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CLASSIFYING SPACES OF DEGENERATING MIXED HODGE STRUCTURES, I: BOREL-SERRE SPACES

Dedicated to Professor Masaki Kashiwara on his sixtieth birthday

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Abstract. Let D be the classifying space of mixed Hodge structures with polarized graded quotients. We construct a real analytic manifold with corners D_{BS} which contains D as a dense open subset. This is the mixed Hodge theoretic version of the Borel-Serre space in the case of pure weight constructed by Borel-Ji and Kato-Usui.

Introduction

0.1. Let D be the classifying space of mixed Hodge structures whose graded quotients of weight filtrations are polarized, with fixed Hodge numbers of the graded quotients, defined in [U]. This space is the mixed Hodge theoretic version of the Griffiths domain [G] in the pure case. In this paper, we construct a real analytic manifold with corners D_{BS} which contains D as a dense open subset. This is a generalization of the Borel-Serre space in the pure case which was constructed in the work of Kato-Usui [KU2]. Note that the Borel-Serre space in the pure case is independently obtained as a consequence of the works of Borel-Ji [BJ1], [BJ2].

As in the theorem below, which is proved in §8–§9, our space D_{BS} has similar properties to those of the original Borel-Serre space $\bar{X}$ in the work of Borel-Serre [BS], which is a real analytic manifold with corners containing, as a dense open subset, the symmetric space X associated to a semi-simple algebraic group over $\mathbf{Q}$. See 1.6 for the definition of the arithmetic discrete group $G_{\mathbf{Z}}$ acting on D in this theorem. Note that in our terminology, "compact" and "locally compact" contain the Hausdorff condition as in Bourbaki [Bn].

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Theorem 0.2. (i) *The action of $G_{\mathbf{Z}}$ on D extends to a real analytic action of $G_{\mathbf{Z}}$ on D_{BS}. For a subgroup Γ of $G_{\mathbf{Z}}$, the action of Γ on D_{BS} is proper and the quotient space $\Gamma\backslash D_{\mathrm{BS}}$ is locally compact. If Γ is a neat subgroup of $G_{\mathbf{Z}}$, the projection $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$ is a local homeomorphism, and $\Gamma\backslash D_{\mathrm{BS}}$ has a unique structure of a real analytic manifold with corners for which $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$ is locally an isomorphism.*

(ii) *If Γ is a subgroup of $G_{\mathbf{Z}}$ of finite index, then $\Gamma\backslash D_{\mathrm{BS}}$ is compact.*

0.3. We shortly describe our construction of D_{BS} and explain our ideas.

In the definition of the classifying space D, we fix a finitely generated free $\mathbf{Z}$-module H_0, a rational increasing filtration W on $H_{0,\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Z}} H_0$, rational non-degenerate bilinear forms $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$ for $w \in \mathbf{Z}$, which is symmetric if w is even and anti-symmetric if w is odd, and non-negative integers $h^{p,q}_w$ with $h^{q,p}_w = h^{p,q}_w$ and with $h^{p,q}_w = 0$ unless $p+q = w$ ($w, p, q \in \mathbf{Z}$). Then D is defined to be the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ for which (H_0, W, F) is a mixed Hodge structure such that for any $w \in \mathbf{Z}$, the Hodge structure $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), F(\mathrm{gr}^W_w))$ of weight w has Hodge numbers $(h^{p,q}_w)_{p,q}$ and is polarized by $\langle\ ,\ \rangle_w$. Let $G_{\mathbf{R}}$ be the group of all automorphisms g of $(H_{0,\mathbf{R}}, W)$ such that the automorphism of gr^W_w induced by g preserves $\langle\ ,\ \rangle_w$ for all $w \in \mathbf{Z}$.

In the pure case, that is, in the case where there is $w \in \mathbf{Z}$ such that $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$, the Borel-Serre space D_{BS} is defined to be the set of all pairs (P, Z) where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and Z is an orbit in D under the Borel-Serre action of A_P. Here A_P is as follows. Let P_u be the unipotent radical of P and let S_P be the largest $\mathbf{Q}$-split torus in the center of P/P_u. Then A_P is the connected component of the group of $\mathbf{R}$-rational points of S_P containing the unity. Hence $A_P \simeq \mathbf{R}^n_{>0}$ where n is the rank of S_P. A point F of D is identified with $(G^\circ_{\mathbf{R}}, \{F\}) \in D_{\mathrm{BS}}$, where $G^\circ_{\mathbf{R}}$ is the connected component of $G_{\mathbf{R}}$ containing the unity in the Zariski topology, which is the largest $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. As a point of the topological space D_{BS}, (P, Z) is the limit point of the points in Z which run to a special direction conducted by P. In §2, we will review the Borel-Serre action and more details in the pure case. This construction is similar to that of the original Borel-Serre space $\bar{X}$ in [BS], which is defined for a semi-simple algebraic group G over $\mathbf{Q}$ and contains the space X of all maximal compact subgroups of $G_{\mathbf{R}}$ as a dense open subset. This $\bar{X}$ is the set of all pairs (P, Z), where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and Z is an orbit in X for the Borel-Serre action of A_P.

Now we consider the mixed case. Assume we are not in the pure case. Then it seems that both the group $G_{\mathbf{R}}$ and the following group $G'_{\mathbf{R}}$ are equally important. Consider the homomorphism

(1) $$\mathbf{R}^\times \to \prod_{w\in\mathbf{Z}} \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w), \quad a \mapsto (a^w)_{w\in\mathbf{Z}},$$

and let $G'_{\mathbf{R}}$ be the algebraic subgroup of $\mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ generated by $G_{\mathbf{R}}$ and a one-dimensional subtorus T of $\mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ whose image in $\prod_{w\in\mathbf{Z}} \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ coincides with the image of the homomorphism (1). This algebraic group $G'_{\mathbf{R}}$ is independent of the choice of T, and is defined over $\mathbf{Q}$. We have the following maps

(2) $$G'_{\mathbf{R}} \hookleftarrow G_{\mathbf{R}} \xrightarrow{\pi} \prod_{w\in\mathbf{Z}} G_{\mathbf{R}}(\mathrm{gr}^W_w),$$

where $G_{\mathbf{R}}(\mathrm{gr}^W_w) = \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w, \langle\ ,\ \rangle_w)$. The map $P' \mapsto P' \cap G_{\mathbf{R}}$ is a bijection from the set of all $\mathbf{Q}$-parabolic subgroups of $G'_{\mathbf{R}}$ to that of $G_{\mathbf{R}}$, whose inverse is given by assigning P to the group generated by P and T. Moreover, the set of all $\mathbf{Q}$-parabolic subgroups P of $G_{\mathbf{R}}$ corresponds bijectively to the set of all families $(P_w)_{w\in\mathbf{Z}}$ of $\mathbf{Q}$-parabolic subgroups P_w of $G_{\mathbf{R}}(\mathrm{gr}^W_w)$ by $P = \pi^{-1}(\prod_w P_w)$, where π is the map in (2). We have $A_P = \prod_w A_{P_w}$. If P' is a $\mathbf{Q}$-parabolic subgroup of $G'_{\mathbf{R}}$ and $P = P' \cap G_{\mathbf{R}}$, then $A_{P'} = \mathbf{R}_{>0} \times A_P$. We will denote $A_{P'}$ by B_P. By a method described below, we define a Borel-Serre action of B_P on D. We define D_{BS} in the mixed case as the set of all pairs (P, Z), where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and Z is either an A_P-orbit or a B_P-orbit in D for the Borel-Serre action. A point F of D is identified with $(G^\circ_{\mathbf{R}}, \{F\}) \in D_{\mathrm{BS}}$. Note that for $P = G^\circ_{\mathbf{R}}$, $A_P = \{1\}$ and $\{F\}$ is an A_P-orbit.

Here, to define the Borel-Serre action of B_P on D, we use the canonical splitting of the weight filtration associated to a mixed Hodge structure, which was defined in [CKS]. This splitting (reviewed in §4 below) played important roles in the studies [CKS] and [KNU] of degeneration of Hodge structures. (It will play key roles also in Part II, Part III, ... of this series of papers.) For $b = (c, a) \in B_P$ with $c \in \mathbf{R}_{>0}$ and $a \in A_P$, we define the Borel-Serre action of b on D by $F \mapsto c_F a_F F$ $(F \in D)$, where $c_F \in \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ is the lifting, by the canonical splitting of W associated to F, of the image of c in $\prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ under the homomorphism (1), and $a_F \in \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ is the lifting, by the canonical splitting of W associated to F, of the Borel-Serre action of the image of a under $A_P \xrightarrow{\simeq} \prod_w A_{P_w}$.

0.4. This paper is the part I of our series of papers in which we will construct various enlargements of D, by adding to D points at infinity corresponding to degenerations of mixed Hodge structures. In the pure case, in [KU2] and [KU3] (a short summary of [KU3] is given in [KU1]), we constructed eight enlargements of D, which are related to each other as in the fundamental diagram

$$(*)\qquad \begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}}. \\
\downarrow & & \downarrow & & & & \\
D_\Sigma & \leftarrow & D^\sharp_\Sigma & & & &
\end{array}$$

In [KU3], for arithmetic subgroups Γ of $\mathrm{Aut}(D)$, we also obtained toroidal partial compactifications of $\Gamma\backslash D$ as the quotients $\Gamma\backslash D_\Sigma$ of D_Σ and proved that $\Gamma\backslash D_\Sigma$ are fine moduli spaces of polarized logarithmic Hodge structures. In our series of papers, we will obtain the mixed Hodge theoretic version of the diagram $(*)$, and study the fine moduli spaces $\Gamma\backslash D_\Sigma$ of logarithmic mixed Hodge structures with polarized graded quotients.

We are very happy to dedicate this paper to Professor Masaki Kashiwara whose study on the degeneration of mixed Hodge structures plays essential roles in these series of papers.

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§1. Classifying spaces of mixed Hodge structures with polarized graded quotients

We review the classifying spaces D of mixed Hodge structures with polarized graded quotients and with fixed Hodge numbers, defined in [U]. These spaces are the mixed Hodge theoretic versions of Griffiths domains. We fix the notation used in this paper. We first review Hodge structures, mixed Hodge structures, and polarized Hodge structures following [D].

1.1. A *Hodge structure* of weight $w \in \mathbf{Z}$ is a pair $H = (H_{\mathbf{Z}}, F)$ where $H_{\mathbf{Z}}$ is a free $\mathbf{Z}$-module of finite rank and F is a decreasing filtration on $H_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ satisfying

$$H_{\mathbf{C}} = \bigoplus_{p+q=w} H_F^{p,q}, \quad \text{where } H_F^{p,q} = F^p \cap \bar{F}^q.$$

Here $\bar{F}^q$ denotes the image of F^q under the complex conjugation $H_{\mathbf{C}} \to H_{\mathbf{C}}\,;\, z \otimes h \mapsto \bar{z} \otimes h$ $(z \in \mathbf{C}, h \in H_{\mathbf{Z}})$.

1.2. A *mixed Hodge structure* is a triple $(H_{\mathbf{Z}}, W, F)$, where $H_{\mathbf{Z}}$ is a free $\mathbf{Z}$-module of finite rank, W is a rational increasing filtration on $H_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ ("rational" means that all W_w are defined over $\mathbf{Q}$), and F is a decreasing filtration on $H_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ such that $W_w = H_{\mathbf{R}}$ for $w \gg 0$, $W_w = 0$ for $w \ll 0$, and $((H_{\mathbf{Z}} \cap W_w)/(H_{\mathbf{Z}} \cap W_{w-1}), F(\mathrm{gr}_w^W))$ is a Hodge structure of weight w for any $w \in \mathbf{Z}$. Here $F(\mathrm{gr}_w^W)$ denotes the filtration on $\mathrm{gr}_{w,\mathbf{C}}^W = \mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}_w^W$ induced by F.

1.3. *Polarization.* Let $H = (H_{\mathbf{Z}}, F)$ be a Hodge structure of weight w. A polarization on H is a rational non-degenerate $\mathbf{R}$-bilinear form

$$\langle\ ,\ \rangle : H_{\mathbf{R}} \times H_{\mathbf{R}} \to \mathbf{R}$$

which is symmetric if w is even and anti-symmetric if w is odd, satisfying the following two conditions.

(1) $\langle F^p, F^q \rangle = 0$ if $p + q > w$. Here and in (2) below, $\langle\ ,\ \rangle$ denotes the $\mathbf{C}$-bilinear form on $H_{\mathbf{C}} \times H_{\mathbf{C}} \to \mathbf{C}$ induced by $\langle\ ,\ \rangle$.

(2) Let $C_F : H_{\mathbf{C}} \to H_{\mathbf{C}}$ be the $\mathbf{C}$-linear map whose restriction to $H_F^{p,q}$ with $p+q=w$ is the multiplication by i^{p-q} (this C_F is called the *Weil operator*). Then the Hermitian form

$$(\ ,\ )_F : H_{\mathbf{C}} \times H_{\mathbf{C}} \to \mathbf{C}, \quad (x, y) \mapsto \langle C_F x, \bar{y} \rangle$$

is positive definite.

A Hodge structure endowed with a polarization is called a *polarized Hodge structure*. The positive definite Hermitian form $(\ ,\ )_F$ in (2) for a polarized Hodge structure is called the Hodge metric.

1.4. To define the space D, we fix a 4-tuple

$$\Phi_0 = (H_0, W, (\langle\ ,\ \rangle_w)_{w\in\mathbf{Z}}, (h^{p,q})_{p,q\in\mathbf{Z}})$$

where

H_0 is a finitely generated free $\mathbf{Z}$-module,

W is a rational increasing filtration on $H_{0,\mathbf{R}}$,

$\langle\ ,\ \rangle_w$ is a rational non-degenerate $\mathbf{R}$-bilinear form $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$ given for each $w \in \mathbf{Z}$ which is symmetric if w is even and anti-symmetric if w is odd, and

$h^{p,q}$ is a non-negative integer given for $p, q \in \mathbf{Z}$ such that $h^{p,q} = h^{q,p}$, $\mathrm{rank}_{\mathbf{Z}}(H_0) = \sum_{p,q} h^{p,q}$, and $\dim_{\mathbf{R}}(\mathrm{gr}^W_w) = \sum_{p+q=w} h^{p,q}$ for all w.

1.5. Let D be the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ for which (H_0, W, F) is a mixed Hodge structure such that, for all $w \in \mathbf{Z}$, $\langle\ ,\ \rangle_w$ are polarizations on $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), F(\mathrm{gr}^W_w))$ and such that, for all $p, q \in \mathbf{Z}$, the dimension of $H^{p,q}$ of $F(\mathrm{gr}^W_{p+q})$ coincides with $h^{p,q}$.

Let $\check{D}$ be the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ satisfying the following two conditions.

(1) $\dim(F^p(\mathrm{gr}^W_{p+q})/F^{p+1}(\mathrm{gr}^W_{p+q})) = h^{p,q}$ for any $p, q \in \mathbf{Z}$.

(2) $\langle\ ,\ \rangle_w$ kills $F^p(\mathrm{gr}^W_w) \times F^q(\mathrm{gr}^W_w)$ for any $p, q, w \in \mathbf{Z}$ such that $p + q > w$.

Then D is an open subset of $\check{D}$.

1.6. We fix notation.

For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$, let G_A be the group of all A-automorphisms g of $H_{0,A}$ which are compatible with W such that $\mathrm{gr}^W_w(g) : \mathrm{gr}^W_w \to \mathrm{gr}^W_w$ are compatible with $\langle\ ,\ \rangle_w$ for all w. Let $G_{A,u} = \{g \in G_A \mid \mathrm{gr}^W_w(g) = 1 \text{ for all } w \in \mathbf{Z}\}$, the unipotent radical of G_A. Then

$$G_A/G_{A,u} = G_A(\mathrm{gr}^W) := \textstyle\prod_w G_A(\mathrm{gr}^W_w),$$

where $G_A(\mathrm{gr}^W_w)$ is "the G_A of $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\ ,\ \rangle_w)$", and G_A is a semi-direct product of $G_{A,u}$ and $G_A(\mathrm{gr}^W)$.

The natural action of $G_{\mathbf{C}}$ on $\check{D}$ is transitive, and $\check{D}$ is a complex homogeneous space under the action of $G_{\mathbf{C}}$. Hence $\check{D}$ is a complex analytic manifold. Furthermore, D is open in $\check{D}$ and it is also a complex analytic manifold.

1.7. For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let $\mathfrak{g}_A = \mathrm{Lie}(G_A)$. We identify $\mathfrak{g}_A$ with the set of all A-linear maps $N : H_{0,A} \to H_{0,A}$ which are compatible with W such that $\langle \mathrm{gr}^W_w(N)(x), y\rangle_w + \langle x, \mathrm{gr}^W_w(N)(y)\rangle_w = 0$ for all w and all x, y. Let $\mathfrak{g}_{A,u}$ be the nilpotent radical $\{N \in \mathfrak{g}_A \mid \mathrm{gr}^W_w(N) = 0 \text{ for all } w \in \mathbf{Z}\}$ of $\mathfrak{g}_A$. Then

$$\mathfrak{g}_A/\mathfrak{g}_{A,u} = \mathfrak{g}_A(\mathrm{gr}^W) := \textstyle\prod_w \mathfrak{g}_A(\mathrm{gr}^W_w),$$

where $\mathfrak{g}_A(\mathrm{gr}^W_w)$ denotes "the $\mathfrak{g}_A$ of $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\ ,\ \rangle_w)$".

1.8. The space D is a natural generalization to the mixed case of a Griffiths domain, i.e., the classifying space of polarized Hodge structures in [G].

Let $D(\mathrm{gr}^W_w)$ be the Griffiths domain of gr^W_w, that is, the D for $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\ ,\ \rangle_w, (h^{p,q})_{p+q=w})$. Let

$$D(\mathrm{gr}^W) = \prod_{w\in\mathbf{Z}} D(\mathrm{gr}^W_w).$$

We have the canonical surjective holomorphic map

$$D \to D(\mathrm{gr}^W), \quad F \mapsto F(\mathrm{gr}^W) := (F(\mathrm{gr}^W_w))_{w\in\mathbf{Z}}.$$

1.9. In the pure case, i.e., in the case where there exists $w \in \mathbf{Z}$ such that $W_w = H_{\mathbf{R}}$ and $W_{w-1} = 0$, the action of $G_{\mathbf{R}}$ on D is transitive. However this transitivity is not true in the mixed case. In Example II (resp. Examples I and III) below, the action of $G_{\mathbf{R}}$ on D is (resp. is not) transitive. The subgroup $G_{\mathbf{R}}G_{\mathbf{C},u}$ of $G_{\mathbf{C}}$ (1.6) acts always transitively on D, and the action of $G_{\mathbf{C},u}$ on each fiber of $D \to D(\mathrm{gr}^W)$ is transitive.

In 1.10–1.12, we give three examples of D. These are the simplest examples for which the set $\{w \in \mathbf{Z} \mid \mathrm{gr}^W_w \neq 0\}$ are $\{0, -2\}$, $\{0, -1\}$, $\{0, -3\}$, respectively.

1.10. Example I. Let $H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, let W be the increasing filtration on $H_{0,\mathbf{R}}$ defined by

$$W_{-3} = 0 \ \subset\ W_{-2} = W_{-1} = \mathbf{R}e_1 \ \subset\ W_0 = H_{0,\mathbf{R}},$$

let $\langle e_2, e_2\rangle_0 = 1$, $\langle e_1, e_1\rangle_{-2} = 1$, and let $h^{0,0} = h^{-1,-1} = 1$, $h^{p,q} = 0$ for all the other (p, q).

We have

$$D = \mathbf{C}.$$

For $z \in \mathbf{C}$, the corresponding $F(z) \in D$ is defined as

$$F(z)^1 = 0 \ \subset\ F(z)^0 = \mathbf{C}(ze_1 + e_2) \ \subset\ F(z)^{-1} = H_{0,\mathbf{C}}.$$

The group $G_{\mathbf{Z},u}$ is isomorphic to $\mathbf{Z}$ and is generated by $\gamma \in G_{\mathbf{Z}}$ which is defined as

$$\gamma(e_1) = e_1, \quad \gamma(e_2) = e_1 + e_2.$$

We have

$$G_{\mathbf{Z},u}\backslash D \simeq \mathbf{C}^\times$$

where $(F(z) \bmod G_{\mathbf{Z},u})$ corresponds to $\exp(2\pi iz) \in \mathbf{C}^\times$.

This space $G_{\mathbf{Z},u}\backslash D$ is the classifying space of extensions of mixed Hodge structures of the form $0 \to \mathbf{Z}(1) \to\ ? \to \mathbf{Z} \to 0$.

In this case, $D(\mathrm{gr}^W)$ is a one point set.

1.11. Example II. Let $H_0 = \mathbf{Z}^3 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$, let

$$W_{-2} = 0 \subset W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}},$$

let $\langle e_3, e_3\rangle_0 = 1$, $\langle e_2, e_1\rangle_{-1} = 1$, and let $h^{0,0} = h^{0,-1} = h^{-1,0} = 1$, $h^{p,q} = 0$ for all the other (p, q).

Then

$$D = \mathfrak{h} \times \mathbf{C}, \quad D(\mathrm{gr}^W) = \mathfrak{h},$$

where $\mathfrak{h}$ is the upper half plane. Here $(\tau, z) \in \mathfrak{h} \times \mathbf{C}$ corresponds to the decreasing filtration F given by

$$F^1 = 0 \subset F^0 = \mathbf{C}(\tau e_1 + e_2) + \mathbf{C}(ze_1 + e_3) \subset F^{-1} = H_{0,\mathbf{C}}.$$

The group $G_{\mathbf{Z},u}$ is isomorphic to $\mathbf{Z}^2$, where $(a, b) \in \mathbf{Z}^2$ corresponds to the element of $G_\mathbf{Z}$ which sends e_j to e_j for $j = 1, 2$ and sends e_3 to $ae_1 + be_2 + e_3$. The quotient space $G_{\mathbf{Z},u}\backslash D$ is the "universal elliptic curves" over the upper half plane $\mathfrak{h}$. For $\tau \in \mathfrak{h}$, the fiber of $G_{\mathbf{Z},u}\backslash D \to D(\mathrm{gr}^W) = \mathfrak{h}$ is identified with the elliptic curve $E_\tau := \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$. The Hodge structure on $H_0 \cap W_{-1}$ corresponding to τ is isomorphic to $H^1(E_\tau)(1)$. Here $H^1(E_\tau)$ denotes the Hodge structure $H^1(E_\tau, \mathbf{Z})$ of weight 1 endowed with the Hodge filtration and (1) here denotes the Tate twist. The fiber of $G_{\mathbf{Z},u}\backslash D \to \mathfrak{h}$ over τ is the classifying space of extensions of mixed Hodge structures of the form

$$0 \to H^1(E_\tau)(1) \to ? \to \mathbf{Z} \to 0.$$

1.12. Example III. Let $H_0 = \mathbf{Z}^3 = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$, let

$$W_{-4} = 0 \subset W_{-3} = W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}},$$

let $\langle e_3, e_3\rangle_0 = 1$, $\langle e_2, e_1\rangle_{-3} = 1$, and let $h^{0,0} = h^{-1,-2} = h^{-2,-1} = 1$, $h^{p,q} = 0$ for all the other (p, q).

Then

$$D = \mathfrak{h} \times \mathbf{C}^2, \quad D(\mathrm{gr}^W) = \mathfrak{h}.$$

Here $(\tau, z_1, z_2) \in \mathfrak{h} \times \mathbf{C}^2$ corresponds to the decreasing filtration F given by

$$F^1 = 0 \subset F^0 = \mathbf{C}(z_1e_1 + z_2e_2 + e_3) \subset F^{-1} = F^0 + \mathbf{C}(\tau e_1 + e_2) \subset F^{-2} = H_{0,\mathbf{C}}.$$

The group $G_{\mathbf{Z},u}$ is the same as in Example II. The Hodge structure on $H_0 \cap W_{-3}$ corresponding to $\tau \in \mathfrak{h} = D(\mathrm{gr}^W_{-3})$ is isomorphic to $H^1(E_\tau)(2)$. The fiber of $G_{\mathbf{Z},u}\backslash D \to D(\mathrm{gr}^W) = \mathfrak{h}$ over $\tau \in \mathfrak{h}$ is the classifying space of extensions of mixed Hodge structures of the form

$$0 \to H^1(E_\tau)(2) \to ? \to \mathbf{Z} \to 0.$$

These three examples will be retreated in §10 to illustrate our results in this paper.

§2. Review of the pure case

In this section, we review the Borel-Serre enlargement D_{BS} of D in the pure case constructed in [KU2]. In [KU2], D_{BS} in the pure case was constructed only as a topological space, but we show that, by refining the work [KU2], we can endow D_{BS} with a structure of a real analytic manifold with corners.

The results in this section are contained also in the works of Borel-Ji [BJ1], [BJ2] on Borel-Serre enlargements of homogeneous spaces by the fact that the space D in the pure case is a homogeneous space over $G_{\mathbf{R}}$. Since our generalization to the mixed case has the style similar to that of the work [KU2], we follow the formulation in [KU2].

Assume that we are in the pure case, that is, $W_w = H_{\mathbf{R}}$ and $W_{w-1} = 0$ for some $w \in \mathbf{Z}$.

2.1. For $F \in D$, we have compact subgroups K_F and K'_F of $G_{\mathbf{R}}$ defined as follows. Let $K'_F = \{g \in G_{\mathbf{R}} \mid gF = F\}$. Let K_F be the subgroup of $G_{\mathbf{R}}$ consisting of all elements which preserve the Hodge metric $(\ ,\ )_F$ (1.3). We have $K'_F \subset K_F$, K_F is a maximal compact subgroup of $G_{\mathbf{R}}$, and

$$K_F = \{g \in G_{\mathbf{R}} \mid C_F g C_F^{-1} = g\} \tag{1}$$

where C_F is the Weil operator of F (1.3). We have

$$D \simeq G_{\mathbf{R}}/K'_F, \quad gF \leftrightarrow (g \bmod K'_F) \quad \text{for } g \in G_{\mathbf{R}}. \tag{2}$$

2.2. As in [BS], a parabolic subgroup of $G_{\mathbf{R}}$ is assumed to be contained in the connected component $G^{\circ}_{\mathbf{R}}$ of $G_{\mathbf{R}}$ containing 1 for the Zariski topology. We allow the improper parabolic $G^{\circ}_{\mathbf{R}}$ itself as a parabolic subgroup of $G_{\mathbf{R}}$.

Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. We identify P with the Lie group of $\mathbf{R}$-valued points of P.

Let S_P be the largest $\mathbf{Q}$-split torus in the center of P/P_u, where P_u denotes the unipotent radical of P. Let $X(S_P)$ be the character group of S_P, which is isomorphic to $\mathbf{Z}^n$ where n is the rank of S_P. Let

$$\begin{aligned} A_P = \mathrm{Hom}(X(S_P), \mathbf{R}_{>0}) &\simeq \mathbf{R}^n_{>0} \\ \subset S_P = \mathrm{Hom}(X(S_P), \mathbf{R}^{\times}) &\simeq (\mathbf{R}^{\times})^n. \end{aligned}$$

Here $\mathbf{R}_{>0} = \{r \in \mathbf{R} \mid r > 0\}$ which we regard as a multiplicative group.

2.3. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. We have an action of the group A_P on D, which we denote by $\circ$ and call the Borel-Serre action, defined as follows.

Let $F \in D$. Then there is a unique homomorphism

$$S_P \to P, \quad a \mapsto a_F$$

of algebraic groups over $\mathbf{R}$ having the following two properties.

(1) $(a_F \bmod P_u) = a$ for any $a \in S_P$.

(2) $C_F a_F C_F^{-1} = a_F^{-1}$ for any $a \in S_P$, where C_F is the Weil operator (1.3) of F.

From 2.1 (1), we have

$$a_{kF} = a_F \quad \text{for any } k \in K_F.$$

We call $a_F \in P$ the Borel-Serre lifting of a at F. The Borel-Serre action of $a \in A_P$ on D is defined by

$$F \mapsto a \circ F := a_F F.$$

For a fixed $\mathbf{r} \in D$, all elements of D are expressed as $pk\mathbf{r}$ with $p \in P$ and $k \in K_{\mathbf{r}}$. The action of $a \in A_P$ on D is described as $a \circ pk\mathbf{r} = pa_{\mathbf{r}}k\mathbf{r}$.

2.4. The Borel-Serre space D_{BS} is defined to be the set of all pairs (P, Z) where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and Z is an A_P-orbit in D with respect to the Borel-Serre action.

2.5. We review the notion real analytic manifold with corners ([BS] Appendix by A. Douady and L. Hérault).

For $m, n \geq 0$ and for an open set U of $\mathbf{R}^m \times \mathbf{R}^n_{\geq 0}$, a function $f : U \to \mathbf{R}$ is called a real analytic function if, for each $x \in U$, there are an open neighborhood U' of x in U, an open set V of $\mathbf{R}^m \times \mathbf{R}^n$ containing U', and a real analytic function g on V such that the restrictions to U' of f and g coincide. We thus have the sheaf of real analytic functions on U. This sheaf is nothing but the inverse image on U of the sheaf of real analytic functions on $\mathbf{R}^m \times \mathbf{R}^n$.

A *real analytic manifold with corners* is a local ringed space over $\mathbf{R}$ which has an open covering whose each member is isomorphic to an open set of $\mathbf{R}^m \times \mathbf{R}^n_{\geq 0}$ with the sheaf of real analytic functions.

2.6. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. Then a real analytic manifold with corners $\bar{A}_P$ is defined as in [BS]. Choose a subtorus $\tilde{S}_P$ of P such that the projection $P \to P/P_u$ induces an isomorphism $\tilde{S}_P \xrightarrow{\simeq} S_P$. Let $X(S_P)^-_{\mathbf{Q}}$ be the $\mathbf{Q}_{\geq 0}$-subcone of $X(S_P)_{\mathbf{Q}} := X(S_P) \otimes \mathbf{Q}$ (i.e., a non-empty subset of $X(S_P)_{\mathbf{Q}}$ which is stable under the addition and under the multiplication by any element of $\mathbf{Q}_{\geq 0}$) generated by all elements of $X(S_P) \simeq X(\tilde{S}_P)$ which appear in the adjoint representation of $\tilde{S}_P$ in $\mathrm{Lie}(P)$. Then $X(S_P)^-_{\mathbf{Q}}$ does not depend on the choice of $\tilde{S}_P$ and $X(S_P)^-_{\mathbf{Q}} \simeq (\mathbf{Q}^{\mathrm{add}}_{\geq 0})^n$. Here $\mathbf{Q}^{\mathrm{add}}_{\geq 0}$ denotes the set $\{r \in \mathbf{Q} \mid r \geq 0\}$ regarded as an additive monoid, and n is the rank of S_P. Let $X(S_P)^+_{\mathbf{Q}}$ be the image of $X(S_P)^-_{\mathbf{Q}}$ under $X(S_P)_{\mathbf{Q}} \xrightarrow{\simeq} X(S_P)_{\mathbf{Q}}$, $\chi \mapsto \chi^{-1}$. Let

$$\bar{A}_P = \mathrm{Hom}(X(S_P)^+_{\mathbf{Q}}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \simeq \mathrm{Hom}((\mathbf{Q}^{\mathrm{add}}_{\geq 0})^n, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \simeq \mathbf{R}^n_{\geq 0},$$

where $\mathbf{R}^{\mathrm{mult}}_{\geq 0}$ denotes the set $\mathbf{R}_{\geq 0} = \{r \in \mathbf{R} \mid r \geq 0\}$ regarded as a multiplicative monoid. Note

$$\begin{aligned} A_P &= \mathrm{Hom}(X(S_P), \mathbf{R}_{>0}) = \mathrm{Hom}(X(S_P)^+_{\mathbf{Q}}, \mathbf{R}_{>0}) \\ &\subset \bar{A}_P = \mathrm{Hom}(X(S_P)^+_{\mathbf{Q}}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}). \end{aligned}$$

The natural action of $\mathbf{R}_{>0}$ on $\mathbf{R}_{\geq 0}$ induces a natural action of A_P on $\bar{A}_P$.

2.7. The space D_{BS} has the following structure of a real analytic manifold with corners.

For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$, let

$$D_{\mathrm{BS}}(P) := \{(Q, Z) \in D_{\mathrm{BS}} \mid Q \supset P\}.$$

Then we have a canonical bijection

$$D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \bar{A}_P, \tag{1}$$

where A_P acts on D by the Borel-Serre action. The definition of the bijection (1) is reviewed in 2.8 below.

By (1), we have a canonical surjection

$$C := \bigsqcup_P \; D \times \bar{A}_P \to D_{\mathrm{BS}},$$

where P ranges over all $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$, and C is a real analytic manifold with corners.

We define the topology of D_{BS} as the quotient of the topology of C via the above surjection.

We define the sheaf of rings of real analytic functions on D_{BS} as follows. For an open set U of D_{BS} and a function $f : U \to \mathbf{R}$, f is real analytic if and only if the composition $U' \to U \xrightarrow{f} \mathbf{R}$, where U' denotes the inverse image of U in C, is real analytic.

With this sheaf of rings over $\mathbf{R}$, D_{BS} is a real analytic manifold with corners as is shown in 2.10.

2.8. We recall here the definition of the bijection (1) in 2.7.

Let P and Q be $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$ and assume $Q \supset P$. Then $Q_u \subset P_u$, and the projection $Q/Q_u \to Q/P_u$ induces an injective homomorphism from S_Q into S_P. The induced homomorphism $A_Q \to A_P$ is compatible with the Borel-Serre actions of A_Q and A_P on D. The homomorphism $X(S_P) \to X(S_Q)$ corresponding to $S_Q \to S_P$ induces a surjective homomorphism $X(S_P)^+_{\mathbf{Q}} \to X(S_Q)^+_{\mathbf{Q}}$.

For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$, there is a bijection $\nu : Q \mapsto \mathrm{Ker}(X(S_P)^+_{\mathbf{Q}} \to X(S_Q)^+_{\mathbf{Q}})$ from the set of all $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$ containing P onto the set of all faces of the $\mathbf{Q}_{\geq 0}$-cone $X(S_P)^+_{\mathbf{Q}}$. (Recall that $X(S_P)^+_{\mathbf{Q}} \simeq (\mathbf{Q}^{\mathrm{add}}_{\geq 0})^n$. A face of $(\mathbf{Q}^{\mathrm{add}}_{\geq 0})^n$ corresponds bijectively to a subset J of $\{1, 2, \ldots, n\}$ by associating J to the face of $(\mathbf{Q}^{\mathrm{add}}_{\geq 0})^n$ consisting of all elements whose j-th components for all $j \in J$ are 0.)

The bijection (1) in 2.7 sends $\mathrm{class}(F, a) \in D \times^{A_P} \bar{A}_P$ ($F \in D$, $a \in \bar{A}_P$) to $(Q, Z) \in D_{\mathrm{BS}}(P)$, where $Q = \nu^{-1}(S)$ with $S = \{\chi \in X(S_P)^+_{\mathbf{Q}} \mid a(\chi) \neq 0\}$, and

$$Z = \{a' \circ F \mid a' \in A_P, a'(\chi) = a(\chi) \text{ for all } \chi \in S\}.$$

Conversely, in the bijection (1) in 2.7, $(Q, Z) \in D_{\mathrm{BS}}(P)$ corresponds to $\mathrm{class}(F, a) \in D \times^{A_P} \bar{A}_P$, where F is any element of Z and $a : X(S_P)^+_{\mathbf{Q}} \to \mathbf{R}_{\geq 0}$ sends $\chi \in X(S_P)^+_{\mathbf{Q}}$ to 1 if $\chi \in \nu(Q)$ and to 0 if $\chi \notin \nu(Q)$.

2.9. For 2.10 below, we give some basic facts (i) and (iii), whose proofs are easy, and their consequences (ii) and (iv). Let P and Q be $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$. Let $P * Q$ be the algebraic subgroup of $G_{\mathbf{R}}$ generated by P and Q, which is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$.

(i) The inverse image of $D_{\mathrm{BS}}(P)$ in $D \times \bar{A}_Q$ under the canonical map $D \times \bar{A}_Q \to D_{\mathrm{BS}}$ (2.7) is the open set $D \times \bar{A}_Q(P * Q)$, where $\bar{A}_Q(P * Q)$ is the open set of $\bar{A}_Q$ consisting of all homomorphisms $X(S_Q)^+_{\mathbf{Q}} \to \mathbf{R}^{\mathrm{mult}}_{\geq 0}$ which send $\mathrm{Ker}(X(S_Q)^+_{\mathbf{Q}} \to X(S_{P*Q})^+_{\mathbf{Q}})$ into $\mathbf{R}_{>0}$.

This shows

(ii) $D_{\mathrm{BS}}(P)$ is open in D_{BS}.

(iii) Let the notation be as in (i). Take a homomorphism $X(S_{P*Q})^+_{\mathbf{Q}} \to X(S_Q)^+_{\mathbf{Q}}$ such that the composition $X(S_{P*Q})^+_{\mathbf{Q}} \to X(S_Q)^+_{\mathbf{Q}} \to X(S_{P*Q})^+_{\mathbf{Q}}$ is the identity, let $A_Q \to A_{P*Q}$ be the corresponding homomorphism, and let H be the kernel of the last homomorphism. Then we have an isomorphism of real analytic manifolds with corners $H \times \bar{A}_{P*Q} \xrightarrow{\simeq} \bar{A}_Q(P * Q)$, $(a, a') \mapsto aa'$. The diagram

$$\begin{array}{ccc} D \times \bar{A}_Q(P * Q) & \to & D \times \bar{A}_P \\ \downarrow & & \downarrow \\ D_{\mathrm{BS}}(P) & = & D_{\mathrm{BS}}(P) \end{array}$$

is commutative, where the upper horizontal arrow is the real analytic map $(F, aa') \mapsto (a \circ F, a')$ $(a \in H, a' \in \bar{A}_{P*Q} \subset \bar{A}_P)$. (Here the inclusion $\bar{A}_{P*Q} \subset \bar{A}_P$ is induced from the surjective homomorphism $X(S_P)^+_{\mathbf{Q}} \to X(S_{P*Q})^+_{\mathbf{Q}}$.)

This shows

(iv) The bijection $D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \bar{A}_P$ in 2.7 (1) is a homeomorphism. Here $D \times^{A_P} \bar{A}_P$ has the topology as a quotient space of $D \times \bar{A}_P$. For an open set U of $D_{\mathrm{BS}}(P)$ and a function $f : U \to \mathbf{R}$, f is real analytic if and only if the composition $U' \to U \xrightarrow{f} \mathbf{R}$ is real analytic where U' is the inverse image of U in $D \times \bar{A}_P$.

2.10. We give a proof of the fact that D_{BS} is a real analytic manifold with corners.

Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. Then there is a real analytic map $f : D \to A_P$ such that $f(a \circ F) = af(F)$ for any $a \in A_P$ and $F \in D$. The existence of f is shown as follows (see [KU2] (2.16)). Let ${}^{\circ}P$ be the intersection of the kernels of $|\chi| : P \to \mathbf{R}_{>0}$ for all homomorphisms of algebraic groups $\chi : P \to \mathbf{R}^{\times}$ defined over $\mathbf{Q}$. Then $P_u \subset {}^{\circ}P$ and the composition $A_P \to P/P_u \to P/{}^{\circ}P$ is an isomorphism ([BS] 1.2). Let $|\ | : P \to A_P$ be the composition $P \to P/{}^{\circ}P \simeq A_P$. Fix $\mathbf{r} \in D$. Then an example of f is defined as $pk\mathbf{r} \mapsto |p|$ $(p \in P,\ k \in K_{\mathbf{r}})$.

Let f be as above and let $D^{(1)} = \{F \in D \mid f(F) = 1\}$. Then $D^{(1)}$ is a closed real analytic submanifold of D (by "closed submanifold", we do not mean "compact submanifold" but just mean "submanifold which is closed") and we have an isomorphism of real analytic manifolds

$$D^{(1)} \times A_P \xrightarrow{\simeq} D, \quad (F, a) \mapsto a \circ F.$$

By the above (iv), this isomorphism extends uniquely to an isomorphism of ringed spaces

$$D^{(1)} \times \bar{A}_P \xrightarrow{\simeq} D_{\mathrm{BS}}(P).$$

Since D_{BS} is covered by open sets $D_{\mathrm{BS}}(P)$ when P varies, this shows that D_{BS} is a real analytic manifold with corners.

2.11. It is easy to see that the action of $G_{\mathbf{Z}}$ on D extends to a real analytic action of $G_{\mathbf{Z}}$ on D_{BS} (i.e., an action of $G_{\mathbf{Z}}$ as automorphisms of the real analytic manifold with corners D_{BS}). Theorem 0.2 in Introduction in the pure case is proved in [KU2] except the part where the structure of real analytic manifold with corners is concerned. This part follows from the other part easily. (Theorem 0.2 in the pure case also follows from the works [BJ1], [BJ2].)

§3. A rough picture of the mixed case

3.1. In this introductory section, we describe roughly the shape of D_{BS} in the mixed case, comparing it with $D_{\mathrm{BS}}(\mathrm{gr}^W) := \prod_{w\in\mathbf{Z}} D_{\mathrm{BS}}(\mathrm{gr}^W_w)$ where $D_{\mathrm{BS}}(\mathrm{gr}^W_w)$ is "the D_{BS} of the pure case $((H_0\cap W_w)/(H_0\cap W_{w-1}), \langle\ ,\ \rangle_w)$" in §2. The proofs of the statements concerning D_{BS} in this section are given later.

3.2. In this paper, the canonical projection $D \to D(\mathrm{gr}^W) := \prod_{w\in\mathbf{Z}} D(\mathrm{gr}^W_w)$ (1.8) will be extended to a surjective morphism $D_{\mathrm{BS}} \to D_{\mathrm{BS}}(\mathrm{gr}^W)$ of real analytic manifolds with corners. We describe roughly the shape of D_{BS} as a fiber space over $D_{\mathrm{BS}}(\mathrm{gr}^W)$.

3.3. Let $\mathrm{spl}(W)$ be the set of all splittings of W. That is, $\mathrm{spl}(W)$ is the set of all isomorphisms

$$s : \mathrm{gr}^W = \bigoplus_w \mathrm{gr}^W_w \xrightarrow{\simeq} H_{0,\mathbf{R}}$$

of $\mathbf{R}$-vector spaces such that for any $w \in \mathbf{Z}$ and $v \in \mathrm{gr}^W_w$, $s(v) \in W_w$ and $v = (s(v) \bmod W_{w-1})$.

For $g \in G_{\mathbf{R},u}$ and $s \in \mathrm{spl}(W)$, the isomorphism $gs : \mathrm{gr}^W \xrightarrow{\simeq} H_{0,\mathbf{R}}$ is also a splitting of W. For this action of $G_{\mathbf{R},u}$ on $\mathrm{spl}(W)$, $\mathrm{spl}(W)$ is a $G_{\mathbf{R},u}$-torsor, that is, for a fixed $s \in \mathrm{spl}(W)$, we have a bijection $G_{\mathbf{R},u} \xrightarrow{\simeq} \mathrm{spl}(W)$, $g \mapsto gs$. Via this bijection, we endow $\mathrm{spl}(W)$ with a structure of a real analytic manifold (which is independent of the choice of s fixed here).

For $F = (F_w)_w \in D(\mathrm{gr}^W)$ ($F_w \in D(\mathrm{gr}^W_w)$) and $s \in \mathrm{spl}(W)$, we have $s(F) \in D$ defined by $s(F)^p = s(\bigoplus_w F^p_w)$ ($p \in \mathbf{Z}$). Let D_{spl} be the subset of D consisting of all elements obtained in this way. Then D_{spl} is a closed real analytic submanifold of D. We have an isomorphism of real analytic manifolds

$$\mathrm{spl}(W) \times D(\mathrm{gr}^W) \xrightarrow{\simeq} D_{\mathrm{spl}}, \quad (s, F) \mapsto s(F).$$

An element of D is said to be $\mathbf{R}$-split if it belongs to D_{spl}.

3.4. As is shown in Proposition 8.7, the following three conditions are equivalent. (1) D is a $G_{\mathbf{R}}$-homogenious space. (2) D is a $G'_{\mathbf{R}}$-homogeneous space. Here $G'_{\mathbf{R}}$ is as in 0.3. (3) $D = D_{\mathrm{spl}}$.

Example II in §1 satisfies these equivalent conditions, but Examples I and III do not.

In the case $D = D_{\mathrm{spl}}$, we have $D = \mathrm{spl}(W) \times D(\mathrm{gr}^W)$. We will have $D_{\mathrm{BS}} = \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ in this case (Proposition 8.7). Thus in this case, for $F \in D(\mathrm{gr}^W)$, the fiber of $D_{\mathrm{BS}} \to D_{\mathrm{BS}}(\mathrm{gr}^W)$ on F is the same as the fiber of $D \to D(\mathrm{gr}^W)$ on F.

However, in the case $D \neq D_{\mathrm{spl}}$, for $F \in D(\mathrm{gr}^W)$, the fiber of $D_{\mathrm{BS}} \to D_{\mathrm{BS}}(\mathrm{gr}^W)$ on F is actually bigger than the fiber of $D \to D(\mathrm{gr}^W)$ on F.

For example, in Example I, $D_{\mathrm{BS}}(\mathrm{gr}^W) = D(\mathrm{gr}^W)$ and it is a one point set, and $D = \mathbf{C}$, $D_{\mathrm{spl}} = \mathbf{R}$. The subgroup $\Gamma := G_{\mathbf{Z},u} \simeq \mathbf{Z}$ of $G_{\mathbf{Z}}$ is of finite index in $G_{\mathbf{Z}}$, and so $\Gamma\backslash D_{\mathrm{BS}}$ should be compact as in (ii) in Theorem 0.2 in Introduction. Since $\Gamma\backslash D = \mathbf{Z}\backslash\mathbf{C}$ is not compact, we need to add new points to D to obtain our D_{BS}. As is explained in 10.1, in this case,

$$D_{\mathrm{BS}} = \{x + iy \mid x \in \mathbf{R}, -\infty \le y \le \infty\} \supset D = \mathbf{C} = \{x + iy \mid x, y \in \mathbf{R}\},$$

and $\Gamma\backslash D_{\mathrm{BS}} \simeq \mathbf{Z}\backslash\mathbf{R} \times [-\infty, \infty]$ is compact.

3.5. In general, as a real analytic manifold, D is an L-bundle over $\mathrm{spl}(W) \times D(\mathrm{gr}^W)$ for some finite-dimensional graded $\mathbf{R}$-vector space L. (With the notation in 4.2, $L = L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$ for $\mathbf{r} \in D(\mathrm{gr}^W)$; all $L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$ are non-canonically isomorphic to each other as graded $\mathbf{R}$-vector spaces.)

We will have a compactification $\bar{L}$ of L (§7) which is a real analytic manifold with corners. As a real analytic manifold with corners, D_{BS} is an $\bar{L}$-bundle over $\mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ (Corollary 8.5). For example, in Example I, the map $D \to \mathrm{spl}(W) \times D(\mathrm{gr}^W)$ is identified with the projection $\mathbf{C} \to \mathbf{R}$, $z \mapsto \mathrm{Re}(z)$, and $L \simeq \mathbf{R}$, $\bar{L} \simeq [-\infty, \infty] \supset \mathbf{R}$ (see 10.1).

3.6. As explained in Introduction, an element of D_{BS} has the form (P, Z) where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and Z is either an A_P-orbit or a B_P-orbit in D for the Borel-Serre action. As is shown in 7.6, Z is an A_P-orbit if and only if in the above local isomorphism between D_{BS} and $\mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W) \times \bar{L}$, the component of (P, Z) in $\bar{L}$ belongs to L.

§4. Canonical splittings of weight filtrations

4.1. In this section, we review the canonical splitting $s = \mathrm{spl}_W(F) \in \mathrm{spl}(W)$ associated to $F \in D$, defined by the theory of Cattani-Kaplan-Schmid [CKS]. This element s played an important role in our previous paper [KNU]. The definition of s was reviewed in detail in section 1 of [KNU].

4.2. Let $F = (F_w)_w \in D(\mathrm{gr}^W)$. Regard F as the filtration $\bigoplus_w F_w$ on $\mathrm{gr}^W_{\mathbf{C}} = \bigoplus_w \mathrm{gr}^W_{w,\mathbf{C}}$, and let $H_F^{p,q} = H_{F_{p+q}}^{p,q} \subset \mathrm{gr}^W_{\mathbf{C}}$. Let

$$L_{\mathbf{R}}^{-1,-1}(F) = \{\delta \in \mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W) \mid \delta(H_F^{p,q}) \subset \textstyle\bigoplus_{p'<p,q'<q} H_F^{p',q'} \text{ for all } p, q \in \mathbf{Z}\}.$$

All elements of $L_{\mathbf{R}}^{-1,-1}(F)$ are nilpotent.

Let $F \in D$. For the canonical splitting $s = \mathrm{spl}_W(F)$ of W associated to F, there is a unique pair (δ, ζ) of elements of $L_{\mathbf{R}}^{-1,-1}(F(\mathrm{gr}^W))$ such that

$$F = s(\exp(-\zeta)\exp(i\delta)F(\mathrm{gr}^W)).$$

Here, $\exp(-\zeta)\exp(i\delta)$ is defined as an automorphism of $\mathrm{gr}^W_{\mathbf{C}}$, and $\exp(-\zeta)\exp(i\delta)F(\mathrm{gr}^W)$ is defined as a decreasing filtration on $\mathrm{gr}^W_{\mathbf{C}}$. This filtration need not be a direct sum of filtrations on $\mathrm{gr}^W_{w,\mathbf{C}}$ for $w \in \mathbf{Z}$.

For $F \in D$, we introduce the definition of the associated element δ first, that of the associated element ζ next, and then give the definition of the associated splitting s.

4.3. For $F \in D$, there is a unique pair $(s', \delta) \in \mathrm{spl}(W) \times L_{\mathbf{R}}^{-1,-1}(F(\mathrm{gr}^W))$ such that

$$F = s'(\exp(i\delta)F(\mathrm{gr}^W)).$$

This is the definition of $\delta = \delta(F)$ associated to F.

The definition of $\zeta = \zeta(F)$ is rather complicated. It is given as a universal Lie polynomial in the (p,q)-Hodge components of δ $(p, q \in \mathbf{Z})$ for $F(\mathrm{gr}^W)$ as is explained below.

The maps

$$D \to \mathrm{End}(\mathrm{gr}^W), \quad F \mapsto \delta(F), \quad F \mapsto \zeta(F)$$

are real analytic.

4.4. Let $b^l_{p,q}$ $(p, q, l \in \mathbf{Z},\ p, q, l \geq 0)$ be the integers determined by $(1-x)^p(1+x)^q = \sum_l b^l_{p,q} x^l$, so that $b^l_{p,q} = 0$ unless $p + q \geq l$.

Define non-commutative polynomials $P_k = P_k(X_2, \ldots, X_{k+1})$ over $\mathbf{Q}$ by $P_0 = 1$, $P_k = -\frac{1}{k}\sum_{j=1}^{k} P_{k-j}X_{j+1}$ $(k \geq 1)$. (So $P_1 = -X_2$, $P_2 = \frac{1}{2}X_2^2 - \frac{1}{2}X_3$, $P_3 = -\frac{1}{6}X_2^3 + \frac{1}{6}X_3X_2 + \frac{1}{3}X_2X_3 - \frac{1}{3}X_4$, etc.)

Let A be the ring of non-commutative polynomials in variables $\delta_{-p,-q}$ $(p \geq 1, q \geq 1)$ over $\mathbf{Q}(i)$. For $p, q \geq 1$, let $A_{-p,-q}$ be the part of A consisting of linear combinations over $\mathbf{Q}(i)$ of products of the form $\delta_{-p_1,-q_1} \cdots \delta_{-p_k,-q_k}$ with $p = \sum_j p_j$, $q = \sum_j q_j$. Then A is the direct sum of the $A_{-p,-q}$ and $\mathbf{Q}(i)$ as a $\mathbf{Q}(i)$-module.

In [CKS] (6.60), it is proved that there exists a unique family of elements $\zeta_{-p,-q}$ and $\eta_{-p,-q}$ of $A_{-p,-q}$ $(p, q \geq 1)$ satisfying the following two conditions (1) and (2).

(1) Let $\hat{A}$ be the formal completion $\varprojlim_k A/I^k$, where I^k denotes the sum of $A_{-p,-q}$ such that $p + q \geq k$. Let $\zeta = \sum_{p,q} \zeta_{-p,-q}$, $\eta = \sum_{p,q} \eta_{-p,-q} \in \hat{A}$. Then we have an identity in $\hat{A}$

$$\exp(-\zeta)\exp(i\delta) = \textstyle\sum_{k\geq 0} P_k(C_2, \ldots, C_{k+1}),$$

where $C_{l+1} = i \sum\limits_{p,q\geq 1} b^{l-1}_{p-1,q-1}\eta_{-p,-q}$.

(2) By the unique ring homomorphism $A \to A$ which sends i to $-i$ and $\delta_{-p,-q}$ to $\delta_{-q,-p}$, the element $\zeta_{-p,-q}$ is sent to $\zeta_{-q,-p}$, and $\eta_{-p,-q}$ is sent to $\eta_{-q,-p}$.

For example, we have

$$\zeta_{-1,-1} = 0,$$
$$\zeta_{-2,-1} = \tfrac{i}{2}\delta_{-2,-1},$$
$$\zeta_{-1,-2} = -\tfrac{i}{2}\delta_{-1,-2}.$$

It can be shown that $\zeta_{p,q}$ are Lie polynomials in $\delta_{k,l}$ $(k, l \leq -1)$.

4.5. For $F \in D(\mathrm{gr}^W)$ and for $\delta \in L_{\mathbf{R}}^{-1,-1}(F)$, let $\delta_{p,q}$ $(p, q \in \mathbf{Z})$ be the (p, q)-Hodge component of δ defined by

$$\delta = \textstyle\sum_{p,q}\ \delta_{p,q} \quad (\delta_{p,q} \in L_{\mathbf{C}}^{-1,-1}(F) = \mathbf{C} \otimes_{\mathbf{R}} L_{\mathbf{R}}^{-1,-1}(F)),$$

$$\delta_{p,q}(H_F^{k,l}) \subset H_F^{k+p,l+q} \text{ for all } k, l \in \mathbf{Z}.$$

We define the element $\zeta \in L_{\mathbf{R}}^{-1,-1}(F)$ associated to the pair (F, δ), as the element whose (p, q)-Hodge component $\zeta_{p,q}$ for F for each $p, q \in \mathbf{Z}$ is the Lie polynomial in $\delta_{k,l}$ $(k, l \leq -1)$ given in 4.4.

4.6. For $F \in D$, the associated δ, ζ, s are as follows. The element δ is already given in 4.3. The element ζ is defined to be the element of $L_{\mathbf{R}}^{-1,-1}(F(\mathrm{gr}^W))$ associated to the pair $(F(\mathrm{gr}^W), \delta)$ as in 4.5. Finally the *canonical splitting* $s = \mathrm{spl}_W(F)$ of W is defined by

$$s = s' \exp(\zeta)$$

where s' is as in 4.3.

4.7. For $F \in D$, the elements $s'\delta(s')^{-1}, s'\zeta(s')^{-1} \in \mathfrak{g}_{\mathbf{R}}$ here are denoted in [CKS] and also in [KNU] as δ, ζ, respectively. Here $s'\delta(s')^{-1}$ is understood as the composition

$$H_{0,\mathbf{R}} \xrightarrow{(s')^{-1}} \mathrm{gr}^W \xrightarrow{\delta} \mathrm{gr}^W \xrightarrow{s'} H_{0,\mathbf{R}}.$$

For these elements, we have

$$F = \exp(is'\delta(s')^{-1})s'(F(\mathrm{gr}^W)) = \exp(is'\delta(s')^{-1})\exp(-s'\zeta(s')^{-1})s(F(\mathrm{gr}^W)).$$

4.8. For $F \in D(\mathrm{gr}^W)$ and $\delta \in L_{\mathbf{R}}^{-1,-1}(F)$, we define a filtration $\theta(F, \delta)$ on $\mathrm{gr}^W_{\mathbf{C}}$ by

$$\theta(F, \delta) = \exp(-\zeta)\exp(i\delta)F,$$

where ζ is the element of $L_{\mathbf{R}}^{-1,-1}(F)$ associated to the pair (F, δ) in 4.5. For $s \in \mathrm{spl}(W)$, the δ, ζ, s associated to $s(\theta(F, \delta))$ are just δ, ζ, s.

Proposition 4.9. *We have an isomorphism of real analytic manifolds*

$$D \simeq \{(s, F, \delta) \in \mathrm{spl}(W) \times D(\mathrm{gr}^W) \times \mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W) \mid \delta \in L_{\mathbf{R}}^{-1,-1}(F)\},$$

$$F \leftrightarrow (\mathrm{spl}_W(F), F(\mathrm{gr}^W), \delta(F)), \quad s(\theta(F,\delta)) \leftrightarrow (s, F, \delta).$$

4.10. For $g = (g_w)_w \in G_{\mathbf{R}}(\mathrm{gr}^W) = \prod_w G_{\mathbf{R}}(\mathrm{gr}^W_w)$, we have

$$g\theta(F,\delta) = \theta(gF, \mathrm{Ad}(g)\delta),$$

where $\mathrm{Ad}(g)\delta = g\delta g^{-1}$.

4.11. For $F \in D(\mathrm{gr}^W)$, $\delta \in L_{\mathbf{R}}^{-1,-1}(F)$ and $s \in \mathrm{spl}(W)$, the element $s(\theta(F,\delta))$ of D belongs to D_{spl} if and only if $\delta = 0$.

§5. Definition of D_{BS}

5.1. Note that a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$ corresponds in one-to-one manner to a family $(P_w)_{w\in\mathbf{Z}}$ of $\mathbf{Q}$-parabolic subgroups P_w of $G_{\mathbf{R}}(\mathrm{gr}^W_w)$. The correspondence is that P is the inverse image of $\prod_{w\in\mathbf{Z}} P_w$ under $G_{\mathbf{R}} \to \prod_{w\in\mathbf{Z}} G_{\mathbf{R}}(\mathrm{gr}^W_w)$. We will denote $\prod_{w\in\mathbf{Z}} P_w$ as $P(\mathrm{gr}^W)$.

Let

$$S_P := \prod_{w\in\mathbf{Z}} S_{P_w}, \quad A_P := \prod_{w\in\mathbf{Z}} A_{P_w} \simeq \mathbf{R}^n_{>0}, \quad \bar{A}_P := \prod_{w\in\mathbf{Z}} \bar{A}_{P_w} \simeq \mathbf{R}^n_{\geq 0}$$

where n is the rank of S_P. Here S_{P_w}, A_{P_w}, and $\bar{A}_{P_w}$ are defined as in §2 for the pure situation $((H_0 \cap W_w)/(H_0 \cap W_{w-1}), \langle\ ,\ \rangle_w, P_w)$.

Let

$$B_P := \mathbf{R}_{>0} \times A_P \simeq \mathbf{R}^{n+1}_{>0}, \quad \bar{B}_P := \mathbf{R}_{\geq 0} \times \bar{A}_P \simeq \mathbf{R}^{n+1}_{\geq 0}.$$

We regard

$$A_P = \{1\} \times A_P \subset B_P, \quad \bar{A}_P = \{1\} \times \bar{A}_P \subset \bar{B}_P.$$

5.2. We have the Borel-Serre action $\circ$ of B_P on D defined as follows. This is the mixed Hodge theoretic version of the Borel-Serre action of A_{P_w} on $D(\mathrm{gr}^W_w)$ in §2.

For $F \in D$ and $b = (c, a) \in B_P$ ($c \in \mathbf{R}_{>0}$, $a = (a_w)_{w\in\mathbf{Z}} \in A_P$ with $a_w \in A_{P_w}$), we define $b \circ F = b_F F$, where $b_F \in \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ is as follows. For the canonical splitting $s : \mathrm{gr}^W \xrightarrow{\simeq} H_{0,\mathbf{R}}$ of W associated to F (§4), b_F acts on the weight w-summand $s(\mathrm{gr}^W_w)$ of $H_{0,\mathbf{R}}$ as $c^w a_{w,F(\mathrm{gr}^W_w)}$ where $a_{w,F(\mathrm{gr}^W_w)} \in P_w$ is the Borel-Serre lifting (2.3) of a_w at $F(\mathrm{gr}^W_w)$.

For $b \in B_P$ and $F \in D$, we call b_F *the Borel-Serre lifting of* b *at* F.

The map $B_P \times D \to D$, $(b, F) \mapsto b \circ F$ is actually an action of B_P on D, as is reduced easily to the pure case.

For $b=(c,a)\in B_P$, $s\in \mathrm{spl}(W)$, $F=(F_w)_w\in D(\mathrm{gr}^W)$ and $\delta\in L_{\mathbf{R}}^{-1,-1}(F)$, we have

(1) $b\circ s(\theta(F,\delta))=a\circ s(\theta(F,c\circ\delta))=s((\bigoplus_w a_{w,F_w})\theta(F,c\circ\delta))=s(\theta(a\circ F,c\circ \mathrm{Ad}(a_F)\delta))$.

Here $c\circ\delta=\sum_w c^w\delta_w$ with δ_w the part of weight w of δ, $a\circ F=(a_w\circ F_w)_w$, and $\mathrm{Ad}(a_F)=\bigoplus_w \mathrm{Ad}(a_{w,F_w})$.

The Borel-Serre action of B_P on D_{spl} factors through the projection $B_P\to A_P$, but not so on the rest of D.

Via the projection $D\to D(\mathrm{gr}^W)$, the Borel-Serre action of B_P on D is compatible with the Borel-Serre action of A_P on $D(\mathrm{gr}^W)$ through the projection $B_P\to A_P$.

If Q is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ containing P, the canonical homomorphism $A_Q\to A_P$ (2.8) and the induced homomorphism $B_Q\to B_P$ are compatible with the Borel-Serre actions of these groups on D.

Definition 5.3. We define D_{BS} to be the set of all pairs (P,Z) where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ and Z is either an A_P-orbit or a B_P-orbit in D for the Borel-Serre action.

Note that for $F\in D$, the A_P-orbit in D containing F and the B_P-orbit in D containing F coincide if and only if $F\in D_{\mathrm{spl}}$.

5.4. We have a canonical map

$$D_{\mathrm{BS}}\to D_{\mathrm{BS}}(\mathrm{gr}^W),\ \ (P,Z)\mapsto (P_w,Z_w)_w$$

where $Z_w=\{F(\mathrm{gr}^W_w)\mid F\in Z\}$, which we denote as $p\mapsto p(\mathrm{gr}^W)$.

5.5. We have a canonical map

$$\mathrm{spl}_W: D_{\mathrm{BS}}\to \mathrm{spl}(W)$$

sending $(P,Z)\in D_{\mathrm{BS}}$ to the canonical splitting of W associated to $F\in Z$ (§4), which is independent of the choice of $F\in Z$ by 5.2 (1).

Combining these, we have a canonical map

$$D_{\mathrm{BS}}\to \mathrm{spl}(W)\times D_{\mathrm{BS}}(\mathrm{gr}^W),\quad p\mapsto (\mathrm{spl}_W(p),\ p(\mathrm{gr}^W)).$$

§6. The real analytic structure with corners

In this section, we define a structure of a ringed space over $\mathbf{R}$ on D_{BS} and lead to the theorem that D_{BS} is a real analytic manifold with corners.

6.1. For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$, let

$$D_{\mathrm{BS}}(P):=\{(Q,Z)\in D_{\mathrm{BS}}\mid Q\supset P\}.$$

Write $P(\mathrm{gr}^W) = \prod_w P_w$. Then $D_{\mathrm{BS}}(P)$ is the inverse image of $D_{\mathrm{BS}}(\mathrm{gr}^W)(P(\mathrm{gr}^W)) := \prod_w D_{\mathrm{BS}}(\mathrm{gr}^W_w)(P_w)$ under the canonical map $D_{\mathrm{BS}} \to D_{\mathrm{BS}}(\mathrm{gr}^W)$.

We have a canonical bijection

$$D_{\mathrm{BS}}(P) \simeq ((D - D_{\mathrm{spl}}) \times^{B_P} \bar{B}_P) \cup (D \times^{A_P} \bar{A}_P). \tag{1}$$

Here the union in the right-hand side is taken in $D \times^{B_P} \bar{B}_P$, where $D \times^{A_P} \bar{A}_P$ is embedded in $D \times^{B_P} \bar{B}_P$ via the identifications $A_P = \{1\} \times A_P \subset B_P$, $\bar{A}_P = \{1\} \times \bar{A}_P \subset \bar{B}_P$. Hence, in the right-hand side, the intersection $((D - D_{\mathrm{spl}}) \times^{B_P} \bar{B}_P) \cap (D \times^{A_P} \bar{A}_P)$ is $(D - D_{\mathrm{spl}}) \times^{A_P} \bar{A}_P$.

The bijection (1) is defined as follows. Let $X(S_P) := \prod_w X(S_{P_w})$ and $X(S_P)^+_{\mathbf{Q}} := \prod_w X(S_{P_w})^+_{\mathbf{Q}}$. For $F \in D$ and $a = (a_w)_w \in \bar{A}_P$ (resp. $F \in D - D_{\mathrm{spl}}$ and $b = (0, a) \in \bar{B}_P$ with $a = (a_w)_w \in \bar{A}_P$), the corresponding $(Q, Z) \in D_{\mathrm{BS}}(P)$ is given as follows. For $w \in \mathbf{Z}$, let (Q_w, Z_w) be the element of $D_{\mathrm{BS}}(\mathrm{gr}^W_w)(P_w)$ corresponding to $(F(\mathrm{gr}^W_w), a_w)$ (2.7). Then Q is the parabolic subgroup of $G_{\mathbf{R}}$ corresponding to $(Q_w)_w$. Z is the A_Q (resp. B_Q)-orbit in D defined by

$$Z = \{a' \circ F \mid a' \in A_P, a'(\chi) = a(\chi) \text{ for all } \chi \in S\}$$

$$(\text{resp. } Z = \{(c, a') \circ F \mid c > 0, a' \in A_P, a'(\chi) = a(\chi) \text{ for all } \chi \in S\}),$$

where $S = \mathrm{Ker}(X(S_P)^+_{\mathbf{Q}} \to X(S_Q)^+_{\mathbf{Q}})$. Conversely, $(Q, Z) \in D_{\mathrm{BS}}(P)$ corresponds to the following element of the right-hand side of (1). Assume Z is an A_Q-orbit (resp. a B_Q-orbit but not an A_Q-orbit). Take $F \in Z$. Then the corresponding element of the right-hand side of (1) is the class of (F, a) (resp. (F, b) with $b = (0, a) \in \bar{B}_P$), where $a \in \bar{A}_P = \mathrm{Hom}(X(S_P)^+_{\mathbf{Q}}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ is defined as follows. For $\chi = (\chi_w)_w \in X(S_P)^+_{\mathbf{Q}}$, $a(\chi)$ is 1 if χ_w belongs to the face of $X(S_{P_w})^+_{\mathbf{Q}}$ corresponding to Q_w (2.8) for any w, and is 0 otherwise.

6.2. We define the topology of D_{BS} and the sheaf of rings of real analytic functions on D_{BS}.

By 6.1 (1), we have a canonical surjection

$$C := \textstyle\bigsqcup_P \ (((D - D_{\mathrm{spl}}) \times \bar{B}_P) \sqcup (D \times \bar{A}_P)) \to D_{\mathrm{BS}},$$

where P ranges over all $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$, and C is a real analytic manifold with corners.

We define the topology of D_{BS} as the quotient of the topology of C via the above surjection.

We define the sheaf of rings of real analytic functions on D_{BS} as follows. For an open set U of D_{BS} and a function $f : U \to \mathbf{R}$, f is real analytic if and only if the composition $U' \to U \xrightarrow{f} \mathbf{R}$, where U' denotes the inverse image of U in C, is real analytic.

The following will be proved in §8.

Theorem 6.3. *With the sheaf of rings over* $\mathbf{R}$ *defined in 6.2,* D_{BS} *is a real analytic manifold with corners.*

Proposition 6.4. (i) *The projection* $D_{\mathrm{BS}} \to \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ *is real analytic, that is, it is a morphism of real analytic manifolds with corners.*

(ii) *Let* P *be a* $\mathbf{Q}$*-parabolic subgroup of* $G_{\mathbf{R}}$*. Then the action of* P *on* D *extends uniquely to a real analytic action of* P *on* $D_{\mathrm{BS}}(P)$*.*

(iii) *Regard the subgroup* $G_{\mathbf{Q}}G_{\mathbf{R},u}$ *of* $G_{\mathbf{R}}$ *as a Lie group in which* $G_{\mathbf{R},u}$ *is an open subgroup with the usual topology. Then the action of* $G_{\mathbf{Q}}G_{\mathbf{R},u}$ *on* D *extends uniquely to a real analytic action of* $G_{\mathbf{Q}}G_{\mathbf{R},u}$ *on* D_{BS}*.*

§7. Compactified graded real vector spaces

This section gives a result needed for §8.

Let V be a finite-dimensional $\mathbf{R}$-vector space endowed with a direct sum decomposition $V = \bigoplus_{w\in\mathbf{Z},w\le -1} V_w$. In this section, we define a compactification $\bar{V}$ of V which is a real analytic manifold with corners. (In fact, $\bar{V}$ is actually a real analytic manifold "with boundary", for the boundary of $\bar{V}$ is smooth. As a differentiable manifold with boundary, $\bar{V}$ is nothing but the spherical compactification of V.) At the end 7.6 of this section, we explain how this compactification is used in this paper.

7.1. Consider the action

$$\mathbf{R}_{>0} \times V \to V, \quad (c, v) \mapsto c \circ v := \textstyle\sum_{w\in\mathbf{Z}} c^w v_w$$

of the group $\mathbf{R}_{>0}$ on V where v_w denotes the V_w-component of v.

Let $\bar{V}$ be the set of all subsets of V which are either a one point set or an $\mathbf{R}_{>0}$-orbit for this action.

7.2. We have a canonical bijection

$$\bar{V} \simeq ((V - \{0\}) \times^{\mathbf{R}_{>0}} \mathbf{R}_{\ge 0}) \cup V, \tag{1}$$

where in the right-hand side $\mathbf{R}_{>0}$ acts on $V - \{0\}$ by $\circ$ as in 7.1. Here the union in the right-hand side is taken in $V \times^{\mathbf{R}_{>0}} \mathbf{R}_{\ge 0}$. So the intersection $((V - \{0\}) \times^{\mathbf{R}_{>0}} \mathbf{R}_{\ge 0}) \cap V$ is $V - \{0\}$. In (1), the one point set $\{v\} \in \bar{V}$ for $v \in V$ corresponds to $v \in V$ in the right-hand side, and the $\mathbf{R}_{>0}$-orbit containing $v \in V - \{0\}$, regarded as an element of $\bar{V}$, corresponds to the class of $(v, 0) \in (V - \{0\}) \times \mathbf{R}_{\ge 0}$ in the right-hand side.

Via the bijection (1), we will identify $\bar{V}$ with a subset of $V \times^{\mathbf{R}_{>0}} \mathbf{R}_{\ge 0}$ and identify $\bar{V} - \{0\}$ with $(V - \{0\}) \times^{\mathbf{R}_{>0}} \mathbf{R}_{\ge 0}$.

7.3. We define a structure of a ringed space over $\mathbf{R}$ on $\bar{V}$. By 7.2, we have a canonical surjection

$$C := ((V - \{0\}) \times \mathbf{R}_{\ge 0}) \sqcup V \to \bar{V},$$

and C is a real analytic manifold with corners.

We define the topology of $\bar{V}$ as the quotient of the topology of C via the above surjection.

We define the sheaf of real analytic functions on $\bar{V}$ is as follows. For an open set U of $\bar{V}$, a function $f : U \to \mathbf{R}$ is real analytic if and only if the composition $U' \to U \xrightarrow{f} \mathbf{R}$, where U' denotes the inverse image of U in C, is real analytic.

As is easily seen, we have:

(i) $\bar{V} - \{0\}$ and V are open in $\bar{V}$.

(ii) The restriction of the topology of $\bar{V}$ to V coincides with the original topology of V. For an open set U of V, a function $f : U \to \mathbf{R}$ is real analytic in the usual sense if and only if it is real analytic in the above sense when U is regarded as an open set of $\bar{V}$.

(iii) The topology of $\bar{V} - \{0\}$ as a subspace of $\bar{V}$ coincides with the quotient of the topology of $(V - \{0\}) \times \mathbf{R}_{\geq 0}$. For an open set U of $\bar{V} - \{0\}$ and for a function $f : U \to \mathbf{R}$, f is real analytic in the above sense if and only if the composition $U' \to U \xrightarrow{f} \mathbf{R}$ is real analytic, where U' denotes the inverse image of U in $(V - \{0\}) \times \mathbf{R}_{\geq 0}$.

Proposition 7.4. *The ringed space $\bar{V}$ over $\mathbf{R}$ is a real analytic manifold with corners. It is compact.*

Proof. Take a positive definite symmetric $\mathbf{R}$-bilinear form $(\ ,\ )_w : V_w \times V_w \to \mathbf{R}$ for each w. Take an integer $m < 0$ which satisfies $m \in w\mathbf{Z}$ for any $w \in \mathbf{Z}$ such that $V_w \neq 0$. Consider the real analytic function

$$f : V - \{0\} \to \mathbf{R}_{>0}\ ,\ v \mapsto (\textstyle\sum_{w \leq -1} (v_w, v_w)_w^{m/w})^{1/2m}.$$

Then $f(c \circ v) = cf(v)$ for any $c \in \mathbf{R}_{>0}$ and $v \in V - \{0\}$.

Let $V^{(1)} = \{v \in V \mid f(v) = 1\}$. Then $V^{(1)}$ is a closed real analytic submanifold of $V - \{0\}$, and we have an isomorphism of real analytic manifolds

$$(1) \qquad V^{(1)} \times \mathbf{R}_{>0} \xrightarrow{\simeq} V - \{0\}, \quad (v, c) \mapsto c \circ v.$$

The inverse map is given by $v \mapsto (f(v)^{-1} \circ v, f(v))$. The isomorphism (1) extends to an isomorphism of ringed spaces over $\mathbf{R}$

$$(2) \qquad V^{(1)} \times \mathbf{R}_{\geq 0} \simeq \bar{V} - \{0\}$$

which sends $(v, 0)$ $(v \in V^{(1)})$ to $\{c \circ v \mid c \in \mathbf{R}_{>0}\} \in \bar{V}$. Hence $\bar{V} - \{0\}$ is a real analytic manifold with corners.

We prove $\bar{V}$ is compact. Note that $V^{(1)}$ is compact. The map (2) extends to a continuous map

$$V^{(1)} \times [0, \infty] \to \bar{V}$$

which sends (v, ∞) $(v \in V^{(1)})$ to 0. Via this map, $\bar{V}$ is homeomorphic to the quotient of the compact space $V^{(1)} \times [0, \infty]$ obtained by identifying all (v, ∞) $(v \in V^{(1)})$. Hence $\bar{V}$ is compact. $\Box$

7.5. Example. Let $w \in \mathbf{Z}$, $w \leq -1$, and consider the simplest case $V = V_w = \mathbf{R}$. Then we have a canonical isomorphism of real analytic manifolds with corners

$$\bar{V} \simeq [-\infty, \infty],$$

which sends the class of (v, c) in $\bar{V}$ $(v \in V = \mathbf{R},\ c \in \mathbf{R}_{\geq 0},\ (v, c) \neq (0, 0))$ to $c^w v$ if $c \neq 0$, to ∞ if $c = 0$ and $v > 0$, and to $-\infty$ if $c = 0$ and $v < 0$. Here we endow $[-\infty, \infty]$ with the following structure of a real analytic manifold with corners. The topology of $[-\infty, \infty]$ is the usual topology, we regard the open set $\mathbf{R}$ of $[-\infty, \infty]$ as a real analytic manifold in the usual way, and we regard $(0, \infty]$ (resp. $[-\infty, 0)$) as a real analytic manifold with corners via the bijection $\mathbf{R}_{\geq 0} \xrightarrow{\simeq} (0, \infty]$, $r \mapsto r^w$ (resp. $\mathbf{R}_{\geq 0} \xrightarrow{\simeq} [-\infty, 0)$, $r \mapsto -r^w$).

The compactified vector space of this section is used in this paper in the following way.

Proposition 7.6. *Let $s \in \operatorname{spl}(W)$ and $(P, Z) \in D_{\mathrm{BS}}(\operatorname{gr}^W)$. Fix $F \in Z$ and let L be the graded vector space $L_{\mathbf{R}}^{-1,-1}(F)$ of weights ≤ -2.*

(i) *There is a bijection from the fiber of the map $D_{\mathrm{BS}} \to \operatorname{spl}(W) \times D_{\mathrm{BS}}(\operatorname{gr}^W)$ (5.5) over $(s, (P, Z))$ onto the compactified vector space $\bar{L}$ given in the following way. An element $(\tilde{P}, \tilde{Z}) \in D_{\mathrm{BS}}$ in this fiber corresponds to the subset $\{\delta \in L \mid s(\theta(F, \delta)) \in \tilde{Z}\}$ of L, which is an element of $\bar{L}$. Here $\tilde{P}$ is the inverse image of $P \subset G_{\mathbf{R}}(\operatorname{gr}^W)$ under $G_{\mathbf{R}} \to G_{\mathbf{R}}(\operatorname{gr}^W)$.*

(ii) *We have the following equivalences. Let $(\tilde{P}, \tilde{Z})$ be an element of this fiber and let $v \in \bar{L}$ be the corresponding element.*

$\tilde{Z}$ is an $A_{\tilde{P}}$-orbit $\iff$ $v \in L$.

$\tilde{Z}$ is a $B_{\tilde{P}}$-orbit but not an $A_{\tilde{P}}$-orbit $\iff$ $v \in \bar{L} - L$.

$\tilde{Z}$ is an $A_{\tilde{P}}$-orbit and also a $B_{\tilde{P}}$-orbit $\iff$ $v = 0$.

Proof. A point of the fiber is a pair $(\tilde{P}, \tilde{Z})$, where $\tilde{Z}$ is either an $A_{\tilde{P}}$-orbit or a $B_{\tilde{P}}$-orbit which contains $s(\theta(F, \delta_1))$ for some $\delta_1 \in L$. If it is an $A_{\tilde{P}}$-orbit, then $\tilde{Z} = \{s(\theta(a \circ F, \operatorname{Ad}(a_F)\delta_1)) \mid a \in A_P\}$, and hence $\{\delta \in L \mid s(\theta(F, \delta)) \in \tilde{Z}\} = \{\delta_1\}$. If it is a $B_{\tilde{P}}$-orbit, then $\tilde{Z} = \{s(\theta(a \circ F, c \circ \operatorname{Ad}(a_F)\delta_1)) \mid a \in A_P, c \in \mathbf{R}_{>0}\}$, and hence $\{\delta \in L \mid s(\theta(F, \delta)) \in \tilde{Z}\} = \{c \circ \delta_1 \mid c \in \mathbf{R}_{>0}\}$. These imply the desired statements. $\Box$

§8. Descriptions of D_{BS}

In this section, we prove results which describe how our space D_{BS} looks like.

8.1. We define two open sets $D_{\rm BS}^{(B)}$ and $D_{\rm BS}^{(A)}$ of $D_{\rm BS}$ such that $D_{\rm BS} = D_{\rm BS}^{(B)} \cup D_{\rm BS}^{(A)}$. Let $D_{\rm BS}^{(B)}$ be the subset of $D_{\rm BS}$ consisting of all elements of the form (P, Z) such that $Z \subset D - D_{\rm spl}$. Let $D_{\rm BS}^{(A)}$ be the subset of $D_{\rm BS}$ consisting of all elements of the form (P, Z) such that Z is an A_P-orbit. Note that $D_{\rm BS}^{(B)}$ is not the set of all $(P, Z) \in D_{\rm BS}$ such that Z is a B_P-orbit, but $D_{\rm BS}^{(B)}$ contains any $(P, Z) \in D_{\rm BS}$ such that Z is a B_P-orbit but not an A_P-orbit. In the bijection of 7.6 between the fiber of $D_{\rm BS} \to {\rm spl}(W) \times D_{\rm BS}({\rm gr}^W)$ and $\bar{L}$, if $p \in D_{\rm BS}$ is in the fiber and if $v \in \bar{L}$ is the image of p, then $p \in D_{\rm BS}^{(B)}$ if and only if $v \neq 0$, and $p \in D_{\rm BS}^{(A)}$ if and only if $v \in L \subset \bar{L}$. For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$, $D_{\rm BS}(P)^{(B)} := D_{\rm BS}(P) \cap D_{\rm BS}^{(B)}$ coincides with the image of $(D - D_{\rm spl}) \times^{B_P} \bar{B}_P \to D_{\rm BS}(P)$ and $D_{\rm BS}(P)^{(A)} := D_{\rm BS}(P) \cap D_{\rm BS}^{(A)}$ coincides with the image of $D \times^{A_P} \bar{A}_P \to D_{\rm BS}(P)$ under the bijection (1) in 6.1. Hence under the canonical maps $(D - D_{\rm spl}) \times \bar{B}_P \to D_{\rm BS}$ and $D \times \bar{A}_P \to D_{\rm BS}$ (6.2), the inverse image of $D_{\rm BS}^{(B)}$ in $(D - D_{\rm spl}) \times \bar{B}_P$ is $(D - D_{\rm spl}) \times \bar{B}_P$ itself, the inverse image of $D_{\rm BS}^{(B)}$ in $D \times \bar{A}_P$ is the open set $(D - D_{\rm spl}) \times \bar{A}_P$, the inverse image of $D_{\rm BS}^{(A)}$ in $(D - D_{\rm spl}) \times \bar{B}_P$ is the open set $(D - D_{\rm spl}) \times \mathbf{R}_{>0} \times \bar{A}_P$, and the inverse image of $D_{\rm BS}^{(A)}$ in $D \times \bar{A}_P$ is $D \times \bar{A}_P$ itself. From these we see that $D_{\rm BS}^{(B)}$ and $D_{\rm BS}^{(A)}$ are open in $D_{\rm BS}$.

We give some basic facts (i) and (iii), whose proofs are easy, and their consequences (ii) and (iv). These are the mixed versions of the corresponding statements in the pure case in 2.9.

Let P and Q be $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}$. Let $P * Q$ be the algebraic subgroup of $G_{\mathbf{R}}$ generated by P and Q, which is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$.

(i) Let $\bar{A}_Q(P*Q)$ be the open set of $\bar{A}_Q$ consisting of all homomorphisms $X(S_Q)^+_{\mathbf{Q}} \to \mathbf{R}^{\rm mult}_{\geq 0}$ which send ${\rm Ker}(X(S_Q)^+_{\mathbf{Q}} \to X(S_{P*Q})^+_{\mathbf{Q}})$ into $\mathbf{R}_{>0}$. Let $\bar{B}_Q(P*Q)$ be the open subset $\mathbf{R}_{\geq 0} \times \bar{A}_Q(P*Q)$ of $\bar{B}_Q$. Then under the canonical maps $(D - D_{\rm spl}) \times \bar{B}_Q \to D_{\rm BS}$ and $D \times \bar{A}_Q \to D_{\rm BS}$ (6.2), the inverse image of $D_{\rm BS}(P)^{(B)}$ in $(D - D_{\rm spl}) \times \bar{B}_Q$ is the open set $(D - D_{\rm spl}) \times \bar{B}_Q(P*Q)$, the inverse image of $D_{\rm BS}(P)^{(B)}$ in $D \times \bar{A}_Q$ is the open set $(D - D_{\rm spl}) \times \bar{A}_Q(P*Q)$, the inverse image of $D_{\rm BS}(P)^{(A)}$ in $(D - D_{\rm spl}) \times \bar{B}_Q$ is the open set $(D - D_{\rm spl}) \times \mathbf{R}_{>0} \times \bar{A}_Q(P*Q)$, and the inverse image of $D_{\rm BS}(P)^{(A)}$ in $D \times \bar{A}_Q$ is the open set $D \times \bar{A}_Q(P*Q)$.

This shows

(ii) $D_{\rm BS}(P)^{(B)}$ and $D_{\rm BS}(P)^{(A)}$ are open in $D_{\rm BS}$.

In fact, (ii) can also be deduced from the fact that $D_{\rm BS}^{(B)}$, $D_{\rm BS}^{(A)}$, and $D_{\rm BS}(P)$ are open sets of $D_{\rm BS}$. (The openness of $D_{\rm BS}(P)$ follows from the openness of $D_{\rm BS}({\rm gr}^W)(P({\rm gr}^W))$ since $D_{\rm BS}(P)$ is the inverse image of $D_{\rm BS}({\rm gr}^W)(P({\rm gr}^W))$ under the canonical map $D_{\rm BS} \to D_{\rm BS}({\rm gr}^W)$ which is continuous.)

(iii) Let the notation be as in (i). Take a homomorphism $X(S_{P*Q})^+_{\mathbf{Q}} \to X(S_Q)^+_{\mathbf{Q}}$ such that the composition $X(S_{P*Q})^+_{\mathbf{Q}} \to X(S_Q)^+_{\mathbf{Q}} \to X(S_{P*Q})^+_{\mathbf{Q}}$ is the identity, let $A_Q \to A_{P*Q}$ be the corresponding homomorphism, and let H be the kernel of the last homomorphism. Then we have an isomorphism of real analytic manifolds with corners $H \times \bar{B}_{P*Q} \xrightarrow{\simeq} \bar{B}_Q(P*Q),\ (a,b) \mapsto ab$ and $H \times \bar{A}_{P*Q} \simeq \bar{A}_Q(P*Q),\ (a,a') \mapsto aa'$. The following diagrams are commutative.

$$\begin{array}{ccc} (D-D_{\mathrm{spl}}) \times \bar{B}_Q(P*Q) & \to & (D-D_{\mathrm{spl}}) \times \bar{B}_P \\ \downarrow & & \downarrow \\ D_{\mathrm{BS}}(P) & = & D_{\mathrm{BS}}(P), \end{array} \qquad \begin{array}{ccc} D \times \bar{A}_Q(P*Q) & \to & D \times \bar{A}_P \\ \downarrow & & \downarrow \\ D_{\mathrm{BS}}(P) & = & D_{\mathrm{BS}}(P), \end{array}$$

where the upper horizontal arrows are the real analytic maps $(F, ab) \mapsto (a \circ F, b)$ $(a \in H, b \in \bar{B}_{P*Q} \subset \bar{B}_P)$ and $(F, aa') \mapsto (a \circ F, a')$ $(a \in H, a' \in \bar{A}_{P*Q} \subset \bar{A}_P)$, respectively. (Here the inclusions $\bar{B}_{P*Q} \subset \bar{B}_P$ and $\bar{A}_{P*Q} \subset \bar{A}_P$ are induced from the surjective homomorphism $X(S_P)^+_{\mathbf{Q}} \to X(S_{P*Q})^+_{\mathbf{Q}}$.)

This shows

(iv) The bijections $D_{\mathrm{BS}}(P)^{(B)} \simeq (D - D_{\mathrm{spl}}) \times^{B_P} \bar{B}_P$ and $D_{\mathrm{BS}}(P)^{(A)} \simeq D \times^{A_P} \bar{A}_P$ induced by the bijection 6.1 (1) are homeomorphisms. Here $(D - D_{\mathrm{spl}}) \times^{B_P} \bar{B}_P$ has the topology as a quotient space of $(D - D_{\mathrm{spl}}) \times \bar{B}_P$, and $D \times^{A_P} \bar{A}_P$ has the topology as a quotient space of $D \times \bar{A}_P$. For an open set U of $D_{\mathrm{BS}}(P)^{(B)}$ and a function $f : U \to \mathbf{R}$, f is real analytic if and only if the composition $U' \to U \xrightarrow{f} \mathbf{R}$ is real analytic, where U' is the inverse image of U in $(D - D_{\mathrm{spl}}) \times \bar{B}_P$. For an open set U of $D_{\mathrm{BS}}(P)^{(A)}$ and a function $f : U \to \mathbf{R}$, f is real analytic if and only if the composition $U' \to U \xrightarrow{f} \mathbf{R}$ is real analytic where U' is the inverse image of U in $D \times \bar{A}_P$.

8.2. We prove Theorem 6.3.

Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$.

From 2.10, for each $w \in \mathbf{Z}$, there is a closed real analytic submanifold $D(\mathrm{gr}^W_w)^{(1)}$ of $D(\mathrm{gr}^W_w)$ such that we have an isomorphism of real analytic manifolds $D(\mathrm{gr}^W_w)^{(1)} \times A_{P_w} \xrightarrow{\simeq} D(\mathrm{gr}^W_w),\ (F,a) \mapsto a \circ F$. Let $D(\mathrm{gr}^W)^{(1)} := \prod_w D(\mathrm{gr}^W_w)^{(1)}$, and let $D^{(1,A)}$ be the inverse image of $D(\mathrm{gr}^W)^{(1)}$ in D. Then $D^{(1,A)}$ is a closed real analytic submanifold of D, and we have an isomorphism of real analytic manifolds

$$D^{(1,A)} \times A_P \xrightarrow{\simeq} D,\ (F,a) \mapsto a \circ F.$$

By 8.1 (iv), this isomorphism extends uniquely to an isomorphism of ringed spaces

$$D^{(1,A)} \times \bar{A}_P \xrightarrow{\simeq} D_{\mathrm{BS}}(P)^{(A)}. \tag{1}$$

Fix $\mathbf{r} \in D(\mathrm{gr}^W)$. For $w \in \mathbf{Z}$, let L_w be the weight w part of $L = L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$, and fix a $K'_{\mathbf{r}_w}$-invariant positive definite symmetric $\mathbf{R}$-bilinear form $(\ ,\ )_w$ on L_w. Define $f : L - \{0\} \to \mathbf{R}_{>0}$ as in the proof of Proposition 7.4, replacing V there by L. For $F \in D(\mathrm{gr}^W)$, taking $g \in G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $F = g\mathbf{r}$, let

$$f_F = f \circ \mathrm{Ad}(g)^{-1} : L_{\mathbf{R}}^{-1,-1}(F) - \{0\} \to \mathbf{R}_{>0}.$$

Then, by the $K'_{\mathbf{r}_w}$-invariance of $(\ ,\ )_w$, this map f_F is independent of the choice of g. Let

$$D^{(1,B)} = \{s(\theta(F,\delta)) \mid F \in D(\mathrm{gr}^W)^{(1)}, \delta \neq 0,\ f_F(\delta) = 1\} \subset D^{(1,A)}.$$

Then $D^{(1,B)}$ is a closed real analytic submanifold of $D - D_{\mathrm{spl}}$, and we have an isomorphism of real analytic manifolds

$$D^{(1,B)} \times B_P \xrightarrow{\simeq} D - D_{\mathrm{spl}}, \quad (F, b) \mapsto b \circ F.$$

By 8.1 (iv), this isomorphism extends uniquely to an isomorphism of ringed spaces

(2) $$D^{(1,B)} \times \bar{B}_P \xrightarrow{\simeq} D_{\mathrm{BS}}(P)^{(B)}.$$

Since D_{BS} is covered by open sets $D_{\mathrm{BS}}(P)^{(A)}$ and $D_{\mathrm{BS}}(P)^{(B)}$ when P varies, (1) and (2) show that D_{BS} is a real analytic manifold with corners. This completes the proof of Theorem 6.3. □

Now Proposition 6.4 is straightforward, and we omit the proof.

In the following theorem, we describe the structure of D_{BS} relative to $D_{\mathrm{BS}}(\mathrm{gr}^W)$.

Theorem 8.3. *Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$, and write $P(\mathrm{gr}^W) = \prod_w P_w$. For each $w \in \mathbf{Z}$, take a closed real analytic submanifold $D(\mathrm{gr}_w^W)^{(1)}$ of $D(\mathrm{gr}_w^W)$ such that $D(\mathrm{gr}_w^W)^{(1)} \times A_{P_w} \xrightarrow{\simeq} D(\mathrm{gr}_w^W)$, $(F, a) \mapsto a \circ F$ (2.10). Let $D(\mathrm{gr}^W)^{(1)} := \prod_w D(\mathrm{gr}_w^W)^{(1)}$, and let $\mu : D(\mathrm{gr}^W)^{(1)} \times \bar{A}_P \xrightarrow{\simeq} D_{\mathrm{BS}}(\mathrm{gr}^W)(P(\mathrm{gr}^W)) := \prod_w D_{\mathrm{BS}}(\mathrm{gr}_w^W)(P_w)$ be the unique isomorphism of real analytic manifolds with corners which extends the isomorphism $D(\mathrm{gr}^W)^{(1)} \times A_P \xrightarrow{\simeq} D(\mathrm{gr}^W)$ of real analytic manifolds. Take $\mathbf{r} \in D(\mathrm{gr}^W)^{(1)}$.*

(i) *There is an open neighborhood U' of $\mathbf{r}$ in $D(\mathrm{gr}^W)$ and a real analytic map $\nu : U' \to G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $F = \nu(F)\mathbf{r}$ for any $F \in U'$.*

(ii) *Let U' and ν be as in* (i) *and let $U := \mu((D(\mathrm{gr}^W)^{(1)} \cap U') \times \bar{A}_P)$ which is an open set of $D_{\mathrm{BS}}(\mathrm{gr}^W)$. Let $L = L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$. Let $\tilde{U}$ be the inverse image of U in D_{BS} under the projection $D_{\mathrm{BS}} \to D_{\mathrm{BS}}(\mathrm{gr}^W)$. Then there is a unique isomorphism*

$$\tilde{U} \xrightarrow{\simeq} \mathrm{spl}(W) \times U \times \bar{L}$$

of real analytic manifolds with corners over $\mathrm{spl}(W) \times U$ *which sends $s(a \circ \theta(F, \delta)) \in D \cap \tilde{U}$ with $s \in \mathrm{spl}(W)$, $a \in A_P$, $F \in D(\mathrm{gr}^W)^{(1)} \cap U'$, and $\delta \in L_{\mathbf{R}}^{-1,-1}(F)$ to $(s, a \circ F, \mathrm{Ad}(\nu(F))^{-1}(\delta))$.* (*For the compactified vector space $\bar{L}$, see §7.*)

Proof. (i) is clear. We prove (ii). We define a map $\tilde{U} \to \bar{L}$ as follows. Consider the maps

(1) $$D_{\mathrm{BS}}(P)^{(B)} \simeq D^{(1,B)} \times \bar{B}_P = D^{(1,B)} \times \mathbf{R}_{\geq 0} \times \bar{A}_P \to D^{(1,B)} \times \mathbf{R}_{\geq 0},$$

(2) $$D_{\mathrm{BS}}(P)^{(A)} \simeq D^{(1,A)} \times \bar{A}_P \to D^{(1,A)},$$

where the isomorphisms are obtained as in 8.2 and the last arrows are the projections. These maps (1) and (2) induce maps

$$\tilde{U} \cap D_{\mathrm{BS}}(P)^{(B)} \to (\tilde{U} \cap D^{(1,B)}) \times \mathbf{R}_{\geq 0} \to (L - \{0\}) \times \mathbf{R}_{\geq 0} \to \bar{L} - \{0\}, \tag{3}$$

$$\tilde{U} \cap D_{\mathrm{BS}}(P)^{(A)} \to \tilde{U} \cap D^{(1,A)} \to L, \tag{4}$$

in which the second arrows are given by $s(\theta(F, \delta)) \mapsto \mathrm{Ad}(\nu(F))^{-1}\delta$ and the last arrow of (3) is induced by the case $V = L$ of the bijection 7.2 (1). The maps (3) and (4) induce the desired map $\tilde{U} \to \bar{L}$. As is easily seen, the maps (3) and (4) induce isomorphisms of real analytic manifolds with corners

$$\tilde{U} \cap D_{\mathrm{BS}}(P)^{(B)} \xrightarrow{\simeq} \mathrm{spl}(W) \times U \times (\bar{L} - \{0\}), \tag{5}$$

$$\tilde{U} \cap D_{\mathrm{BS}}(P)^{(A)} \xrightarrow{\simeq} \mathrm{spl}(W) \times U \times L. \tag{6}$$

These (5) and (6) show that the map $\tilde{U} \to \mathrm{spl}(W) \times U \times \bar{L}$ is an isomorphism of real analytic manifolds with corners. □

Corollary 8.4. *The isomorphism class of the graded* $\mathbf{R}$*-vector space* $L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$ *for* $\mathbf{r} \in D(\mathrm{gr}^W)$ *is independent of* $\mathbf{r}$. *Let* $L = L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$ *for a fixed* $\mathbf{r}$. *Then* $D_{\mathrm{BS}}(\mathrm{gr}^W)$ *is covered by open subsets* U *such that the inverse image of* U *in* D_{BS} *is isomorphic to* $\mathrm{spl}(W) \times U \times \bar{L}$ *as a real analytic manifold with corners over* $\mathrm{spl}(W) \times U$.

Proof. This follows from Theorem 8.3. □

By Corollary 8.4, we have

Corollary 8.5. *The canonical map* $D_{\mathrm{BS}} \to \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ *is an* $\bar{L}$*-bundle. In particular, it is proper and surjective.*

Sometimes, this bundle is canonically trivialized as follows.

8.6. We will consider the following three cases.

(a) The case where $h^{p,q} = 0$ unless $p = q$. (Example I in 1.10 is contained in (a).)

(b) The case where there is k such that $\mathrm{gr}_w = 0$ unless $w \in \{k, k-1\}$. (Example II in 1.11 is contained in (b).)

(c) The case where there is an odd integer k such that $\mathrm{gr}_w^W = 0$ unless $w \in \{k-1, k, k+1\}$, and $h^{p,q} = 0$ if $p + q \in \{k-1, k+1\}$ and $p \neq q$. (Example I in 1.10 and Example II in 1.11 are contained in (c). Many connected mixed Shimura varieties are D in the case (c).)

Proposition 8.7. (i) *The following three conditions are equivalent. (1) D is a $G_{\mathbf{R}}$-homogenious space. (2) D is a $G'_{\mathbf{R}}$-homogeneous space. Here $G'_{\mathbf{R}}$ is as in 0.3. (3) $D = D_{\mathrm{spl}}$.*

(ii) *In the case* (b)*, the equivalent conditions in* (i) *are satisfied.*

(iii) *If the equivalent conditions in* (i) *are satisfied, then*

$$D_{\mathrm{BS}} \xrightarrow{\simeq} \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W).$$

Proof. We prove (i) and (iii). Assume that the action of $G'_{\mathbf{R}}$ on D is transitive. We show that $D = D_{\mathrm{spl}}$. Let $F \in D$. Take any element F' of D_{spl}. Then $F = gF'$ for some $g \in G'_{\mathbf{R}}$ by assumption. Hence $F \in D_{\mathrm{spl}}$. Conversely assume $D = D_{\mathrm{spl}}$. Then, by Corollary 8.4, we have (iii). Hence the transitivity of the action of $G_{\mathbf{R}}$ on D follows from the transitivity of the action of $G_{\mathbf{R}}(\mathrm{gr}^W)$ on $D(\mathrm{gr}^W)$ (2.1) and the transitivity of the action of $G_{\mathbf{R},u}$ on $\mathrm{spl}(W)$ (3.3).

Finally it is well known that $D = D_{\mathrm{spl}}$ in the case (b) (see, for example, [KNU] 1.5 for the proof). □

Proposition 8.8. *Assume we are either in the case* (a) *or in the case* (c)*. Fix $\mathbf{r} \in D(\mathrm{gr}^W)$ and let $L = L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$. Then we have a global isomorphism*

$$D_{\mathrm{BS}} \simeq \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W) \times \bar{L}$$

of real analytic manifolds with corners.

Proof. In these cases (a), (c), the adjoint action of $G_{\mathbf{R}}(\mathrm{gr}^W)$ on $\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$ is trivial. Hence the subspace $L_{\mathbf{R}}^{-1,-1}(F)$ of $\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$ is independent of $F \in D(\mathrm{gr}^W)$ and the isomorphism in Theorem 8.3 (ii), which is given locally on $D_{\mathrm{BS}}(\mathrm{gr}^W)$, glues together to an isomorphism in this proposition. □

§9. Arithmetic quotients

The purpose of this section is to prove the following Theorem 9.1.

Theorem 9.1. (i) *For a subgroup Γ of $G_{\mathbf{Z}}$, the action of Γ on D_{BS} is proper, and the quotient space $\Gamma\backslash D_{\mathrm{BS}}$ is locally compact (in particular, it is Hausdorff). If Γ is neat, the projection $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$ is a local homeomorphism, and $\Gamma\backslash D_{\mathrm{BS}}$ has a unique structure of a real analytic manifold with corners for which $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$ is locally an isomorphism.*

(ii) *If Γ is a subgroup of $G_{\mathbf{Z}}$ of finite index, the quotient space $\Gamma\backslash D_{\mathrm{BS}}$ is compact.*

(iii) *If Γ is a subgroup of $G_{\mathbf{Z}}$ such that $\Gamma_u := \Gamma \cap G_{\mathbf{Z},u}$ is of finite index in $G_{\mathbf{Z},u}$, the projection $\Gamma\backslash D_{\mathrm{BS}} \to (\Gamma/\Gamma_u)\backslash D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper. In particular, the map $G_{\mathbf{Z},u}\backslash D_{\mathrm{BS}} \to D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper.*

Here in (i), the meaning of "neat" is as follows. A subgroup Γ of $G_{\mathbf{Z}}$ is said to be neat if for any $\gamma \in \Gamma$, the subgroup of $\mathbf{C}^\times$ generated by all eigenvalues of the action of γ on $H_{0,\mathbf{C}}$ is torsion free. If Γ is neat, then Γ is torsion free. There exists a neat subgroup of $G_{\mathbf{Z}}$ of finite index (cf. [B]).

The proof of Theorem 9.1 is given in 9.2–9.7.

9.2. Since $D_{\mathrm{BS}} \to \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper (8.5) and since $D_{\mathrm{BS}}(\mathrm{gr}^W)$ is Hausdorff (2.11) as well as $\mathrm{spl}(W)$ (3.3), D_{BS} is a Hausdorff space.

9.3. We prove that for any subgroup Γ of $G_{\mathbf{Z}}$, the action of Γ on D_{BS} is proper. The canonical continuous map $D_{\mathrm{BS}} \to \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ is compatible with the actions of Γ, where $\gamma \in \Gamma$ acts on $\mathrm{spl}(W)$ as $s \mapsto \gamma s\, \mathrm{gr}^W(\gamma)^{-1}$ ($s \in \mathrm{spl}(W)$), and acts on $D_{\mathrm{BS}}(\mathrm{gr}^W)$ through $\Gamma \to \Gamma/\Gamma_u \to G_{\mathbf{Z}}(\mathrm{gr}^W)$ in the natural way. Since $\mathrm{spl}(W)$ is a $G_{\mathbf{R},u}$-torsor (3.3), the action of Γ_u on $\mathrm{spl}(W)$ is proper. Since the action of Γ/Γ_u on $D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper (2.11) and the action of Γ_u on $\mathrm{spl}(W)$ is proper, the action of Γ on $\mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper. Since D_{BS} is Hausdorff (9.2), this shows that the action of Γ on D_{BS} is proper.

9.4. Since the action of Γ on D_{BS} is proper (9.3), it follows that the quotient space $\Gamma\backslash D_{\mathrm{BS}}$ is Hausdorff. Since D_{BS} is locally compact, this quotient space is also locally compact.

9.5. We prove that if Γ is a neat subgroup of $G_{\mathbf{Z}}$, then the map $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$ is a local homeomorphism. This will show that $\Gamma\backslash D_{\mathrm{BS}}$ has a unique structure of a real analytic manifold with corners for which $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$ is locally an isomorphism.

By 9.3, it is sufficient to prove that if $p \in D_{\mathrm{BS}}$ and $\gamma \in \Gamma$ satisfy $\gamma p = p$, then $\gamma = 1$. We have $\gamma p(\mathrm{gr}^W) = p(\mathrm{gr}^W)$ in $D_{\mathrm{BS}}(\mathrm{gr}^W)$. Since $\bar\gamma\bar p = \bar p$ with $\bar\gamma \in \Gamma/\Gamma_u$ and $\bar p \in D_{\mathrm{BS}}(\mathrm{gr}^W)$ implies $\bar\gamma = 1$ (2.11), we have $\gamma \in \Gamma_u$. By applying $\mathrm{spl}_W : D_{\mathrm{BS}} \to \mathrm{spl}(W)$ to $\gamma p = p$, we have $\gamma\, \mathrm{spl}_W(p) = \mathrm{spl}_W(p)$. Since $\mathrm{spl}(W)$ is a $G_{\mathbf{R},u}$-torsor, we have $\gamma = 1$.

Thus we have proved (i) of Theorem 9.1.

9.6. We prove (iii) of Theorem 9.1. Let Γ be a subgroup of $G_{\mathbf{Z}}$ such that Γ_u is of finite index in $G_{\mathbf{Z},u}$. The quotient space $\Gamma_u\backslash G_{\mathbf{R},u}$ is compact as is easily seen. Since $\mathrm{spl}(W)$ is a $G_{\mathbf{R},u}$-torsor, the quotient space $\Gamma_u\backslash \mathrm{spl}(W)$ is also compact. Hence the map $\Gamma_u\backslash(\mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)) = (\Gamma_u\backslash \mathrm{spl}(W)) \times D_{\mathrm{BS}}(\mathrm{gr}^W) \to D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper. Since $D_{\mathrm{BS}} \to \mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper, the map $\Gamma_u\backslash D_{\mathrm{BS}} \to \Gamma_u\backslash(\mathrm{spl}(W) \times D_{\mathrm{BS}}(\mathrm{gr}^W))$ is also proper. Hence the composition $\Gamma_u\backslash D_{\mathrm{BS}} \to D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper. Dividing by Γ/Γ_u, we have that the map $\Gamma\backslash D_{\mathrm{BS}} \to (\Gamma/\Gamma_u)\backslash D_{\mathrm{BS}}(\mathrm{gr}^W)$ is proper.

9.7. Theorem 9.1 (ii) follows from (iii) and from the fact that $\Gamma'\backslash D_{\mathrm{BS}}(\mathrm{gr}^W)$ is compact for any subgroup Γ' of $G_{\mathbf{Z}}(\mathrm{gr}^W)$ of finite index (2.11).

§10. Examples

10.1. Consider Example I in 1.10. The space D_{BS} is described as follows.

10.1.1. We have a commutative diagram of real analytic manifolds with corners

$$\begin{array}{ccc} D & = & \mathbf{C} \\ \cap & & \cap \\ D_{\mathrm{BS}} & \simeq & X := \{x+iy \mid x \in \mathbf{R}, -\infty \le y \le \infty\} \end{array}$$

which extends the identification $D = \mathbf{C}$ in 1.10. Here X is regarded as a real analytic manifold with corners via the bijection $\mathbf{R} \times [-\infty, \infty] \xrightarrow{\simeq} X,\ (x, y) \mapsto x + iy$, where $[-\infty, \infty]$ has the structure of a real analytic manifold with corners defined in 7.5 with $w = -2$.

10.1.2. The projection $D_{\mathrm{BS}} \to \mathrm{spl}(W)$ corresponds to $X \to \mathrm{spl}(W),\ x + iy \mapsto s_x$, where $s_x(e_2 \bmod W_{-1}) = xe_1 + e_2$.

10.1.3. Let $P = G_{\mathbf{R},u}$. Note that P is the unique parabolic subgroup of $G_{\mathbf{R}}$, $A_P = \{1\}$ and $B_P = \mathbf{R}_{>0}$. The Borel-Serre action of $b \in \mathbf{R}_{>0} = B_P$ on D corresponds to the action $x + iy \mapsto x + ib^{-2}y$ on X.

10.1.4. The element of D_{BS} corresponding to $x + i\infty \in X$ $(x \in \mathbf{R})$ is (P, Z), where Z is the B_P-orbit $x + i\mathbf{R}_{>0}$ in $\mathbf{C} = D$, and the element of D_{BS} corresponding to $x - i\infty \in X$ is (P, Z), where Z is the B_P-orbit $x - i\mathbf{R}_{>0}$ in $\mathbf{C} = D$.

10.2. Consider Example II in 1.11. In this case, $W_0 = H_{0,\mathbf{R}}$ and $W_{-2} = 0$. Let P be the parabolic subgroup of $G_{\mathbf{R}}$ consisting of all elements g of $G_{\mathbf{R}}$ such that $\mathrm{gr}^W_0(g) = 1$ and such that $\mathrm{gr}^W_{-1}(g)$ preserves $\mathbf{R}e_1$. The space $D_{\mathrm{BS}}(P)$ is described as follows.

10.2.1. We have a commutative diagram of real analytic manifolds with corners

$$\begin{array}{ccccc} D & = & \mathfrak{h} \times \mathbf{C} & \simeq & \mathbf{R}^3 \times \mathbf{R}_{>0} \\ \cap & & & & \cap \\ D_{\mathrm{BS}}(P) & & \simeq & & \mathbf{R}^3 \times \mathbf{R}_{\ge 0} \end{array}$$

which extends the identification $D = \mathfrak{h} \times \mathbf{C}$ in 1.11, where the right upper horizontal isomorphism $\mathfrak{h} \times \mathbf{C} \simeq \mathbf{R}^3 \times \mathbf{R}_{>0}$ sends $(s_1, s_2, x, r) \in \mathbf{R}^3 \times \mathbf{R}_{>0}$ to $(x + ir^{-2}, s_1 - (x + ir^{-2})s_2) \in \mathfrak{h} \times \mathbf{C}$. As an enlargement of $D(\mathrm{gr}^W) = \mathfrak{h}$, $D_{\mathrm{BS}}(\mathrm{gr}^W)(P(\mathrm{gr}^W))$ is identified with $\{x + iy \mid x \in \mathbf{R}, 0 < y \le \infty\}$ whose structure as a real analytic manifold with corners is defined by the bijection $(x, r) \mapsto x + ir^{-2}$ from $\mathbf{R} \times \mathbf{R}_{\ge 0}$, and the projection $D_{\mathrm{BS}}(P) \to D_{\mathrm{BS}}(\mathrm{gr}^W)(P(\mathrm{gr}^W))$ corresponds to the map $\mathbf{R}^3 \times \mathbf{R}_{\ge 0} \to \{x + iy \mid x \in \mathbf{R}, 0 < y \le \infty\}$ defined by $(s_1, s_2, x, r) \mapsto x + ir^{-2}$ $(s_1, s_2, x \in \mathbf{R}, r \in \mathbf{R}_{\ge 0})$.

10.2.2. The projection $D_{\mathrm{BS}}(P) \to \mathrm{spl}(W)$ corresponds to $(s_1, s_2, x, r) \mapsto s$, where $s(e_3 \bmod W_{-1}) = s_1e_1 + s_2e_2 + e_3$.

10.2.3. We have $A_P \simeq \mathbf{R}_{>0}$ and $B_P = \mathbf{R}_{>0} \times A_P = \mathbf{R}_{>0} \times \mathbf{R}_{>0}$. The Borel-Serre action of $(r_1, r_2) \in B_P$ $(r_1, r_2 \in \mathbf{R}_{>0})$ on D corresponds to $\mathbf{R}^3 \times \mathbf{R}_{>0} \to \mathbf{R}^3 \times \mathbf{R}_{>0},\ (s_1, s_2, x, r) \mapsto (s_1, s_2, x, r_2r)$.

10.2.4. The element of $D_{\mathrm{BS}}(P)$ corresponding to $(s_1, s_2, x, 0)$ is (P, Z), where Z is the A_P-orbit in D (which is also a B_P-orbit) corresponding to $(s_1, s_2, x, \mathbf{R}_{>0})$.

10.3. Consider Example III in 1.12.

Let P be the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ consisting of all elements g such that $\mathrm{gr}_0^W(g) = 1$ and such that $\mathrm{gr}_{-3}^W(g)$ preserves $\mathbf{R}e_1$. Let $L = \mathbf{R}^2$. The space $D_{\mathrm{BS}}(P)$ is described as follows.

10.3.1. We have a commutative diagram of real analytic manifolds with corners

$$\begin{array}{ccccc} D & = & \mathfrak{h} \times \mathbf{C}^2 & \simeq & \mathbf{R}^3 \times \mathbf{R}_{>0} \times L \\ \cap & & & & \cap \\ D_{\mathrm{BS}}(P) & & \simeq & & \mathbf{R}^3 \times \mathbf{R}_{\geq 0} \times \bar{L} \end{array}$$

which extends the identification $D = \mathfrak{h} \times \mathbf{C}^2$ in 1.12. Here the right upper horizontal isomorphism sends $(s_1, s_2, x, r, d) \in \mathbf{R}^3 \times \mathbf{R}_{>0} \times L$ to

$$(x + ir^{-2},\ s_1 + r^{-1}(id_1 - \tfrac{1}{2}d_2) + xr(id_2 + \tfrac{1}{2}d_1),\ s_2 + r(id_2 + \tfrac{1}{2}d_1)) \in \mathfrak{h} \times \mathbf{C}^2$$

($s_1, s_2, x \in \mathbf{R}$, $r \in \mathbf{R}_{>0}$, $d = (d_1, d_2) \in L$ with $d_1, d_2 \in \mathbf{R}$), and $\bar{L}$ is the compactification of L in §7 regarding L as being of pure weight -3.

The projection $D_{\mathrm{BS}}(P) \simeq \mathbf{R}^3 \times \mathbf{R}_{\geq 0} \times \bar{L} \to D_{\mathrm{BS}}(\mathrm{gr}^W)(P(\mathrm{gr}^W)) = \{x + iy \mid x \in \mathbf{R}, 0 < y \leq \infty\}$ is given by $(s_1, s_2, x, r, d) \mapsto x + ir^{-2}$ ($s_1, s_2, x \in \mathbf{R}, r \in \mathbf{R}_{\geq 0}, d \in \bar{L}$).

10.3.2. The projection $D_{\mathrm{BS}}(P) \to \mathrm{spl}(W)$ corresponds to $(s_1, s_2, x, r, d) \mapsto s$, where $s(e_3 \bmod W_{-1}) = s_1e_1 + s_2e_2 + e_3$.

10.3.3. We have $A_P = \mathbf{R}_{>0}$, $B_P = \mathbf{R}_{>0} \times \mathbf{R}_{>0}$. The Borel-Serre action of $(r_1, r_2) \in B_P$ ($r_1, r_2 \in \mathbf{R}_{>0}$) on $D_{\mathrm{BS}}(P)$ corresponds to $(s_1, s_2, x, r, d) \mapsto (s_1, s_2, x, r_2r, r_1^{-3}d)$.

10.3.4. The element of $D_{\mathrm{BS}}(P)$ corresponding to $(s_1, s_2, x, 0, d)$ ($d \in L$) is (P, Z), where Z is the A_P-orbit in D corresponding to $(s_1, s_2, x, \mathbf{R}_{>0}, d)$. The element of $D_{\mathrm{BS}}(P)$ corresponding to $(s_1, s_2, x, r, \infty d)$ ($r \in \mathbf{R}_{>0}$, $d \in L$, $d \neq 0$, ∞d denotes the limit in $\bar{L}$ of $td \in L$ for $t \in \mathbf{R}_{>0}$, $t \to \infty$) is (Q, Z), where $Q = \{g \in G_{\mathbf{R}} \mid \mathrm{gr}_0^W(g) = 1\}$ and Z is the B_Q-orbit in D corresponding to $(s_1, s_2, x, r, \mathbf{R}_{>0}d)$. The element of $D_{\mathrm{BS}}(P)$ corresponding to $(s_1, s_2, x, 0, \infty d)$ ($d \in L$, $d \neq 0$) is (P, Z), where Z is the B_P-orbit in D corresponding to $(s_1, s_2, x, \mathbf{R}_{>0}, \mathbf{R}_{>0}d)$.

We prove these assertions 10.3.1–10.3.4.

Let $\mathbf{r} \in D(\mathrm{gr}^W) = \mathfrak{h}$ be the point $i \in \mathfrak{h}$. We have $L_{\mathbf{R}}^{-1,-1}(\mathbf{r}) \simeq L = \mathbf{R}^2$, where $(d_1, d_2) \in L$ ($d_j \in \mathbf{R}$) corresponds to the element $\delta \in L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$ which sends (e_3 mod W_{-1}) to $d_1e_1 + d_2e_2$ and sends e_1, e_2 to 0. We will identify L with $L_{\mathbf{R}}^{-1,-1}(\mathbf{r})$ via this isomorphism. From the formula

$$\zeta = \tfrac{i}{2}\delta_{-2,-1} - \tfrac{i}{2}\delta_{-1,-2},$$

we see that $\theta(\mathbf{r}, d)$ is given by

$$\theta(\mathbf{r}, d)^1 = 0 \ \subset\ \theta(\mathbf{r}, d)^0 = \mathbf{C}(id_1e_1 + id_2e_2 + \tfrac{1}{2}(d_1e_2 - d_2e_1) + (e_3 \bmod W_{-1}))$$

$$\subset\ \theta(\mathbf{r}, d)^{-1} = \theta(\mathbf{r}, d)^0 + \mathbf{C}(ie_1 + e_2)\ \subset\ \theta(\mathbf{r}, d)^{-2} = \mathrm{gr}_{\mathbf{C}}^W\,.$$

From this we see that the composition of the upper horizontal isomorphisms in the diagram in 10.3.1 sends $(s_1, s_2, x, r, d) \in \mathbf{R}^3 \times \mathbf{R}_{>0} \times L$ to

$$s(g_x t(r)\theta(\mathbf{r}, d)),$$

where s is the splitting of W corresponding to (s_1, s_2), and $g_x, t(r)$ are the elements of $P(\mathrm{gr}^W)$ defined by

$$g_x(e_1) = e_1, \quad g_x(e_2) = xe_1 + e_2, \quad g_x(e_3 \bmod W_{-1}) = (e_3 \bmod W_{-1}),$$

$$t(r)(e_1) = r^{-1}e_1, \quad t(r)(e_2) = re_2, \quad t(r)(e_3 \bmod W_{-1}) = (e_3 \bmod W_{-1}).$$

This proves the assertions.

In Examples II, III, any $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ other than $\{g \in G_{\mathbf{R}} \mid \mathrm{gr}_0^W(g) = 1\}$ is $G_{\mathbf{Q}}$-conjugate to P. Hence the above 10.2 and 10.3 give local descriptions of D_{BS} at all points of D_{BS} for these examples.

In 10.4 and 10.5, in the cases of Example I and Example II, respectively, we introduce shapes of other enlargements of D or $\Gamma\backslash D$ in Introduction.

10.4. Remark. In the rather simple case Example I in 1.10, the mixed Hodge theoretic version of the diagram $(*)$ in Introduction will be seen to be just:

$$\begin{array}{ccccccc}
 & & & & X & = & X \\
 & & & & \| & & \| \\
\mathbf{P}^1(\mathbf{C}) & \leftarrow & X & = & X & & X \\
\| & & \| & & & & \\
\mathbf{P}^1(\mathbf{C}) & \leftarrow & X. & & & &
\end{array}$$

Here X is as in 10.1, $\Gamma = G_{\mathbf{Z},u}$, and Σ is chosen suitably. The map $X \to \mathbf{P}^1(\mathbf{C})$ sends $z \in \mathbf{C} \subset X$ to $\exp(2\pi iz) \in \mathbf{C}^\times \subset \mathbf{P}^1(\mathbf{C})$, the point $x + i\infty \in X$ $(x \in \mathbf{R})$ to $0 \in \mathbf{P}^1(\mathbf{C})$, and the point $x - i\infty \in X$ to $\infty \in \mathbf{P}^1(\mathbf{C})$.

10.5. Remark. In the case of Example II in 1.11, for a torsion free subgroup of Γ' of $\mathrm{SL}(2, \mathbf{Z})$ which contains the kernel of $\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/N\mathbf{Z})$ for some $N \geq 1$, and for the inverse image $\Gamma \subset G_{\mathbf{Z}}$ of Γ' under $G_{\mathbf{Z}} \to G_{\mathbf{Z}}(\mathrm{gr}^W) = \mathrm{SL}(2, \mathbf{Z})$, $\Gamma\backslash D$ is the universal elliptic curve over the modular curve $\Gamma'\backslash\mathfrak{h}$, and for a suitable Σ, $\Gamma\backslash D_\Sigma$ is a toroidal compactification of $\Gamma\backslash D$. In this case, $D_{\mathrm{SL}(2)} = D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS},\mathrm{val}} = D_{\mathrm{BS}} = D_{\mathrm{BS}}(\mathrm{gr}^W) \times \mathbf{R}^2$. We do not discuss here the other spaces in the diagram $(*)$ in Introduction in this case.

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