

Proposition 10.6. *Let the notation be as in Theorem 0.5.*

(1) $\varepsilon_m^{[\mu]} = 0$ *unless* $m \equiv \mu \mod 2$.

(2) Let $\mathbf{C}\{T_1, \dots, T_n\} \subset \mathbf{C}[[T_1, \dots, T_n]]$ *be the ring of convergent series in* n *variables. As an element of* $\mathbf{C}\{(y_2/y_1)^{1/2}, \dots, (y_{n+1}/y_n)^{1/2}\}[y_1^{1/2}, \dots, y_n^{1/2}] \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$, $\mathrm{Ad}(t(y))\varepsilon(y)$ *belongs to* $\mathbf{C}\{y_2/y_1, \dots, y_{n+1}/y_n\}[y_1, \dots, y_n] \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$.

Proof. Since $\varepsilon(y) = \mathrm{Ad}(g(y)t(y))^{-1}\varepsilon(W, \exp(\sum_{j=1}^n iy_jN_j)F)$ and $g(y)$ is the exponential of an element of $\mathbf{C}\{y_2/y_1, \dots, y_{n+1}/y_n\} \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$, (1) follows from Lemma 10.5.

(2) follows from (1). □

Proposition 10.7. *Let the notation be as in Theorem 0.5. For* $y_j > 0$ *and* $y_{j+1}/y_j \ll 1$ $(1 \le j \le n$, y_{n+1} *denotes* $1)$, $\mathrm{Ad}(t(y))^{-1}\delta(W, \exp(\sum_{j=1}^n iy_jN_j)F)$, $\mathrm{Ad}(t(y))^{-1}\zeta(W, \exp(\sum_{j=1}^n iy_jN_j)F)$, $\delta(W, t(y)^{-1}g(y)^{-1}\exp(\sum_{j=1}^n iy_jN_j)F)$, *and* $\zeta(W, t(y)^{-1}g(y)^{-1}\exp(\sum_{j=1}^n iy_jN_j)F)$ (*resp.* $\mathrm{Ad}(t(y))^{-1}\varepsilon(W, \exp(\sum_{j=1}^n iy_jN_j)F)$) *are* (*resp. is*) *expressed as convergent series in* $(y_{j+1}/y_j)^{1/2}$ $(1 \le j \le n)$ *with coefficients in* $\mathfrak{g}_{\mathbf{R}}$ (*resp.* $\mathfrak{g}_{\mathbf{C}}$).

Proof. Since $\mathrm{Ad}(t(y))^{-1}\varepsilon(W, \exp(\sum_{j=1}^n iy_jN_j)F) = \mathrm{Ad}({}^e g(y))\varepsilon(y)$, the assertion for ε is deduced from Theorem 0.5 (3), (4). By 1.2 (1), the assertions for δ and ζ follow from the above result for $\mathrm{Ad}(t(y))^{-1}\varepsilon(W, \exp(\sum_{j=1}^n iy_jN_j)F)$ and from Theorem 0.5 (4). □

For real analytic manifolds A, B and $c > 0$, and for a map

$$f : A \times \{\lambda \in \mathbf{R}^n \mid 0 < \lambda_j < c \text{ for } 1 \le j \le n\} \to B,$$

we say f is real analytic at $\lambda = 0$ if the following condition is satisfied: For each $\alpha \in A$, there are an open neighborhood U of α in A, a real number c'such that $0 < c' < c$, and a real analytic map

$$f' : U \times \{\lambda \in \mathbf{R}^n \mid -c' < \lambda_j < c' \text{ for } 1 \le j \le n\} \to B$$

which coincides with f on $U \times \{\lambda \in \mathbf{R}^n \mid 0 < \lambda_j < c' \text{ for } 1 \le j \le n\}$.

The following is a generalization of [CKS] Remark (4.65) (ii) to the mixed case.

Proposition 10.8. *Let $(V, W, (\langle\ ,\ \rangle_w)_w)$ and $G_{w,\mathbf{R}}$, $G_{\mathbf{R}}$, $\mathfrak{g}_{\mathbf{R}}$, $D^\vee$ be as in 0.2. Let A be a real analytic manifold, and let*

$$A \to \mathfrak{g}_{\mathbf{R}}\ ;\ \alpha \mapsto N_{j,\alpha} \quad (1 \le j \le n), \qquad A \to D^\vee\ ;\ \alpha \mapsto F_\alpha$$

be real analytic maps satisfying the following conditions (i) and (ii).

(i) For any $\alpha \in A$, $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$ generates a mixed nilpotent orbit.

(ii) For $1 \le j \le n$, the relative monodromy filtration $M(N_{1,\alpha} + \cdots + N_{j,\alpha}, W)$ is independent of $\alpha \in A$.

Then:

(1) Locally on A, the number $c > 0$ in Theorem 0.5 (2) for $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$ can be taken to be common for any $\alpha \in A$. Let $u_{m,\alpha}$ ($m \in \mathbf{N}^n, \alpha \in A$) be u_m in Theorem 0.5 (2) for $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$. Then the map $\alpha \mapsto u_{m,\alpha}$ is real analytic.

(2) Locally on A, let ${}^e g_{w,\alpha}(y)$ ($w \in \mathbf{Z}, \alpha \in A$) be ${}^e g(y)$ in 0.1 for $(\mathrm{gr}^W_w(N_{1,\alpha}), \dots, \mathrm{gr}^W_w(N_{n,\alpha}), F_\alpha(\mathrm{gr}^W_w))$ which we choose (note that ${}^e g(y)$ in 0.1 is not unique) in the way that for any w, the function

$$A \times \{\lambda \in \mathbf{R}^n \mid 0 < \lambda_j < c\} \to G_{w,\mathbf{R}}\ ;\ (\alpha, \lambda) \mapsto {}^e g_{w,\alpha}(y), \quad \text{where } \lambda_j = \sqrt{y_{j+1}}/\sqrt{y_j}$$

is real analytic at $\lambda = 0$ (this is possible by [CKS] *Remark (4.65) (ii)). Then locally on A, the numbers $c > 0$ in Theorem 0.5 (3) and (4) for $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$ can be taken to be common for any $\alpha \in A$. Let $b_{m,\alpha}$ (resp. $\varepsilon_{m,\alpha}$) ($m \in \mathbf{N}^n, \alpha \in A$) be b_m (resp. ε_m) in Theorem 0.5 (3) (resp. (4)) for $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$. Then the maps $\alpha \mapsto b_{m,\alpha}$, $\alpha \mapsto \varepsilon_{m,\alpha}$ are real analytic.*

Proof. For $0 \le j \le n$, the map

$$A \to D^\vee; \alpha \mapsto \hat{F}_{\alpha,(j)}$$

is real analytic. This is shown step by step, by downward induction on j using the real analycity of $\varepsilon(W^{(j)}, -)$.

Hence for $0 \le j \le n$, the splitting $s^{(j)}_\alpha$ of $W^{(j)}$ defined by the $\mathbf{R}$-split mixed Hodge structure $(W^{(j)}, \hat{F}_{\alpha,(j)})$ is real analytic in α.

Let $\mathfrak{g}_{\mathbf{R}} = \bigoplus_{\mu \in \mathbf{Z}^n} \mathfrak{g}_{\mathbf{R}}^{[\mu,\alpha]}$ be the decomposition in 10.2 defined by $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$. Since this decomposition is determined by $s^{(j)}_\alpha$ ($1 \le j \le n$) as in 10.2 and since the $s^{(j)}_\alpha$ are real analytic in α, this decomposition of $\mathfrak{g}_{\mathbf{R}}$ is real analytic in α.

Now as in the proof of 10.4, consider

$$\begin{aligned} &t_\alpha(y)^{-1} \exp(\textstyle\sum_{j=1}^n iy_j N_{j,\alpha}) F_\alpha \\ &= \exp(\mathrm{Ad}(t_\alpha(y))^{-1} \textstyle\sum_{j=1}^n iy_j N_{j,\alpha}) \exp(\mathrm{Ad}(t_\alpha(y))^{-1} \varepsilon^{(n)}_\alpha) \hat{F}_{\alpha,(n)}, \end{aligned}$$

where $t_\alpha(y)$ is $t(y)$ in Theorem 0.5 (3) for $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$ and $\varepsilon^{(n)}_\alpha = \varepsilon(W^{(n)}, F_\alpha)$. As functions in (α, λ),

$$\mathrm{Ad}(t_\alpha(y))^{-1}(y_j N_{j,\alpha}) = \textstyle\sum_{\mu \in \mathbf{Z}^n} \prod_{k=1}^{j-1} (\frac{y_k}{y_{k+1}})^{\mu(k)/2} N_{j,\alpha}^{[\mu,\alpha]},$$

$$\mathrm{Ad}(t_\alpha(y))^{-1}(\varepsilon^{(n)}_\alpha) = \textstyle\sum_{\mu \in \mathbf{Z}^n} \prod_{k=1}^{n} (\frac{y_k}{y_{k+1}})^{\mu(k)/2} (\varepsilon^{(n)}_\alpha)^{[\mu,\alpha]}$$

are real analytic at $\lambda = 0$. Hence we have

Claim 1. As a function in (α, λ), $t_\alpha(y)^{-1} \exp(\sum_{j=1}^n iy_j N_{j,\alpha})F_\alpha$ is real analytic at $\lambda = 0$.

We prove (1). Let $u_\alpha(y)$ be $u(y)$ in Theorem 0.5 (2) for $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$. Let ${}^e u_\alpha(y) = t_\alpha(y)^{-1} u_\alpha(y) t_\alpha(y)$. Then ${}^e u_\alpha(y) = {}^e s_\alpha(y) s_\alpha^{-1}$, where ${}^e s_\alpha(y)$ is the splitting of W associated to the mixed Hodge structure $(W, t_\alpha(y)^{-1} \exp(\sum_{j=1}^n iy_j N_{j,\alpha})F_\alpha)$ (we assume $y_j/y_{j+1} \gg 0$). By Claim 1, as a function in (α, λ), ${}^e s_\alpha(y)$ is real analytic at $\lambda = 0$. Hence we have

Claim 2. As a function in (α, λ), ${}^e u_\alpha(y)$ is real analytic at $\lambda = 0$.

Hence we can write ${}^e u_\alpha(y) = \exp(\sum_{m \in \mathbf{N}^n} {}^e u_{m,\alpha} \prod_{j=1}^n \lambda_j^{m(j)})$, there is $c > 0$ locally on A such that $\sum_{m \in \mathbf{N}^n} {}^e u_{m,\alpha} \prod_{j=1}^n \lambda_j^{m(j)}$ absolutely converges for any $\alpha \in A$ and $0 \le \lambda_j < c$ $(1 \le j \le n)$, and the map $\alpha \mapsto {}^e u_{m,\alpha}$ is real analytic for any $m \in \mathbf{N}^n$. Since $u_{m,\alpha} = \sum_{\mu \in \mathbf{Z}^n} ({}^e u_{2m-\mu,\alpha})^{[\mu,\alpha]}$, the map $\alpha \mapsto u_{m,\alpha}$ is also real analytic.

For each $\mu \in \mathbf{Z}^n$, by

$$\sum_{m \in \mathbf{N}^n} (u_{m,\alpha})^{[\mu,\alpha]} \prod_{j=1}^n \lambda_j^{2m(j)} = \prod_{j=1}^n \lambda_j^{\mu(j)} \sum_{m \in \mathbf{N}^n} ({}^e u_{m,\alpha})^{[\mu,\alpha]} \prod_{j=1}^n \lambda_j^{m(j)},$$

we see that $\sum_{m \in \mathbf{N}^n} (u_{m,\alpha})^{[\mu,\alpha]} \prod_{j=1}^n \lambda_j^{2m(j)}$ absolutely converges when $0 \le \lambda_j < c$ $(1 \le j \le n)$. Hence the function $(\alpha, \lambda) \mapsto u_\alpha(y)$ is real analytic at $\lambda = 0$. This proves (1).

We prove (2). Let ${}^e g_\alpha(y)$ (resp. $\varepsilon_\alpha(y)$) be ${}^e g(y)$ (resp. $\varepsilon(y)$) in Theorem 0.5 (3) (resp. (4)) for $(N_{1,\alpha}, \dots, N_{n,\alpha}, F_\alpha)$. Since ${}^e g_\alpha(y) = {}^e u_\alpha(y) s_\alpha(\bigoplus_w {}^e g_{w,\alpha}(y)) s_\alpha^{-1}$, ${}^e g_\alpha(y)$ is real analytic at $\lambda = 0$ as a function in (α, λ) by Claim 2. Hence by Claim 1, $\varepsilon_\alpha(y) = \varepsilon(W, {}^e g_\alpha(y)^{-1} t_\alpha(y)^{-1} \exp(\sum_{j=1}^n iy_j N_{j,\alpha})F_\alpha)$ is real analytic at $\lambda = 0$ as a function in (α, λ). $\square$

§11. Relationship with work of Pearlstein [P3]

11.1. Assume that $(V, W, (\langle\ ,\ \rangle_w)_w, N, F)$ generates a mixed nilpotent orbit with $n = 1$.

Consider the $\mathbf{R}$-split mixed Hodge structures $\hat{F} = \exp(\zeta)\exp(-i\delta)F$, $\tilde{F} = \exp(-i\delta)F$ with respect to $M(N, W)$, where $\zeta = \zeta(M(N, W), F)$, $\delta = \delta(M(N, W), F)$.

11.2. In [P3], Pearlstein proved an SL(2)-orbit theorem for the cases (I), (II) in 0.7, which contains the following result. Assume that we are either in the case (I) or in the case (II). Then

$$\exp(iyN)F = g_P(y)\exp(iyN)\tilde{F} \quad \text{with} \quad g_P(y) = \exp(\textstyle\sum_{m=0}^\infty g_m y^{-m})$$

when $y \gg 0$, for some $g_m \in \mathfrak{g}_{\mathbf{C}}$ $(m \ge 0)$ such that $g_m : W_w/W_{w-2} \to W_w/W_{w-2}$ is real for any $w \in \mathbf{Z}$ and such that $\sum_{m=0}^\infty g_m T_m$ is a convergent series. This result is the part (a) and the first part of (b) in his Theorem 4.2. Note that the above formula for $\exp(iyN)F$ is also written as

$$\exp(iyN)F = g_P(y)\exp(-\zeta)\exp(iyN)\hat{F}$$

by using $\hat{F}$ instead of $\tilde{F}$, for N and ζ commute.

In this section, we will reprove this result by using our Theorem 0.5. The authors of the present paper do not know whether their method would work well also to reprove the rest (the latter half of (b), and (c)) of his Theorem 4.2.

In what follows, unless explicitly stated, we do not assume (I) nor (II).

Lemma 11.3. $\exp(iN)\hat{F} = \exp(\varepsilon(\infty))\mathbf{r}.$

Here $\mathbf{r}$ *and* $\varepsilon(y) = \exp(\sum_{m\geq 0} \varepsilon_m y^{-m/2})$ *for* $y \gg 0$ *is as in Theorem 0.5.*

Proof. This is a special case of 10.4 (2). □

By Theorem 0.5 (4), Lemma 11.3 and $t(y)Nt(y)^{-1} = yN$ (10.3), we have

$$\begin{aligned}\exp(iyN)F &= g(y)t(y)\exp(\varepsilon(y))\exp(\varepsilon(\infty))^{-1}\exp(iN)\hat{F}\\ &= g(y)t(y)\exp(\varepsilon(y))\exp(\varepsilon(\infty))^{-1}t(y)^{-1}\exp(iyN)\hat{F}.\end{aligned}$$

Hence we have

Proposition 11.4. *Define*

$$g_P(y) := g(y)e(y)\exp(\zeta), \quad \text{with } e(y) := t(y)\exp(\varepsilon(y))\exp(\varepsilon(\infty))^{-1}t(y)^{-1},$$

where $g(y)$ *is as in Theorem 0.5 and* $\zeta = \zeta(M(N,W),F)$*. Then* $g_P(y) : W_w/W_{w-2} \to W_w/W_{w-2}$ *is real for any* $w \in \mathbf{Z}$ *(cf. 1.5), and we have*

$$\exp(iyN)F = g_P(y)\exp(iyN)\tilde{F}.$$

Lemma 11.5. *When* $y \to \infty$*,* $g_P(y)$ *converges if and only if* $e(y)$ *converges.*

Proof. By Theorem 0.5 (2), $u(y)$ converges to 1. By the result in the pure case in one variable introduced in 0.1, $g_w(y)$ converges to 1 for each w. Hence $g(y) = u(y)s(\bigoplus_w g_w(y))s^{-1}$ converges to 1. This implies the lemma. □

Since $\varepsilon(y)$ is a convergent series in $y^{-1/2}$, $e(y)$ and $g_P(y)$ are the exponentials of elements of $\mathbf{C}\{y^{-1/2}\}[y] \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$, where $\mathbf{C}\{T\}$ denotes the ring of convergent series in one variable T.

Lemma 11.6. *Let* $e(y)$ *be as in 11.4. Then* $e(y)$ *and* $g_P(y)$ *are the exponentials of elements of* $\mathbf{C}\{y^{-1}\}[y] \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$*.*

Proof. We have $\mathrm{Ad}(t(y))\varepsilon(y) \in \mathbf{C}\{y^{-1}\}[y] \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$ by 10.6 (2), and $\mathrm{Ad}(t(y))\varepsilon(\infty) \in \mathbf{C}[y^{\pm}] \otimes_{\mathbf{R}} \mathfrak{g}_{\mathbf{R}}$ by 10.6 (1). These prove 11.6. □

By 11.6, to reprove the statements in 11.2 in the cases (I) and (II), it is sufficient to prove that $g_P(y)$ defined in 11.4 converges when $y \to \infty$ in these cases.

11.7. Assume that we are in Case (I).

We deduce the convergence of $g_P(y)$ from Theorem 0.5. In case (I), $\varepsilon(y) = 0$ (1.5) and so $e(y) = 1$. Hence $g_P(y) = g(y)\exp(\zeta)$, and this converges to $\exp(\zeta)$. $\square$

In this case, we have

$$\exp(iyN)F = g(y)\exp(iyN)\hat{F}$$

with $g(y)$ in Theorem 0.5.

11.8. Assume that we are in Case (II).

We deduce the convergence of $g_P(y)$ from Theorem 0.5.

By twist, we assume that gr^W_w are zero if $w \neq 0, -1, -2$, and gr^W_0 (resp. gr^W_{-2}) is of Hodge type $(0, 0)$ (resp. $(-1, -1)$).

Write $\exp(\varepsilon(y))\exp(\varepsilon(\infty))^{-1} = \exp(\sum_{m\geq 1} c_m y^{-m/2})$. Since $W_0 = V$ and $W_{-3} = 0$ and since $c_m W_w \subset W_{w-2}$ for any $w \in \mathbf{Z}$, we have $c_m V \subset W_{-2}$ and $c_m W_{-1} = 0$. By the assumption of the Hodge numbers of gr^W_0 and gr^W_{-2}, we have $t_w(y) = 1$ for $w = 0, -2$. Hence $e(y) = \exp(\sum_{m\geq 1} c_m y^{1-(m/2)})$. By Lemma 11.6, we have $c_m = 0$ for any odd m. Hence $e(y) = \exp(\sum_{m\geq 0} c_{2m+2} y^{-m})$, which converges to $\exp(c_2)$. By 11.5, $g_P(y)$ also converges. $\square$

11.9. In general, the $g_P(y)$ defined in 11.4 need not converge when $y \to \infty$. This is explained in Example 13.4.

In that example, one can even verify that there is no $f(y) \in \mathrm{Aut}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}})$ $(y \gg 0)$ satisfying the following conditions 11.9.1–11.9.3.

11.9.1. $\exp(iyN)F = f(y)\exp(iyN)\tilde{F}$.

11.9.2. $f(y) = \exp(\sum_{m=0}^{\infty} a_m y^{-m})$ with $a_m \in \mathrm{End}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}})$, where the series inside exp is a convergent series.

11.9.3. For each $w \in \mathbf{Z}$, $\mathrm{gr}^W_w(f(y))$ is real and preserves $\langle\ ,\ \rangle_w$.

This is explained in 13.5.

However if we replace 11.9.3 by the following weaker condition 11.9.3′, then for that example, we can find $f(y)$ satisfying 11.9.1, 11.9.2, 11.9.3′. See 13.6.

11.9.3′. For each $w \in \mathbf{Z}$, $\mathrm{gr}^W_w(f(y))$ is real and preserves $\langle\ ,\ \rangle_w$ up to non-zero multiples.

The authors wonder whether there is always $f(y)$ satisfying 11.9.1, 11.9.2, 11.9.3′, or at least 11.9.1, 11.9.2, 11.9.3″, where

11.9.3″. For each $w \in \mathbf{Z}$, $\mathrm{gr}^W_w(f(y))$ is real.

Note that if we do not require 11.9.3″, we always have the expression of $\exp(iyN)F$ as $\exp(iyN)F = \exp(i\delta)\exp(iyN)\tilde{F}$.

11.10. The above question in 11.9 has the following version for several variables. Let $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \cdots, N_n, F)$ be as in Theorem 0.5. We wonder whether there is $h(y) \in \mathrm{Aut}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}})$ which is exp of a convergent power series in y_{j+1}/y_j $(1 \leq j \leq n)$ with coefficients in $\mathfrak{g}_{\mathbf{C}}$ such that, when $y_j > 0$ and $y_{j+1}/y_j \ll 1$, $\exp(\sum_{j=1}^{n} iy_j N_j)F = h(y)\exp(\sum_{j=1}^{n} iy_j N_j^{\Delta})\hat{F}_{(n)}$ and the action of $h(y)$ on gr^W_w is real for any w, where N_j^{Δ}

and $\hat{F}_{(n)}$ are as in 10.1–10.2. Can we furthermore find such $h(y)$ which also preserves $\langle\ ,\ \rangle_w$ up to non-zero multiples for each w? Note that

$$\exp(\textstyle\sum_{j=1}^{n} iy_j N_j^{\Delta})\hat{F}_{(n)} = t(y)\exp(\sum_{j=1}^{n} iN_j^{\Delta})\hat{F}_{(n)} = t(y)\exp(\varepsilon_0)\mathbf{r}.$$

§12. Norm estimate

The aymptotic behavior of the Hodge metric in degeneration of polarized Hodge structure was studied in [Sc], [CKS], [K1]. We generalize the theory to degeneration of mixed Hodge structure with polarized graded quotients. Results on this subject were obtained by Pearlstein in [P3] Theorem 4.7 in the cases I, II in 0.7 with $n = 1$ (his definition of the norm ([P3] 2.6) differs from ours).

12.1. Let $(V, W, (\langle\ ,\ \rangle_w)_{w\in\mathbf{Z}}, F)$ be a mixed Hodge structure with polarized graded quotients. Here V is a finite dimensional $\mathbf{R}$-vector space, W is an increasing filtration on V such that $W_w = V$ for $w \gg 0$ and $W_w = 0$ for $w \ll 0$, $\langle\ ,\ \rangle_w$ for each $w \in \mathbf{Z}$ is a non-degenerate $\mathbf{R}$-bilinear form $\mathrm{gr}_w^W \times \mathrm{gr}_w^W \to \mathbf{R}$ which is $(-1)^w$-symmetric, and F is a decreasing filtration on $V_{\mathbf{C}}$ such that (W, F) is a mixed Hodge structure and that $F(\mathrm{gr}_w^W)$ is polarized by $\langle\ ,\ \rangle_w$ for each w.

We will consider the Hodge metric on $V_{\mathbf{C}}$. For this, we lift the Hodge metrics of the graded quotients to $V_{\mathbf{C}}$ by using the canonical splitting of W in the following way. For $c > 0$, we define a Hermitian form

$$(\ ,\ )_{F,c}\ :\ V_{\mathbf{C}} \times V_{\mathbf{C}} \to \mathbf{C}$$

as follows.

For each $w \in \mathbf{Z}$, let

$$(\ ,\ )_{F(\mathrm{gr}_w^W)} : \mathrm{gr}_{w,\mathbf{C}}^W \times \mathrm{gr}_{w,\mathbf{C}}^W \to \mathbf{C}$$

be the Hodge metric on $\mathrm{gr}_{w,\mathbf{C}}^W$ defined by $\langle\ ,\ \rangle_w$ and $F(\mathrm{gr}_w^W)$ (8.2). For $v \in V_{\mathbf{C}}$ and for $w \in \mathbf{Z}$, let $v_{w,F}$ be the image in $\mathrm{gr}_{w,\mathbf{C}}^W$ of the w-component of v with respect to the canonical splitting of W associated to the mixed Hodge structure (W, F). Define

$$(v, v')_{F,c} = \textstyle\sum_{w\in\mathbf{Z}} c^w (v_{w,F}, v'_{w,F})_{F(\mathrm{gr}_w^W)} \quad (v, v' \in V_{\mathbf{C}}).$$

For $v \in V_{\mathbf{C}}$, define

$$\|v\|_{F,c} = (v, v)_{F,c}^{1/2}.$$

12.2. Let

$$\Delta = \{q \in \mathbf{C} \mid |q| < 1\},\ \ \Delta^* = \Delta \setminus \{0\},\ \ S = \Delta^{n+r},\ \ S^* = (\Delta^*)^n \times \Delta^r.$$

Let $S \to \mathbf{C}$; $q \mapsto q_j$ $(1 \le j \le n + r)$ be the coordinate functions. For $1 \le j \le n$ and for $q \in S$ such that $q_j \ne 0$, define $y_j(q) > 0$ by

$$y_j(q) = -(2\pi)^{-1}\log(|q_j|).$$

Define

$$y_{n+1} = 1$$

(so y_{n+1} does not mean $-(2\pi)^{-1}\log(|q_{n+1}|)$).

Fix non-empty open intervals I_j in $\mathbf{R}$ of lengths < 1 for $1 \le j \le n$, and define an open set U of S^* by

$$U = \{q \in S^* \mid \tfrac{q_j}{|q_j|} \in \exp(2\pi i I_j) \text{ for } 1 \le j \le n\}.$$

Define maps $z_j : U \to \mathbf{C}$ and $x_j : U \to I_j$ $(1 \le j \le n)$ by

$$q_j = \exp(2\pi i z_j(q)), \quad z_j = x_j + iy_j.$$

Let

$$\mathbf{0} = (0, \dots, 0) \in S.$$

For $q \in S$, let

$$J(q) = \{j \mid 1 \le j \le n,\, q_j = 0\}, \quad J'(q) = \{j \mid 1 \le j \le n,\, q_j \ne 0\}.$$

12.3. Fix $(V, W, (\langle\ ,\ \rangle_w)_w)$ as in 0.2, and let D and $D^\vee$ be as in 0.2. Fix nilpotent linear maps $N_j : V \to V$ $(1 \le j \le n)$ such that $N_j N_k = N_k N_j$ for any j, k, and $N_j W_w \subset W_w$ for any j, w.

We consider a holomorphic map

$$\Psi : S \to D^\vee$$

satisfying the following condition 12.3.1.

12.3.1. For any $q \in S$, $\big(V, W, (\langle\ ,\ \rangle_w)_w, (N_j)_{j \in J(q)}, \exp(\sum_{j \in J'(q)} iy_j(q)N_j)\Psi(q)\big)$ generates a mixed nilpotent orbit.

As will be reviewed in 12.10 below, such Ψ appears once we are given an admissible variation of mixed Hodge structure on S^* with polarized graded quotients and with unipotent local monodromy.

For $q \in U$, let $z_j = z_j(q), x_j = x_j(q), y_j = y_j(q)$, and consider

$$F(q) = \exp(\textstyle\sum_{j=1}^n z_j N_j)\Psi(q) \in D.$$

As will be explained in 12.10 below, in the case Ψ comes from an admissible variation of mixed Hodge structure H on S^* with polarized graded quotients and with unipotent local monodromy, $F(q)$ is the fiber at q of the Hodge filtration of H.

In the following Theorem 12.4, for $v \in V_{\mathbf{C}}$, we consider the asymptotic behavior of the norm $\|v\|_{F(q), y_1}$ (12.1) when $q \in U$, $q \to \mathbf{0}$, $y_j/y_{j+1} \to \infty$ for $1 \le j \le n$.

Theorem 12.4. *Let the notation be as above. Let $W^{(j)} = M(N_1 + \cdots + N_j, W)$ for $1 \le j \le n$. Fix $v \in V_{\mathbf{C}}$ and let $\mu \in \mathbf{Z}^n$.*

(1) Assume $v \in \bigcap_{j=1}^n W^{(j)}_{\mu(j),\mathbf{C}}$. Then there are constants $c, C > 0$ such that

$$\|v\|_{F(q), y_1} \le C \cdot \textstyle\prod_{j=1}^n (\frac{y_j}{y_{j+1}})^{\mu(j)/2}$$

for any $q \in U$ satisfying $y_j/y_{j+1} > c$ for $1 \le j \le n$. (Here $y_j = y_j(q)$ for $1 \le j \le n$ and $y_{n+1} = 1$.)

(2) Fix $a_j \in I_j$ $(1 \le j \le n)$. Define $\mathbf{r} \in D$ and the decomposition $V = \bigoplus_{\mu \in \mathbf{Z}^n} V^{[\mu]}$ with respect to

$$(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, \exp(\textstyle\sum_{j=1}^n a_j N_j)\Psi(\mathbf{0})),$$

which generates a mixed nilpotent orbit, as in 0.5 and 10.2, respectively.

Assume $v \in \bigcap_{j=1}^n W^{(j)}_{\mu(j),\mathbf{C}}$. Then, when $q \in U$ converges to $\mathbf{0}$ satisfying $y_j/y_{j+1} \to \infty$ and $x_j \to a_j$ for $1 \le j \le n$, we have the convergence

$$(\textstyle\prod_{j=1}^n (\frac{y_j}{y_{j+1}})^{\mu(j)/2})^{-1} \|v\|_{F(q),y_1} \to \|v^{[\mu]}\|_{\mathbf{r},1}.$$

This limit is zero if and only if $v^{[\mu]} = 0$, that is, if and only if v belongs to the sum $\sum_{\mu' < \mu} \bigcap_{j=1}^n W^{(j)}_{\mu'(j),\mathbf{C}}$.

(3) Assume $v \notin \bigcap_{j=1}^n W^{(j)}_{\mu(j),\mathbf{C}}$. Let a_j $(1 \le j \le n)$ be as in (2). Then it can happen that $q \in U$ converges to $\mathbf{0}$ satisfying $y_j/y_{j+1} \to \infty$ and $x_j \to a_j$ for $1 \le j \le n$ but $(\prod_{j=1}^n (\frac{y_j}{y_{j+1}})^{\mu(j)/2})^{-1} \|v\|_{F(q),y_1}$ tends to ∞.

12.5. For the proof of Theorem 12.4, it is enough to prove (2) and (3) assuming $a_j = 0$ for $1 \le j \le n$. In fact, if (2) and (3) are proved in the case $a_j = 0$ for $1 \le j \le n$, it implies that (2) and (3) are true in general. Assume $v \in \bigcap_{j=1}^n W^{(j)}_{\mu(j),\mathbf{C}}$. Then (2) implies that for each $a = (a_j)_j \in \mathbf{R}^n$, we find open intervals $I_{a,j}$ in $\mathbf{R}$ of lengths < 1 and $c_a, C_a > 0$ such that $a_j \in I_{a,j}$ $(1 \le j \le n)$ and such that the estimate in 12.4 (1) holds when we take $((I_{a,j})_j, c_a, C_a)$ as $((I_j)_j, c, C)$ in (1). Since the closure of $\prod_{j=1}^n I_j$ in $\mathbf{R}^n$ is compact, it is contained in $\bigcup_{a \in A} \prod_{j=1}^n I_{a,j}$ for some finite set A of $\mathbf{R}^n$. 12.4 (1) then holds for $c = \max_{a \in A} c_a$, $C = \max_{a \in A} C_a$.

For the proofs of Theorem 12.4 (2) and (3) assuming $a_j = 0$ for $1 \le j \le n$, the key ideas are, roughly speaking, as in the following (1)–(3). The method is similar to that in the pure case in [Sc], [CKS], [K1].

(1) By Theorem 0.5, $t(y)^{-1} \exp(\sum_{j=1}^n iy_j N_j)\Psi(\mathbf{0})$ converges when $y_j/y_{j+1} \to \infty$ $(1 \le j \le n)$.

(2) $t(y)^{-1}F(q)$ converges (Corollary 12.8 below). This is deduced from (1), as follows. Since $\Psi(q)$ is very near to $\Psi(\mathbf{0})$ (12.6 below), $t(y)^{-1}F(q) = t(y)^{-1} \exp(\sum_{j=1}^n z_j N_j)\Psi(q)$ is near to $t(y)^{-1} \exp(\sum_{j=1}^n iy_j N_j)\Psi(\mathbf{0})$, which converges by (1).

(3) As in Theorem 0.5, $t(y)$ is related to the canonical splitting of the weight filtration which was used in 12.1 for the construction of the Hodge metric. By this, the convergence of $t(y)^{-1}F(q)$ in (2) shows that the Hodge metric of $F(q)$ twisted by $t(y)$ converges. Since the action of $t(y)$ on V is understood well, this gives the estimate of the Hodge metric of $F(q)$ (this is done in 12.9 below).

12.6. For $q \in S$ and for $0 \le j \le n$, let $q^{(j)}$ be the point of S obtained from q by replacing the first j coordinates of q by 0. Hence $q^{(0)} = q$. When $q \to \mathbf{0}$, for $0 \le j \le n$,

$\Psi(q^{(j)})$ converges to $\Psi(\mathbf{0})$. Furthermore if $q \to \mathbf{0}$ and $y_j/y_{j+1} \to \infty$ for $1 \le j \le n$, since the map $q \mapsto \Psi(q); S \to D^\vee$ is holomorphic, we have

$$d(\Psi(q^{(j)}), \Psi(q^{(k)})) \le O(|q_k|) \quad \text{if} \quad 0 \le j < k \le n.$$

Here d is a metric on a neighborhood of $\Psi(\mathbf{0})$ in $D^\vee$ which is compatible with the analytic structure. (This means that on a neighborhood of $\Psi(\mathbf{0})$ in $D^\vee$, for a metric d' defined by using complex analytic coordinates, $\log(d/d')$ is bounded below and bounded above.)

Lemma 12.7. *Fix j such that $1 \le j \le n$. When $q \to \mathbf{0}$, $\mathrm{Re}(q_k) > 0$ and $x_k \to 0$ for $1 \le k \le n$, and $y_k/y_{k+1} \to \infty$ for $j \le k \le n$ ($y_{n+1} = 1$), then we have the convergence*

$$A_j(q)\Psi(q^{(j-1)}) \to \exp(\textstyle\sum_{k=j}^n iN_{k,j})\hat{\Psi}(\mathbf{0})$$
$$\textit{with } A_j(q) = \textstyle\prod_{k=j}^n t^{(k)}(y_k/y_{k+1})^{-1} \cdot \exp(\sum_{k=j}^n iy_kN_k)\exp(\sum_{k=1}^n x_kN_k),$$

where $(W^{(n)}, \hat{\Psi}(\mathbf{0}))$ is the $\mathbf{R}$-split mixed Hodge structure associated to $(W^{(n)}, \Psi(\mathbf{0}))$, and $N_{k,j}$ is the sum of $N_k^{[\mu]}$ for $\mu \in \mathbf{Z}^n$ such that $\mu(l) = 0$ for $j \le l < k$. Here, $W^{(n)}$, $t^{(k)}$, $N_k^{[\mu]}$ are those in 10.1–10.3 with respect to $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, \Psi(\mathbf{0}))$ which generates a mixed nilpotent orbit.

Proof. We use downward induction on j. If $j < n$, by induction, we may assume

$(1)_j$ $\quad A_{j+1}(q)\Psi(q^{(j)}) \to \exp(\sum_{k=j+1}^n iN_{k,j+1})\hat{\Psi}(\mathbf{0})$.

In the case $j = n$, consider the fact

$(1)_n$ $\quad A_{n+1}(q)\Psi(q^{(n)}) \to \Psi(\mathbf{0})$, where $A_{n+1}(q) = \exp(\sum_{k=1}^n x_kN_k)$.

Since $(1)_j$ is a convergence of mixed Hodge structures for the weight filtration $W^{(j)}$, $t^{(j)}(y_j/y_{j+1})^{-1} \cdot$(l.h.s. of $(1)_j$) converges to the limit of $t^{(j)}(y_j/y_{j+1})^{-1} \cdot$(r.h.s. of $(1)_j$) which is $\exp(\sum_{k=j+1}^n iN_{k,j})\hat{\Psi}(\mathbf{0})$.

We apply $\exp(iN_j)$ to this. Then since $\exp(iN_j)t^{(j)}(y_j/y_{j+1})^{-1}A_{j+1}(q) = A_j(q)$ by 10.3, we have $A_j(q)\Psi(q^{(j)}) \to \exp(\sum_{k=j}^n iN_{k,j})\hat{\Psi}(\mathbf{0})$.

Finally we can replace $\Psi(q^{(j)})$ in the last convergence by $\Psi(q^{(j-1)})$. In fact by 12.6, we have $\Psi(q^{(j-1)}) = f(q)\Psi(q^{(j)})$ with $f(q) \in G_{\mathbf{C}}$, $d(f(q), 1) = O(|q_j|)$. Here d is a metric on a neighborhood of 1 in $G_{\mathbf{C}}$ which is compatible with the analytic structure. Hence $A_j(q)\Psi(q^{(j-1)}) = A_j(q)f(q)\Psi(q^{(j)}) = f'(q)A_j(q)\Psi(q^{(j)})$, where $f'(q) = A_j(q)f(q)A_j(q)^{-1}$. By the shape of $A_j(q)$, the norm of the operator $\mathrm{Ad}(A_j(q))$ on $\mathfrak{g}_{\mathbf{C}}$ is $O(|y_j|^l)$ for some $l > 0$, and hence by $d(f(q), 1) = O(|q_j|)$, we have $f'(q) \to 1$. Thus we obtain $A_j(q)\Psi(q^{(j-1)}) \to \exp(\sum_{k=j}^n iN_{k,j})\hat{\Psi}(\mathbf{0})$. □

Corollary 12.8. $t(y)^{-1}F(q) \to \exp(\varepsilon_0)\mathbf{r}$. *Here ε_0 is as in Theorem 0.5 (4) for $(V, W, (\langle\ ,\ \rangle_w)_w, N_1, \dots, N_n, \Psi(\mathbf{0}))$.*

Proof. The case $j = 1$ of Lemma 12.7 shows

$$t(y)^{-1}F(q) = t(y)^{-1}\exp(\textstyle\sum_{j=1}^n z_jN_j)\Psi(q) \to \exp(\sum_{j=1}^n iN_j^{\Delta})\hat{\Psi}(\mathbf{0}) = \exp(\varepsilon_0)\mathbf{r},$$

where the last equality follows from 10.4 (2). □

12.9. Proofs of Theorem 12.4 (2) and (3) assuming $a_j = 0$ for $1 \le j \le n$. Let $F(q)' = t(y)^{-1}F(q)$. This converges to $\exp(\varepsilon_0)\mathbf{r}$ (12.8). Let $u_q = s_{F(q)}s^{-1} \in G_{\mathbf{R}}$, where s is the splitting of W in Theorem 0.5 (1) (that is, s is the canonical splitting of W associated to the mixed Hodge structure $(W, \mathbf{r})$) and $s_{F(q)}$ is the canonical splitting of W associated to $(W, F(q))$. Let ${}^e u_q = t(y)^{-1}u_q t(y) = s_{F(q)'}s^{-1}$. By 12.8, $s_{F(q)'}$ converges to $s_{\exp(\varepsilon_0)\mathbf{r}} = s_{\mathbf{r}} = s$. Hence ${}^e u_q$ converges to 1. We have

$$\begin{aligned}
\|v\|^2_{F(q),y_1} &= \textstyle\sum_{w\in\mathbf{Z}} y_1^w \|v_{w,F(q)}\|^2_{F(q)(\mathrm{gr}^W_w)} \\
&= \textstyle\sum_{w\in\mathbf{Z}} y_1^w \|(u_q^{-1}v)_{w,\mathbf{r}}\|^2_{F(q)(\mathrm{gr}^W_w)} \\
&= \textstyle\sum_{w\in\mathbf{Z}} y_1^w \|t_w(y)^{-1}(u_q^{-1}v)_{w,\mathbf{r}}\|^2_{F(q)'(\mathrm{gr}^W_w)} \\
&= \textstyle\sum_{w\in\mathbf{Z}} \|(t(y)^{-1}u_q^{-1}v)_{w,\mathbf{r}}\|^2_{F(q)'(\mathrm{gr}^W_w)} \\
&= \textstyle\sum_{w\in\mathbf{Z}} \|({}^e u_q{}^{-1}t(y)^{-1}v)_{w,\mathbf{r}}\|^2_{F(q)'(\mathrm{gr}^W_w)} \\
&= \textstyle\sum_{w\in\mathbf{Z}} \|\sum_{\nu\in\mathbf{Z}^n} {}^e u_q^{-1} \prod_{j=1}^n (\frac{y_j}{y_{j+1}})^{\nu(j)/2} v^{[\nu]}_{w,\mathbf{r}}\|^2_{F(q)'(\mathrm{gr}^W_w)}.
\end{aligned}$$

Assume $v \in \bigcap_{j=1}^n W^{(j)}_{\mu(j),\mathbf{C}}$. Then $v^{[\nu]} = 0$ unless $\nu \le \mu$. Furthermore by 12.8, $F(q)'(\mathrm{gr}^W_w)$ converges to $\exp(\varepsilon_0)\mathbf{r}(\mathrm{gr}^W_w) = \mathbf{r}(\mathrm{gr}^W_w)$ for each w. Hence by the last expression of $\|v\|^2_{F(q),y_1}$, we have 12.4 (2).

Next we prove 12.4 (3). Assume $v \notin \bigcap_{j=1}^n W^{(j)}_{\mu(j),\mathbf{C}}$. Then there are a family of integers $a(j) > 0$ $(1 \le j \le n)$ and $\mu' \in \mathbf{Z}^n$ such that $\mu' \ne \mu$, $v^{[\mu']} \ne 0$ and such that $\sum_{j=1}^n a(j)\mu'(j) > \sum_{j=1}^n a(j)\nu(j)$ for any $\nu \in \mathbf{Z}^n$ satisfying $v^{[\nu]} \ne 0$, $\nu \ne \mu'$. When $q \in S^*$ converges to $\mathbf{0}$ satisfying $y_j/y_{j+1} = \lambda^{a(j)}$ with $\lambda \to \infty$ $(1 \le j \le n$, y_{n+1} denotes 1), then $\lambda^{-\sum_{j=1}^n a(j)\mu'(j)}\|v\|^2_{F(q),y_1}$ converges to the non-zero real number $\|v^{[\mu']}\|^2_{\mathbf{r},1}$. Hence

$$\textstyle\prod_{j=1}^n (\frac{y_j}{y_{j+1}})^{-\mu(j)}\|v\|^2_{F(q),y_1} = \lambda^{-\sum_{j=1}^n a(j)\mu(j)}\|v\|^2_{F(q),y_1} \to \infty. \quad \square$$

12.10. We review the fact that Ψ as in 12.3 appears when an admissible variation of mixed Hodge structure with polarized graded quotients and with unipotent local monodromy on S^* is given, and review how the latter object appears in geometry.

For a complex analytic manifold M, a variation of mixed Hodge structure with polarized graded quotients (VPMH for short) on M is a 4-ple $H = (H_{\mathbf{R}}, W, (\langle\ ,\ \rangle_w)_w, F)$, where

- $H_{\mathbf{R}}$ is a locally constant sheaf of finite dimensional $\mathbf{R}$-vector spaces on M,
- W is an increasing filtration on $H_{\mathbf{R}}$ by locally constant $\mathbf{R}$-submodules,
- $\langle\ ,\ \rangle_w$ for each $w \in \mathbf{Z}$ is a non-degenerate $(-1)^w$-symmetric pairing $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$,
- F is a decreasing filtration on $H_{\mathcal{O}} := \mathcal{O}_M \otimes_{\mathbf{C}} H_{\mathbf{C}}$ $(H_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} H_{\mathbf{R}})$ by $\mathcal{O}_M$-submodules such that the $\mathcal{O}_M$-modules F^p and $H_{\mathcal{O}}/F^p$ are locally free of finite rank for all $p \in \mathbf{Z}$,

satisfying the following conditions 12.10.1 and 12.10.2.

12.10.1. For any $q \in M$, $(H_{\mathbf{R},q}, W_q, (\langle\ ,\ \rangle_w)_w, F(q))$ is a mixed Hodge structure with polarized graded quotients. Here $F(q) = \mathbf{C} \otimes_{\mathcal{O}_{M,q}} F_q$ with $\mathcal{O}_{M,q} \to \mathbf{C}$ the evaluation $f \mapsto f(q)$ at q.

12.10.2. The connection $\nabla : H_{\mathcal{O}} \to \Omega^1_M \otimes_{\mathcal{O}_M} H_{\mathcal{O}}$; $f \otimes v \mapsto df \otimes v$ ($f \in \mathcal{O}_M$, $v \in H_{\mathbf{C}}$) satisfies $\nabla(F^p) \subset \Omega^1_M \otimes_{\mathcal{O}_M} F^{p-1}$ for any $p \in \mathbf{Z}$. Here Ω^1_M is the sheaf of holomorphic differential 1-forms.

Assume that a VPMH H on S^* is given.

Fix non-empty open intervals I_j of lengths < 1 in $\mathbf{R}$ for $1 \leq j \leq n$ and define the open set U of S^* to be as in 12.2. The restriction of $H_{\mathbf{R}}$ to U is a constant sheaf associated to a finite dimensional $\mathbf{R}$-vector space V, and the restrictions of W and $(\langle\ ,\ \rangle_w)_w$ to U are also constant. This gives $(V, W, (\langle\ ,\ \rangle_w)_w)$ as in 0.2. For $q \in U$, $F(q)$ is regarded as a filtration on $V_{\mathbf{C}}$, and we have a holomorphic map

$$F : U \to D\ :\ q \mapsto F(q).$$

For $q \in S^*$, if a is a continuous map from the closed interval $[0,1]$ to S^* such that $a(0) \in U$ and $a(1) = q$, then a induces an isomorphism $\beta_a : H_{\mathbf{R},q} \xrightarrow{\sim} V$.

For $1 \leq j \leq n$, let $\gamma_j : V \to V$ be the j-th monodromy defined by $\gamma_j = \beta_a$, where a is a continuous map $[0,1] \to S^*$ satisfying $a(0) = a(1) \in U$, $a(t)_k = a(0)_k$ for any $t \in [0,1]$ and for $1 \leq k \leq n+r$ such that $k \neq j$, and $a(t)_j = a(0)_j \exp(2\pi i t)$ for any $t \in [0,1]$ (β_a is independent of the choice of such a). Then the γ_j commute each other. Let $\Gamma \subset \mathrm{Aut}(V)$ be the group generated by γ_j $(1 \leq j \leq n)$. Then Γ acts on D in the natural way. The projection $D \to \Gamma\backslash D$ is a local homeomorphism and hence $\Gamma\backslash D$ has a structure of a complex manifold.

We have a holomorphic map

$$\Phi : S^* \to \Gamma\backslash D$$

defined by $\Phi(q) = (\beta_a(F(q)) \bmod \Gamma)$, where a is a continuous map $[0,1] \to S^*$ satisfying $a(0) \in U$ and $a(1) = q$ (then $(\beta_a(F(q)) \bmod \Gamma)$ is independent of the choice of a). We have

$$\Phi(q) = (F(q) \bmod \Gamma) \quad \text{for } q \in U.$$

Let

$$\tau : \mathfrak{h}^n \times \Delta^r \to S^*$$

be the surjective holomorphic map defined by $\tau(p)_j = \exp(2\pi i p_j)$ for $1 \leq j \leq n$ and $\tau(p)_j = p_j$ for $n+1 \leq j \leq n+r$. Define a holomorphic map

$$\tilde{\Phi} : \mathfrak{h}^n \times \Delta^r \to D$$

by $\tilde{\Phi}(p) = \beta_a(F(\tau(p)))$, where a is a continuous map $[0,1] \to S^*$ which satisfies $a(t)_j = p_j$ for any $t \in [0,1]$ and $n+1 \leq j \leq n+r$ and satisfies the following condition. If $1 \leq j \leq n$, then there is $c_j \in \mathbf{C}$ such that $\mathrm{Re}(c_j) \in I_j$, $a(0)_j = \exp(2\pi i c_j)$, and $a(t)_j = a(0)_j \exp(2\pi i t(p_j - c_j))$ for any $t \in [0,1]$. Such a satisfies $a(0) \in U$ and $a(1) = \tau(p)$, and $\beta_a(F(\tau(p))) \in D$ is independent of the choice of such a. We have

$$\Phi(\tau(p)) = (\tilde{\Phi}(p) \bmod \Gamma) \ \text{ for any } p \in \mathfrak{h}^n \times \Delta^r.$$

Now assume :

12.10.3. The monodromy $\gamma_j : V \to V$ is unipotent for every $1 \le j \le n$.

Let $N_j = \log(\gamma_j) : V \to V$ for $1 \le j \le n$. Then we have

$$\tilde{\Phi}(p + 1_j) = \gamma_j \tilde{\Phi}(p) = \exp(N_j)\tilde{\Phi}(p)$$

for any $p \in \mathfrak{h}^n \times \Delta^r$ and $1 \le j \le n$, where $p + 1_j$ is the element of $\mathfrak{h}^n \times \Delta^r$ defined by $(p + 1_j)_k = p_k$ for any $1 \le k \le n + r$ such that $k \ne j$, and $(p + 1_j)_j = p_j + 1$. Hence there is a unique holomorphic map

$$\Psi : S^* \to D^\vee$$

satisfying $\Psi(\tau(p)) = \exp(-\sum_{j=1}^n p_j N_j)\tilde{\Phi}(p)$ for any $p \in \mathfrak{h}^n \times \Delta^r$. We have

$$F(q) = \exp(\textstyle\sum_{1 \le j \le n} z_j N_j)\Psi(q) \ \text{ for } q \in U.$$

For these facts, see, e.g., [Sc] §4.

The VPMH H is said to be admissible if the following 12.10.4 and 12.10.5 are satisfied. (See Steenbrink-Zucker [SZ], Kashiwara [K2] for the admissibility.)

12.10.4. Ψ extends to a holomorphic map $S \to D^\vee$.

12.10.5. $(V, W, N_1, \dots, N_n)$ satisfies the condition (iv) in 0.2.

If these conditions are satisfied, then the condition 12.3.1 is satisfied. In fact, for $(V, W, (\langle\ ,\ \rangle_w)_w, (N_j)_{j \in J(q)}, \exp(\sum_{j \in J'(q)} i y_j N_j)\Psi(q))$ for each $q \in S$, the condition (i) in 0.2 is clearly satisfied, (iii) follows from the Griffiths transversality 12.10.2, (iv) follows from 12.10.5, and furthermore, (ii) is satisfied by the nilpotent orbit theorem of Schmid [Sc] applied to the graded quotients $F(\mathrm{gr}^W_w)$.

We describe how an admissible VPMH on S^* with unipotent local monodromy arises from geometry.

Let X be a complex analytic space with a projective morphism $f : X \to S$. Let X^* be the inverse image of S^*. Assume that a divisor E on X is given and assume the following (i) and (ii).

(i) The restriction $g : X^* \to S^*$ of f is smooth.

(ii) The restriction of E to X^* is a divisor with normal crossings and any intersection of any family of irreducible components of E is smooth over S^*.

Then by section 5 of Steenbrink-Zucker [SZ] (the case $n = 1, r = 0$) and by Kashiwara [K2] and Saito [Sa] (cf. [Fn] 3.1.2), after pulling back by

$$S \to S\ ;\ (q_1, \dots, q_{n+r}) \mapsto (q_1^e, \dots, q_n^e, q_{n+1}, \dots, q_{n+r})$$

for some $e \ge 1$, we have an admissible VPMH $H = (H_{\mathbf{R}}, W, (\langle\ ,\ \rangle_w)_{w \in \mathbf{Z}}, F)$ with unipotent local monodromy on S^* for each $m \in \mathbf{Z}$ as follows. Let $h : X^* \setminus E \to S^*$ be the restriction of g. Then, $H_{\mathbf{R}} = R^m h_*(\mathbf{R})$, W is the weight filtration defined by the method of Deligne [D1], and the $\langle\ ,\ \rangle_w$ are defined by using a polarization of X. Let $H_{\mathcal{O}} = \mathcal{O}_{S^*} \otimes_{\mathbf{C}} H_{\mathbf{C}}$. Then we have a canonical isomorphism $H_{\mathcal{O}} \simeq R^m g_*(\Omega^{\bullet}_{X^*/S^*}(\log E))$, where $\Omega^{\bullet}_{X^*/S^*}(\log E)$ is the de Rham complex of X^* over S^* with logarithmic poles along E. For $p \in \mathbf{Z}$, the p-th Hodge filtration F^p on $H_{\mathcal{O}}$ is the image of the injective homomorphism

$$R^m g_*(\Omega^{\bullet \geq p}_{X^*/S^*}(\log E)) \to R^m g_*(\Omega^{\bullet}_{X^*/S^*}(\log E)) \simeq H_{\mathcal{O}}.$$

As is said above, this VPMH is indeed admissible, in particular, satisfies the condition 12.10.4. This is proved in [Fn] 3.1.2. For readers' convenience, we write here a sketch of a slightly different proof.

Let P be the product of ($D^{\vee}$ of gr^W_w) for all $w \in \mathbf{Z}$. Then by the nilpotent orbit theorem of Schmid [Sc], after the base change $q_j \mapsto q_j^e$ $(1 \leq j \leq n)$ for some $e \geq 1$, we have:

(1) The composition $S^* \xrightarrow{\Psi} D^{\vee} \to P$ extends to a holomorphic map $S \to P$.

On the other hand, in the case $n = 1$ and $r = 0$, by [SZ] and by semi-stable reduction theorem, $\Psi : S^* \to D^{\vee}$ extends to $S \to D^{\vee}$ after the base change $q_j \mapsto q_j^e$ $(1 \leq j \leq n)$ for some $e \geq 1$. The method in [SZ] in fact proves the following (2) after the base change $q_j \mapsto q_j^e$ $(1 \leq j \leq n)$ for some $e \geq 1$.

(2) There is a closed analytic subspace Y of S of codimension ≥ 2 such that $S^* \cap Y$ is empty and such that $\Psi : S^* \to D^{\vee}$ extends to a holomorphic map $S - Y \to D^{\vee}$.

We have also ([U]):

(3) Locally on P, $D^{\vee} \to P$ has a structure of a vector bundle on P.

For a vector bundle on a complex analytic manifold, a section defined on the complement of a closed analytic subspace of codimension ≥ 2 in the base space extends to a section on the whole base space. Hence by (1)–(3), Ψ extends to a holomorphic map $S \to D^{\vee}$.

§13. Examples

Example 13.1. Let V be a 3 dimensional $\mathbf{R}$-vector space with basis (e_1, e_2, e_3), let W be the increasing filtration on V defined by $W_0 = V$, $W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2$, and $W_{-2} = 0$. Hence $\mathrm{gr}^W_w = 0$ unless $w = 0$ or -1. Define $\langle\ ,\ \rangle_w$ by $\langle e_3, e_3\rangle_0 = 1$, $\langle e_2, e_1\rangle_{-1} = 1$. Take $c_j \in \mathbf{R}$ $(j = 1, 2)$, and let N_j $(j = 1, 2)$ be the elements of $\mathfrak{g}_{\mathbf{R}}$ defined by

$$N_j(e_1) = 0,\ N_j(e_2) = e_1,\ N_j(e_3) = c_j e_1.$$

Let F be the decreasing filtration on $V_{\mathbf{C}}$ defined by $F^{-1} = V_{\mathbf{C}}$, $F^0 = \mathbf{C}e_2 + \mathbf{C}e_3$, $F^1 = 0$. Then (N_1, N_2, F) generates a mixed nilpotent orbit (0.2). If $y_1 + y_2 > 0$, $\exp(iy_1N_1 + iy_2N_2)F \in D$ and the Hodge type of $\exp(iy_1N_1 + iy_2N_2)F(\mathrm{gr}^W_w)$ is $(0, 0)$ for $w = 0$, and $(0, -1) + (-1, 0)$ for $w = -1$. In the notation in Theorem 0.5, we have:

(1) $s(y)(e_3 \bmod W_{-1}) = e_3 - (c_1y_1 + c_2y_2)(y_1 + y_2)^{-1}e_2$, $s(y)$ converges to s which is defined by $s(e_3 \bmod W_{-1}) = e_3 - c_1e_2$.

(2) $s(y) = u(y)s$ with $u(y)(e_3) = e_3 + (c_2 - c_1)(\sum_{m=1}^{\infty}(-y_2/y_1)^m)e_2$ when $y_2/y_1 < 1$.

(3) $t(y)e_1 = y_1e_1,\ t(y)e_j = e_j$ for $j = 2, 3$,
$\mathbf{r}^{-1} = V_{\mathbf{C}},\ \mathbf{r}^0 = \mathbf{C}(ie_1 + e_2) + \mathbf{C}(e_3 - c_1e_2),\ \mathbf{r}^1 = 0.$
As is noted in 0.4, $(g_w(y))_w$ is not unique. One choice is $g_w(y) = 1$ for any w, and for this choice, $g(y) = {}^eg(y) = u(y)$.

(4) $\varepsilon(y) = 0$ for the above choice of $(g_w(y))_w$.

These (1)–(4) are obtained as follows. Since

$$\begin{aligned}\exp(iy_1N_1 + iy_2N_2)F^0 &= \mathbf{C}(e_2 + i(y_1 + y_2)e_1) + \mathbf{C}(e_3 + i(c_1y_1 + c_2y_2)e_1)\\ &= \mathbf{C}(e_2 + i(y_1 + y_2)e_1) + \mathbf{C}(e_3 - (c_1y_1 + c_2y_2)(y_1 + y_2)^{-1}e_2),\end{aligned}$$

we have (1) by 1.5. We have (4) also by 1.5. (2) and (3) follow from (1).

We consider the norm estimate. In 12.3, take $n = 2, r = 0$, and take $\Psi : S = \Delta^2 \to D^{\vee}$ to be the constant function with value F. Then the condition 12.3.1 is satisfied. Let $v = e_3$ in Theorem 12.4. Then Theorem 12.4 says that when $q \in S^* = (\Delta^*)^2$ converges to $(0, 0) \in \Delta^2$ satisfying $y_1/y_2 \to \infty$ and $x_1, x_2 \to 0$ (recall that $q = (q_1, q_2)$ with $q_j = \exp(2\pi i(x_j + iy_j))$ with x_j, y_j real), $\|e_3\|_{F(q),y_1}$ converges. Here we show

$$\|e_3\|_{F(q),y_1} \to (1 + c_1^2)^{1/2}$$

directly. For simplicity, assume that q_1, q_2 are positive real numbers (that is, $q = (\exp(-2\pi y_1), \exp(-2\pi y_2))$). For the canonical splitting of W associated to $(W, F(q))$, the 0-component of e_3 is $e_3 - (c_1y_1 + c_2y_2)(y_1 + y_2)^{-1}e_2$ and the (-1)-component of e_3 is $(c_1y_1 + c_2y_2)(y_1 + y_2)^{-1}e_2$. Hence

$$(e_3, e_3)_{F(q),y_1} = 1 + y_1^{-1}((c_1y_1 + c_2y_2)(y_1 + y_2)^{-1}e_2, (c_1y_1 + c_2y_2)(y_1 + y_2)^{-1}e_2)_{F(q)(\mathrm{gr}^W_{-1})}.$$

We have $(e_2, e_2)_{F(q)(\mathrm{gr}^W_{-1})} = y_1 + y_2$. Hence $(e_3, e_3)_{F(q),y_1} = 1 + y_1^{-1}((c_1y_1 + c_2y_2)(y_1 + y_2)^{-1})^2(y_1 + y_2) \to 1 + c_1^2$.

This example appears in geometry in the following way. If $y_1 + y_2 > 0$, $\exp(iy_1N_1 + iy_2N_2)F(\mathrm{gr}^W_{-1})$ is isomorphic to the Hodge structure $H^1(E_y, \mathbf{R})(1)$, where E_y is the elliptic curve $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}(iy_1 + iy_2))$. If p_y denotes the point $ic_1y_1 + ic_2y_2 \bmod \mathbf{Z} + \mathbf{Z}(iy_1 + iy_2)$ of E_y and if $p_y \neq 0$, $\exp(iy_1N_1 + iy_2N_2)F$ is isomorphic to the mixed Hodge structure $H^1(E_y \setminus \{p_y, 0\}, \mathbf{R})(1)$.

Example 13.2. Let V be a 2 dimensional $\mathbf{R}$-vector space with basis (e_1, e_2), let W be the increasing filtration on V defined by $W_0 = V$, $W_{-1} = W_{-2} = \mathbf{R}e_1$, and $W_{-3} = 0$. Hence $\mathrm{gr}^W_w = 0$ unless $w = 0$ or -2. Define $\langle\ ,\ \rangle_w$ by $\langle e_2, e_2\rangle_0 = 1$, $\langle e_1, e_1\rangle_{-2} = 1$. Let N be the element of $\mathfrak{g}_{\mathbf{R}}$ defined by

$$N(e_1) = 0, N(e_2) = e_1.$$

Let $\alpha \in \mathbf{C}$, and let F be the decreasing filtration on $V_{\mathbf{C}}$ defined by $F^{-1} = V_{\mathbf{C}}$, $F^0 = \mathbf{C}(\alpha e_1 + e_2)$, $F^1 = 0$. Then (N, F) generates a mixed nilpotent orbit, and $\exp(iyN)F \in$

D for any $y \in \mathbf{R}$. The Hodge type of $\exp(iyN)F(\mathrm{gr}^W_w)$ is $(0,0)$ for $w=0$, and $(-1,-1)$ for $w=-2$. We have:

(1) $s(y)(e_2 \bmod W_{-1}) = \mathrm{Re}(\alpha)e_1 + e_2$, hence $s(y) = s$ is constant.

(2) $u(y) = 1$.

(3) $t(y)e_1 = ye_1$, $t(y)(\mathrm{Re}(\alpha)e_1 + e_2) = \mathrm{Re}(\alpha)e_1 + e_2$.
$\mathbf{r}^{-1} = V_{\mathbf{C}}$, $\mathbf{r}^0 = \mathbf{C}(\mathrm{Re}(\alpha)e_1 + e_2)$, $\mathbf{r}^1 = 0$.
$g_w(y) = 1$ for any w, $g(y) = {}^e g(y) = 1$.

(4) $\varepsilon(y) = i(1 + \mathrm{Im}(\alpha)y^{-1})N$.

(5) Let $g_P(y)$ be as in 11.4. Then $g_P(y) = \exp(i\,\mathrm{Im}(\alpha)N)$.

These are obtained as follows. We have $\exp(iyN)F^0 = \mathbf{C}((iy+\alpha)e_1 + e_2)$, and by the definition of δ, we see $\delta(W, \exp(iyN)F) = (y + \mathrm{Im}(\alpha))N$ and $\delta(W, \exp(iyN)F)$ has only the $(-1,-1)$-Hodge component. Since $\zeta_{-1,-1} = 0$ (see Appendix), we have $\zeta(W, \exp(iyN)F) = 0$. Hence we have (1). (2) and (3) follow from (1). We have $t(y)^{-1}g(y)^{-1}\exp(iyN)F^0 = \mathbf{C}((\mathrm{Re}(\alpha) + i(1+\mathrm{Im}(\alpha)y^{-1}))e_1 + e_2)$. Hence $\delta(W, t(y)^{-1}g(y)^{-1}\exp(iyN)F)$ is $(1+\mathrm{Im}(\alpha)y^{-1})N$, and $\zeta(W, t(y)^{-1}g(y)^{-1}\exp(iyN)F)$ is 0 by $\zeta_{-1,-1} = 0$. This shows (4).

We have $M(N,W) = W$, $\delta(M(N,W), F) = \mathrm{Im}(\alpha)N$, and $\delta(M(N,W),F)$ has only $(-1,-1)$-Hodge component. Hence $\zeta(M(N,W),F) = 0$. By this, we obtain (5) from (3) and (4).

In this example, as is mentioned at the end of 10.1, there is no $f(y) \in \mathrm{Aut}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}})$ such that

$$\exp(iyN)F = f(y)\varphi(iy), \quad f(y) \to 1 \quad (y \to \infty).$$

Here $\varphi(iy)$ is as in 10.1. In fact, $\varphi(iy) = t(y)\mathbf{r} = \mathbf{r}$, and hence $f(y)$ should send $\mathrm{Re}(\alpha)e_1 + e_2$ to $(iy+\alpha)e_1 + e_2$. Hence $f(y)$ can not converge when $y \to \infty$.

This nilpotent orbit appears at $0 \in \Delta$ from the variation of mixed Hodge structure on Δ^* obtained by the pull-back of the exponential sequence $0 \to \mathbf{Z}(1) \to \mathcal{O}_{\Delta^*} \to \mathcal{O}^{\times}_{\Delta^*} \to 0$ with respect to $\mathbf{Z} \to \mathcal{O}^{\times}_{\Delta^*}$; $1 \mapsto e^{-2\pi i\alpha}q^{-1}$.

Example 13.3. In this example, we see that for the convergence in Theorem 0.5 (1), it is crucial that the canonical splitting of W is defined by using ζ, not only δ, as in §1, and that the values of ζ like

$$\zeta_{-1,-5} = -\frac{15i}{16}\delta_{-1,-5} \quad \text{(see Appendix)},$$

which may seem strange, are exactly necessary for the convergence.

Let V be a 6 dimensional $\mathbf{R}$-vector space with basis $(e_1, \dots, e_6)$, let W be the increasing filtration on V defined by $W_0 = V$, $W_{-1} = W_{-6} = \mathbf{R}e_1 + \cdots + \mathbf{R}e_5$, and $W_{-7} = 0$. Hence $\mathrm{gr}^W_w = 0$ unless $w = 0$ or -6.

Define $\langle\ ,\ \rangle_w$ as follows: $\langle e_6, e_6\rangle_0 = 1$. For $1 \le j \le 5$ and $1 \le k \le 5$, $\langle e_j, e_k\rangle_{-6}$ is $(-1)^{k+1}$ if $j + k = 6$, and is 0 otherwise.

Let N be the element of $\mathfrak{g}_{\mathbf{R}}$ defined by $N(e_1) = 0, N(e_{j+1}) = e_j$ for $1 \le j \le 5$.

Let F be the decreasing filtration on $V_{\mathbf{C}}$ defined by $F^p = V_{\mathbf{C}}$ for $p \le -5$, $F^p = \sum_{j=p+6}^{6} \mathbf{C}e_j$ for $-5 \le p \le 0$, and $F^p = 0$ for $p \ge 1$. Then (N, F) generates a mixed nilpotent orbit. For $y > 0$, $\exp(iyN)F \in D$ and the Hodge type of $\exp(iyN)F(\mathrm{gr}^W_w)$ is $(0,0)$ for $w = 0$, and $(-1,-5)+(-2,-4)+(-3,-3)+(-4,-2)+(-5,-1)$ for $w = -6$.

Let R and I be the real part and the imaginary part of $\exp(iyN)e_6$, respectively. Then $\delta(W, \exp(iyN)F)$ annihilates W_{-1} and sends e_6 to I, and the splitting of W by the mixed Hodge structure $(W, \exp(-i\delta(W, \exp(iyN)F))F)$ is given by $(e_6 \bmod W_{-1}) \mapsto R$.

Let $g := \exp(iyN)e_5$. Let $I = 2\mathrm{Re}\,(v^{-5,-1} + v^{-4,-2}) + v^{-3,-3}$ be the Hodge decomposition. Then we have $v^{-5,-1} = \dfrac{1}{5}y\bar{g}$ and $v^{-5,-1} + v^{-4,-2} = \dfrac{2}{5}y\bar{g} + \dfrac{i}{10}y^2 N\bar{g}$. Further, by Lemma in Appendix, we have

$$(*) \qquad \begin{cases} \zeta_{-5,-1} = \frac{i}{2^5}\left(\binom{5}{1} + \cdots + \binom{5}{4}\right)\delta_{-5,-1} = \frac{15i}{16}\delta_{-5,-1}, \\ \zeta_{-4,-2} = \frac{i}{2^5}\left(\binom{5}{2} + \binom{5}{3}\right)\delta_{-4,-2} = \frac{5i}{8}\delta_{-4,-2}, \\ \zeta_{-3,-3} = 0\delta_{-3,-3}. \end{cases}$$

Hence the canonical $\mathbf{R}$-splitting of W associated to the mixed Hodge structure $(W, \exp(iyN)F)$ sends $(e_6 \bmod W_{-1})$ to

$$\begin{aligned} & R + 2\,\mathrm{Re}\left\{\tfrac{i}{2^5}\left(\tbinom{5}{1} + \cdots + \tbinom{5}{4}\right)v^{-5,-1} + \tfrac{i}{2^5}\left(\tbinom{5}{2} + \tbinom{5}{3}\right)v^{-4,-2}\right\} + 0v^{-3,-3} \\ &= R - \tfrac{1}{2^3}\,\mathrm{Im}\left\{\tbinom{5}{1}v^{-5,-1} + \tbinom{5}{2}(v^{-5,-1} + v^{-4,-2})\right\} \\ &= R - \tfrac{1}{8}\,\mathrm{Im}(5y\bar{g} + iy^2 N\bar{g}) \\ &= \left(\tfrac{y^4}{24}e_2 - \tfrac{y^2}{2}e_4 + e_6\right) - \tfrac{1}{8}\left\{5y\left(\tfrac{y^3}{6}e_2 - ye_4\right) + y^2\left(-\tfrac{y^2}{2}e_2 + e_4\right)\right\} \\ &= e_6, \end{aligned}$$

which is constant and hence converges. Thus we proved the convergence $s(y)$ in Theorem 0.5 directly here by the formula $(*)$. Note that in the above last equality, we can observe that all the divergent terms are cancelled in virtue of the fact that the coefficients of $\delta_{-5,-1}$, $\delta_{-4,-2}$, $\delta_{-3,-3}$ in the formula $(*)$ are nothing but $\frac{15i}{16}$, $\frac{5i}{8}$, 0, respectively. In fact, these coefficients of the formula $(*)$ are even determined conversely by using the fact that the canonical splitting of W associated to $(W, \exp(iyN)F)$ converges, as is seen by a similar computation as above.

Example 13.4. In 13.4–13.6, we consider an example mentioned in 11.9.

Let V be a 3 dimensional $\mathbf{R}$-vector space with basis (e_1, e_2, e_3), and let W be the increasing filtration on V defined by $W_0 = V$, $W_{-1} = W_{-3} = \mathbf{R}e_1 + \mathbf{R}e_2$, and $W_{-4} = 0$. Hence $\mathrm{gr}^W_w = 0$ unless $w = 0$ or -3. Define $\langle\ ,\ \rangle_w$ by $\langle e_3, e_3\rangle_0 = 1$, $\langle e_2, e_1\rangle_{-3} = 1$. Let N be the element of $\mathfrak{g}_{\mathbf{R}}$ defined by

$$N(e_1) = 0,\ N(e_2) = e_1,\ N(e_3) = e_2.$$

Let F' be the decreasing filtration on $V_{\mathbf{C}}$ defined by $(F')^{-2} = V_{\mathbf{C}}$, $(F')^{-1} = \mathbf{C}e_2 + \mathbf{C}e_3$, $(F')^0 = \mathbf{C}e_3$, $(F')^1 = 0$, let $a \in \mathbf{R}$, and let $F = \exp(iaN)F'$. Then (N, F) generates a mixed nilpotent orbit. For $y > -a$, $\exp(iyN)F \in D$ and the Hodge type of $\exp(iyN)F(\mathrm{gr}^W_w)$ is $(0,0)$ for $w = 0$, and $(-1,-2) + (-2,-1)$ for $w = -3$. We have:

(1) $s(y)(e_3 \bmod W_{-1}) = e_3$, hence $s(y) = s$ is constant.

(2) $u(y) = 1$.

(3) $t(y)(e_1) = y^2 e_1,\ t(y)(e_2) = ye_2,\ t(y)(e_3) = e_3$,
$\mathbf{r}^{-2} = V_{\mathbf{C}},\ \mathbf{r}^{-1} = \mathbf{C}(ie_1 + e_2) + \mathbf{C}e_3,\ \mathbf{r}^0 = \mathbf{C}e_3,\ \mathbf{r}^1 = 0$,
$g(y)(e_1) = (1+\frac{a}{y})^{1/2}e_1$, $g(y)(e_2) = (1+\frac{a}{y})^{-1/2}e_2$, $g(y)e_3 = e_3$, ${}^e g(y) = g(y)$,
where we have chosen $(g_w(y))_w$ as $g_w(y) = (\mathrm{gr}^W_w$ of the above $g(y))$.

(4) $\varepsilon(y)(e_1) = \varepsilon(y)(e_2) = 0,\ \varepsilon(y)(e_3) = -\frac{1}{2}(1+\frac{a}{y})^{3/2}e_1 + i(1+\frac{a}{y})^{3/2}e_2$.

(5) $g_P(y)(e_1) = (1+\frac{a}{y})^{1/2}e_1,\ g_P(y)(e_2) = (1+\frac{a}{y})^{-1/2}e_2$,
$g_P(y)(e_3) = \frac{1}{2}y^2((1+\frac{a}{y})^{1/2} - (1+\frac{a}{y})^2)e_1 + iy((1+\frac{a}{y}) - (1+\frac{a}{y})^{-1/2})e_2 + e_3$.

This (5) shows that when $y \to \infty$, $g_P(y)$ converges if and only if $a = 0$.

We obtain (1)–(5) as follows.

Let $\delta^{(0)}(y)$, $\zeta^{(0)}(y)$ and $\varepsilon^{(0)}(y)$ be δ, ζ and ε for $(W, \exp(iyN)F)$, respectively. Let $\delta^{(1)}$, $\zeta^{(1)}$, $\varepsilon^{(1)}$, be δ, ζ and ε for $(M(N,W), F)$, respectively.

We have $\exp(iyN)F = \exp(i(a+y)N)F'$ so that

$$\begin{aligned}
&\exp(iyN)F^{-2} = V_{\mathbf{C}},\\
&\exp(iyN)F^{-1} = \mathbf{C}(i(a+y)e_1 + e_2) + \mathbf{C}\left(-\tfrac{(a+y)^2}{2}e_1 + i(a+y)e_2 + e_3\right),\\
&\exp(iyN)F^{0} = \mathbf{C}\left(-\tfrac{(a+y)^2}{2}e_1 + i(a+y)e_2 + e_3\right),\\
&\exp(iyN)F^{1} = 0.
\end{aligned}$$

By 1.3, $\delta^{(0)}(y)e_1 = \delta^{(0)}(y)e_2 = 0$, $\delta^{(0)}(y)e_3 = (a+y)e_2$. We have $\delta^{(0)}(y) = \delta^{(0)}(y)_{-1,-2} + \delta^{(0)}(y)_{-2,-1}$, where $\delta^{(0)}(y)_{-1,-2}(e_j) = \delta^{(0)}(y)_{-2,-1}(e_j) = 0$ for $j = 1,2$, and $\delta^{(0)}(y)_{-1,-2}(e_3) = i\frac{(a+y)^2}{2}e_1 + \frac{a+y}{2}e_2$, $\delta^{(0)}(y)_{-2,-1}(e_3) = -i\frac{(a+y)^2}{2}e_1 + \frac{a+y}{2}e_2$.

Since (W,F) is of the Hodge type $(0,0) + (-1,-2) + (-2,-1)$, we have $\zeta^{(0)}(y) = \zeta^{(0)}(y)_{-1,-2} + \zeta^{(0)}(y)_{-2,-1} = -\frac{i}{2}\delta^{(0)}(y)_{-1,-2} + \frac{i}{2}\delta^{(0)}(y)_{-2,-1}$ (see Appendix for the last equality), and hence $\zeta^{(0)}(y)(e_1) = \zeta^{(0)}(y)(e_2) = 0, \zeta^{(0)}(y)(e_3) = \frac{(a+y)^2}{2}e_1$. Hence $\varepsilon^{(0)}(y)(e_1) = \varepsilon^{(0)}(y)(e_2) = 0, \varepsilon^{(0)}(y)(e_3) = -\frac{(a+y)^2}{2}e_1 + i(a+y)e_2$.

Therefore the $\mathbf{R}$-split mixed Hodge structure associated to $(W, \exp(iyN)F)$ is given by $\exp(-\varepsilon^{(0)}(y))\exp(iyN)F^0 = \mathbf{C}e_3$, and the canonical splitting $s(y)$ of W associated to $\exp(iyN)F$ is given as (1).

(2) and (3) follow from (1).

(4) follows by $\varepsilon(y) = \varepsilon(W, t(y)^{-1}g(y)^{-1}\exp(iyN)F) = t(y)^{-1}g(y)^{-1}\varepsilon^{(0)}(y)g(y)t(y)$ (see 0.3) from the computation of $\varepsilon^{(0)}(y) = \varepsilon(W, \exp(iyN)F)$ in the proof of (1).

To show (5), let $M = M(N,W)$. Then $M_0 = V$; $M_{-1} = M_{-2} = \mathbf{R}e_1 + \mathbf{R}e_2$; $M_{-3} = M_{-4} = \mathbf{R}e_1$; $M_{-5} = 0$. We have $\delta^{(1)} = aN$. Since $\delta^{(1)}$ coincides with its $(-1,-1)$-Hodge component, we have $\zeta^{(1)} = 0$. Now (5) follows from (3), (4) and $\zeta^{(1)} = 0$.

13.5. As mentioned in 11.9, in the previous example, there is no $f(y)$ satisfying 11.9.1–11.9.3 unless $a = 0$.

To see this, we first compute $\tilde{F}$ and $\exp(iyN)\tilde{F}$. By 13.4, we have $\tilde{F} = \exp(-i\delta^{(1)})F = F'$. Hence we have $\exp(iyN)\tilde{F}^{-2} = V_{\mathbf{C}}$, $\exp(iyN)\tilde{F}^{-1} = \mathbf{C}(iye_1 + e_2) + \mathbf{C}(-\frac{y^2}{2}e_1 + iye_2 + e_3)$, $\exp(iyN)\tilde{F}^0 = \mathbf{C}(-\frac{y^2}{2}e_1 + iye_2 + e_3)$, $\exp(iyN)\tilde{F}^1 = 0$.

Now assume that there is a real analytic function $f(y)$ satisfying the above conditions. Define $a_j(y)$ $(1 \le j \le 4)$ and $b_j(y)$ $(j = 1, 2)$ by

$$\begin{aligned} f(y)(e_1) &= a_1(y)e_1 + a_3(y)e_2, \\ f(y)(e_2) &= a_2(y)e_1 + a_4(y)e_2, \\ f(y)(e_3) &= b_1(y)e_1 + b_2(y)e_2 + e_3. \end{aligned}$$

Then all the $a_j(y)$ and the $b_j(y)$ are convergent power series in y^{-1}. Further the $a_j(y)$'s are real and satisfy

$$a_1(y)a_4(y) - a_2(y)a_3(y) = 1,$$

because $\mathrm{gr}^W_{-3}(f(y))$ preserves $\langle\ ,\ \rangle_{-3}$.

Together with the computation of $\exp(iyN)\tilde{F}$ in the above, we have

$$\begin{aligned} f(y)\exp(iyN)\tilde{F}^{-1} &= \mathbf{C}((iya_1(y) + a_2(y))e_1 + (iya_3(y) + a_4(y))e_2) \\ &\quad + \mathbf{C}((-\tfrac{y^2}{2}a_1(y) + iya_2(y) + b_1(y))e_1 \\ &\quad + (-\tfrac{y^2}{2}a_3(y) + iya_4(y) + b_2(y))e_2 + e_3), \\ f(y)\exp(iyN)\tilde{F}^0 &= \mathbf{C}((-\tfrac{y^2}{2}a_1(y) + iya_2(y) + b_1(y))e_1 \\ &\quad + (-\tfrac{y^2}{2}a_3(y) + iya_4(y) + b_2(y))e_2 + e_3). \end{aligned}$$

Comparing this with the description of $\exp(iyN)F$ in the previous subsection via the equality $\exp(iyN)F = f(y)\exp(iyN)\tilde{F}$, we have two equalities. The first one is $\mathbf{C}(i(a+y)e_1 + e_2) = \mathbf{C}((iya_1(y) + a_2(y))e_1 + (iya_3(y) + a_4(y))e_2)$, or equivalently,

$$\begin{vmatrix} i(a+y) & iya_1(y) + a_2(y) \\ 1 & iya_3(y) + a_4(y) \end{vmatrix} = 0. \tag{1}$$

The second one is $\mathbf{C}(-\frac{(a+y)^2}{2}e_1 + i(a+y)e_2 + e_3) = \mathbf{C}((-\frac{y^2}{2}a_1(y) + iya_2(y) + b_1(y))e_1 + (-\frac{y^2}{2}a_3(y) + iya_4(y) + b_2(y))e_2 + e_3)$, or equivalently,

$$\begin{aligned} -\tfrac{(a+y)^2}{2} &= -\tfrac{y^2}{2}a_1(y) + iya_2(y) + b_1(y), \\ i(a+y) &= -\tfrac{y^2}{2}a_3(y) + iya_4(y) + b_2(y). \end{aligned} \tag{2}$$

We deduce a contradiction from these equalities. Taking the imaginary part of (1), we see

$$ya_1(y) = (a+y)a_4(y). \tag{3}$$

Taking the real (resp. imaginary) part of (2), we see

(4) the constant terms of $a_1(y)$ and $a_3(y)$ (resp. $a_2(y)$ and $a_4(y)$) are 1 and 0 (resp. 0 and 1), respectively.

Comparing the coefficients of y in the first equality of (2), we see that the coefficient of y^{-1} in the expansion of $a_1(y)$ is $2a$, that is,

(5) $$a_1(y) = 1 + 2ay^{-1} + \cdots .$$

By (3) and (5), we have also

(6) $$a_4(y) = 1 + ay^{-1} + \cdots .$$

Combining (4), (5), and (6), we have $a_1(y)a_4(y) - a_2(y)a_3(y) = 1 + 3ay^{-1} + \cdots$, which contradicts $a_1(y)a_4(y) - a_2(y)a_3(y) = 1$ unless $a = 0$.

13.6. In the above example, we have

$$\exp(iyN)F = \exp(i(y+a)N)\tilde{F} = t(y+a)\exp(iN)\tilde{F} = t(1+ay^{-1})\exp(iyN)\tilde{F}.$$

Note that $t(1+ay^{-1})$ does not preserve the intersection forms of gr^W, but it multiplies the intersection forms. Hence if we define $f(y) := t(1+ay^{-1})$, it satisfies the conditions 11.9.1, 11.9.2, 11.9.3′.

Appendix: Computation of $\zeta_{-p,-q}$

Here for reader's convenience, we review how it is seen that $\zeta_{-p,-q}$ are Lie polynomials and how they are computed. These are included in [CKS], but we think it is nice that a summary is written here. Let the notation be as in 1.4.

Let

$$Q := \sum_{k\geq 1} Q_k(X_2, \ldots, X_{k+1})y^{-k} := \log\left(\sum_{k\geq 0} P_k y^{-k}\right).$$

Then

$$Q' + \frac{[Q',Q]}{2!} + \frac{[[Q',Q],Q]}{3!} + \frac{[[[Q',Q],Q],Q]}{4!} + \cdots = \sum_{k\geq 2} X_k y^{-k},$$

where $Q' := \frac{dQ}{dy}$ formally. Comparing the coefficients $y^{-2}, y^{-3}, y^{-4}, \ldots$, we have $-Q_1 = X_2, -2Q_2 = X_3, -3Q_3 + \frac{1}{2}[Q_1, Q_2] = X_4, \ldots$, and $Q_1 = -X_2, Q_2 = -\frac{1}{2}X_3, Q_3 = -\frac{1}{3}X_4 + \frac{1}{12}[X_2, X_3], \ldots$, and in this way we see inductively that $Q_1, Q_2, Q_3, \ldots$ are Lie polynomials. By Campbell-Hausdorff formula applied to

$$\exp(-\zeta)\exp(i\delta) = \exp\left(\sum_{k\geq 1} Q_k(C_2, \ldots, C_{k+1})\right),$$

we can see inductively that $\zeta_{-p,-q}$ are Lie polynomials and can calculate them as

$$\begin{aligned}
\zeta_{-1,-1} &= 0,\\
\zeta_{-1,-2} &= -\tfrac{i}{2}\delta_{-1,-2},\\
\zeta_{-1,-3} &= -\tfrac{3i}{4}\delta_{-1,-3},\\
\zeta_{-2,-2} &= 0,\\
\zeta_{-1,-4} &= -\tfrac{7i}{8}\delta_{-1,-4},\\
\zeta_{-2,-3} &= -\tfrac{3i}{8}\delta_{-2,-3} - \tfrac{1}{8}[\delta_{-1,-1}, \delta_{-1,-2}],\\
\zeta_{-1,-5} &= -\tfrac{15i}{16}\delta_{-1,-5},\\
\zeta_{-2,-4} &= -\tfrac{5i}{8}\delta_{-2,-4} + \tfrac{7}{32}[\delta_{-1,-1}, \delta_{-1,-3}],\\
\zeta_{-3,-3} &= -\tfrac{1}{8}[\delta_{-1,-1}, \delta_{-2,-2}], \ \ldots .
\end{aligned}$$

(Note that $\zeta_{-q,-p}$ is obtained from $\zeta_{-p,-q}$ by 1.4 (ii) for any p, q.)

The following lemma is used in §13.

Lemma. *For $p \geq q \geq 1$, we have*

$$\eta_{-p,-q} \equiv -\tfrac{(p+q-1)!}{2^{p+q-2}(p-1)!(q-1)!}\delta_{-p,-q},$$
$$\zeta_{-p,-q} \equiv \tfrac{i}{2^{p+q-1}}\left(\textstyle\sum_{q\leq k<p}\binom{p+q-1}{k}\right)\delta_{-p,-q},$$

modulo Lie polynomials in the $\delta_{-r,-s}$ for $1 \leq r < p$, $1 \leq s < q$. Here in the second congruence we understand $\zeta_{-p,-p} \equiv 0$. $\eta_{-q,-p}$, $\zeta_{-q,-p}$ are given as the conjugations over $\mathfrak{g}_{\mathbf{R}}$ of $\eta_{-p,-q}$, $\zeta_{-p,-q}$, respectively.

Proof. From the proof of Lemma (6.60) in [CKS], we have

$$d_{p-1,q-1}\eta_{-p,-q} + i\zeta_{-p,-q} \equiv -\delta_{-p,-q}, \tag{1}$$

modulo Lie polynomials in the $\delta_{-r,-s}$ for $1 \leq r < p$, $1 \leq s < q$, where

$$d_{p-1,q-1} = \int_0^1 (1-t)^{p-1}(1+t)^{q-1}dt. \tag{2}$$

Exchanging p and q in (1), and taking the conjugate over $\mathfrak{g}_{\mathbf{R}}$, we have

$$d_{q-1,p-1}\eta_{-p,-q} - i\zeta_{-p,-q} \equiv -\delta_{-p,-q}. \tag{3}$$

From (1) and (3), we obtain

$$\eta_{-p,-q} \equiv -\tfrac{2}{d_{p-1,q-1}+d_{q-1,p-1}}\delta_{-p,-q},$$
$$\zeta_{-p,-q} \equiv \tfrac{-d_{p-1,q-1}+d_{q-1,p-1}}{d_{p-1,q-1}+d_{q-1,p-1}}i\delta_{-p,-q}.$$

On the other hand, from (2) we compute

$$-d_{p-1,q-1} + d_{q-1,p-1} = \tfrac{(p-1)!(q-1)!}{(p+q-1)!}\left(\textstyle\sum_{q\leq k<p}\binom{p+q-1}{k}\right),$$
$$d_{p-1,q-1} + d_{q-1,p-1} = \tfrac{(p-1)!(q-1)!}{(p+q-1)!}2^{p+q-1}.$$

Substituting these, we obtain the assertion. □

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